An infinite swampland of $\mathrm{U}(1)$ charge spectra in 6D supergravity theories

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ABSTRACT: We analyze the anomaly constraints on 6D supergravity theories with a single abelian $\mathrm{U}(1)$ gauge factor. For theories with charges restricted to $q = \pm1, \pm2$ and no tensor multiplets, anomaly-free models match those models that can be realized from F-theory compactifications almost perfectly. For theories with tensor multiplets or with larger charges, the F-theory constraints are less well understood. We show, however, that there is an infinite class of distinct massless charge spectra in the "swampland" of theories that satisfy all known quantum consistency conditions but do not admit a realization through F-theory or any other known approach to string compactification. We also compare the spectra of charged matter in abelian theories with those that can be realized from breaking nonabelian $\mathrm{SU}(2)$ and higher rank gauge symmetries.

Contents

1 Introduction

The largest spacetime dimension in which a supersymmetric theory can have matter fields in any representation of the gauge group other than the adjoint is six. This makes six dimensions a natural place to begin to try to systematically understand the structure of matter in physical theories with supersymmetry. Furthermore, in six dimensions, gravitational, gauge, and mixed anomalies strongly constrain the gauge group and matter content [1, 2], and additional quantum consistency constraints [2–4] further limit the set of possible low-energy supergravity theories. The theories that satisfy these constraints match fairly closely in certain regimes with the set of theories that can be realized from string theory compactifications through F-theory, suggesting that any consistent low-energy 6D supergravity theory may arise from string theory [5]. While there are still a number of ways in which the set of theories that satisfy all known quantum consistency theories is larger than the set of known string vacua, understanding the "swampland" [6, 7] of apparently consistent theories with no description in terms of any known class of string theories has been a productive approach to developing our understanding of both the set of quantum consistency conditions on gravity theories and the structure of string theory vacua.

In this paper we investigate the constraints that 6D anomaly cancellation conditions place on the charged matter content of theories with a single U(1) gauge field. For theories with only nonabelian gauge fields, it is known that (at least for theories with $T < 9$ tensor multiplets) there are a finite number of distinct massless spectra of gauge fields and charged matter representations that are consistent with anomaly cancellation and other simple constraints such as the proper sign of the gauge kinetic term for all gauge fields [3, 8]. One of the goals of this paper is to inquire whether a similar finiteness bound can be demonstrated for 6D theories with abelian gauge factors. The short answer is that it seems that it cannot, at least without some further constraints. We find that there indeed are infinite families of theories with distinct sets of U(1)-charged matter fields that appear consistent from anomaly cancellation and other known constraints, even restricting to the simplest class of models with $T = 0$ tensor multiplets. On the other hand, it is known that F-theory, the most general approach known to constructing string vacua in six dimensions, gives only a finite number of distinct possible massless spectra [8]. Thus, it seems that there is an infinite swampland of U(1) theories in six dimensions.

While in this paper we focus primarily on quantum consistency constraints from anomalies, independent of any UV completion of the theory, we also consider at various places in the paper additional constraints that may limit the set of theories from the F-theory point of view. In particular, for theories with an abelian U(1) gauge group and

matter restricted to charges $q = \pm 1, \pm 2$, and for corresponding unHiggsed theories with SU(2) gauge groups, standard F-theory constructions give constraints that are stronger than but similar in structure to the constraints from anomaly cancellation. One particular question that we address more generally for many of the anomaly-free U(1) spectra we find is whether a given spectrum of U(1)-charged matter fields can be realized by breaking through the Higgs mechanism a nonabelian theory with $SU(N)$ or other gauge group with matter charged in various representations. This is a useful perspective in particular because nonabelian theories are much better understood, both in terms of low-energy constraints and in terms of F-theory realizations. In addition to providing insight on swampland-type questions, understanding the allowed structure of abelian matter charges and their relationship with nonabelian theories may also provide new insight into the difficult problem of understanding how F-theory solutions with U(1) factors and various matter charges can arise through explicit Weierstrass model constructions.

Note that different authors use the term "swampland" in slightly different ways. Here, to be precise, we define the swampland to be the set of theories that obey all *known* quantum consistency conditions yet cannot be realized in any *known* approach to string compactification. We focus here on massless spectra of 6D $\mathcal{N} = 1$ supergravity theories. Note that with this definition, the swampland is a time-dependent class of theories; discovery of a new quantum consistency condition or a new string construction can reduce the swampland in scope. Note also that while for some spectra it can be argued that no construction is possible in F-theory (as currently formulated) or in any other known approach to 6D string compactification, for other theories it is not known whether an F-theory construction is possible; in the latter situation, we refer to such models here as "possible" swampland models, where the adjective "possible" reflects specifically on our incomplete understanding of how to answer the well-defined mathematical question of whether one of the finite set of distinct Weierstrass models over an allowed F-theory base surface can realize a specific massless spectrum.

We systematically investigate the structure of 6D supergravity theories with a single U(1) gauge field and their possible matter spectra. We begin with some background on nonabelian and abelian anomaly conditions in 6D in section 2, where we characterize generic spectra for theories with a single $SU(N)$ or U(1) gauge group in terms of a pair of parameters appearing in the anomaly cancellation conditions. In section 3, we show that the $SU(N)$ and U(1) anomaly conditions can be directly related, giving closely parallel constraints to the charged spectra. In section 4, we focus on models with abelian matter with charges $q = \pm 1, \pm 2$ only. We show that for theories with no tensor multiplets ($T = 0$), the constraints from anomaly cancellation directly match simple UV constraints from F-theory and there is no swampland except for one specific model where the global structure of the gauge group becomes relevant. For theories with more tensor multiplets the anomaly cancellation conditions of the low-energy theory appear in a stronger form for standard F-theory constructions, though this stronger form can be violated for more exotic F-theory constructions. In section 5, we include charges $|q| > 2$ and identify an infinite swampland of charge spectra, even for theories without tensor multiplets. We then consider how the number of anomaly-consistent spectra grows as the natural parameter from the anomaly

constraints increases. In section 6, we consider more explicitly theories with charge $q = \pm 3, \pm 4, \pm 5$ matter, in order to further understand the types of models that appear in the "swampland." In section 7, we summarize our results on the spectra that inhabit the swampland, and we finish with concluding remarks in section 8.

As this work was being completed, we received a preliminary version of [9], in which general constraints on 6D supergravity theories from anomaly cancellation are analyzed, and which has some overlap with the analysis of this paper — in particular, some of the conditions we identify and use in this paper are special cases of the more general conditions analyzed in that paper.

2 Background

In six dimensions, gauge, gravitational, and mixed anomaly cancellation imposes fairly strict constraints on the spectrum of gauge fields and matter content that can be coupled to gravity in a consistent quantum theory [1, 2]. We briefly review in this section the local anomaly cancellation conditions for a 6D supergravity theory with a single SU(N) or U(1) gauge field. We use these conditions to parameterize the theories with the simplest, generic matter content, and describe how theories containing matter with higher charges can be related to generic models.

2.1 Anomaly cancellation conditions

For the purposes of this paper, we are interested in single SU(N) or U(1) gauge group factors, so we will make use of the anomaly cancellation (AC) conditions only for these cases.

2.1.1 Anomalies for SU(N) models

Consider a single SU(N) gauge group factor, and let x_R be the number of hypermultiplets transforming in the irreducible representation R of dimension d_R. The numbers of massless vector multiplets and hypermultiplets, denoted V and H respectively, are then given by

$$V = d_{\text{Adj}} = N^2 - 1, \quad H = \sum_R x_R d_R, \tag{2.1}$$

The number of tensor multiplets is denoted by T.

The structure of the anomaly polynomial for 6D supergravity gives the gravitational and nonabelian AC conditions [1, 2], which in the notation of [8] can be written as

$$273 = H - V + 29T, \tag{2.2a}$$

$$a \cdot a = 9 - T, \tag{2.2b}$$

$$a \cdot b = -\frac{1}{6}\left(\sum_R x_R A_R - A_{\text{Adj}}\right), \tag{2.2c}$$

$$0 = \sum_R x_R B_R - B_{\text{Adj}}, \tag{2.2d}$$

$$b \cdot b = \frac{1}{3}\left(\sum_R x_R C_R - C_{\text{Adj}}\right). \tag{2.2e}$$

R	Dimension	A_R	B_R	C_R
□	N	1	1	0
Adjoint	N^2-1	$2N$	$2N$	6
⊟	$\frac{N(N-1)}{2}$	$N-2$	$N-8$	3
⊞	$\frac{N(N+1)(N-1)}{3}$	N^2-3	N^2-27	$6N$
⊡⊡	$\frac{N(N+1)}{2}$	$N+2$	$N+8$	3
⊡⊡⊡	$\frac{N(N+1)(N+2)}{6}$	$\frac{(N+2)(N+3)}{2}$	$\frac{N^2+17N+54}{2}$	$3(N+4)$
⊡⊡⊡⊡	$\frac{N(N+1)(N+2)(N+3)}{24}$	$\frac{(N+2)(N+3)(N+4)}{6}$	$\frac{(N+4)\left(N^2+23N+96\right)}{6}$	$\frac{3(N+4)(N+5)}{2}$

Table 1. Values of the group theory coefficients A_R, B_R, and C_R, and the dimensions, associated with a variety of representations R of SU(N), for $N \geq 4$. For SU(2) and SU(3), no quartic Casimir exists, so $B_R = 0$. In this case, the value of A_R is as given, and the value of C_R is obtained by taking $C_R + B_R/2$ using the values given here.

Here, a and b are vectors in a $(1+T)$-dimensional real vector space that carries a symmetric bilinear form Ω of signature $(1, T)$; the notation $x \cdot y$ denotes the associated SO$(1, T)$-invariant product $\Omega_{\alpha\beta} x^\alpha y^\beta$. For each representation R of SU(N), the group theory coefficients A_R, B_R, and C_R are defined by

$$\operatorname{tr}_R F^2 = A_R \operatorname{tr} F^2\,, \quad \operatorname{tr}_R F^4 = B_R \operatorname{tr} F^4 + C_R \left(\operatorname{tr} F^2\right)^2\,, \tag{2.3}$$

where tr denotes the trace in the fundamental representation and tr_R denotes the trace in an arbitrary representation R. These group theory coefficients can be computed for any SU(N) in a manner described in [3, 10, 11], among other sources. Values for several representations are given in table 1.

Note that the anomaly equations really depend only on the local structure of the gauge algebra. Thus, for example, the anomaly conditions are the same for SU(2) and SO(3). For the most part we ignore the global structure of the gauge group, but a case where this distinction becomes relevant is discussed in section 4.3.

2.1.2 Anomalies for U(1) models

Consider now a single U(1) gauge factor. The irreducible representations of U(1) have charges $q \in \mathbb{Z}$; we define x_q to be the number of hypermultiplets transforming in the charge-q irreducible representation of U(1). The anomaly polynomial in this case yields the abelian AC equations [10, 12],

$$a \cdot \tilde{b} = -\frac{1}{6} \sum_{n>0} x_q q^2\,, \tag{2.4a}$$

$$\tilde{b} \cdot \tilde{b} = \frac{1}{3} \sum_{n>0} x_q q^4\,. \tag{2.4b}$$

Here, $\tilde{b}$ is once again an $SO(1, T)$ vector. We sum over only $q > 0$ because each charge-q hypermultiplet contains a field with charge $+q$ and a field with charge $-q$. Note that in these equations, $q = 1$ is not necessarily the fundamental unit charge for the U(1). We discuss this further in section 2.2.

2.1.3 Geometric interpretation in F-theory

From the F-theory perspective, the quantities appearing in the nonabelian AC conditions (2.2) have a geometric interpretation [8, 13]. For an F-theory compactification on a base B with canonical class K and nonabelian gauge group $SU(N)$ associated with a codimension one fiber singularity over a divisor in the homology class Σ in B, we can identify a with the canonical class K and b with the divisor class Σ. In this case, the $SO(1, T)$ product is identified with the intersection product on divisor classes. Thus, we have

$$a \cdot a = K \cdot K \,, \tag{2.5a}$$

$$a \cdot b = K \cdot \Sigma \,, \tag{2.5b}$$

$$b \cdot b = \Sigma \cdot \Sigma \,. \tag{2.5c}$$

Defining the self-intersection of Σ to be $n := \Sigma \cdot \Sigma$, we can then use the Riemann-Roch formula,

$$2(g - 1) = \Sigma \cdot (K + \Sigma) = n + K \cdot \Sigma \,, \tag{2.6}$$

where g is the genus of curve Σ, to rewrite (2.2) as

$$273 = H - V + 29T \,, \tag{2.7a}$$

$$K \cdot K = 9 - T \,, \tag{2.7b}$$

$$2(g - 1) - n = -\frac{1}{6}\left(\sum_R x_R A_R - A_{\text{Adj}}\right) , \tag{2.7c}$$

$$0 = \sum_R x_R B_R - B_{\text{Adj}} \,, \tag{2.7d}$$

$$n = \frac{1}{3}\left(\sum_R x_R C_R - C_{\text{Adj}}\right) . \tag{2.7e}$$

When the gauge group is abelian and the only gauge group factor is a single U(1), the geometric interpretation of $\tilde{b}$, as elaborated in [14], is

$$\tilde{b} = 2(-K + [z]) \,, \tag{2.8}$$

where $[z]$ is the divisor class of the z-component of the section that generates the Mordell-Weil group of rational sections of the elliptic fibration. In an F-theory model, therefore, $\tilde{b}$ must be both effective and even. In analogy with the nonabelian case, we define $\tilde{n}, \gamma$ through

$$\tilde{b} \cdot \tilde{b} = \tilde{n} \,, \quad a \cdot \tilde{b} = -\gamma \,. \tag{2.9}$$

The abelian AC conditions (2.4) then become

$$6\gamma = \sum_q x_q q^2 \,, \tag{2.10a}$$

$$3\tilde{n} = \sum_q x_q q^4 \,. \tag{2.10b}$$

A unified geometric description of the abelian and nonabelian anomaly conditions in the framework of F-theory is given in [15].

2.2 Additional constraints

In addition to the local anomaly cancellation conditions, we trivially have integrality and non-negativity conditions on x_R, x_q: the number of hypermultiplets transforming in a given representation must be a non-negative integer. The exception is in the case of quaternionic representations of a nonabelian gauge group. We can have so-called "half-hypermultiplets" transforming in these representations, which have half the field content of a full hypermultiplet; a full hypermultiplet transforming in the representation R contains an equal number of fields transforming in both R and in $\bar{R}$, while a half-hypermultiplet has half the field content, all transforming only under R. For our purposes, this amounts to allowing x_R to be a (non-negative) half-integer for such representations. (Note that all even-dimensional irreducible representations of SU(2) are quaternionic.)

In addition to the local anomaly cancellation conditions we have described above, there are also global anomalies that must cancel. In particular [9, 16–19], for SU(2) and SU(3) theories with only fundamental and adjoint matter fields, the number of fields in the fundamental representation must be congruent to 4 and 0 mod 6, respectively. This condition is automatically satisfied by the generic spectra described in the next section. There are no global anomalies for larger SU(N) groups. Since, as described in [16], the global anomaly constraints depend only on the net contribution from all matter representations to the group theory coefficients A_R, B_R, C_R, we expect that global anomalies are therefore also satisfied in all theories we consider here that satisfy the local anomaly cancellation conditions.

We now consider integrality constraints on the parameters $n, g, \tilde{n}, \gamma$, and on $a, b, \tilde{b}$. We begin with the nonabelian parameters. By simply using the properties of the group theory coefficients A_R, B_R, C_R, it was shown in [8] that $n = b \cdot b$ and $a \cdot b$ must take integer values in any theory satisfying the AC conditions. Thus, n must be an integer for any anomaly-free theory. A similar argument [9] shows that $a \cdot b + b \cdot b \in 2\mathbb{Z}$ directly from the group theory coefficients, so that g is always an integer. In [4], it was shown that the lattice Γ of string charges coupled to the two-form fields of the theory, with Dirac product defined by Ω, is a unimodular lattice. The structure of the Green-Schwarz terms in the 6D supergravity action suggests that a, b, which are a priori simply vectors in the real vector space $\mathbb{R}^{1,T}$ with inner product Ω, must in fact be elements of the string charge lattice Γ, as they are associated with the string charges of gauge and gravitational instantons, respectively; a similar consideration in the abelian sector suggests that $\tilde{b}$ should also be an element of the lattice Γ, and hence that $\tilde{n}$ and γ are integers, although the anomaly conditions only directly constrain, e.g., that $3\tilde{n}$ is an integer. (For example, the abelian equations are

solved by $\tilde{n} = 19/3, \gamma = 7/6, x_1 = 3, x_2 = 1$, although there is no theory with an integral lattice Γ and $\tilde{b} \in \Gamma$ that gives these values). The string charge lattice also has associated with it a positive cone supporting BPS strings, and we expect that $-a$, b, and $\tilde{b}$ lie in this positive cone. The consistency constraints on this positive cone are not well understood from the supergravity point of view, although in this paper this condition is primarily only relevant in the discussion of $T = 1$ models in section 4.4. Note that in most of the examples we consider here, we take $T = 0$, where, as discussed in section 4.3, $\tilde{b}$ automatically takes (even) integer values for models with generic abelian charges $\pm 1, \pm 2$, and the positivity condition that $-a, b, \tilde{b} > 0$ also arises naturally.

Finally, note that for every solution $a, \tilde{b}, x_q$ of the abelian AC equations (2.4), there is an infinite family of solutions $a' = a, \tilde{b}' = k^2 \tilde{b}, x'_{kq} = x_q$, for $k = 1, 2, \ldots$. At the level of massless spectra, these solutions may seem equivalent up to a scaling of the overall charge, but because $\tilde{b}$ has an absolute scaling relative to the lattice Γ, these models must be considered as physically distinct. A distinct question is the fundamental unit of charge under the U(1) gauge group. Even though the models with e.g. $\tilde{b}' = 4\tilde{b}, x'_{2q} = x_q$ are physically distinct, the fundamental unit of charge in the $\tilde{b}', x'$ theory may naturally be taken to be $q = 2$, or may be $q = 1$, in particular in a scenario where there are massive states in the spectrum with that charge. We give an explicit example of a pair of distinct physical theories with charges related by scaling in this way in section 4.3, and return to this general question in the discussion of the infinite apparent swampland for U(1) models later in the paper.

2.3 Generic and non-generic SU(N) models

We can now determine all anomaly-free models for an SU(N) gauge theory that contains only fundamentals, adjoints, and two-index antisymmetric matter fields (in the case of SU(2) and SU(3), which have no quartic Casimir, we consider only fundamentals and adjoints). In terms of the self-intersection n and genus g as defined above, solving the SU(N) AC conditions (2.7) yields models of the form

$$[6n + 2(N + 6)(1 - g)] \times \square + g \times \text{Adj}, \quad N = 2, 3, \qquad (2.11a)$$

$$[(8 - N)n + 16(1 - g)] \times \square + g \times \text{Adj} + [n + 2(1 - g)] \times \boxminus, \quad N \geq 4. \qquad (2.11b)$$

These are in several senses the most generic types of SU(N) spectra for 6D supergravity theories. For any values of n, g for which this massless spectrum is possible (non-negative), the resulting model will have more uncharged scalar fields than any other mass spectrum satisfying anomaly cancellation with the same values of n, g. This can be understood in F-theory from the fact that the representations used here come from the most generic types of codimension two singularities; in general, more exotic representations involve tuning to special points in the moduli space where the number of uncharged scalar fields is smaller [20–22]. Note, however, that for some values of n, g some of the representations in these models have negative multiplicity; in such cases, this "generic" type of matter is not possible and if there are solutions to the anomaly equations with non-negative multiplicities x_R they must include other matter representations.

From the generic models (2.11), we can determine the other possible spectra for given values of n, g. The group coefficients A_R, B_R, and C_R for larger representations will be given

by a linear combination of those for the fundamental, adjoint, and antisymmetric, so we can always exchange this linear combination of representations in (2.11) for the corresponding representation R, and the resulting spectrum will automatically satisfy the AC conditions.

For example, consider the $\square\square\square$ representation of SU(2); the anomaly coefficients of this representation can be related to those of the generic matter representations by a linear system of equations,

$$A_{\square\square\square} = c_1 A_{\square} + c_2 A_{\square\square}, \tag{2.12a}$$

$$C_{\square\square\square} = c_1 C_{\square} + c_2 C_{\square\square}. \tag{2.12b}$$

Using the values from table 1 gives $c_1 = -14$ and $c_2 = 6$; thus, we can freely make the exchange

$$14 \times \mathbf{1} + 6 \times \square\square \longleftrightarrow 1 \times \square\square\square + 14 \times \square \tag{2.13}$$

in any SU(2) model satisfying the AC conditions to yield another model satisfying the AC conditions. Note that we must check that the non-negativity constraint is satisfied after the exchange, but note also that in some cases such an exchange may take an inconsistent model, such as one with more than 6 fields in the $\square\square$ representation but a negative number of fields in the $\square$ representation, to a consistent model.

Representations that are related in this way are termed *anomaly equivalent* [20, 23]. Here we refer to the transformation between one model and another using anomaly equivalent representations as an *exchange*. Note that these exchanges need not be realizable by any physical process; here we simply use these exchanges as an organizational tool for classifying solutions to the anomaly equations. There are many instances, however, in which these exchanges can actually be realized as physical transitions [21, 22].

2.4 Generic and non-generic U(1) models

We can similarly compute all anomaly-free models for a U(1) gauge theory that contains only charges $q = \pm 1, \pm 2$. Solving the U(1) AC conditions (2.10) yields models of the form

$$(8\gamma - \tilde{n}) \times (\pm \mathbf{1}) + \left(\frac{\tilde{n} - 2\gamma}{4} \right) \times (\pm \mathbf{2}), \tag{2.14}$$

where $(\pm \mathbf{n})$ indicates a representation with charge $\pm n$.

As in the SU(N) case, we can, in principle, use this result to construct all other anomaly-free U(1) models by way of exchanges: comparing (2.10) and (2.7), we see that the U(1) analogues of A_R and C_R are simply q^2 and q^4, allowing us to determine an anomaly equivalence between any higher charge and a linear combination of fields of charges $\pm 1, \pm 2$. For charge $q = \pm 3$, for example, solving the system

$$3^2 = c_1 1^2 + c_2 2^2, \tag{2.15a}$$

$$3^4 = c_1 1^4 + c_2 2^4, \tag{2.15b}$$

gives $c_1 = -15, c_2 = 6$, so we can freely make the exchange

$$10 \times (\mathbf{0}) + 6 \times (\pm \mathbf{2}) \longleftrightarrow 1 \times (\pm \mathbf{3}) + 15 \times (\pm \mathbf{1}) \tag{2.16}$$

in an anomaly-free U(1) model to yield another anomaly-free U(1) model (again, subject to non-negativity). As with the nonabelian theories, we treat these exchanges as a formal way of relating low-energy theories with distinct spectra, though we expect that in many cases there are corresponding physical transitions (see, e.g., the discussion in [24]).

Note that when $T \geq 9$, there are possible values of $a, \tilde{b}$ that satisfy $a \cdot \tilde{b} = \tilde{b} \cdot \tilde{b} = 0$. These correspond to non-Higgsable U(1) models with no matter content, and $\tilde{n} = \gamma = 0$. Explicit examples of such theories have been identified in F-theory [25, 26]. Such models are generally associated with Higgsing SU(2) models that have only a single adjoint representation; for example, this describes any F-theory model with $T = 9$ where the gauge factor is supported on a curve in the genus one class $\Sigma = -K$.

3 Relating the U(1) and SU(N) anomaly cancellation equations

Since nonabelian gauge groups and matter are easier to understand and constrain than abelian gauge groups and matter, both from the low-energy point of view and from F-theory, it is often helpful to relate allowed abelian structures to nonabelian models that can be broken to reproduce abelian gauge groups and matter through Higgsing (see, e.g., [14, 27, 28]). In particular, if a nonabelian gauge group and matter spectrum can be realized in F-theory and leads in field theory to a given abelian gauge group and matter spectrum after a Higgsing process, then there must also be an F-theory realization of the resulting abelian theory (though the explicit realization of the Higgsing process in an F-theory Weierstrass model can be rather tricky [24]). Since there are strict bounds on the finite set of nonabelian gauge groups and matter that can be realized in F-theory, only a limited set of abelian models can be unHiggsed to a nonabelian model with an F-theory realization. Note that there are also U(1) models that can be realized in F-theory that cannot be unHiggsed to a good SU(2) model in F-theory; we encounter some examples of this later in the paper. Note also that there are only a finite number of massless spectra with a single SU(N) gauge factor that satisfy the anomaly conditions for any value of T, since the number of hypermultiplets is bounded above for any given N in such a situation, so only a finite number of U(1) models can be unHiggsed to an anomaly-free SU(N) theory.

In this section, we directly relate the anomaly cancellation conditions and the structure of matter spectra in theories with a U(1) gauge group to the AC conditions and matter spectra in theories with an SU(N) gauge group, and we describe a general class of Higgs transitions that break SU(N) to U(1).

We first compare the AC conditions for the SU(N) and U(1) cases directly. The U(1) AC conditions (2.4) for charges $q = \pm 1, \pm 2$ are

$$-6a \cdot \tilde{b} = x_1 + 4x_2 \,, \tag{3.1a}$$

$$3\tilde{b} \cdot \tilde{b} = x_1 + 16x_2 \,. \tag{3.1b}$$

If we do not impose non-negativity conditions, then any anomaly-free U(1) model can be exchanged to a model with only these charges, and the matter content will satisfy these two constraints.

Similarly, any SU(N) model can be exchanged to a model with only fundamentals, adjoints, and antisymmetrics if we do not impose the non-negativity conditions. For such a model, the SU(N) AC conditions (2.2) yield

$$-6a \cdot b = x_\square + 2N(x_{\text{Adj}} - 1) + (N-2)x_\boxminus, \tag{3.2a}$$

$$3b \cdot b = \frac{1}{2}x_\square + (N+6)(x_{\text{Adj}} - 1) + \frac{1}{2}(N-2)x_\boxminus, \tag{3.2b}$$

for $N > 3$. In the case of $N = 2, 3$, we take $x_\boxminus = 0$ here; for $N > 3$, (3.2b) follows from adding eq. (2.2e) to half of eq. (2.2d).

We now assume that $\tilde{b} = kb$, for $k \in \mathbb{Z}$. In such circumstances, we can equate the right-hand sides of (3.1) with those of (3.2), appropriately scaled, yielding

$$x_1 + 4x_2 = kx_\square + 2kN(x_{\text{Adj}} - 1) + k(N-2)x_\boxminus, \tag{3.3a}$$

$$x_1 + 16x_2 = \frac{k^2}{2}x_\square + k^2(N+6)(x_{\text{Adj}} - 1) + \frac{k^2}{2}(N-2)x_\boxminus. \tag{3.3b}$$

From these equations, we find

$$x_1 = \frac{k}{6}(8-k)x_\square + \frac{k}{3}[8N - k(N+6)](x_{\text{Adj}} - 1) + \frac{k}{6}(8-k)(N-2)x_\boxminus, \tag{3.4a}$$

$$x_2 = \frac{k}{24}(k-2)x_\square + \frac{k}{12}[k(N+6) - 2N](x_{\text{Adj}} - 1) + \frac{k}{24}(k-2)(N-2)x_\boxminus. \tag{3.4b}$$

By inspection, if k is 0 or 2 modulo 6, then $x_1, x_2 \in \mathbb{Z}$ for any $N, x_\square, x_{\text{Adj}}, x_\boxminus \in \mathbb{Z}$. We thus expect $k \equiv 0, 2 \pmod 6$, for any valid Higgsing from SU(N) to U(1) that gives $\tilde{b} = kb$ for some k. Note that the relationship (3.4) was found by considering only the AC conditions, without any reference to the explicit symmetry-breaking pattern.

For an SU(N) gauge theory, we can explicitly construct a Higgsing pattern to U(1) with $\tilde{b} = m(m-1)b$, for $2 \leq m \leq N$. To see this, first note that one possible Higgsing of an SU(N) to a U(1) using two adjoints is achieved by giving a VEV to a Cartan generator in one adjoint matter field that breaks SU(N) → U(1)$^{N-1}$, and then giving VEVs to $N-2$ of the U(1) charges coming from the second adjoint of SU(N) in order to break $N-2$ of the remaining U(1) factors. Without loss of generality, let the surviving U(1) generator be diag($1, 1, \cdots, 1, -N+1$); then we see that under this Higgsing, the fundamental, adjoint, and antisymmetric representations decompose as

$$\begin{aligned}
\square &\to (N-1) \times (\pm\boldsymbol{1}) + 1 \times (\pm(\boldsymbol{N-1})), \\
\text{Adj} &\to (N-1)^2 \times (\boldsymbol{0}) + 2(N-1) \times (\pm\boldsymbol{N}), \\
\boxminus &\to \frac{(N-1)(N-2)}{2} \times (\pm\boldsymbol{2}) + (N-1) \times (\pm(\boldsymbol{N-2})).
\end{aligned} \tag{3.5}$$

We can then determine the relationship between $\tilde{b}$ and b by comparing the AC equations for the two theories. After the Higgsing, the resulting U(1) model must satisfy

$$-6a \cdot \tilde{b} = x_1 + 4x_2 + (N-2)^2 x_{N-2} + (N-1)^2 x_{N-1} + N^2 x_N, \tag{3.6}$$

and we see from eq. (3.5) that

$$x_1 = (N-1)x_\square, \quad x_2 = \frac{(N-1)(N-2)}{2}x_\square,$$
$$x_{N-2} = (N-1)x_\square, \quad x_{N-1} = x_\square, \quad x_N = 2(N-1)(x_{\text{Adj}} - 1),$$

(3.7)

noting that Higgsing on an adjoint charge uses up one of the adjoints. Plugging these into eq. (3.6) and comparing with eq. (3.2a) yields

$$-6a \cdot \tilde{b} = N(N-1)\left[x_\square + 2N(x_{\text{Adj}} - 1) + (N-2)x_\square\right] = N(N-1)(-6a \cdot b).$$

(3.8)

Thus, this Higgsing gives $\tilde{b} = N(N-1)b$.

We could instead achieve $\tilde{b} = (N-1)(N-2)b$ by first Higgsing on an adjoint charge that breaks $\mathrm{SU}(N) \to \mathrm{SU}(N-1) \times \mathrm{U}(1)$, and then carrying out the Higgsing described above. By iterating this step, we can thus achieve Higgsings of $\mathrm{SU}(N) \to \mathrm{U}(1)$ with $\tilde{b} = m(m-1)b$ for any $2 \leq m \leq N$. Note that, as anticipated above, $m(m-1) \equiv 0, 2 \pmod 6$ for any m.

This analysis gives us a way of relating a broad class of generic U(1) theories to Higgsed $\mathrm{SU}(N)$ models with generic matter spectra.

4 Charge $|q| \leq 2$

We now consider the explicit classification of U(1) models and their unHiggsings. We begin by considering models with U(1) charges $|q| \leq 2$, i.e., $x_q = 0$ for $|q| \geq 3$. For arbitrary T, models will be parameterized by $\gamma, \tilde{n}$ in the case of U(1) models and g, n in the case of SU(2) models, where these parameters are constrained by the condition that the numbers of matter fields x_q, x_R in any consistent model are non-negative.

4.1 U(1) models

Restricting ourselves to $|q| \leq 2$, we have U(1) models of the form (2.14),

$$(8\gamma - \tilde{n}) \times (\pm \mathbf{1}) + \left(\frac{\tilde{n} - 2\gamma}{4}\right) \times (\pm \mathbf{2}).$$

(4.1)

In this case, we have $H = x_0 + x_1 + x_2$, with x_0 the number of trivial representations, and $V = 1$, so the gravitational condition (2.2a) becomes

$$274 = x_0 + x_1 + x_2 + 29T.$$

(4.2)

Considering only the charged matter, this yields the condition

$$274 \geq 29T - \frac{3(\tilde{n} - 10\gamma)}{4}.$$

(4.3)

The anomaly equations and integrality/non-negativity of charges impose the conditions

$$3\tilde{n} \in \mathbb{Z}, \quad \tilde{n} - 2\gamma \in 4\mathbb{Z}, \quad 2\gamma \leq \tilde{n} \leq 8\gamma,$$

(4.4)

though, as discussed in section 2.2, $\tilde{n}$ is presumably constrained to be an integer in any quantum-consistent low-energy theory. Note that the second of these conditions matches the constraint numbered 5 in [9] in the case of a single abelian gauge factor. Note also that the last condition in (4.4) can be written as

$$- 2a \cdot \tilde{b} \le \tilde{b} \cdot \tilde{b} \le -8a \cdot \tilde{b}\,. \tag{4.5}$$

This form of the constraint will be helpful in connecting to a related stronger condition on certain classes of F-theory models in section 4.5. Note that we can also derive the constraints (4.5) directly from (3.1); the first bound follows from the condition $4x_2 \le 16x_2$, and the second can be found by multiplying the first equation by 4 and noting that $4x_1 \ge x_1$.

4.2 SU(2) models

As we have seen, the anomaly-free $SU(N)$ models with only generic matter representations are given by (2.11). Specifically, for SU(2) the anomaly-free models are of the form

$$[6n + 16(1 - g)] \times \square + g \times \square\square\,. \tag{4.6}$$

We can obtain models containing larger representations by using the group theory coefficients to exchange these additional representations for some number of fundamentals and adjoints. However, models with only fundamentals and adjoints will Higgs down to U(1) models with charges $q = \pm 1, \pm 2$, which is the current case of interest. We have $H = x_0 + 2x_\square + 3x_{\square\square}$ and $V = 3$, so the gravitational condition (2.2a) becomes

$$276 \ge 2x_\square + 3x_{\square\square} + 29T\,, \tag{4.7}$$

or

$$244 \ge 12n + 29(T - g)\,. \tag{4.8}$$

We can now Higgs these down to U(1) models. We Higgs on an adjoint charge that leaves the generator T_3 unbroken (by this, we mean that we give an adjoint hypermultiplet a VEV equal to this adjoint charge). The resulting U(1) charges are twice the eigenvalues of the generator T_3 in the given representation. Thus,

$$\square \to 2 \times (\pm\mathbf{1})\,, \tag{4.9a}$$
$$\square\square \to 1 \times (\mathbf{0}) + 2 \times (\pm\mathbf{2})\,. \tag{4.9b}$$

Higgsing uses up one of the adjoints, so we find models of the form

$$[12n + 32(1 - g)] \times (\pm\mathbf{1}) + 2(g - 1) \times (\pm\mathbf{2})\,. \tag{4.10}$$

Matching these with the models from section 4.1, we find

$$\tilde{n} = 4n\,, \quad \gamma = 2n + 4(1 - g)\,. \tag{4.11}$$

Note that $\tilde{n} = 4n$ corresponds to $\tilde{b} = 2b$, which agrees with our expectation from section 3 for an SU(2) Higgsing.

We know that both n and g are integers such that the number of each representation in (4.6) is non-negative (the number of each representation is always integral due to the integrality of n and g). By contrast, for the abelian theories we only know that the number of each representation in (2.14) must be a non-negative integer, which gives us the conditions (4.4).

Thus, it is not guaranteed from these considerations that every anomaly-free U(1) model with matter of charge at most ± 2 can be unHiggsed to a consistent SU(2) model. In particular, even if we assume that $\tilde{n}$ is an integer, there are solutions of the second and third conditions in (4.4) (such as $\tilde{n} = 2, \gamma = 1$) that do not take the form of (4.11). On the other hand, if we assume that $\tilde{b}$ is an even element of the string charge lattice Γ (i.e., $\tilde{b}/2 \in \Gamma$), then $\tilde{n} \in 4\mathbb{Z}$ and there is always an anomaly-free SU(2) model with $b = \tilde{b}/2$ that gives the corresponding U(1) model with charges $\pm 1, \pm 2$ after Higgsing, up to the condition that the U(1) theory has sufficient uncharged scalars to match the charge 0 contribution from (4.9). Recently the condition that $\tilde{b}$ is an even element of the string charge lattice was proven under some mild assumptions in [9]. We discuss this condition further in the context of F-theory constructions and the swampland in sections 4.5 and 7.

4.3 $T = 0$

We now specialize to the case of models with no tensor multiplets, $T = 0$. In this case, the vectors a, b, and $\tilde{b}$ simply become numbers. In particular, we have $a = -3$; in order for the kinetic terms of the gauge fields to have the appropriate sign, b must be positive, and so the $a \cdot b$ condition fixes the sign of a to be negative. In this case, it is useful to use the AC conditions in the form (2.4) rather than (2.10).

We restrict the maximum U(1) charge to be $q = \pm 2$. The abelian AC conditions (2.4) give us

$$18\tilde{b} = x_1 + 4x_2 \,, \tag{4.12a}$$

$$3\tilde{b}^2 = x_1 + 16x_2 \,, \tag{4.12b}$$

yielding models of the form

$$\tilde{b}\left(24 - \tilde{b}\right) \times (\pm \mathbf{1}) + \frac{\tilde{b}\left(\tilde{b} - 6\right)}{4} \times (\pm \mathbf{2}) \,. \tag{4.13}$$

Note that requiring $\frac{\tilde{b}(\tilde{b}-6)}{4} \in \mathbb{N} = \mathbb{Z}_{\geq 0}$ implies that $\tilde{b}$ is an even integer. We see that $\tilde{b} \geq 0$ from eq. (4.12a), and $6 \leq \tilde{b} \leq 24$ from the condition that the numbers of fields of charge ± 1 and ± 2 is non-negative. In this set of models, we have $H = x_0 + x_1 + x_2$, $V = 1$, and $T = 0$, so the gravitational condition becomes $274 \geq x_1 + x_2$, which all otherwise anomaly-consistent models satisfy. Note that in the case $\tilde{b} = 24$, there are actually two distinct U(1) models with the spectrum (4.13), depending on the choice of unit charge; we discuss this issue further below.

Now we consider SU(2) models containing only fundamentals and adjoints, which we can Higgs to U(1) models with charges $q = \pm 1, \pm 2$ when there is at least one adjoint field.

Since b is simply a number, we have $n = b^2$. The corresponding value of g is

$$g = \frac{(b-1)(b-2)}{2}\,. \tag{4.14}$$

From (2.11), we find models of the form

$$2b(12-b) \times \square + \frac{(b-1)(b-2)}{2} \times \square\square. \tag{4.15}$$

Note that we must have $b \leq 12$ in order to have a non-negative integer number of each representation.

In general, we can include half-hypermultiplets of any quaternionic (pseudoreal) irreducible representation. In this case, however, the requirement that $b \in \mathbb{Z}$ (which follows from the integrality of n) implies that $x_\square \in \mathbb{Z}$. Furthermore, the gravitational condition $273 = H - V + 29T$ does not reduce the number of solutions in this case (as in the U(1) case above): we have $H = x_0 + 2x_\square + 3x_{\square\square}$, $V = 3$, and $T = 0$, so the condition becomes $276 \geq 2x_\square + 3x_{\square\square}$, which all otherwise anomaly-consistent models satisfy.

We can now Higgs the models (4.15) down to U(1) models when $b \geq 3$ so that there is at least one adjoint field. We Higgs on a charge that leaves the generator T_3 unbroken, as before, and find models of the form

$$4b(12-b) \times (\pm\mathbf{1}) + b(b-3) \times (\pm\mathbf{2})\,. \tag{4.16}$$

Note that we have an exact matching between these models and those we found by solving (2.4) if we take $\tilde{b} = 2b$, once again matching the expectation from section 3. This relies on the fact that $\tilde{b}$ must be an even integer, as discussed previously.

Thus, we have determined that every anomaly-free U(1) massless spectrum with $T = 0$ and matter of charge at most ± 2 can be unHiggsed to a model with an $\mathfrak{su}(2)$ gauge algebra. As we discuss in more detail in section 4.5, in fact each of the spectra of the form (4.13) has an explicit realization in F-theory. Except for some further subtleties that are relevant for the case $\tilde{b} = 24, b = 12$, this indicates that in this simple context there is no swampland.

For the case $\tilde{b} = 24$, there are some additional subtleties that are useful to discuss. Note that, as discussed in section 2.2, models with only hypermultiplets of charge $q = \pm 2$ are equivalent at the level of the anomaly conditions to those with the same number of hypermultiplets of charge $q = \pm 1$. In the case under consideration here, the model with $\tilde{b} = 6$ has $x_1 = 108, x_2 = 0$ and the model with $\tilde{b} = 24$ has $x_1 = 0, x_2 = 108$. However, each of these models can be unHiggsed in this case to an anomaly-free SU(2) model with a distinct spectrum:

$$\begin{aligned}
54 \times \square + 1 \times \square\square &\longleftrightarrow 108 \times (\pm\mathbf{1})\,, \\
55 \times \square\square &\longleftrightarrow 108 \times (\pm\mathbf{2})\,.
\end{aligned} \tag{4.17}$$

This illustrates the point mentioned earlier that models with massless fields of charges that are all multiples of a non-unit integer k can be physically distinct from those that satisfy the condition that the GCD of the charges is 1. There is a further issue, however, which is that the $\tilde{b} = 24$ model with 108 charges $q = \pm 2$ can have two distinct realizations; in one, $q = \pm 1$ is the fundamental U(1) charge, and in the other, $q = \pm 2$ is the fundamental U(1)

charge. The latter is still distinct from the model with 108 charges ± 1 and $\tilde{b} = 6$ due to the difference in values of $\tilde{b}$. The two models with $\tilde{b} = 24$ unHiggs to two distinct nonabelian models, the first of which has gauge group SU(2) and the second of which has gauge group SO(3). As we discuss further in section 4.5, only the latter of these two models is realized in F-theory (at least using known constructions).

4.4 $T = 1$

To classify $T = 1$ U(1) and SU(2) models from the supergravity point of view, we must determine not only the possible values of $\tilde{n}, \gamma, n, g$ that are consistent with anomaly cancellation but also which of these values can be realized by vectors $a, b, \tilde{b}$ in the $(1, 1)$ signature string charge lattice Γ. We know that the string charge lattice of a 6D $\mathcal{N} = 1$ theory must be unimodular [4]. There are two unimodular lattices of signature $(1, 1)$, the odd lattice with inner product

$$\Omega_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.18}$$

and the even lattice with inner product

$$\Omega_0 = U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.19}$$

We denote the corresponding string charge lattices by Γ_1, Γ_0. For Γ_1, the only possible choice for a so that $a \cdot a = 9 - T = 8$ is $-a = -a_1 = (3, -1)$, up to symmetries. For Γ_0 there are two possibilities: $-a = -a_0 = (2, 2)$ and $-a = -a'_0 = (4, 1)$. For brevity, we denote the possible combinations of string charge lattice and a vector by Γ_1, Γ_0, and Γ'_0. Note that while both Γ_1 and Γ_0 are realized in string theory with a variety of different choices of positive cone (as we discuss further in the next section), there is no known realization from string theory of models of type Γ'_0, i.e., the lattice Γ_0 with $-a = (4, 1)$.

For each of the possible classes of models $\Gamma_1, \Gamma_0, \Gamma'_0$, we have carried out a complete enumeration of the set of possible U(1) models, including values of $\tilde{n}, \gamma$ and associated values of $\tilde{b}$. For the 3 classes of models there are $195, 195, 314$ distinct allowed spectra (i.e., integer non-negative values of x_1, x_2), respectively, that satisfy the AC conditions, with $251, 383, 370$ possible distinct combinations of spectra and $\tilde{b}$ values. Note that the number of spectra and corresponding $\tilde{b}$ values includes a multiplicity for symmetries, i.e., $\tilde{b} = (b_1, b_2)$ and $\tilde{b} = (b_2, b_1)$ are both counted.

In this enumeration we have not imposed the positivity condition on $\tilde{b}$; this is discussed further in the following section, but the basic observation is that for the Γ_0 and Γ_1 lattices, there is a choice of positive cone compatible with F-theory, generated by the elements $(1, -1)$ and $(0, 1)$, which all anomaly-free solutions satisfy. (In fact, all solutions on the even lattice are in the stricter positive cone generated by $(1, 0)$ and $(0, 1)$.) Note that for Γ_1 and Γ_0 an even $\tilde{b}$ always gives an even value of x_2, so an odd x_2 means that $\tilde{b}$ is not even. This relation does not hold for Γ'_0, where we can have $\tilde{b}$ even and x_2 odd.

The sets of possible U(1) spectra on the lattices Γ_1 and Γ_0 are exactly the same; each anomaly-free U(1) model on the even lattice has a corresponding anomaly-free U(1) model

(or several) on the odd lattice with the same spectrum, and vice versa. There are, however, many U(1) models that satisfy the AC conditions that cannot be unHiggsed to an SU(2) model. In some cases, a spectrum can be associated with a value of $\tilde{b}$ on one lattice that admits an SU(2) unHiggsing, but cannot be unHiggsed on the other lattice. For example, the spectrum

$$x_1 = 8, \quad x_2 = 142 \tag{4.20}$$

can be realized on the Γ_1 lattice by $\tilde{b} = (29, 9)$ or $\tilde{b} = (43, -33)$, neither of which can be unHiggsed since $\tilde{b}$ is not even in either case. On the other hand, this spectrum can be realized on the even lattice Γ_0 by $\tilde{b} = (10, 38)$, which can be unHiggsed to the SU(2) model $x_\square = 4, x_{\square\square} = 72, b = (5, 19)$.

Of the 251 (383) anomaly-free U(1) models (including spectrum and $\tilde{b}$) on the lattice Γ_1 (Γ_0), 150 (280) models (corresponding to 105 (140) spectra) are not unHiggsable to an SU(2) model. Ninety of these spectra are not unHiggsable on either lattice. For many (40 spectra) of the models that cannot be unHiggsed on either lattice, the unHiggsing cannot occur because they would violate the SU(2) gravitational anomaly bound — this condition is independent of $\tilde{b}$. For all the remaining models that are not unHiggsable on either lattice (50 spectra), x_2 is odd, implying that one component of $\tilde{b}$ is odd for both choices of lattice. An example of such a model is

$$x_1 = 8, \quad x_2 = 97, \tag{4.21}$$

which can be realized on the lattice Γ_1 by $\tilde{b} = (23, -3)$ and on Γ_0 by $\tilde{b} = (13, 20)$. An example of a model that cannot be unHiggsed due to the SU(2) gravitational anomaly bound is

$$x_1 = 4, \quad x_2 = 236, \tag{4.22}$$

which can be realized on the lattice Γ_1 by $\tilde{b} = (44, 26)$ and on Γ_0 by $\tilde{b} = (9, 70)$. Note that there are 18 spectra that cannot be unHiggsed on either lattice for both of these reasons.

There are 370 models on Γ_0', giving 314 spectra, including all 195 of the spectra from the other choices of lattice and a. They all satisfy positivity for both cones described above. None of these models have F-theory realizations, and we do not explore them further here.

4.5 F-theory constraints and the swampland for charge $\pm 1, \pm 2$ models

We now consider explicitly the set of U(1) models with $|q| = 1, 2$ that can be realized in F-theory both directly as a U(1) Weierstrass model and indirectly through Higgsing of an SU(2) model. This section assumes some basic familiarity with F-theory; for more background see [8, 29, 30].

In general, 6D supergravity models come from compactifying F-theory on a complex surface B. The 6D string lattice is then associated with $H_{1,1}(B, \mathbb{Z})$, with the inner product arising from intersection of divisor classes and the positive cone being the cone of effective divisors. For $T = 0$ the surface B is $\mathbb{P}^1$, and the canonical class is $-K = 3H$. For $T = 1$ the surface B is a Hirzebruch surface $\mathbb{F}_m, m \leq 12$. When $m \geq 3$, there are non-Higgsable nonabelian gauge groups everywhere in the moduli space [31], so a pure U(1) or SU(2) theory is only possible for $m = 0, 1, 2$. In the cases $m = 0, 2$ we have the 6D charge

lattice Γ_0 with $-K = (2,2)$ and for $m = 1$ we have Γ_1 with $-K = (3,-1)$. In the case of $m = 1$, the effective cone on Γ_1 is generated by the vectors $(0,1)$ and $(1,-1)$, so for example $\tilde{b} = (23,-3)$, as encountered in the example (4.21), lies in the positive cone of the corresponding 6D supergravity theory. For $m = 0, 2$, the effective cone on Γ_0 is generated by $\{(1,0),(0,1)\}$ and $\{(1,-1),(0,1)\}$, respectively.

A 6D F-theory model with SU(2) gauge group realized on a divisor associated with the vanishing locus of a function (really a section of a line bundle) $\sigma = 0$ of divisor class $b = [\sigma]$ has a Weierstrass model

$$y^2 = x^3 + fx + g\,, \tag{4.23}$$

where the discriminant vanishes to quadratic order in σ,

$$\Delta = 4f^3 + 27g^2 = \sigma^2(\tilde{\Delta})\,. \tag{4.24}$$

A fairly general class of F-theory SU(2) models can be constructed by choosing f, g to have the form

$$f = -\frac{1}{48}\phi^2 + f_1\sigma + f_2\sigma^2\,, \tag{4.25a}$$

$$g = \frac{1}{864}\phi^3 - \frac{1}{12}\phi f_1\sigma + g_2\sigma^2\,, \tag{4.25b}$$

which can be arranged either using the Tate tuning of a general Weierstrass model [30] or an order by order tuning [20] so that the discriminant vanishes to order σ^2. Here, f, g are sections of line bundles $\mathcal{O}(-4K)$ and $\mathcal{O}(-6K)$. Thus, f_1 is a section of $\mathcal{O}(-4K - b)$, f_2 is a section of $\mathcal{O}(-4K - 2b)$, and so forth. If $-4K - b$ is not effective, that is if $b \leq -4K$ does not hold,[1] then f_1, f_2, and g_2 all vanish and Δ vanishes identically, which gives a globally singular and unphysical F-theory model. So an SU(2) model of the form (4.25) cannot exist unless $b \leq -4K$. Furthermore, the condition on b for the curve to have genus one or higher so that an adjoint Higgsing is possible is that $-K \leq b$. Thus, for a good F-theory model with an SU(2) realized through the form (4.25) on a smooth divisor of at least genus one, corresponding to the existence of at least one adjoint field, we need

$$-K \leq b \leq -4K\,. \tag{4.26}$$

The Morrison-Park direct construction gives, as in (2.8),

$$\tilde{b} = 2(-K + [z])\,. \tag{4.27}$$

In this construction, $[z]$ is effective, so $\tilde{b}$ is even, and $-2K \leq \tilde{b}$. The construction also requires $\tilde{b} \leq -8K$, or as in the nonabelian SU(2) model above the discriminant would vanish identically, so we have

$$-2K \leq \tilde{b} \leq -8K\,. \tag{4.28}$$

Furthermore, $b = \tilde{b}/2 = -K + [z]$ can be explicitly described in the Morrison-Park model as the locus that supports an SU(2) after an explicit unHiggsing, and satisfies exactly

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[1]Note that for divisors, the notation $A \leq B$ is used to indicate that $B - A$ is effective.

the conditions (4.26). In the absence of additional nonabelian gauge groups, which might cause singularities upon unHiggsing of the U(1), therefore, there is a precise necessary and sufficient condition for the existence of a Morrison-Park U(1) model, which is equivalent to the necessary and sufficient condition for the existence of an SU(2) model of the form (4.25). The condition is that there exist a divisor $\tilde{b} = 2b$ satisfying (4.26) in the nonabelian case and (4.28) in the abelian case.

Note that F-theory constructions of SU(2) models and U(1) models that do not satisfy the conditions (4.26) and (4.28) are possible. In particular, as described in [22], if the divisor supporting the SU(2) is singular, the leading terms in (4.25) can take a more general "non-UFD" form where ϕ is not in the ring of global functions on the divisor, so the upper bound from (4.26) on b does not hold. Similar considerations motivated by the construction of U(1) models that can have higher charges [24] show that the upper bound on $\tilde{b}$ from (4.28) also is not necessary when the U(1) is realized in a fashion that does not take the Morrison-Park form. Thus, these stronger constraints only hold for the standard F-theory constructions described above and are not universal constraints on all F-theory models. In fact, the ring of functions on σ can fail globally to be a UFD even when σ is smooth, when the genus of σ is nonzero. For brevity in the subsequent discussion we refer to SU(2) models of the form (4.25), where the constraint (4.26) necessarily holds, as "UFD SU(2) models," and models not of this form as "non-UFD SU(2) models."[2] Note that for any SU(2) model in F-theory, the constraint

$$b \leq -6K \tag{4.29}$$

must be satisfied since $-12K - \sigma^2$ must be effective for the discriminant to have the form (4.24). This is a simple example of the "Kodaira constraint," discussed more generally in [8].

For $T = 0$, b for an SU(2) model is simply the degree of the curve on which the SU(2) is tuned. The genus of this curve is $g = (b-1)(b-2)/2$. The SU(2) constraint (4.26) states that $3 \leq b \leq 12$, and the corresponding condition for the Morrison-Park U(1) model is (4.28). Because $a, \tilde{b}$ are simply numbers, (4.28) is precisely equivalent to the condition (4.5), which states

$$-2a \cdot \tilde{b} \leq \tilde{b} \cdot \tilde{b} \leq -8a \cdot \tilde{b}. \tag{4.30}$$

Since the anomaly cancellation conditions already impose the constraint that $\tilde{b}$ is even in the case $T = 0$, we see that for $T = 0$ there is a match between anomaly-allowed U(1) models with charges $q = \pm 1, \pm 2$ and the models that can be constructed from F-theory.

As mentioned in section 4.3, the $\tilde{b} = 24$ model potentially has two distinct realizations, one with fundamental charge $q = \pm 1$ and one with fundamental charge $q = \pm 2$. These two models unHiggs to models with nonabelian gauge groups SU(2) and SO(3),

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[2]We would like to thank Yinan Wang for pointing out that some models of the form constructed in [24] have no charges $|q| > 2$. These models can be unHiggsed to non-UFD SU(2) models, so that in fact there exist F-theory models that violate the constraints (4.26) and (4.28). Such exotic constructions exist even in some cases where the curve supporting the gauge factor is smooth.

respectively. Only the latter of the two models has a natural realization in F-theory through the Morrison-Park construction.[3] After the unHiggsing of the Morrison-Park model, the resulting Weierstrass model has an $\mathfrak{su}(2)$ gauge algebra. The group is, however, $SO(3) = SU(2)/\mathbb{Z}_2$ in this case due to the presence of $\mathbb{Z}_2$ torsion; this can be checked by noting that the Weierstrass model takes a Tate form with $a_2, a_3, a_6 = 0$, which as shown in [32] gives the $SO(3)$ gauge group. This is natural, from the absence of massless fields that transform nontrivially under the $\mathbb{Z}_2$ center of $SU(2)$ in this case. On the other hand, there is no clear reason from the low-energy theory that the massless spectrum of the $U(1)$ theory cannot consist only of charges $q = \pm 2$ even if the fundamental charge is $q = \pm 1$, or that there cannot be an $SU(2)$ model with only adjoints in the massless spectrum. While we do not have an F-theory realization of these models, and suspect they do not exist, we also cannot completely rule out an exotic non-UFD type F-theory realization. So these models are likely but not certain candidates for the swampland. We discuss some further aspects of this in connection with the "completeness hypothesis" in section 7.8.

For larger values of T, the anomaly cancellation conditions from 6D supergravity are weaker than the conditions from F-theory. These two sets of conditions do seem, however to have some interesting parallels. For $U(1)$ models with only charges $q = \pm 1, \pm 2$ and no additional gauge groups, the AC conditions (4.4) impose the constraints

$$\tilde{b} \cdot \tilde{b} + 2a \cdot \tilde{b} \in 4\mathbb{Z}, \quad -2a \cdot \tilde{b} \le \tilde{b} \cdot \tilde{b} \le -8a \cdot \tilde{b}. \tag{4.31}$$

The existence of an F-theory model either from the Morrison-Park construction or from Higgsing a UFD $SU(2)$ model with only fundamentals and adjoints imposes the constraints

$$\tilde{b} \in 2\mathbb{Z}, \quad -2a \le \tilde{b} \le -8a. \tag{4.32}$$

Similarly, the anomaly constraints on an $SU(2)$ model with only fundamental and adjoint charges, no other gauge groups, and at least one adjoint field are

$$-a \cdot b \le b \cdot b \le -4a \cdot b, \tag{4.33}$$

where the first inequality comes from $g > 0$ and the second from the constraint that the number of fundamental fields in (4.6) is non-negative, $6n + 16(1-g) \ge 0$. As in the abelian case, these constraints are closely related to the conditions for the existence of a UFD $SU(2)$ model in F-theory,

$$-a \le b \le -4a. \tag{4.34}$$

In both the abelian and nonabelian cases, the Morrison-Park and UFD $SU(2)$ F-theory constraints are strictly stronger than but closely parallel to the constraints from anomalies. While the signature of the inner product is indefinite, in the cases we consider here we must have $-a \cdot \tilde{b} \ge 0$ and $\tilde{b} \cdot \tilde{b} \ge 0$. This follows directly from the anomaly relations (3.1). The analogous inequalities for b also hold for $SU(2)$ theories with at least one adjoint, from (3.2). Thus, while the anomaly cancellation conditions are a necessary consequence of these F-theory conditions on specific models, not all models satisfying the anomaly conditions

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[3]We would like to thank Ling Lin for discussions on this point.

satisfy these F-theory conditions. The fact that exotic F-theory constructions can allow the constraints (4.32) and (4.34) to be violated makes it clear that these cannot really be low-energy constraints. But it does suggest some interesting structure for certain generic classes of models, and raises a question of whether Morrison-Park U(1) and UFD SU(2) models have some special characteristic structure that can be identified in the low-energy supergravity theory. We return to these questions in section 7.

The fact that the anomaly and F-theory constraints precisely agree for $q = \pm 1, \pm 2$, $T = 0$ theories follows from the fact that the inner product is simply a product of numbers in this case, with $\tilde{b}, b$ non-negative integers and $\tilde{b}$ automatically even, so that the anomaly constraints immediately imply the Morrison-Park U(1) and UFD SU(2) constraints. For $T > 0$, the story becomes more complicated. From our analysis of $T = 1$ models, we have identified anomaly-free U(1) models for which $\tilde{b}$ is not even, and models for which $\tilde{b} \leq -8a$ is not satisfied, even though in both cases the conditions (4.31) are satisfied. If we assume, however, that $\tilde{b}$ must be even, it is unclear whether or not the charge spectra with $T = 1$, $q = \pm 1, \pm 2$ that violate the condition $\tilde{b} \leq -8a$ can be realized in F-theory. For most, but not all, of those spectra that admit an unHiggsing to an anomaly-free SU(2) model, the Kodaira constraint (4.29) is satisfied for some choice of effective cone that is allowed from F-theory. Understanding which, if any, of the models that violate the condition $\tilde{b} \leq -8a$ and can or cannot be unHiggsed to an SU(2) model are in the swampland would require a more complete understanding of non-UFD or non-Morrison-Park F-theory constructions, likely building on the approach of [24].

There are some spectra that arise, such as $(x_1, x_2) = (0, 150)$, which are particularly interesting. This spectrum can be associated with $\tilde{b} = (10, 40)$ on the even lattice Γ_0, which does not satisfy $\tilde{b} \leq -8a$ on any effective cone from F-theory. Furthermore, the model can be unHiggsed to an SU(2) model that satisfies the anomaly conditions, but has $b = (5, 20)$ and violates the Kodaira bound for any effective cone from F-theory. This model can also be realized on the odd lattice Γ_1, where there is a similar issue. Thus, for both lattices, the resulting SU(2) model is in the "swampland" of theories that do not violate any known quantum consistency condition based on the low-energy spectrum but definitely cannot be realized in F-theory. In fact, there are eight other U(1) models, such as $(x_1, x_2) = (4, 146)$ and $(x_1, x_2) = (12, 138)$, that have SU(2) unHiggsings on one of Γ_0 or Γ_1 and that violate the Kodaira bound for any effective cone from F-theory; each of the associated SU(2) models is therefore in the swampland. It seems likely that these U(1) models are also not realizable from F-theory and are in the swampland, but we do not have any way of proving this at this time.

5 Asymptotics and infinite anomaly-free charge families

In this section, we construct some infinite families of anomaly-free charge spectra, and study the growth of the number of anomaly-free U(1) models in the limit of large $\tilde{b}$, for $T = 0$. In this way, we can further understand the proliferation of apparently consistent U(1) models at larger charges.

5.1 Large charge families

Before moving to asymptotics, we can directly address the question of whether the total number of different spectra that satisfy the anomaly equations is finite or infinite. As we saw at the end of section 2.2, each solution to the abelian AC equations trivially gives rise to an infinite family of models by scaling the charges and the associated $\tilde{b}$, and thus we already know that the number of distinct anomaly-free U(1) spectra is infinite. However, there are additional nontrivial infinite families of U(1) models satisfying the AC conditions. One such family is given by

$$54 \times (\pm q) + 54 \times (\pm r) + 54 \times (\pm(q+r)), \quad \tilde{b} = 6\left(q^2 + qr + r^2\right), \quad q, r \in \mathbb{Z}. \quad (5.1)$$

In analogy with (5.1), we can similarly construct an infinite family of four-charge models in the following fashion. Choose distinct $m, n \in \mathbb{Z}_+$ with $n \le m/2$, and define

$$
\begin{aligned}
a &= m^2 - 2mn, \\
b &= 2mn - n^2, \\
c &= m^2 - n^2, \\
d &= 2\left(m^2 - mn + n^2\right).
\end{aligned}
\quad (5.2)
$$

Then,

$$54 \times (\pm a) + 54 \times (\pm b) + 54 \times (\pm c) + 54 \times (\pm d), \quad \tilde{b} = 12\left(m^2 - mn + n^2\right)^2 \quad (5.3)$$

is an anomaly-free model. Note that if $m \equiv -n \pmod 3$, then $\gcd(a,b,c,d) = 3$, and $\gcd(a,b,c,d) = 1$ otherwise. These numbers have a geometric interpretation, as shown in figure 1: c is the integral side length of an equilateral triangle such that there exists an integral cevian (a line segment connecting a vertex of the triangle with any point on the opposite side) of length $d/2$ dividing c into integral parts, $a + b = c$. The set of (a,b,c,d) generated in this way (after dividing by the GCD, if necessary) is precisely the set of primitive integer tuples satisfying this property.

These infinite families of anomaly-free U(1) spectra are particularly interesting, because it is known that the number of distinct anomaly-consistent spectra for nonabelian gauge groups and matter representations in supergravity theories with $T < 9$ is finite. It is also known that the number of consistent spectra (abelian or nonabelian) that can arise from F-theory is finite. This indicates that there is an infinite swampland of abelian models. The size of this swampland could be reduced either by finding further quantum consistency constraints that rule out some of these models, or by finding some other approach to string compactifications that does not come from F-theory.

Note that there cannot be an infinite number of anomaly-free models with charges below any given upper bound $q \le Q$, since the total number of charged hypermultiplet fields is bounded by the gravitational anomaly condition $\sum_q x_q \le 274$, and there are thus a finite number of spectra with $q \le Q$.

Note also that issues related to the fundamental unit charge under the U(1) do not affect these infinite families. For example, the family (5.1) with $q = 1$ and arbitrary r gives an infinite family with unit charge 1.
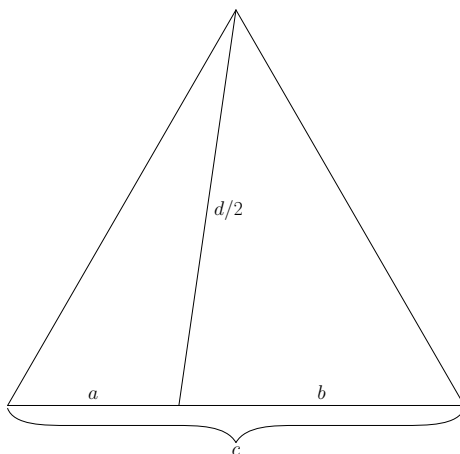
Figure 1. An integer tuple (a, b, c, d) such that there exists an equilateral triangle of side length c with an integral cevian of length $d/2$ dividing the side into integral parts, $a + b = c$. Such tuples produce anomaly-free models of the form (5.3).

5.2 A continuous approximation

The abelian AC conditions (2.4) can be written as

$$\sum_{i=1}^{h} q_i^2 = 18\tilde{b} \,, \tag{5.4a}$$

$$\sum_{i=1}^{h} q_i^4 = 3\tilde{b}^2 \,, \tag{5.4b}$$

where the sum runs over all charged matter representations, $i = 1, \ldots, h$, with the nonzero charges q_i possibly degenerate.

To estimate the number of solutions of these equations for any given value of $\tilde{b}$, we can consider this as a special case of the problem of finding the number of solutions of the equations

$$\sum_{i=1}^{h} q_i^2 = B \,, \tag{5.5a}$$

$$\sum_{i=1}^{h} q_i^4 = C \,, \tag{5.5b}$$

where B and C are integers. We further define the quantity

$$x = \frac{C}{B^2} = \frac{\sum_i q_i^4}{\left(\sum_i q_i^2\right)^2} \,. \tag{5.6}$$

In our case of interest, $B = 18\tilde{b}$, and $x = 1/108$, so $C = xB^2$ is automatically an integer for any integer $\tilde{b}$ and this value of x.

A simple scaling argument suggests that the number of solutions of the equations (5.5) should go roughly as $B^{(h-6)/2}$ for fixed h at large B, for generic values of x in the range

$1/h \leq x \leq 1$. This can be seen as follows: the number of charge combinations q_i that satisfy $\sum_i q_i^2 \leq B$ clearly scales as $B^{h/2}$, as in the continuum approximation it is the volume of an h-dimensional ball of radius $\sqrt{B}$. The number of solutions that satisfy the equality $\sum_i q_i^2 = B$ will go as the derivative of the number of solutions of the inequality, so as $B^{(h-2)/2}$. The set of charge combinations q_i gives a distribution on the set of x that should approach a smooth distribution in the large B limit, where the limiting values $x = 1/h, 1$ are realized by the charge combinations $(q, q, \ldots, q)$ and $(q, 0, 0, \ldots, 0)$, respectively. The possible values of $\lfloor xB^2 \rceil$ (rounded integer value) are then distributed across the range from B^2/h to B^2, so for any given x the number of solutions of (5.5) should scale as $B^{(h-2)/2}/B^2 = B^{(h-6)/2}$. This scaling can be understood geometrically as arising from the intersection of the $(h-1)$-dimensional sphere and quartic hypersurface defined by the equations (5.5).

This simple scaling argument suggests that when $h \geq 108$, the number of solutions of (5.4) will scale as $\tilde{b}^{(h-6)/2}$, suggesting large infinite families of solutions, as the total number of solutions up to an arbitrary $\tilde{b}$ should go as the integrated value, $\tilde{b}^{(h-4)/2}$. By choosing simpler charge combinations, where a large multiple of each of a smaller number of charges arises, this estimate suggests, for example, that the number of solutions with N copies each of m charges $q_1, \ldots, q_m$ should scale as $\tilde{b}^{(m-6)/2}$, as long as $108 \leq mN \leq 274$. The number of integrated solutions up to a maximum value $\tilde{b}$ will then go as

$$\mathcal{N}_m^{\text{approximate}} \sim \tilde{b}^{(m-4)/2} \,. \tag{5.7}$$

So, for example, there should be a logarithmically divergent number of solutions with an equal number of four charges, a linearly divergent number of solutions with an equal number of five charges, etc.

Note, however, that the infinite families identified in section 5.1 both exceed this estimate, with the integrated number of models in the three-charge family scaling as $\tilde{b}^1$, and the integrated number of models of the four-charge family scaling as $\tilde{b}^{1/2}$. Indeed, this continuous scaling approximation misses out on several features that modify this simplified analysis. First, the continuous distribution on possible values of x can have singular peaks that enhance the number of solutions; second, number-theoretic considerations can be relevant in affecting the number of solutions for particular values of B, x.

We may also wish to restrict to cases where the q_i do not all have a common factor, in order to suppress the contribution of the trivial infinite families produced by scaling of charges. This only affects the analysis by an overall factor, and is only really relevant for small h, m.

The result of these features is that the scaling estimate just given is generally an underestimate of the total number of solutions that can arise at favorable combinations of charge multiplicities. To understand these issues more clearly, we consider a simplified example in further detail.

5.3 Two charges

As a simple toy example to illustrate the issues involved, we consider the case of two charges, $h = 2$, and the system of equations

$$q^2 + r^2 = B \,, \quad q^4 + r^4 = xB^2 \,. \tag{5.8}$$

In the continuous approximation, we can estimate the number of solutions for fixed B, x by the integral

$$I_2(x) = \int_0^\infty dq\, dr\, \delta\big(q^2 + r^2 - B\big)\delta\big(q^4 + r^4 - xB^2\big)\,. \tag{5.9}$$

This gives the area in the (q, r)-plane that maps to a unit area in the (B, xB^2)-plane, corresponding to the Jacobian of the transformation between these spaces, which estimates the number of integer solutions for (q, r) corresponding to a given (B, x) when $B \gg 1$. Using the relation

$$\int dx\, g(x)\delta(f(x)) = \sum_{x_i} \frac{g(x_i)}{|f'(x_i)|}\,, \tag{5.10}$$

where the x_i are the zeroes of $f(x)$ in the relevant integration range, we can write (5.9) as

$$\begin{aligned} I_2(x) &= \int_0^\infty dq\, \frac{\theta\big(B - q^2\big)}{2\sqrt{B - q^2}}\delta\Big(q^4 + \big(B - q^2\big)^2 - xB^2\Big) \\ &= \frac{\theta(2x - 1)\theta(1 - x)}{2\sqrt{2}B^2\sqrt{2x - 1}\sqrt{1 - x}}\,, \end{aligned} \tag{5.11}$$

with

$$\theta(x) = \begin{cases} 1\,, & x > 0 \\ 0\,, & x < 0 \end{cases} \tag{5.12}$$

the Heaviside step function. Here, we have assumed $B > 0$.

In accord with the general analysis of the preceding section, the number of solutions in this continuous approximation thus scales as $B^{(2-6)/2} = B^{-2}$. Note, however, that the function $I_2(x)$ diverges at the values $x = 1/2, x = 1$. Thus, in the continuous approximation, while for generic values of x the number of solutions of the integer equations (5.8) should scale as B^{-2}, the number of solutions for $x = 1/2$ appears to diverge. From the fact that $C = xB^2$ is expected to be an integer, we can estimate the effect of this divergence by integrating the divergent integrand over the range from $C = B^2/2$ to $C = B^2/2 + 1$. Noting that $dC = B^2\, dx$, this yields

$$\int_{B^2/2}^{B^2/2+1} \frac{dC}{2B^2\sqrt{2x - 1}} = \int_{1/2}^{1/2+1/B^2} \frac{dx}{2\sqrt{2x - 1}} = \int_0^{1/B^2} \frac{d\epsilon}{2\sqrt{2\epsilon}} = \frac{1}{\sqrt{2}B}\,, \tag{5.13}$$

where we have approximated $\sqrt{2}\sqrt{1 - x} \approx 1$ over the entire integration range. Thus, we may expect that the number of integer solutions of (5.8) with $x = 1/2$ scales as $1/B$, i.e., with an extra factor of B. In the 3-charge case, a similar extra factor of B contributes to the scaling of the infinite family of charges (5.1), which scales $\tilde{b}^{3/2}$ faster than the asymptotic estimate (5.7).

Let us consider the more precise number-theoretic aspects of this question with $n = 2$. In fact, the solutions to (5.8) for $x = 1/2$ are those where $r = q, B = 2q^2$, so the number of solutions with $q^2 + r^2 \leq B$ and $x = 1/2$ is $\sqrt{B/2}$, and the average number of solutions of the equality goes as the derivative of this, or $1/\sqrt{8B}$. Thus, the number of solutions at
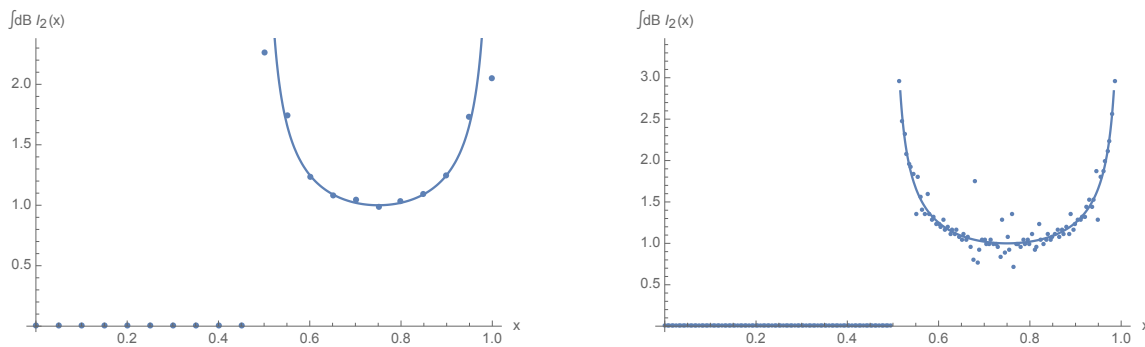
Figure 2. The prediction for the distribution of values of $x = (q^4 + r^4)/(q^2 + r^2)^2$ compared with numerical values for all (q, r) with $q^2 + r^2 \leq 10\,000$, where numerical values are averaged over bins of size $1/20$ (left) and $1/200$ (right). Irregularities in the right diagram indicate number-theoretical patterns that become relevant for bins smaller than $1/\sqrt{B}$, such as the large peak at $x = 0.68$, as mentioned in the text.

$x = 1/2$ exceeds even the above continuum estimate adjusted to take account for the singularity. Note, however, that this family of solutions seems to be unrelated to the singularity. There is another family of solutions with $r = 2q, B = 5q^2, x = 17/25 = 0.68$ that provides $\sim \sqrt{4B/5}$ solutions to the equations with the inequality, including a factor of 2 to include solutions with $q = 2r$. These families of solutions arise for every relatively prime pair of integers (a, b), where $(q, r) = (na, nb)$, and always give sets of order $\sqrt{B}$ of the total B solutions to the inequality $q^2 + r^2 \leq B$. This pattern represents a breakdown of the continuous approximation. Thus, we expect that a histogram of values of x for the set of all $q^2 + r^2 \leq B$ will look fairly smooth if the bins are larger than $1/\sqrt{B}$, but exhibit number-theoretic irregularities when the number of bins increases beyond $\sqrt{B}$. This is illustrated in figure 2.

Related to this, we may wish to restrict our attention to theories where (q, r) are relatively prime, to suppress the contribution of the trivial infinite families seen in section 2.2. Note that in the large B limit, using the fact that the probability that a given pair of integers is relatively prime is $6/\pi^2$, the overall distribution should be multiplied by this factor when considering only relatively prime charges.

We can carry out a similar analysis for the case of three distinct charges, four distinct charges, and so on. As in the case of two charges, there can be enhancements to the naïve growth coming from singular peaks in the distribution on values of x as well as from number-theoretic effects.

5.4 Fixed number of distinct charges

From the preceding analyses, we can now consider the more general problem of estimating the number of solutions of the anomaly equations with k distinct charges $q_1, \ldots, q_k$ with

multiplicities $n_1, \ldots, n_k$. The equations that must be solved are then

$$\sum_{i=1}^{k} n_i q_i^2 = B \,, \tag{5.14a}$$

$$\sum_{i=1}^{k} n_i q_i^4 = \frac{1}{108} B^2 \,, \tag{5.14b}$$

where $B = 18\tilde{b}$ is an integer divisible by 18. Since the total number of charges $\sum_i n_i$ is bounded above by 274, for each k there are a finite number of possible relevant combinations of the multiplicities n_i. The asymptotic number of solutions at large B should then scale as the number of solutions for any combination of multiplicities. For each distinct combination of k and multiplicities n_i, we have a similar set of equations to those studied above. This suggests that for generic choices of k, n_i the integrated number of solutions will scale as $\tilde{b}^{(k-4)/2}$, and that at specific values associated with singularities like those described above, the number of solutions will scale as $\tilde{b}^{(k-2)/2}$. Thus, we would expect based on these analyses that the total number of solutions with 3 distinct charges with any $18\tilde{b} \leq B$ should scale as $B^{1/2}$, the number of solutions with 4 distinct charges should scale as B, and the number of solutions with 5 distinct charges should scale as $B^{3/2}$. In this analysis, we have not accounted for number-theoretic effects like those encountered in section 5.3, which are needed to account for the scaling of families like (5.1).

Summarizing the results of the analysis, the continuum estimates from section 5.2 show that infinite families of charges are expected, with increasing rates of growth in $\tilde{b}$ as the number of distinct charges increases, and singularities and number theoretic effects explain why certain infinite families like those identified in section 5.1 arise with slightly larger powers in the growth rates than those predicted by the continuous approximation.

6 Classifying models with charges $|q| > 2$

As we saw in section 5, there are, in fact, an infinite number of anomaly-free spectra for low-energy U(1) models. As the number of consistent spectra that can arise from F-theory is finite, this means that there is an infinite swampland of distinct massless spectra for U(1) models. As discussed in the previous section, however, it is difficult to show that any specific U(1) model does not have an F-theory realization. In order to better understand this infinite swampland, therefore, it is illustrative to consider particular cases. To this end, in this section we first consider models with maximum U(1) charge $|q| \leq 3$, i.e., $x_q = 0$ for $|q| \geq 4$, and $T = 0$. We analyze the types of models that arise, and note mechanisms by which we can identify models that are clearly *not* in the swampland. We then carry out the same analysis for models with charges up to ± 4 and ± 5.

One of our primary tools for identifying a model as not being in the swampland is unHiggsing; if a U(1) model can be unHiggsed to a nonabelian model that is realizable by an F-theory compactification, then we can realize the U(1) model in F-theory by Higgsing the associated nonabelian F-theory model. In particular, the most direct unHiggsing of a U(1) model is to an SU(2) model, and so we will classify the U(1) models we find, in part, by

their ability to be unHiggsed to an anomaly-free SU(2) model. We will also see cases where models that cannot be unHiggsed to an anomaly-free SU(2) model may be unHiggsed to an anomaly-free nonabelian model with a larger gauge group. If this nonabelian model can be realized in F-theory, this similarly identifies the model as not being part of the swampland. We additionally present arguments that may help identify cases where a model cannot be unHiggsed to any nonabelian model.

There are also cases, however, where we can directly construct a Weierstrass model for a U(1) theory [24], despite it apparently not being unHiggsable to any anomaly-consistent nonabelian model. Thus, while having an unHiggsing to a nonabelian model with an F-theory realization is sufficient to show that a U(1) model is not in the swampland, showing that there does not exist such an unHiggsing is not a guarantee that the model cannot be found in F-theory, and is therefore not a complete criterion for identifying swampland models.

Ultimately, we have not managed to identify any specific U(1) models for which we can definitively say, "This anomaly-free model cannot be realized in F-theory and therefore is in the swampland," though we know that as we continue to classify models with arbitrarily high charges, infinitely many (in fact, cofinitely many) models must be in the swampland. Nonetheless, the analysis here gives us an idea of the kinds of models that may arise as we consider ever larger charges, and may give some hint as to what the potential obstructions are to finding F-theory compactifications that realize them.

Throughout this section we restrict our attention to models with $T = 0$.

6.1 Charge $|q| \leq 3$ U(1) models

Anomaly-free U(1) models with $T = 0$ can be found via exchanges of the form (2.16) from the $|q| \leq 2$ models (4.13). From this, we find that anomaly-free U(1) models with charges $q = \pm 1, \pm 2, \pm 3$ are of the form

$$\left[\tilde{b}\left(24 - \tilde{b}\right) + 15x_3\right] \times (\pm \mathbf{1}) + \left[\frac{\tilde{b}\left(\tilde{b} - 6\right)}{4} - 6x_3\right] \times (\pm \mathbf{2}) + x_3 \times (\pm \mathbf{3}). \quad (6.1)$$

The requirement that the number of hypermultiplets transforming in each irreducible representation must be a non-negative integer tells us that $\tilde{b}$ is a non-negative, even integer. The number of such models for which x_1, x_2, x_3 are all non-negative integers is 260. The gravitational condition in this case becomes $274 \geq x_1 + x_2 + x_3$, which reduces the number of anomaly-free models to 245. Note that these numbers include models with $x_3 = 0$.

6.2 SU(2) models with 3-symmetric matter

Now we consider SU(2) models containing fundamentals, adjoints, and triple-symmetrics, which we can Higgs to U(1) models with charges $q = \pm 1, \pm 2, \pm 3$. We allow for half-hypermultiplets of both $\square$ and $\square\square\square$, although we will once again find that all models have full hypermultiplets in the fundamental. Anomaly-free models can be found via exchanges of the form (2.13) from the generic matter models (2.11). Thus, we have models of the

form

$$\left[2b(12-b)+14x_{\square\square\square}\right]\times\square+\left[\frac{(b-1)(b-2)}{2}-6x_{\square\square\square}\right]\times\square\square+x_{\square\square\square}\times\square\square\square. \quad (6.2)$$

The number of such models for which $b, 2x_{\square}, x_{\square\square}, 2x_{\square\square\square}$ are all non-negative integers and $x_{\square\square} \geq 1$ is 223 (we require the number of adjoints to be at least 1 so that we can Higgs on an adjoint). In this case, the gravitational condition becomes $276 \geq 2x_{\square}+3x_{\square\square}+4x_{\square\square\square}$, which reduces the number of anomaly-free models to 199. Note that these numbers include models with $x_{\square\square\square} = 0$.

We can see already (since $245 > 199$) that there are U(1) models with $|q| \leq 3$ that cannot be obtained by Higgsing an anomaly-free SU(2) model. To explore this further, we need to look at the Higgsing of these SU(2) models down to U(1) models. We again Higgs on an adjoint charge leaving the generator T_3 unbroken, which gives models of the form

$$\left[4b(12-b)+30x_{\square\square\square}\right]\times(\pm\mathbf{1})+\left[b(b-3)-12x_{\square\square\square}\right]\times(\pm\mathbf{2})+2x_{\square\square\square}\times(\pm\mathbf{3}). \quad (6.3)$$

As in section 4.3, these solutions match the form coming from the abelian AC conditions by taking $\tilde{b} = 2b$, as well as $x_3 = 2x_{\square\square\square}$.

6.3 Classification of charge $|q| \leq 3$ U(1) models

We can now divide the anomaly-free U(1) models with charge $|q| \leq 3$ and integer non-negative spectra into five classes, grouped into two broad types, as shown in table 2.

The total set of U(1) models we are considering are those that satisfy the AC conditions, including the gravitational bound. In classifying these models, it is useful to distinguish the gravitational bound from the other AC conditions, which we will loosely refer to as non-gravitational (NG), despite the fact that they include the mixed gauge-gravitational anomaly condition. Along with the NG conditions, we include the requirement that the number of hypermultiplets transforming under any representation must be a non-negative integer. We collectively refer to these conditions by NGIN (Non-Gravitational, Integrality, Non-negativity). Note that we will also use the initialism NGIN to refer to these conditions for SU(N), weakening the integrality condition to allow quaternionic irreducible representations to have half-hypermultiplets.

Each anomaly-free U(1) model can be naïvely unHiggsed to some (potentially inconsistent) SU(2) model; we say that a U(1) model satisfies the SU(2) NGIN conditions if its corresponding unHiggsed SU(2) satisfies these conditions. Similarly, we say that a U(1) model satisfies the SU(2) gravitational bound if its corresponding unHiggsed SU(2) satisfies this bound. Note that we consider all models with $|q| \leq 3$, which includes all models with $|q| \leq 2$.

6.3.1 UnHiggsable U(1)s

First, we consider the models that can be unHiggsed to an anomaly-free SU(2) model. These are subdivided into two classes.

Type	#	Exchangeable	SU(2) NGIN	SU(2) Grav
UnHiggsable	72	✓	✓	✓
	127		✓	✓
Non-unHiggsable	37			✓
	2	✓	✓	
	7		✓	

Table 2. Classes of $|q| \leq 3$ models satisfying the U(1) AC conditions, as well as the integrality and non-negativity conditions. Classes are grouped into two types: those unHiggsable to an anomaly-free SU(2) model and those that are not.

Exchangeable, unHiggsable U(1)s. First, there are those models that can be reached from a U(1) model with $|q| \leq 2$ using exchanges of the form given in (2.16). This U(1) exchange corresponds exactly to the SU(2) exchange (2.13), and in this case none of the corresponding SU(2) exchanges results in a half-integer number of adjoints, thus allowing all U(1) models in this class to be unHiggsed to SU(2). There are 72 such models. Note that these models all satisfy the SU(2) Tate bound $\tilde{b} \leq 24$; in fact, the exchangeable U(1) models (which include two non-unHiggsable models, as discussed in the next section) are precisely those models that satisfy the SU(2) Tate bound.

Non-exchangeable, unHiggsable U(1)s. Second, we have models that cannot be reached by U(1) exchanges from an acceptable $|q| \leq 2$ model, but can nevertheless be unHiggsed to an anomaly-free SU(2) model. Such models have $\tilde{b}(24 - \tilde{b}) < 0$ in (6.1), but have a positive number of ($\pm\mathbf{1}$) representations due to the value of x_3; thus, they satisfy AC, but cannot be exchanged to a maximum charge $q = \pm 2$ model because the result would have a negative number of ($\pm\mathbf{1}$) representations. There are 127 such models.

6.3.2 Non-unHiggsable U(1)s

Next, we consider the models that cannot be unHiggsed to an anomaly-free SU(2) model. These are further subdivided into three classes. Such models are the most interesting because we have the least understanding of how to show whether or not they can be realized in F-theory.

NGIN non-unHiggsable U(1)s. First, we have models that cannot be unHiggsed to an SU(2) model because the resultant SU(2) model would have a negative number of fundamentals (thus violating NGIN). There are 37 such U(1) models, listed in table 3. These models are precisely those that have $x_3 > x_1$; we can see from (6.3) that any U(1) model obtained by Higgsing an SU(2) must have $x_3 \leq x_1$. None of these models are U(1) exchangeable, and they all satisfy the SU(2) gravitational bound.

We can obtain at least some of these models by Higgsing from larger gauge groups. For example, consider Higgsing SU(3) down to a single U(1). To do so, we must Higgs so that exactly one of the generators is left unbroken. Take the Cartan generators T_1, T_2 to be such that the fundamental has charges $(1, 1), (-1, 0), (0, -1)$ and the adjoint has charges

$\tilde{b}$	x_1	x_2	x_3
30	0	108	12
32	14	100	18
34	5	100	23
34	20	94	24
36	3	96	29
36	18	90	30
38	8	88	36
38	23	82	37
40	5	82	43
40	20	76	44
40	35	70	45
42	9	72	51
42	24	66	52
42	39	60	53
44	5	64	59
44	20	58	60
44	35	52	61
44	50	46	62
46	8	52	68
46	23	46	69
46	38	40	70
46	53	34	71
46	68	28	72
48	3	42	77
48	18	36	78
48	33	30	79
48	48	24	80
48	63	18	81
48	78	12	82
50	5	28	87
50	20	22	88
50	35	16	89
50	50	10	90
50	65	4	91
52	14	10	98
52	29	4	99
54	0	0	108

Table 3. Anomaly-free $T = 0$ U(1) models with maximum charge $q = \pm 3$ that cannot be unHiggsed to an anomaly-free SU(2) model because of the NGIN conditions.

$\tilde{b}$	x_1	x_2	x_3
22	254	4	14
24	240	12	16

Table 4. Anomaly-free, exchangeable $T = 0$ U(1) models with $|q| \leq 3$ that cannot be unHiggsed to an anomaly-free SU(2) model because of the gravitational bound.

$(\pm 2, \pm 1), (\pm 1, \pm 2), (\pm 1, \mp 1), 2 \times (0, 0)$ under these generators. We can then Higgs on a charge such that SU(3) breaks down to U(1) × U(1) with T_1, T_2 the generators of the two factors, and then we can further Higgs on the charge $(1, -1)$ so that only the generator $T_1 + T_2$ survives. Under this Higgsing, we have

$$\square \to 2 \times (\pm \mathbf{1}) + 1 \times (\pm \mathbf{2}), \tag{6.4a}$$

$$\begin{array}{c}\square\square\\\square\end{array} \to 4 \times (\mathbf{0}) + 4 \times (\pm \mathbf{3}). \tag{6.4b}$$

We can then Higgs the anomaly-free SU(3) model

$$b = 8: \quad 24 \times \square + 21 \times \begin{array}{c}\square\square\\\square\end{array} \tag{6.5}$$

to yield the U(1) model

$$\tilde{b} = 48: \quad 48 \times (\pm \mathbf{1}) + 24 \times (\pm \mathbf{2}) + 80 \times (\pm \mathbf{3}), \tag{6.6}$$

which appears in the list of 37 U(1) models that cannot be unHiggsed to SU(2) models due to the NGIN conditions. Note that $\tilde{b} = 6b$, as expected from section 3.

In general, for some larger gauge group of rank r, we can Higgs down to a single U(1) by choosing some $r - 1$ linearly independent roots (adjoint charges) in the root space and projecting these out to see how the irreducible representations split.

Exchangeable, gravity non-unHiggsable U(1)s. Second, we have exchangeable U(1) models that satisfy the SU(2) NGIN conditions, but cannot be unHiggsed to an anomaly-free SU(2) model due to the gravitational bound. In other words, these models do not contain enough uncharged scalars to perform the unHiggsing. There are two such models, listed in table 4.

The second of these models, with $\tilde{b} = 24$, can be found in F-theory by a non-UFD construction, as described in [24], with $[\eta_a] \cdot [\eta_b] = 16$.

Non-exchangeable, gravity non-unHiggsable U(1)s. Finally, we have non-exchangeable U(1) models that nevertheless satisfy the SU(2) NGIN conditions, but cannot be unHiggsed to an anomaly-free SU(2) model due to the gravitational bound. There are seven such models, listed in table 5.

We believe that the gravitational bound for SU(2) models that Higgs to U(1) models with maximum charge $q = \pm 3$, $276 \geq 2x_\square + 3x_{\square\square} + 34x_{\square\square\square}$, is less strict than the gravitational bounds associated with other nonabelian groups, which would imply that the nine U(1) models in this and the previous class cannot be unHiggsed to any nonabelian

$\tilde{b}$	x_1	x_2	x_3
26	233	16	19
28	233	16	23
30	225	18	27
32	224	16	32
34	215	16	37
36	213	12	43
38	218	4	50

Table 5. Anomaly-free, non-exchangeable $T = 0$ U(1) models with $|q| \leq 3$ that cannot be unHiggsed to an anomaly-free SU(2) model because of the gravitational condition.

gauge group. We do not have a completely rigorous and exhaustive proof of this assertion, but we give a partial analysis here.

We can see that some of these models cannot be unHiggsed to any SU(N) model via adjoint Higgsings by considering the number of uncharged scalars. If we begin with an SU(N) gauge group, then the "quickest" route (in the sense of producing the fewest uncharged scalars) to a single U(1) unbroken symmetry group is to Higgs on an adjoint charge that breaks to U(1)$^{N-1}$, and then Higgs $N - 2$ more times on charges coming from the original SU(N) adjoints to break to a single U(1). This is the same Higgsing procedure we saw in section 3. In order to carry out this process, the original model must have at least two adjoints. The adjoint to which we give a VEV in the first step splits into the would-be Goldstone bosons that are eaten to give the gauge bosons mass, and the remaining modes are uncharged; we must have at least one other adjoint in order to have adjoint charges on which to Higgs away all but one of the remaining U(1)s.

In the first step, the Cartan generators of the SU(N) are those that become the generators of the unbroken U(1)$^{N-1}$. Every adjoint (including the one given the VEV) contributes $N - 1$ uncharged scalars, which are precisely Cartan charges. In each of the subsequent Higgsings, because we give a VEV to a charge coming originally from an SU(N) adjoint, the negative of this charge also exists; one of these becomes the would-be Goldstone that is eaten, and the other becomes an uncharged scalar. Both such charges from any additional adjoints become uncharged scalars.

Thus, in the entire process, we gain $x_{\mathrm{Adj}}(N - 1)$ uncharged scalars from Higgsing the first adjoint, and then $(2x_{\mathrm{Adj}} - 3)(N - 2)$ additional uncharged scalars from the subsequent Higgsings. Because $x_{\mathrm{Adj}} \geq 2$, the number of uncharged scalars must then be at least

$$x_0 \geq 2(N - 1) + (N - 2) = 3N - 4 \,. \tag{6.7}$$

Type	$\#_4$	$\#_5$	Exchangeable	SU(2) NGIN	SU(2) Grav
UnHiggsable	88	89	✓	✓	✓
	1 157	11 320		✓	✓
Non-unHiggsable	42	43	✓		✓
	3 204	81 320			✓
	7	11	✓	✓	
	358	7 344		✓	
	7	8	✓		
	426	22 182			

Table 6. Classes of $|q| \leq 4$ and $|q| \leq 5$ models satisfying the U(1) AC conditions, as well as the integrality and non-negativity conditions. Classes are grouped into two types: those unHiggsable to an anomaly-free SU(2) model and those that are not.

The models

$$
\begin{aligned}
\tilde{b} = 22 : \quad & 254 \times (\pm\mathbf{1}) + 4 \times (\pm\mathbf{2}) + 14 \times (\pm\mathbf{3}) \,, \\
\tilde{b} = 28 : \quad & 233 \times (\pm\mathbf{1}) + 16 \times (\pm\mathbf{2}) + 23 \times (\pm\mathbf{3}) \,, \\
\tilde{b} = 32 : \quad & 224 \times (\pm\mathbf{1}) + 16 \times (\pm\mathbf{2}) + 32 \times (\pm\mathbf{3}) \,, \\
\tilde{b} = 38 : \quad & 218 \times (\pm\mathbf{1}) + 4 \times (\pm\mathbf{2}) + 50 \times (\pm\mathbf{3}) \,,
\end{aligned}
\tag{6.8}
$$

all have $x_0 = 2$ and are not unHiggsable to SU(2) models, and thus cannot be reached by adjoint Higgsings of any SU(N) model, by the argument above.

6.4 Charges $|q| \leq 4, 5$

We can carry out the same classification for U(1) models with charges $|q| \leq 4$ and $|q| \leq 5$. We classify these U(1) models by which of three properties they satisfy: anomaly equivalence with a $|q| \leq 2$ model ("U(1) exchangeability"), the SU(2) NGIN conditions, and the SU(2) gravitational bound, as explained in section 6.3. There are potentially eight distinct classifications. As we saw in section 6.3, for $|q| \leq 3$ only five of these classes actually contain models.

For $|q| \leq 4$ and $|q| \leq 5$, there exist models in each of the eight possible classes, as shown in table 6. Note that the number of exchangeable models is relatively small. Note also that as the maximum charge grows, the number of models that satisfy NGIN but violate the U(1) gravitational bound (not listed in table 6) grows rapidly. Indeed, the gravitational anomaly bound for U(1) enforces a cutoff on the allowed number of hypermultiplets, and so we expect that as we include higher and higher charges, a vast majority of models satisfying NGIN will not satisfy these bounds. Nonetheless, as discussed in the previous section, there are still large infinite families of models that satisfy NGIN and also the gravitational anomaly condition.

7 F-theory and the swampland

One of the principal questions of this paper is to ascertain the scope of the set of models that appear consistent from the point of view of anomalies and other known quantum consistency conditions, but that are not realized in F-theory. In this section we summarize some of the main results on such "swampland" models.

The main conclusion is that we have found there is an infinite swampland of apparently consistent massless charge spectra for 6D U(1) supergravity models that cannot be realized in F-theory. However, there is no single specific U(1) model that we can identify for which we can prove definitively that there is no F-theory realization.

7.1 Anomaly constraints on U(1) and SU(N) models

For U(1) models with charges $q = \pm 1, \pm 2$, the anomaly constraints imply the condition

$$- 2a \cdot \tilde{b} \le \tilde{b} \cdot \tilde{b} \le -8a \cdot \tilde{b}, \tag{7.1}$$

where $x \le y$ means that $y - x$ is in the "effective" positivity cone of the theory. When the charges are in the range $q \le Q$, following the same logic as described just following (4.5), the anomaly equations imply that any U(1) model must satisfy the constraint

$$\tilde{b} \cdot \tilde{b} \le -2Q^2 a \cdot \tilde{b}. \tag{7.2}$$

For $Q = 3$, this gives the constraint

$$\tilde{b} \cdot \tilde{b} \le 18a \cdot \tilde{b}. \tag{7.3}$$

For SU(2) models with only fundamental and adjoint matter, and at least one adjoint field, the anomaly constraints imply the condition

$$- a \cdot b \le b \cdot b \le -4a \cdot b. \tag{7.4}$$

For SU(3), the analogous constraint is

$$- a \cdot b \le b \cdot b \le -3a \cdot b. \tag{7.5}$$

These constraints match nicely with the observation that Higgsing an SU(N) model on adjoint fields in a natural way gives a U(1) model with $\tilde{b} = N(N-1)b$.

7.2 Evenness condition on $\tilde{b}$

For $T = 0$, the anomaly conditions imply that $\tilde{b}$ is an even integer. For $T > 0$, the condition that $\tilde{b}$ be an even element of the charge lattice is necessary for an F-theory realization, but is not imposed by anomalies. Recently, [9] showed that the condition that $\tilde{b}$ is even follows from some fairly mild assumptions (specifically, they assume that the theory can be compactified on any spin manifold, and that there is an allowed gauge configuration in every topological class). Thus, it seems that the evenness of $\tilde{b}$ is likely a necessary condition for the consistency of the quantum gravity theory, and models with non-even $\tilde{b}$ are presumably inconsistent and therefore not really of interest or in the swampland.

7.3 $T = 0$, $q = \pm 1, \pm 2$

In the limited class of U(1) models where there are no tensor multiplets ($T = 0$), and charges are restricted to $q = \pm 1, \pm 2$, there is a precise match between the set of models that satisfy the anomaly conditions and those that can be realized in F-theory, except for the model with $x_1 = 0, x_2 = 108$, where only the model with fundamental U(1) charge $q = \pm 2$ has a known realization in F-theory. We discuss this exceptional case further below in section 7.8. Other than this case, there is no swampland for $T = 0$, $q = \pm 1, \pm 2$.

7.4   Bounds from F-theory

For $T > 0$ models with tensor multiplets, and for models with massless fields having U(1) charges $|q| > 2$, while all known F-theory models satisfy the anomaly constraints, there are in general a variety of low-energy models that have no known F-theory realization, and infinite classes of anomaly-consistent models of which only a finite number can be realized in F-theory. So in general F-theory constraints are stronger than the anomaly constraints.

The only completely clear explicit constraint that we are aware of, however, on a and b that holds for all F-theory models and is stronger than the anomaly constraints is the "Kodaira bound." In [8], it was pointed out that any F-theory model with an SU(N) gauge symmetry associated with a divisor b (giving the corresponding anomaly coefficient) must satisfy the bound

$$Nb \leq -12a, \tag{7.6}$$

associated with the Kodaira condition that the discriminant is in the class $-12K$. The constraint (7.6) also implies a relation between the R^2 and F^2 terms in the low-energy theory. For SU(2), SU(3), the Kodaira bound becomes, respectively, $b \leq -6a$ and $b \leq -4a$, which have a similar form to, but are slightly weaker than, the upper bounds on b from (7.4) and (7.5). As discussed in section 4.5, when $T > 1$ there are anomaly-free SU(2) models that violate this Kodaira constraint and are therefore in the swampland. There are associated U(1) models that arise from Higgsing these SU(2) models, but we do not have any proof that these cannot be realized in F-theory, though it seems likely that they are also in the swampland.

7.5   Morrison-Park U(1) bounds and UFD SU(N) bounds

F-theory models with U(1) or SU(N) gauge groups that are constructed using standard methodology obey an interesting set of constraints closely parallel to (7.1), (7.4), and (7.5). F-theory models of the Morrison-Park form [14] all satisfy the condition

$$-2a \leq \tilde{b} \leq -8a, \tag{7.7}$$

This condition implies (7.1) in the physically allowed cases, but is stronger than that condition. In fact, the evenness of $\tilde{b}$ and the condition (7.7) seem to be sufficient conditions for an F-theory model to exist. The condition (7.7) is not, however, necessary for an F-theory model to exist; exotic constructions using the methodology of [24] can violate this condition, even for models with only charges $q = \pm 1, \pm 2$.

For any SU(2) model that is realized in F-theory using the standard Tate/UFD construction (4.25), the inequality

$$b \leq -4a \tag{7.8}$$

must hold. For $T = 0$ this is equivalent to the anomaly constraint, but for $T > 0$ this is a stronger condition. Similarly, for any UFD SU(3) model we have the inequality

$$b \leq -3a \,, \tag{7.9}$$

which is again closely analogous to but stronger than the SU(3) anomaly constraint (7.5). As in the abelian case, it is very suggestive that standard F-theory constructions give constraints that are so closely related to the anomaly constraints. Nonetheless, models that violate these UFD constraints can be constructed using the non-UFD methodology of [22].

The connection between these Morrison-Park and UFD constraints and the anomaly constraints for abelian and nonabelian theories is quite intriguing. It raises the natural question of whether there is some signature in the low-energy theory of models that come from a UFD type construction (including Morrison-Park) that would distinguish these models from more exotic models that involve non-UFD algebraic structure in the F-theory realization. We leave this as a provocative question for further research.

7.6 Models with U(1) charges $|q| > 2$

We have identified infinite families of anomaly-consistent charge spectra for U(1) theories. In these families, the charges become arbitrarily large; indeed, there are only a finite number of anomaly-consistent models with any fixed upper bound on the charge.

On the other hand, our understanding of F-theory models with higher U(1) charges is quite limited. Some special cases of U(1) models with charge $|q| = 3$ matter were studied in [33, 34], and a more systematic construction was described in [24]. At this point only a few models with $|q| = 4$ have been explicitly constructed [24], and no models with larger q have been explicitly constructed. The first examples with $q = \pm 3$ matter were found in [33] by systematically studying the set of possible toric hypersurface fiber types. One approach to trying to identify new models with larger U(1) charges would be to investigate more fully the set of toric and complete intersection fiber types using methodology such as that developed in [35, 36]. Another approach would be to develop further the non-UFD approach to describing U(1) models with higher charges, either directly or through Higgsing of nonabelian models with exotic matter as in [22, 24].

In principle, models with up to $|q| = 6$ must be realizable in F-theory; for example, one can get a model with $|q| = 6$ matter by tuning an SU(6) on a quartic on $\mathbb{P}^2$, which has three adjoints, and then Higgsing on two of the adjoints to get a U(1). (A similar construction with SU(5) gives matter with $|q| = 5$.) It is unknown whether any F-theory model can in principle be constructed with U(1) matter having charge $|q| > 6$, but there is clearly a finite upper bound on q as the number of distinct spectra for F-theory models is itself finite. It does not seem straightforward to get a charge greater than $|q| = 6$ in F-theory by breaking an SU(N), and there are some hints [37], based on the approach to charge $|q| = 2$ constructions in recent work [38], that it may be impossible to construct consistent

F-theory models with singularities carrying U(1) matter with $|q| > 6$. Further progress on the F-theory side is clearly needed to understand these kinds of constraints more clearly.

7.7 Constraints on the charge lattice and positivity

We have focused here on explicit questions about the spectrum and constraints related to anomaly constraints. There is a further set of constraints imposed by F-theory, discussed partly in [8], which are a bit more difficult to assess from the point of view of the low-energy theory. In particular, F-theory allows only certain positive cones for charged strings, associated with the cone of effective divisors in the F-theory compactifications space. At present we have no understanding of how the positivity cone of the low-energy theory may be constrained, outside the framework of F-theory. Even assuming that the positivity cone is fixed to be one that comes from F-theory, F-theory has the constraint that whenever a curve of self-intersection $C \cdot C \leq -3$ is in the positive cone, there must be an associated non-Higgsable nonabelian gauge group [31]. No analogue of this argument is known from the point of view of the low-energy theory, though it is natural to imagine that a careful treatment of the worldvolume theory of the strings arising in the charge lattice may provide such constraints. Here we have mostly left these questions to the side, but they certainly must be addressed for a complete understanding or clearing of the 6D supergravity swampland.

7.8 The completeness hypothesis

Another constraint on supergravity models that plays a role in this discussion is the "completeness hypothesis," which states that for a gauge theory coupled to gravity, all possible charges of the gauge group are realized by some state in the Hilbert space [39]. This is related to the idea that there are no global symmetries in a quantum theory coupled to gravity. While there are various arguments for these statements based on black holes and related physics, and the statements have been proven in the AdS/CFT context [40], there is no complete proof of these statements in the 6D flat-space supergravity context relevant for this paper. Thus, this is a hypothesis or folk theorem in this context. Some of the models we encounter should be ruled out by this hypothesis. In particular, we have discussed the model with $T = 0$ and 108 massless fields of charge $q = \pm 2$ with $\tilde{b} = 24$, where the fundamental U(1) unit charge can be chosen to be $q = \pm 1$. This model appears not to be realized in F-theory, and at least naively violates the completeness hypothesis. There is no principle we are aware of, however, that rules out the existence of a massless spectrum with these features in a theory that still has massive fields of charge $q = \pm 1$. Indeed, there are "non-Higgsable" U(1) models that can be realized in F-theory [25, 26, 41] that have no massless charged fields in the low-energy spectrum. Nonetheless, these models should contain massive charged fields, for example associated with massive adjoint fields in an unHiggsing to SU(2). This shows that the massless spectrum does not always completely determine the allowed charge spectrum of the theory. In any case, models such as the $\tilde{b} = 24$ model with fundamental charge $q = \pm 1$ do not appear to have F-theory realizations and seem to be in the swampland; a further study of these models and others that do not obviously satisfy the completeness hypothesis seems like an interesting direction for further research. Note, however, that models such as the model with $x_1 = 0, x_2 = 108, x_3 = 12$ listed in table 3,

which have no fields of charge $q = \pm 1$ but for which the GCD of massless fields is 1, are not in tension with the completeness hypothesis. For such a model, combining massless states of charges $+3$ and -2 will generally give a massive state of charge $+1$. So there is no reason that these spectra should not be possible for consistent theories of quantum gravity.

8 Conclusions

We have explored the anomaly constraints on 6D $\mathcal{N} = 1$ supergravity models with a single U(1) gauge factor. We have found that there are infinite families of distinct spectra that satisfy the anomaly constraints and all other known quantum consistency conditions for these theories. Since F-theory can only give a finite set of distinct spectra, this means that there is an infinite swampland of apparently consistent low-energy models with no UV realization at present.

We have observed some constraints on certain standard F-theory constructions that are closely parallel to anomaly constraints for U(1) models with small charges and SU(N) models with small N. These F-theory constraints are violated by more exotic constructions, but the close relationship between these formulae suggests that there may be some physical meaning to these F-theory constraints. For SU(N) models, the Kodaira condition $Nb \leq -12a$ provides a universal constraint on models that can arise from F-theory, which is violated by some anomaly-consistent low-energy spectra. We do not have a clear understanding of any analogous constraint from F-theory for models with only an abelian U(1) gauge group.

Further work is needed in several directions to explore these questions. It is important to determine whether some new consistency constraint may be identified that would limit the set of consistent U(1) models to a finite space of theories. It would be interesting to look for explicit arguments for the inequalities expected from F-theory associated with the Kodaira constraint, or the stronger constraints on certain classes of models that parallel the low-energy anomaly constraints. All these constraints might be understood in the low-energy theory by identifying inconsistencies in theories where the higher derivative R^2 terms have coefficients that are too small relative to the gauge couplings.

Finally, more work is needed on the F-theory side. We do not at this time have any clear explicit construction of F-theory U(1) models with charges $|q| > 4$, or any systematic construction of models with charges $|q| > 3$. We do not know what the upper bound on U(1) charges in F-theory models may be, although as discussed in the previous section, it may be $|q| \leq 6$. It would also be interesting to further explore the set of consistent U(1) charge spectra in the presence of an additional nonabelian gauge factor, relating anomaly constraints to some of the F-theory structure studied in, e.g., [42]. Some progress in this direction has been made in [43, 44]. A variety of models of this type have been studied using explicit toric constructions [35, 36], which as in the pure U(1) case may help provide useful data for such analyses.

All of these questions suggest that further exploration of the matter fields in U(1) models in 6D supergravity and in F-theory may be a rich arena for further research in the near future.

Acknowledgments

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