

Understanding the cancelation of double poles in the Pfaffian of CHY-formulism

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ABSTRACT: For a physical field theory, the tree-level amplitudes should possess only single poles. However, when computing amplitudes with Cachazo-He-Yuan (CHY) formulation, individual terms in the intermediate steps will contribute higher-order poles. In this paper, we investigate the cancelation of higher-order poles in CHY formula with Pfaffian as the building block. We develop a diagrammatic rule for expanding the reduced Pfaffian. Then by organizing diagrams in appropriate groups and applying the cross-ratio identities, we show that all potential contributions to higher-order poles in the reduced Pfaffian are canceled out, i.e., only single poles survive in Yang-Mills theory and gravity. Furthermore, we show the cancelations of higher-order poles in other field theories by introducing appropriate truncations, based on the single pole structure of Pfaffian.

KEYWORDS: Scattering Amplitudes, Gauge Symmetry

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²The unusual ordering of authors instead of the standard alphabet ordering is for young researchers to get proper recognition of contributions under the current out-dated practice in China.

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1 Introduction

The Cachazo-He-Yuan(CHY) formula [1–5] provides a new perspective to understand scattering amplitudes for massless particles in arbitrary dimensions. The skeleton of CHY-formula consists of so-called scattering equations

$$\mathcal{E}_a = \sum_{b \neq a, b=1}^n \frac{s_{ab}}{z_a - z_b} = 0, \quad a = 1, \dots, n, \quad (1.1)$$

where z_a 's, $a = 1, \dots, n$ are complex variables. The Mandelstam variable s_{ab} is defined by $s_{ab} = 2k_a \cdot k_b$ and k_a denotes the momentum of external particle a . Möbius invariance of scattering equations allows us to reduce the number of independent equations to $(n - 3)$, while the equations (1.1) have $(n - 3)!$ independent solutions. Based on the scattering equations (1.1), CHY formula expresses an n -point tree-level scattering amplitude A_n for massless particles as follows,

$$A_n = \int \frac{dz_1 \dots dz_n}{\text{Vol}[\text{SL}(2, \mathbb{C})]} \prod_a' \delta \left(\sum_{b \neq a} \frac{s_{ab}}{z_{ab}} \right) \mathcal{I}_n^{\text{CHY}}. \quad (1.2)$$

The scattering equations and the measure of CHY-integrals are universal for all theories while $\mathcal{I}_n^{\text{CHY}}$ encodes all the information, including external polarizations, for a specific field theory.

Although the CHY-formulation is simple and beautiful, the evaluation of amplitude is difficult. It is almost impossible to perform computation beyond five points by directly solving the scattering equations due to the complicated nature of algebraic system. A few studies on the direct solutions can be found in [6–15], but restricted to four-dimension and at special kinematics. Alternative methods for evaluating CHY-integrals without explicitly solving scattering equations are proposed by several groups from different approaches. Some of the methods borrow the ideas from computational algebraic geometry, by use of Vieta formula [16], elimination theory [17–19], companion matrix [20], Bezoutian matrix [21]. Based on the polynomial form [22] of scattering equations, polynomial reduction techniques are also introduced in this problem [23, 24]. In [25, 26], differential operators are applied to the evaluation of CHY-integrals. More computational efficient methods are also developed recently. Techniques for contour integration of the CHY-integrals are proposed and sharpened in [27–29], where the concept of Λ scattering equations is introduced, and it results to a recursive computation of CHY-integrand from lower-point sub-CHY-integrands until end up with some simple building blocks. In [30, 31], Feynman-like diagrams are introduced for the evaluation of CHY-integral by some kind of rules, and in [32], Berends-Giele recursions are also applied to the situations where CHY-integrands are products of two Parke-Taylor factors. While along the other routine, string theory inspired method has been developed systematically in [33–35] and [36], named integration rule method. The discovery of cross-ratio identity further sharpens the computational power of integration rule method [37–39], making it simple and automatic for evaluating any generic CHY-integrand.¹

The integration rule method combined with cross-ratio identity has now become an ideal tool for evaluating CHY-integrals. The evaluation of amplitudes in the CHY-formalism can be performed with only the knowledge of CHY-integrands, ignoring the solutions of scattering equations, the CHY-integral measure, etc. However, some issues do need further clarification. For the integration rule method to be valid, the terms to be evaluated should be Möbius invariant. In the current case, it means the CHY-integrand behaves as $\frac{1}{z_i^4}$ under $z_i \rightarrow \infty$. While all the CHY-integrands for the known field theories so far are by construction Möbius invariant, each single term in the expansion of CHY-integrands is not apparently Möbius invariant, which causes trouble for the intermediate computation. This issue is not yet completely solved, but for most CHY-integrands containing the reduced Pfaffian of matrix Ψ , a rewriting of certain entries of matrix Ψ would suffice to make every term in the expansion of CHY-integrands Möbius invariant. For the situations where integration rule method is applicable, we then confront the double pole (or more generically, higher-order pole) problem. For field theories, the physical amplitudes should possess only single poles. However in the setup of CHY-framework, terms in the expansion of CHY-integrands would be evaluated to results of higher-order poles in

¹By private communication, Yong Zhang provides a mathematica code for CHY evaluation. Readers who are interested in the code can contact the email address yongzhang@itp.ac.cn.

the intermediate steps. Of course summing over all results the higher-order poles should be canceled by factors in the numerator, but in most computations we would get a large size of data which makes it impossible to simplify further in a normal desktop. Hence the cancelation of higher-order poles is inexplicit in the final result generated by integration rule method. It is not unexpected that, the origin of higher-order poles can be traced back to the CHY-integrand level, and a thorough understanding of how the cancelation works out in the CHY-integrand level would be a crucial step towards the generalization of CHY-formalism.

In this paper, we systematically study the cancelation of potential higher-order poles in various field theories described by CHY-integrands. This paper is organized as follows. In section 2, we provide a review on the CHY-integrands in various field theories, the expansion of Pfaffian and cross-ratio identities. A diagrammatical expansion of reduced Pfaffian is provided in section 3. The cancelation of double poles in Yang-Mills theory and gravity are investigated in section 4, where explicit examples are provided. General discussions on the cancelation of double poles for other field theories are given in section 5. Conclusion can be found in section 6, and in appendix we give detailed studies on the off-shell and on-shell identities of CHY-integrands and illustrate their applications to simplify complicated CHY-integrands.

2 A review of CHY-integrand, the expansion of Pfaffian and cross-ratio identity

In this section, we provide a review on the CHY-integrand of various field theories and the related knowledge, e.g., the expansion of Pfaffian, the cross-ratio identity and integration rules, which is useful for later discussions.

The CHY-integrands: the field theory is fully described by its corresponding CHY-integrand \mathcal{I}^{CHY} , and in the concern of integration rule method, only CHY-integrand is necessary for the evaluation of amplitude. The building block of CHY-integrands are Parke-Taylor(PT) factor

$$\text{PT}(\alpha) := \frac{1}{z_{\alpha_1\alpha_2}z_{\alpha_2\alpha_3} \cdots z_{\alpha_{n-1}\alpha_n}z_{\alpha_n\alpha_1}}, \quad z_{ij} = z_i - z_j, \quad (2.1)$$

and the Pfaffian and reduced Pfaffian of certain matrix. For n -particle scattering, let us define the following four $n \times n$ matrices A, B, C, X with entries

$$\begin{aligned} A_{a \neq b} &= \frac{k_a \cdot k_b}{z_{ab}}, & B_{a \neq b} &= \frac{\epsilon_a \cdot \epsilon_b}{z_{ab}}, & C_{a \neq b} &= \frac{\epsilon_a \cdot k_b}{z_{ab}}, & X_{a \neq b} &= \frac{1}{z_{ab}}, \\ A_{a=b} &= 0, & B_{a=b} &= 0, & C_{a=b} &= -\sum_{c \neq a} \frac{\epsilon_a \cdot k_c}{z_{ac}}, & X_{a=b} &= 0. \end{aligned} \quad (2.2)$$

Special attention should be paid to the diagonal entries of matrix C since they will break the Möbius invariance of terms in the expansion of CHY-integrands. In the practical computation, the definition

$$C_{aa} = \sum_{i \neq a, t} (\epsilon_a \cdot k_i) \frac{z_{it}}{z_{ia}z_{at}} \quad (2.3)$$

is adopted, which is equivalent to the original definition by momentum conservation and scattering equations. This definition provides a better Möbius covariant representation, i.e., it is uniform weight-2 for z_a and weight-0 for others. The z_t is a gauge choice and can be chosen arbitrary. With matrices A, B, C , we can define a $2n \times 2n$ matrix Ψ ,

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \tag{2.4}$$

where C^T is the transpose of matrix C .

With these building blocks (2.1), (2.2) and (2.4), we are able to construct CHY-integrands for a great number of theories. For such purpose, the Pfaffian of skew-symmetric matrix is introduced. The determinant of an anti-symmetric matrix Ψ is a perfect square of some polynomial, and the Pfaffian $\text{Pf } \Psi$ is defined as the square root of the determinant. In the solution of scattering equations, the $2n \times 2n$ matrix Ψ is degenerate, so we need further to introduce the reduced Pfaffian $\text{Pf}' \Psi$ defined as

$$\text{Pf}' \Psi := \frac{2(-)^{i+j}}{z_{ij}} \text{Pf } \Psi_{(ij)}^{(ij)}, \tag{2.5}$$

where $\Psi_{(ij)}^{(ij)}$ stands for the matrix Ψ with the i -th and j -th column and rows removed. Of course the definition of Pfaffian and reduced Pfaffian applies to any skew-symmetric matrices, for instance the matrix A defined in (2.2).

With above definitions, we list the CHY-integrand for various theories [40] as,

The described theory	\mathcal{I}_L	\mathcal{I}_R
Bi-adjoint scalar	$\text{PT}_n(\alpha)$	$\text{PT}_n(\beta)$
Yang-Mills theory	$\text{PT}_n(\alpha)$	$\text{Pf}' \Psi_n$
Einstein gravity	$\text{Pf}' \Psi_n$	$\text{Pf}' \tilde{\Psi}_n$
Einstein-Yang-Mills theory (single trace)	$\text{PT}_m(\beta) \text{Pf } \Psi_{n-m}$	$\text{Pf}' \tilde{\Psi}_n$
Born-Infeld theory	$(\text{Pf}' A_n)^2$	$\text{Pf}' \Psi_n$
Nonlinear sigma model	$\text{PT}_n(\alpha)$	$(\text{Pf}' A_n)^2$
Yang-Mills-scalar theory	$\text{PT}_n(\alpha)$	$(\text{Pf } X_n) (\text{Pf}' A_n)$
Einstein-Maxwell-scalar theory	$(\text{Pf } X_n) (\text{Pf}' A_n)$	$(\text{Pf } X_n)(\text{Pf}' A_n)$
Dirac-Born-Infeld theory	$(\text{Pf}' A_n)^2$	$(\text{Pf } X_n) (\text{Pf}' A_n)$
Special Galileon theory	$(\text{Pf}' A_n)^2$	$(\text{Pf}' A_n)^2$

where we have used the fact that the CHY-integrands \mathcal{I}^{CHY} is a weight-4 rational functions of z_i 's which can usually be factorized as product of two weight-2 ingredients $\mathcal{I}^{\text{CHY}} = \mathcal{I}_L \times \mathcal{I}_R$.

The expansion of CHY-integrand: the difficulty of evaluation comes from the terms of Pfaffian in the CHY-integrands, which would produce higher-order poles. So a genuine expansion of Pfaffian is possible to simplify our discussion. In [31], it is pointed out that

the reduced Pfaffian $\text{Pf}'\Psi$ can be expanded into cycles as,

$$\text{Pf}'\Psi = -2^{n-3} \sum_{p \in S_n} (-1)^p \frac{W_I U_J \cdots U_K}{z_I z_J \cdots z_K}, \quad (2.6)$$

where the permutation p has been written into the cycle form with cycles I, J, \dots, K . The z_I for a given cycle $I = (i_1, i_2, \dots, i_m)$ is defined as $z_{i_1 i_2} z_{i_2 i_3} \cdots z_{i_m i_1}$. For length- m cycle I , a constant factor $(-1)^{m+1}$ should be considered, which sums together to give the $(-)^p$ factor in (2.6). The open cycle W is defined as

$$W_I = \epsilon_{i\lambda} \cdot (F_{i_2} \cdot F_{i_3} \cdots F_{i_{m-1}}) \cdot \epsilon_{j\nu}, \quad (2.7)$$

in which $\epsilon_{i\lambda}$ and $\epsilon_{j\nu}$ denote the polarizations of particles i, j respecting to the deleted rows and columns (i.e., the gauge choice). The closed cycle U is defined as

$$\begin{aligned} U_I &= \frac{1}{2} \text{Tr}(F_{i_1} \cdot F_{i_2} \cdots F_{i_m}), & (\text{for } I \text{ containing more than one element}), \\ U_I &= C_{ii} & (\text{for } I \text{ contains only one element } i). \end{aligned} \quad (2.8)$$

In eqs. (2.7) and (2.8), $F_a^{\mu\nu}$ is defined as

$$F_a := k_a^\mu \epsilon_a^\nu - \epsilon_a^\mu k_a^\nu. \quad (2.9)$$

The Pfaffian which is used in e.g., EYM theory also have a similar expansion,

$$\text{Pf } \Psi_m = (-1)^{\frac{1}{2}m(m+1)} \sum_{p \in S_m} (-1)^p \frac{U_I U_J \cdots U_K}{z_I z_J \cdots z_K}, \quad (2.10)$$

where Ψ_m is an $2m \times 2m$ sub-matrix of Ψ_n by deleting the rows and columns corresponding to a set of $(n - m)$ external particles.

For presentation purpose, we would use the following notation for open and closed cycles,

$$\begin{aligned} [a_1, a_2, \dots, a_n] &:= z_{a_1 a_2} z_{a_2 a_3} \cdots z_{a_{n-1} a_n}, \\ \langle a_1, a_2, \dots, a_n \rangle &:= z_{a_1 a_2} z_{a_2 a_3} \cdots z_{a_{n-1} a_n} z_{a_n a_1}. \end{aligned} \quad (2.11)$$

The cross-ratio identity and others: to expand the terms of Pfaffian with higher-order poles into terms with single poles, we shall apply various identities on the CHY-integrands [37–39]. Some identities are algebraic, for instance

$$\frac{z_{ab} z_{dc}}{z_{ac} z_{bc}} = \frac{z_{ad}}{z_{ac}} - \frac{z_{bd}}{z_{bc}}, \quad (2.12)$$

which does not require the z to be the solutions of scattering equations. We will call them *off-shell identities*. The other identities are valid only on the solutions of scattering equations, and we will call them *on-shell identities*. An important on-shell identity is the cross-ratio identity,

$$-1 = \sum_{i \in A/\{a\}, j \in A^c/\{b\}} \frac{s_{ij} z_{ia} z_{jb}}{s_A z_{ij} z_{ab}}, \quad (2.13)$$

where A is a subset of $\{1, 2, \dots, n\}$ and A^c is its complement subset. s_A is the Lorentz invariant scalar product of momenta labeled by set A , i.e., $s_A = (\sum_{i \in A} k_i)^2$. Because of momentum conservation we have $s_A = s_{A^c}$. The choice of (a, b) is called the gauge choice of cross-ratio identity, and different gauge choice will end up with different but equivalent explicit expressions.

In the appendix A we will give detailed studies on the various identities and their applications to reduce complicated CHY-integrands to simple ones.

The order of poles: during the process of evaluation, the CHY-integrand is expanded into many Möbius invariant terms, with the generic form,

$$\frac{f(\epsilon, k)}{\prod_{1 \leq i < j \leq n} z_{ij}^{\alpha_{ij}}}, \tag{2.14}$$

where $f(\epsilon, k)$ is kinematic factors, which is irrelevant for the evaluation. The integration rule method provides a way of examining the poles that are to appear in the final result after evaluation as well as the order of poles. The Möbius invariant term (2.14) can be represented by *4-regular graph*, where each z_i is a node and a factor z_{ij} in denominator is represented by a solid line from z_i to z_j while a factor z_{ij} in numerator is represented by a dashed line. We would generically express the factor z_{ij} in numerator as z_{ij} with negative α_{ij} . In this setup, the possible poles of a term (2.14) is characterized by the **pole index** $\chi(A)$ [33–35]:

$$\chi(A) := \mathbb{L}[A] - 2(|A| - 1). \tag{2.15}$$

Here, the linking number $\mathbb{L}[A]$ is defined as the number of solid lines minus the number of dashed lines connecting the nodes inside set A and $|A|$ is the length of set A . For a set $A = \{a_1, a_2, \dots, a_m\}$ with pole index $\chi(A)$, the pole behaves as $1/(s_A)^{\chi(A)+1}$ in the final result. If $\chi(A) < 0$, s_A will not be a pole, while if $\chi(A) = 0$, s_A will appear as a single pole, and if $\chi(A) > 0$, it will contribute to higher-order poles. The higher-order poles do appear term by term in the expansion of CHY-integrals. For example, in Yang-Mills theory with a single reduced Pfaffian, we can have double poles in some terms. While in Gravity theory with two reduced Pfaffian, we can have triple poles in some terms.

As mentioned, the weight-4 CHY-integrand \mathcal{I}^{CHY} has a factorization $\mathcal{I}^{\text{CHY}} = \mathcal{I}_L \times \mathcal{I}_R$ where $\mathcal{I}_L, \mathcal{I}_R$ are weight-2 objects. We can also define the pole index for them as

$$\chi_L(A) := \mathbb{L}[A]_{\mathcal{I}_L} - (|A| - 1), \quad \chi_R(A) := \mathbb{L}[A]_{\mathcal{I}_R} - (|A| - 1), \tag{2.16}$$

and

$$\chi(A) = \chi_L(A) + \chi_R(A), \tag{2.17}$$

where the linking number is now counted inside each \mathcal{I}_L or \mathcal{I}_R . It is easy to see that, for PT-factor given in (2.1) we will always have $\chi(A) \leq 0$. For the reduced Pfaffian or Pfaffian of sub-matrix given in (2.6) and (2.10), we have $\chi(A) \leq 1$. The condition $\chi(A) = 1$ happens when and only when the set A contains one or more cycles (i.e., a cycle belongs to A or their intersection is empty). This explains the observation mentioned above that, for CHY-integrands given by the product of PT-factor and reduced Pfaffian, individual terms

can contribute to double poles, while for gravity theory with CHY-integrands given by the product of two reduced Pfaffian, individual terms can contribute to triple poles.

3 Diagrammatic rules for the expansion of Pfaffian

To evaluate amplitudes via CHY-formula, we should expand the (reduced) Pfaffian as shown in (2.6) and (2.10). In this expansion, there are two pieces of information. One is the variables z_i 's and the other one is the kinematics (k_i 's, ϵ_i 's). The W_I, U_I factors given in (2.7), (2.8) are compact collection of many terms and since each term has its individual character, further expansion of W and U -cycles into terms of products of $(k_i \cdot k_j)$, $(\epsilon_i \cdot k_j)$ and $(\epsilon_i \cdot \epsilon_j)$ is needed. In this section, we establish a diagrammatic rule for representing this expansion.

3.1 Rearranging the expansion of Pfaffian

In (2.6) and (2.10) we sum over all possible permutations p of n elements. This sum can be rearranged as follows. We sum over the distributions of n elements into possible subsets and then sum over all permutations for each subset in a given distribution. Then, for any given term $\cdots U_I \cdots$ containing a cycle $U_I = \frac{1}{2} \text{Tr}(F_{i_1} \cdot F_{i_2} \cdots F_{i_m})$ ($m > 2$), we can always find another term which only differs from the former one by reflecting the U_I -cycle. For example, for $n = 4$, we can have a (1)(234) and also a (1)(432) which are related by a reflection of the second cycle (234). Since both the U -cycle and PT-factor satisfy the same reflection relation,

$$\begin{aligned} U_I &:= \frac{1}{2} \text{Tr}(F_{i_1} F_{i_2} \cdots F_{i_m}) = (-1)^m \frac{1}{2} \text{Tr}(F_{i_1} F_{i_2} \cdots F_{i_m}) := (-1)^m U_{I'}, \\ z_I &:= \frac{1}{z_{i_1 i_2} z_{i_2 i_3} \cdots z_{i_m i_1}} = (-1)^m \frac{1}{z_{i_1 i_m} \cdots z_{i_3 i_2} z_{i_2 i_1}} := (-1)^m z_{I'}, \end{aligned} \quad (3.1)$$

we can pair them together as

$$\left[(-1)^I \frac{U_I}{z_I} + (-1)^{I'} \frac{U_{I'}}{z_{I'}} \right] (-1)^J \cdots (-1)^K \frac{U_J \cdots U_K}{z_J \cdots z_K} = (-1)^{I+J+\cdots+K} \frac{\widetilde{U}_I U_J \cdots U_K}{z_I z_J \cdots z_K}, \quad (3.2)$$

where the sign $(-1)^I$ is 1 when I has odd number of elements and (-1) when I has even number of elements. The \widetilde{U}_I is defined as

$$\widetilde{U}_I := 2U_I = \text{Tr}(F_{i_1} F_{i_2} \cdots F_{i_m}), \quad m > 2. \quad (3.3)$$

The cases with $m = 1$ and $m = 2$ are not included since the reflections of cycles (i_1) and $(i_1 i_2)$ are themselves. So we define $\widetilde{U} = U$ for $m = 1, 2$. The W -cycle is not included since its two ends have been fixed. Using this manipulation, we rewrite the expansion of reduced Pfaffian and Pfaffian of sub-matrix (2.6), (2.10) as

$$\text{Pf}' \Psi = -2^{n-3} \sum_{\tilde{p} \in S_n} (-1)^{\tilde{p}} \frac{W_I \widetilde{U}_J \cdots \widetilde{U}_K}{z_{\tilde{p}}}, \quad z_p := z_I z_J \cdots z_K, \quad (3.4)$$

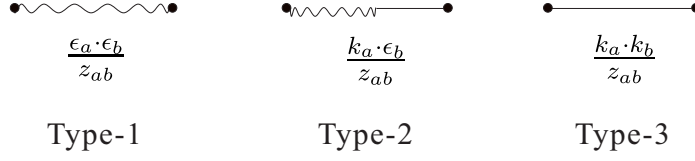


Figure 1. Three types of lines.



Figure 2. If a node i) belongs to a W -cycle but is not an end of W -cycle or ii) belongs to an U -cycle with more than one element, it gets contribution from the corresponding W - or U -cycle as shown by figures (a) and (b). The two structures are related by flipping a sign because they are corresponding to the $p^\mu \epsilon^\nu$ and $-p^\nu \epsilon^\mu$ of $F^{\mu\nu}$.

and

$$\text{Pf } \Psi_m = (-1)^{\frac{1}{2}m(m+1)} \sum_{\tilde{p} \in S_m} (-1)^{\tilde{p}} \frac{\tilde{U}_I \tilde{U}_J \cdots \tilde{U}_K}{z_{\tilde{p}}}, \quad z_p := z_I z_J \cdots z_K. \quad (3.5)$$

Here we sum over all possible partitions of m elements into subsets and for given partition, we sum over reflection independent permutations for each subset. Remember that $\tilde{U}_I = U_I$ when I only contains one or two elements. For example, if $m = 4$, the cycles \tilde{p} of $\text{Pf } \Psi_m$ are given by

$$\left\{ (1)(234), (2)(134), (3)(124), (4)(123), (1)(2)(34), (1)(3)(24), (1)(4)(23), (2)(3)(14), (2)(4)(13), (3)(4)(12), (12)(34), (13)(24), (14)(23), (1234), (1243), (1324), (1)(2)(3)(4) \right\}. \quad (3.6)$$

In the rest of this paper, we will always mention the U -cycles as the \tilde{U} -cycles and use the rearranged expansions (3.4) and (3.5).

3.2 Diagrammatic rules

Now let us establish the diagrammatic rules for writing Pfaffian or reduced Pfaffian explicitly. To do this, we expand each W and \tilde{U} -cycle in terms of products of factors $(\epsilon \cdot \epsilon)$, $(k \cdot k)$ and $(\epsilon \cdot k)$. A diagrammatic interpretation for this expansion can be established as follows,

- We associate nodes with external particles. Two nodes a and b can be connected by (1) type-1 line if we have $\frac{(\epsilon_a \cdot \epsilon_b)}{z_{ab}}$, or (2) type-2 line if we have $\frac{(\epsilon_a \cdot k_b)}{z_{ab}}$, or (3) type-3 line if we have $\frac{(k_a \cdot k_b)}{z_{ab}}$, as shown in figure 1. In this definition, the direction of lines would matter and we will fix the convention of direction later.
- *Contributions from W -cycle:* terms of a W -cycle always have two ends. The two nodes play as the ends of W -cycle should be connected with curved lines, i.e., type-1 line or the curved part of type-2 line. This means if one end of such a line is node a , we only have $(\epsilon_a \cdot \epsilon_i)$ or $(\epsilon_a \cdot k_i)$ but do not have $(k_a \cdot \epsilon_i)$ and $(k_a \cdot k_b)$. Other nodes on W -cycle between the two ends get contributions which are shown by figure 2.

We should also have another type of line connecting the two nodes a and b , which represents $\frac{1}{z_{ba}}$ (although in this paper, we will not deal with W -cycle).

- *Contributions from U -cycles with more than one element:* a U -cycle with more than one element produces loop structures. Each node belongs to an U -cycle also gets two kinds of contributions from this cycle, as shown in figure 2.b. An important point is that the two lines connecting to the node must be one straight line and one wavy line. In the definition of \tilde{U} (3.3), we have required that there are at least three elements. When there are only two elements, we have instead

$$\frac{1}{2} \text{Tr} ((k_a \epsilon_a - \epsilon_a k_a)(k_b \epsilon_b - \epsilon_b k_b)) = (\epsilon_a \cdot k_b)(\epsilon_b \cdot k_a) - (\epsilon_a \cdot \epsilon_b)(k_b \cdot k_a). \quad (3.7)$$

The disappearance of factor $\frac{1}{2}$ is the reason that we can treat U -cycle with at least two elements uniformly. Another thing is that, the U -cycle contains many terms with relative \pm sign. The diagrams with only type-2 lines will have (+) sign, and the sign of others shall be determined from it. We will address the sign rule soon after.

- *Contributions from U -cycles with only one element:* if a node a belongs to a U -cycle with only one element (i.e., C_{aa}), it could be connected with all other nodes via $\frac{(\epsilon_a \cdot k_i)}{z_{ai}}$. More precisely speaking, using (2.3) one line connecting node a and i from C_{aa} should be $\left(\frac{z_{it}}{z_{at}}\right) \frac{(k_i \cdot \epsilon_a)}{z_{ia}}$, where t is the gauge choice. Thus this cycle contributes type-2 lines whose curved part is connected to node a , multiplied by a factor $\left(\frac{z_{it}}{z_{at}}\right)$. This type of cycles can contribute to either loop structure or tree structure.
- *Directions of lines:* for a loop diagram, we read it clockwise. For tree structures (which coming from C_{aa}) connected to loop diagrams, we always read a (type-2) line from the straight part (k) to the curved part (ϵ).
- *Overall signs:* remember that each cycle is associated by a factor 1 when it contains odd number of elements and (-1) when it contains even number of elements. This is the overall sign.

With this diagrammatic interpretation, Pfaffian can be expanded as tree structures rooted at loops. This diagrammatic rule can be regarded as a generalization of spanning tree expression for MHV gravity [41] and EYM amplitudes [10].

3.3 Examples

Now let us take the expansion of Pfaffian $\text{Pf}(\Psi_4)$ as an example to illustrate. There are five types of cycles: $(abcd)$, $(a)(bcd)$, $(ab)(cd)$, $(a)(b)(cd)$ and $(a)(b)(c)(d)$, where a, b, c, d can label as permutations of 1, 2, 3, 4. All reflection independent cycles are already given by (3.6). The $(abcd)$ contains only U -cycle with more than one element, while the $(a)(b)(c)(d)$ only gets contribution from C_{aa} 's. We consider these cycles one by one.

For the U -cycle (1234), we have four possible structures, as shown by the diagrams A(1), A(2), A(3) and A(4) in figure 3. We consider each diagram as a function of ϵ_a, k_a ,

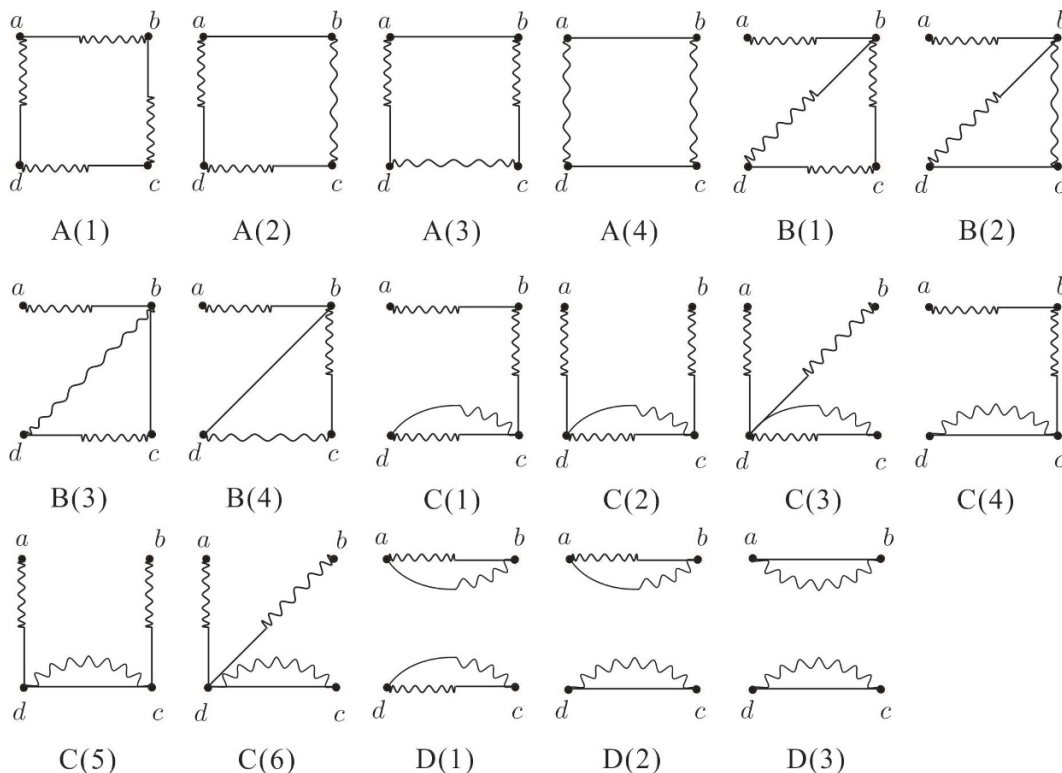


Figure 3. All possible structures in the four-element example.

$\epsilon_b, k_b, \epsilon_c, k_c$ and ϵ_d, k_d , and denote e.g. A(1) by

$$A_1(abcd) := (k_a \cdot \epsilon_b)(k_b \cdot \epsilon_c)(k_c \cdot \epsilon_d)(k_d \cdot \epsilon_a). \quad (3.8)$$

With this notation, $\tilde{U}(1234)$ is given by

$$\begin{aligned} \tilde{U}(1234) = & (-1) \left[A_1(1234) - A_2(1234) - A_2(4123) - A_2(3412) - A_2(2341) \right. \\ & + A_1(1432) - A_2(1432) - A_2(2143) - A_2(3214) - A_2(4321) \\ & \left. + A_3(1234) + A_3(4123) + A_3(3412) + A_3(2341) + A_4(1234) + A_4(4123) \right]. \end{aligned} \quad (3.9)$$

Let us pause a little bit to explain (3.9). With $\text{Tr}(FFFF)$, after expanding we will get 16 terms as in (3.9). However, some terms share the same pattern and in current case, there are four patterns. Now we present a trick to find these patterns for a loop diagram:

- First let us assign a number to three types of lines: 0 for the type-1, 1 for the type-2 and 2 for the type-3. In fact, this number is the mass dimension of these lines. With this assignment, we can write down the cyclic ordered lists for A_i as

$$\begin{aligned} A_1 &\rightarrow (1, 1, 1, 1), & A_2 &\rightarrow (2, 0, 1, 1), \\ A_3 &\rightarrow (2, 1, 0, 1), & A_4 &\rightarrow (2, 0, 2, 0). \end{aligned} \quad (3.10)$$

- Now we can see the construction of patterns for $\tilde{U}(1234)$. First we split 4 into four number n_i to construct the ordered list (n_1, n_2, n_3, n_4) , such that: (1) $n_i \in \{0, 1, 2\}$, (2) $\sum_{i=1}^4 n_i = 4$, (3) If $n_i = 2$, then both n_{i-1}, n_{i+1} can not be 2. Similarly If $n_i = 0$, then both n_{i-1}, n_{i+1} can not be 0. After getting the ordered list, we compare them. If two ordered lists (n_1, n_2, n_3, n_4) and $(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4)$ are the same either by cyclic rotation or by order-reversing, we will say they have defined the same pattern.

With above rule, it is easy to see that, (1) When there are two n_i taking value 2, the only allowed list is $A_4 : (2, 0, 2, 0)$, (2) When there is only one n_i taking value 2, for example, $n_2 = 2$, there are four possibilities $(0, 2, 0, *)$, $(1, 2, 1, *)$, $(0, 2, 1, *)$ and $(1, 2, 0, *)$. However, the sum to be 4 picks only the latter three $(1, 2, 1, 0)$, $(0, 2, 1, 1)$ and $(1, 2, 0, 1)$. Since the last two are related by order-reversing, we get the patterns $A_3 : (1, 2, 1, 0)$ and $A_2 : (1, 2, 0, 1)$, (3) When there is no $n_i = 2$, the only possibility is $A_1 : (1, 1, 1, 1)$.

- In fact, we can get all patterns and their relative sign starting from the fundamental pattern $A_1 : +(1, 1, 1, 1)$,² by the so called *flipping action*. The flipping action is defined as taking two nearby (n_i, n_{i+1}) and changing it to $(n_i - 1, n_{i+1} + 1)$ or $(n_i + 1, n_{i+1} - 1)$. It is worth to notice that the allowed flipping action must satisfy that (1) obtain new $n_i \in \{0, 1, 2\}$, (2) no two 2 or two 0 are nearby. If a pattern is obtained from fundamental pattern by odd number of flipping actions, its sign is negative, while if a pattern is obtained from fundamental pattern by even number of flipping actions, its sign is positive.

Using above rule, it is easy to see that the sign for A_2 is $(-)$ and for A_3, A_4 , $(+)$.

Having done the (1234), we move to the (1)(234) case and the result is

$$\begin{aligned} \tilde{U}(1)\tilde{U}(234) &= \frac{z_{1t}}{z_{2t}} \left[B(1)(1234) + B(1)(1243) - B(2)(1234) - B(2)(1243) \right. \\ &\quad \left. - B(3)(1234) - B(3)(1243) - B(4)(1234) - B(4)(1243) \right] \\ &\quad + \text{Cyclic}\{2, 3, 4\}. \end{aligned} \tag{3.11}$$

Again let us give some explanations. Unlike the case (1234), because of the cycle (1), the node of 2, 3, 4 connecting to node 1 is special, so cyclic symmetry is lost although the order reversing symmetry is still kept. Using the algorithm, we split 3 into three ordered positions to get $B_1 = (1, 1, 1)$, $B_4 = (2, 1, 0)$, $B_3 = (0, 2, 1)$, $B_2 = (1, 0, 2)$ (another three $(2, 0, 1)$, $(1, 2, 0)$, $(0, 1, 2)$ are order reversing comparing to previous three, so we do not count). Furthermore, B_2, B_3, B_4 are obtained from B_1 by one flipping action, so their relative sign is $(-)$.

For the third case (12)(34), the result is

$$\tilde{U}(12)\tilde{U}(34) = D(1)(1234) - D(2)(3412) - D(2)(1234) + D(3)(1234). \tag{3.12}$$

Since each (ab) gives $+(1, 1), -(2, 0)$ patterns, when multiplying together, we get $D_1 = +(1, 1)(1, 1)$, $D_2 = -(1, 1)(2, 0)$, $D_3 = +(2, 0)(2, 0)$ three patterns as shown in figure 3.

²The fundamental pattern is the one with all $n_i = 1$ and its sign is always $+1$. One must be careful for the rule of type-2 line.

The fourth case (1)(2)(3)(4) is a little bit different. Unlike the loop diagram (i.e., cycle with at least two elements) with three types of lines, here we can have only type-2 line for single cycle C_{aa} . Thus the problem is reduced to find the 2-regular graph, i.e., each node has two and only two lines connecting to it. Thus there are only two patterns: D_1 and A_1 . One complication for the single cycle is that there is an extra factor $\frac{z_{it}}{z_{at}}$ attaching to the type-2 line.

For the last case $(a)(b)(cd)$, the situation is the most complicated. The (cd) cycle gives (1, 1) and (2, 0) two patterns, but depending on how single cycles are attached, we can have (1) for a, b attached to each other, it reduces to D_1, D_2, D_3 , (2) for a attached to b , but b attached to, for example, c , it gives C_1, C_4 , (3) for a, b attached to same point, for example, d , it gives C_3, C_6 , (4) for a, b attached to different points, it gives C_2, C_5 .

4 The cancelation of double poles in Yang-Mills theory and gravity

In this section, we investigate the cancelation of higher-order poles in Yang-Mills and gravity theories. The building blocks of these two theories are PT-factor and the reduced Pfaffian. Since for PT-factor, we always have $\chi_L(A) \leq 0$, and the trouble comes from the reduced Pfaffian, where $\chi_R(A) = 1$ do appear. For instance, if we consider higher-order pole with three elements $\{a, b, c\}$,³ we only need to consider the terms with cycles (abc) , $(a)(bc)$, $(b)(ac)$, $(c)(ab)$ and $(a)(b)(c)$. When summing together, it is easy to see that these terms having the form $\dots \text{Pf}(\Psi_{abc})$. According to the definition of linking number, $\mathbb{L} = 3$ for the cycle (abc) because we have $z_{ab}z_{bc}z_{ca}$ in the numerator. Thus $\chi[(abc)] = 3 - 2(3 - 1) = 1$. The cycles $(a)(bc)$ contains a one-element cycle C_{aa} . When expanding C_{aa} by (2.3), we can find two terms $(\epsilon_a \cdot k_b) \frac{z_{bt}}{z_{ba}z_{at}}$ and $(\epsilon_a \cdot k_c) \frac{z_{ct}}{z_{ca}z_{at}}$, each contributes 1 to linking number. Together with (bc) , which contributes 2 to linking number, such a term provides $\mathbb{L} = 1 + 2 = 3$. Then the corresponding pole index is again 1. Similarly, $(c)(ab)$ and $(a)(b)(c)$ also contribute terms with $\chi = 1$. Therefore potential double poles exist in $\dots \text{Pf}(\Psi_{abc})$. This pattern is general, thus for possible double pole s_A , our focus will be $\text{Pf}(\Psi_A)$. We will show by some examples that after using various on-shell and off-shell identities, $\text{Pf}(\Psi_A)$ could effectively have $\chi(A) = 0$, by either terms with explicit $\chi = 1$ having numerator factor s_A or when summing some terms together, the $\chi = 1$ is reduced to $\chi = 0$.

4.1 The cancelation of higher-order poles with two elements

This is the first non-trivial case, and we will study it from different approaches to clarify some conceptual points.

Let us start with explicit evaluation of $\text{Pf}(\Psi_{12})$. There are two cycles (12) and (1)(2). For (12) cycle, the contribution is

$$\frac{\frac{1}{2} \text{Tr}((k_1 \epsilon_1 - \epsilon_1 k_1)(k_2 \epsilon_2 - \epsilon_2 k_2))}{\langle 12 \rangle} = \frac{(k_1 \cdot \epsilon_2)(k_2 \cdot \epsilon_1) - (k_1 \cdot k_2)(\epsilon_1 \cdot \epsilon_2)}{\langle 12 \rangle}, \quad (4.1)$$

³We will not consider the case where elements of the subset A have been chosen as gauge for reduced Pfaffian. We will discuss this situation later.



Figure 4. Diagrams contributing to double pole in s_{12} channel.

where for simplicity the notation (2.11) has been applied. For the cycle (1)(2), when using (2.3), we take the same gauge choice $t = n$, and get

$$\left\{ (\epsilon_1 \cdot k_2) \frac{z_{2t}}{z_{21}z_{1t}} + \sum_{j=3}^{n-1} (\epsilon_1 \cdot k_j) \frac{z_{jt}}{z_{j1}z_{1t}} \right\} \left\{ (\epsilon_2 \cdot k_1) \frac{z_{1t}}{z_{12}z_{2t}} + \sum_{q=3}^{n-1} (\epsilon_2 \cdot k_q) \frac{z_{qt}}{z_{q2}z_{2t}} \right\}. \quad (4.2)$$

It is easy to see that the other terms will have $\chi(\{1, 2\}) \leq 0$, except the following part

$$\left\{ (\epsilon_1 \cdot k_2) \frac{z_{2t}}{z_{21}z_{1t}} \right\} \left\{ (\epsilon_2 \cdot k_1) \frac{z_{1t}}{z_{12}z_{2t}} \right\} = \frac{(k_1 \cdot \epsilon_2)(k_2 \cdot \epsilon_1)}{\langle 12 \rangle}. \quad (4.3)$$

Now we need to combine these two terms with the proper sign and get

$$\frac{(k_1 \cdot \epsilon_2)(k_2 \cdot \epsilon_1)}{\langle 12 \rangle} - \frac{(k_1 \cdot \epsilon_2)(k_2 \cdot \epsilon_1) - (k_1 \cdot k_2)(\epsilon_1 \cdot \epsilon_2)}{\langle 12 \rangle} = \frac{(k_1 \cdot k_2)(\epsilon_1 \cdot \epsilon_2)}{\langle 12 \rangle}. \quad (4.4)$$

We see immediately that, although the denominator $\langle 12 \rangle$ gives $\chi(\{1, 2\}) = 1$, the explicit numerator factor $(k_1 \cdot k_2) = \frac{1}{2}s_{12}$ will reduce double pole to single pole.

Above calculation is correct but a little too rough. We need to show that the result should not depend on the gauge choice for the single cycle. Now let us present a systematic discussion on this issue,

- Firstly, from the expansion (2.3) we see that there are two choices for the gauge t . In the first choice, we choose $t \in A$. In this case, no matter which j is, the linking number is always +1, so we need to sum over all j . In the second choice, we choose $t \notin A$, thus only when $j \in A$, we get the linking number one which contributes to double pole. This tells us that to simplify the calculation, we should take $t \notin A$.
- In our previous calculation, although we have taken $t \notin A$, we have made the special choice to set the same t for both C_{11}, C_{22} . In general we could take two different gauge choices, so we are left with (again, with such gauge choice, only those $j \in A$ are needed to be summed over)

$$\begin{aligned} & \left\{ (\epsilon_1 \cdot k_2) \frac{z_{2t}}{z_{21}z_{1t}} \right\} \left\{ (\epsilon_2 \cdot k_1) \frac{z_{1\tilde{t}}}{z_{12}z_{2\tilde{t}}} \right\} = \frac{(k_1 \cdot \epsilon_2)(k_2 \cdot \epsilon_1)}{\langle 12 \rangle} \frac{z_{2t}z_{1\tilde{t}}}{z_{1t}z_{2\tilde{t}}} \\ & = \frac{(k_1 \cdot \epsilon_2)(k_2 \cdot \epsilon_1)}{\langle 12 \rangle} \frac{z_{2\tilde{t}}z_{1t} + z_{21}z_{t\tilde{t}}}{z_{1t}z_{2\tilde{t}}} \rightarrow \frac{(k_1 \cdot \epsilon_2)(k_2 \cdot \epsilon_1)}{\langle 12 \rangle}. \end{aligned} \quad (4.5)$$

Among the two terms at the second line, since the numerator z_{21} in the second term has decreased the linking number by one, we are left with only the first term, which is the same result as (4.3).

- Now we consider the gauge choice $t \in A$, for example $t = 1$ for C_{22} . Then we will have

$$\begin{aligned} & \left\{ (\epsilon_1 \cdot k_2) \frac{z_{2t}}{z_{21}z_{1t}} \right\} \left\{ \sum_{j=3}^n (\epsilon_2 \cdot k_j) \frac{z_{j1}}{z_{j2}z_{21}} \right\} \\ &= \sum_{j=3}^n \frac{(\epsilon_1 \cdot k_2)(\epsilon_2 \cdot k_j)}{-\langle 12 \rangle} \frac{z_{j1}z_{2t}}{z_{j2}z_{1t}} = \sum_{j=3}^n \frac{(\epsilon_1 \cdot k_2)(\epsilon_2 \cdot k_j)}{-\langle 12 \rangle} \frac{z_{j2}z_{1t} + z_{jt}z_{21}}{z_{j2}z_{1t}}. \end{aligned} \quad (4.6)$$

Again, after dropping the second term, we are left with

$$\sum_{j=3}^n \frac{(\epsilon_1 \cdot k_2)(\epsilon_2 \cdot k_j)}{-\langle 12 \rangle} = \frac{(\epsilon_1 \cdot k_2)(\epsilon_2 \cdot k_1)}{\langle 12 \rangle}, \quad (4.7)$$

which is the same result as (4.3).

By the above detailed discussions, we see that after properly using the various (such as Schouten) identities, momentum conservation and on-shell conditions, we do get the same answer for arbitrary gauge choices. With this clarification, in the latter computations we will take proper gauge choice without worrying the independence with the gauge choice.

Now we will use our diagrammatic rules to re-do above calculation. The purpose of presenting both calculations is to get familiar with our new technique and find the general pattern for later examples. The potential contribution of double poles with two elements 1 and 2 comes from the cycles (12) and (1)(2). There are two kinds of diagrams (see figure 4).⁴ The first diagram gets contribution from both (12) cycle and (1)(2) cycle. Particularly, it reads (noticing that each two-element cycle contains a (-1) and one element cycle contains 1)

$$-\frac{(k_1 \cdot \epsilon_2)(k_2 \cdot \epsilon_1)}{\langle 12 \rangle} + \frac{z_{1t}}{z_{2t}} \frac{(k_1 \cdot \epsilon_2)}{z_{12}} \frac{z_{2t}}{z_{1t}} \frac{(k_2 \cdot \epsilon_1)}{z_{21}} = 0, \quad (4.8)$$

where we choose the same gauge z_t for C_{11} and C_{22} . The second diagram evaluates to

$$\frac{(k_1 \cdot k_2)(\epsilon_1 \cdot \epsilon_2)}{\langle 12 \rangle}. \quad (4.9)$$

Thus we have simply reproduced (4.3).

4.2 The cancelation of higher-order pole with three elements

Now let us consider the cancelation of double poles with three elements using the diagrammatic rules developed in this paper. There are cycles (123), (1)(23), (12)(3), (13)(2) and (1)(2)(3) contributes. We collect their contributions according to the pattern of kinematic factors,

⁴As we have explained above, for single cycle C_{aa} to contribute to $\chi(A) = 1$, we must have $j \in A$, which can be seen clearly if we take gauge $t \notin A$. Thus for cycle (1)(2), we need to consider only the case when node 1 is connected to node 2.

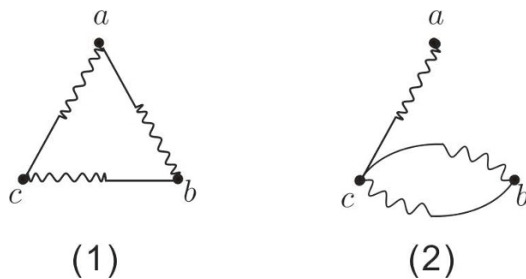


Figure 5. Diagrams contributing to double pole, which only contain type-2 lines.

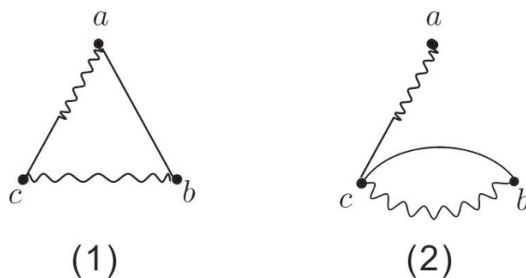


Figure 6. Diagrams contributing to double pole, which contain type-1 and type-3 lines.

- Diagrams containing at least one type-2 loop (loops constructed by only type-2 lines) are shown by figure 5. This is the complicated case since both cases, i.e., U -cycle with at least two elements and single cycle merging, will contribute. Thus figure 5.1 gets contribution from (123) and (1)(2)(3) cycles and can be evaluated to

$$-\frac{(k_a \cdot \epsilon_b)(k_b \cdot \epsilon_c)(k_c \cdot \epsilon_a)}{\langle abc \rangle} + \frac{z_{at}(k_a \cdot \epsilon_b)}{z_{bt}} \frac{z_{bt}(k_b \cdot \epsilon_c)}{z_{ct}} \frac{z_{ct}(k_c \cdot \epsilon_a)}{z_{at}} \frac{1}{z_{ca}} = 0. \quad (4.10)$$

Similarly, figure 5.2 gets contribution from $(a)(bc)$ and $(a)(b)(c)$ cycles and can be evaluated to

$$\frac{(\epsilon_a \cdot k_c)}{z_{ac}} \left[-\frac{(k_a \cdot \epsilon_b)(k_b \cdot \epsilon_a)}{\langle bc \rangle} + \frac{z_{at}(k_a \cdot \epsilon_b)}{z_{bt}} \frac{z_{bt}(k_b \cdot \epsilon_a)}{z_{at}} \frac{1}{z_{ba}} \right] = 0. \quad (4.11)$$

Thus all diagrams containing a type-2 loop are canceled.

- Diagrams do not contain any type-2 loop. In this case, all loop structures should also contain type-1 and type-3 lines. For the case with three elements, we have two kinds of typical diagrams, as shown in figure 6. The first one comes from the cycle (abc) while the second diagram comes from $(a)(bc)$. Thus the two diagrams in figure 6 gives

$$\begin{aligned} & -\frac{\epsilon_a \cdot k_c}{z_{ac}} \frac{\epsilon_c \cdot \epsilon_b}{z_{cb}} \left[\frac{k_b \cdot k_a}{z_{ba}} + \frac{z_{ct} k_b \cdot k_c}{z_{at} z_{bc}} \right] = -\frac{\epsilon_a \cdot k_c}{z_{ac}} \frac{\epsilon_c \cdot \epsilon_b}{z_{cb}} \left[\frac{k_b \cdot k_a}{z_{ba}} \left(1 + \frac{k_b \cdot k_c}{k_b \cdot k_a} \frac{z_{ba} z_{ct}}{z_{bc} z_{at}} \right) \right] \\ & = \frac{\epsilon_a \cdot k_c}{z_{ac}} \frac{\epsilon_c \cdot \epsilon_b}{z_{cb}} \left[\frac{k_b \cdot k_a}{z_{ba}} \sum_{i \neq a, b, c, t} \frac{k_b \cdot k_i}{k_b \cdot k_a} \frac{z_{ba} z_{it}}{z_{bi} z_{at}} \right] = \frac{\epsilon_a \cdot k_c}{z_{ac}} \frac{\epsilon_c \cdot \epsilon_b}{z_{cb}} \left[\sum_{i \neq a, b, c, t} \frac{k_b \cdot k_i z_{it}}{z_{bi} z_{at}} \right], \quad (4.12) \end{aligned}$$

where the cross-ratio identity (2.13) has been used with the subset $A = \{a, b\}$ and gauge choice (a, t) . The part inside the bracket has the following features, (1) cancelation of $(k_a \cdot k_b)$ between numerator and denominator, (2) cancelation of z_{ba} between numerator and denominator. In the final form, the r.h.s. of above expression is still weight-2 graph for all nodes, but the linking number contribution is effectively reduced by one, i.e., $\mathbb{L}(\{a, b, c\}) = 3 - 1 = 2$ by $z_{ac}z_{cb}$.

Before finishing this subsection, let us compare this example with the one in the previous subsection. These two examples have shown two different patterns of removing double poles. In the first example, it is the explicit numerator factor s_{ab} that removes the double pole but the linking number is not changed. In the second example, after using the cross-ratio identity, linking number is effectively decreased by one but there is no s_{abc} factor in the numerator.

4.3 The cancelation of higher-order poles with four elements

It is natural to generalize the discussions above to more complicated cases, which can be summarized as, (1) all diagrams containing at least one type-2 loop should be canceled, (2) the other diagrams (containing type-1 and type-3 lines) are grouped together if they give the same diagram when all type-3 lines in them are removed (e.g., the two diagrams in figure 6). The cancelation of double poles in these diagrams are results of cross-ratio identity. Now let us take the cancelation of double poles with four elements as a more general example to see these two kinds of cancelations.

The cancelation between diagrams containing type-2 loops. A pure type-2 loop can come from either U -cycle with more than one element or a product of U -cycles each containing one element. In the four-element case, the diagrams containing type-2 loops are given by (A1), (B1), (C1), (C2), (C3), (D1) and (D2) in figure 3. We take the (A1) diagram in figure 3 as an example. The diagram (A1) of figure 3 receives the following contribution from U -cycle $(abcd)$ with four elements,

$$-\frac{1}{\langle abcd \rangle} (k_a \cdot \epsilon_b)(k_b \cdot \epsilon_c)(k_c \cdot \epsilon_d)(k_d \cdot \epsilon_a), \quad (4.13)$$

and a contribution from U -cycles $(a)(b)(c)(d)$, each containing one element, i.e.,

$$\frac{1}{\langle abcd \rangle} \frac{z_{at}}{z_{bt}} (k_a \cdot \epsilon_b) \frac{z_{bt}}{z_{ct}} (k_b \cdot \epsilon_c) \frac{z_{ct}}{z_{dt}} (k_c \cdot \epsilon_d) \frac{z_{dt}}{z_{at}} (k_d \cdot \epsilon_a) = \frac{1}{\langle abcd \rangle} (k_a \cdot \epsilon_b)(k_b \cdot \epsilon_c)(k_c \cdot \epsilon_d)(k_d \cdot \epsilon_a), \quad (4.14)$$

where all one-element cycles taking the same gauge choice $t \neq a, b, c, d$. These two contributions are canceled with each other.

This cancelation is easily generalized to cases containing at least one type-2 loop. If we consider a diagram containing a type-2 loop with nodes a_1, a_2, \dots, a_m on it, the U -cycle with m elements $(a_1 a_2 \dots a_m)$ and the product of m one-element U -cycles $(a_i), i = 1, \dots, m)$ contribute to this loop. According to (3.3) and (3.1), the U -cycle $(a_1 a_2 \dots a_m)$ contributes

$$(-1)^{m+1} \text{Tr}(F_{a_1} F_{a_2} \dots F_{a_m}) \quad (4.15)$$

where the prefactor $(-1)^{m+1}$ is -1 for even number of elements, while $+1$ for odd number of elements. A type-2 loop

$$(-1)^{m+1} \frac{1}{\langle a_1 a_2 \cdots a_m \rangle} (-1)^m (k_{a_1} \cdot \epsilon_{a_2}) (k_{a_2} \cdot \epsilon_{a_3}) \cdots (k_{a_m} \cdot \epsilon_{a_1}), \quad (4.16)$$

is obtained by picking the contraction of $-k_{a_i} \epsilon_{a_i}$ from $F_{a_i}^{\mu\nu} = k_{a_i}^\mu \epsilon_{a_i}^\nu - k_{a_i}^\nu \epsilon_{a_i}^\mu$ ($i = 1, \dots, m$) in the trace. According to (2.3), a one-element U -cycle $C_{a_i a_i}$ contains a term

$$(k_{a_{i-1}} \cdot \epsilon_{a_i}) \frac{z_{a_{i-1}t}}{z_{a_{i-1}a_i} z_{a_i t}}, \quad (4.17)$$

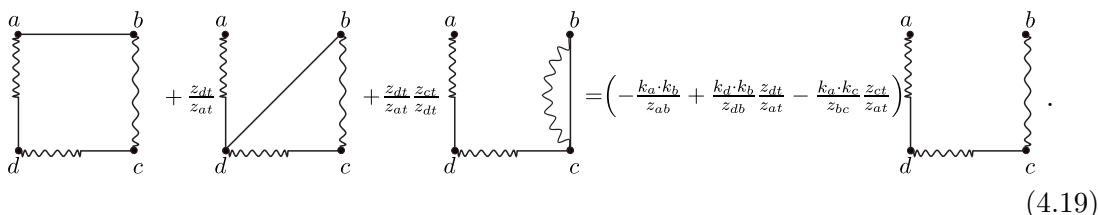
where we define $k_{a_0} \equiv k_n$. The product of one-element U -cycles $C_{a_1 a_1} C_{a_2 a_2} \cdots C_{a_m a_m}$ (see (2.3)) then contributes a type-2 loop

$$\begin{aligned} & \frac{1}{\langle a_1 a_2 \cdots a_m \rangle} \frac{z_{a_1 t}}{z_{a_2 t}} (k_{a_1} \cdot \epsilon_{a_2}) \frac{z_{a_2 t}}{z_{a_3 t}} (k_{a_2} \cdot \epsilon_{a_3}) \cdots \frac{z_{a_m t}}{z_{a_1 t}} (k_{a_m} \cdot \epsilon_{a_1}) \\ &= \frac{1}{\langle a_1 a_2 \cdots a_m \rangle} (k_{a_1} \cdot \epsilon_{a_2}) (k_{a_2} \cdot \epsilon_{a_3}) \cdots (k_{a_m} \cdot \epsilon_{a_1}), \end{aligned} \quad (4.18)$$

where we have chosen the t ($t \neq a_1, \dots, a_m$) for all C_{ii} to be the same. This expression is precisely canceled with the corresponding contribution (4.16) from m -element U -cycle. After such cancelations, the diagrams (A1), (B1), (C1), (C2), (C3), (D1) and (D2) in figure 3 are all canceled. Thus only those diagrams which do not contain any type-2 loop survive.

The cancelation of double poles in diagrams which do not contain any type-2 loop. Now let us turn to the diagrams with no type-2 loop. As shown in the case of three-element poles, we should group together those diagrams which are the same after removing all type-3 lines. The cancelation of double poles can be found by applying cross-ratio identity. In the four-element case, we have the following types of cancelations,

- (1) The first type of cancelation happens between diagrams (A2), (B4) and (C4) with respect to the cycles $(abcd)$, $(a)(dbc)$ and $(a)(d)(bc)$. The potential contributions to four-element higher-order poles are collected as



$$\left(-\frac{k_a \cdot k_b}{z_{ab}} + \frac{k_d \cdot k_b}{z_{db}} \frac{z_{dt}}{z_{at}} - \frac{k_b \cdot k_c}{z_{bc}} \frac{z_{ct}}{z_{at}} \right) \frac{\epsilon_b \cdot \epsilon_c}{z_{bc}} \frac{k_c \cdot \epsilon_d}{z_{cd}} \frac{k_d \cdot \epsilon_a}{z_{da}} \quad (4.19)$$

The r.h.s. of above equation reads

$$\begin{aligned} & \left(-\frac{k_a \cdot k_b}{z_{ab}} + \frac{k_d \cdot k_b}{z_{db}} \frac{z_{dt}}{z_{at}} - \frac{k_b \cdot k_c}{z_{bc}} \frac{z_{ct}}{z_{at}} \right) \frac{\epsilon_b \cdot \epsilon_c}{z_{bc}} \frac{k_c \cdot \epsilon_d}{z_{cd}} \frac{k_d \cdot \epsilon_a}{z_{da}} \\ &= -\frac{k_a \cdot k_b}{z_{ab}} \left(1 + \frac{s_{bd}}{s_{ba}} \frac{z_{ba}}{z_{bd}} \frac{z_{dt}}{z_{at}} + \frac{s_{bc}}{s_{ba}} \frac{z_{ba}}{z_{bc}} \frac{z_{ct}}{z_{at}} \right) \frac{\epsilon_b \cdot \epsilon_c}{z_{bc}} \frac{k_c \cdot \epsilon_d}{z_{cd}} \frac{k_d \cdot \epsilon_a}{z_{da}}. \end{aligned} \quad (4.20)$$

Applying cross-ratio identity (2.13), this contribution becomes

$$\begin{aligned} & \frac{k_a \cdot k_b}{z_{ab}} \left(\sum_{e \neq a,b,c,d,t} \frac{s_{be} z_{ba} z_{et}}{s_{ba} z_{be} z_{at}} \right) \frac{\epsilon_a \cdot k_d \epsilon_d \cdot k_c \epsilon_c \cdot \epsilon_b}{z_{ad} z_{dc} z_{cb}} \\ &= - \left(\sum_{e \neq a,b,c,d,t} \frac{k_b \cdot k_e z_{et}}{z_{be} z_{at}} \right) \frac{\epsilon_a \cdot k_d \epsilon_d \cdot k_c \epsilon_c \cdot \epsilon_b}{z_{ad} z_{dc} z_{cb}}, \end{aligned} \quad (4.21)$$

in which, while keeping the weight-2 conditions for every nodes, the linking number for this part is decreased to $m - 1 = 4 - 1 = 3$.

(2) The second type of cancelation happens between diagrams (A3), (B3) and (C5). Particularly, we have

$$\begin{aligned} & \left(-\frac{k_a \cdot k_b}{z_{ab}} - \frac{k_d \cdot k_b z_{dt}}{z_{db} z_{at}} + \frac{k_a \cdot k_c z_{ct}}{z_{ac} z_{bt}} + \frac{k_d \cdot k_c z_{dt} z_{ct}}{z_{dc} z_{at} z_{bt}} \right) \\ & \left(\text{Diagram with vertices } a, b, c, d \text{ and wavy lines on vertical sides} \right), \end{aligned} \quad (4.22)$$

whose r.h.s. is explicitly written as

$$\left[-\frac{k_a \cdot k_b}{z_{ab}} \left(1 + \frac{s_{bd} z_{ba} z_{dt}}{s_{ba} z_{bd} z_{at}} \right) + \frac{k_a \cdot k_c z_{ct}}{z_{ac} z_{bt}} \left(1 + \frac{s_{cd} z_{ca} z_{dt}}{s_{ca} z_{cd} z_{at}} \right) \right] \frac{\epsilon_a \cdot k_d \epsilon_d \cdot \epsilon_c k_c \cdot \epsilon_b}{z_{ad} z_{dc} z_{cb}}. \quad (4.23)$$

According to the cross-ratio identity (2.13), we have

$$1 + \frac{s_{bd} z_{ba} z_{dt}}{s_{ba} z_{bd} z_{at}} = -\frac{s_{bc} z_{ba} z_{ct}}{s_{ba} z_{bc} z_{at}} - \sum_{e \neq a,b,c,d,t} \frac{s_{be} z_{ba} z_{et}}{s_{ba} z_{be} z_{at}}, \quad (4.24)$$

$$1 + \frac{s_{cd} z_{ca} z_{dt}}{s_{ca} z_{cd} z_{at}} = -\frac{s_{cb} z_{ca} z_{bt}}{s_{ca} z_{cb} z_{at}} - \sum_{e \neq a,b,c,d,t} \frac{s_{ce} z_{ca} z_{et}}{s_{ca} z_{ce} z_{at}}. \quad (4.25)$$

Plugging these identities into (4.23), we immediately arrive at

$$-\frac{1}{2} \left[\sum_{e \neq a,b,c,d,t} \frac{s_{be} z_{et}}{z_{be} z_{at}} - \sum_{e \neq a,b,c,d,t} \frac{s_{ce} z_{et}}{z_{ce} z_{at}} \right] \frac{\epsilon_a \cdot k_d \epsilon_d \cdot \epsilon_c k_c \cdot \epsilon_b}{z_{ad} z_{dc} z_{cb}}. \quad (4.26)$$

Again, while keeping the weight-2 conditions for all nodes, the linking number is decreased to 3.

(3) The third kind of cancelation happens among diagrams (A4) and (D3). In particular,

$$\left(-\frac{k_a \cdot k_b k_c \cdot k_d}{z_{ab} z_{cd}} + \frac{k_a \cdot k_c k_b \cdot k_d}{z_{ac} z_{bd}} + \frac{k_a \cdot k_d k_c \cdot k_b}{z_{ad} z_{cb}} \right) \epsilon_b \cdot \epsilon_c \epsilon_d \cdot \epsilon_a \frac{z_{bc}}{z_{da}}, \quad (4.27)$$

where the r.h.s. reads

$$\left(-\frac{k_a \cdot k_b k_c \cdot k_d}{z_{ab} z_{cd}} + \frac{k_a \cdot k_c k_b \cdot k_d}{z_{ac} z_{bd}} + \frac{k_a \cdot k_d k_c \cdot k_b}{z_{ad} z_{cb}} \right) \frac{\epsilon_b \cdot \epsilon_c \epsilon_d \cdot \epsilon_a}{z_{bc} z_{da}}. \quad (4.28)$$

According to the cross-ratio identity (2.13), we have the following expressions

$$-\frac{k_a \cdot k_b}{z_{ab}} = (k_a \cdot k_c) \frac{z_{ct}}{z_{ac} z_{bt}} + (k_a \cdot k_d) \frac{z_{dt}}{z_{ad} z_{bt}} + \sum_{i \neq a, b, c, d, t} (k_a \cdot k_i) \frac{z_{it}}{z_{ai} z_{bt}}, \quad (4.29)$$

$$-\frac{k_d \cdot k_b}{z_{db}} = (k_d \cdot k_c) \frac{z_{ct}}{z_{dc} z_{bt}} + (k_d \cdot k_a) \frac{z_{at}}{z_{da} z_{bt}} + \sum_{i \neq a, b, c, d, t} (k_d \cdot k_i) \frac{z_{it}}{z_{di} z_{bt}}, \quad (4.30)$$

$$-\frac{k_c \cdot k_b}{z_{cb}} = (k_c \cdot k_a) \frac{z_{at}}{z_{ca} z_{bt}} + (k_c \cdot k_d) \frac{z_{dt}}{z_{cd} z_{bt}} + \sum_{i \neq a, b, c, d, t} (k_c \cdot k_i) \frac{z_{it}}{z_{ci} z_{bt}}, \quad (4.31)$$

where $t \neq a, b, c, d$. Plugging these equations into (4.28), we finally obtain

$$\sum_{i \neq a, b, c, d, t} \left[(k_a \cdot k_i) \frac{z_{it}}{z_{ai} z_{bt}} \frac{k_c \cdot k_d}{z_{cd}} + (k_d \cdot k_i) \frac{z_{it}}{z_{di} z_{bt}} \frac{k_a \cdot k_c}{z_{ac}} - (k_c \cdot k_i) \frac{z_{it}}{z_{ci} z_{bt}} \frac{k_a \cdot k_d}{z_{ad}} \right] \frac{\epsilon_b \cdot \epsilon_c \epsilon_d \cdot \epsilon_a}{z_{bc} z_{da}}, \quad (4.32)$$

which decrease the linking number to $m - 1 = 3$ while keeping the weight-2 conditions for all nodes.

(4) The fourth type of cancelation is between diagrams (B2) and (C6). All such contributions are collected as

$$\left(\frac{k_c \cdot k_d z_{bt}}{z_{cd} z_{at}} + \frac{k_c \cdot k_a z_{bt}}{z_{ca} z_{dt}} - \frac{k_c \cdot k_b z_{bt} z_{bt}}{z_{cb} z_{at} z_{dt}} \right) \epsilon_b \cdot \epsilon_c \epsilon_d \cdot \epsilon_a \frac{z_{bc}}{z_{da}}, \quad (4.33)$$

The r.h.s. is given by

$$\frac{k_c \cdot k_d z_{bt}}{z_{cd} z_{at}} \left(1 + \frac{s_{ca} z_{cd} z_{at}}{s_{cd} z_{ca} z_{dt}} + \frac{s_{cb} z_{cd} z_{bt}}{s_{cd} z_{cb} z_{dt}} \right) \frac{k_b \cdot \epsilon_a \epsilon_d \cdot k_b \epsilon_b \cdot \epsilon_c}{z_{ba} z_{db} z_{bc}}. \quad (4.34)$$

Using the cross-ratio identity (2.13), we arrive at

$$\frac{k_c \cdot k_d}{z_{cd}} \frac{z_{bt}}{z_{at}} \left(- \sum_{e \neq a,b,c,d,t} \frac{s_{ce} z_{cd} z_{et}}{s_{cd} z_{ce} z_{dt}} \right) \frac{k_b \cdot \epsilon_a}{z_{ba}} \frac{\epsilon_d \cdot k_b}{z_{db}} \frac{\epsilon_b \cdot \epsilon_c}{z_{bc}}, \quad (4.35)$$

which decreases the linking number to $m - 1 = 3$.

5 No higher-order poles by more general consideration

In the above section, we have used explicit calculations to show the cancelation of higher-order poles when summing over all contributions. In this section, we will take a different approach to study the same problem. Comparing with the previous method, this new approach is simpler and general, which is the advantage of this method. However, it can not present the explicit picture of how the cancelation happens, which is an advantage of the first method.

Our starting point is to show that the reduced Pfaffian does not contribute to the double pole. The key for this conclusion is that, the expansion of reduced Pfaffian (2.6) is independent of the gauge choice of removing the two rows and columns [2]. Bearing this in mind, we then present the arguments. For a given subset A of n -elements, we can always take the gauge choice (μ, ν) of the reduced Pfaffian, such that $\mu \in A$ and $\nu \notin A$. From (2.16), it is known that for $\chi(A) = 1$, we need subset A to be given as the union of cycles of permutation $p \in S'_n$. However, with our special gauge choice, the cycle W_I does not belong to A , thus we have shown that $\chi(A) \leq 0$ for all terms in the (2.6). Since for any pole (i.e., any subset A), we can always make the gauge choice to show the absence of the double pole as above and the whole result is independent of the gauge choice, we have shown the absence of all possible double poles in the reduced Pfaffian. It is worth to emphasize that in the above argument, the independence of gauge choice for the reduced Pfaffian has played crucial role. However, this fact is true based on both the gauge invariance and scattering equations, so it is the on-shell property.

With above scenario, we can show immediately that when (2.10) appears as a factor in the CHY-integrand, it will not contribute double pole s_A , where A is the subset of these m -particles. The argument is very easy. If we choose the gauge $\mu, \nu \notin A$, the reduced Pfaffian can be written as

$$\text{Pf}' \Psi_n = \text{Pf} \Psi_m \left(\sum \dots \right) + \dots, \quad (5.1)$$

where possible double pole contribution for s_A comes only from the first term at the right handed side. However, since $\text{Pf}' \Psi_n$ does not contain double pole s_A and terms inside $(\sum \dots)$ have different structures of ϵ, k contractions, consistency at both sides will immediately imply that the factor $\text{Pf} \Psi_m$ will not give double poles either by providing an overall factor s_A in numerator or by decreasing the linking number by one after using various on-shell or off-shell identities. This claim can be used to explain the following facts,

- For the single trace part of Einstein-Yang-Mills theory [4] given by

$$\mathcal{I}_{r,s}^{\text{EYM}} = \text{PT}_r(\alpha) \text{Pf} \Psi_S \text{Pf}' \Psi_n, \quad (5.2)$$

the naive counting indicates the $\chi(\mathcal{S}) = 1$. However, as we have argued, the factor $\text{Pf } \Psi_{\mathcal{S}}$ will provide a factor $P_{\mathcal{S}}^2$ in numerator or decrease the linking number by one, so this double pole does not appear.

As a comparison, the double trace of gluons without gravitons in Einstein-Yang-Mills theory has the CHY-integrand [4]

$$\mathcal{I}_{r+s}^{\text{EYM}} = s_r \text{PT}_r(\alpha) \text{PT}_s(\beta) \text{Pf } \Psi_n. \quad (5.3)$$

The double cycle $\text{PT}_r(\alpha) \text{PT}_s(\beta)$ will generate manifest double pole s_r^2 when one integrates z_i 's, thus the explicit kinematic factor s_r is needed to make it to be physical amplitude.

- For Yang-Mills-Scalar theory with q scalars and $r = n - q$ gluons the CHY-integrand is given by [4]

$$\mathcal{I}^{\text{YMs}} = \text{PT}_n(\alpha) (\text{PT}_q(\alpha) \text{Pf } \Psi_r). \quad (5.4)$$

The naive double pole s_q^2 from z -integration will be canceled by the kinematic numerator factor s_q^2 provided by $\text{Pf } \Psi_r$ (the part with effectively reduced linking number will not give double pole after z -integration). Similar argument holds for more general CHY-integrand with q scalars, r gluons and $s = n - q - r$ gravitons,

$$\mathcal{I} = \left(\text{PT}_{q+r}(\alpha) \text{Pf } \Psi_s \right) \times \left(\text{PT}_q(\beta) \text{Pf } \Psi_{r+s} \right). \quad (5.5)$$

So naive double poles of P_s^2 and P_{s+r}^2 will not appear.

5.1 Dimensional reduction to EYM theory

The argument given in (5.1) has shown that $\text{Pf } \Psi_A$ will contribute double pole of s_A . However, it is not obvious that the double poles $s_{B \subset A}$ in $\text{Pf } \Psi_m$ (for example, the CHY-integrand (5.2)) will not appear. To understand this point, we can use the technique of dimensional reduction.

To demonstrate the method, let us focus on the single trace part of Einstein-Yang-Mills theory given in (5.2). We start from gravity CHY-integrand $\text{Pf } \Psi_n(k_i, \epsilon_i, z_i) \text{Pf } \Psi_n(k_i, \tilde{\epsilon}_i, z_i)$, which gives result containing only single poles for all allowed physical configurations. Now we divide n particles into two subsets, $1, 2, \dots, m \in \{\mathfrak{g}\}$ and $m+1, m+2, \dots, n \in \{\mathfrak{h}\}$ and assign the particular physical configurations as follows. Firstly, all momenta in $(D+d)$ -dimensions are split into D -dimensional part and d -dimensional part as

$$\{(k_1, \eta), (k_2, -\eta), (k_i, 0)\}, \quad i = 3, \dots, n, \quad k_i \in R^{1, D-1}, \quad \eta \in R^d, \quad (5.6)$$

where on-shell conditions require

$$\eta^2 = 0, \quad k_i^2 = 0, \quad i = 1, 2, \dots, n, \quad \sum_{i=1}^n k_i = 0. \quad (5.7)$$

Secondly, the polarization vectors are taken as

$$\{(0, \tilde{\epsilon}_1), (0, \tilde{\epsilon}_2), (0, \tilde{\epsilon}_i), (\tilde{\epsilon}_j, 0)\}, \quad i = 3, 4, \dots, m, \quad j = m+1, m+2, \dots, n, \quad (5.8)$$

which satisfy

$$\begin{aligned} \eta \cdot \tilde{\epsilon}_1 = \eta \cdot \tilde{\epsilon}_2 = 0, & \quad \tilde{\epsilon}_i = \tilde{\epsilon}, & \quad i = 3, \dots, m, & \quad \tilde{\epsilon}^2 = 0, \\ \tilde{\epsilon}_1 \cdot \tilde{\epsilon} = 0, & \quad \tilde{\epsilon}_j \cdot k_j = 0, & \quad j = m+1, \dots, n. & \end{aligned} \quad (5.9)$$

This condition can always be achieved when d is large enough. It is obvious that when we do the dimensional reduction from $(D+d)$ to D , polarization assignment in (5.8) means that, particles $\{1, \dots, m\}$ will become the gluons while particles $\{m+1, \dots, n\}$ will remain to be gravitons. Having imposed these conditions, we can see,

- The scattering equation in the full $(D+d)$ -dimensions also implies the scattering equations in D -dimensions since all $K_i \cdot K_j = k_i \cdot k_j$.
- The C_{ii} for gluon subset are given by

$$C_{11} = C_{22} = 0, \quad C_{ii} = \tilde{\epsilon} \cdot \eta \frac{z_{12}}{z_{1i} z_{i2}}, \quad i = 2, 3, \dots, m, \quad (5.10)$$

where we have chosen the gauge $t = 2$ for $i = 3, \dots, m$. The C_{kk} for graviton subset are given as

$$C_{kk} = \sum_{j \neq k} \tilde{\epsilon}_k \cdot k_j \frac{z_{jt}}{z_{jk} z_{kt}}, \quad k = m+1, \dots, n, \quad (5.11)$$

which are nothing but those in Pfaffian of gravitons in D -dimensions.

Now we evaluate the $(D+d)$ -dimensional reduced Pfaffian $\text{Pf}' \Psi_n(k_i, \tilde{\epsilon}_i, z_i)$ by choosing the gauge (1,2). For this choice, the allowed permutations will be the following cycle structures,

$$(1\alpha_1 2)(\alpha_2) \cdots (\alpha_m) C_{j_1 j_1} \cdots C_{j_t j_t}, \quad (5.12)$$

and we consider these cycles one by one as,

- For W -cycle, the numerator is

$$\epsilon_1 \cdot U_{\alpha_1(1)} U_{\alpha_2(2)} \cdots \epsilon_2. \quad (5.13)$$

Now using $U_i = k_i \epsilon_i - \epsilon_i k_i$ and imposing conditions (5.8), (5.9), we see that for all subsets $\alpha_1 \subset \{3, 4, \dots, n\}$, the contraction is zero. So the only non-zero contribution from W -cycle is when $\alpha_1 = \emptyset$ with factor

$$\frac{\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2}{\langle 12 \rangle}. \quad (5.14)$$

- For U -cycle with at least two elements, if $i \in \{3, \dots, r\}$ inside an U -cycle, the combination

$$U_k \cdot (k_i \tilde{\epsilon}_i - \tilde{\epsilon}_i k_i) \cdot U_t \quad (5.15)$$

will be zero, since by our reduction conditions, for any $k \in \{3, 4, \dots, n\}$ we will always have

$$U_k \cdot \epsilon_i = 0. \tag{5.16}$$

In other words, any $i \in \{3, 4, \dots, r\}$ can not be inside an U -cycle with at least two elements.

With above discussions, we see that non-zero contributions are

$$\text{Pf}' \Psi_n(k_i, \tilde{\epsilon}_i, z_i) \rightarrow \frac{\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2}{\langle 12 \rangle} C_{33} \cdots C_{mm} \left\{ \sum \cdots \right\} = \frac{\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2}{\langle 12 \rangle} C_{33} \cdots C_{mm} \text{Pf}' \Psi_G. \tag{5.17}$$

Result (5.17) is not the form (5.2) we are looking for. To reach that, we must use (5.10) and the insertion relation (A.11). Thus

$$\frac{\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2}{\langle 12 \rangle} C_{33} \cdots C_{rr} = \sum_{\alpha} (\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2) (\tilde{\epsilon} \cdot \eta)^{m-2} \text{PT}(1\alpha(3, \dots, m)2). \tag{5.18}$$

Combining (5.17) and (5.18), we see that the $\text{Pf}' \Psi_n(k_i, \tilde{\epsilon}_i, z_i)$ indeed has been reduced to the sum of the form $\text{PT}(1\alpha(3, \dots, m)2) \text{Pf}' \Psi_G$.

Having finished the part $\text{Pf}' \Psi_n(k_i, \tilde{\epsilon}_i, z_i)$, we are left with the part $\text{Pf}' \Psi_n(k_i, \epsilon_i, z_i)$, which is in $(D+d)$ -dimension. To reduce to D -dimension, we must impose proper choice of polarization vectors $\epsilon_i^{(D+d)}$. It is easy to see that the choice $\epsilon_i^{(D+d)} = (\epsilon_i, 0)$ will do the job.

Putting all together we see that, starting from $(D+d)$ -dimensional gravity theory, we are able to reduce to single trace part of EYM theory with CHY-integrand (5.2).⁵ Since the gravity theory does not contain any double poles, so is (5.2). This finishes our general proof.

5.2 Dimensional reduction to $(\text{Pf}' A_n)^2$

In effective theories, such as non-linear sigma model and Dirac-Born-Infeld theory, we also encounter $(\text{Pf}' A_n)^2$. This $(\text{Pf}' A_n)^2$ can also be obtained from $\text{Pf}' \Psi$ by taking appropriate dimensional reduction. Specifically, we impose momenta and polarization vectors in $(d+d+d)$ -dimensions as follows,

$$\begin{aligned} K_a &= (k_a; 0; 0), & \hat{\epsilon}_{a \neq \alpha, \beta} &= (0; \epsilon_a; 0), & \hat{\epsilon}_i &= (0; 0; \epsilon_i), \\ i &= \alpha, \beta, & a &= 1, 2, \dots, n, \end{aligned} \tag{5.19}$$

where α, β are the gauge choice for reduced Pfaffian. With this assignment $K_a \cdot \hat{\epsilon}_b = 0$, so transverse condition of polarization vector has kept and the C -block of matrix Ψ is zero. Thus we have

$$\text{Pf}' \Psi^{(d+d+d)} = \text{Pf}' A^{(d+d+d)} \text{Pf}' B^{(d+d+d)}. \tag{5.20}$$

Furthermore, with the choice in (5.19), we see two facts. First, the $\text{Pf}' A$ in $(d+d+d)$ -dimension is in fact in d -dimension. Second, $\hat{\epsilon}_i \cdot \hat{\epsilon}_a = 0$ when $i = \alpha, \beta$ and $a \neq \alpha, \beta$, thus we have

$$\text{Pf}' B = \frac{(-)^{\alpha+\beta} \epsilon_{\alpha} \cdot \epsilon_{\beta}}{z_{\alpha\beta}} \text{Pf}' B_{\alpha\beta}^{\alpha\beta} \sim \ell^2 k_{\alpha} \cdot k_{\beta} \text{Pf}' A|_{\epsilon_a \rightarrow \ell k_a}, \tag{5.21}$$

⁵If taking the graviton subset to be empty, we have reduced the gravity theory to Yang-Mills theory.

where in the last step, we have set $\epsilon_i = \ell k_i$ for $i = 1, \dots, n$. Putting all together, we see that up to factor $\ell^2 k_\alpha \cdot k_\beta$, we do dimensionally reduce the reduced Pfaffian to $(\text{Pf}' A_n)^2$.

There is one obvious generalization. Instead of just two α, β , we divide all n -particles into m groups, and polarization vectors of each group belongs to independent subspace. Then we can take $\epsilon_a \sim k_a$, so

$$\text{Pf } B \rightarrow \prod_{i=1}^m \text{Pf } A_i. \tag{5.22}$$

6 Conclusions

In this paper, we systematically discuss the cancelation of higher-order poles in CHY-formula. By expanding the cycles of (reduced) Pfaffian into pieces we established a digrammatic representations. Grouping diagrams appropriately and applying cross-ratio identity, we show that the linking number for a pole s_A receives a value of $|A| - 1$ from the Pfaffian. This means there isn't any higher-order poles in Yang-Mills theory and gravity. We then developed the dimensional reduction procedures, by which integrands of other theories can be produced from gravity theory. Thus higher-order poles will not exist in these theories by the consistent reduction.

Inspired by results in this paper, there are several interesting questions worth to investigate. The first thing is that although with explicit examples of two, three, four points, we have shown the pattern how the explicit cancelation of double poles work, writing down the general explicit argument is still welcome.

Another thing is that, in papers [5, 40], CHY-integrands for various field theories have been proposed through various techniques, such as compactifying, generalized dimensional reduction, generalizing, squeezing and extension from soft limit, etc. Starting from a physical meaningful mother theory,⁶ some techniques guarantee physically meaningful daughter theories at the end, such as the compactifying and generalized dimensional reduction. This is exactly the aspect we are using in this paper. However, some techniques, such as squeezing and extension from soft limits, are not so obvious to produce physical meaningful daughter theories at the end. Thus it is definitely important to study these techniques further and to see if all these different techniques can be unified from a single picture. Furthermore, finding the algorithm to read out the daughter theory (i.e., its field contents and Lagrangian) from the known mother theory in various construction techniques is also an interesting question.

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⁶Here the physical meaningful theory means the corresponding tree-level amplitudes possess only single pole, correct factorization and soft limits, etc.

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A The on-shell and off-shell identities of CHY-integrands

In the decomposition of reduced Pfaffian as sum of Parke-Taylor factors, we have taken advantages of many non-trivial relations between rational functions of z_i 's. Some relations are valid at the algebraic level, and we call them off-shell relations. The others are valid only when z_i takes values of the solutions of scattering equations and we call them on-shell ones. The most important one of the latter case is the cross-ratio identities (2.13), derived from the original scattering equations. Any others can be derived from them. For the off-shell identities, we borrow the name from amplitude relations and have (recall the notation (2.11))

- The Schouten identity,

$$\frac{[a b]}{[a c][c b]} = \frac{[a d]}{[a c][c d]} + \frac{[d b]}{[d c][c b]}, \quad (\text{A.1})$$

- The U(1)-decoupling relation,

$$\sum_{\sqcup} \frac{1}{\langle a_1, \{a_2, \dots, a_n\} \sqcup \{b\} \rangle} = 0, \quad (\text{A.2})$$

- The KK-relation,

$$\frac{1}{\langle a_1, \alpha, a_n, \beta \rangle} = (-)^{n_\beta} \sum_{\sqcup} \frac{1}{\langle a_1, \alpha \sqcup \beta^T, a_n \rangle}, \quad (\text{A.3})$$

where n_β is the number of elements in set β , and β^T is the reverse of set β .

The Schouten identity is trivial, by understanding that

$$[a b][c d] = [a c][b d] + [a d][c b].$$

The proof of U(1)-relation: to prove the U(1)-decoupling relation, let us start from the Schouten identity

$$\frac{[a_1, a_n]}{[a_1, b][b, a_n]} = \frac{[a_1, a_{n-1}]}{[a_1, b][b, a_{n-1}]} + \frac{[a_{n-1}, a_n]}{[a_{n-1}, b][b, a_n]}. \quad (\text{A.4})$$

Repeatedly using the Schouten identity

$$\frac{[a_1, a_k]}{[a_1, b][b, a_k]} = \frac{[a_1, a_{k-1}]}{[a_1, b][b, a_{k-1}]} + \frac{[a_{k-1}, a_k]}{[a_{k-1}, b][b, a_k]} \quad (\text{A.5})$$

until $k = 2$, we get

$$\frac{[a_1, a_n]}{[a_1, b][b, a_n]} = \sum_{k=2}^n \frac{[a_{k-1}, a_k]}{[a_{k-1}, b][b, a_k]}. \quad (\text{A.6})$$

Then

$$\begin{aligned} \frac{1}{\langle a_1, a_2, \dots, a_n, b \rangle} &= -\frac{1}{\langle a_1, a_2, \dots, a_n \rangle} \left(\frac{[a_1, a_n]}{[a_1, b][b, a_n]} \right) \\ &= -\sum_{k=2}^n \frac{1}{\langle a_1, a_2, \dots, a_n \rangle} \frac{[a_{k-1}, a_k]}{[a_{k-1}, b][b, a_k]}, \end{aligned} \quad (\text{A.7})$$

which is nothing but the U(1)-relation after canceling the numerator $[a_{k-1}, a_k]$ with its corresponding factor in the $\langle a_1, \dots, a_n \rangle$.

The proof of the KK-relation: the KK-relation can be proven by induction.⁷ For $n_\beta = 1$, it is the U(1)-relation, which has already been proven. Assuming (A.3) is valid for $\beta = \{\beta_1, \beta_2, \dots, \beta_m\}$, when $n_\beta = m + 1$, we have

$$\begin{aligned} \frac{1}{\langle a_1, \alpha, a_n, \beta_1, \beta_2, \dots, \beta_m, \beta_{m+1} \rangle} &= \frac{1}{\langle a_1, \alpha, a_n, \beta_1, \beta_2, \dots, \beta_m \rangle} \frac{[\beta_m, a_1]}{[\beta_m, \beta_{m+1}][\beta_{m+1}, a_1]} \\ &= \sum_{\sqcup} (-)^m \frac{1}{\langle a_1, \alpha \sqcup \{\beta_m, \beta_{m-1}, \dots, \beta_1\}, a_n \rangle} \frac{[\beta_m, a_1]}{[\beta_m, \beta_{m+1}][\beta_{m+1}, a_1]}. \end{aligned} \quad (\text{A.8})$$

The second line can be rewritten as

$$\begin{aligned} &\sum_{s=1}^{n_\alpha} \sum_{\sqcup} (-)^m \frac{1}{[a_1, \alpha_1, \dots, \alpha_s, \beta_m][\beta_m, \{\alpha_{s+1}, \dots, \alpha_{n_\alpha}\} \sqcup \{\beta_{m-1}, \dots, \beta_1\}, a_n][a_n, a_1]} \frac{[\beta_m, a_1]}{[\beta_m, \beta_{m+1}][\beta_{m+1}, a_1]} \\ &= \sum_{s=1}^{n_\alpha} \sum_{\sqcup} (-)^m \frac{[\beta_m, a_1]}{\langle a_1, \alpha_1, \dots, \alpha_s, \beta_m, \beta_{m+1} \rangle [\beta_m, \{\alpha_{s+1}, \dots, \alpha_{n_\alpha}\} \sqcup \{\beta_{m-1}, \dots, \beta_1\}, a_n][a_n, a_1]} \\ &= \sum_{s=1}^{n_\alpha} \sum_{\sqcup, \sqcup_1} (-)^{m+1} \frac{[\beta_m, a_1]}{\langle a_1, \{\alpha_1, \dots, \alpha_s\} \sqcup_1 \{\beta_{m+1}\}, \beta_m \rangle [\beta_m, \{\alpha_{s+1}, \dots, \alpha_{n_\alpha}\} \sqcup \{\beta_{m-1}, \dots, \beta_1\}, a_n][a_n, a_1]} \\ &= \sum_{\sqcup} (-)^{m+1} \frac{1}{\langle a_1, \alpha \sqcup \{\beta_{m+1}, \beta_m, \dots, \beta_1\}, a_n \rangle} = \sum_{\sqcup} (-)^{n_\beta} \frac{1}{\langle a_1, \alpha \sqcup \beta^T, a_n \rangle}, \end{aligned} \quad (\text{A.9})$$

where in the third line we have used the U(1)-relation for $1/\langle a_1, \alpha_1, \dots, \alpha_s, \beta_m, \beta_{m+1} \rangle$. This ends the induction proof.

Some implications can be deduced from the U(1)-relation (A.2) and KK-relation (A.3). From the U(1)-relation with the expression (A.7),

$$\frac{1}{[a_1, a_2, \dots, a_n][a_n, b][b, a_1]} = -\sum_{k=2}^n \frac{1}{[a_n, a_1][a_1, a_2, \dots, a_n]} \frac{[a_{k-1}, a_k]}{[a_{k-1}, b][b, a_k]}, \quad (\text{A.10})$$

we immediately get the so called insertion relation,

$$\frac{1}{[a_1, a_2, \dots, a_n]} \frac{[a_1, a_n]}{[a_1, b][b, a_n]} = \sum_{i=1}^{n-1} \frac{1}{[a_1, \dots, a_i, b, a_{i+1}, \dots, a_n]}, \quad (\text{A.11})$$

where we have inserted the node b between a_1 and a_n . From the KK-relation (A.3), we get the so called open-up relation,

$$\frac{[a_1, a_n]}{\langle a_1, \alpha, a_n, \beta \rangle} = (-)^{n_\beta+1} \sum_{\sqcup} \frac{1}{[a_1, \alpha \sqcup \beta^T, a_n]}, \quad (\text{A.12})$$

⁷Similar discussions can be found in [42].

which opens a closed cycle $\langle a_1, \alpha, a_n, \beta \rangle$ to a sum of open cycles. This relation can be trivially seen by applying KK-relation (A.3) for the denominator. We can diagrammatically abbreviate it as,

$$a_1 \text{---} \bigcirc \text{---} a_n = a_1 \text{---} \bigcirc \text{---} \bigcirc \text{---} \bigcirc \text{---} a_n, \quad (\text{A.13})$$

where a line with white dots means there are other z_i 's locating along the line, with its explicit definition in (A.12).

A sketch of expanding into PT-factors: having presented the off-shell and on-shell identities, now we show how to use them to simplify the CHY-integrand to the PT-factors, which are easily evaluated by integration rule method [33–36] without referring to the scattering equations. This algorithm has been laid out in [39], but here we provide an alternative understanding. It is trivial to see that any weight-2 CHY-integrand can be written as product of a PT-factor with n nodes and cross-ratio factors such as $\frac{[a_i a_j][a_k a_\ell]}{[a_i a_k][a_j a_\ell]}$. Thus by showing the reduction of one cross-ratio factor is suffice to explain any situations. Let us focus on the following CHY-integrand, given as

$$\frac{1}{\langle a_1, a_2, \dots, a_n \rangle} \frac{[a_i a_j][a_k a_\ell]}{[a_i a_k][a_j a_\ell]} = \left(\frac{[a_i a_j]}{\langle a_1, a_2, \dots, a_n \rangle} \right) \frac{[a_k a_\ell]}{[a_i a_k][a_j a_\ell]}. \quad (\text{A.14})$$

Applying (A.12) to the expression in the bracket, we will get two possible results for (A.14), as

$$a_i \text{---} \bigcirc \text{---} a_l \text{---} \bigcirc \text{---} a_k \text{---} \bigcirc \text{---} a_j \quad a_i \text{---} \bigcirc \text{---} a_k \text{---} \bigcirc \text{---} a_l \text{---} \bigcirc \text{---} a_j \quad (\text{A.15})$$

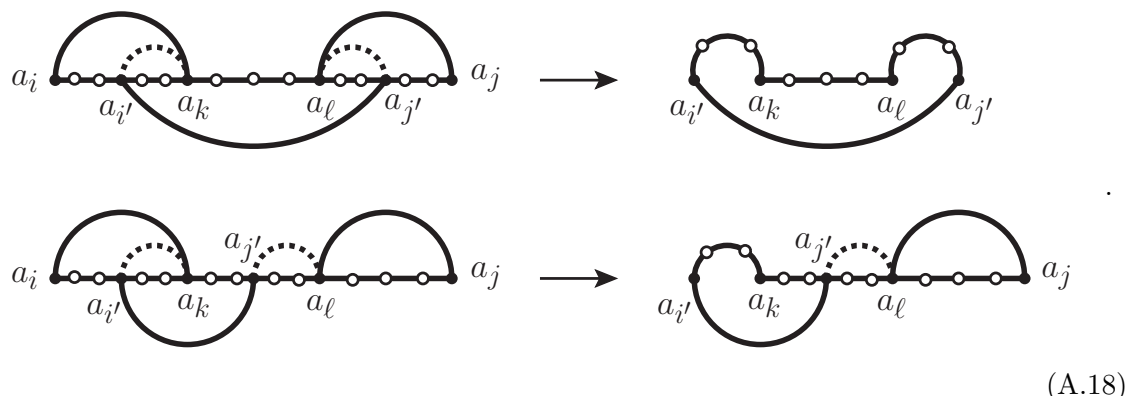
where the expression in the bracket leads to the line with white dots from z_i to z_j , and the other factors denoted by half circles. For the first situation in (A.15), we can again apply (A.13) to the up-half plane, which ends up with

$$a_l \text{---} \bigcirc \text{---} a_k \text{---} \bigcirc \text{---} a_j, \quad (\text{A.16})$$

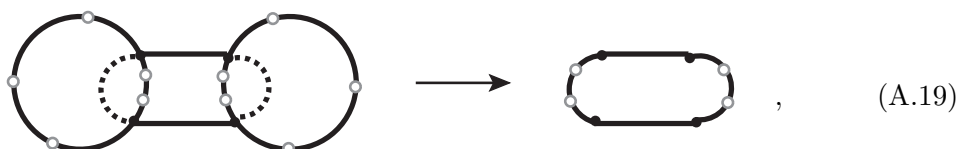
which is a PT-factor. For the second situation in (A.15), we shall use the cross-ratio identity

$$-1 = \sum_{\substack{i' \in A/\{k\} \\ j' \in A^c/\{\ell\}}} \frac{s_{i'j'} [a_{i'} a_k][a_{j'} a_\ell]}{s_A [a_{i'} a_{j'}][a_k a_\ell]}, \tag{A.17}$$

where we choose set A to be collection of z_i 's in the left-most cycle, so j' runs over white dots in between a_k, a_ℓ or those in between a_ℓ, a_j . The factor $[a_k a_\ell]$ in denominator cancels the dashed line, so after multiplying (A.17) to the second figure of (A.15), we get the following contributions depending on the location of $a_{j'}$,



The result in the first line is already PT-factor, while the result in the second line has the same structure as the second figure in (A.15), but with fewer z_i 's in between $a_{j'}, a_\ell$. Recursively applying cross-ratio identity, we will end up with the situation where there is no z_i in between $a_{j'}, a_\ell$, hence the dashed line is canceled and we get two disjoint cycles. In such case, we can apply cross-ratio identity again as



where for the cross-ratio identity (A.17) we have chosen $a_{i'}, a_k$ in one cycle and $a_{j'}, a_\ell$ in the other cycle. Then applying the open-up identity (A.13) in both sides, we get the desired result.

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