

# Cut-touching linear functionals in the conformal bootstrap

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**ABSTRACT:** The modern conformal bootstrap program often employs the method of linear functionals to derive the numerical or analytical bounds on the CFT data. These functionals must have a crucial “swapping” property, allowing to swap infinite summation with the action of the functional in the conformal bootstrap sum rule. Swapping is easy to justify for the popular functionals involving finite sums of derivatives. However, it is far from obvious for “cut-touching” functionals, involving integration over regions where conformal block decomposition does not converge uniformly. Functionals of this type were recently considered by Mazáč in his work on analytic derivation of optimal bootstrap bounds. We derive general swapping criteria for the cut-touching functionals, and check in a few explicit examples that Mazáč’s functionals pass our criteria.

**KEYWORDS:** Conformal Field Theory, Conformal and W Symmetry

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## 1 Introduction

Any conformal field theory (CFT) is characterized by the spectrum of local primary operators and by their operator product expansion (OPE) coefficients, called collectively the CFT data. This data has to satisfy a consistency conditions following from the OPE associativity. The program of constraining or solving the CFT data using the OPE associativity is known as the conformal bootstrap [1–4]. The conformal bootstrap equations are mathematically well defined and can be studied on a computer [5]. This approach has led in the last 10 years to a wealth of rigorous numerical results about CFTs, many of which are currently out of reach of analytical methods.

Following [5], the numerical conformal bootstrap analysis is usually formulated in terms of *linear functionals*, as we will now review (see also [6, 7]). Let us specialize to the case of a one-dimensional (1d) CFT, by which we mean here a theory of local operators in 1d whose correlation functions transform covariantly under the fractional linear transformations  $x \rightarrow (ax+b)/(cx+d)$ . These form the group  $SL(2, \mathbb{R})$ , the 1d counterpart of the group of global conformal transformations in  $d$  dimensions. Consider in such a theory a four point (4pt) correlation function of a primary operator  $\phi$  of scaling dimension  $\Delta_\phi$ . Conformal invariance constrains this correlator to have the form:

$$\langle \phi(x_4)\phi(x_1)\phi(x_2)\phi(x_3) \rangle = |x_{12}|^{-2\Delta_\phi} |x_{34}|^{-2\Delta_\phi} \mathcal{G}(z), \quad (1.1)$$

where  $x_{ij} = x_i - x_j$  and  $z = x_{12}x_{34}/(x_{13}x_{24})$  is the conformally invariant cross ratio. We are assuming the operators to be cyclically ordered on the conformally compactified real axis (as appropriate in 1d). If we put the operators at  $0, z, 1, \infty$  then we have<sup>1</sup>

$$\langle \phi(\infty)\phi(0)\phi(z)\phi(1) \rangle = z^{-2\Delta_\phi} \mathcal{G}(z). \tag{1.2}$$

The function  $\mathcal{G}(z)$  is initially defined on the real interval  $0 < z < 1$  (its analytic continuation will be discussed below). Since we are dealing with a 4pt function of four identical operators, this function satisfies on this interval the following *crossing relation*

$$z^{-2\Delta_\phi} \mathcal{G}(z) = (1-z)^{-2\Delta_\phi} \mathcal{G}(1-z), \tag{1.3}$$

which is the simplest example of a conformal bootstrap equation. Near the endpoints of the interval, we have asymptotic behavior

$$\mathcal{G}(z) \rightarrow 1 \quad (z \rightarrow 0), \quad \mathcal{G}(z) \sim \frac{1}{(1-z)^{2\Delta_\phi}} \quad (z \rightarrow 1). \tag{1.4}$$

This is given by the unit operator contribution in the OPE  $\phi \times \phi$ , and is clearly consistent with the crossing relation.

Furthermore, the function  $\mathcal{G}(z)$  can be expanded into conformal blocks [8–10]:<sup>2</sup>

$$\mathcal{G}(z) = \sum_{i=0}^{\infty} p_i G_{\Delta_i}(z), \quad G_{\Delta}(z) = z^{\Delta} {}_2F_1(\Delta, \Delta, 2\Delta; z). \tag{1.5}$$

Here  $\Delta_i$  are the scaling dimension of all primary operators appearing in the OPE  $\phi \times \phi$ . The  $p_i$  are the squares of the OPE coefficients. We have  $\Delta_0 = 1, p_0 = 1$  corresponding to the unit operator. We will assume that our theory is unitary. In such theories all subsequent operators satisfy the unitarity bound  $\Delta_i > 0$ . Also the OPE coefficients are real in unitary theories, implying  $p_i \geq 0$ . We will also assume for simplicity that the spectrum of operators is discrete without accumulation points, so that there is a finite number of operators below any fixed dimension. This assumption is not crucial and can be relaxed; see appendix B.

Eqs. (1.3), (1.5) can be rewritten as a sum rule

$$\sum p_i F_{\Delta_i}(z) = 0, \quad F_{\Delta}(z) = z^{-2\Delta_\phi} G_{\Delta}(z) - (1-z)^{-2\Delta_\phi} G_{\Delta}(1-z). \tag{1.6}$$

( $F$ 's also depend on  $\Delta_\phi$  but we omit this dependence in the notation.) This equation can be used to put constraints on the allowed unitary CFT spectra [5]. The strategy is to look for a linear functional

$$\omega : f \mapsto \omega(f) \tag{1.7}$$

which satisfies the conditions

$$\omega(F_{\Delta_0}) > 0, \quad \omega(F_{\Delta_i}) \geq 0 \quad (i > 1) \tag{1.8}$$

<sup>1</sup>Defining as usual  $\phi(\infty) = \lim_{z \rightarrow \infty} |z|^{2\Delta_\phi} \phi(z)$ .

<sup>2</sup>For  $d > 1$  conformal blocks depend also on the spin of the exchanged primary and on the second cross ratio  $\bar{z}$ .

for all  $\Delta_i$  in some putative spectrum. Applying  $\omega$  to (1.6) and *assuming we can swap the order of summation with the action of the functional* (a nontrivial requirement since the number of operators in the OPE is always infinite), we have

$$\sum p_i \omega(F_{\Delta_i}) = 0, \tag{1.9}$$

which is impossible in view of (1.8) and of  $p_0 = 1, p_i \geq 0$  in unitary CFTs. The putative spectrum is thus ruled out.

The linear functionals used in [5] and in essentially all subsequent numerical work were linear combinations of a finite number of derivatives at the midpoint  $z = \frac{1}{2}$ :

$$\omega(f) = \sum_{n \leq N} c_n f^{(n)}\left(\frac{1}{2}\right). \tag{1.10}$$

As we will review below, for these functionals the above-mentioned crucial swapping assumption is very easy to justify, basically because the conformal block decomposition converges uniformly near  $z = \frac{1}{2}$ .<sup>3</sup>

Recently Mazáč [12] introduced a new class of linear functionals. Unlike the functionals of [5], his functionals involve integrals of  $f$  over regions of the  $z$  space approaching the analyticity cuts where the conformal block decomposition ceases to converge. The use of such “cut-touching” functionals raises anew the problem of justifying swapping. His functionals are very interesting because, as explained in his paper, they lead to an analytic understanding of some optimal conformal bootstrap bounds previously conjectured by extrapolating numerical results. He verified the relevant conditions analogous to (1.8) in his paper. However, he has not discussed nor even mentioned swapping. This is an unfortunate gap in his otherwise beautiful analysis.

Although some amends were made in the online presentation [13], we consider this issue not fully clarified, and sufficiently important to dedicate this short note to it. The functionals like in [12], or even more complicated ones, may well become widespread in the conformal bootstrap. Anticipating these developments, we will show here a minimal standard of rigor in dealing with such functionals. Following this standard is important to ensure that the results are technically correct. While in this note we focus on the case  $d = 1$ , the standard we impose has a natural extension to  $d > 1$ , and we hope it will be followed there as well.

We start in section 2 by discussing the analytic continuation of the 4pt function and of its conformal block decomposition into the plane of complex  $z$ . In section 3 we formalize the swapping property (along with the more obvious finiteness) which the linear functionals used in the conformal bootstrap must have. For the usual functionals these properties are trivially verified. Then in section 4 we turn to the cut-touching functionals. We derive some general criteria guaranteeing that such functionals obey finiteness and swapping. In section 5 we use our criteria to prove swapping for the functionals used in [12], at least for

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<sup>3</sup>This is also true for functionals in [11] using points away from  $z = \frac{1}{2}$  but within the region of uniform convergence.

the particular cases of low-lying  $\Delta_\phi$  where [12] provides sufficient details. The general case remains incomplete.

Appendix A contains a simple counterexample, showing that taking swapping for granted and proceeding formally can lead to manifestly wrong results. Appendix B deals with the case when the operator spectrum is continuous or has accumulation points.

## 2 Analytic continuation

The function  $\mathcal{G}(z)$ , while originally defined on the interval  $0 < z < 1$ , allows an analytic continuation into the complex plane of  $z$  with cuts along  $(-\infty, 0)$  and  $(1, +\infty)$  (“cut plane”). Analytic continuation is provided by the series (1.5). Clearly, the individual terms in the series are analytic functions in the cut plane. In addition, the series converges in the cut plane. To show this latter fact, it is convenient to work in the  $\rho$  coordinate [14, 15]:

$$\rho(z) = \frac{z}{(1 + \sqrt{1-z})^2}, \quad z(\rho) = \frac{4\rho}{(1+\rho)^2} \tag{2.1}$$

The cut  $z$  plane is thus mapped to the disk  $|\rho| < 1$ .

Consider the series (1.5) transformed to the  $\rho$  coordinate:

$$\tilde{\mathcal{G}}(\rho) = \sum p_i \tilde{G}_{\Delta_i}(\rho), \quad \tilde{\mathcal{G}}(\rho) = \mathcal{G}(z(\rho)), \quad \tilde{G}_\Delta(\rho) = G_\Delta(z(\rho)). \tag{2.2}$$

Using hypergeometric function identities, we have [15]:

$$\tilde{G}_\Delta(\rho) = (4\rho)^\Delta {}_2F_1\left(\frac{1}{2}, \Delta, \Delta + \frac{1}{2}; \rho^2\right). \tag{2.3}$$

We will not actually need this exact formula but three properties of conformal blocks that it implies:

$$\tilde{G}_\Delta(r) \geq 0 \quad (0 \leq r < 1), \tag{2.4}$$

$$\tilde{G}_\Delta(r) = O\left(\log \frac{1}{1-r}\right) \quad (r \rightarrow 1), \tag{2.5}$$

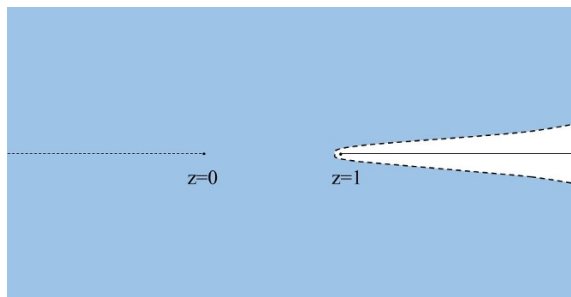
and

$$|\tilde{G}_\Delta(re^{i\theta})| \leq \tilde{G}_\Delta(r). \tag{2.6}$$

The first property is obvious, the second is a standard hypergeometric asymptotics. The last property follows by expanding the hypergeometric function in (2.3) in a power series and noticing that all coefficients are positive (if  $\Delta \geq 0$  as demanded by unitarity).

Now we can finish the convergence proof. For real  $\rho = r$ ,  $0 < r < 1$ , the function  $\tilde{\mathcal{G}}(\rho)$  is finite, and all terms in its series (2.2) are positive, so the series does converge. On the other hand, for complex  $\rho = re^{i\theta}$ ,  $r < 1$ , each term in the series is dominated by its value at  $\rho = r$ . So the series converges in the disk  $|\rho| < 1$ . The original series (1.5) then converges in the cut  $z$  plane.

The above argument is a 1d adaptation of the general  $d$ -dimensional argument from [14]. The argument is robust and can be extended in several directions. For example,



**Figure 1.** A region where the series (1.5) converges uniformly (the image of the disk  $|\rho| \leq 1 - \epsilon$  in the  $z$  plane).

the same argument shows that the convergence in any subdisk  $|\rho| \leq 1 - \epsilon$  is uniform. It is also easy to argue that the convergence in any such subdisk is exponentially fast [14]. For a precise formulation, let  $\kappa$  be any real number in the range  $1 - \epsilon < \kappa < 1$ . Then there is a constant  $C$  such that for any  $\Delta_*$  the tail of the series (2.2) corresponding to summing over  $\Delta_i \geq \Delta_*$  satisfies the uniform bound:

$$\left| \sum_{\Delta_i \geq \Delta_*} p_i \tilde{G}_{\Delta_i}(\rho) \right| \leq C \kappa^{\Delta_*} \quad \text{for all } |\rho| \leq 1 - \epsilon. \quad (2.7)$$

The r.h.s. of this bound becomes exponentially small for large  $\Delta_*$ .<sup>4</sup> The subdisks  $|\rho| \leq 1 - \epsilon$  are mapped onto the subregions of the  $z$  plane shown in figure 1. In any such subregion the series (1.5) converges uniformly and exponentially fast.

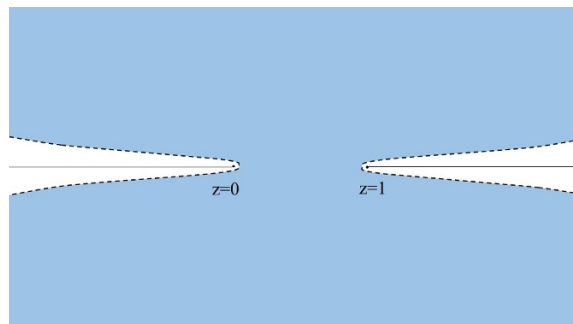
Also notice that the cut through  $(-\infty, 0)$  is present in the above argument only because the factors  $z^\Delta$  and  $\rho^\Delta$  in the conformal blocks have this cut. The convergence is not spoiled by the presence of this cut. In fact the argument proves that the function  $\mathcal{G}(z)$  can be analytically extended through this cut, and one can circle around the origin through a second, third etc sheet. The same is of course true for the cut  $(1, +\infty)$  because the function  $\mathcal{G}(z)$  is crossing symmetric, eq. (1.3) (or because we can equivalently run the argument around  $z = 1$ ). In this way one can explore the full domain of analyticity of  $\mathcal{G}(z)$ , which is an infinitely-sheeted Riemann surface if  $\Delta_\phi$  is an irrational number. In this work we will stay on the first sheet.

A comment is in order concerning the origin of the positivity property of the conformal blocks and of their power series coefficients, which played an important role in the above proof. In terms of the 4pt uncton, passing to the  $\rho$  coordinate corresponds to mapping it conformally to the configuration

$$\langle \phi(-1)\phi(-\rho)\phi(\rho)\phi(1) \rangle. \quad (2.8)$$

For real  $\rho < 1$ , the new configuration is reflection positive. This explains why all terms in the power series expansion of  $\tilde{\mathcal{G}}(\rho)$  have to be positive [14].

<sup>4</sup>One can also put  $\kappa = 1 - \epsilon$  at the cost of making the constant  $C$  grow as a power of  $\Delta_*$  [14], but we will not need this sharper estimate here.



**Figure 2.** A region of uniform convergence of the series in the crossing relation (1.6).

### 3 Functionals: general considerations

So let us go back to the crossing relation (1.6) satisfied by a 4pt function of some 1d CFT. Based on the discussion of the previous section, the following facts are true:

- Functions  $F_\Delta$  are analytic in the cut plane.
- The series converges in the cut plane.<sup>5</sup>
- The convergence is uniform in the subregions where both conditions  $|\rho(z)| < 1 - \epsilon$  and  $|\rho(1 - z)| < 1 - \epsilon$  are satisfied (see figure 2).

We would like to consider linear functionals  $\omega(f)$  which have the following two properties:

- P1. (Finiteness)  $\omega(F_\Delta)$  is finite for any  $\Delta \geq 0$ .
- P2. (Swapping) For any possible 4pt function of an operator of dimension  $\Delta_\phi$ , eq. (1.6) implies that

$$\sum p_i \omega(F_{\Delta_i}) = 0, \tag{3.1}$$

the series converging in the usual sense.

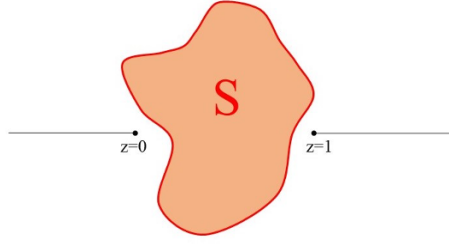
It's important to emphasize that the functional should be defined not just on the functions  $F_\Delta$  and on their finite linear combinations, but on a wider class of functions.<sup>6</sup> This class should at the very least include the functions  $\mathcal{F}_{\Delta_*}(z)$  which will be introduced shortly.

In practice, the functional  $\omega(f)$  will be given by some sort of integral or a combination of derivatives and property P1 should be relatively easy to check, especially given that

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<sup>5</sup>The following comment may be useful. As mentioned in the previous section, the 4pt function can be analytically extended through the branch cuts to an infinitely-sheeted Riemann surface. The original crossing relation in the form (1.3) is then true everywhere on this surface with appropriate identification  $z \rightarrow 1 - z$ , not just on the first sheet. However, it has to be emphasized that the crossing relation in the series form (1.6) really stops converging on the branch cuts. This precludes any straightforward use of the series relation beyond the first sheet.

<sup>6</sup>In this respect the notation of eqs. (2.8) and (2.22) of [12] is confusing, while that in [13] is OK.



**Figure 3.** Support of integration in the functional of example 1.

the conformal blocks in 1d are known explicitly. Property P2 is more tricky. It can be “derived” by applying functional  $\omega(f)$  to both sides of (1.6). However this is formal since it requires interchanging the action of the functional with infinite summation. Sometimes this formal argument is easy to justify, sometimes more work is needed. We will see examples in a second.

Assuming that P1 holds, the strategy to establish P2 is as follows. Split (1.6) into two parts (we switch from summing over  $i$  to summing over the discrete set of occurring  $\Delta$ ’s):

$$\sum_{\Delta < \Delta_*} p_\Delta F_\Delta(z) + \mathcal{F}_{\Delta_*}(z) = 0, \quad \mathcal{F}_{\Delta_*}(z) \equiv \sum_{\Delta \geq \Delta_*} p_\Delta F_\Delta(z). \quad (3.2)$$

Now we can apply  $\omega$  and get:

$$\sum_{\Delta < \Delta_*} p_\Delta \omega(F_\Delta) + \omega(\mathcal{F}_{\Delta_*}) = 0. \quad (3.3)$$

Notice that here we interchanged the functional with a *finite* summation, which is always a legal operation. Furthermore, the function  $\mathcal{F}_{\Delta_*}$  goes to zero in the cut plane as  $\Delta_* \rightarrow \infty$ , uniformly so in the regions shown in figure 2. So we may expect that, under wide conditions on the functional  $\omega$ ,

$$\omega(\mathcal{F}_{\Delta_*}) \rightarrow 0 \quad (\Delta_* \rightarrow \infty). \quad (3.4)$$

If we can show this rigorously, then (3.1) follows and we are done. This is what it takes to justify the formal argument.

Let us consider two examples where (3.4) is immediate.

**Example 1.** Suppose the functional  $\omega$  is given by an integral over some integrable measure  $d\mu$  whose support  $S$  is a bounded set, which is fully contained in the cut plane and does not touch the cuts (see figure 3):

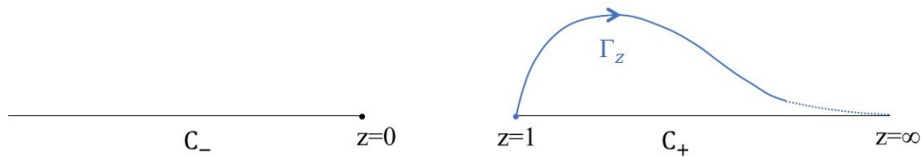
$$\omega(f) = \int_S d\mu f(z). \quad (3.5)$$

Then (3.4) follows trivially from the uniform convergence of (1.6) on  $S$ .

**Example 2.** Suppose the functional  $\omega(f)$  is a derivative of a finite order  $n$  at a point  $z_0$  lying strictly inside the cut plane:

$$\omega(f) = f^{(n)}(z_0). \quad (3.6)$$





**Figure 4.** Contour  $\Gamma_z$ . Also shown are the two cuts of the cut plane.

This example can be reduced to the previous one, by representing the derivative via Cauchy’s formula as a contour integral over a circle fully contained in the cut plane.

Clearly, a finite linear combination of derivatives will do as well. The functionals (1.10) used in the numerical bootstrap belong to this class. The simplicity of verification of (3.1) in this case explains why it was left implicit in the literature. For example, the authors of ref. [14] carefully established the convergence of the conformal block decomposition in the cut plane and stated that this puts the numerical conformal bootstrap results on “mathematically solid ground”. What they had in mind was a kind of the above argument.

#### 4 Cut-touching functionals

We will now consider a functional of the following form:

$$\omega(f) = \text{Im} \int_{\Gamma_z} dz H(z) f(z), \quad (4.1)$$

where  $H(z)$  is a fixed analytic function in the upper half-plane. The function  $f(z)$  on which the functional acts is also assumed analytic in the upper half-plane (in fact it will be analytic in the cut plane). The contour  $\Gamma_z$  starts at  $z = 1$  and ends at  $z = +\infty$ , as shown in figure 4. Of course since the functions are analytic we may deform the contour. For example, we may want to make it run along the cut. Such contour deformations may be useful in actual explicit calculations, but for the proof of properties P1, P2 it will be convenient to keep the contour in the bulk of the upper half-plane, touching its boundary only at two points as shown.

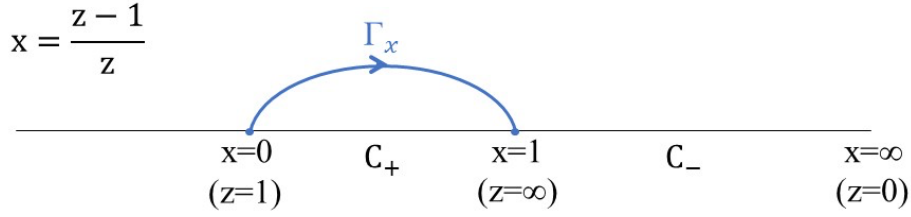
As in [12], let us pass from the coordinate  $z$  to coordinate

$$x(z) = \frac{z-1}{z}, \quad z(x) = \frac{1}{1-x}. \quad (4.2)$$

The upper half-plane of  $z$  is mapped to the upper half-plane of  $x$  with points  $0, 1, \infty$  and the contour mapped as in figure 5. It is equivalent but more convenient to analyze the functional in terms of the  $x$  coordinate:

$$\omega(f) = \text{Im} \int_{\Gamma_x} dx h(x) (1-x)^{-2\Delta_\phi} f(z(x)). \quad (4.3)$$

The function  $h(x) = H(z(x)) z'(x) (1-x)^{2\Delta_\phi}$  may have some singularities on the real axis but we will assume it is analytic in the upper half-plane. The factor  $(1-x)^{-2\Delta_\phi}$  is factored



**Figure 5.** Contour  $\Gamma_x$ . Also shown are the images of the two cuts of the cut plane under the transformation from  $z$  to  $x$ . The function  $h(x)$  and the functions  $f(z(x))$  on which the functional is evaluated will be analytic in the upper half-plane.

out for future convenience, as in [12]. We will assume in our analysis that contour  $\Gamma_x$  approaches  $x = 0, 1$  not tangentially to the real axis.

It's clear that for such functionals the proof of swapping given above for examples 1, 2 cannot be applied. The problem is that the convergence of the series (1.6) near  $z = 1, +\infty$  (which map to  $x = 0, 1$ ) is not uniform. To establish (3.4), we will need to understand how  $\mathcal{F}_{\Delta_*}$  behaves near these points. The condition for swapping, whatever it is, will depend in a nontrivial way on  $\Delta_\phi$  and on the asymptotics of  $h(x)$  near  $x = 0, 1$ . Our goal here will be to work out this condition.

#### 4.1 Finiteness

To check finiteness, we need to estimate how  $F_\Delta(z(x))$  behaves near  $x = 0, 1$ . For  $x \rightarrow 0$  we have  $z \approx 1 + x$ ,

$$G_\Delta(z) = O(\log 1/|x|), \quad G_\Delta(1-z) = O(|x|^\Delta). \quad (4.4)$$

For  $x = 1 + \epsilon$ ,  $\epsilon \rightarrow 0$  we have  $z \approx -1/\epsilon$ . To estimate  $G_\Delta(z)$  we pass to the  $\rho$  coordinate:

$$\rho = \rho(z) \approx \rho(-1/\epsilon) \approx -1 + 2\sqrt{\epsilon}. \quad (4.5)$$

For the crossed channel we have:

$$\rho' = \rho(1-z) \approx \rho(1/\epsilon) \approx -1 - 2i\sqrt{\epsilon}. \quad (4.6)$$

Since we are assuming that  $\epsilon$  is not parallel to the real axis we have both

$$|\rho|, |\rho'| = 1 - O(\sqrt{|\epsilon|}). \quad (4.7)$$

Using the estimate (2.6), we have

$$|\tilde{G}_\Delta(\rho)| \leq \tilde{G}_\Delta(|\rho|) = O(\log 1/|\epsilon|), \quad (4.8)$$

and analogously for  $\tilde{G}_\Delta(\rho')$ .

Combining the above estimates for  $G_\Delta$ 's we can estimate  $F_\Delta$ . We have:

$$x \rightarrow 0: \quad (1-x)^{-2\Delta_\phi} F_\Delta(z(x)) = O(\log 1/|x|) + O(|x|^{\Delta-2\Delta_\phi}) = O(|x|^{-2\Delta_\phi}), \quad (4.9)$$

where we used  $\Delta \geq 0, \Delta_\phi > 0$ . Further

$$\begin{aligned}
 x = 1 + \epsilon : \quad (1-x)^{-2\Delta_\phi} F_\Delta(z(x)) &= z^{2\Delta_\phi} F_\Delta(z(x)) \\
 &= O(G_\Delta(z)) + O(G_\Delta(1-z)) = O(\log 1/|\epsilon|). \quad (4.10)
 \end{aligned}$$

Finiteness will hold if the following integrals along  $\Gamma_x$  involving these bounds are absolutely convergent:

$$\int dx h(x) |x|^{-2\Delta_\phi} \quad \text{over the part of } \Gamma_x \text{ near } x = 0, \quad (4.11)$$

$$\int dx h(x) \log \frac{1}{|1-x|} \quad \text{over the part of } \Gamma_x \text{ near } x = 1. \quad (4.12)$$

Notice that these are sufficient conditions for finiteness. However, it's easy to see that these conditions are also necessary, unless one considers functionals with  $h(x)$  rapidly oscillating near endpoints  $x = 0, 1$ , which can be rendered finite by cancellations.

## 4.2 Swapping

Let us split the contour  $\Gamma_x$  into three parts, two “end parts”, one close to  $x = 0$  and one close to  $x = 1$ , and the “bulk part”. As  $\Delta_* \rightarrow \infty$ , the function  $\mathcal{F}_{\Delta_*}(z(x))$  goes to zero uniformly (and exponentially fast) on the bulk part. So that part of the integral can be made arbitrarily small by choosing a sufficiently large  $\Delta_*$ .

On the end parts, we will estimate  $\mathcal{F}_{\Delta_*}$  as follows. First of all we bound all terms by absolute value:

$$|\mathcal{F}_{\Delta_*}(z(x))| \leq |z|^{-2\Delta_\phi} |\mathcal{G}_{\Delta_*}(z)| + |1-z|^{-2\Delta_\phi} |\mathcal{G}_{\Delta_*}(1-z)|. \quad (4.13)$$

Here  $\mathcal{G}_{\Delta_*}$  is the tail of the conformal block decomposition, defined as  $\mathcal{F}_{\Delta_*}$  in (3.2) but summing over  $G_\Delta$ .

Using (2.6), we can estimate these tails by the whole function  $\mathcal{G}$  evaluated at the absolute value of the  $\rho$  coordinate:

$$|\mathcal{G}_{\Delta_*}(z)| \leq \tilde{\mathcal{G}}(|\rho(z)|) \leq \frac{\text{const.}}{(1-|\rho(z)|)^{4\Delta_\phi}}. \quad (4.14)$$

The second estimate can be understood for example by estimating the 4pt function using the OPEs  $\phi(\rho) \times \phi(1)$  and  $\phi(-\rho) \times \phi(-1)$  in (2.8). Alternatively it just follows from the second of the asymptotics (1.4). There is also an analogous estimate for  $\mathcal{G}_{\Delta_*}(1-z)$  with  $\rho(1-z)$ .

Now, for  $x \rightarrow 0, z \approx 1+x$  (recall that the contour is not along the real axis) this strategy gives us:

$$|\rho(z)| \approx 1 - \text{const.} \sqrt{|x|} \quad (4.15)$$

and

$$\tilde{\mathcal{G}}(|\rho(z)|) = O(|x|^{-2\Delta_\phi}), \quad \tilde{\mathcal{G}}(|\rho(1-z)|) = O(1). \quad (4.16)$$

On the other hand, for  $x = 1 + \epsilon$ , using the estimates on  $\rho$ 's from the previous section we find:

$$\tilde{\mathcal{G}}(|\rho(z)|), \tilde{\mathcal{G}}(|\rho(1-z)|) = O(|\epsilon|^{-2\Delta_\phi}) \tag{4.17}$$

We now combine these estimates on  $\tilde{\mathcal{G}}$  to get estimates on  $\mathcal{F}_{\Delta_*}$ . We have:

$$\begin{aligned} x \rightarrow 0 : \quad & (1-x)^{-2\Delta_\phi} \mathcal{F}_{\Delta_*}(z(x)) = O(|x|^{-2\Delta_\phi}), \\ x = 1 + \epsilon : \quad & (1-x)^{-2\Delta_\phi} \mathcal{F}_{\Delta_*}(z(x)) = z^{2\Delta_\phi} \mathcal{F}_{\Delta_*}(z(x)) \\ & = O(\mathcal{G}_{\Delta_*}(z)) + O(\mathcal{G}_{\Delta_*}(1-z)) = O(|\epsilon|^{-2\Delta_\phi}). \end{aligned} \tag{4.18}$$

It's crucial for what follows that the r.h.s. of these estimates does not depend on  $\Delta_*$ .

Suppose now that the following integrals of  $h(x)$  against these bounds are absolutely convergent:

$$\int dx h(x) |x|^{-2\Delta_\phi} \quad \text{over the part of } \Gamma_x \text{ near } x = 0, \tag{4.19}$$

$$\int dx h(x) |1-x|^{-2\Delta_\phi} \quad \text{over the part of } \Gamma_x \text{ near } x = 1. \tag{4.20}$$

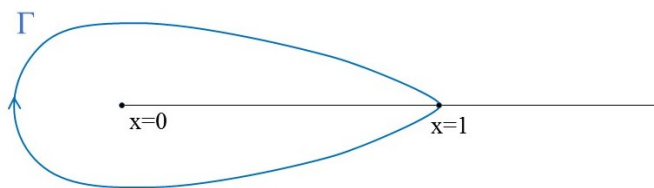
Then we claim that the swapping property holds.

To show (3.4) we argue as follows. Pick any  $\delta > 0$ . Take the end parts of the contour sufficiently short so that those parts of the integral, for any  $\Delta_*$ , are smaller in absolute value than  $\delta$ . This is possible by the conditions (4.19), (4.20). The bulk part of (3.4) tends to zero as  $\Delta_* \rightarrow \infty$ , since the integrand uniformly converges to zero there. We conclude that the large  $\Delta_*$  limit of (3.4), in absolute value, is smaller than  $\delta$ . Since  $\delta$  is arbitrary, the limit is zero. This completes the proof.

Conditions (4.19), (4.20) are sufficient conditions for swapping. Moreover we believe that, just like the finiteness conditions (4.11), (4.12), these conditions are also best possible (i.e. necessary) for  $h(x)$  which do not rapidly oscillate near the endpoints  $x = 0, 1$ . This may not be totally obvious from our proof. For example, one might think that the estimates (4.13) and (4.14) are too crude. However, we think that in general there cannot be a full cancellation between the terms whose absolute values are summed in the r.h.s. of (4.13). As for (4.14), we only use this estimate near the endpoints of the contour, where it's basically best possible.

Notice that while (4.19) is identical to (4.11), the other condition is stricter than (4.12). So, the fact that the functional is finite on each  $F_\Delta$  does not yet guarantee swapping.

We would like to finish this section with the following comment. The problem of justifying the swap of integration and summation is of course standard in mathematics. One powerful result is Lebesgue's dominated convergence theorem. There are several reasons why we chose not to appeal to it in our exposition, but to deduce everything from scratch. First, Lebesgue's theorem is very general (it deals with an almost everywhere convergent sequence of measurable functions), and it's not good practice to shoot sparrows with a cannon. Second, if we did appeal to this theorem, we could eliminate but the paragraph following the conditions (4.19), (4.20). The estimates (4.18) would still have to be derived ("dominated convergence"), and this is what constitutes anyway the bulk of our argument. Finally, we believe that there is an added value in seeing what actually goes into the proof.



**Figure 6.** The contour used in the definition of basis functionals (5.1).

## 5 Relation to the work of Mazáč

The cut-touching functionals from the previous section are closely related to the functionals constructed in [12], with the purpose to give an analytic proof of a certain optimal bootstrap bound involving operators of dimension  $\Delta_\phi \in \mathbb{N}/2$ . Let us review this connection in detail.

Mazáč begins by considering a family of basis functionals of the form

$$\omega(f) = \frac{1}{2\pi i} \int_{\Gamma} dx h(x) (1-x)^{-2\Delta_\phi} f(z(x)), \quad (5.1)$$

with  $h(x) = p_n(x)$  a Legendre polynomial. The function  $f$  is assumed analytic in the cut plane. He chooses the contour  $\Gamma$  to run as in figure 6, staying away from the point  $x = 0$ . Conditions for the finiteness<sup>7</sup> and swapping of these functionals can be examined exactly as above. It's clear that only conditions at  $x = 1$  need to be imposed. The finiteness condition (4.12) is satisfied. On the other hand, the swapping condition (4.20) is not satisfied, because  $p_n(1) \neq 0$ .

That's not a problem because he does not act with the basis functionals themselves on the sum rule. Instead he considers their linear combinations, corresponding to

$$h(x) = \sum_n a_n p_n(x). \quad (5.2)$$

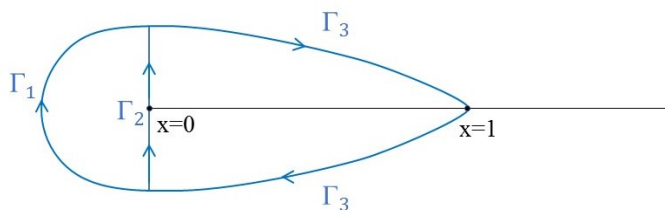
The coefficients  $a_n$  have to be chosen so that several conditions are satisfied. First of all, since his goal is to prove an optimal bootstrap bound, the functional has to be *extremal*, which means that it has to satisfy certain positivity conditions closely related to (1.8). These conditions have been discussed in detail in his work and we will not discuss them here.

Then the functional has to have the swapping property. This was not actually discussed in [12]. Near  $x = 1$  we must have (4.20), which should arise as a result of cancelation between various terms in the sum defining  $h(x)$ .<sup>8</sup>

Additional complications arise near  $x = 0$ . Namely, as a result of the infinite summation, the function  $h(x)$  develops a cut over the negative real axis  $x < 0$ . For this reason the

<sup>7</sup>Ref. [12] actually works out  $\omega(F_\Delta)$  for all functionals in closed form. So their finiteness is not in doubt. We will still discuss finiteness for completeness, but our focus is on justifying swapping.

<sup>8</sup>The talk [13] (29m30s) cites the condition  $h(x) = O((x-1)^{\Delta_\phi})$  as needed “for the functional to be defined on infinite sums of blocks bounded at infinity”. This is not far from our condition (4.20), although a bit stronger than necessary. We emphasize however that the functional has to be not just “defined”, but has to satisfy eq. (3.4) from which the swapping property follows.



**Figure 7.** The contours used in (5.4). It is important that  $\Gamma_1$  goes around  $x = 0$ , while  $\Gamma_2$  passes through it.

contour in figure 6 is no longer appropriate, and has to be modified. In fact, the behavior of his  $h(x)$  near  $x = 0$  can be described by the formula

$$h(x) = h_1(x) + h_2(x), \tag{5.3}$$

where  $h_1(x)$  is analytic near  $x = 0$ , while  $h_2(x)$  has a cut along  $x < 0$ . The total functional can then be defined as a sum of three terms

$$\omega(f) = \frac{1}{2\pi i} \left( \int_{\Gamma_1} h_1(x) + \int_{\Gamma_2} h_2(x) + \int_{\Gamma_3} h(x) \right) \times (1-x)^{-2\Delta_\phi} f(z(x)) dx, \tag{5.4}$$

where the three parts of the contour are chosen as in figure 7. The finiteness and swapping conditions for this functional are (4.12) and (4.20) imposed on  $h(x)$  and (4.19) imposed on  $h_2(x)$ , while  $h_1(x)$  does not have to satisfy any condition near  $x = 0$ . This contour prescription is equivalent to the one discussed in [12] below eq. (5.19).

After this introduction, let's see how the functionals of [12] fare with respect to all these conditions. To be more precise, his functionals correspond to

$$h(x) = \tilde{h}(x) + c(x), \tag{5.5}$$

where  $\tilde{h}(x)$  is a sum as in (5.2) with summation over even/odd  $n$  depending if  $\Delta_\phi$  is integer or halfinteger:

$$\tilde{h}(x) = \sum_{n \in 2\mathbb{N}} a_n p_n(x) \quad (\Delta_\phi \in \mathbb{N}), \tag{5.6}$$

$$\tilde{h}(x) = \sum_{n \in 2\mathbb{N}-1} a_n p_n(x) \quad \left( \Delta_\phi \in \mathbb{N} - \frac{1}{2} \right). \tag{5.7}$$

Using the properties of Legendre polynomials, this implies that

$$\tilde{h}(1-x) = -\tilde{h}(x) \quad (\Delta_\phi \in \mathbb{N}), \tag{5.8}$$

$$\tilde{h}(1-x) = \tilde{h}(x) \quad \left( \Delta_\phi \in \mathbb{N} - \frac{1}{2} \right). \tag{5.9}$$

On the other hand,  $c(x)$  is a finite degree polynomial which can be used to make the total  $h(x)$  vanish at  $x = 1$  sufficiently fast.

Consider first  $\Delta_\phi \in \mathbb{N}$ . In this case Mazáč says in section 5.2 (we translate his eq. (5.18) and others into our notation) that near  $x = 0$  we can decompose  $\tilde{h}(x)$  as

$$\tilde{h}(x) = \tilde{h}_1(x) + \tilde{h}_2(x), \tag{5.10}$$

where  $\tilde{h}_1(x)$  is analytic near  $x = 0$ , while  $\tilde{h}_2(x)$  has a cut along  $x < 0$  and satisfies

$$\tilde{h}_2(x) = O(x^{2\Delta_\phi} \log x) \tag{5.11}$$

Then by antisymmetry (5.8) we have the behavior near  $x = 1$ :

$$\tilde{h}(x) = -\tilde{h}_1(1-x) - \tilde{h}_2(1-x). \tag{5.12}$$

In section 5.3 he uses the freedom to add  $c(x)$  to set the behavior of

$$-\tilde{h}_1(1-x) + c(x) = O((x-1)^{2\Delta_\phi}). \tag{5.13}$$

He conjectures that it's always possible although he only checked it up to  $\Delta_\phi = 5$ . If so, we can define this functional as in (5.3), (5.4) with  $h_1(x) = \tilde{h}_1(x) + c(x)$  and  $h_2(x) = \tilde{h}_2(x)$  and have the conditions for the finiteness and swapping satisfied near both  $x = 0$  and  $x = 1$ .

Consider next  $\Delta_\phi \in \mathbb{N} - \frac{1}{2}$ , discussed in [12], section 5.4 and appendix A. For the particular values  $\Delta_\phi = 1/2, 3/2, 5/2$  he provides explicit  $\tilde{h}(x)$ , see his (4.29), (A.12)–(A.14). In these three cases we checked that near  $x = 0$  one has decomposition (5.10) with the nonanalytic part satisfying (5.11). The behavior near  $x = 1$  is given by symmetry (5.9):

$$\tilde{h}(x) = \tilde{h}_1(1-x) + \tilde{h}_2(1-x). \tag{5.14}$$

He says that he's able, at least for  $\Delta_\phi \leq 9/2$ , to use the freedom of adding  $c(x)$  to set:

$$\tilde{h}_1(1-x) + c(x) = O((x-1)^{2\Delta_\phi}). \tag{5.15}$$

If that's the case then the conditions for the finiteness and swapping are indeed satisfied, just as for  $\Delta_\phi \in \mathbb{N}$ .

The bottom line is that in the cases of low-lying  $\Delta_\phi$ , where ref. [12] provides sufficient information, we are able to apply our criteria and to prove swapping. A more detailed understanding and an extension of his argument would be needed to establish this for general  $\Delta_\phi$ . This is beyond the scope of our work.

## 6 Conclusions

Conformal field theories are both physically relevant and mathematically well defined. They satisfy precise axioms, which can be used to derive rigorous bounds separating the possible from the impossible. These bounds are usually argued by contradiction, employing the method of linear functionals. The non-explicit character of such arguments requires special care, otherwise one risks to throw out the baby with the bathwater. In this note we proposed a blueprint which needs to be followed to guarantee that this does not happen. As an application, we checked that the functionals recently constructed in [12] can be safely used in the conformal bootstrap.

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## A Toy counterexample

Mathematics textbooks are full of examples when one cannot swap integration with summation. We give one here so that you don't have to go look for it yourself. The example is based on simple power series expansions. However, the mechanism is general, and one should beware of falling into similar traps when working with conformal block expansions.

Consider the following functions on the real interval  $0 < t < 1$ :

$$\phi_0(t) = 1, \quad \phi_n(t) = (n+1)t^n - nt^{n-1} \quad (n = 1, 2, \dots). \quad (\text{A.1})$$

The series of these functions sums to zero:

$$\sum_{n=0}^{\infty} \phi_n(t) = 0 \quad (0 < t < 1). \quad (\text{A.2})$$

Indeed, it was designed so that the subsequent terms cancel telescopically, so that the partial sums

$$\sum_{n=0}^N \phi_n(t) = (N+1)t^N \rightarrow 0 \quad (0 < t < 1). \quad (\text{A.3})$$

Now consider formally integrating the series against some function  $w(t)$ :

$$\sum_{n=0}^{\infty} I_n = 0, \quad I_n = \int_0^1 dt w(t) \phi_n(t). \quad (\text{A.4})$$

Let us check this in a couple of examples. If we take  $w(t) = 1 - t$ , then things work nicely:

$$I_0 = \frac{1}{2}, \quad I_n = \frac{1}{n+2} - \frac{1}{n+1} \quad (n = 1, 2, \dots), \quad (\text{A.5})$$

and the series in (A.4) does converge to zero. On the other hand, for  $w(t) = 1$  we have

$$I_0 = 1, \quad I_n = 0 \quad (n > 0), \quad (\text{A.6})$$

in manifest contradiction with (A.4).

To understand this “paradox”, consider the tails of the series (A.2):

$$\Phi_N(t) = \sum_{n=N+1}^{\infty} \phi_n(t) = -(N+1)t^N. \quad (\text{A.7})$$

To swap integration and summation, we must have a condition analogous to (3.4):

$$\int dt w(t) \Phi_N(t) \rightarrow 0 \quad (N \rightarrow \infty). \quad (\text{A.8})$$

This condition is satisfied for  $w(t) = 1 - t$  but not for  $w(t) = 1$ .



## B Spectra with accumulation points

In the main text we made an assumption that the spectrum of operators appearing in the conformal block decomposition (1.5) is discrete without accumulation points. However, there exist 2d and 1d CFTs with continuous spectrum, such as the Liouville theory and its associated boundary CFTs (although in  $d > 2$  there are no known examples showing such behavior). Here we will show that our main results remain unchanged if the spectrum is continuous or has accumulation points.

In such a general situation, eq. (1.5) should be replaced by an indefinite Stieltjes integral

$$\mathcal{G}(z) = \int_0^\infty dP(\Delta) G_\Delta(z), \tag{B.1}$$

associated with a monotonically increasing function  $P(\Delta)$ ,  $P(0) = 0$ . Convergence of this integral is understood in two steps. First one defines the integral for a finite upper limit:

$$\int_0^{\Delta_*} dP(\Delta) G_\Delta(z). \tag{B.2}$$

This is defined as the  $N \rightarrow \infty$  limit of the Riemann-Stieltjes (RS) sums:

$$\sum_{i=0}^{N-1} [P(\Delta_{i+1}) - P(\Delta_i)] G_{\Delta_i}(z), \tag{B.3}$$

corresponding to finer and finer subdivisions of the interval  $[0, \Delta_*]$ :

$$\Delta_0 = 0 < \Delta_1 < \dots < \Delta_N = \Delta_*. \tag{B.4}$$

Let  $z$  vary over a region where  $|\rho(z)| < 1 - \epsilon$ . For such  $z$ , the functions  $G_\Delta(z)$  depend uniformly continuously on  $\Delta \in [0, \Delta_*]$ . This is enough to guarantee that the RS sums have a uniform limit. Since the individual RS sums are analytic, their limit (B.2) is analytic as well.

The second step is to define the integral in (B.1) as the limit of (B.2) as  $\Delta_* \rightarrow \infty$ . Since (B.2) monotonically grows with  $\Delta_*$  for  $0 < z < 1$ , the limit does exist on this interval. Then one argues as in section 2, using the property (2.6) of conformal blocks, that the convergence as  $\Delta_* \rightarrow \infty$  is uniform in the regions  $|\rho(z)| < 1 - \epsilon$ . This shows that the function  $\mathcal{G}(z)$  is analytic in the cut complex plane, just as before.

By the given argument, we have the following approximation of  $\mathcal{G}(z)$  by finite sums of conformal blocks with two error terms:

$$\mathcal{G}(z) = \sum_{i=0}^{N-1} [P(\Delta_{i+1}) - P(\Delta_i)] G_{\Delta_i}(z) + \mathcal{G}_{\Delta_*}(z) + \mathcal{G}_{\Delta_*, N}^{\text{RS}}(z). \tag{B.5}$$

The first error term  $\mathcal{G}_{\Delta_*}(z)$  is the difference between (B.1) and (B.2), while the second error term  $\mathcal{G}_{\Delta_*, N}^{\text{RS}}(z)$  is the difference between (B.2) and the RS sum. This is to be compared with the situation in the main text, where we had only the first error term.

The first error term has the same properties as before: it goes uniformly to zero with  $\Delta_* \rightarrow \infty$  in the region  $|\rho(z)| < 1 - \epsilon$ , and it can be uniformly in  $\Delta_*$  bounded by the full 4pt function, as in eq. (4.14).

On the other hand, as discussed above, the second error term can be made uniformly small in the same region  $|\rho(z)| < 1 - \epsilon$ , by taking  $N \rightarrow \infty$  (for any fixed  $\Delta_*$ ). Outside of this region we can use a crude upper bound:

$$|\mathcal{G}_{\Delta_*, N}^{\text{RS}}(z)| \leq \text{const} \left( 1 + \log \frac{1}{1 - |\rho(z)|} \right). \quad (\text{B.6})$$

Here const may depend on  $\Delta_*$  but is independent of  $N$ . This bound follows from the fact that each individual conformal block satisfies such a bound.

Now we are in a position to repeat the analysis of section 3. Eq. (3.3) is replaced by:

$$\sum_{i=0}^{N-1} [P(\Delta_{i+1}) - P(\Delta_i)] \omega(F_{\Delta_i}) + \omega(\mathcal{F}_{\Delta_*}) + \omega(\mathcal{F}_{\Delta_*, N}^{\text{RS}}) = 0. \quad (\text{B.7})$$

When we take the limit  $N \rightarrow \infty$  and then  $\Delta_* \rightarrow \infty$ , this will become the desired equation

$$\int_0^\infty dP(\Delta) \omega(F_\Delta) = 0, \quad (\text{B.8})$$

provided that we can show (3.4) (which is done exactly as before) and, in addition, that

$$\omega(\mathcal{F}_{\Delta_*, N}^{\text{RS}}) \rightarrow 0 \quad (N \rightarrow \infty, \Delta_* \text{ fixed}). \quad (\text{B.9})$$

This extra condition is obvious for the simple functionals (3.5), (3.6) since  $\mathcal{F}_{\Delta_*, N}^{\text{RS}}$  goes uniformly to zero in the relevant region of  $z$ . For the cut-touching functionals, a little thought has to be given to what happens near the points  $x = 0, 1$ . Since this error term satisfies the same crude bound (B.6) as the conformal blocks, one can recycle the estimates from section 4.1. Conditions (4.11), (4.12) are then sufficient to guarantee (B.9).

The conclusion of this discussion is that the sufficient conditions for the finiteness and swapping derived in the main text remain valid when the spectrum is continuous or discrete with accumulation points.

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