(p, q)-five brane and (p, q)-string solutions, their bound state and its near horizon limit

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ABSTRACT: We determine (p, q)-string and (p, q)-five brane solutions of type IIB supergravity using $\mathrm{SL}(2, \mathbb{Z})$-symmetry of the full type IIB superstring theory. We also determine $\mathrm{SL}(2, \mathbb{Z})$-transformed solution corresponding to the bound state of NS5-branes and fundamental strings. Then we analyze its near horizon limit and we show that it leads to the $\mathrm{AdS}_3 \times S^3$ with mixed fluxes.

Contents

1   Introduction

It is conjectured that the type IIB superstring theory possesses $SL(2, \mathbb{Z})$ non-perturbative duality. The first evidence follows from the manifestly $SL(2, \mathbb{R})$-invariance of type IIB supergravity effective action [1, 2] for recent excellent review see [3]. Further evidence follows from the spectrum of non-perturbative objects that are present in type IIB theory:Dp-branes with $p = 1, 3, 5, 7, 9$ [4], fundamental string and NS5-brane.[1] It was argued that under S-duality fundamental string maps to D1-brane, D5-brane maps to NS5-brane and so on. On the other hand we know that macroscopic extend objects are sources of supergravity fields and hence it is possible to find corresponding background solutions that solve the supergravity equations of motions. Well known examples of such solutions are fundamental string solution [9] or NS5-brane solution [10]. Then with the help of the $SL(2, \mathbb{R})$-covariance of type IIB supergravity action new solutions corresponding to (p, q)-string were found in [7]. In fact, the existence of this solution serves as a further evidence of $SL(2, \mathbb{Z})$-duality of type IIB string theory. The main idea of this construction is to start with fundamental string solution and performs $SL(2, \mathbb{R})$ rotation. Then the requirement that the resulting configuration has to have integer charge in some units fixes entries of this matrix as functions of these charges and asymptotic values of dilaton and Ramond-Ramond one form. It is important to stress that this solution depends on one harmonic function with manifestly $SL(2, \mathbb{R})$-covariant coefficient. Then this method was applied for the construction of (p, q)-five brane backgrounds in [17]. These very interesting backgrounds were analyzed recently from the (m, n)-string probe point of view in [8] and it was shown that when we perform $SL(2, \mathbb{R})$ rotation that maps the macroscopic (p, q)-string background to

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[1]For a review, see [5, 6].

the fundamental string background the probe string does not transform in the expected way since now it carries non-integer charge with respect to NSNS two form. The same situation also occurs in case of (p,q)-five brane background. It was suggested there that the resolution of this puzzle could be found when we consider a full type IIB superstring theory that is invariant under $SL(2,\mathbb{Z})$-subgroup of $SL(2,\mathbb{R})$. This paper is devoted to this analysis.

We propose that it is natural to search for the (p,q)-macroscopic string solution with the presence of the source which is manifestly $SL(2,\mathbb{R})$ covariant (p,q)-string action [18–22]. However the fact that there is no fractional string or D-brane charges demands that the proper invariant group of type superstring theory is $SL(2,\mathbb{Z})$ rather than $SL(2,\mathbb{R})$ group and this group *should* be used for the construction of (p,q)-macroscopic string and five brane solutions.[2] With the help of this argument we will be able to find supergravity solutions corresponding to (p,q)-macroscopic string and five branes that have correct NSNS and RR charges and where the (m,n)-string probe has the expected properties. We show that these solutions depend on two harmonic functions which is different from the solutions found in [7, 17] which however reflects the fact that (p,q)-string can be considered as the bound state of p-fundamental string and q-D1-branes even if this bound state is not threshold and hence harmonic superposition rules cannot be applied for it [23].

As the next step we extend this analysis to the case of NS5-brane whose supergravity solution has been known for a long time [10]. We perform $SL(2,\mathbb{Z})$ transformation of this solution and find new solution corresponding to $(d,-b)$-five brane. This solution is characterized by two harmonic functions whose parameters depend on the charges of $(d,-b)$-five brane and on the asymptotic values of moduli. We also analyze (m,n)-string probe in this background and we show that it is equivalent to the dynamics of (m',n')-string in original NS5-brane background where m',n' are integers that again explain the puzzle found in [8].

Finally we consider $SL(2,\mathbb{Z})$ transformed solution of a bound state of Q_5 NS5-branes wrapped on four torus and Q_1 fundamental strings that are smeared over this four torus [11]. We find solution that is characterized by four harmonic functions that depend on the moduli of this solution and charges with respect to RR and NSNS two forms. This solution has also an interesting near horizon limit which is $AdS_3 \times S^3$ with mixed three form fluxes with integer charges. Integrability of superstring in this background was studied recently in [28]. The main idea of this paper is to start with the pure RR background when the new WZ term that represents the coupling to the NSNS flux is added. Using this construction many new interesting results were derived [29–35]. It is important to stress that in all these works the value of NSNS flux can be interpreted as the deformation parameter that takes any real value from the interval $(0,1)$. This is perfectly consistent from the point of view of perturbative string theory since classical string does not couple to the dilaton. On the other hand this approximation certainly breaks down when we consider D1-brane in this background that couples to the dilaton through Dirac-Born-Infeld action. In order to analyze D1-brane in the $AdS_3 \times S^3$ with mixed three form fluxes we have to have

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[2]Even if five brane solutions are source free we can consider solutions where the source is covariant (p,q)-five brane action [22] that electrically couples to doublets of six forms that are dual to NSNS and RR two forms.

background with explicit values od dilaton and RR zero form too. It is natural to presume that such a background arises as the near horizon limit of the $\mathrm{SL}(2,\mathbb{Z})$-transformed solution corresponding to the bound state of NS-five branes and fundamental strings. We show that this is really true. More explicitly, we show that $\mathrm{SL}(2,\mathbb{Z})$-transformation and near horizon limit commutes which is a generalization of the commutativity of S-duality and near horizon limit found in [14].[3] Then we study (m,n)-string in $\mathrm{AdS}_3 \times S^3$ background with mixed three form fluxes and using the fact that the near horizon limit and $\mathrm{SL}(2,\mathbb{Z})$-transformation commutes we can map this (m,n)-string to the (m',n')-string in $\mathrm{AdS}_3 \times S^3$ background with NSNS two form flux. We show that for the special value of (m,n) charges the (m,n)-string in the $\mathrm{AdS}_3 \times S^3$ background is equivalent to the fundamental string in $\mathrm{AdS}_3 \times S^3$ background with NSNS three form flux that can be described by standard CFT techniques [12, 14–16]. Of course, this result does not solve the problem of the analysis of fundamental string in $\mathrm{AdS}_3 \times S^3$ with mixed fluxes which is very complicated and deserves very special treatment [36].

This paper is organized as follows. In the next section 2 we review the basic fact about type IIB low energy effective theory and suggest the main idea how to derive $\mathrm{SL}(2,\mathbb{Z})$-transformed solutions of type IIB supergravity equations of motion. In section 3 we apply this procedure to the case of (p,q)-string background. In section 4 we perform the same analysis in case of (p,q)-five brane background and we extend this analysis to the bound state of NS5-branes and fundamental strings in section 5. In section 6 we take the near horizon limit of this solution and analyze its properties. Finally in conclusion 7 we outline our results and suggest possible extension of this work.

2 $\mathrm{SL}(2,\mathbb{Z})$ covariance of type IIB string theory

In this section we review the basic facts about bosonic content of the type IIB low energy effective action and we present the general idea of how to find $\mathrm{SL}(2,\mathbb{Z})$ transformed solution.

The type IIB theory has two three-form field strengths $H = dB, F = dC^{(2)}$, where H corresponds to NSNS three form while F belongs to RR sector and does not couple to the usual string world-sheet. Type IIB theory has also two scalar fields that can be combined into a complex field $\tau = \chi + ie^{-\Phi}$. The dilaton Φ is in the NSNS sector while χ belongs to the RR sector. The other bosonic fields are the metric g_{MN} in Einstein frame and self-dual five form field strength F_5. However this field can be consistently set to zero for solutions that we study in this paper and hence we do not include it to the action. Then it is possible

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[3]In some way near horizon limit conforms to Geroch's criteria of limiting spacetimes [38] which could imply that the limiting process and duality transformations commute. On the other hand it is not completely clear whether $\mathrm{SL}(2,\mathbb{Z})$-duality of type IIB theory could be considered as an isometry of this space-time so that it is useful to check explicitly the commutativity $\mathrm{SL}(2,\mathbb{Z})$-duality transformations and the near horizon limit. On the other hand $\mathrm{SL}(2,\mathbb{Z})$-duality of type IIB theory can be considered as a symmetry of the compactification of M-theory on two-torus so that it is possible that by uplifting NS5-brane and funamdental string background solution to the corresponding M-theory background with manifest $\mathrm{SL}(2,\mathbb{Z})$-isometry Geroch's criteria of limiting space-time can be applied.

to write down $SL(2,\mathbb{R})$ covariant form of the bosonic part of type IIB effective action

$$S_{\text{IIB}} = \frac{1}{2\tilde{\kappa}_{10}^2} \int d^{10}x \sqrt{-g} \left(R + \frac{1}{4}\text{Tr}(\partial_M \mathcal{M} \partial^M \mathcal{M}^{-1}) - \frac{1}{12}\mathbf{H}_{MNP}^T \mathcal{M} \mathbf{H}^{MNP} \right),$$
$$2\tilde{\kappa}_{10}^2 = (2\pi)^7 \alpha'^4, \tag{2.1}$$

where we have combined $B, C^{(2)}$ into

$$\mathbf{H} = d\mathbf{B} = \begin{pmatrix} dB \\ dC^{(2)} \end{pmatrix} \tag{2.2}$$

and where

$$\mathcal{M} = e^\Phi \begin{pmatrix} \tau\tau^* & \chi \\ \chi & 1 \end{pmatrix} = e^\Phi \begin{pmatrix} \chi^2 + e^{-2\Phi} & \chi \\ \chi & 1 \end{pmatrix}, \quad \det\mathcal{M} = 1. \tag{2.3}$$

This action has manifest invariance under the global $SL(2,\mathbb{R})$ transformation

$$\hat{\mathcal{M}} = \Lambda \mathcal{M} \Lambda^T, \hat{\mathbf{B}} = (\Lambda^T)^{-1}\mathbf{B}, \tag{2.4}$$

where

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det\Lambda = ad - bc = 1. \tag{2.5}$$

Let us now introduce an action for (m,n)-string that couples electrically to NSNS and RR two form and hence can be considered as a source for corresponding fields. The action for (m,n)-string has the form[4]

$$S_{(p,q)} = -T_{\text{D1}} \int d\tau d\sigma \sqrt{\mathbf{m}^T \mathcal{M}^{-1} \mathbf{m}} \sqrt{-\det g_{MN} \partial_\alpha x^M \partial_\beta x^N} +$$
$$+ T_{\text{D1}} \int d\tau d\sigma \mathbf{m}^T \mathbf{B}_{MN} \partial_\tau x^M \partial_\sigma x^N, \quad \mathbf{m} = \begin{pmatrix} m \\ n \end{pmatrix}, \tag{2.6}$$

where $T_{\text{D1}} = \frac{1}{2\pi\alpha'}$. The action (2.6) is invariant under global transformations (2.4) on condition that $\mathbf{m}$ transforms as

$$\hat{\mathbf{m}} = \Lambda \mathbf{m}. \tag{2.7}$$

In this notation m counts the number of fundamental strings while n counts the number of D1-branes and hence they have to be integer. This fact implies that $\Lambda \in SL(2,\mathbb{Z})$. In other words $SL(2,\mathbb{R})$ invariance of type IIB low energy effective action is broken to its $SL(2,\mathbb{Z})$ subgroup due to the charge quantization condition. As a consequence the action that includes both type IIB effective action and string probe action

$$S = S_{\text{IIB}} + S_{(p,q)} \tag{2.8}$$

is invariant under $SL(2,\mathbb{Z})$ group rather than under $SL(2,\mathbb{R})$ group. This fact will have a crucial consequence for the construction of (p,q)- string and five brane solutions.

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[4]For recent discussion, see [8].

Using this basic result we now present the main idea how to derive supergravity solution with (p,q)-fundamental string as a source. Observe that we can write

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p & b \\ q & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \mathbf{m}(p,q) = \Lambda(p,q)\mathbf{m}_F \,,\, pd - bq = 1 \tag{2.9}$$

and hence (p,q)-string action has the form

$$S_{(p,q)} = -T_{\mathrm{D}1} \int d\tau d\sigma \sqrt{\mathbf{m}_F^T \tilde{\mathcal{M}}^{-1} \mathbf{m}_F} \sqrt{-\det g_{MN} \partial_\alpha x^M \partial_\beta x^N} + $$
$$+ T_{\mathrm{D}1} \int d\tau d\sigma \mathbf{m}_F^T \tilde{\mathbf{B}}_{MN}(p,q) \partial_\tau x^M \partial_\sigma x^N \,, \tag{2.10}$$

where

$$\tilde{\mathcal{M}} = \Lambda^{-1}(p,q)\mathcal{M}(p,q)(\Lambda^T(p,q))^{-1} \,, \quad \tilde{\mathbf{B}} = \Lambda^T(p,q)\mathbf{B}(p,q) \,. \tag{2.11}$$

Then with the manifest $\mathrm{SL}(2,\mathbb{R})$ invariance of the Type IIB effective action we find that it has the form

$$S_{\mathrm{IIB}} = \frac{1}{2\tilde{\kappa}_{10}^2} \int d^{10}x \sqrt{-g} \left(R + \frac{1}{4}\mathrm{Tr}(\partial_M \tilde{\mathcal{M}} \partial^M \tilde{\mathcal{M}}^{-1}) - \frac{1}{12}\tilde{\mathbf{H}}_{MNP}^T \tilde{\mathcal{M}} \tilde{\mathbf{H}}^{MNP} \right) \tag{2.12}$$

so that $\tilde{\mathcal{M}}$ and $\tilde{\mathbf{B}}$ have the same functional form as corresponding fields in case of fundamental string as the source so that we denote its value with superscript F and omit tilde over them. In other words we find following components of $\mathcal{M}(p,q)$ and $\mathbf{B}(p,q)$ corresponding to the (p,q)-string as a source:

$$\mathcal{M}(p,q) = \Lambda(p,q)\mathcal{M}_F \Lambda^T(p,q) \,, \mathbf{B}(p,q) = (\Lambda^T)^{-1}(p,q)\mathbf{B}_F \,. \tag{2.13}$$

In this case the electric charge corresponding to this background has the form

$$\mathbf{q}_{(p,q)} = \frac{1}{2\tilde{\kappa}_{10}^2} \int_{S^8} \mathcal{M} \star \mathbf{H} = \Lambda(p,q)\frac{1}{2\tilde{\kappa}_{10}^2} \int_{S^8} \mathcal{M}_F \star \mathbf{H}_F = \begin{pmatrix} p \\ q \end{pmatrix} q_F \,, \tag{2.14}$$

where $q_F = \frac{1}{2\pi\alpha'}$ is the electric charge of the fundamental string.

In case of solitonic (p,q)-brane we can argue in the similar way with the difference that this is a magnetic solution that is source free. Further, the charge transforms in the same way as corresponding field strength

$$\mathbf{q}_{(d,-b)}^5 = \frac{1}{2\tilde{\kappa}_{10}^2} \int_{S^3} \mathbf{H}(p,q) = (\Lambda^T)^{-1}(p,q)\frac{1}{2\tilde{\kappa}_{10}^2} \int_{S^3} \mathbf{H}_F = \begin{pmatrix} d \\ -b \end{pmatrix} q_{\mathrm{NS}5} \,, \tag{2.15}$$

where $q_{\mathrm{NS}5} = \frac{1}{(2\pi)^5 \alpha'^3}$ is the magnetic charge of NS5-brane. After the outline of this general procedure we proceed in next sections to the explicit construction of the $\mathrm{SL}(2,\mathbb{Z})$-transformed solutions.

3   SL(2, ℤ)-string solution

Let us start with the fundamental string solution [9]

$$ds^2 = G^F_{MN} dx^M dx^N = \frac{1}{H_F} dx^2_{\mathrm{II}} + dx^2_\perp\,, \quad H_{m01} = \partial_m H_F^{-1}\,,$$

$$e^\Phi = g_s \frac{1}{\sqrt{H_F}}\,, \quad H_F = 1 + \frac{\alpha g_s^2}{r^6}\,, \quad \alpha = \frac{(2\pi)^6 \alpha'^3}{6\Omega_7}\,, \quad B_{01} = \frac{1}{H_F} - 1\,, \qquad (3.1)$$

where $dx^2_{\mathrm{II}} = -dt^2 + dx_1^2\,, dx^2_T = dx_m dx^m\,, m = 2, \ldots, 9\,,\ r^2 = x_m x^m$ and where the line element is expressed in string frame. We use the notation where small g_{MN} corresponds to the Einstein frame metric while G_{MN} corresponds to the string frame metric. Note that these two metrics are related by rescaling

$$g_{MN} = e^{-\Phi/2} G_{MN}\,. \qquad (3.2)$$

Since g_{MN} is invariant under SL(2, ℤ) transformation we derive relation between transformed and original string frame metrics

$$\hat{G}_{MN} = e^{\frac{1}{2}(\hat\Phi - \Phi)} G_{MN}\,. \qquad (3.3)$$

Now we are ready to find solution that will be defined as SL(2, ℤ) transformation of the solution (3.1) where the matrix Λ has the form $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\hat\chi = \frac{ace^{-2\Phi} + bd}{c^2 e^{-2\Phi} + d^2}\,, \qquad\qquad e^{-\hat\Phi} = \frac{e^{-\Phi}}{c^2 e^{-2\Phi} + d^2}\,,$$

$$\hat{G}_{MN} = \sqrt{c^2 e^{-2\Phi} + d^2}\, G_{MN}\,,$$

$$\hat{B}_{MN} = d B_{MN}\,, \qquad\qquad \hat{C}^{(2)}_{MN} = -b B_{MN}\,, \quad ad - bc = 1\,. \qquad (3.4)$$

As we argued in the previous section the new solutions have the charges

$$\begin{pmatrix} q^F_{\mathrm{NS}} \\ q^F_{\mathrm{RR}} \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} g_F\,. \qquad (3.5)$$

Now we write explicitly SL(2, ℤ)-transformed line element

$$d\hat{s}^2 = \frac{\sqrt{c^2 + d^2 g_s^2}}{g_s} \sqrt{1 + \frac{c^2}{c^2 + d^2 g_s^2} \frac{\alpha g_s^2}{r^6}} (H_F^{-1} dx^2_{\mathrm{II}} + dx^2_\perp)\,. \qquad (3.6)$$

We see that it is natural to perform rescaling of the coordinates

$$\frac{(c^2 + d^2 g_s^2)^{1/4}}{\sqrt{g_s}} x^M = \hat{x}^M \Rightarrow r^6 = \hat{r}^6 \frac{g_s^3}{(c^2 + d^2 g_s^2)^{3/2}}\,. \qquad (3.7)$$

On the other hand the solution found above still depends on the string coupling g_s of the original solution and non-physical parameters b, d that appear in Λ. However we would like to express the new solution using the assymptotic values of $\hat\Phi$ and $\hat\chi$ together with a

and c that correspond to q_{NS}^F and q_{NS}^R charges. In order to do this we take the limit $r \to \infty$ in $e^{-\hat{\Phi}}$ and $\hat{\chi}$ given in (3.4) and we obtain

$$\lim_{r \to \infty} e^{-\hat{\Phi}} = \frac{1}{\hat{g}_s} = \frac{g_s}{c^2 + d^2 g_s^2},$$

$$\lim_{r \to \infty} \hat{\chi} \equiv \chi_0 = \frac{ac + bdg_s^2}{c^2 + d^2 g_s^2}. \tag{3.8}$$

If we multiply the last equation with c and use the fact that $ad - bc = 1$ we obtain

$$g_s d = \hat{g}_s(a - c\chi_0), \quad g_s = \frac{1}{\hat{g}_s}(c^2 + \hat{g}_s^2(a - c\chi_0)^2). \tag{3.9}$$

Then we define two harmonic functions

$$\hat{H} \equiv H_F(\hat{r}) = 1 + \frac{\alpha \hat{g}_s \sqrt{c^2 + \hat{g}_s^2(a - c\chi_0)^2}}{\hat{r}^6},$$

$$\hat{H}' = 1 + \frac{c^2}{\sqrt{c^2 + \hat{g}_s^2(a - c\chi_0)^2}} \frac{\alpha \hat{g}_s}{\hat{r}^6} \tag{3.10}$$

so that the line element has the final form

$$d\hat{s}^2 = \sqrt{\hat{H}'}(\hat{H}^{-1} d\hat{x}_{\mathrm{II}}^2 + d\hat{x}_\perp^2). \tag{3.11}$$

As a check note that for $a = \chi_0 = 0$ we obtain that $\hat{H} = \hat{H}' \equiv H_{(0,1)}$ and hence

$$d\hat{s}^2 = \frac{1}{\sqrt{H_{(0,1)}}} d\hat{x}_{\mathrm{II}}^2 + \sqrt{H_{(0,1)}} d\hat{x}_\perp^2, \quad H_{(0,1)} = 1 + \frac{\alpha \hat{g}_s}{\hat{r}^6} \tag{3.12}$$

which corresponds to the line element of the D1-brane which is S-dual to the fundamental string solution.

Finally we express dilaton as a function of $\hat{H}$ and $\hat{H}'$

$$e^{-\hat{\Phi}} = \frac{1}{\hat{g}_s} \frac{\sqrt{\hat{H}}}{\hat{H}'} \tag{3.13}$$

and find components of NSNS and RR two forms in the new coordinates $\hat{x}$. To do this we use the fact that

$$\hat{B} = dB_{MN} dx^M \wedge dx^N = \frac{\hat{g}_s(a - c\chi_0)}{\sqrt{c^2 + \hat{g}_s^2(a - c\chi_0)^2}} B_{MN} d\hat{x}^M \wedge d\hat{x}^M \tag{3.14}$$

and consequently

$$\hat{B}_{01} = \frac{\hat{g}_s(a - c\chi_0)}{\sqrt{c^2 + \hat{g}_s^2(a - c\chi_0)^2}} \left(\frac{1}{\hat{H}(\hat{x})} - 1\right). \tag{3.15}$$

In the same way we find

$$\hat{C}_{01} = \frac{c - \hat{g}_s^2(a - c\chi_0)\chi_0}{\hat{g}_s \sqrt{c^2 + \hat{g}_s^2(a - c\chi_0)^2}} \left(\frac{1}{\hat{H}(\hat{x})} - 1\right)$$

using $b = \frac{ad-1}{c}$. In summary, we claim that the supergravity solution corresponding to (a,c)-string has the form

$$ds^2 = \sqrt{\hat{H}'}(\hat{H}^{-1}d\hat{x}_{\rm II}^2 + d\hat{x}_{\perp}^2)\,, \qquad e^{-\hat{\Phi}} = \frac{1}{\hat{g}_s}\frac{\sqrt{\hat{H}}}{\hat{H}'}\,, \qquad (3.16)$$

$$\hat{B}_{01} = \frac{\hat{g}_s(a-c\chi_0)}{\sqrt{c^2 + \hat{g}_s^2(a-c\chi_0)^2}}\left(\frac{1}{\hat{H}(\hat{x})} - 1\right)\,, \quad \hat{C}_{01} = \frac{c - \hat{g}_s^2(a-c\chi_0)\chi_0}{\hat{g}_s\sqrt{c^2 + \hat{g}_s^2(a-c\chi_0)^2}}\left(\frac{1}{\hat{H}(\hat{x})} - 1\right)$$

$$\hat{H} = 1 + \frac{\alpha\hat{g}_s\sqrt{c^2 + \hat{g}_s^2(a-c\chi_0)^2}}{\hat{r}^6}\,, \qquad \hat{H}' = 1 + \frac{c^2}{\sqrt{c^2 + \hat{g}_s^2(a-c\chi_0)^2}}\frac{\alpha\hat{g}_s}{\hat{r}^6}\,.$$

It is instructive to compare this solution with (a,c)-fundamental string solution found in [7]. The main difference is that our solution depends on two harmonic functions as opposite to the solution derived in [7]. In some way this is a reflection of the fact that we have a bound state of D1-brane and fundamental string even if this superposition does not correspond to the harmonic superposition rule [23] as this bound state is not marginal. Further, the arguments of the harmonic functions are different from the expression used in [7] which however implies that our solution is defined with the help of $\mathrm{SL}(2,\mathbb{Z})$ matrix rather than $\mathrm{SL}(2,\mathbb{R})$ matrix that was used in [7]. As a consequence the solution (3.16) behaves consistently from the probe (m,n)-string point of view. To see this explicitly let us consider probe (m,n)-string in this background when the action has the form

$$S_{(m,n)} = -T_{\rm D1}\int d\tau d\sigma \sqrt{\mathbf{m}^T\hat{\mathcal{M}}^{-1}\mathbf{m}}\sqrt{-\det\hat{g}_{MN}\partial_\alpha\hat{x}^M\partial_\beta\hat{x}^N} + $$
$$ + T_{\rm D1}\int d\tau d\sigma \mathbf{m}^T\hat{\mathbf{B}}_{MN}\partial_\tau\hat{x}^M\partial_\sigma\hat{x}^N\,, \qquad (3.17)$$

where $\hat{g}_{MN}$ is the Einstein frame metric in rescaled coordinates

$$\hat{g}_{MN} = \frac{\hat{g}_s}{\sqrt{c^2 + \hat{g}_s^2(a-c\chi_0)^2}}g_{MN}(\hat{r})\,. \qquad (3.18)$$

However due to the fact that the pullback of the Einstein metric and two forms is invariant under rescaling by definition we can easily use the original variables x instead of $\hat{x}$. Then the probe action has the form

$$S_{(m,n)} = -T_{\rm D1}\int d\tau d\sigma \sqrt{m'^2 + n'^2 e^{-2\Phi_F}}\sqrt{-\det G_{MN}^F\partial_\alpha x^M\partial_\beta x^N} + $$
$$ + T_{\rm D1}\int d\tau d\sigma m' B_{MN}^F\partial_\tau x^M\partial_\sigma x^N\,, \qquad (3.19)$$

where

$$\mathbf{m}' = \begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} dm - bn \\ -cm + an \end{pmatrix}\,. \qquad (3.20)$$

From the previous action we see that the problem of the analysis of the dynamics of (m,n)-string in (a,c)-string background is reduced to the analysis of (m',n')-string in the fundamental string background where m', n' are evaluated at (3.20). The beautiful

analysis of this problem was performed in [24] and we will not reviewed it here. We also see that for $m = a, n = c$ we obtain that $m' = 1, n' = 0$ which is again consistent with the picture of probe F-string in fundamental string background that is rotated by $SL(2, \mathbb{Z})$ transformation. In other words our solution solves the issue that was found in our previous paper [8].

4 $(d, -b)$-five brane solution

In this section we find $(d, -b)$-five brane solution when we perform $SL(2, \mathbb{Z})$ transformation of NS5-brane supergravity solution. Recall that this solution has the form [10]

$$ds^2 = G_{MN}^{\mathrm{NS5}} dx^M dx^N = dx_{\mathrm{II}}^2 + H_{\mathrm{NS5}} dx_\perp^2 \,, \quad e^\Phi = g_s H_{\mathrm{NS5}}^{1/2} \,,$$

$$H_{\mathrm{NS5}} = 1 + \frac{\alpha'}{r^2} \,, \quad H_{mnp}^{\mathrm{NS5}} = \epsilon_{mnpq} \partial_q H_{\mathrm{NS5}} \,, \tag{4.1}$$

where $dx_{\mathrm{II}}^2 = \eta_{\mu\nu} dx^\mu dx^\nu \,, \mu, \nu = 0, \ldots, 5 \,, dx_\perp^2 = dx_m dx^m \,, m = 6, \ldots, 9 \,, r^2 = x_m x^m$. As in previous section we perform $SL(2, \mathbb{Z})$ transformations

$$\hat{\chi} = \frac{ace^{-2\Phi} + bd}{c^2 e^{-2\Phi} + d^2} \,, \qquad e^{-\hat{\Phi}} = \frac{e^{-\Phi}}{c^2 e^{-2\Phi} + d^2} \,,$$

$$\hat{G}_{MN} = \sqrt{c^2 e^{-2\Phi} + d^2} G_{MN}^{\mathrm{NS5}} \,, \quad \hat{B}_{MN} = d B_{MN}^{\mathrm{NS5}} \,, \quad \hat{C}_{MN}^{(2)} = -b B_{MN}^{\mathrm{NS5}} \,. \tag{4.2}$$

We will argue below that d and b are proportional to $NSNS$ and RR magnetic charges of five brane. For that reason we would like to express transformed solution as a function of d, b together with the string coupling constant $\hat{g}_s$ and assymptotic values of $\hat{\chi}_{(0)}$.

To begin with we note that the transformed line element has the form

$$d\hat{s}^2 = \sqrt{c^2 + d^2 g_s^2 H_{\mathrm{NS5}}} \frac{1}{g_s} (H_{\mathrm{NS5}}^{-1/2} dx_{\mathrm{II}}^2 + H_{\mathrm{NS5}}^{1/2} dx_\perp^2) \,. \tag{4.3}$$

Now we observe that we can write

$$\sqrt{c^2 + d^2 g_s^2 H_{\mathrm{NS5}}} = \sqrt{c^2 + d^2 g_s^2} \sqrt{1 + \frac{d^2}{c^2 + d^2 g_s^2} \frac{N\alpha' g_s^2}{r^2}} = \sqrt{c^2 + d^2 g_s^2} \hat{H}'^{1/2} \,. \tag{4.4}$$

For $r \to \infty$ we obtain

$$\lim_{r \to \infty} e^{-\hat{\Phi}} = \frac{1}{\hat{g}_s} = \frac{g_s}{c^2 + d^2 g_s^2} \,,$$

$$\lim_{r \to \infty} \chi \equiv \chi_0 = \frac{ac + bdg_s^2}{c^2 + d^2 g_s^2} \,. \tag{4.5}$$

Now if we multiply the first expression with d and use the fact that $ad - bc = 1$ we can express c in terms of $\chi_0, \hat{g}_s, d, b$ as

$$c = \hat{g}_s g_s (d\chi_0 - b) \tag{4.6}$$

and consequently

$$g_s = \frac{\hat{g}_s}{\hat{g}_s^2 (d\chi_0 - b)^2 + d^2} \,. \tag{4.7}$$

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We further rescale coordinates as

$$(\hat{g}_s^2(d\chi_0 - b)^2 + d^2)^{1/4}x^M = \hat{x}^M \Rightarrow r^2 = \frac{\hat{r}^2}{\sqrt{\hat{g}_s^2(d\chi_0 - b)^2 + d^2}} \tag{4.8}$$

and hence $\hat{H}$ and $\hat{H}'$ have the form

$$\hat{H} = 1 + \frac{N\alpha'}{\hat{r}^2}\sqrt{\hat{g}_s^2(d\chi_0 - b)^2 + d^2}\,,$$

$$\hat{H}' = 1 + \frac{N\alpha'}{\hat{r}^2}\frac{d^2}{\sqrt{\hat{g}_s^2(d\chi_0 - b)^2 + d^2}}\,. \tag{4.9}$$

Finally we determine components of RR and NSNS two forms. It is useful to express them in covariant independent formulation and we obtain

$$\hat{\mathbf{H}} = 2\begin{pmatrix} d \\ -b \end{pmatrix}\alpha'\epsilon_3\,, \tag{4.10}$$

where ϵ_3 is the volume of three sphere. As a check let us calculate NSNS magnetic charge of $(d, -b)$-five brane

$$q_{\mathrm{NS}}^5 = \frac{1}{2\tilde{\kappa}_{10}^2}\int_{S^3}\hat{H} = dq_{\mathrm{NS5}}\,. \tag{4.11}$$

In the same way we determine RR charge

$$q_{\mathrm{RR}}^5 = -bq_{\mathrm{NS5}} \tag{4.12}$$

which is in agreement with the general result (2.15). Finally we determine the space-time dependence of the dilaton

$$e^{-\hat{\Phi}} = \frac{1}{\hat{g}_s}\frac{\sqrt{\hat{H}}}{\hat{H}'}\,. \tag{4.13}$$

Now we proceed to the analysis of the probe (m, n)-string in this background. In the same way as in previous section we find that the action has the form

$$S_{(m,n)} = -T_{\mathrm{D1}}\int d\tau d\sigma\sqrt{m'^2 + n'^2 e^{-2\Phi_{\mathrm{NS5}}}}\sqrt{-\det G_{MN}^{\mathrm{NS5}}\partial_\alpha x^M\partial_\beta x^N} +$$

$$+ T_{\mathrm{D1}}\int d\tau d\sigma m' B_{MN}^{\mathrm{NS5}}\partial_\tau x^M\partial_\sigma x^N\,, \tag{4.14}$$

where

$$\mathbf{m}' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} dm - bn \\ -cm + an \end{pmatrix}\,. \tag{4.15}$$

As we could expect the (m, n)-action in $(d, -b)$-five brane background is equivalent to the action of (m', n') string in the background of $NS5$-brane. The most interesting case occurs for

$$m = a\,, \quad n = c \tag{4.16}$$

that implies $m' = 1$, $n' = 0$ and hence the action corresponds to the fundamental string in NS5-brane action. In order to analyze main properties of this configuration we consider

string stretched along x^0, x^1 directions, impose the static gauge $x^0 = \tau, x^1 = \sigma$ and finally consider time dependent radial coordinate only. Then the induced metric has the form

$$g_{\tau\tau} = \frac{1}{\sqrt{g_s}}(-H_{\mathrm{NS5}}^{-1/4} + H_{\mathrm{NS5}}^{3/4}\dot{R}^2)\,, \quad g_{\sigma\sigma} = \frac{1}{\sqrt{g_s}}H^{-1/4}\,, \dot{R} = \frac{dR}{dt} \qquad (4.17)$$

and hence the action (4.14) has the form

$$S_{(a,c)} = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{1 - H_{\mathrm{NS5}}\dot{R}^2}\,. \qquad (4.18)$$

From (4.18) we see that there is no potential for the fundamental string probe in the NS5-brane background which nicely demonstrates the fact that fundamental string together with NS5-brane background can form marginal bound state. Expressing this fact in the original variables, we see that there is no potential for (a,c)-string in $(d,-b)$-five brane background whenever $ad - bc = 1$.

Another interesting case occurs for

$$m = b\,, \quad n = d \qquad (4.19)$$

that implies $m' = 0, n' = 1$. In other words the dynamics of (b,d)-string in $(d,-b)$-five brane background is equivalent to the motion of D1-brane in NS5-brane background. The dynamics of this configuration was analyzed in [25] and we will not repeat it here. Finally note that the motion of the general (m',n')-string in NS5-brane background is simple generalization of the case of the electrified brane [26, 27]. Since this generalization is trivial we will not repeat it here and recommend the original papers for more details.

5   SL(2, ℤ) transformation of NS5-brane and F-string bound state

In this section we perform $\mathrm{SL}(2,\mathbb{Z})$ transformation of the supergravity solution with Q_1 fundamental strings and Q_5-five branes that has the form [11]

$$e^{-2\Phi} = \frac{1}{g_s^2}f_5^{-1}f_1\,, \quad B_{05} = \frac{1}{f_1} - 1\,, \quad H_{mnp} = \epsilon_{mnpq}\partial_q f_5\,, m,n,p,q = 1,2,3,4\,,$$
$$ds^2 = f_1^{-1}(-dt^2 + dx_5^2) + f_5(dx_1^2 + \cdots + dx_4^2) + (dx_6^2 + \cdots + dx_9^2)\,, \qquad (5.1)$$

and where

$$f_1 = 1 + \frac{r_1^2}{r^2} = 1 + \frac{16\pi^4 g_s^2 \alpha'^3 Q_1}{V_4 r^2}\,, \quad f_5 = 1 + \frac{r_5^2}{r^2} = 1 + \frac{\alpha' Q_5}{r^2}\,, \qquad (5.2)$$

where $r^2 = \sum_{m=1}^4 x_m^2$, where x_m are coordinates in the space transverse to NS5-branes wrapped over four torus with volume $V_4 = (2\pi)^4\alpha'^2 v$. This fact implies that each $x^i, i = 6, \ldots, 9$ are identified with period $2\pi v^{1/4}\alpha'^{1/2}$. Note also that the fundamental strings are smeared over this four torus.

Let us now perform $\mathrm{SL}(2,\mathbb{Z})$ transformation of the background (5.1) and we find that the charges corresponding to the fundamental strings and NS5-branes are equal to

$$\begin{pmatrix} q_{\mathrm{NS}}^F \\ q_{\mathrm{RR}}^F \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} Q_1 q_F\,, \quad \begin{pmatrix} q_{\mathrm{NS}}^5 \\ q_{\mathrm{RR}}^5 \end{pmatrix} = \begin{pmatrix} d \\ -b \end{pmatrix} Q_5 q_{\mathrm{NS5}} \qquad (5.3)$$

while the line element has the form

$$d\hat{s}^2 = \frac{\sqrt{c^2 + d^2 g_s^2}}{g_s} \sqrt{1 + \frac{c^2 r_1^2 + d^2 g_s^2 r_5^2}{c^2 + d^2 g_s^2} \frac{1}{r^2}} \times$$
$$\times \left(\frac{1}{f_1 \sqrt{f_5}} (-dt^2 + dx_5^2) + \sqrt{f_5}(dx_1^2 + \cdots + dx_4^2) + \frac{1}{\sqrt{f_5}} (dx_6^2 + \cdots + dx_9^2) \right). \quad (5.4)$$

We see that it is again natural to perform rescaling

$$\frac{(c^2 + d^2 g_s^2)^{1/4}}{\sqrt{g_s}} x^M = \hat{x}^M \Rightarrow r^2 = \hat{r}^2 \frac{g_s}{\sqrt{c^2 + d^2 g_s^2}} \, . \quad (5.5)$$

Further, in the limit $r \to \infty$ we have

$$\frac{1}{\hat{g}_s} = \frac{g_s}{c^2 + d^2 g_s^2} \, , \quad \chi_0 = \frac{ac + bd g_s^2}{c^2 + d^2 g_s^2} \quad (5.6)$$

and we again want to express f_1 and f_5 as functions of $\hat{g}_s$ and χ_0 together with the numbers that are proportional to the corresponding charges. If we proceed in the same way as in previous two sections we find

$$\hat{f}_1 = 1 + \frac{16\pi^4 \alpha^3 \hat{g}_s Q_1}{\hat{V}_4 \hat{r}^2} \sqrt{c^2 + \hat{g}_s^2 (a - c\chi_0)^2} \, , \quad (5.7)$$

where we also used the fact that under rescaling given above the coordinates $\hat{x}_i, i = 6, \ldots, 9$ have identifications $\frac{2\pi v^{1/4} \alpha' (c^2 + d^2 g_s^2)^{1/4}}{\sqrt{g_s}}$ and hence

$$V_4 = \hat{V}_4 \frac{c^2 + \hat{g}_s^2 (a - c\chi_0)^2}{\hat{g}_s^2} \, . \quad (5.8)$$

In case of f_5 we proceed as in section 3 and we obtain

$$f_5 = 1 + \frac{\alpha' Q_5 \sqrt{\hat{g}_s^2 (d\chi_0 - b)^2 + d^2}}{\hat{r}^2} \, . \quad (5.9)$$

Finally we write

$$1 + \frac{c^2 r_1^2 + d^2 g_s^2 r_5^2}{c^2 + d^2 g_s^2} \frac{1}{r^2} = \hat{f}_1' + \hat{f}_5' - 1$$
$$\hat{f}_1' = 1 + \frac{16\pi^4 \alpha'^3 Q_1 c^2 \hat{g}_s}{\hat{r}^2 \hat{V}_4 \sqrt{c^2 + \hat{g}_s^2 (a - c\chi_0)^2}} \, ,$$
$$f_5' = 1 + \frac{\alpha' Q_5}{\hat{r}^2} \frac{d^2}{\sqrt{d^2 + \hat{g}_s^2 (d\chi_0 - b)^2}} \, . \quad (5.10)$$

Collecting all these results together we obtain the line element in the form

$$d\hat{s}^2 = \sqrt{\hat{f}_1' + \hat{f}_5' - 1} \left(\frac{1}{\hat{f}_1 \sqrt{\hat{f}_5}} (-d\hat{x}_0^2 + d\hat{x}_5^2) + \sqrt{\hat{f}_5}(d\hat{x}_1^2 + \cdots + d\hat{x}_4^2) + \frac{1}{\sqrt{\hat{f}_5}} (d\hat{x}_6^2 + \cdots + d\hat{x}_9^2) \right)$$
$$(5.11)$$

while the dilaton and RR zero form are equal to

$$e^{-\hat{\Phi}} = \frac{1}{\hat{g}_s} \frac{\sqrt{\hat{f}_1 \hat{f}_5}}{\hat{f}_1' + \hat{f}_5' - 1},$$

$$\hat{\chi} = \frac{1}{c^2 + \hat{g}_s^2(a - c\chi_0)^2} \frac{ac\hat{f}_1 + \frac{bd}{\hat{g}_s^2}(c^2 + \hat{g}_s^2(a - c\chi_0)^2)\hat{f}_5}{\hat{f}_1' + \hat{f}_5' - 1}. \tag{5.12}$$

Finally the non-zero RR and NSNS two and three forms have the form

$$\hat{C}_{05} = -\frac{b}{\sqrt{c^2 + \hat{g}_s^2(a - c\chi_0)^2}} \left(\frac{1}{\hat{f}_1} - 1\right),$$

$$\hat{B}_{05} = \frac{d}{\sqrt{c^2 + \hat{g}_s^2(a - c\chi_0)^2}} \left(\frac{1}{\hat{f}_1} - 1\right). \tag{5.13}$$

For the contribution from 5-branes we obtain

$$\hat{H}_{mnp} = 2dQ_5\alpha'(\epsilon_{S^3})_{mnp}, \quad \hat{F}_{mnp} = -2bQ_5\alpha'(\epsilon_{S^3})_{mnp}. \tag{5.14}$$

As the check of the validity of our solution let us consider S-duality transformation when $a = b = 0$ and $c = 1, b = -1$ that also implies $\chi_0 = 0$. Then it is easy to see that the harmonic functions defined above have the form

$$\hat{f}_1 = \hat{f}_1' = 1 + \frac{16\pi^4\alpha'^3\hat{g}_s Q_1}{\hat{V}_4\hat{r}^2}, \quad \hat{f}_5 = 1 + \frac{\alpha'Q_5\hat{g}_s}{\hat{r}^2}, \quad \hat{f}_5' = 1 \tag{5.15}$$

and hence the line element has the form

$$d\hat{s}^2 = \frac{1}{\sqrt{\hat{f}_1\hat{f}_5}}(-d\hat{x}_0^2 + d\hat{x}_5^2) + \sqrt{\hat{f}_1\hat{f}_5}(d\hat{x}_1^2 + \cdots + d\hat{x}_4^2) + \frac{\sqrt{\hat{f}_1}}{\sqrt{\hat{f}_5}}(d\hat{x}_6^2 + \cdots + d\hat{x}_9^2) \tag{5.16}$$

and dilaton

$$e^{-\hat{\Phi}} = \frac{1}{\hat{g}_s}\sqrt{\frac{\hat{f}_5}{\hat{f}_1}} \tag{5.17}$$

which is precisely the D1-D5-brane background [12], for nice review see [13].

We are mainly interested in the near horizon limit of the background specified by the equations (5.11), (5.12), (5.13) and (5.14) since we expect that it leads to AdS$_3 \times S^3$ background with mixed fluxes. We shall do it in the next section.

6 Near horizon limit

We consider two ways how to perform the near horizon limit of SL$(2, \mathbb{Z})$ transformed solution of the bound state of NS-five branes and fundamental strings. In the first case we firstly take the near horizon limit and then perform the SL$(2, \mathbb{Z})$ transformation. In the second one we consider opposite situation when we firstly perform the SL$(2, \mathbb{Z})$ transformation and then we take the near horizon limit. Finally we compare these two results.

In the first case we start with the solution (5.1) and take its near horizon limit $r \ll r_{1,5}$ and we obtain the background in the form

$$ds^2 = \frac{r^2}{r_1^2}(-dt^2 + dx_5^2) + \frac{r_5^2}{r^2}(dx_1^2 + \cdots + dx_4^2) + (dx_6^2 + \cdots + dx_9^2) . \tag{6.1}$$

Let us rescale the time and x_5 coordinate as

$$\frac{r_5}{r_1}dr = d\hat{t} , \quad \frac{r_5}{r_1}dx_5 = d\hat{x}_5 \tag{6.2}$$

so that the line element has the form

$$ds^2 = \frac{r^2}{L^2}(-d\hat{t}^2 + d\hat{x}_5^2) + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_3 + (dx_6^2 + \ldots dx_9^2) , L^2 = r_5^2 \tag{6.3}$$

while the dilaton is constant and equal to

$$e^{-2\Phi} = \frac{1}{g_s^2}\frac{r_1^2}{r_5^2} . \tag{6.4}$$

Note that thanks to the rescaling we have

$$H_{r05} = 2\frac{r}{L^2} \tag{6.5}$$

so that the form H can be written using the volume element of AdS$_3$ as

$$H = \frac{2}{L}\epsilon_{\text{AdS}_3} , \epsilon_{\text{AdS}_3} = \sqrt{\det g_{\text{AdS}_3}}\, dr \wedge d\hat{t} \wedge d\hat{x}_5 . \tag{6.6}$$

The flux from the NS5-brane source has the form

$$H = 2L^2 \epsilon_{S^3} . \tag{6.7}$$

Finally we introduce dimensionless coordinates on AdS$_3$ in the following way

$$\hat{t} = L\tilde{t} , \quad \hat{x}_5 = L\tilde{x}_5 \tag{6.8}$$

so that $\epsilon_{\text{AdS}_3} = L^3 \tilde{\epsilon}_{\text{AdS}_3}$ and hence we obtain the result

$$H = 2L^2(\tilde{\epsilon}_{\text{AdS}_3} + \epsilon_{S^3}) \tag{6.9}$$

and the line element in the form

$$ds^2 = L^2\left[\tilde{r}^2(-d\tilde{t}^2 + d\tilde{x}^2) + \frac{d\tilde{r}^2}{\tilde{r}^2} + d\Omega_3\right] + ds_T^2 =$$
$$= L^2\left[ds_{\widetilde{\text{AdS}}_3}^2 + ds_{\Omega^3}^2\right] + ds_T^2 , \tag{6.10}$$

where $ds_T^2 = dx_6^2 + \cdots + dx_9^2$ and where $ds_{\widetilde{\text{AdS}}_3}^2$ is the line element of AdS$_3$ space expressed using dimensionless variables.

Since the solution given above is a consistent solution of type IIB supergravity it is possible to perform $SL(2,\mathbb{Z})$ transformation of this solution. As a result we obtain

$$d\tilde{s}^2 = \sqrt{\frac{c^2}{g_s^2}\frac{r_1^2}{r_5^2} + d^2}\left[L^2\left[ds^2_{\widetilde{\text{AdS}_3}} + ds^2_{\Omega^3}\right] + ds^2_T\right].\tag{6.11}$$

We see that the new solution has the curvature radius

$$\tilde{L}^2 = \sqrt{\frac{c^2}{g_s^2}\frac{r_1^2}{r_5^2} + d^2}L^2 = \frac{r_5}{g_s}\sqrt{c^2 r_1^2 + d^2 g_s^2 r_5^2}.\tag{6.12}$$

Further, there are following NSNS and RR three forms

$$\tilde{H} = 2d\alpha' Q_5(\tilde{\epsilon}_{\text{AdS}_3} + \epsilon_{S^3}), \quad \tilde{F} = -2b\alpha' Q_5(\tilde{\epsilon}_{\text{AdS}_3} + \epsilon_{S^3})\tag{6.13}$$

and dilaton and zero RR form are equal to

$$e^{-\tilde{\Phi}} \equiv \frac{1}{\hat{g}_s} = \frac{\sqrt{Q_1 Q_5 v}}{c^2 Q_1 + d^2 Q_5 v}, \quad \tilde{\chi} = \frac{acQ_1 + bdvQ_5}{c^2 Q_1 + d^2 Q_5 v}.\tag{6.14}$$

Let us now consider the case when we firstly perform $SL(2,\mathbb{Z})$ duality transformation and then take the near horizon limit. Explicitly, we begin with the $SL(2,\mathbb{Z})$ transformed background

$$d\hat{s}^2 = \frac{1}{g_s}\sqrt{c^2 r_1^2 + d^2 g_s^2 r_5^2}\left(\frac{r^2}{r_1^2 r_5}(-dt^2 + dx_5^2) + \frac{r_5}{r^2}dr^2 + r_5 d\Omega_3 + \frac{1}{r_5}(dx_6^2 + \cdots + dx_9^2)\right).\tag{6.15}$$

We rescale t and x^5 coordinates as

$$t\frac{1}{g_s r_1}\sqrt{c_1^2 r_1^2 + d^2 g_s^2 r_5^2} = \hat{t}, \quad x^5\frac{1}{g_s}\sqrt{c_1^2 r_1^2 + d^2 g_s^2 r_5^2} = \hat{x}^5,\tag{6.16}$$

so that the line element has the form

$$d\hat{s}^2 = r^2\frac{g_s}{r_5\sqrt{c^2 r_1^2 + d^2 g_s^2 r_5^2}}(-d\hat{t}^2 + d\hat{x}_5^2) + \frac{r_5}{g_s}\sqrt{c^2 r_1^2 + d^2 g_s^2 r_5^2}\frac{dr^2}{r^2} +$$
$$+ \frac{r_5}{g_s}\sqrt{c^2 r_1^2 + d^2 g_s^2 r_5^2}d\Omega_3^2 + \frac{1}{g_s r_5}\sqrt{c^2 r_1^2 + d^2 g_s^2 r_5^2}(dx_6^2 + \ldots dx_9^2),\tag{6.17}$$

where the expression on the first line corresponds to the line element of $\text{AdS}_3 \times S^3$ with the curvature radius

$$\hat{L}^2 = \frac{r_5\sqrt{c^2 r_1^2 + d^2 g_s^2 r_5^2}}{g_s} = \alpha'\sqrt{\frac{c^2 Q_1 Q_5 + d^2 Q_5^2 v}{v}}\tag{6.18}$$

and we see that $\hat{L}$ and $\tilde{L}$ coincide. In the same way we obtain

$$e^{-\hat{\Phi}} = \frac{\sqrt{Q_1 Q_5 v}}{c^2 Q_1 + d^2 Q_5 v}, \quad \hat{\chi} = \frac{acQ_1 + bdvQ_5}{c^2 Q_1 + d^2 Q_5 v}\tag{6.19}$$

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and we again see that these expressions coincide with $\tilde{\tilde{\Phi}}$ and $\tilde{\chi}$. Now we focus on the near horizon limit of forms. In case of $\hat{B}_{05}$ we obtain

$$\hat{B}_{05} = \frac{g_s^2}{c_1^2 r_1^2 + d^2 g_s^2 r_5^2} r^2 \tag{6.20}$$

so that using the rescaled coordinates

$$\hat{t} = \hat{L}\tilde{t}, \quad \hat{r} = \hat{L}\tilde{r}, \quad \hat{x}_5 = \hat{L}\tilde{x}_5 \tag{6.21}$$

we obtain

$$\hat{H} = 2dQ_5\alpha'\left(\epsilon_{\widetilde{\mathrm{AdS}}_3} + \epsilon_{S^3}\right), \quad \hat{F} = -2bQ_5\alpha'\left(\epsilon_{\widetilde{\mathrm{AdS}}_3} + \epsilon_{S^3}\right) \tag{6.22}$$

that agree with $\tilde{H}$ and $\tilde{F}$. In summary we have shown the important result that the near horizon limit and $\mathrm{SL}(2,\mathbb{Z})$ transformation commutes which implies that the $\mathrm{AdS}_3 \times S^3$ background with mixed fluxes can be derived from $\mathrm{AdS}_3 \times S^3$ background through $\mathrm{SL}(2,\mathbb{Z})$ rotation.

We will now analyze the consequences of this result on the dynamics of a probe string in this background. Using the same arguments as in previous section we find the action in the form

$$S = -T_{\mathrm{D1}} \int d\tau d\sigma \sqrt{m'^2 + n'^2 e^{-2\Phi_{\mathrm{NS}}}} \sqrt{-\det(G_{MN}^{\mathrm{NS}}\partial_\alpha x^M \partial_\beta x^N)} +$$
$$+ T_{\mathrm{D1}} \int d\tau d\sigma m' B_{MN}^{\mathrm{NS}} \partial_\tau x^M \partial_\sigma x^N, \tag{6.23}$$

where

$$\mathbf{m'} = \begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} dm - bn \\ -cm + an \end{pmatrix}, \tag{6.24}$$

and where $\Phi_{\mathrm{NS}}, G_{MN}^{\mathrm{NS}}$ and B_{MN}^{NS} correspond to the $\mathrm{AdS}_3 \times S^3$ with NSNS flux.

Now the equations of motion that follow from the action have the form

$$0 = \partial_\alpha \left[G_{MN}^{\mathrm{NS}} \partial_\beta x^N g^{\beta\alpha} \sqrt{-\det g_{\alpha\beta}} \sqrt{m'^2 + n'^2 e^{-2\Phi_{\mathrm{NS}}}} \right]$$
$$- \frac{1}{2}\partial_M G_{KL}^{\mathrm{NS}} \partial_\alpha x^K \partial_\beta x^L \sqrt{m'^2 + n'^2 e^{-2\Phi_{\mathrm{NS}}}}$$
$$+ m' H_{MKN}^{\mathrm{NS}} \partial_\tau x^K \partial_\sigma x^N, \tag{6.25}$$

where

$$H^{\mathrm{NS}} = dB^{\mathrm{NS}}. \tag{6.26}$$

To proceed further we use the fact that $\mathrm{AdS}_3 \times S^3$ is isomorphic to the group manifold $G = \mathrm{SU}(1,1) \times \mathrm{SU}(2)$. Explicitly, let g be the group element from G. Then it is possible to write the metric (6.10) as

$$G_{MN}^{\mathrm{NS}} = L^2 E_M^A E_N^B K_{AB}, \tag{6.27}$$

where for the group element $g \in G$ we have

$$J \equiv g^{-1}dg = E_M^A T_A dx^M, \tag{6.28}$$

where T_A is the basis of Lie Algebra $\mathcal{G}$ of the group G. Note that $K_{AB} = \text{Tr}(T_A T_B)$. Further, from the definition (6.28) we obtain

$$dJ + J \wedge J = 0 \tag{6.29}$$

that implies

$$\partial_M E_N^A - \partial_N E_M^A + f^A_{BC} E_M^B E_N^C = 0\,, \tag{6.30}$$

where

$$[T_B, T_C] = T_A f^A_{BC}\,. \tag{6.31}$$

In case of the flux (6.9) we have following relation:

$$H^{\text{NS}}_{MNK} E^M_A E^N_B E^K_C = L^2 f_{ABC}\,. \tag{6.32}$$

With the help of (6.32) we can write

$$E^M_C H_{MKL} \partial_\tau x^K \partial_\sigma x^L = L^2 f_{CAB} J^A_\tau J^B_\sigma\,, \tag{6.33}$$

where E^M_A is inverse to E^B_M defined as

$$E^M_A E^B_M = \delta^B_A\,, \quad E^M_A E^A_N = \delta^M_N\,. \tag{6.34}$$

Now with the help of (6.30) and (6.33) we can rewrite the equations of motion (6.25) to the form that contains the current $J^A_\alpha = E^A_M \partial_\alpha x^M$

$$L^2 T_{\text{D1}} K_{AB} \partial_\alpha \left[J^B_\beta g^{\beta\alpha} \sqrt{-\det g_{\alpha\beta}} \sqrt{m'^2 + n'^2 e^{-2\Phi_{\text{NS}}}} \right] +$$
$$+ L^2 T_{\text{D1}} m' f_{ABC} J^B_\tau J^C_\sigma = 0\,, \tag{6.35}$$

where

$$g_{\alpha\beta} = K_{AB} J^A_\alpha J^B_\beta\,.$$

The equation (6.35) can be rewritten into the form

$$\partial_\alpha \hat{J}^{A\alpha} = 0\,, \quad \hat{J}^{A\alpha} = L^2 T_{\text{D1}} \left(J^A_\beta g^{\beta\alpha} \sqrt{-\det g_{\alpha\beta}} \sqrt{m'^2 + n'^2 e^{-2\Phi_{\text{NS}}}} + m' \epsilon^{\alpha\beta} J^A_\beta \right)\,. \tag{6.36}$$

We see that the current $\hat{J}^{A\alpha}$ is conserved. Following [35] we introduce an auxiliary metric $\gamma_{\alpha\beta}$ that obeys the equation

$$T_{\alpha\beta} \equiv \frac{1}{2} \gamma_{\alpha\beta} \gamma^{\mu\nu} g_{\mu\nu} - g_{\alpha\beta} = 0\,. \tag{6.37}$$

It is easy to see that this equation has solution $\gamma_{\alpha\beta} = g_{\alpha\beta}$. If we further introduce light-cone coordinates

$$\sigma^+ = \frac{1}{2}(\tau + \sigma)\,, \quad \sigma^- = \frac{1}{2}(\tau - \sigma) \tag{6.38}$$

we can rewrite the equation (6.36) into the form

$$\partial_+ \hat{J}^{A+} + \partial_- \hat{J}^{A-} = 0\,, \quad \partial_\pm = \frac{\partial}{\partial \sigma^\pm}\,, \tag{6.39}$$

where

$$
\begin{aligned}
\hat{J}^{A+} = \frac{1}{2}(\hat{J}^{A\tau} + \hat{J}^{A\sigma}) = \\
= \frac{\sqrt{\lambda}}{2}\left[\sqrt{m'^2 + e^{-2\Phi_{\mathrm{NS}}}n'^2}\sqrt{-\gamma}\left(\gamma^{\tau\alpha}J_\alpha^A + \gamma^{\sigma\alpha}J_\alpha^A\right) + m'(J_\sigma^A - J_\tau^A)\right], \\
\hat{J}^{A-} = \frac{1}{2}(\hat{J}^{A\tau} - \hat{J}^{A\sigma}) = \\
= \frac{\sqrt{\lambda}}{2}\left[\sqrt{m'^2 + e^{-2\Phi_{\mathrm{NS}}}n'^2}\sqrt{-\gamma}(\gamma^{\tau\alpha}J_\alpha^A - \gamma^{\sigma\alpha}J_\alpha^A) + m'(J_\sigma^A + J_\tau^A)\right], \quad (6.40)
\end{aligned}
$$

where $\sqrt{\lambda} = \frac{L^2}{2\pi\alpha'}$. As the next step we fix the auxiliary metric to have the form $\gamma_{\alpha\beta} = \eta_{\alpha\beta}, \eta_{\alpha\beta} = \mathrm{diag}(-1,1)$ keeping in mind that currents still have to obey the equation (6.37). In this gauge $\hat{J}_\pm^A$ simplify considerably and we obtain

$$
\begin{aligned}
\hat{J}^{A+} = -\frac{1}{2}\hat{J}_-^A = \frac{\sqrt{\lambda}}{2}\left[J_\sigma^A\left(\sqrt{m'^2 + n'^2 e^{-2\Phi_{\mathrm{NS}}}} + m'\right)\right. \\
\left. - J_\tau^A\left(\sqrt{m'^2 + e^{-2\Phi_{\mathrm{NS}}}n'^2} + m'\right)\right], \\
\hat{J}^{A-} = -\frac{1}{2}\hat{J}_+^A = -\frac{T_{\mathrm{D1}}}{2}\left[J_\tau^A\left(\sqrt{m'^2 + e^{-2\Phi_{\mathrm{NS}}}n'^2} - m'\right)\right. \\
\left. + J_\sigma^A\left(\sqrt{m'^2 + n'^2 e^{-2\Phi_{\mathrm{NS}}}} - m'\right)\right], \quad (6.41)
\end{aligned}
$$

where we introduced the light-cone metric with $\eta_{+-} = \eta_{-+} = -2$, $\eta^{+-} = \eta^{-+} = -\frac{1}{2}$ so that $\hat{J}^{A+} = \eta^{+-}\hat{J}_-^A = -\frac{1}{2}\hat{J}_-^A$, $\hat{J}^{A-} = \eta^{-+}\hat{J}_-^A = -\frac{1}{2}\hat{J}_+^A$. We see that for

$$
n' = 0 \quad (6.42)
$$

the current $\hat{J}_+^A$ vanishes identically and the equation (6.39) gives

$$
\partial_+ \hat{J}_-^A = 0, \quad \hat{J}_-^A = 2\sqrt{\lambda}m'(J_\tau^A - J_\sigma^A). \quad (6.43)
$$

Note that we can write $\hat{J}_- = \hat{J}_-^A T_A = 2g^{-1}\partial_- g$. Then from (6.43) we obtain

$$
\frac{1}{2}\partial_+ \hat{J}_- = -g^{-1}\partial_+ g g^{-1}\partial_- g + g^{-1}\partial_-\partial_+ g g^{-1}g = g^{-1}\partial_-[\partial_+ g g^{-1}]g = 0 \quad (6.44)
$$

so that there is second current $\hat{J}_+ = \partial_+ g g^{-1}$ that obeys the equation

$$
\partial_- \hat{J}_+ = 0. \quad (6.45)
$$

Our result shows that (a,c)-string in the $(d,-b)$-mixed flux background has two holomorphic and anti-holomorphic currents and can be analyzed in the same way as WZW model [37] using powerful conformal field theory techniques. On the other hand it is important to stress that (m,n)-string in $(d,-b)$-background is still classically integrable for any values of m,n [35].

7 Conclusion

Let us outline results derived in this paper. We found type IIB supergravity solutions corresponding to (p,q)-string and (p,q)-five brane backgrounds using $\mathrm{SL}(2,\mathbb{Z})$ covariance of type IIB superstring theory. We showed that these solutions have the correct values of charges and also that the probe (m,n)-string in this background can be mapped to the (m',n')-string in the original fundamental or NS5-brane background where m',n' are integers whose values are predicted by $\mathrm{SL}(2,\mathbb{Z})$ duality of type IIB superstring theory. We also derived background that arises by $\mathrm{SL}(2,\mathbb{Z})$ transformation of the bound state of Q_5 NS5-branes and Q_1 fundamental strings. Then we considered its near horizon limit and argued that it leads to the $\mathrm{AdS}_3 \times S^3$ background with mixed three form fluxes. Then we analyzed (m,n)-string in given background and we argued that for $m = a, n = c$ the string equations of motion are equivalent to the conservation of two holomorphic and antiholomorphic currents and hence this particular case can be analyzed using two dimensional conformal field theory. This very interesting result shows that (a,c)-string is the natural probe of $\mathrm{AdS}_3 \times S^3$ background with $(d,-b)$-fluxes.

The extension of this work is as follows. It would be nice to analyze solutions of (m,n)-string equations of motion in $\mathrm{AdS}_3 \times S^3$ background with mixed fluxes. background. It would be also nice to analyze integrability of general (m,n)-string in this background in more details using the manifest covariant form of (m,n)-string action. We hope to return to these problems in future.

Acknowledgments

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