

# A Jordanian deformation of AdS space in type IIB supergravity

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**ABSTRACT:** We consider a Jordanian deformation of the  $\text{AdS}_5 \times \text{S}^5$  superstring action by taking a simple  $R$ -operator which satisfies the classical Yang-Baxter equation. The metric and NS-NS two-form are explicitly derived with a coordinate system. Only the AdS part is deformed and the resulting geometry contains the 3D Schrödinger spacetime as a subspace. Then we present the full solution in type IIB supergravity by determining the other field components. In particular, the dilaton is constant and a R-R three-form field strength is turned on. The symmetry of the solution is  $[\text{SL}(2, \mathbb{R}) \times \text{U}(1)^2] \times [\text{SU}(3) \times \text{U}(1)]$  and contains an anisotropic scale symmetry.

**KEYWORDS:** AdS-CFT Correspondence, Integrable Field Theories, Sigma Models

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## 1 Introduction

One of the most intriguing subjects in string theory is the AdS/CFT correspondence [1–3] and it has well been studied from various aspects with an enormous number of works. Although it is often supposed to hold as a matter of course in the recent studies, it is still important to elaborate the original form, the duality between type IIB string theory on  $\text{AdS}_5 \times \text{S}^5$  and the  $\mathcal{N}=4$  super Yang-Mills (SYM) theory, to gain deeper insight for the basic origin of AdS/CFT. In this direction, the integrability behind this duality would play an important role (For a comprehensive review, see [4]).

We will concentrate on the string-theory side, type IIB string theory on  $\text{AdS}_5 \times \text{S}^5$  here. The Green-Schwarz type string action can be constructed based on a supercoset [5],

$$PSU(2, 2|4) / [SO(1, 4) \times SO(5)] .$$

This coset enjoys the  $\mathbb{Z}_4$ -grading property and it leads to the classical integrability [6].<sup>1</sup> The classification of possible supercosets, which lead to the classically integrable, consistent string theories, is performed in [10].<sup>2</sup>

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<sup>1</sup>There is another formulation of the  $\text{AdS}_5 \times \text{S}^5$  superstring action [7]. For the classical integrability in this formalism, see [8, 9].

<sup>2</sup>Some new examples of  $\text{AdS}_2$  and  $\text{AdS}_3$  have been found in [11].

The next task is to consider integrable deformations. Although there are some kinds of integrable deformations, we will focus upon  $q$ -deformations of the  $\text{AdS}_5 \times \text{S}^5$  superstring. Deformations of this type are the standard  $q$ -deformations of Drinfeld-Jimbo type [12–14] (For a nice review, see [15]). The work along this direction was initiated by elaborating the classical integrability of squashed  $\text{S}^3$  sigma models [16–30]. Then the result was generalized to higher-dimensional cases [31] with the help of the Yang-Baxter sigma model (YBsM) description [19, 20] (For recent progress on YBsM, see [32, 33]).

Then, by applying the YBsM description to the  $\text{AdS}_5 \times \text{S}^5$  superstring and the  $q$ -deformed classical action was presented in an abstract form with the group-theoretical language [34]. In the YBsM description, a linear  $R$ -operator is a key ingredient. It is constructed from a skew-symmetric, classical  $r$ -matrix satisfying the modified classical Yang-Baxter equation (mCYBE). The coordinate system has been introduced in [35] and the metric in the string frame and NS-NS two-form have been determined. However, the complete gravitational solution has not been fixed yet in type IIB supergravity.

There is another kind of  $q$ -deformations, which are called Jordanian deformations [36–38] or sometimes non-standard  $q$ -deformations (For the case of Lie superalgebras, see [39–42]). In the previous work [43], we have considered Jordanian deformations of the  $\text{AdS}_5 \times \text{S}^5$  superstring action by using linear  $R$ -operators satisfying the classical Yang-Baxter equation (CYBE), rather than mCYBE. The action presented in [43] is also written abstractly in terms of a group element, and explicit examples have not been provided yet.

In this paper, we consider a Jordanian deformation of the  $\text{AdS}_5 \times \text{S}^5$  with a simple  $R$ -operator. The metric and NS-NS two-form are explicitly derived with a coordinate system. Only the AdS part is deformed and the resulting geometry contains the 3D Schrödinger spacetime as a subspace. In this sense, this study can be regarded as a generalization of the previous works [44–46]. Then we present the full solution in type IIB supergravity by determining the other field components. In particular, the dilaton is constant and a R-R three-form field strength is turned on. The symmetry of the solution is given by  $[\text{SL}(2, \mathbb{R}) \times \text{U}(1)^2] \times [\text{SU}(3) \times \text{U}(1)]$  and contains an anisotropic scale symmetry.

This paper is organized as follows. In section 2 we give a short review of Jordanian deformations of the  $\text{AdS}_5 \times \text{S}^5$  superstring action. Then, by taking a simple  $R$ -operator satisfying CYBE, the metric and NS-NS two-form are explicitly derived with a coordinate system. The resulting geometry is given by the product of a deformed AdS space and round  $\text{S}^5$ . Also for a slightly generalized  $R$ -operator, the string action is derived. The resulting metric represents a time-dependent background. In section 3 we present the gravitational solution in type IIB supergravity by finding out the other field components. In particular, the dilaton is constant. Section 4 is devoted to conclusion and discussion. In appendix A, our notation and convention is summarized. In appendix B, we list some classical  $r$ -matrices and the associated string actions.

## 2 Jordanian deformations of $\text{AdS}_5 \times \text{S}^5$

In this section, we first introduce Jordanian deformations of the  $\text{AdS}_5 \times \text{S}^5$  superstring. Then by taking a simple example of skew-symmetric, classical  $r$ -matrix, the string action

is obtained with a coordinate system. Then the metric and NS-NS two-form are derived explicitly. The resulting metric contains the 3D Schrödinger spacetime as a subspace. A more general example is also presented.

**2.1 Setup**

First of all, we will give a short summary of the work [43]. One may consider Jordanian deformations of the  $\text{AdS}_5 \times S^5$  superstring action with linear  $R$  operators satisfying CYBE. The construction follows basically [34] with the help of the YBSM description [19, 20].

The deformed Green-Schwarz string action is given by [43]

$$S = -\frac{1}{2} \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma P_{-}^{\alpha\beta} \text{Str} \left( A_{\alpha} d \circ \frac{1}{1 - \eta [R_{\text{Jor}}]_g \circ d} (A_{\beta}) \right), \tag{2.1}$$

where the left-invariant one-form  $A_{\alpha}$  is given by

$$A_{\alpha} \equiv g^{-1} \partial_{\alpha} g, \quad g \in \text{SU}(2, 2|4). \tag{2.2}$$

The projection operators  $P_{\pm}^{\alpha\beta}$  are defined as linear-combinations of the metric  $\gamma^{\alpha\beta}$  and the anti-symmetric tensor  $\epsilon^{\alpha\beta}$  on the string world-sheet like

$$P_{\pm}^{\alpha\beta} \equiv \frac{1}{2} \left( \gamma^{\alpha\beta} \pm \epsilon^{\alpha\beta} \right) \tag{2.3}$$

and satisfy the following properties,

$$P_{\pm}^{\alpha\gamma} \gamma_{\gamma\delta} P_{\pm}^{\delta\beta} = P_{\pm}^{\alpha\beta}, \quad P_{\pm}^{\alpha\gamma} \gamma_{\gamma\delta} P_{\mp}^{\delta\beta} = 0. \tag{2.4}$$

In the action (2.1) the projection  $P_{-}^{\alpha\beta}$  is utilized.

Recall that the Lie superalgebra  $\mathfrak{su}(2, 2|4)$  has a  $\mathbb{Z}_4$  automorphism,  $\Omega$  with  $\Omega^4 = 1$ . This automorphism leads to the decomposition of  $\mathfrak{su}(2, 2|4)$  as follows:

$$\mathfrak{su}(2, 2|4) = \mathfrak{su}(2, 2|4)^{(0)} \oplus \mathfrak{su}(2, 2|4)^{(1)} \oplus \mathfrak{su}(2, 2|4)^{(2)} \oplus \mathfrak{su}(2, 2|4)^{(3)}. \tag{2.5}$$

Here the operation of  $\Omega$  is defined for an element of  $\mathfrak{su}(2, 2|4)^{(n)}$  as

$$\Omega(X^{(n)}) = i^n X^{(n)} \quad \text{for } X^{(n)} \in \mathfrak{su}(2, 2|4)^{(n)}. \tag{2.6}$$

and  $\mathfrak{su}(2, 2|4)^{(n)}$  satisfies the following relation,

$$\left[ \mathfrak{su}(2, 2|4)^{(n)}, \mathfrak{su}(2, 2|4)^{(m)} \right] \subset \mathfrak{su}(2, 2|4)^{(n+m)} \pmod{4}.$$

In particular,  $\mathfrak{su}(2, 2|4)^{(0)}$  is nothing but  $\mathfrak{so}(1, 4) \times \mathfrak{so}(5)$ .

One can introduce projections  $P_n$  from  $\mathfrak{su}(2, 2|4)$  onto  $\mathfrak{su}(2, 2|4)^{(n)}$  ( $n = 0, 1, 2, 3$ ). Then the operator  $d$  is defined as

$$d \equiv P_1 + 2P_2 - P_3. \tag{2.7}$$

The remaining task is to introduce the operation  $[R_{\text{Jor}}]_g$ . In the first place, we introduce a linear  $R$ -operator,  $R_{\text{Jor}}$ ,

$$R_{\text{Jor}} : \mathfrak{gl}(4|4) \rightarrow \mathfrak{gl}(4|4), \tag{2.8}$$

which satisfies the following three properties,

1) the classical Yang-Baxter equation (CYBE) :

$$[R_{\text{Jor}}(M), R_{\text{Jor}}(N)] - R_{\text{Jor}}([R_{\text{Jor}}(M), N] + [M, R_{\text{Jor}}(N)]) = 0,$$

2) the nilpotency :  $(R_{\text{Jor}})^n(M) = 0 \quad (n \geq 3),$

3) the skew-symmetric property (the unitarity condition):

$$\text{Str}(MR_{\text{Jor}}(N)) = -\text{Str}(R_{\text{Jor}}(M)N).$$

The property 3) is not necessary for the definition of Jordanian deformations, but for the classical integrability of the string theory.

Note that  $R_{\text{Jor}}$  does not preserve the real-form condition of  $\mathfrak{su}(2, 2|4)$  in general, even if the domain is restricted to  $\mathfrak{su}(2, 2|4)$ . The real-form condition is not necessary for the classical integrability (i.e., the construction of Lax pair) as shown in [43]. One may expect that the string actions become complex if the real-form condition is not preserved. However, it is not always the case. In fact, some examples of  $R_{\text{Jor}}$ , which break the real-form condition, give rise to real string actions, as we will see later. That is, the real-form condition should be regarded as a sufficient condition for the reality. Still, we have no general criterion to specify the linear operators that lead to the real string actions. It is an important issue to argue the criterion in the future.

Then the operator  $[R_{\text{Jor}}]_g$  is defined as a sequence of the adjoint operation  $\text{Ad}_g$  by  $g$ , the  $R$ -operation and the inverse of the adjoint :

$$[R_{\text{Jor}}]_g(M) \equiv \text{Ad}_g^{-1} \circ R_{\text{Jor}} \circ \text{Ad}_g(M) = g^{-1} R_{\text{Jor}}(gMg^{-1})g. \tag{2.9}$$

This operation is intrinsic to the coset case [19, 20, 31].

**The tensorial notation of the  $R$ -operator.** It is helpful to see the tensorial notation of  $R_{\text{Jor}}$ . In the present case,  $r_{\text{Jor}}$  is skew-symmetric due to the property 3). Hence  $r_{\text{Jor}}$  can be represented by using a skew-symmetrized tensor product of two elements of  $\mathfrak{gl}(4|4)$ ,

$$r_{\text{Jor}} = \sum_i (a_i \otimes b_i - b_i \otimes a_i) \equiv \sum_i a_i \wedge b_i. \tag{2.10}$$

The linear  $R$  operator action is associated with the tensorial notation as follows:

$$R_{\text{Jor}}(M) \equiv \text{Tr}_2[r_{\text{Jor}}(1 \otimes M)] = \sum_i (a_i \text{Tr}(b_i M) - b_i \text{Tr}(a_i M)). \tag{2.11}$$

In the tensorial notation, the property 1) is recast into the familiar expression of CYBE,

$$[r_{\text{Jor},12}, r_{\text{Jor},13}] + [r_{\text{Jor},12}, r_{\text{Jor},23}] + [r_{\text{Jor},13}, r_{\text{Jor},23}] = 0. \tag{2.12}$$

Here the subscripts of  $r_{\text{Jor}}$  specify vector spaces on which  $r_{\text{Jor}}$  acts.

Thus a skew-symmetric solution of CYBE is associated with a linear  $R$ -operator and is related to an integrable deformation of  $\text{AdS}_5 \times \text{S}^5$ . So far, the string action (2.1) is written in an abstract form with a group-theoretical language. In the next subsection, we will take an example of skew-symmetric classical  $r$ -matrix and express explicitly the action with a coordinate system.

## 2.2 A simple example of the string action

Let us consider an explicit example of Jordanian deformations by taking a skew-symmetric classical  $r$ -matrix,<sup>3</sup>

$$r_{\text{Jor}} = \frac{1}{\sqrt{2}} E_{24} \wedge (E_{22} - E_{44}), \quad (2.13)$$

where  $E_{ij}$  is a  $4 \times 4$  matrix defined as

$$(E_{ij})_{kl} \equiv \delta_{ik} \delta_{jl}.$$

The normalization of  $r_{\text{Jor}}$  is absorbed by rescaling of  $\eta$ , as one can see from the action (2.1). Here it is fixed for later convenience.

The  $r$ -matrix (2.13) induces the action of the associated  $R$ -operator as

$$R_{\text{Jor}}(E_{22}) = -R_{\text{Jor}}(E_{44}) = \frac{1}{\sqrt{2}} E_{24}, \quad R_{\text{Jor}}(E_{42}) = -\frac{1}{\sqrt{2}} (E_{22} - E_{44}). \quad (2.14)$$

This mapping rule is obtained from the relation (2.11). Note that the  $r$ -matrix (2.13) does not preserve the real-form condition of  $\mathfrak{su}(2, 2|4)$ . However, it leads to a real string action, as we will see later.

Let us evaluate the string action (2.1). For simplicity, we focus on the bosonic part by restricting a group element  $g$  to the bosonic subsector. In addition, only the  $\text{AdS}_5$  part is deformed in the present example and hence the coset construction for the  $S^5$  part is the usual. Therefore, we concentrate on the coset construction for the  $\text{AdS}_5$  part. Then it is convenient to consider the following coset representative,

$$g = e^{p_0 x^0 + p_1 x^1 + p_2 x^2 + p_3 x^3} e^{\gamma_5 \rho / 2} \in \text{SU}(2, 2) / \text{SO}(1, 4). \quad (2.15)$$

Here  $p_\mu$  ( $\mu = 0, 1, 2, 3$ ) are defined as

$$\begin{aligned} p_0 &= \frac{1}{2} (\gamma_0 - 2n_{05}), & p_1 &= \frac{1}{2} (\gamma_1 - 2n_{15}), \\ p_2 &= \frac{1}{2} (\gamma_2 - 2n_{25}), & p_3 &= \frac{1}{2} (\gamma_3 - 2n_{35}). \end{aligned} \quad (2.16)$$

For the definition of the  $\mathfrak{so}(2, 4)$  generators  $\gamma_\mu$ ,  $\gamma_5$ ,  $n_{\mu\nu}$  and  $n_{\mu 5}$ , see appendix A.

Later, we often use the following quantities

$$x^\pm \equiv \frac{1}{\sqrt{2}} (x^0 \pm x^3), \quad z \equiv e^\rho, \quad (2.17)$$

instead of  $x^0$ ,  $x^3$  and  $\rho$ .

With this setup, the classical action (2.1) can be rewritten as

$$\begin{aligned} S &= S_{\text{AdS}} + S_S, \\ S_{\text{AdS}} &= -\frac{1}{2} \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma (\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \text{Tr} \left( A_\alpha P_2 \circ \frac{1}{1 - 2\eta [R_{\text{Jor}}]_g \circ P_2} (A_\beta) \right). \end{aligned} \quad (2.18)$$

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<sup>3</sup>The term ‘‘Jordanian’’ comes from the fact that  $r_{\text{Jor}}$  is represented by an upper triangular matrix.

Here  $A_\alpha$  is restricted to  $\mathfrak{su}(2, 2)$ . Then  $S_S$  represents the usual  $S^5$  part of the string action and we will not touch on this in the present section.

From now on, let us compute the explicit form of  $A_\alpha$  and

$$J_\alpha \equiv \frac{1}{1 - 2\eta [R_{\text{Jor}}]_g \circ P_2}(A_\alpha). \quad (2.19)$$

After that, the bosonic part of the classical action can be determined explicitly with the coordinate system introduced with the parametrization (2.15).

First of all,  $P_2(A_\alpha)$  can be evaluated as

$$P_2(A_\alpha) = \gamma_0 a_\alpha^0 + \gamma_1 a_\alpha^1 + \gamma_2 a_\alpha^2 + \gamma_3 a_\alpha^3 + \gamma_5 a_\alpha^5, \quad (2.20)$$

where each of the coefficients is given by

$$\begin{aligned} a_\alpha^1 &= \frac{\partial_\alpha x^1}{2z}, & a_\alpha^2 &= \frac{\partial_\alpha x^2}{2z}, & \frac{1}{\sqrt{2}}(a_\alpha^3 + a_\alpha^0) &= \frac{\partial_\alpha x^+}{2z}, \\ \frac{1}{\sqrt{2}}(a_\alpha^3 - a_\alpha^0) &= -\frac{\partial_\alpha x^-}{2z}, & a_\alpha^5 &= \frac{\partial_\alpha z}{2z}. \end{aligned}$$

The next task is to evaluate  $P_2(J_\alpha)$ . The relation (2.19) can be inverted as

$$A_\alpha = \left(1 - 2\eta [R_{\text{Jor}}]_g \circ P_2\right)(J_\alpha). \quad (2.21)$$

By acting  $P_2$  on both sides, the following expression is obtained,

$$\begin{aligned} P_2(A_\alpha) &= P_2 \circ \left(1 - 2\eta [R_{\text{Jor}}]_g \circ P_2\right)(J_\alpha) \\ &= P_2(J_\alpha) - 2\eta P_2 \circ [R_{\text{Jor}}]_g (P_2(J_\alpha)). \end{aligned} \quad (2.22)$$

This is just a linear equation for  $P_2(J_\alpha)$  and hence it is straightforward to evaluate  $P_2(J_\alpha)$ . Note that  $P_2(J_\alpha)$  can be expanded as

$$P_2(J_\alpha) = \gamma_0 j_\alpha^0 + \gamma_1 j_\alpha^1 + \gamma_2 j_\alpha^2 + \gamma_3 j_\alpha^3 + \gamma_5 j_\alpha^5. \quad (2.23)$$

Here  $j_\alpha^\mu$  and  $j_\alpha^5$  are unknown functions to be determined. With this expansion, the right-hand side of (2.22) can be rewritten as

$$\begin{aligned} P_2(A_\alpha) &= \gamma_1 \frac{2z^2 j_\alpha^1 + \sqrt{2}\eta x^1 (j_\alpha^3 + j_\alpha^0)}{2z^2} + \gamma_2 \frac{2z^2 j_\alpha^2 + \sqrt{2}\eta x^2 (j_\alpha^3 + j_\alpha^0)}{2z^2} \\ &+ \gamma_3 \frac{2z^2 j_\alpha^3 - \sqrt{2}\eta (x^1 j_\alpha^1 + x^2 j_\alpha^2 + z j_\alpha^5)}{2z^2} + \gamma_0 \frac{2z^2 j_\alpha^0 + \sqrt{2}\eta (x^1 j_\alpha^1 + x^2 j_\alpha^2 + z j_\alpha^5)}{2z^2} \\ &+ \gamma_5 \frac{2z j_\alpha^5 + \sqrt{2}\eta (j_\alpha^3 + j_\alpha^0)}{2z}. \end{aligned} \quad (2.24)$$

By comparing (2.24) with (2.20), the expressions of  $j_\alpha^\mu$  and  $j_\alpha^5$  are determined as

$$j_\alpha^1 = \frac{z^2 \partial_\alpha x^1 - \eta x^1 \partial_\alpha x^+}{2z^3}, \quad j_\alpha^2 = \frac{z^2 \partial_\alpha x^2 - \eta x^2 \partial_\alpha x^+}{2z^3},$$

$$\frac{1}{\sqrt{2}}(j_\alpha^3 - j_\alpha^0) = \frac{-z^2 \partial_\alpha x^- + \eta(x^1 \partial_\alpha x^1 + x^2 \partial_\alpha x^2 + z \partial_\alpha z) - \eta^2 \left(1 + \frac{(x^1)^2 + (x^2)^2}{z^2}\right) \partial_\alpha x^+}{2z^3},$$

$$\frac{1}{\sqrt{2}}(j_\alpha^3 + j_\alpha^0) = \frac{\partial_\alpha x^+}{2z}, \quad j_\alpha^5 = \frac{z \partial_\alpha z - \eta \partial_\alpha x^+}{2z^2}.$$

Thus the classical action has been obtained as

$$S_{\text{AdS}} = -\frac{1}{2} \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \gamma^{\alpha\beta} \left[ \frac{1}{z^2} (-2\partial_\alpha x^+ \partial_\beta x^- + \partial_\alpha x^1 \partial_\beta x^1 + \partial_\alpha x^2 \partial_\beta x^2 + \partial_\alpha z \partial_\beta z) \right. \\ \left. - \frac{\eta^2}{z^4} \left(1 + \frac{(x^1)^2 + (x^2)^2}{z^2}\right) \partial_\alpha x^+ \partial_\beta x^+ \right] \\ + \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \epsilon^{\alpha\beta} \frac{\eta}{z^4} (x^1 \partial_\alpha x^+ \partial_\beta x^1 + x^2 \partial_\alpha x^+ \partial_\beta x^2 + z \partial_\alpha x^+ \partial_\beta z), \quad (2.25)$$

where the last term in the coupling to NS-NS two-form is a surface term and it can be dropped off. This action is real. From this action and the  $S^5$  part, one can read off the metric in the string frame and NS-NS two-form. In order to determine the string background completely, it is still necessary to fix the other field components by solving the field equations of motion in type IIB supergravity. This will be the issue in the next section.

So far, the  $r$ -matrix (2.13) has been considered. Note that the following four  $r$ -matrices

$$r_{\text{Jor}}^{(1)} = \frac{1}{\sqrt{2}} E_{13} \wedge (E_{11} - E_{33}), \quad r_{\text{Jor}}^{(2)} = \frac{1}{\sqrt{2}} E_{23} \wedge (E_{22} - E_{33}), \\ r_{\text{Jor}}^{(3)} = \frac{1}{\sqrt{2}} E_{14} \wedge (E_{11} - E_{44}), \\ r_{\text{Jor}}^{(4)} = \frac{1}{\sqrt{2}} [E_{14} \wedge (E_{11} - E_{44}) - 2(E_{12} \wedge E_{24} + E_{13} \wedge E_{34})]$$

lead to the same string action (2.25), up to double Wick rotations and coordinate transformations (For the detail, see appendix B). Note that  $r_{\text{Jor}}^{(2)}$  and  $r_{\text{Jor}}^{(4)}$  are obtained by performing adjoint operations with  $\Delta(E_{23})$  and  $\Delta(E_{14})$ , respectively, to the classical  $r$ -matrix of Drinfeld-Jimbo type satisfying mCYBE, as argued in [43]. Thus the present example may be regarded as Jordanian twists [36–38], though this fact is not manifest from the expression of the  $r$ -matrix (2.13).

### 2.3 Other examples

Before closing this section, let us present a generalized example<sup>4</sup> of skew-symmetric  $r$ -matrix satisfying CYBE,

$$r_{\text{Jor}} = \frac{i}{\sqrt{2}} [E_{24} \wedge (E_{22} - E_{44}) - 2E_{23} \wedge E_{34}]. \quad (2.26)$$

This  $r$ -matrix is also regarded as a Jordanian twist [36–38]. Note that the  $r$ -matrix (2.26) does not preserve the real-form condition of  $\mathfrak{su}(2, 2|4)$ . However, this is also an example which gives rise to a real string action, as we will see below.

<sup>4</sup>The imaginary unit  $i$  is multiplied so that the resulting NS-NS two-form should be real.



We will not show the derivation in detail. With the  $r$ -matrix (2.26), only the AdS<sub>5</sub> part is deformed again and the algorithm of the derivation is the same.

The resulting action for the deformed AdS<sub>5</sub> part is given by

$$\begin{aligned}
S_{\text{AdS}} = & -\frac{1}{2} \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \gamma^{\alpha\beta} \frac{z^2}{z^4 + 4\eta^2(x^+)^2} \\
& \times \left[ -2\partial_\alpha x^+ \partial_\beta x^- + \partial_\alpha x^1 \partial_\beta x^1 + \partial_\alpha x^2 \partial_\beta x^2 + \partial_\alpha z \partial_\beta z \right. \\
& + \frac{\eta^2}{z^4} (4x^+ \partial_\alpha x^+ (x^1 \partial_\beta x^1 + x^2 \partial_\beta x^2 - 2x^+ \partial_\beta x^-) + 4(x^+)^2 \partial_\alpha z \partial_\beta z \\
& \left. + (z^2 - (x^1)^2 - (x^2)^2) \partial_\alpha x^+ \partial_\beta x^+ \right] + 4 \frac{\eta^4}{z^6} (x^+)^2 \partial_\alpha x^+ \partial_\beta x^+ \\
& + \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \epsilon^{\alpha\beta} \frac{\eta}{z^4 + 4\eta^2(x^+)^2} \\
& \times \left[ x^2 \partial_\alpha x^+ \partial_\beta x^1 - x^1 \partial_\alpha x^+ \partial_\beta x^2 + 2x^+ \partial_\alpha x^1 \partial_\beta x^2 \right], \tag{2.27}
\end{aligned}$$

where surface terms have already been ignored. This action is real again. Note that the NS-NS two-form contains an imaginary part if the surface term cannot be dropped off, for example, by considering the open string case. In the present case, the metric depends on the light-cone time  $x^+$  explicitly. This metric may have an interesting feature as a dynamical brane background.

**A two-parameter deformation.** It is also interesting to see a two-parameter deformation of AdS<sub>5</sub>. It can be considered by taking the following  $r$ -matrix,

$$r_{\text{Jor}} = \frac{s_1}{\sqrt{2}} E_{24} \wedge (E_{22} - E_{44}) + \frac{s_2}{\sqrt{2}} E_{13} \wedge (E_{11} - E_{33}), \tag{2.28}$$

where  $s_1$  and  $s_2$  are constant parameters. The resulting string action is given by

$$\begin{aligned}
S_{\text{AdS}} = & -\frac{1}{2} \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \gamma^{\alpha\beta} \frac{z^2}{z^4 + 2\eta^2 s_1 s_2 [z^2 + (x^1)^2 + (x^2)^2]} \\
& \times \left( -2\partial_\alpha x^+ \partial_\beta x^- + \partial_\alpha x^1 \partial_\beta x^1 + \partial_\alpha x^2 \partial_\beta x^2 + \partial_\alpha z \partial_\beta z \right. \\
& - \frac{\eta^2}{z^2} \left[ \left( 1 + \frac{(x^1)^2 + (x^2)^2}{z^2} \right) (s_1 \partial_\alpha x^+ + s_2 \partial_\alpha x^-) (s_1 \partial_\beta x^+ + s_2 \partial_\beta x^-) \right. \\
& - 2s_1 s_2 \left\{ \left( 1 + \frac{(x^1)^2 + (x^2)^2}{z^2} \right) (\partial_\alpha x^1 \partial_\beta x^1 + \partial_\alpha x^2 \partial_\beta x^2 + \partial_\alpha z \partial_\beta z) \right. \\
& \left. \left. - \left( \partial_\alpha z + \frac{x^1}{z} \partial_\alpha x^1 + \frac{x^2}{z} \partial_\alpha x^2 \right) \left( \partial_\beta z + \frac{x^1}{z} \partial_\beta x^1 + \frac{x^2}{z} \partial_\beta x^2 \right) \right\} \right] \Big) \\
& + \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \epsilon^{\alpha\beta} \frac{\eta}{z^4 + 2s_1 s_2 \eta^2 [z^2 + (x^1)^2 + (x^2)^2]} \\
& \times \left[ x^1 (s_1 \partial_\alpha x^+ - s_2 \partial_\alpha x^-) \partial_\beta x^1 + x^2 (s_1 \partial_\alpha x^+ - s_2 \partial_\alpha x^-) \partial_\beta x^2 \right. \\
& \left. + z (s_1 \partial_\alpha x^+ - s_2 \partial_\alpha x^-) \partial_\beta z \right]. \tag{2.29}
\end{aligned}$$

The action is complicated but it is still real.

### 3 A solution in type IIB supergravity

In this section we will present a solution in type IIB supergravity containing the metric and NS-NS two-form obtained from (2.25).<sup>5</sup>

#### 3.1 The action of type IIB supergravity

Let us first introduce the equations of motion of type IIB supergravity [47]. Here we will follow the notation of [48]. The action of the bosonic part is given by

$$S_{\text{IIB}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} R - \frac{1}{4\kappa^2} \int \left( d\Phi \wedge *d\Phi + e^{2\Phi} dC \wedge *dC \right. \\ \left. + e^{-\Phi} H_3 \wedge *H_3 + e^\Phi \tilde{F}_3 \wedge *\tilde{F}_3 + \frac{1}{2} \tilde{F}_5 \wedge *\tilde{F}_5 + C_4 \wedge H_3 \wedge F_3 \right). \quad (3.1)$$

Here  $G_{MN}$  is the 10D metric in the Einstein frame and  $R$  is its Ricci scalar. The constant parameter  $\kappa$  is related to the 10D Newton constant  $G_{10}$  like  $2\kappa^2 \equiv 16\pi G_{10}$ . The symbol  $*$  denotes the 10D Hodge dual operator.  $\Phi$  is the fluctuation of the dilaton field and  $C$  is the axion field. Then  $B_2$ ,  $C_2$  and  $C_4$  are the NS-NS two-form, the R-R two-form and the R-R four-form. Their field strengths are defined as

$$H_3 \equiv dB_2, \quad F_3 \equiv dC_2, \quad F_5 \equiv dC_4 \quad (3.2)$$

The modified field strengths  $\tilde{F}_3$  and  $\tilde{F}_5$  are defined as

$$\tilde{F}_3 \equiv F_3 - CH_3, \quad \tilde{F}_5 \equiv F_5 - C_2 \wedge H_3. \quad (3.3)$$

Note that  $\tilde{F}_5$  has to satisfy the self-dual condition,

$$\tilde{F}_5 = *\tilde{F}_5. \quad (3.4)$$

By taking variations of the action (3.1), the equations of motion are obtained as

$$R_{MN} = \frac{1}{2} \partial_M \Phi \partial_N \Phi + \frac{e^{2\Phi}}{2} \partial_M C \partial_N C + \frac{1}{96} \tilde{F}_{MPQRS} \tilde{F}_N{}^{PQRS} \\ + \frac{1}{4} \left( e^{-\Phi} H_{MPQ} H_N{}^{PQ} + e^\Phi \tilde{F}_{MPQ} \tilde{F}_N{}^{PQ} \right) \\ - \frac{1}{48} G_{MN} \left( e^{-\Phi} H_{PQR} H^{PQR} + e^\Phi \tilde{F}_{PQR} \tilde{F}^{PQR} \right), \quad (3.5)$$

$$d*\tilde{F}_5 = -F_3 \wedge H_3, \quad (3.6)$$

$$\nabla^2 \Phi = e^{2\Phi} \partial^M C \partial_M C - \frac{e^{-\Phi}}{12} H_{MNP} H^{MNP} + \frac{e^\Phi}{12} \tilde{F}_{MNP} \tilde{F}^{MNP}, \quad (3.7)$$

$$\nabla^M (e^{2\Phi} \partial_M C) = -\frac{e^\Phi}{6} H_{MNP} \tilde{F}^{MNP}, \quad (3.8)$$

$$d*(e^{-\Phi} H_3 - e^\Phi C \tilde{F}_3) = -F_5 \wedge F_3, \quad (3.9)$$

$$d*(e^\Phi \tilde{F}_3) = F_5 \wedge H_3. \quad (3.10)$$

<sup>5</sup>Note that the solutions with (2.27) and (2.29) will not be discussed hereafter, because the metric is intricate and we have not succeeded to determine the other field components.

The Bianchi identities are given by

$$dH_3 = 0, \quad dF_3 = 0, \quad dF_5 = 0, \quad (3.11)$$

$$d\tilde{F}_3 = -dC \wedge H_3, \quad d\tilde{F}_5 = -F_3 \wedge H_3. \quad (3.12)$$

With this setup, we will consider a gravitational solution in the next subsection.

### 3.2 A Jordanian deformed solution

It is a turn to present a solution corresponding to a Jordanian deformation. From the construction of the string action, the metric is given by

$$ds^2 = L^2 \left[ \frac{-2dx^+ dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} - \frac{\eta^2}{z^4} \left( 1 + \frac{(x^1)^2 + (x^2)^2}{z^2} \right) (dx^+)^2 + ds_{S^5}^2 \right], \quad (3.13)$$

$$ds_{S^5}^2 = ds_{CP^2}^2 + (d\chi + \omega)^2.$$

Here the metric of round  $S^5$  is expressed as a  $U(1)$  fibration over  $CP^2$ , where  $\chi$  is the local coordinate on the Hopf fibre and  $\omega$  is the one-form potential for the Kähler form on  $CP^2$ . The metric of  $CP^2$  and  $\omega$  are given by<sup>6</sup>

$$ds_{CP^2}^2 = d\mu^2 + \sin^2 \mu (\Sigma_1^2 + \Sigma_2^2 + \cos^2 \mu \Sigma_3^2), \quad \omega = \sin^2 \mu \Sigma_3, \quad (3.14)$$

where  $\Sigma_a$  ( $a = 1, 2, 3$ ) are defined as

$$\Sigma_1 \equiv \frac{1}{2} (\cos \psi d\theta + \sin \psi \sin \theta d\phi), \quad \Sigma_2 \equiv \frac{1}{2} (\sin \psi d\theta - \cos \psi \sin \theta d\phi),$$

$$\Sigma_3 \equiv \frac{1}{2} (d\psi + \cos \theta d\phi).$$

Note that the metric contains the 3D Schrödinger spacetime<sup>7</sup> as a subspace with  $x^1 = x^2 = 0$ , while the deformed  $AdS_5$  part itself is not the 5D Schrödinger spacetime.<sup>8</sup>

Note that the metric which appear in the string action is represented in the string frame. However, the metric for the deformed  $AdS_5$  part is invariant under the following scaling

$$x^+ \rightarrow \lambda^2 x^+, \quad x^- \rightarrow x^-, \quad x^i \rightarrow \lambda x^i, \quad z \rightarrow \lambda z, \quad (3.15)$$

and one may expect that the dilaton should be constant (i.e.,  $\Phi = 0$ ). Thus the metric can be regarded as the one in the Einstein frame. For simplicity, we set  $C = 0$ .

By considering the  $S^5$  components of the equation of motion for the metric (3.5) and taking account of the self-duality condition (3.4), the five-form field-strength is fixed as

$$F_5 = 4L^4 \left[ -\frac{1}{z^5} dx^+ \wedge dx^- \wedge dx^1 \wedge dx^2 \wedge dz + \text{vol}(S^5) \right]. \quad (3.16)$$

Thus  $F_5$  is not modified under the deformation.

<sup>6</sup>We follow appendix A.2 of [49].

<sup>7</sup>Therefore, the result of [50] on the fast-moving limit [51] is directly applicable for this background. Note that the NS-NS two-form also vanishes at  $x^1 = x^2 = 0$ .

<sup>8</sup>One may consider whether the deformed  $AdS_5$  can be represented by a coset by following [52].

Then the NS-NS two-form  $B_2$  has also been derived as

$$B_2 = \frac{L^2\eta}{z^4} (x^1 dx^+ \wedge dx^1 + x^2 dx^+ \wedge dx^2), \quad (3.17)$$

and the associated field strength is given by

$$H_3 = -\frac{4L^2\eta}{z^5} (x^1 dx^+ \wedge dx^1 \wedge dz + x^2 dx^+ \wedge dx^2 \wedge dz). \quad (3.18)$$

From the equation of motion for  $H_3$ , (3.9), one can notice that  $F_3$  has to be turned on.

The remaining task is to find out  $F_3$  so as to satisfy all of the equations of motion. The resulting  $F_3$  is give by

$$F_3 = \frac{4L^2\eta}{z^5} \left[ x^2 dx^+ \wedge dx^1 \wedge dz - x^1 dx^+ \wedge dx^2 \wedge dz - \frac{z}{2} dx^+ \wedge dx^1 \wedge dx^2 \right] \\ - \frac{2L^2\eta}{z^3} \left[ dx^+ \wedge dz \wedge (d\chi + \omega) - \frac{z}{2} dx^+ \wedge d\omega \right], \quad (3.19)$$

where the associated R-R two-form  $C_2$  is given by

$$C_2 = -\frac{L^2\eta}{z^4} (x^2 dx^+ \wedge dx^1 - x^1 dx^+ \wedge dx^2) - \frac{L^2\eta}{z^2} dx^+ \wedge (d\chi + \omega). \quad (3.20)$$

In summary, the gravitational solution in the Einstein frame is given by

$$ds^2 = L^2 \left[ \frac{-2dx^+ dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} \right. \\ \left. - \frac{\eta^2}{z^4} \left( 1 + \frac{(x^1)^2 + (x^2)^2}{z^2} \right) (dx^+)^2 + ds_{\text{CP}^2}^2 + (d\chi + \omega)^2 \right], \quad (3.21)$$

$$F_5 = 4L^4 \left[ -\frac{1}{z^5} dx^+ \wedge dx^- \wedge dx^1 \wedge dx^2 \wedge dz + \text{vol}(S^5) \right],$$

$$H_3 = -\frac{4L^2\eta}{z^5} (x^1 dx^+ \wedge dx^1 \wedge dz + x^2 dx^+ \wedge dx^2 \wedge dz),$$

$$F_3 = \frac{4L^2\eta}{z^5} \left[ x^2 dx^+ \wedge dx^1 \wedge dz - x^1 dx^+ \wedge dx^2 \wedge dz - \frac{z}{2} dx^+ \wedge dx^1 \wedge dx^2 \right] \\ - \frac{2L^2\eta}{z^3} \left[ dx^+ \wedge dz \wedge (d\chi + \omega) - \frac{z}{2} dx^+ \wedge d\omega \right].$$

where  $\Phi = C = 0$ . The R-R scalar field  $C$  may take a non-vanishing constant  $C \neq 0$ . In this case, the R-R two-form  $C_2$  has to be shifted as  $C_2 \rightarrow C_2 + CB_2$ .

Here we should argue the consistency of the solution obtained here as a string background. It is instrumental to see that a similar solution has already been presented in appendix C of [53]. The deformed AdS<sub>5</sub> part of the solution in [53] is the same as that of our solution, while fluxes are different. An important point is that the deformed part is realized through a null Melvin twist [54], and our solution may also be obtained by performing null Melvin twists and S-duality for the original AdS<sub>5</sub> × S<sup>5</sup> (For the detail including other examples, see [55]). Thus the consistency would automatically be ensured by construction.

It is also worth noting that other consistent string backgrounds are obtained with our method with classical  $r$ -matrices. For example,  $\gamma$ -deformations of  $\text{AdS}_5 \times \text{S}^5$  [56, 57] are reproduced in [58] and the gravity duals for non-commutative gauge theories [59–61] are obtained in [62]. Probably, various kinds of consistent string backgrounds would be realized from classical  $r$ -matrices satisfying CYBE. The relation between gravity solutions and classical  $r$ -matrices should be called “the gravity/CYBE correspondence”, as proposed in [58].

Finally, it should be noted that the Green-Schwarz string action on this solution (at the quadratic order of fermions) is easily obtained by substituting the solution (3.21) into (3.27) of [63]. Recently, the quartic-order action has been derived in [64]. It would also be useful for further studies. As a matter of course, the total action is real, including the fermionic sector. It would be interesting to argue the world-sheet S-matrix by using the obtained action.

**The symmetry of the solution.** Let us check the symmetry of the solution (3.21).

We first concentrate on the symmetry of the deformed AdS part. It is obvious to see that the solution is invariant under two translations:

$$H : x^+ \rightarrow x^+ + a^+, \quad M : x^- \rightarrow x^- + a^-, \quad (3.22)$$

where  $a^\pm$  are constant parameters. The invariance under the rotation in the 1-2 plane is also manifest. Recall that the solution is invariant under the anisotropic scaling,

$$D : x^+ \rightarrow \lambda^2 x^+, \quad x^- \rightarrow x^-, \quad x^i \rightarrow \lambda x^i, \quad z \rightarrow \lambda z \quad (\lambda : \text{a constant}). \quad (3.23)$$

A less obvious one is the special conformal transformation,<sup>9</sup>

$$\begin{aligned} C : x^+ &\rightarrow (1 - ax^+)x^+, & x^- &\rightarrow x^- - \frac{a}{2}(x^i x^i + z^2), \\ x^i &\rightarrow (1 - ax^+)x^i, & z &\rightarrow (1 - ax^+)z, \end{aligned} \quad (3.24)$$

where  $a$  is an infinitesimal parameter. Note that the solution (3.21) is not invariant under spatial translations and Galilean boosts due to the deformation. The symmetries  $H$ ,  $D$  and  $C$  generate  $\text{SL}(2, \mathbb{R})$ . Then  $M$  and the rotation in the 1-2 plane generates two  $\text{U}(1)$ 's.

For the sphere part, the  $\text{SO}(6)$  symmetry is broken to  $\text{SU}(3) \times \text{U}(1)$  due to the presence of the R-R three-form field strength, where  $\text{SU}(3)$  is the isometry of  $\mathbb{CP}^2$  and  $\text{U}(1)$  corresponds to a shift symmetry of  $\chi$ .

In total, the resulting symmetry is given by

$$[\text{SL}(2, \mathbb{R}) \times \text{U}(1)^2] \times [\text{SU}(3) \times \text{U}(1)]. \quad (3.25)$$

It seems likely that the solution (3.21) is not supersymmetric because the  $F_3$  flux is the same type of the one considered in [66], where the  $H_3$  flux is considered but the mechanism to break supersymmetries would be identical. It might be interesting to consider a brane-wave deformation, instead of the  $F_3$  flux, as in [67]. Some of the original supersymmetries may be preserved, while the integrability would become unclear.

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<sup>9</sup>For the derivation of this transformation law, for example, see [65].

In comparison to the Jordanian deformed solution (3.21), it seems quite difficult to find out the full gravitational solution corresponding to the standard deformation in type IIB supergravity. The metric in the string frame is obtained in [35], but it involves a curvature singularity and the dilaton would be very complicated.

### 3.3 The tidal force

It is also important to check whether the solution (3.21) involves a singularity or not. The solution is just regarded as a pp-wave like deformation of the  $\text{AdS}_5 \times \text{S}^5$  background. Hence no obvious curvature singularity is not found by computing curvature invariants. However, there may be another kind of singularity called pp-singularity [68]. In order to discuss this singularity, it is necessary to check the tidal force.

First of all, one needs to take a time-like world-line and its tangent vector is

$$t^m = \dot{x}^+ \left( \frac{\partial}{\partial x^+} \right)^m + \dot{x}^- \left( \frac{\partial}{\partial x^-} \right)^m + \dot{x}^1 \left( \frac{\partial}{\partial x^1} \right)^m + \dot{x}^2 \left( \frac{\partial}{\partial x^2} \right)^m + \dot{z} \left( \frac{\partial}{\partial z} \right)^m,$$

where the index  $m$  runs only for the deformed  $\text{AdS}_5$  part and “dot” denotes the derivative with respect to the affine parameter  $\lambda$ . Assume that the affine parameter is chosen so that the tangent vector becomes a unit vector:

$$G_{mn} t^m t^n = -1. \tag{3.26}$$

The dynamics of a particle moving on the solution (3.21) is described by the action

$$S = \frac{1}{2} \int d\lambda \frac{1}{z^2} \left[ -2\dot{x}^+ \dot{x}^- + (\dot{x}^1)^2 + (\dot{x}^2)^2 + \dot{z}^2 - \frac{\eta^2}{z^2} \left( 1 + \frac{(x^1)^2 + (x^2)^2}{z^2} \right) (\dot{x}^+)^2 \right].$$

The equations of motion for  $x^\pm$  provide two constants of motion,  $P_-$  and  $E$ ,

$$P_- = -\frac{\dot{x}^+}{z^2}, \quad E = \frac{1}{z^2} \left[ -\dot{x}^- - \frac{\eta^2}{z^2} \left( 1 + \frac{(x^1)^2 + (x^2)^2}{z^2} \right) \dot{x}^+ \right]. \tag{3.27}$$

Solving  $P_-$  and  $E$  with respect to  $\dot{x}^\pm$  leads to the following expressions,

$$\dot{x}^+ = -z^2 P_-, \quad \dot{x}^- = -z^2 E + \eta^2 \left( 1 + \frac{(x^1)^2 + (x^2)^2}{z^2} \right) P_-. \tag{3.28}$$

The equations of motion for  $x^1$  and  $x^2$  are given by

$$\frac{d}{d\lambda} \left( \frac{\dot{x}^1}{z^2} \right) = -\frac{\eta^2 x^1}{z^6} (\dot{x}^+)^2, \quad \frac{d}{d\lambda} \left( \frac{\dot{x}^2}{z^2} \right) = -\frac{\eta^2 x^2}{z^6} (\dot{x}^+)^2. \tag{3.29}$$

Noting that the normalization condition (3.26) is explicitly written as

$$\frac{1}{z^2} \left[ -2\dot{x}^+ \dot{x}^- + (\dot{x}^1)^2 + (\dot{x}^2)^2 + \dot{z}^2 - \frac{\eta^2}{z^2} \left( 1 + \frac{(x^1)^2 + (x^2)^2}{z^2} \right) (\dot{x}^+)^2 \right] = -1, \tag{3.30}$$

one can solve (3.26) for  $\dot{z}$  and obtain the following expression,

$$\dot{z} = \sqrt{-z^2 + 2EP_- z^4 - \eta^2 (z^2 + (x^1)^2 + (x^2)^2) P_-^2 - (\dot{x}^1)^2 - (\dot{x}^2)^2}. \tag{3.31}$$

Here we have used the expressions of  $\dot{x}^\pm$  given in (3.28).

To evaluate the tidal force, it is not necessary to solve equations of motion explicitly. The tidal force is represented by the components of Riemann tensor in an orthonormal frame which is parallelly transported along the world-line. Thus one just needs to identify a basis  $e^m$  for the orthonormal frame

$$\frac{d}{d\lambda} e^m = \Gamma_{np}^m t^n e^p. \quad (3.32)$$

The orthonormal system is given by

$$\begin{aligned} n_1^m &= -\frac{\dot{x}^1}{P_- z} \left( \frac{\partial}{\partial x^-} \right)^m + z \left( \frac{\partial}{\partial x^1} \right)^m, \\ n_2^m &= -\frac{\dot{x}^2}{P_- z} \left( \frac{\partial}{\partial x^-} \right)^m + z \left( \frac{\partial}{\partial x^2} \right)^m, \\ p^m &= -\sin \lambda \left[ \dot{x}^+ \left( \frac{\partial}{\partial x^+} \right)^m + \left( \dot{x}^- + \frac{1}{P_-} \right) \left( \frac{\partial}{\partial x^-} \right)^m + \dot{x}^1 \left( \frac{\partial}{\partial x^1} \right)^m + \dot{x}^2 \left( \frac{\partial}{\partial x^2} \right)^m \right. \\ &\quad \left. + \dot{z} \left( \frac{\partial}{\partial z} \right)^m \right] + \cos \lambda \left[ \frac{\dot{x}^-}{P_- z} \left( \frac{\partial}{\partial x^-} \right)^m - z \left( \frac{\partial}{\partial z} \right)^m \right], \\ q^m &= \cos \lambda \left[ \dot{x}^+ \left( \frac{\partial}{\partial x^+} \right)^m + \left( \dot{x}^- + \frac{1}{P_-} \right) \left( \frac{\partial}{\partial x^-} \right)^m + \dot{x}^1 \left( \frac{\partial}{\partial x^1} \right)^m + \dot{x}^2 \left( \frac{\partial}{\partial x^2} \right)^m \right. \\ &\quad \left. + \dot{z} \left( \frac{\partial}{\partial z} \right)^m \right] + \sin \lambda \left[ \frac{\dot{x}^-}{P_- z} \left( \frac{\partial}{\partial x^-} \right)^m - z \left( \frac{\partial}{\partial z} \right)^m \right]. \end{aligned}$$

Then the tidal force is defined as

$$R_{(t)(e_1)(t)(e_2)} \equiv R_{pqr}^m G_{mn} t^n e_1^p t^q e_2^r, \quad (3.33)$$

and the components of the tidal force are listed below :

$$\begin{aligned} R_{(t)(1)(t)(1)} &= 1 + 2\eta^2 \left( 1 + \frac{(x^1)^2 + (x^2)^2}{z^2} \right) P_-^2, & R_{(t)(1)(t)(2)} &= 0, \\ R_{(t)(2)(t)(2)} &= 1 + 2\eta^2 \left( 1 + \frac{(x^1)^2 + (x^2)^2}{z^2} \right) P_-^2, \\ R_{(t)(1)(t)(p)} &= 4\eta^2 \frac{x^1}{z} P_-^2 \cos \lambda, & R_{(t)(1)(t)(q)} &= 4\eta^2 \frac{x^1}{z} P_-^2 \sin \lambda, \\ R_{(t)(2)(t)(p)} &= 4\eta^2 \frac{x^2}{z} P_-^2 \cos \lambda, & R_{(t)(2)(t)(q)} &= 4\eta^2 \frac{x^2}{z} P_-^2 \sin \lambda, \\ R_{(t)(p)(t)(p)} &= 1 + 4\eta^2 \left( 1 + 3 \frac{(x^1)^2 + (x^2)^2}{z^2} \right) P_-^2 \cos^2 \lambda, \\ R_{(t)(p)(t)(q)} &= 4\eta^2 \left( 1 + 3 \frac{(x^1)^2 + (x^2)^2}{z^2} \right) P_-^2 \sin \lambda \cos \lambda, \\ R_{(t)(q)(t)(q)} &= 1 + 4\eta^2 \left( 1 + 3 \frac{(x^1)^2 + (x^2)^2}{z^2} \right) P_-^2 \sin^2 \lambda. \end{aligned}$$

From the tidal force, one can see that the solution (3.21) is regular at the horizon,  $z = \infty$ , while it hits on a singular at the boundary,  $z = 0$ , except at  $x^1 = x^2 = 0$ .

It is worth noting the similarity to the 5D Schrödinger spacetime with the dynamical critical exponent  $z_c$ . When  $z_c = 2$ , there is no divergence of the tidal force at the horizon and the boundary [69]. But, when  $z_c = 3$ , the tidal force diverges at the boundary [69]. The solution (3.21) exhibits an isotropic scaling with  $z_c = 2$ , while its asymptotic behavior around the boundary is close to the one with  $z_c = 3$ . The divergence of the tidal force at the boundary in the solution (3.21) is similar to the one of the Schrödinger spacetime with  $z_c = 3$ .

## 4 Conclusion and discussion

We have considered a Jordanian deformation of the  $\text{AdS}_5 \times \text{S}^5$  superstring action with a simple  $R$  operator satisfying CYBE. The metric and NS-NS two-form have explicitly been derived with a coordinate system. Only the  $\text{AdS}_5$  part is deformed and the resulting geometry contains the 3D Schrödinger spacetime as a subspace. Then we have presented a solution in type IIB supergravity by determining the other field components. In particular, the dilaton is constant and a R-R three-form field strength is turned on. The symmetry of the solution is given by  $[\text{SL}(2, \mathbb{R}) \times \text{U}(1)^2] \times [\text{SU}(3) \times \text{U}(1)]$  and contains an anisotropic scale symmetry. Though the curvature invariants are not singular, the tidal force diverges at the boundary, except a certain point.

There are many open problems now. The first is to consider a relation to deformed S-matrices on the string world-sheet. The standard  $q$ -deformations of the S-matrices are studied in [70–77], but Jordanian deformed S-matrices have not been argued yet. It would be interesting to study them and compare the results with the string world-sheet S-matrices as in [35]. It is significant to investigate the deformation of  $\mathcal{N}=4$  SYM corresponding to the gravitational solution presented here. It is argued to be a light-like non-commutative gauge theory [53]. It would be nice to consider an interpretation of the classical  $r$ -matrix in the context of the dual gauge theory.

Although we have considered integrable deformations of the  $\text{AdS}_5$  part, it is possible to consider those of  $\text{S}^5$ , similarly. In fact, real  $\gamma$ -deformations of  $\text{AdS}_5 \times \text{S}^5$  [56, 57] have been reproduced in [58]. Various kinds of consistent string backgrounds would be reproduced from classical  $r$ -matrices satisfying CYBE.

The solution presented here is just an example (For other examples of deformed  $\text{AdS}_5$ , which have been found in the later progress, see [62]). We expect that many interesting gravitational solutions would be found through Jordanian deformations. The recipe to look for them is given in [43] and this paper. We hope that many integrable solutions are discovered.

## Acknowledgments

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## A Our notation and convention

Our notation and convention is summarized here by basically following [78].

An element of Lie superalgebra  $\mathfrak{su}(2, 2|4)$  is represented by an  $8 \times 8$  supermatrix:

$$M = \begin{bmatrix} m & \xi \\ \zeta & n \end{bmatrix}. \quad (\text{A.1})$$

Here  $m$  and  $n$  are  $4 \times 4$  matrices with Grassmann even elements, while  $\xi$  and  $\zeta$  are  $4 \times 4$  matrices with Grassmann odd elements. These matrices satisfy an appropriate reality condition. As a result, it turns out that  $m$  and  $n$  belong to  $\mathfrak{su}(2, 2) = \mathfrak{so}(2, 4)$  and  $\mathfrak{su}(4) = \mathfrak{so}(6)$ , respectively.

For our purpose, it is helpful to prepare an explicit basis of  $\mathfrak{su}(4)$  and  $\mathfrak{su}(2, 2)$ . Let us first introduce the following  $\gamma$  matrices:

$$\begin{aligned} \gamma_1 &= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, & \gamma_2 &= \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, & \gamma_3 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\ \gamma_4 &= \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, & \gamma_5 &= -\gamma_1\gamma_2\gamma_3\gamma_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \end{aligned} \quad (\text{A.2})$$

Then  $n_{ij}$  ( $i, j = 1, 2, 3, 4, 5$ ) are given by

$$n_{ij} = \frac{1}{4} [\gamma_i, \gamma_j]. \quad (\text{A.3})$$

It is easy to see that  $\gamma_i$ 's generate the Clifford algebra of  $\mathfrak{so}(5)$ :

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}. \quad (\text{A.4})$$

Thus  $n_{ij}$ 's generate the Lie algebra  $\mathfrak{so}(5)$ . Note that

$$n_{ij}, \quad n_{i6} = \frac{i}{2}\gamma_i$$

are regarded as the generators of  $\mathfrak{so}(6)$ .

On the other hand,  $\gamma_1, \gamma_2, \gamma_3, \gamma_0 = i\gamma_4$  and  $\gamma_5$

$$\gamma_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \quad \gamma_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\gamma_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (\text{A.5})$$

generate the Clifford algebra of  $\mathfrak{so}(1, 4)$ :

$$\begin{aligned} \{\gamma_\mu, \gamma_\nu\} &= 2\eta_{\mu\nu} \quad (\mu, \nu = 0, 1, 2, 3), \\ \{\gamma_\mu, \gamma_5\} &= 0, \quad (\gamma_5)^2 = 1. \end{aligned} \quad (\text{A.6})$$

Then the generators

$$n_{\mu\nu} = \frac{1}{4} [\gamma_\mu, \gamma_\nu], \quad n_{\mu 5} = \frac{1}{4} [\gamma_\mu, \gamma_5] \quad (\text{A.7})$$

satisfy the defining relations of  $\mathfrak{so}(1, 4)$ . In addition,

$$n_{\mu\nu}, \quad n_{\mu 5}, \quad \gamma_\mu, \quad \gamma_5$$

are regarded as the spinor representation of  $\mathfrak{so}(2, 4)$ .

## B A list of $r$ -matrices and deformed string actions

This appendix gives a list of some possible  $r$ -matrices and the associated string actions.

The AdS part of the Jordanian deformed action can be rewritten as

$$\begin{aligned} S_{\text{AdS}} &= -\frac{1}{2} \int d\sigma^2 (\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \text{Tr} \left( A_\alpha P_2 \circ \frac{1}{1 - 2\eta[R_{\text{Jor}}]_g \circ P_2} A_\beta \right) \\ &= -\frac{1}{2} \int d\sigma^2 (\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \text{Tr} (A_\alpha P_2 (J_\beta)) \\ &= -\frac{1}{2} \int d\sigma^2 (-\text{Tr}(A_t P_2 (J_t)) + \text{Tr}(A_x P_2 (J_x))) \\ &\quad + \frac{1}{2} \int d\sigma^2 (\text{Tr}(A_t P_2 (J_x)) - \text{Tr}(A_x P_2 (J_t))) \\ &= \int d\sigma^2 (L_G + L_B), \end{aligned} \quad (\text{B.1})$$

where the sigma model part  $L_G$  and the coupling to NS-NS two-form  $L_B$  are given by

$$\begin{aligned} L_G &\equiv \frac{1}{2} [\text{Tr}(A_t P_2 (J_t)) - \text{Tr}(A_x P_2 (J_x))], \\ L_B &\equiv \frac{1}{2} [\text{Tr}(A_t P_2 (J_x)) - \text{Tr}(A_x P_2 (J_t))]. \end{aligned} \quad (\text{B.2})$$

The undeformed AdS<sub>5</sub> part is represented by

$$L_G^{\eta=0} = -\frac{\gamma^{\alpha\beta}}{2z^2} (-2\partial_\alpha x^+ \partial_\beta x^- + \partial_\alpha x^1 \partial_\beta x^1 + \partial_\alpha x^2 \partial_\beta x^2 + \partial_\alpha z \partial_\beta z). \quad (\text{B.3})$$

This part is common for all of the deformation, and  $L_B$  always vanishes in the  $\eta \rightarrow 0$  limit.

It would be interesting to classify possible  $r$ -matrices and the associated string actions, though the classification here is focussed upon some simple examples and not complete. Remarkably, all of the string actions contained in the list are real, up to surface terms appearing in  $L_B$ , after performing appropriate Wick rotations.

The deformed string actions are classified into the three classes:

1. Class A =  $\{(0), (1), (2), (3), (4)\}$ ,
2. Class B =  $\{(5), (6)\}$ ,
3. Class C =  $\{(7), (8)\}$ .

Each of the classes has the identical action, up to double Wick rotations and coordinate transformations. The class A corresponds to the case of (2.13) discussed in the body. The class B is the one discussed in subsection 2.3. The class C seems unphysical because two time directions appear after performing double Wick rotations to make the actions real.

The three classes are listed below.

**Class A.**

$$(0) \quad r_{\text{Jor}}^{(0)} = \frac{1}{\sqrt{2}} E_{24} \wedge (E_{22} - E_{44})$$

The deformed Lagrangian:

$$\begin{aligned} L_G &= L_G^{\eta=0} + \eta^2 \gamma^{\alpha\beta} \frac{(x^1)^2 + (x^2)^2 + z^2}{2z^6} \partial_\alpha x^+ \partial_\beta x^+, \\ L_B &= \epsilon^{\alpha\beta} \frac{\eta}{z^4} \partial_\alpha x^+ (x^1 \partial_\beta x^1 + x^2 \partial_\beta x^2 + z \partial_\beta z). \end{aligned} \tag{B.4}$$

This is the case with (2.13) considered in the body. The last term in  $L_B$  is a surface term. It can be ignored without boundaries. The Lagrangian (B.4) is invariant under  $\text{SL}(2, \mathbb{R}) \times \text{U}(1)^2$ , which contains the anisotropic scaling invariance under

$$x^+ \rightarrow \lambda^2 x^+, \quad x^- \rightarrow x^-, \quad x^i \rightarrow \lambda x^i, \quad z \rightarrow \lambda z, \tag{B.5}$$

where  $\lambda$  is a constant. For the detail, see subsection 3.2.

$$(1) \quad r_{\text{Jor}}^{(1)} = \frac{1}{\sqrt{2}} E_{13} \wedge (E_{11} - E_{33})$$

The deformed Lagrangian:

$$\begin{aligned} L_G &= L_G^{\eta=0} + \eta^2 \gamma^{\alpha\beta} \frac{(x^1)^2 + (x^2)^2 + z^2}{2z^6} \partial_\alpha x^- \partial_\beta x^-, \\ L_B &= -\epsilon^{\alpha\beta} \frac{\eta}{z^4} \partial_\alpha x^- (x^1 \partial_\beta x^1 + x^2 \partial_\beta x^2 + z \partial_\beta z). \end{aligned} \tag{B.6}$$

This can be obtained from the case (0) by exchanging  $x^\pm \rightarrow x^\mp$  and flipping  $\eta \rightarrow -\eta$ . Thus this case is equivalent to the case (0).

$$(2) \quad r_{\text{Jor}}^{(2)} = \frac{1}{\sqrt{2}} E_{23} \wedge (E_{22} - E_{33})$$

The deformed Lagrangian:

$$\begin{aligned} L_G &= L_G^{\eta=0} + \eta^2 \gamma^{\alpha\beta} \frac{-2x^+x^- + z^2}{2z^6} \partial_\alpha \left( \frac{x^1 - ix^2}{\sqrt{2}} \right) \partial_\beta \left( \frac{x^1 - ix^2}{\sqrt{2}} \right), \\ L_B &= -\epsilon^{\alpha\beta} \frac{\eta}{z^4} \partial_\alpha \left( \frac{x^1 - ix^2}{\sqrt{2}} \right) (x^+ \partial_\beta x^- + x^- \partial_\beta x^+ - z \partial_\beta z). \end{aligned} \quad (\text{B.7})$$

Note that  $x^+ \partial_\beta x^- + x^- \partial_\beta x^+ = x^0 \partial_\beta x^0 - x^3 \partial_\beta x^3$ . After performing the double Wick rotation  $x^2 \rightarrow ix^2$  and  $x^0 \rightarrow ix^0$  and redefining the light-cone coordinates like  $\tilde{x}^\pm = (x^2 \pm x^1)/\sqrt{2}$ , this case is identical to the case (0), up to the total derivative.

$$(3) \quad r_{\text{Jor}}^{(4)} = \frac{1}{\sqrt{2}} E_{14} \wedge (E_{11} - E_{44})$$

The deformed Lagrangian:

$$\begin{aligned} L_G &= L_G^{\eta=0} + \eta^2 \gamma^{\alpha\beta} \frac{-2x^+x^- + z^2}{2z^6} \partial_\alpha \left( \frac{x^1 + ix^2}{\sqrt{2}} \right) \partial_\beta \left( \frac{x^1 + ix^2}{\sqrt{2}} \right), \\ L_B &= \epsilon^{\alpha\beta} \frac{\eta}{z^4} \partial_\alpha \left( \frac{x^1 + ix^2}{\sqrt{2}} \right) (x^+ \partial_\alpha x^- + x^- \partial_\alpha x^+ - z \partial_\alpha z). \end{aligned} \quad (\text{B.8})$$

After flipping  $x^2 \rightarrow -x^2$  and  $\eta \rightarrow -\eta$ , this case is equivalent to the case (2). Thus the Lagrangian (B.8) is also equivalent to the case (0), up to the total derivative.

$$(4) \quad r_{\text{Jor}}^{(3)} = \frac{1}{\sqrt{2}} \left[ E_{14} \wedge (E_{11} - E_{44}) - 2(E_{12} \wedge E_{24} + E_{13} \wedge E_{34}) \right]$$

The deformed Lagrangian:

$$\begin{aligned} L_G &= L_G^{\eta=0} + \eta^2 \gamma^{\alpha\beta} \frac{-2x^+x^- + z^2}{2z^6} \partial_\alpha \left( \frac{x^1 + ix^2}{\sqrt{2}} \right) \partial_\beta \left( \frac{x^1 + ix^2}{\sqrt{2}} \right), \\ L_B &= -\epsilon^{\alpha\beta} \frac{\eta}{z^4} \partial_\alpha \left( \frac{x^1 + ix^2}{\sqrt{2}} \right) (x^+ \partial_\beta x^- + x^- \partial_\beta x^+ + z \partial_\beta z). \end{aligned} \quad (\text{B.9})$$

After flipping  $x^2 \rightarrow -x^2$ , this case is equivalent to the case (2), up to the total derivative. Thus the Lagrangian (B.9) is also equivalent to the case (0).

Note that the actions in the class A are identical, up to the total derivative. If boundaries are taken into account, the class A should be divided into subclasses. But we are interested in closed strings here and will not argue such subclasses.

### Class B.

$$(5) \quad r_{\text{Jor}}^{(5)} = \frac{i}{\sqrt{2}} \left[ E_{24} \wedge (E_{22} - E_{44}) - 2E_{23} \wedge E_{34} \right]$$

The deformed Lagrangian:

$$L_G = \frac{z^4}{z^4 + 4\eta^2(x^+)^2} \left[ L_G^{\eta=0} - \frac{\eta^2}{2z^6} \gamma^{\alpha\beta} (4x^+ \partial_\alpha x^+ [x^1 \partial_\beta x^1 + x^2 \partial_\beta x^2 - 2x^+ \partial_\beta x^-]) \right]$$

$$\begin{aligned}
 & +4(x^+)^2\partial_\alpha z\partial_\beta z + [z^2 - (x^1)^2 - (x^2)^2] \partial_\alpha x^+ \partial_\beta x^+ \\
 & -2\frac{\eta^4}{z^8}(x^+)^2\gamma^{\alpha\beta}\partial_\alpha x^+ \partial_\beta x^+ \Big], \\
 L_B = & \frac{\eta}{z^4 + 4\eta^2(x^+)^2}\epsilon^{\alpha\beta} \left[ x^2\partial_\alpha x^+ \partial_\beta x^1 - x^1\partial_\alpha x^+ \partial_\beta x^2 + 2x^+ \partial_\alpha x^1 \partial_\beta x^2 \right] \\
 & + i\frac{\eta}{z^3}\epsilon^{\alpha\beta}\partial_\alpha x^+ \partial_\beta z. \tag{B.10}
 \end{aligned}$$

The last term in  $L_B$  is imaginary but just a surface term. Thus the Lagrangian (B.10) is real without boundaries. Note that the Lagrangian (B.10) is invariant under the anisotropic scaling (B.5), the rotation in the 1-2 plane and the shift of  $x^-$ , i.e.,  $U(1)^3$ .

$$(6) \quad r_{\text{Jor}}^{(6)} = \frac{i}{\sqrt{2}} \left[ E_{13} \wedge (E_{11} - E_{33}) - 2E_{12} \wedge E_{23} \right]$$

The deformed Lagrangian:

$$\begin{aligned}
 L_G = & \frac{z^4}{z^4 + 4\eta^2(x^-)^2} \left[ L_G^{\eta=0} - \frac{\eta^2}{2z^6}\gamma^{\alpha\beta} \left( 4x^- \partial_\alpha x^- [x^1 \partial_\beta x^1 + x^2 \partial_\beta x^2 - 2x^+ \partial_\beta x^-] \right. \right. \\
 & \left. \left. + 4(x^-)^2 \partial_\alpha z \partial_\beta z + [z^2 - (x^1)^2 - (x^2)^2] \partial_\alpha x^- \partial_\beta x^- \right) \right. \\
 & \left. - 2\frac{\eta^4}{z^8}(x^-)^2\gamma^{\alpha\beta}\partial_\alpha x^- \partial_\beta x^- \right], \\
 L_B = & \frac{-\eta}{z^4 + 4\eta^2(x^-)^2}\epsilon^{\alpha\beta} \left[ x^2\partial_\alpha x^- \partial_\beta x^1 - x^1\partial_\alpha x^- \partial_\beta x^2 + 2x^- \partial_\alpha x^1 \partial_\beta x^2 \right] \\
 & - i\frac{\eta}{z^3}\epsilon^{\alpha\beta}\partial_\alpha x^- \partial_\beta z. \tag{B.11}
 \end{aligned}$$

Through exchanging  $x^\pm \rightarrow x^\mp$  and flipping  $\eta \rightarrow -\eta$ , this is equivalent to the case (5).

The class B corresponds to the case discussed in subsection 2.3.

### Class C.

$$(7) \quad r_{\text{Jor}}^{(7)} = \frac{1}{\sqrt{2}} E_{34} \wedge (E_{33} - E_{44})$$

The deformed Lagrangian:

$$\begin{aligned}
 L_G = & L_G^{\eta=0} - \eta^2\gamma^{\alpha\beta} \frac{(x^1)^2 + (x^2)^2 - 2x^+x^-}{2z^6} \left[ x^+ \partial_\alpha (x^1 + ix^2) - (x^1 + ix^2) \partial_\alpha x^+ \right] \\
 & \times \left[ x^+ \partial_\beta (x^1 + ix^2) - (x^1 + ix^2) \partial_\beta x^+ \right], \\
 L_B = & \epsilon^{\alpha\beta} \frac{\eta}{z^4} \left[ (x^1 + ix^2)x^+ (\partial_\alpha x^- \partial_\beta x^+ + i\partial_\alpha x^1 \partial_\beta x^2) \right. \\
 & + (x^1 + ix^2)\partial_\alpha x^+ (ix^2\partial_\beta x^1 - ix^1\partial_\beta x^2) \\
 & \left. - x^+ (x^+ \partial_\alpha x^- - x^- \partial_\alpha x^+) \partial_\beta (x^1 + ix^2) \right]. \tag{B.12}
 \end{aligned}$$

By performing a Wick rotation  $x^2 \rightarrow -ix^2$ , the Lagrangian (B.12) becomes real but contain two time directions. Thus it seems to be unphysical. Note that, in comparison

to the other cases, the Lagrangian (B.12) is invariant under the isotropic scaling

$$x^+ \rightarrow \lambda x^+, \quad x^- \rightarrow \lambda x^-, \quad x^i \rightarrow \lambda x^i, \quad z \rightarrow \lambda z,$$

where  $\lambda$  is a constant. After the Wick rotation, the Lagrangian (B.12) is invariant also under the transformation,

$$x^+ \rightarrow \lambda' x^+, \quad x^- \rightarrow \lambda'^{-1} x^-, \quad \tilde{x}^+ \rightarrow \lambda'^{-1} \tilde{x}^+, \quad \tilde{x}^- \rightarrow \lambda' \tilde{x}^-, \quad z \rightarrow z,$$

where  $\tilde{x}^\pm = (x^2 \pm x^1)/\sqrt{2}$  and  $\lambda'$  is a constant. This can be understood as the diagonal part of the two Lorentz boosts. In addition, it has the invariance under a “rotation” in the  $(x^+, \tilde{x}^+)$  and  $(x^-, \tilde{x}^-)$  planes,

$$x^\pm \rightarrow \cos \theta x^\pm - \sin \theta \tilde{x}^\pm, \quad \tilde{x}^\pm \rightarrow \sin \theta x^\pm + \cos \theta \tilde{x}^\pm.$$

Thus the resulting symmetry is  $U(1)^3$ .

$$(8) \quad r_{\text{Jor}}^{(8)} = \frac{1}{\sqrt{2}} E_{12} \wedge (E_{11} - E_{22})$$

The deformed Lagrangian:

$$\begin{aligned} L_G &= L_G^{\eta=0} - \eta^2 \gamma^{\alpha\beta} \frac{(x^1)^2 + (x^2)^2 - 2x^+ x^-}{2z^6} [x^- \partial_\alpha (x^1 + ix^2) - (x^1 + ix^2) \partial_\alpha x^-] \\ &\quad \times [x^- \partial_\beta (x^1 + ix^2) - (x^1 + ix^2) \partial_\beta x^-], \\ L_B &= \epsilon^{\alpha\beta} \frac{\eta}{z^4} \left[ (x^1 + ix^2) x^- (\partial_\alpha x^+ \partial_\beta x^- + i \partial_\alpha x^1 \partial_\beta x^2) \right. \\ &\quad \left. + (x^1 + ix^2) \partial_\alpha x^- (ix^2 \partial_\beta x^1 - ix^1 \partial_\beta x^2) \right. \\ &\quad \left. - x^- (x^- \partial_\alpha x^+ - x^+ \partial_\alpha x^-) \partial_\beta (x^1 + ix^2) \right]. \end{aligned} \quad (B.13)$$

By exchanging  $x^\pm \rightarrow x^\mp$ , this case is equivalent to the case (7).

The class C seems to be unphysical because of two time directions. It would be interesting to figure out a general criterion for the physical metric so as to exclude the class C.

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