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The large $\mathcal{N} = 4$ superconformal $\mathcal{W}_\infty$ algebra

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Abstract: The most general large $\mathcal{N} = 4$ superconformal $\mathcal{W}_\infty$ algebra, containing in addition to the superconformal algebra one supermultiplet for each integer spin, is analysed in detail. It is found that the $\mathcal{W}_\infty$ algebra is uniquely determined by the levels of the two $\mathfrak{su}(2)$ algebras, a conclusion that holds both for the linear and the non-linear case. We also perform various cross-checks of our analysis, and exhibit two different types of truncations in some detail.

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1 Introduction

The duality between higher spin theories on AdS_3 [1, 2] and large N limits of 2d CFTs, see [3] for a review, can be understood and tested in quite some detail. This applies, in particular, to the bosonic example of [4], thus suggesting that supersymmetry is not a crucial

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ingredient for these types of dualities. On the other hand, it is believed that the vector-like higher spin/CFT dualities arise from a full stringy AdS/CFT correspondence upon taking the tensionless limit and concentrating on the states belonging to the leading Regge trajectory [5–7]. In this context the supersymmetric versions of the dualities naturally arise, and thus the supersymmetric examples deserve special attention. There have been some attempts to understand in detail the way in which the higher spin/CFT dualities fit into string theory, see e.g. [8] for a review as well as the proposal in [9]; however, it is fair to say that there are still many open questions. The 3d/2d case seems to be a very promising arena to explore these issues in more detail since both sides of the duality are under very good quantitive control.

With this vision in mind, the analysis of the $\mathcal{N} = 4$ supersymmetric version of the higher spin/CFT duality was initiated in [10]. It relates the higher spin theory based on the Lie algebra $\mathfrak{shs}_2[\lambda]$ to the Wolf space cosets

$$\frac{\mathfrak{su}^{(1)}(N+2)_{k+N+2}}{\mathfrak{su}(N)^{(1)}_{N+k+2} \oplus \mathfrak{u}(1)} \qquad \text{with} \qquad \lambda = \frac{N}{N+k+2} \; . \tag{1.1}$$

These theories have 'large' $\mathcal{N} = 4$ superconformal symmetry, which is the expected superconformal symmetry of the dual to string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$. In a sense this case is more restrictive than the better explored $\mathrm{AdS}_3 \times \mathrm{S}^3 \times M_4$ case with $M_4 = \mathbb{T}^4$ or $M_4 = \mathrm{K3}$, in which case only the small $\mathcal{N} = 4$ superconformal algebra is expected to appear. In particular, the large $\mathcal{N} = 4$ superconformal algebra contains two affine $\mathfrak{su}(2)$ algebras, and the small $\mathcal{N} = 4$ superconformal algebra can be obtained as a contraction in the limit in which one of the levels is sent to infinity. The other reason for studying the case with large $\mathcal{N} = 4$ superconformal symmetry is that the dual CFT of string theory is unknown [11] (see however [12] for a recent proposal), and one may hope that the novel higher spin perspective may also suggest new avenues for overcoming this impasse. Finally, it would be very interesting to make contact with the approach based on the integrable spin chain viewpoint of [13, 14].

The proposal of [10] was subsequently explored further. In particular, the spectrum of the two descriptions was matched in [15], see also [16] for an earlier analysis, and the asymptotic symmetry algebra of the higher spin theory was shown to agree with the 't Hooft limit of the Wolf space coset $\mathcal{W}$ algebras [17]. While many of the features of this duality mirror precisely what happens for the original bosonic proposal [4] and its $\mathcal{N} = 2$ supersymmetric generalisation [18, 19], there is one intriguing difference that was already noticed in [10]: while the quantum $\mathcal{W}_\infty$ algebras underlying the bosonic and the $\mathcal{N} = 2$ version exhibit a triality or quadrality relation [20, 21], respectively, that explains the identification of the quantisation of the asymptotic symmetry algebra with the dual coset algebra even at finite N, a similar relation does not seem to exist in the large $\mathcal{N} = 4$ case. It is therefore interesting to understand the structure of the large $\mathcal{N} = 4$ quantum $\mathcal{W}_\infty$ algebra in detail.

This is what will be done in this paper. As we shall see, the relevant quantum $\mathcal{W}_\infty$ algebra is uniquely determined in terms of the levels of the two affine $\mathfrak{su}(2)$ algebras. As a consequence, the quantisation of the asymptotic symmetry algebra of the higher spin theory must coincide with the coset algebra provided that the levels of the two $\mathfrak{su}(2)$ algebras agree, thus explaining the agreement of the symmetries without a triality-like relation. The absence of such a relation only implies that the quantisation of the asymptotic symmetry algebra of the higher spin theory based on the finite dimensional higher spin algebra $\mathfrak{shs}_2[\lambda]$ with $\lambda = M$ integer is *not* isomorphic to the Wolf space coset (1.1) with $N = M$. In fact, as we shall also explain in detail, while both algebras truncate to some finitely generated quantum algebras at integer M, the precise structure of the truncation is rather different in the two cases.

The paper is organised as follows. In section 2 and 3 we study the structure of the non-linear large $\mathcal{N} = 4$ $\mathcal{W}_\infty$ algebra. In particular, we explain our conventions for the supermultiplets in section 2, and make the most general ansatz for the various OPEs in section 3.2. We then study the constraints that follow from imposing the associativity of the OPEs, and describe our results in section 3.3 (as well as appendix B). In section 3.4 we analyse the different truncation patterns of this $\mathcal{W}_\infty$ algebra, and explain how the finitely generated symmetry algebras associated to $\mathfrak{shs}_2[M]$ and the coset algebra at finite N, respectively, fit into this picture. In section 4 we repeat the analysis for the case of the linear $\mathcal{N} = 4$ $\mathcal{W}_\infty$ algebra, and find essentially the same structure. As a non-trivial consistency check of our analysis we explain in detail in section 4.3 and 4.4 how the two sets of results are related to one another upon going from the linear to the non-linear description. Section 5 contains our conclusions, and some of the more technical material has been relegated to three appendices.

2 The non-linear large $\mathcal{N} = 4$ superconformal algebra

In this section we explain our conventions for the description of the large $\mathcal{N} = 4$ superconformal algebra, its superprimaries and their descendants.

2.1 The OPEs of the superconformal algebra

The non-linear large $\mathcal{N} = 4$ superconformal algebra is generated by the stress energy tensor

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\,, \tag{2.1}$$

six spin 1 currents $A^{\pm i}$, $i = 1,2,3$, which are primary with respect to T and generate an $\mathfrak{su}(2)_{k_+} \oplus \mathfrak{su}(2)_{k_-}$ subalgebra

$$A^{\pm i}(z)A^{\pm j}(w) \sim \frac{k_\pm \eta^{ij}}{(z-w)^2} + \frac{f^{ij}{}_l A^{\pm l}(w)}{z-w}\,, \tag{2.2}$$

as well as four spin $\frac{3}{2}$ supercharges $G^{\alpha\beta}$ which are primary with respect to both T and the currents $A^{\pm i}$

$$A^{+i}(z)G^{\alpha\beta}(w) \sim \frac{\rho^i_{\gamma\alpha} G^{\gamma\beta}}{z-w}\,, \qquad A^{-i}(z)G^{\alpha\beta}(w) \sim \frac{\rho^i_{\gamma\beta} G^{\alpha\gamma}}{z-w}\,. \tag{2.3}$$

Here ρ^i denotes the spin $j = \frac{1}{2}$ representation of $\mathfrak{su}(2)$, and the $\mathfrak{su}(2)$ invariant bilinear form η in eq. (2.2) is defined by $\eta^{ij} = \operatorname{tr} \rho^i \rho^j$. Global $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ symmetry constrains

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the OPEs of the supercharges to take the following most general quadratic form

$$G^{\alpha\beta}(z)G^{\gamma\delta}(w) \sim \frac{b\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}}{(z-w)^3} + \left[\frac{1}{(z-w)^2} + \frac{\partial}{2(z-w)}\right]\left(s^+\epsilon_{\beta\delta}\ell_{i,\alpha\gamma}A^{+i} + s^-\epsilon_{\alpha\gamma}\ell_{i,\beta\delta}A^{-i}\right)(w)$$
$$+\frac{1}{z-w}\left[\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}(-4T + s^{++}\eta_{ij}A^{+i}A^{+j} + s^{--}\eta_{ij}A^{-i}A^{-j}) + s^{+-}\ell_{i,\alpha\gamma}\ell_{j,\beta\delta}A^{+i}A^{-j}\right](w)\,, \tag{2.4}$$

where $\epsilon_{\alpha\beta}$ is the antisymmetric matrix with $\epsilon_{12}=1$, and the matrices ℓ_i are defined by

$$\ell_{i,\alpha\beta} = \epsilon_{\alpha\gamma}\,\rho^j_{\gamma\beta}\,\eta_{ji}\ . \tag{2.5}$$

Here η_{ij} is the inverse of η^{ij}, and in the following we shall routinely use these two matrices to raise and lower the indices in the adjoint representation. The Jacobi identities fix the structure constants in eqs. (2.1) and (2.4) to [22] (see also [23])

$$c = \frac{3(k_+ + k_- + 2k_+k_-)}{k_+ + k_- + 2} = \frac{6(k_+ + 1)(k_- + 1)}{k_+ + k_- + 2} - 3\,, \qquad b = -\frac{8k_+k_-}{2+k_++k_-}\,, \tag{2.6}$$

$$s^{\pm} = \frac{8k_{\mp}}{2+k_++k_-}\,, \qquad s^{\pm\pm} = \frac{2}{2+k_++k_-}\,, \qquad s^{+-} = -\frac{8}{2+k_++k_-}\ . \tag{2.7}$$

In the limit $k_{\pm}\to\infty$ with the ratio

$$\alpha = \frac{k_-}{k_+} \qquad \text{kept fixed,} \tag{2.8}$$

the wedge modes of the the non-linear large $\mathcal{N}=4$ superconformal algebra generate the exceptional Lie superalgebra $D(2,1;\alpha)$. Conversely, the non-linear large $\mathcal{N}=4$ superconformal algebra can be constructed as the Drinfel'd-Sokolov reduction of $D(2,1;\alpha)$ [24].

2.2 Superprimaries and their descendants

We call a field $\mathcal{N}=4$ superprimary provided that it is primary with respect to the stress-energy tensor T, as well as the currents $A^{\pm i}$. In addition, we require that the OPEs with the supercharges $G^{\alpha\beta}$ only have first order poles; in terms of the corresponding state these conditions are equivalent to requiring that it is annihilated by the positive modes of the stress-energy tensor, the currents and the supercharges, respectively.

In general, an $\mathcal{N}=4$ superprimary then transforms in an (irreducible) representation of the zero modes $A^{\pm i}_0$ of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$; in the following we shall consider the case where this representation is the singlet representation. We then denote the superconformal descendants of the superconformal primary $V^{(s)}$ by

component	$V^{(s)}_0$	$V^{(s)\alpha\beta}_{1/2}$	$V^{(s)\pm i}_1$	$V^{(s)\alpha\beta}_{3/2}$	$V^{(s)}_2$
conformal spin	s	$s+1/2$	$s+1$	$s+3/2$	$s+2$
$\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ spin	$(0,0)$	$(1/2,1/2)$	$(1,0)\oplus(0,1)$	$(1/2,1/2)$	$(0,0)$

. (2.9)

Here s is the conformal dimension of the superprimary field $V^{(s)}_0$, and the structure of the multiplet is as described in [10], see also [25].

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The precise form of the OPEs of these component fields with the fields of the large $\mathcal{N}=4$ superconformal algebra depend, to a certain extent, on our conventions.[1] We have chosen to work with a quasiprimary basis, and the guiding principle for our conventions has been to minimise the number of non-linear terms. For example, for the OPEs of the component fields with the stress-energy tensor we make the ansatz

$$\begin{aligned}
T(z)V_0^{(s)}(w) &\sim \frac{sV_0^{(s)}(w)}{(z-w)^2} + \frac{\partial V_0^{(s)}(w)}{z-w}\,, \\
T(z)V_{1/2}^{(s)\alpha\beta}(w) &\sim \frac{(s+\frac{1}{2})V_{1/2}^{(s)\alpha\beta}(w)}{(z-w)^2} + \frac{\partial V_{1/2}^{(s)\alpha\beta}(w)}{z-w}\,, \\
T(z)V_1^{(s)\pm i}(w) &\sim \frac{(s+1)V_1^{(s)\pm i}(w)}{(z-w)^2} + \frac{\partial V_1^{(s)\pm i}(w)}{z-w}\,, \\
T(z)V_{3/2}^{(s)}(w) &\sim \frac{(s+\frac{3}{2})V_{3/2}^{(s)\alpha\beta}(w)}{(z-w)^2} + \frac{\partial V_{3/2}^{(s)\alpha\beta}(w)}{z-w}\,, \\
T(z)V_2^{(s)}(w) &\sim \frac{tV_0^{(s)}(w)}{(z-w)^4} + \frac{sV_2^{(s)}(w)}{(z-w)^2} + \frac{\partial V_2^{(s)}(w)}{z-w}\,.
\end{aligned} \tag{2.10}$$

Note that these fields are Virasoro primary, except for $V_2^{(s)}$, which is only quasi-primary if $t \neq 0$ (as will be generically the case, see below). Similarly, as regards their behaviour under the current algebra, we postulate

$$\begin{aligned}
&A^{\pm i}(z)V_0^{(s)}(w) \sim 0\,, \\
&A^{+i}(z)V_{1/2}^{(s)\alpha\beta}(w) \sim \frac{\rho^i_{\gamma\alpha}V_{1/2}^{(s)\gamma\beta}(w)}{z-w}\,, \qquad A^{-i}(z)V_{1/2}^{(s)\alpha\beta}(w) \sim \frac{\rho^i_{\gamma\beta}V_{1/2}^{(s)\alpha\gamma}(w)}{z-w}\,, \\
&A^{\pm i}(z)V_1^{(s)\pm j}(w) \sim \frac{a_1^{\pm}\eta^{ij}V_0^{(s)}(w)}{(z-w)^2} + \frac{f^{ij}{}_l V_1^{(s)\pm l}(w)}{z-w}\,, \quad A^{\pm i}(z)V_1^{(s)\mp j}(w) \sim 0\,, \\
&A^{+i}(z)V_{3/2}^{(s)\alpha\beta}(w) \sim \frac{a_{3/2}^{+}\rho^i_{\gamma\alpha}V_{1/2}^{(s)\gamma\beta}(w)}{(z-w)^2} + \frac{\rho^i_{\gamma\alpha}V_{3/2}^{(s)\gamma\beta}(w)}{z-w}\,, \\
&A^{-i}(z)V_{3/2}^{(s)\alpha\beta}(w) \sim \frac{a_{3/2}^{-}\rho^i_{\gamma\beta}V_{1/2}^{(s)\alpha\gamma}(w)}{(z-w)^2} + \frac{\rho^i_{\gamma\beta}V_{3/2}^{(s)\alpha\gamma}(w)}{z-w}\,, \\
&A^{\pm i}(z)V_2^{(s)}(w) \sim \frac{a_2^{\pm}V_1^{(s)\pm i}(w)}{(z-w)^2}\,.
\end{aligned} \tag{2.11}$$

Thus $V_0^{(s)}$ and $V_{1/2}^{(s)}$ are affine-primary, but the higher component fields are not (since there are double poles in the OPEs with the currents). Our conventions for the OPEs with the supercharges are given in appendix A, and the associativity of this ansatz with the $\mathcal{N}=4$

[1] This is to be contrasted with the case of the linear superconformal algebras where requiring that the defining OPEs are linear usually leads to a unique choice. In the present case, a linear basis does not exist, and we need to fix this ambiguity differently.

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fields then implies that we have to choose

$$t = -\frac{48s(1+s)(k_+ - k_-)}{(1+2s)(2+k_+ + k_-)}\,, \tag{2.12}$$

$$a_1^{\pm} = 4s\,, \quad a_{3/2}^{\pm} = \pm\frac{8(1+s)[1+k_{\pm} + s(2+k_+ + k_-)]}{(1+2s)(2+k_+ + k_-)}\,, \quad a_2^{\pm} = \pm 4(1+s)\,, \tag{2.13}$$

as well as the values given in eq. (A.2).

3 Non-linear large $\mathcal{N} = 4$ $\mathcal{W}_\infty$ algebra

With these preparations we are now ready to study the structure of the $\mathcal{W}$ algebra that contains in addition to the non-linear large $\mathcal{N} = 4$ superconformal algebra higher spin multiplets $V^{(s)}$ of spin $s = 1, 2, 3, \ldots$ — one multiplet for every positive integer spin.

We shall use the same methods as in [20, 26–28]: first we write down the most general ansatz for the OPEs between the higher spin currents that are allowed by the basic requirements of conformal symmetry. Then we impose the Jacobi identities to solve for the structure constants in these OPEs. Our primary goal is to understand how many non-equivalent such $\mathcal{W}_\infty$ algebras exist, i.e., whether there are any further free parameters, in addition to $k_\pm$, that characterise these algebras.

In this section we shall write all OPEs in a quasiprimary basis. The OPEs then take the general form [29]

$$\Phi^i(z)\,\Phi^j(w) = \sum_k \frac{C^{ij}{}_k}{(z-w)^{h_i+h_j-h_k}} \sum_{n=0}^{\infty} \frac{(h_i - h_j + h_k)_n}{n!(2h_k)_n} (z-w)^n\, \partial^n \Phi^k(w)\,, \tag{3.1}$$

where Φ^i, Φ^j, Φ^k are quasi-primary operators of conformal dimension h_i, h_j and h_k, respectively, $C^{ij}{}_k$ are the structure constants and $(x)_n = \Gamma(x+n)/\Gamma(x)$ denotes the Pochhammer symbol. In order to improve the readability of the following formulas, we shall always use the shorthand notation for the singular part of the OPEs of type (3.1)

$$\Phi^i \times \Phi^j \sim \sum_{k\,:\,h_k < h_i + h_j} C^{ij}{}_k \Phi^k\,. \tag{3.2}$$

It should be obvious how to recover the actual singular part of the OPE (3.1) from the shorthand expression (3.2).

3.1 Composite fields

In order to be able to write down the most general ansatz for the OPEs of the higher spin fields in a quasiprimary basis we first need to find all the quasiprimary operators at every spin. A convenient (albeit somewhat formal) way of doing this is as follows. We introduce a 'mark' for every field of the algebra

component	$A^{\pm i}$	$G^{\alpha\beta}$	T	$V_0^{(s)}$	$V_{1/2}^{(s)\alpha\beta}$	$V_1^{(s)\pm i}$	$V_{3/2}^{(s)\alpha\beta}$	$V_2^{(s)}$
mark	$y_{0,1}^{\pm}$	$y_{0,3/2}$	$y_{0,2}$	$y_{s,0}$	$y_{s,1/2}$	$y_{s,1}^{\pm}$	$y_{s,3/2}$	$y_{s,2}$

(3.3)

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Then, the marked character of the full $\mathcal{W}_\infty$ algebra takes the form

$$\chi_\infty = \chi_0 \cdot \chi_{\rm hs}\,, \tag{3.4}$$

where χ_0 is the character of the large $\mathcal{N}=4$ superconformal algebra

$$\chi_0 = \prod_{n=1}^{\infty} \frac{\prod_{m,m'=-\frac{1}{2}}^{\frac{1}{2}}(1+y_{0,\frac{3}{2}} z_+^{2m} z_-^{2m'} q^{n+\frac{1}{2}})}{(1-y_{0,2}q^{n+1})\prod_{m=-1}^{1}(1-y_{0,1}^{+} z_+^{2m} q^n)(1-y_{0,1}^{-} z_-^{2m} q^n)}\,, \tag{3.5}$$

$z_\pm$ are the chemical potentials for the two $\mathfrak{su}(2)$ algebras, and $\chi_{\rm hs}$ counts the states generated by the higher spin fields

$$\chi_{\rm hs} = \prod_{s=1}^{\infty}\prod_{n=s}^{\infty} \frac{\prod_{m,m'=-\frac{1}{2}}^{\frac{1}{2}}(1+y_{s,\frac{1}{2}} z_+^{2m} z_-^{2m'} q^{n+\frac{1}{2}})(1+y_{s,\frac{3}{2}} z_+^{2m} z_-^{2m'} q^{n+\frac{3}{2}})}{(1-y_{s,0}q^n)(1-y_{s,2}q^{n+2})\prod_{m=-1}^{1}(1-y_{s,1}^{+} z_+^{2m} q^{n+1})(1-y_{s,1}^{-} z_-^{2m} q^{n+1})}\,. \tag{3.6}$$

The quasiprimary fields at spin s are then counted by the 'multiplicities' d_s, where

$$\chi_\infty = 1 + \sum_{s\in\mathbb{N}/2} \frac{d_s q^s}{1-q}\,. \tag{3.7}$$

The first few d_s are explicitly

$$\begin{aligned}
d_1 &= y_{1,0} + y_{0,1}^{+}{\rm ch}_1(z_+) + y_{0,1}^{-}{\rm ch}_1(z_-)\,,\\
d_{\frac{3}{2}} &= \big(y_{0,\frac{3}{2}} + y_{1,\frac{1}{2}}\big){\rm ch}_{\frac{1}{2}}(z_+){\rm ch}_{\frac{1}{2}}(z_-)\,,\\
d_2 &= \big[y_{0,2} + y_{2,0} + (y_{1,0})^2 + (y_{0,1}^{+})^2 + (y_{0,1}^{-})^2\big] + \big(y_{1,1}^{+} + y_{0,1}^{+}y_{1,0}\big){\rm ch}_1(z_+) +\\
&\quad + (y_{1,1}^{-} + y_{0,1}^{-}y_{1,0}){\rm ch}_1(z_-) + y_{1,1}^{+}y_{1,1}^{-}{\rm ch}_1(z_+){\rm ch}_1(z_-)\,,\\
d_{\frac{5}{2}} &= \big[y_{1,\frac{3}{2}} + y_{2,\frac{1}{2}} + (y_{1,0} + y_{0,1}^{+} + y_{0,1}^{-})(y_{0,\frac{3}{2}} + y_{1,\frac{1}{2}})\big]{\rm ch}_{\frac{1}{2}}(z_+){\rm ch}_{\frac{1}{2}}(z_-) +\\
&\quad + y_{0,1}^{+}(y_{0,\frac{3}{2}} + y_{1,\frac{1}{2}}){\rm ch}_{\frac{3}{2}}(z_+)ch_{\frac{1}{2}}(z_-) + y_{0,1}^{-}(y_{0,\frac{3}{2}} + y_{1,\frac{1}{2}}){\rm ch}_{\frac{1}{2}}(z_+){\rm ch}_{\frac{3}{2}}(z_-)\,,\\
d_3 &= \big\{y_{3,0} + y_{1,2} + y_{1,0}\big[y_{0,2} + y_{2,0} + (y_{1,0})^2 + (y_{0,1}^{+})^2 + (y_{0,1}^{-})^2\big] + y_{0,1}^{+}y_{1,1}^{+} + y_{0,1}^{-}y_{1,1}^{-} +\\
&\quad + y_{0,\frac{3}{2}}y_{1,\frac{1}{2}}\big\} + \big\{y_{2,1}^{+} + y_{1,0}\big(y_{1,1}^{+} + y_{0,1}^{+} + y_{1,0}y_{0,1}^{+}\big) + y_{0,1}^{+}[y_{0,2} + y_{2,0} + y_{0,1}^{+} + y_{1,1}^{+} +\\
&\quad + (y_{0,1}^{+})^2 + (y_{0,1}^{-})^2] + y_{0,\frac{3}{2}} + y_{0,\frac{3}{2}}y_{1,\frac{1}{2}} + (y_{1,\frac{1}{2}})^2\big\}{\rm ch}_1(z_+) + \big\{y_{2,1}^{-} + y_{1,0}\big(y_{1,1}^{-} + y_{0,1}^{-} +\\
&\quad + y_{1,0}y_{0,1}^{-}\big) + y_{0,1}^{-}[y_{0,2} + y_{2,0} + y_{0,1}^{-} + y_{1,1}^{-} + (y_{0,1}^{-})^2 + (y_{0,1}^{+})^2] + y_{0,\frac{3}{2}} + y_{0,\frac{3}{2}}y_{1,\frac{1}{2}} +\\
&\quad + (y_{1,\frac{1}{2}})^2\big\}{\rm ch}_1(z_-) + \big[y_{0,\frac{3}{2}}y_{1,\frac{1}{2}} + y_{1,1}^{+}y_{0,1}^{-} + y_{0,1}^{+}y_{1,1}^{-} + y_{0,1}^{+}y_{0,1}^{-}y_{1,0} + y_{0,1}^{+}y_{0,1}^{-}\big] \times\\
&\quad \times {\rm ch}_1(z_+){\rm ch}_1(z_-) + \big[y_{1,1}^{+}y_{0,1}^{+} + (y_{0,1}^{+})^2 y_{1,0}\big]{\rm ch}_2(z_+) + \big[y_{1,1}^{-}y_{0,1}^{-} + (y_{0,1}^{-})^2 y_{1,0}\big] \times\\
&\quad \times {\rm ch}_2(z_-) + (y_{0,1}^{+})^3{\rm ch}_3(z_+) + (y_{0,1}^{-})^3{\rm ch}_3(z_-) + (y_{0,1}^{+})^2 y_{0,1}^{-}{\rm ch}_2(z_+){\rm ch}_1(z_-) +\\
&\quad + (y_{0,1}^{-})^2 y_{0,1}^{+}{\rm ch}_1(z_+){\rm ch}_2(z_-)\,,
\end{aligned}$$

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where $\mathrm{ch}_j(z) = \sum_{m=-j}^{j} z^{2m}$ is the character of the $\mathfrak{su}(2)$ representation of spin j. From the explicit expressions for d_s we can verify that all quasiprimaries up to spin 3 are given by

$$\begin{aligned}
s=1: &\quad V_0^{(1)}\,,\ A^{\pm i}\,,\\
s=3/2: &\quad G^{\alpha\beta}\,,\ V_{1/2}^{(1)\alpha\beta}\,,\\
s=2: &\quad T\,, V_0^{(2)}\,,\ V_1^{(1)\pm i}\,,\ [V_0^{(1)}V_0^{(1)}]\,,\ [A^{\pm i}V_0^{(1)}]\,,\ [A^{\pm i}A^{\pm j}]\,,\ [A^{+i}A^{-j}]\,,\\
s=5/2: &\quad V_{3/2}^{(1)\alpha\beta}\,,\ V_{1/2}^{(2)\alpha\beta}\,,\ [V_0^{(1)}V_{1/2}^{(1)\alpha\beta}]\,,\ [V_0^{(1)}G^{\alpha\beta}]\,,\ [A^{\pm i}V_{1/2}^{(1)\alpha\beta}]\,,\ [A^{\pm i}G^{\alpha\beta}]\,,\\
s=3: &\quad V_0^{(3)}\,,\ V_1^{(2)\pm i}\,,\ V_2^{(1)}\,,\ [V_0^{(1)}V_0^{(2)}]\,,\ [V_0^{(1)}V_1^{(1)\pm i}]\,,\ [V_0^{(1)}[V_0^{(1)}V_0^{(1)}]]\,,\\
&\quad [V_{1/2}^{(1)\alpha\beta}V_{1/2}^{(1)\gamma\delta}]\,,\ [G^{\alpha\beta}V_{1/2}^{(1)\gamma\delta}]\,,\ [G^{\alpha\beta}G^{\gamma\delta}]\,,\ [A^{\pm i}V_0^{(2)}]\,,\ [A^{\pm i}V_1^{(1)\pm j}]\,,\\
&\quad [A^{\pm i}V_1^{(1)\mp j}]\,,\ [A^{\pm i}V_0^{(1)}]_{-1}\,,\ [A^{\pm i}[V_0^{(1)}V_0^{(1)}]]\,,\ [A^{\pm i}[A^{\pm j}V_0^{(1)}]]\,,\\
&\quad [A^{+i}[A^{-j}V_0^{(1)}]]\,,\ [TV_0^{(1)}]\,,\ [A^{\pm i}[A^{\pm j}A^{\pm l}]]\,,\ [A^{\pm i}[A^{\pm j}A^{\mp l}]]\,,\ [TA^{\pm i}]\,,\\
&\quad [A^{\pm i}A^{\pm j}]_{-1}\,,\ [A^{+i}A^{-j}]_{-1}\,.
\end{aligned}$$

Here we have introduced a modified normal ordered product $[\Phi^i\Phi^j]$, which is characterised by the property that it defines a quasiprimary operator provided that Φ^i and Φ^j are quasiprimary. More precisely, this modified normal ordered product differs from the standard normal ordered product $(\Phi^i\Phi^j)$ by the descendants of the quasiprimary operators appearing in the singular part of the OPE (3.1)

$$(\Phi^i\Phi^j) = [\Phi^i\Phi^j] + \sum_k C^{ij}{}_k \binom{2h_i-1}{h_i+h_j-h_k} \frac{\Gamma(2h_k)}{\Gamma(h_i+h_j+h_k)}\,\partial^{h_i+h_j-h_k}\Phi^k\,. \tag{3.8}$$

We have also introduced the following quasiprimary fields[2]

$$\begin{aligned}
[A^{\pm i}V_0^{(1)}]_{-1} &= \frac{1}{2}(\partial A^{\pm i}V_0^{(1)}) - \frac{1}{2}(A^{\pm i}\partial V_0^{(1)})\,,\\
[A^{\pm i}A^{\pm j}]_{-1} &= \frac{1}{2}(\partial A^{\pm i}A^{\pm j}) - \frac{1}{2}(A^{\pm i}\partial A^{\pm j}) - \frac{1}{12}f^{ij}{}_l\partial^2 A^{\pm l}\,,\\
[A^{+i}A^{-j}]_{-1} &= \frac{1}{2}(\partial A^{+i}A^{-j}) - \frac{1}{2}(A^{+i}\partial A^{-j})\,.
\end{aligned}$$

We can also deduce from the marked character the number of (composite) $\mathcal{N}=4$ superprimary fields that transform in the singlet representation $(0;0)$ of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ at spin s. To this end we expand the marked character with $y_{0,1}^{\pm} = y_{0,3/2} = y_{0,2} = 1$ and $y_{s,0} = y_{s,1/2} = y_{s,1}^{\pm} = y_{s,3/2} = y_{s,2} = y_s$, in terms of characters of $\mathcal{N}=4$ superprimaries

$$\chi_\infty = \chi_0 + \sum_{s\in\mathbb{N}/2} e_s q^s \times \prod_{n=1}^{\infty} \frac{\prod_{m,m'=-\frac{1}{2}}^{\frac{1}{2}}(1+z_+^{2m}z_-^{2m'}q^{n-\frac{1}{2}})}{(1-q^n)\prod_{m=-1}^{1}(1-z_+^{2m}q^n)(1-z_-^{2m}q^n)}\,, \tag{3.9}$$

where e_s is the 'multiplicity' of the $\mathcal{N}=4$ superprimaries at spin s. We can further decompose e_s into $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ characters to get the 'multiplicity' of the superprimaries

[2] These fields can be rewritten in terms of the normal ordered product $\mathscr{N}(\Phi^i, \partial^n\Phi^j)$ defined in [30].

in a given representation

$$e_s = \sum_{l_+,l_-} e_s(l_+,l_-)\,\mathrm{ch}_{l_+}(z_+)\,\mathrm{ch}_{l_-}(z_-)\ . \tag{3.10}$$

The first few values of $e_s(0,0)$ are then

$$\begin{aligned} e_1(0,0) &= y_1\,,\\ e_2(0,0) &= y_2 + y_1^2\,,\\ e_3(0,0) &= y_3 + y_1^3 + y_1 y_2\,, \end{aligned} \tag{3.11}$$

and it is not hard to convince oneself that $e_s(0,0)=0$ for all half-integer values of s. Thus, at spin $s=2$ there is a single composite superprimary of the form $[V_0^{(1)}V_0^{(1)}]+\cdots$, which can be used to redefine $V_0^{(2)}$, while at spin $s=3$ there are two composite superprimaries of the form $[V_0^{(1)}[V_0^{(1)}V_0^{(1)}]]+\cdots$ and $[V_0^{(1)}V_0^{(2)}]+\cdots$, which can be used to redefine $V_0^{(3)}$.

3.2 Ansatz for OPEs

With these preparations we can now make the most general ansatz for the OPEs between the various higher spin fields (up to total spin 4). Our ansatz will obviously need to respect the $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ symmetry (coming from the zero modes of the currents). At total spin 2 and $\frac{5}{2}$, the most general ansatz is then

$$V_0^{(1)}\times V_0^{(1)} \sim n_1 I + 0\cdot V_0^{(1)}\,, \qquad V_0^{(1)}\times V_{1/2}^{(1)\alpha\beta} \sim w_1 G^{\alpha\beta} + 0\cdot V_{1/2}^{(1)\alpha\beta}\ . \tag{3.12}$$

Here the coefficient in front of $V_0^{(1)}$ vanishes because a single spin one current can only generate an abelian Kac-Moody algebra. It is also clear that the coefficient in front of $V_{1/2}^{(1)\alpha\beta}$ must vanish because, by conformal symmetry, the 3-point function

$$\langle V_0^{(1)}(z)\,V_{1/2}^{(1)\alpha\beta}(w)\,V_{1/2}^{(1)\gamma\delta}(v)\rangle \tag{3.13}$$

is symmetric under the exchange of w and v, which however is incompatible with the fermionic nature of these fields.

At total spin 3 the most general ansatz for the OPEs is then

$$\begin{aligned} V_0^{(1)}\times V_1^{(1)+i} &\sim w_2 A^{+i} + w_3[A^{+i}V_0^{(1)}] + w_4 V_1^{(1)+i}\,,\\ V_0^{(1)}\times V_1^{(1)-i} &\sim w_5 A^{-i} + w_6[A^{-i}V_0^{(1)}] + w_7 V_1^{(1)-i}\,,\\ V_{1/2}^{(1)\alpha\beta}\times V_{1/2}^{(1)\gamma\delta} &\sim \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\big(w_8 I + w_9 V_0^{(1)} + w_{10}T + w_{11}[A^{+i}A^{+}{}_i] + w_{12}[A^{-i}A^{-}{}_i] +\\ &\quad + w_{13}[V_0^{(1)}V_0^{(1)}] + w_{14}V_0^{(2)}\big) + \epsilon_{\beta\delta}\ell_{i,\alpha\gamma}\big(w_{15}A^{+i} + w_{16}[A^{+i}V_0^{(1)}] +\\ &\quad + w_{17}V_1^{(1)+i}\big) + \epsilon_{\alpha\gamma}\ell_{i,\beta\delta}\big(w_{18}A^{-i} + w_{19}[A^{-i}V_0^{(1)}] + w_{20}V_1^{(1)-i}\big) +\\ &\quad + \ell_{i,\alpha\gamma}\ell_{j,\beta\delta}w_{21}[A^{+i}A^{-j}]\,,\\ V_0^{(1)}\times V_0^{(2)} &\sim w_{22}V_0^{(1)} + w_{23}T + w_{24}[A^{+i}A^{+}{}_i] + w_{25}[A^{-i}A^{-}{}_i] + w_{26}[V_0^{(1)}V_0^{(1)}] +\\ &\quad + w_{27}V_0^{(2)}\,, \end{aligned} \tag{3.14}$$

where the identity operator I cannot appear in the last OPE because the two point function $\langle V_0^{(1)}(z)V_0^{(2)}(w)\rangle$ vanishes.

Similarly, the most general ansatz for the OPEs of total spin $\frac{7}{2}$ is

$$\begin{aligned}
V_0^{(1)} \times V_{3/2}^{(1)\alpha\beta} \sim\ & w_{28}G^{\alpha\beta} + w_{29}V_{1/2}^{(1)\alpha\beta} + w_{30}V_{3/2}^{(1)\alpha\beta} + w_{31}V_{1/2}^{(2)\alpha\beta} + w_{32}[V_0^{(1)}G^{\alpha\beta}] + \\
& + w_{33}[V_0^{(1)}V_{1/2}^{(1)\alpha\beta}] + \rho_{i,\gamma\alpha}\big(w_{34}[A^{+i}V_{1/2}^{(1)\gamma\beta}] + w_{35}[A^{+i}G^{\gamma\beta}]\big) + \\
& + \rho_{i,\gamma\beta}\big(w_{36}[A^{-i}V_{1/2}^{(1)\alpha\gamma}] + w_{37}[A^{-i}G^{\alpha\gamma}]\big)\,, \\
V_0^{(1)} \times V_{1/2}^{(2)\alpha\beta} \sim\ & 0 \cdot G^{\alpha\beta} + w_{38}V_{1/2}^{(1)\alpha\beta} + \cdots + \rho_{i,\gamma\beta}\big(w_{45}[A^{-i}V_{1/2}^{(1)\alpha\gamma}] + w_{46}[A^{-i}G^{\alpha\gamma}]\big)\,, \\
V_{1/2}^{(1)\alpha\beta} \times V_0^{(2)} \sim\ & 0 \cdot G^{\alpha\beta} + w_{47}V_{1/2}^{(1)\alpha\beta} + \cdots + \rho_{i,\gamma\beta}\big(w_{54}[A^{-i}V_{1/2}^{(1)\alpha\gamma}] + w_{55}[A^{-i}G^{\alpha\gamma}]\big)\,, \\
V_{1/2}^{(1)\alpha\beta} \times V_1^{(1)+i} \sim\ & \rho^i_{\gamma\alpha}\big\{w_{56}G^{\gamma\beta} + w_{57}V_{1/2}^{(1)\gamma\beta} + w_{58}V_{3/2}^{(1)\gamma\beta} + w_{59}V_{1/2}^{(2)\gamma\beta} + w_{60}[V_0^{(1)}G^{\gamma\beta}] + \\
& + w_{61}[V_0^{(1)}V_{1/2}^{(1)\gamma\beta}] + \rho_{j,\delta\gamma}\big(w_{62}[A^{+j}V_{1/2}^{(1)\delta\beta}] + w_{63}[A^{+j}G^{\delta\beta}]\big) + \\
& + \rho_{j,\delta\beta}\big(w_{64}[A^{-j}V_{1/2}^{(1)\gamma\delta}] + w_{65}[A^{-j}G^{\gamma\delta}]\big)\big\} + \\
& + w_{66}[A^{+i}V_{1/2}^{(1)\alpha\beta}] + w_{67}[A^{+i}G^{\alpha\beta}]\,, \\
V_{1/2}^{(1)\alpha\beta} \times V_1^{(1)-i} \sim\ & \rho^i_{\gamma\beta}\big\{w_{68}G^{\alpha\gamma} + w_{69}V_{1/2}^{(1)\alpha\gamma} + w_{70}V_{3/2}^{(1)\alpha\gamma} + w_{71}V_{1/2}^{(2)\alpha\gamma} + w_{72}[V_0^{(1)}G^{\alpha\gamma}] + \\
& + w_{73}[V_0^{(1)}V_{1/2}^{(1)\alpha\gamma}] + \rho_{j,\delta\gamma}\big(w_{74}[A^{-j}V_{1/2}^{(1)\alpha\delta}] + w_{75}[A^{-j}G^{\alpha\delta}]\big) + \\
& + \rho_{j,\delta\alpha}\big(w_{76}[A^{+j}V_{1/2}^{(1)\delta\gamma}] + w_{77}[A^{+j}G^{\delta\gamma}]\big)\big\} + \\
& + w_{78}[A^{-i}V_{1/2}^{(1)\alpha\beta}] + w_{79}[A^{-i}G^{\alpha\beta}]\,.
\end{aligned} \tag{3.15}$$

In order to explain the above notation we note that the general ansatz for the OPEs $V_0^{(1)} \times V_{3/2}^{(1)\alpha\beta}$, $V_0^{(1)} \times V_{1/2}^{(2)\alpha\beta}$ and $V_{1/2}^{(1)\alpha\beta} \times V_0^{(2)}$ all have the same form, except that the actual structure constants will in general be different; we have therefore labelled the structure constants of the latter two OPEs using the same ordering as for the first. We hope this compact notation does not lead to any confusion. We should also mention that the coefficient in front of $G^{\alpha\beta}$ in the OPE $V_0^{(1)} \times V_{1/2}^{(2)\alpha\beta}$ must vanish because the two operators belong to different superprimary multiplets and hence cannot generate the superconformal family of the identity. The same remark applies to the OPE $V_{1/2}^{(1)\alpha\beta} \times V_0^{(2)}$.

The general ansatz for the OPEs of total spin 4 is given in appendix B.

3.3 Jacobi identities

Next we want to determine the actual structure constants, using the requirement that the $\mathcal{W}$ algebra must have associative OPEs, i.e., $(A(z)B(w))C(v) = A(z)(B(w)C(v))$. Using usual contour deformation arguments, see e.g. [31], this amounts to the condition that for all triplets A, B, C of $\mathcal{W}$ algebra generators we have the identity

$$[A[BC]_p]_q - (-1)^{|A||B|}[B[AC]_q]_p = \sum_{l=1}^{\infty} \binom{q-1}{l-1} [[AB]_l C]_{p+q-l}\,, \quad p, q > 0\,, \tag{3.16}$$

where $[AB]_p$ is the operator that multiplies the p-th order pole in the OPE of A with B, etc. This condition is believed to be equivalent to the requirement that the corresponding

Jacobi identities are satisfied, and we shall denote the set of equations (3.16) by $A \times B \times C$. To compute these identities we use the packages `OPEdefs` and `OPEconf` of Thielemans, see [31, 32].

We shall proceed level by level. First we solve all the Jacobi identities that can be computed with the OPEs of section 3.2. The first two OPEs (3.12) allow one to analyse the Jacobi identities

$$\begin{aligned} &T \times V_0^{(1)} \times V_0^{(1)}\,, \quad A^{\pm i} \times V_0^{(1)} \times V_0^{(1)}\,, \quad V_0^{(1)} \times V_0^{(1)} \times V_0^{(1)}\,, \quad G^{\alpha\beta} \times V_0^{(1)} \times V_0^{(1)}\,, \\ &T \times V_0^{(1)} \times V_{1/2}^{(1)\alpha\beta}\,, \quad A^{\pm i} \times V_0^{(1)} \times V_{1/2}^{(1)\alpha\beta}\,, \quad V_0^{(1)} \times V_0^{(1)} \times V_{1/2}^{(1)\alpha\beta}\,. \end{aligned}$$

It turns out that all of these are trivially satisfied. At one level higher, i.e. with the OPEs (3.14), one can compute the next group of Jacobi identities

$$\begin{aligned} &T\times V_0^{(1)} \times V_1^{(1)\pm i}\,, \quad A^{\pm i} \times V_0^{(1)} \times V_1^{(1)\pm j}\,, \quad V_0^{(1)} \times V_0^{(1)} \times V_1^{(1)\pm j}\,, \\ &T\times V_{1/2}^{(1)\alpha\beta} \times V_{1/2}^{(1)\gamma\delta}\,, \quad A^{\pm i} \times V_{1/2}^{(1)\alpha\beta} \times V_{1/2}^{(1)\gamma\delta}\,, \quad V_0^{(1)} \times V_{1/2}^{(1)\alpha\beta} \times V_{1/2}^{(1)\gamma\delta}\,, \\ &T\times V_0^{(1)} \times V_0^{(2)}\,, \quad A^{\pm i} \times V_0^{(1)} \times V_0^{(2)}\,, \quad V_0^{(1)} \times V_0^{(1)}\times V_0^{(2)}\,, \quad G^{\alpha\beta} \times V_0^{(1)}\times V_{1/2}^{(1)\gamma\delta}. \end{aligned}$$

These are satisfied provided the only non-zero structure constants in the OPEs (3.14) are

$$\begin{aligned} n_1 &= -\frac{2k_-k_+}{2+k_-+k_+}\,, & w_1 &= 1\,, & w_2 &= -\frac{8k_-}{2+k_-+k_+}\,, \\ w_5 &= -\frac{8k_+}{2+k_-+k_+}\,, & w_8 &= -\frac{8k_-k_+}{2+k_-+k_+}\,, & w_{10} &= -4\,, \\ w_{11} &= \frac{2}{2+k_-+k_+}\,, & w_{12} &= \frac{2}{2+k_-+k_+}\,, & w_{15} &= \frac{8k_-}{2+k_-+k_+}\,, \\ w_{18} &= \frac{8k_+}{2+k_-+k_+}\,, & w_{21} &= -\frac{8}{2+k_-+k_+}\,, \end{aligned} \tag{3.17}$$

where we have chosen to normalise $V_0^{(1)}$, and consequently all the other fields in the supermultiplet $V^{(1)}$, by fixing $w_1 = 1$. We remark that the only structure constant that is at this level not fixed is w_{22}. In fact, w_{22} cannot be determined in this manner because it reflects the freedom of redefining $V_0^{(2)}$ by a multiple of $[V_0^{(1)}V_0^{(1)}] + \cdots$, see eq. (3.11). We shall therefore, in the following, use this freedom to set

$$w_{22} = 0\,. \tag{3.18}$$

Note that it follows from the structure of the OPEs (3.14) and the form of the structure constants (3.17) that no simple operator of spin 2 appears on the r.h.s. of these OPEs. As a consequence, we can also already now compute the special Jacobi identity

$$V_{1/2}^{(1)\alpha\beta} \times V_{1/2}^{(1)\gamma\delta} \times V_{1/2}^{(1)\mu\nu}\,. \tag{3.19}$$

However, as it turns out, this identity is automatically satisfied.

Next we turn to the Jacobi identities that can be computed with the OPEs (3.15). In order to proceed efficiently, we first impose for all OPEs $A \times B$ the Jacobi identity

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$T \times A \times B$, i.e., we ensure that the conformal symmetry is respected. Then it follows from eq. (3.16) that in order to compute a Jacobi identity for a triplet of generators A, B, C for which the spins sum up to s, it is sufficient to know the OPEs between all pairs of generators for which the spins sum up to $s-1$. Thus, with the OPEs (3.15) (as well as the OPEs from above), we can compute the Jacobi identities for all triplets of generators for which the spins sum up to $\frac{9}{2}$. Solving these identities we find that the non-zero structure constants in the OPEs (3.15) must equal

$$
\begin{aligned}
w_{28} &= -\frac{16\,(-k_- + k_+)}{3\,(2 + k_- + k_+)}\,, \\
w_{30} &= w_{70}\,, \\
w_{31} &= 1\,, \\
w_{32} &= -4\,(-k_- + k_+)\,(5 + 4k_- + 4k_+ + 2k_-k_+)\,K w_{70}\,, \\
w_{33} &= -4\,(-k_- + k_+)\,(5 + 4k_- + 4k_+ + 2k_-k_+)\,K\,, \\
w_{34} &= -\frac{8\,(2 + k_- + 2k_+)\left(-2 - k_- - k_+ + 2k_-k_+ + 2k_-^2 k_+\right) K w_{70}}{2 + k_- + k_+}\,, \\
w_{35} &= \frac{8\,(2 + k_- + 2k_+)\left(-2 - k_- - k_+ + 2k_-k_+ + 2k_-^2 k_+\right) K}{2 + k_- + k_+}\,, \\
w_{36} &= \frac{8\,(2 + 2k_- + k_+)\left(-2 - k_- - k_+ + 2k_-k_+ + 2k_-k_+^2\right) K w_{70}}{2 + k_- + k_+}\,, \\
w_{37} &= -\frac{8\,(2 + 2k_- + k_+)\left(-2 - k_- - k_+ + 2k_-k_+ + 2k_-k_+^2\right) K}{2 + k_- + k_+}\,, \\
w_{39} &= -1 - w_{70}^2\,, \\
w_{40} &= -w_{70}\,, \\
w_{41} &= 4\,(-k_- + k_+)\,(5 + 4k_- + 4k_+ + 2k_-k_+)\left(1 + w_{70}^2\right) K\,, \\
w_{43} &= \frac{8\,(2 + k_- + 2k_+)\left(-2 - k_- - k_+ + 2k_-k_+ + 2k_-^2 k_+\right)\left(1 + w_{70}^2\right) K}{2 + k_- + k_+}\,, \\
w_{45} &= -\frac{8\,(2 + 2k_- + k_+)\left(-2 - k_- - k_+ + 2k_-k_+ + 2k_-k_+^2\right)\left(1 + w_{70}^2\right) K}{2 + k_- + k_+}\,, \\
w_{48} &= 1 + w_{70}^2\,, \\
w_{49} &= w_{70}\,, \\
w_{50} &= -4\,(-k_- + k_+)\,(5 + 4k_- + 4k_+ + 2k_-k_+)\left(1 + w_{70}^2\right) K\,, \\
w_{52} &= -\frac{8\,(2 + k_- + 2k_+)\left(-2 - k_- - k_+ + 2k_-k_+ + 2k_-^2 k_+\right)\left(1 + w_{70}^2\right) K}{2 + k_- + k_+}\,, \\
w_{54} &= \frac{8\,(2 + 2k_- + k_+)\left(-2 - k_- - k_+ + 2k_-k_+ + 2k_-k_+^2\right)\left(1 + w_{70}^2\right) K}{2 + k_- + k_+}\,, \\
w_{56} &= \frac{4\,(1 + k_- + 2k_+)}{2 + k_- + k_+}\,, \\
w_{58} &= -w_{70}\,, \\
w_{59} &= -1\,,
\end{aligned}
$$

$$
\begin{aligned}
w_{60} &= 4\,(-k_- + k_+)\,(5 + 4k_- + 4k_+ + 2k_-k_+)\,w_{70}K\,,\\
w_{61} &= 4\,(-k_- + k_+)\,(5 + 4k_- + 4k_+ + 2k_-k_+)\,K\,,\\
w_{62} &= \frac{8\,(2 + k_- + 2k_+)\left(-2 - k_- - k_+ + 2k_-k_+ + 2k_-^2k_+\right)Kw_{70}}{2 + k_- + k_+}\,,\\
w_{63} &= -\frac{8\,(2 + k_- + 2k_+)\left(-2 - k_- - k_+ + 2k_-k_+ + 2k_-^2k_+\right)K}{2 + k_- + k_+}\,,\\
w_{64} &= -\frac{8\,(2 + 2k_- + k_+)\left(-2 - k_- - k_+ + 2k_-k_+ + 2k_-k_+^2\right)Kw_{70}}{2 + k_- + k_+}\,,\\
w_{65} &= \frac{8k_-\,(-1 + k_+)\,(1 + k_+)\,(2 + k_- + 2k_+)\,K}{2 + k_- + k_+}\,,\\
w_{67} &= \frac{4}{2 + k_- + k_+}\,,\\
w_{68} &= \frac{4\,(1 + 2k_- + k_+)}{2 + k_- + k_+}\,,\\
&w_{70}\,,\\
w_{71} &= 1\,,\\
w_{72} &= -4\,(-k_- + k_+)\,(5 + 4k_- + 4k_+ + 2k_-k_+)\,Kw_{70}\,,\\
w_{73} &= -4\,(-k_- + k_+)\,(5 + 4k_- + 4k_+ + 2k_-k_+)\,K\,,\\
w_{74} &= \frac{8\,(2 + 2k_- + k_+)\left(-2 - k_- - k_+ + 2k_-k_+ + 2k_-k_+^2\right)Kw_{70}}{2 + k_- + k_+}\,,\\
w_{75} &= -\frac{8\,(2 + 2k_- + k_+)\left(-2 - k_- - k_+ + 2k_-k_+ + 2k_-k_+^2\right)K}{2 + k_- + k_+}\,,\\
w_{76} &= -\frac{8\,(2 + k_- + 2k_+)\left(-2 - k_- - k_+ + 2k_-k_+ + 2k_-^2k_+\right)Kw_{70}}{2 + k_- + k_+}\,,\\
w_{77} &= \frac{8\,(-1 + k_-)\,(1 + k_-)\,k_+\,(2 + 2k_- + k_+)\,K}{2 + k_- + k_+}\,,\\
w_{79} &= \frac{4}{2 + k_- + k_+}\,,
\end{aligned}
\tag{3.20}
$$

where K is a shorthand notation for the frequently occurring expression

$$
K = \frac{1}{-4 - 4k_- - k_-^2 - 4k_+ + 3k_-k_+ + 4k_-^2k_+ - k_+^2 + 4k_-k_+^2 + 3k_-^2k_+^2}\,, \tag{3.21}
$$

and we have chosen to normalize $V^{(2)}$ by fixing $w_{31} = 1$. Notice that the structure constants in the OPEs (3.15) are uniquely determined by $k_\pm$ and w_{70}; there are no field redefinitions that render the structure constant w_{70} redundant so, in principle it can either get fixed by the higher Jacobi identities or, if it does not, describe a genuine parameter of the $\mathcal{W}_\infty$ algebra.

With the next set of OPEs (B.1) we can compute the Jacobi identities $A \times B \times C$ for all triplets of generators for which the spins sum up to 5. Solving these identities we find in particular that

$$
w_{129} = w_{161} = 0\,,
$$

which means that no $\mathcal{W}_\infty$ algebra generator of spin 3 can appear in the singular part of the OPEs $V_0^{(2)} \times V_1^{(1)\pm i}$ and, obviously, also in $V_1^{(1)+i} \times V_1^{(1)-j}$ and $V_0^{(2)} \times V_0^{(2)}$. For this reason, the OPEs (B.1) are also sufficient to compute the special Jacobi identities

$$V_0^{(2)} \times V_0^{(2)} \times V_0^{(2)} \,, \quad V_0^{(2)} \times V_0^{(2)} \times V_1^{(1)\pm i} \,, \quad V_0^{(2)} \times V_1^{(1)+i} \times V_1^{(1)-j} \,. \tag{3.22}$$

Solving in addition these identities we find, first of all that

$$w_{70} = 0 \,, \tag{3.23}$$

and, secondly, that the following structure constants remain undetermined

$$w_{80} \,, \quad w_{81} \,, \quad w_{85} \,, \quad w_{89} \,, \quad w_{90} \,, \tag{3.24}$$

while all the other structure constants in the OPEs (B.1) are uniquely fixed in terms of these and $k_\pm$; the explicit expressions for those structure constants that are non-zero are given in appendix B.2.

Let us now try to understand the meaning of the free parameters in eq. (3.24). Firstly, just like w_{22}, the structure constants w_{80} and w_{81} are redundant because they can be set to any value by absorbing into $V_0^{(3)}$ a linear combination of the two composite $\mathcal{N} = 4$ superprimary fields at spin 3, see the discussion following eq. (3.11); we shall fix this redefinition freedom of $V_0^{(3)}$ by setting

$$w_{80} = w_{81} = 0 \,. \tag{3.25}$$

Secondly, we note that there is a similarity between w_{70} and w_{85}, w_{89}, w_{90} — they all appear in front of operators that violate the parity 'symmetry' of the OPEs

$$V^{(s)} \mapsto (-1)^s V^{(s)} \,, \tag{3.26}$$

that is a natural symmetry of the underlying higher spin algebra $\mathfrak{shs}_2[\lambda]$. In fact, a careful inspection of the structure constants (3.17), (3.20), and (3.23) shows that w_{85}, w_{89}, w_{90} are the *only* structure constants that violate this symmetry. We have gone one level higher with the ansatz for the OPEs and verified that, in perfect analogy with what happened to w_{70}, these parity violating structure constants are required to vanish by the next set of Jacobi identities for which the spins sum up to $\frac{9}{2}$

$$w_{85} = w_{89} = w_{90} = 0 \,. \tag{3.27}$$

We interpret this fact as evidence for a mechanism by which the consistency of the $\mathcal{W}_\infty$ algebra imposes (dynamically) the parity symmetry (3.26) on all the OPEs. Furthermore, if we assume that the parity symmetry (3.26) holds generally, then all the structure constants could again be determined uniquely in terms of $k_\pm$, modulo the redefinition freedom of the generators. We take this as a strong indication that the most general $\mathcal{N} = 4$ $\mathcal{W}_\infty$ algebra with the above field content does not have any other parameters except for $k_\pm$.

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3.4 Truncations

Given the higher spin/CFT duality of [10] we expect the large $\mathcal{N}=4$ $\mathcal{W}_\infty$ algebra to exhibit two kinds of truncations. First, for suitable values of μ (namely $\mu \in \mathbb{Z}\backslash\{0,1\}$), the underlying higher spin algebra $\mathfrak{shs}_2[\mu]$ can be truncated to a finite dimensional Lie algebra, and one may therefore expect that this will also be reflected in the corresponding $\mathcal{W}_\infty$ algebra. Second, the dual Wolf space cosets should be finitely generated, and thus we should expect the $\mathcal{W}_\infty$ algebra to truncate for positive integer values of $k_\pm$. Unlike the situation with less supersymmetry [20, 21], these two truncation phenomena seem to be of different nature (see also the discussion in [10]), and we shall therefore study them separately.

3.4.1 The higher spin truncation

As already explained in [10], if we set $\mu = -N$ or $\mu = N+1$ with $N \in \mathbb{N}$, then the higher spin algebra $\mathfrak{shs}_2[\mu]$ can be truncated to an algebra that is generated by $D(2,1;\alpha)$, the first $N-1$ supermultiplets $V^{(s)}$ with $s = 1,\ldots,N-1$, as well as 'half' of the N-th supermultiplet

$$\hat{V}^{(N)\pm} = \{\, V_0^{(N)}\,,\ V_{1/2}^{(N)\alpha\beta}\,,\ V_1^{(N)\pm i}\,\}\,, \tag{3.28}$$

where the plus case arises for $\mu = -N$ and the minus case for $\mu = N+1$.

Let us concentrate in the following on the case $\mu = N+1$ for which the minus truncation of (3.28) arises; the other case works similarly. In order for the multiplet to truncate in the actual $\mathcal{W}_\infty$ algebra the missing states, i.e., the states that would be there in $V^{(N)}$ but are absent in $V^{(N)-}$, must actually be null; thus the higher spin analysis predicts null-vectors which turn out to be of the form (see also [17])

$$\begin{aligned}
s = N+1: &\quad V_1^{(N)-i} + \kappa[A^{-i}V_0^{(N)}]\,,\\
s = N+\tfrac{3}{2}: &\quad V_{3/2}^{(N)\alpha\beta} + \kappa[G^{\alpha\beta}V_0^{(N)}] - 2\kappa\rho_{i,\gamma\alpha}[A^{+i}V_{1/2}^{(N)\gamma\beta}]\,,\\
s = N+2: &\quad V_2^{(N)} - 4\kappa[TV_0^{(N)}] - \kappa\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}[G^{\alpha\beta}V_{1/2}^{(N)\gamma\delta}] + \tfrac{1}{2}\kappa^2[[A^+{}_iA^{+i}]V_0^{(N)}] +\\
&\quad + \tfrac{1}{2}\kappa^2[[A^-{}_iA^{-i}]V_0^{(N)}] - \kappa[A^+{}_iV_1^{(N)+i}]\,,
\end{aligned} \tag{3.29}$$

where $\kappa = -4N/k_-$. These solutions appear for k_- given by[3]

$$k_- = -\frac{N(2+k_+)}{1+N}\ . \tag{3.30}$$

Here the κ-dependent terms are required to make the states in eq. (3.29) primary with respect to the stress-energy tensor and the current fields. In the 't Hooft limit, $k_\pm \to \infty$ with the ratio $\alpha = k_-/k_+$ kept fixed, $\alpha \to -N/(1+N)$ which corresponds precisely to $\mu = N+1$. In this limit the constant κ vanishes and we recover the truncation exhibited in [10].

We have checked that these vectors are singular with respect to the (non-linear) large $\mathcal{N}=4$ superconformal algebra, but we expect that they actually lie in an ideal of the full $\mathcal{W}_\infty$ algebra (that can be consistently quotiened out). Given our detailed understanding

[3] There is a second solution $k_- = 1$ to which we will return below. This fact was also noticed in [17].

of the latter, we can check this at least for $N = 1$ and $N = 2$. To illustrate these checks, consider first the case $N = 1$. It follows from (3.14) that

$$V_0^{(1)} \times \left(V_1^{(1)-i} + \kappa[A^{-i}V_0^{(1)}]\right) = (w_5 - \kappa n_1)A^{-i}\,, \tag{3.31}$$

where we have used that $[A^{-i}V_0^{(1)}] = (A^{-i}V_0^{(1)})$, as well as $w_6 = w_7 = 0$, see eq. (3.17). Since the left-hand-side should lie in the ideal (but A^{-i} does not) consistency requires that the prefactor vanishes, $w_5 - \kappa n_1 = 0$; this turns out to be true, using eq. (3.17) for $N = 1$.

A somewhat more trivial test is that in the OPE $V_0^{(1)} \times V_0^{(2)}$ the coefficient w_{22} of $V_0^{(1)}$ vanishes; this is automatically the case for our definition of $V_0^{(2)}$, see the discussion around eq. (3.18). Less trivially, in the OPEs

$$V_0^{(1)} \times V_{1/2}^{(2)}\,, \qquad \text{and} \qquad V_{1/2}^{(1)} \times V_0^{(2)} \tag{3.32}$$

in eq. (3.15), the right-hand-side is indeed null because the coefficients of the terms that do not break parity, i.e., $V_{1/2}^{(1)}$ and $[A^{-i}V_{1/2}^{(1)}]$ vanish, and $V_{3/2}^{(1)}$ enters only in the combination (3.29), i.e. $w_{41}/w_{39} = w_{50}/w_{48} = \kappa$ and $w_{43}/w_{39} = w_{52}/w_{48} = -2\kappa$. The fact that the whole multiplet $V^{(2)}$ is null also follows from the vanishing of the central term n_2 in the OPEs $V_0^{(2)} \times V_0^{(2)}$, see appendix B.2.

The analysis for $N = 2$ is similar, except for one interesting subtlety. The only OPE on which we can test this truncation is $V_0^{(1)} \times V_1^{(2)-i}$, for which we find

$$V_0^{(1)} \times \left(V_1^{(2)-i} + \kappa[A^{-i}V_0^{(2)}]\right) = w_{143}V_1^{(1)-i} + (w_{144} + \kappa w_{22})[A^{-i}V_0^{(1)}] + w_{152}\epsilon_{\alpha\gamma}r^i_{\beta\delta}[G^{\alpha\beta}V_{1/2}^{(1)\gamma\delta}]\,. \tag{3.33}$$

The right hand side does not depend on κ because $V_0^{(1)}$ has a regular OPE with $V_0^{(2)}$, and, on the face of it, it does not vanish. This is a consequence of the fact that the actual null-vector of the full $\mathcal{W}_\infty$ algebra requires a specific choice for $V_0^{(2)}$, which in the above conventions corresponds not to $w_{22} = 0$ (see eq. (3.18)), but rather to

$$w_{22} = -\frac{32(k_+ - 1)(1 + 2k_+)}{3k_+(2 + k_+)}\,. \tag{3.34}$$

With this choice of $V_0^{(2)}$ and setting k_- to equal eq. (3.30) with $N = 2$, the right-hand-side of (3.33) is indeed zero, i.e., $w_{143} = w_{144} + \kappa w_{22} = w_{152} = 0$.

3.4.2 The coset truncation

Recall that the Wolf space coset algebra (written in bosonic form)

$$\frac{\mathfrak{su}(N+2)_k \oplus \mathfrak{so}(4N)_1}{\mathfrak{su}(N)_{k+2} \oplus \mathfrak{u}(1)} \tag{3.35}$$

has a non-linear large $\mathcal{N} = 4$ superconformal symmetry with $k_+ = k$ and $k_- = N$. For k and N large, the higher spin content of the above coset algebra agrees with the large $\mathcal{N} = 4$ $\mathcal{W}_\infty$ algebra for spins sufficiently small compared to k and N, see [10] for a simple higher spin counting argument or [15] for a more involved proof based on characters. Furthermore, it was confirmed in [17], that the asymptotic symmetry algebras match. It is therefore very natural to expect that the $\mathcal{N}=4$ $\mathcal{W}_\infty$ algebra truncates to the above coset algebra at positive integer levels $k_\pm$. The case with $N=3$ was also discussed in [33].

The first hint on the form of the coset truncation can be obtained by comparing the vacuum character of the coset algebra (3.35) to the $\mathcal{W}_\infty$ algebra. The first deviation can be computed with the help of eq. (3.24) of [15] and the $\mathfrak{su}(N)$ modification rules of [34], and for large enough k one finds

$$\chi_{\text{coset}} = \chi_\infty - q^{N+1} \sum_{l_-=0}^{N+1} \text{ch}_{l_-}(z_-) + \mathcal{O}(q^{N+\frac{3}{2}}) \ . \tag{3.36}$$

For example, for $k_- = N = 1$ these null vectors appear at conformal dimension 2 and are given by

$$\begin{aligned}
(l_+,l_-) = (0,2): &\qquad [A^{-i}A^{-j}] + [A^{-j}A^{-i}] - \tfrac{2}{3}\eta^{ij}[A^{-}{}_{l}A^{-l}]\,, \\
(l_+,l_-) = (0,1): &\qquad V_1^{(1)-i} - 4[A^{-i}V_0^{(1)}]\,, \\
(l_+,l_-) = (0,0): &\qquad V_0^{(2)}\,,
\end{aligned} \tag{3.37}$$

where the first vector corresponds to the lowest affine null vector of the $\mathfrak{su}(2)_1$ vacuum representation, while the fact that $V_0^{(2)}$ is null follows from the vanishing of its 2-point function, i.e., n_2 vanishes (without inducing poles in any other structure constants). The existence of the second null vector, explained by the fact that the representation of $\mathfrak{su}(2)_{k_-}$ with spin $l_- = 1$ is not integrable at $k_- = 1$, implies that $V^{(1)}$ truncates to a short representation $\hat{V}^{(1)}$ (and, in fact, all supermultiplets $V^{(s)}$ truncate this way). The null vectors (3.37) and the ideal generated by them suggest that only the generators of the superconformal algebra and $\hat{V}^{(1)}$ survive the truncation, although, in contradistinction with the situation in the previous subsection, the remaining generators must satisfy infinitely many additional constraints to account for the affine $\mathfrak{su}(2)_1$ null vectors.

For $k_- = 2$ the first set of null vectors that are predicted by eq. (3.36) appear at conformal dimension 3, and they correspond to the lowest null vector of the $\mathfrak{su}(2)_2$ vacuum representation, which has spin $(l_+,l_-) = (0,3)$, the lowest null vector of the $\mathfrak{su}(2)_2$ representation generated by the affine primary $V_1^{(2)-i} - 4[A^{-i}V_0^{(2)}]$, which has spin $(l_+,l_-) = (0,2)$, the unique superprimary with spin $(l_+,l_-) = (0,1)$ at conformal dimension 3

$$\begin{aligned}
&(V_1^{(2)-i} - 4[A^{-i}V_0^{(2)}]) + \tfrac{4(k_++1)(10+11k_+)}{19k_+^2+18k_+-16}\Big([V_0^{(1)}(V_1^{(1)-i} - 2[A^{-i}V_0^{(1)}])]\Big) + \\
&+ \tfrac{8(k_++6)(2k_+^2-5)\epsilon_{\alpha\gamma}r^i_{\beta\delta}}{(k_++4)(19k_+^2+18k_+-16)}\Big([V_{1/2}^{(1)\alpha\beta}V_{1/2}^{(1)\gamma\delta}] - \tfrac{k_+^2+3k_+-1}{2k_+^2-5}[G^{\alpha\beta}G^{\gamma\delta}]\Big) + \\
&+ \tfrac{32(k_+-4)(k_++1)(k_++6)}{(k_++4)(19k_+^2+18k_+-16)}[TA^{-i}] - \tfrac{16(k_+-4)k_+(k_++1)(k_++6)}{5(k_++4)^2(19k_+^2+18k_+-16)}f^i{}_{jl}[A^{-j}A^{-l}]_{-1} - \\
&- \tfrac{16(k_+-4)(k_++1)(k_++5)(k_++6)}{5(k_++4)^2 19k_+^2+18k_+-16)}\Big([[A^-{}_jA^{-j}]A^{-i}] + \tfrac{5}{(k_++5)}[[A^+{}_jA^{+j}]A^{-i}]\Big)\,,
\end{aligned} \tag{3.38}$$

as well as $V_0^{(3)}$ (with spin $(l_+,l_-) = (0,0)$). A non-trivial check of the fact that the latter two superprimaries are null is that their OPEs with $V_0^{(1)}$ vanish indeed. Again, it is tempting to believe that only the superconformal algebra and the first two supermultiplets $V^{(1)}$ and $V^{(2)}$ survive the truncation, but it is clear from the affine representation theory that they must satisfy infinitely many additional constraints.

In general, we expect that for arbitrary integer values of $k_\pm$ the coset algebra is generated by $A^{\pm i}$, T, $G^{\alpha\beta}$ as well as the first $\min(k_-, k_+)$ supermultiplets. Furthermore, there are infinitely many additional constraints of spin $s \geq \min(k_-, k_+)$, accounting for the various $\mathfrak{su}(2)_{k_\pm}$ null vectors.

4 Linear large $\mathcal{N} = 4$ $\mathcal{W}_\infty$ algebra

In the previous sections we have discussed the structure of the 'non-linear' large $\mathcal{N} = 4$ $\mathcal{W}_\infty$ algebra that contains, in addition to the non-linear large $\mathcal{N} = 4$ superconformal algebra $\tilde{A}_\gamma$, muliplets of spin $s = 1, 2, \ldots$. The non-linear large $\mathcal{N} = 4$ superconformal algebra $\tilde{A}_\gamma$ can be obtained, upon quotienting out the free fermions and the $\mathfrak{u}(1)$ current [22] from the linear A_γ algebra, see [22, 35–39] for some early literature on the subject. The same construction can also be applied to the 'linear version' of the full $\mathcal{W}_\infty$ algebra. One may therefore suspect that the structure of the 'linear' $\mathcal{W}_\infty$ algebra will also be characterised just by the levels $k_\pm$ of the two affine $\mathfrak{su}(2)$ algebras.[4]

In order to confirm this we shall, in this section, repeat the above analysis for the linear case. Since the techniques are largely the same, we shall be relatively brief.

4.1 The linear $\mathcal{N} = 4$ superconformal algebra A_γ

The linear large $\mathcal{N} = 4$ superconformal algebra A_γ contains in addition to the energy momentum tensor, the current algebra

$$\mathfrak{su}(2)_{k_+} \oplus \mathfrak{su}(2)_{k_-} \oplus \mathfrak{u}(1)\,, \tag{4.1}$$

as well as four supercharges G^a and four free fermions Q^a both of which transform in the $\left(\frac{1}{2}, \frac{1}{2}\right)_0$ with respect to the above current algebra. We shall denote the $\mathfrak{u}(1)$ current by U, while the currents of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ are $A^{\pm,i}$ with $i = 1, 2, 3$. The central charge of the Virasoro algebra equals

$$c = \frac{6\,k_+\,k_-}{k_+ + k_-}\,, \quad \text{and} \quad \gamma = \frac{k_-}{k_+ + k_-}\,, \qquad (\overline{\gamma} = 1 - \gamma)\ . \tag{4.2}$$

Apart from the standard TT OPE, the additional OPEs defining A_γ are

$$\begin{aligned} G^a(z)\,G^b(w) &\sim \frac{2c}{3}\,\frac{\delta^{ab}}{(z-w)^3} - 8\,\frac{\gamma\,\alpha^{+,i}_{ab}\,A^{+,i} + \overline{\gamma}\,\alpha^{-,i}_{ab}\,A^{-,i}}{(z-w)^2} \\ &\quad -4\,\frac{\gamma\,\alpha^{+,i}_{ab}\,\partial A^{+,i} + \overline{\gamma}\,\alpha^{-}_{ab}\,\partial A^{-,i}}{z-w} + \frac{2\,\delta^{ab}\,T}{z-w}\,, \end{aligned} \tag{4.3}$$

$$A^{\pm,i}(z)\,A^{\pm,j}(w) \sim -\frac{k^\pm}{2}\,\frac{\delta^{ij}}{(z-w)^2} + \frac{\epsilon^{ijk}\,A^{\pm,k}}{z-w}\,, \tag{4.4}$$

$$Q^a(z)\,Q^b(w) \sim -\frac{k^+ + k^-}{2}\,\frac{\delta^{ab}}{z-w}\,, \tag{4.5}$$

[4]To avoid confusion we should stress that the full 'linear' $\mathcal{W}_\infty$ algebra is in fact also non-linear — by the qualifier 'linear' we only mean that it contains the linear large $\mathcal{N} = 4$ superconformal algebra as a subalgebra (rather than the non-linear $\tilde{A}_\gamma$ algebra). The fact that this algebra cannot be completely linearised was already noticed, on the level of the dual asymptotic symmetry algebra, in [17].

$$U(z)\,U(w) \sim -\frac{k^+ + k^-}{2}\,\frac{1}{(z-w)^2}\,, \tag{4.6}$$

$$A^{\pm,i}(z)\,G^a(w) \sim \mp\frac{2\,k^\pm}{k^+ + k^-}\,\frac{\alpha^{\pm,i}_{ab}\,Q^b}{(z-w)^2} + \frac{\alpha^{\pm,i}_{ab}\,G^b}{z-w}\,, \tag{4.7}$$

$$A^{\pm,i}(z)\,Q^a(w) \sim \frac{\alpha^{\pm,i}_{ab}\,Q^b}{z-w}\,, \tag{4.8}$$

$$Q^a(z)\,G^b(w) \sim 2\,\frac{\alpha^{+,i}_{ab}\,A^{+,i} - \alpha^{-,i}_{ab}\,A^{-,i}}{z-w} + \frac{\delta^{ab}\,U}{z-w}\,, \tag{4.9}$$

$$Q^a(z)\,U(w) \sim 0\,, \tag{4.10}$$

$$U(z)\,G^a(w) \sim \frac{Q^a}{(z-w)^2}\,. \tag{4.11}$$

Here the matrices $\alpha^{\pm,i}_{ab}$ are the $\mathfrak{so}(4)$ generators[5]

$$\alpha^{\pm,i}_{ab} = \tfrac{1}{2}(\pm\delta^i_a\,\delta^4_b \mp \delta^i_b\,\delta^4_a + \epsilon_{iab})\,, \tag{4.12}$$

obeying the (anti)-commutation relations

$$[\alpha^{\pm,i}, \alpha^{\pm,j}] = -\epsilon^{ijk}\,\alpha^{\pm,k}\,, \qquad [\alpha^{\pm,i}, \alpha^{\mp,j}] = 0\,, \qquad \{\alpha^{\pm,i}, \alpha^{\pm,j}\} = -\tfrac{1}{2}\delta^{ij}\,. \tag{4.13}$$

4.2 The general linear multiplet

For the description of the linear $\mathcal{W}_\infty$ algebra we now need to add a linear $\mathcal{N}=4$ multiplet whose components close under the OPE with A_γ [40, 41] (see also [42]). As before, we only need the special case of a scalar multiplet, i.e., one whose lowest component is $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ invariant. The multiplet components can be labelled as in eq. (2.9), except that we use a different convention to label the $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ indices

component	$V^{(s)}_0$	$V^{(s),a}_{1/2}$	$V^{(s),\pm,i}_1$	$V^{(s),a}_{3/2}$	$V^{(s)}_2$
conformal spin	s	$s+\frac{1}{2}$	$s+1$	$s+\frac{3}{2}$	$s+2$
$\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ spin	$(0,0)$	$(\frac{1}{2},\frac{1}{2})$	$(1,0)\oplus(0,1)$	$(\frac{1}{2},\frac{1}{2})$	$(0,0)$

(4.14)

The OPEs of the A_γ fields with the various component fields are given in appendix A.

We can then proceed as in the analysis of the non-linear algebra. We make the most general ansatz for the OPEs between the various higher spin fields, and then impose Jacobi identities, i.e., the associativity of the OPEs, to determine the structure constants recursively. We have performed this analysis for the OPEs up to total spin $\frac{7}{2}$. We have again found that, apart from $k_\pm$, there are no free parameters — except for those one would expect to be determined by imposing higher Jacobi identities.

[5] In our conventions $\epsilon_{123} = \epsilon^{123} = 1$ and $\epsilon_{ab4} = 0$.

4.3 From the linear to the non-linear description

According to [22], it is possible to decouple the free fermions Q^a, and the $\mathfrak{u}(1)$ field U from the linear $\mathcal{W}_\infty$ algebra by effectively performing a coset construction. On the level of the linear superconformal A_γ algebra, this amounts to redefining the stress-energy tensor, the supercharges and the affine currents as

$$\widetilde{T} = T + \frac{1}{k_+ + k_-}\left[-(Q^c\partial Q^c) + (UU)\right], \tag{4.15}$$

$$\begin{aligned}\widetilde{G}^a &= G^a + \frac{2}{k_+ + k_-}\Big[(UQ^a) - 2\,\alpha^{+,i}_{ab}(A^{+,i}Q^b) + 2\,\alpha^{-,i}_{ab}(A^{-,i}Q^b)\\ &\quad + \frac{2}{3}\frac{1}{k_+ + k_-}\,\epsilon_{abcd}(Q^bQ^cQ^d)\Big],\end{aligned} \tag{4.16}$$

$$\widetilde{A}^{\pm,i} = A^{\pm,i} - \frac{1}{k_+ + k_-}\,\alpha^{\pm,i}_{ab}(Q^aQ^b)\,. \tag{4.17}$$

The modified fields then obey the non-linear $\widetilde{A}_\gamma$ algebra and have regular OPEs with the decoupling fields

$$U(z)\,\{\widetilde{A}^{\pm,i}(w),\ \widetilde{G}^a(w),\ \widetilde{T}(w)\} \sim 0\,, \tag{4.18}$$

$$Q^a(z)\,\{\widetilde{A}^{\pm,i}(w),\ \widetilde{G}^b(w),\ \widetilde{T}(w)\} \sim 0\,. \tag{4.19}$$

The redefined currents still satisfy an affine $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ algebra, but the levels are now shifted to

$$\tilde{k}_\pm = k_\pm - 1\,. \tag{4.20}$$

We can similarly decouple the free fermions and the $\mathfrak{u}(1)$ field from the rest of the linear $\mathcal{W}_\infty$ algebra. For the lowest spin component there is nothing to be done,

$$\widetilde{V}^{(s)}_0 = V^{(s)}_0\,, \tag{4.21}$$

and the remaining components can be obtained by repeatedly applying the supercurrents $\widetilde{G}^a$; this leads to

$$\widetilde{V}^{(s),a}_{1/2} = V^{(s),a}_{1/2}\,, \tag{4.22}$$

$$\widetilde{V}^{(s),\pm,i}_1 = V^{(s),\pm,i}_1 \pm \frac{4}{k_+ + k_-}\,\alpha^{\pm,i}_{ab}(Q^a\,V^{(s),b}_{1/2})\,, \tag{4.23}$$

$$\begin{aligned}\widetilde{V}^{(s),a}_{3/2} &= V^{(s),a}_{3/2} + \frac{4}{k_+ + k_-}\Big[2\,s\,(\partial Q^a\,V^{(s)}_0) - (Q^a\,\partial V^{(s)}_0) - (U\,V^{(s),a}_{1/2})\\ &\quad -2\alpha^{+,i}_{ab}(A^{+,i}\,V^{(s),b}_{1/2}) + 2\alpha^{-,i}_{ab}(A^{-,i}\,V^{(s),b}_{1/2}) - \alpha^{+,i}_{ab}(Q^b\,V^{(s),+,i}_1) - \alpha^{-,i}_{ab}(Q^b\,V^{(s),-,i}_1)\Big],\end{aligned} \tag{4.24}$$

$$\begin{aligned}\widetilde{V}^{(s)}_2 &= V^{(s)}_2 + \frac{4}{k_+ + k_-}\Big[-(2s+1)\,(\partial Q^a\,V^{(s),a}_{1/2}) + (Q^a\,\partial V^{(s),a}_{1/2})\\ &\quad +2\,s\,(\partial U\,V^{(s)}_0) - 2\,(U\,\partial V^{(s)}_0)\Big]\,.\end{aligned} \tag{4.25}$$

By construction, these component fields then have regular OPEs with the free fermions and the $\mathfrak{u}(1)$ field, as one may also check directly,

$$U(z)\,\widetilde{V}^{(s)}(w) \sim 0\,, \qquad Q^a(z)\,\widetilde{V}^{(s)}(w) \sim 0\,. \tag{4.26}$$

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4.4 Comparison of the structure constants

As a cross-check of our results we should now be able to reproduce the OPEs of the non-linear $\mathcal{W}_\infty$ algebra from those of the linear analysis. Up to the level to which we have determined the linear algebra[6] we have performed this analysis, and we have found perfect agreement, thus giving a highly non-trivial consistency check on our analysis. In order to illustrate the nature of the comparison, let us give two specific examples.

The simplest case is the fusion of the first multiplet $V^{(1)}$ with itself. Up to the level considered below, only the conformal block of the identity appears, and the first few cases are explicitly

$$V_0^{(1)}\,V_0^{(1)} \sim \frac{1}{z^2}\,\mathcal{O}_0 + \frac{1}{z}\,\mathcal{O}_1\,, \tag{4.27}$$

$$V_0^{(1)}\,V_{1/2}^{(1),a} \sim \frac{1}{z^2}\,\mathcal{O}_{1/2}^a + \frac{1}{z}\,\mathcal{O}_{3/2}^a\,, \tag{4.28}$$

$$V_{1/2}^{(1),a}\,V_{1/2}^{(1),b} \sim \frac{1}{z^3}\,\mathcal{O}_0^{ab} + \frac{1}{z^2}\,\mathcal{O}_1^{ab} + \frac{1}{z}\,\mathcal{O}_2^{ab}\,, \tag{4.29}$$

$$V_0^{(1)}\,V_1^{(1),\pm,i} \sim \frac{1}{z^2}\,\mathcal{O}_1^{\pm,i} + \frac{1}{z}\,\mathcal{O}_2^{\pm,i}\,, \tag{4.30}$$

where $\mathcal{O}_s$ is an operator of dimension s built with the components of A_γ. The solution of the Jacobi identities predicts that the operators on the right hand side are

$$\begin{aligned}
\mathcal{O}_0 &= n_1\,\mathbb{I}\,, \\
\mathcal{O}_1 &= 0\,, \\
\mathcal{O}_{1/2}^a &= 0\,, \\
\mathcal{O}_{3/2}^a &= z_1\,G^a + z_2\,(UQ^a) + z_3\,\alpha_{ab}^{+,i}(A^{+,i}Q^b) + z_4\,\alpha_{ab}^{-,i}(A^{-,i}Q^b) + z_5\,\epsilon_{abcd}(Q^bQ^cQ^d)\,, \\
\mathcal{O}_0^{ab} &= z_6\,\delta^{ab}\,\mathbb{I}\,, \\
\mathcal{O}_1^{ab} &= z_7\,\alpha_{ab}^{-,i}A^{-,i} + z_8\,\alpha_{ab}^{+,i}A^{+,i} + z_9\,(Q^aQ^b) + z_{10}\,\epsilon_{abcd}(Q^cQ^d)\,, \\
\mathcal{O}_2^{ab} &= z_{11}\,\delta^{ab}(A^{-,i}A^{-,i}) + z_{12}\,\alpha_{ac}^{-,i}\alpha_{cb}^{+,j}\,(A^{-,i}A^{+,j}) + z_{13}\,\delta^{ab}(A^{+,i}A^{+,i}) + z_{14}\,\alpha_{ab}^{-,i}\partial A^{-,i} \\
&\quad + z_{15}\,\alpha_{ac}^{-,i}(A^{-,i}Q^cQ^b) + z_{16}\,\alpha_{ac}^{-,i}\epsilon_{cbde}(A^{-,i}Q^dQ^e) + z_{17}\,\delta^{ab}\alpha_{cd}^{-,i}(A^{-,i}Q^cQ^d) \\
&\quad + z_{18}\,\alpha_{ac}^{+,i}(A^{+,i}Q^cQ^d) + z_{19}\,\alpha_{ac}^{+,i}\epsilon_{cbde}(A^{+,i}Q^dQ^e) + z_{20}\,\delta^{ab}\alpha_{cd}^{+,i}(A^{+,i}Q^cQ^d) \\
&\quad + z_{21}\,\alpha_{ab}^{+,i}\partial A^{+,i} + z_{22}\,(Q^a\partial Q^b) + z_{23}\,(\partial Q^aQ^b) + z_{24}\,\delta^{ab}(Q^c\partial Q^c) \\
&\quad + z_{25}\,\epsilon_{abcd}(Q^c\partial Q^d) + z_{26}\,\delta^{ab}T + z_{27}\,\delta^{ab}(UU)\,, \\
\mathcal{O}_1^{+,i} &= z_{28}\,A^{+,i} + z_{29}\,\alpha_{ab}^{+,i}(Q^aQ^b)\,, \\
\mathcal{O}_2^{+,i} &= z_{30}\,\epsilon^{ijk}\alpha_{cd}^{+,j}(A^{+,k}Q^cQ^d) + z_{31}\,\alpha_{ab}^{+,i}(\partial Q^aQ^b) + z_{32}\,\alpha_{ab}^{+,i}(Q^aG^b) + z_{33}\,\alpha_{ab}^{+,i}(UQ^aQ^b)\,, \\
\mathcal{O}_1^{-,i} &= z_{34}\,A^{-,i} + z_{35}\,\alpha_{ab}^{-,i}(Q^aQ^b)\,, \\
\mathcal{O}_2^{-,i} &= z_{36}\,\epsilon^{ijk}\alpha_{cd}^{-,j}(A^{-,k}Q^cQ^d) + z_{37}\,\alpha_{ab}^{-,i}(\partial Q^aQ^b) + z_{38}\,\alpha_{ab}^{-,i}(Q^aG^b) + z_{39}\,\alpha_{ab}^{-,i}(UQ^aQ^b)\,,
\end{aligned} \tag{4.31}$$

where the constants $z_1, \ldots, z_{39}$ are listed in appendix C. Upon redefining the currents of A_γ and the component fields as in (4.15)–(4.17) and (4.21)–(4.25), respectively, these OPEs

[6] Since the linear algebra contains more fields, it is harder to push the analysis to the same level as for the non-linear case.

take the same form as in the non-linear calculation, i.e., as in (3.12) and (3.14),

$$\widetilde{V}_0^{(1)}\, \widetilde{V}_0^{(1)} \sim \frac{1}{z^2}\, n_1\, \mathbb{I}\,, \tag{4.32}$$

$$\widetilde{V}_0^{(1)}\, \widetilde{V}_{1/2}^{(1),a} \sim \frac{1}{z}\, z_1\, \widetilde{G}^a\,, \tag{4.33}$$

$$\begin{aligned}\widetilde{V}_{1/2}^{(1),a}\, \widetilde{V}_{1/2}^{(1),b} \sim\ & \frac{1}{z^3}\, z_6\, \delta^{ab}\, \mathbb{I} + \frac{1}{z^2}\left(z_7\, \alpha_{ab}^{-,i} \widetilde{A}^{-,i} + z_8\, \alpha_{ab}^{+,i} \widetilde{A}^{+,i}\right) \\ & + \frac{1}{z}\Big[z_{11}\, \delta^{ab}(\widetilde{A}^{-,i}\, \widetilde{A}^{-,i}) + z_{12}\, \alpha_{ac}^{-,i}\alpha_{cb}^{+,j}\, (\widetilde{A}^{-,i}\, \widetilde{A}^{+,j}) + z_{13}\, \delta^{ab}(\widetilde{A}^{+,i}\, \widetilde{A}^{+,i}) \\ & + z_{14}\, \alpha_{ab}^{-,i} \partial \widetilde{A}^{-,i} + z_{21}\, \alpha_{ab}^{+,i} \partial \widetilde{A}^{+,i} + z_{26}\, \delta^{ab}\, \widetilde{T}\Big]\,, \end{aligned} \tag{4.34}$$

$$\widetilde{V}_0^{(1)}\, \widetilde{V}_1^{(1),+,i} \sim \frac{1}{z^2}\, z_{28}\, \widetilde{A}^{+,i}\,, \tag{4.35}$$

$$\widetilde{V}_0^{(1)}\, \widetilde{V}_1^{(1),+,i} \sim \frac{1}{z^2}\, z_{34}\, \widetilde{A}^{-,i}\,. \tag{4.36}$$

Indeed, comparing the relevant coefficients leads to

OPE	field	$\widetilde{A}_\gamma$	A_γ
$V_0^{(1)} \times V_{1/2}^{(1)}$	$\widetilde{G}^a$	$\frac{w_1}{n_1} = -\frac{\tilde{k}_+ + \tilde{k}_- + 2}{2\,\tilde{k}_+\,\tilde{k}_-}$	$\frac{z_1}{n_1} = -\frac{k_+ + k_-}{2\,(k_+ - 1)\,(k_- - 1)}$
$V_0^{(1)} \times V_1^{(1)}$	$\widetilde{A}^{+,i}$	$\frac{w_2}{n_1} = \frac{4}{\tilde{k}_+}$	$\frac{z_{28}}{n_1} = -\frac{4}{k_+ - 1}$
$V_{1/2}^{(1)} \times V_{1/2}^{(1)}$	$\widetilde{A}^{+,i}$	$\frac{w_{15}}{n_1} = -\frac{4}{\tilde{k}_+}$	$\frac{z_8}{n_1} = \frac{4}{k_+ - 1}$
$V_{1/2}^{(1)} \times V_{1/2}^{(1)}$	$(\widetilde{A}^{+,i}\, \widetilde{A}^{+,i})$	$\frac{w_{11}}{n_1} = -\frac{1}{\tilde{k}_+\,\tilde{k}_-}$	$\frac{z_{11}}{n_1} = -\frac{1}{(k_+ - 1)(k_- - 1)}$
$V_{1/2}^{(1)} \times V_{1/2}^{(1)}$	$\widetilde{T}$	$\frac{w_{10}}{n_1} = \frac{2(\tilde{k}_+ + \tilde{k}_- + 2)}{\tilde{k}_+\,\tilde{k}_-}$	$\frac{z_{26}}{n_1} = -\frac{k_+ + k_-}{(k_+ - 1)(k_- - 1)}$

(4.37)

Here $\tilde{k}_\pm$ are the levels of the non-linear realisation that are related to the levels $k_\pm$ of the linear realisation as in (4.20). The ratios (that are normalisation independent) match precisely once the various signs and factors of 2 (that are a consequence of the different conventions we have employed for the two calculations) have been taken into account. For example, the normalisation of the supercharges differs effectively by a factor of $\sqrt{2}i$, as follows by comparing eq. (2.4) to eq. (4.3).[7] This also leads to a similar rescaling of the $V_{1/2}^{(s)}$ components of the multiplets. Furthermore, for example the coefficient z_8 in (4.34) multiplies a matrix, which differs be a normalisation factor from the corresponding matrix in (3.14) that is multiplied by w_{15}. Taking all of these factors carefully into account, the two calculations match exactly.

Another example comes from the OPE of the first and second multiplet. Up to the level we considered only the $V^{(1)}$ multiplet appears in the OPE, and for example, the Jacobi identities of the linear $\mathcal{W}_\infty$ algebra predict that we have

$$\widetilde{V}_{1/2}^{(1),a}\, \widetilde{V}_0^{(2)} \sim \frac{1}{z}\, \Phi_{5/2}^a\,, \tag{4.38}$$

[7] In addition, there is a change of basis since we have used a $\mathfrak{su}(2)$-bispinor notation in the non-linear analysis, while for the linear analysis we have worked with $\mathfrak{so}(4)$ vectors.

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where $\Phi^a_{5/2}$ is an operator of conformal dimension 5/2, transforming in the $(\frac{1}{2},\frac{1}{2})$ representation of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$. Its explicit form turns out to be

$$\begin{aligned}\Phi^a_{5/2} =\; & w_1\, V^{(1),a}_{3/2} + w_2\,\partial V^{(1),a}_{1/2} + w_3\,(\partial Q^a V^{(1)}_0) + w_4\,(Q^a\partial V^{(1)}_0)\\ & + w_5\,(G^a V^{(1)}_0) + w_6\,(U V^{(1),a}_{1/2}) + w_7\,(U Q^a V^{(1)}_0)\\ & + w_8\,\alpha^{+,i}_{ab}\,(A^{+,i}V^{(1),b}_{1/2}) + w_9\,\alpha^{-,i}_{ab}\,(A^{-,i}V^{(1),b}_{1/2}) +\\ & + w_{10}\,\alpha^{+,i}_{ab}\,(A^{+,i}Q^b V^{(1)}_0) + w_{11}\,\alpha^{-,i}_{ab}\,(A^{-,i}Q^b V^{(1)}_0)\\ & + w_{12}\,\alpha^{+,i}_{ab}\,(Q^b V^{(1),+,i}_1) + w_{13}\,\alpha^{-,i}_{ab}\,(Q^b V^{(1),-,i}_1) + w_{14}\epsilon_{abcd}(Q^bQ^cQ^dV^{(1)}_0)\\ & + w_{15}\,\alpha^{+,i}_{ab}\,\alpha^{+,i}_{cd}(Q^cQ^dV^{(1),b}_{1/2}) + w_{16}\,\alpha^{-,i}_{ab}\,\alpha^{-,i}_{cd}(Q^cQ^dV^{(1),b}_{1/2})\,,\end{aligned}\tag{4.39}$$

where the values of the coefficients are given explicitly in appendix D. In terms of the decoupled fields we can write $\Phi^a_{5/2}$ as

$$\begin{aligned}\Phi^a_{5/2} =\; & w'_1\,\widetilde{V}^{(1),a}_{3/2} + w'_2\,\partial\widetilde{V}^{(1),a}_{1/2} + w'_3\,(\widetilde{G}^a\,\widetilde{V}^{(1)}_0)\\ & + w'_4\,\alpha^{+,i}_{ab}\,(\widetilde{A}^{+,i}\,\widetilde{V}^{(1),b}_{1/2}) + w'_5\,\alpha^{-,i}_{ab}\,(\widetilde{A}^{-,i}\,\widetilde{V}^{(1),b}_{1/2})\,,\end{aligned}\tag{4.40}$$

where

$$w'_1 = w_1,\quad w'_2 = w_2,\quad w'_3 = w_5,\quad w'_4 = w_8 + \frac{8\,w_1}{k_+ + k_-},\quad w'_5 = w_9 - \frac{8\,w_1}{k_+ + k_-}\,.\tag{4.41}$$

In fact, this expression is (for generic coefficients w') the most general solution of the decoupling conditions

$$U(z)\,\Phi^a_{5/2}(w)\sim 0\,,\qquad Q^a(z)\,\Phi^a_{5/2}(w)\sim 0\,.\tag{4.42}$$

We can finally bring it into the same form as the corresponding formula in (3.15) using the $[\cdots]$ bracket

$$\Phi^a_{5/2} = w'_1\,\widetilde{V}^{(1),a}_{3/2} + w'_3\,\Big[\widetilde{V}^{(1)}_0\,\widetilde{G}^a\Big] + w'_4\,\alpha^{+,i}_{ab}\,\Big[\widetilde{A}^{+,i}\,\widetilde{V}^{(1),b}_{1/2}\Big] + w'_5\,\alpha^{-,i}_{ab}\,\Big[\widetilde{A}^{-,i}\,\widetilde{V}^{(1),b}_{1/2}\Big]\,.\tag{4.43}$$

As regards the structure constants, it only makes sense to compare ratios since the normalisation of $V^{(2)}_0$ is arbitrary. For example, the coefficient of $\Big[\widetilde{V}^{(1)}_0\,\widetilde{G}^a\Big]$ relative to $\widetilde{V}^{(1),a}_{3/2}$ equals (see appendix D)

$$\begin{aligned}\frac{w'_3}{w'_1} &= -\frac{16(2\gamma-1)(c(c+6)+18(\gamma-1)\gamma)}{36(c+2)\gamma^2-36(c+2)\gamma+c(24-(c-4)c)}\\ &= \frac{4\,(k_- - k_+)\,(2k_+k_- + 2k_- + 2k_+ - 1)}{3k_+^2k_-^2 - 2k_+k_-^2 - 2k_-^2 - 2k_+^2k_- - k_+k_- + k_- - 2k_+^2 + k_+}\,.\end{aligned}\tag{4.44}$$

This must be compared with the analogous quantity in the non-linear computation which is

$$\frac{w_{50}}{w_{48}} = \frac{4(\tilde{k}_- - \tilde{k}_+)(2\tilde{k}_+\tilde{k}_- + 4\tilde{k}_- + 4\tilde{k}_+ + 5)}{3\tilde{k}_+^2\tilde{k}_-^2 + 4\tilde{k}_+\tilde{k}_-^2 - \tilde{k}_-^2 + 4\tilde{k}_+^2\tilde{k}_- + 3\tilde{k}_+\tilde{k}_- - 4\tilde{k}_- - \tilde{k}_+^2 - 4\tilde{k}_+ - 4}\,,\tag{4.45}$$

and one checks that they agree precisely, using again the relation between the levels (4.20). Similarly, the coefficient of of $\left[\widetilde{A}^{+,i}\,\widetilde{V}^{(1),b}_{1/2}\right]$ relative to $\widetilde{V}^{(1),a}_{3/2}$ reads

$$\begin{aligned}\frac{w_4'}{w_1'} &= -\frac{64\gamma(c(\gamma-2)-6(\gamma-1)\gamma)((c-6)c-18(\gamma-1)(2\gamma-1))}{c\,(-36(c+2)\gamma^2+36(c+2)\gamma+c((c-4)c-24))}\\ &= \frac{16\,(k_-+2k_+-1)\left(2k_+k_-^2-2k_-^2-2k_+k_-+k_--k_+\right)}{(k_-+k_+)\left(3k_+^2k_-^2-2k_+k_-^2-2k_-^2-2k_+^2k_--k_+k_-+k_--2k_+^2+k_+\right)}\,, \end{aligned} \tag{4.46}$$

and this matches precisely, using eq. (4.20), the analogous ratio in the non-linear computation (again there is a relative factor of -2 because of the different conventions that were used, see the comment below (4.37))

$$-2\frac{w_{52}}{w_{48}} = \frac{16(\tilde{k}_-+2\tilde{k}_++2)(2\tilde{k}_+\tilde{k}_-^2+2\tilde{k}_+\tilde{k}_--\tilde{k}_--\tilde{k}_+-2)}{(\tilde{k}_-+\tilde{k}_++2)(3\tilde{k}_+^2\tilde{k}_-^2+4\tilde{k}_+\tilde{k}_-^2-\tilde{k}_-^2+4\tilde{k}_+^2\tilde{k}_-+3\tilde{k}_+\tilde{k}_--4\tilde{k}_--\tilde{k}_+^2-4\tilde{k}_+-4)}\,. \tag{4.47}$$

The analysis for the coefficient of $\left[\widetilde{A}^{-,i}\,\widetilde{V}^{(1),b}_{1/2}\right]$ is the same since it can be obtained from (4.46) upon exchanging $k_\pm \to k_\mp$.

These comparisons give rise to pretty non-trivial consistency checks of our analysis, and it is very satisfying that they work out precisely. In summary, the results of this and the previous section therefore give strong indications that the $\mathcal{N}=4$ superconformal $\mathcal{W}_\infty$ algebra consisting of the large $\mathcal{N}=4$ superconformal algebra as well as one $\mathcal{N}=4$ supermultiplet for each integer spin, are uniquely characterised in terms of the levels of the two $\mathfrak{su}(2)$ algebras. This statement applies both to the linear as well as the non-linear version of the algebra.

5 Conclusions

In this paper we have studied the structure of the most general large $\mathcal{N}=4$ superconformal $\mathcal{W}_\infty$ algebra that contains, in addition to the superconformal algebra, one $\mathcal{N}=4$ multiplet for each integer spin $s=1,2,\ldots$. We have found strong evidence in favour of the claim that this family of algebras is uniquely characterised in terms of the levels of the two $\mathfrak{su}(2)$ algebras (that are a part of the large $\mathcal{N}=4$ superconformal algebra). Among other things, this shows that the Wolf space cosets account essentially for all such $\mathcal{W}_\infty$ algebras. While this is natural from the perspective of these cosets, it is a little surprising that the complete structure of the algebra is essentially fixed by the large $\mathcal{N}=4$ algebra itself — this is to be compared with, say, the bosonic situation where the free parameter corresponding to λ encodes how the different (conformal) multiplets couple to one another.

Another consequence of this analysis is that the quantisation of the higher spin theory is essentially unique. Indeed, both levels $k_\pm$ can be identified with parameters of the (classical) higher spin theory,

$$\lambda = \frac{k_-}{k_++k_-+2}\,, \quad \text{and} \quad c=\frac{6k_+k_-}{k_++k_-+2}\,, \tag{5.1}$$

where λ is the parameter that appears in the underlying higher spin algebra $\mathfrak{shs}_2[\lambda]$, while c is the central charge that is determined in terms of the size of the AdS space. Note that our result is compatible with the explicit analysis of [17] where the asymptotic symmetry algebra of the higher spin theory was matched with the 't Hooft limit of the Wolf space cosets — both are the 't Hooft limit of a unique quantum $\mathcal{W}_\infty$ algebra, and hence must agree.

In the limit where one of the levels of the two $\mathfrak{su}(2)$ algebras goes to infinity, the large $\mathcal{N}=4$ superconformal algebra can be truncated to the small $\mathcal{N}=4$ superconformal algebra. Thus our analysis predicts that there is at least one family of $\mathcal{W}_\infty$ algebras with small $\mathcal{N}=4$ superconformal algebra that are labelled by the level of the surviving $\mathfrak{su}(2)$ algebra (or equivalently by the central charge). It would be interesting to see whether this accounts for all small $\mathcal{N}=4$ algebras with this multiplet spectrum, or whether there are additional constructions that cannot be obtained as a limit of a large $\mathcal{N}=4$ superconformal $\mathcal{W}$ algebra. In particular, one may expect that the $\mathcal{W}$ algebra that is relevant for string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{K3}$ should not appear in this fashion.

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A The structure of the supermultiplet

In this appendix we specify our conventions for the OPEs of the superconformal generators with the various component fields of the $\mathcal{N}=4$ supermultiplet.

A.1 The non-linear case

For the case of the non-linear $\tilde{A}_\gamma$ algebra, the OPEs of the stress-energy tensor and the affine currents were given already in eqs. (2.10) and (2.11). Our ansatz for the OPEs of the supercharges with the component fields of the $\mathcal{N}=4$ supermultiplet is

$$\begin{aligned}
G^{\alpha\beta}(z)V_0^{(s)}(w) &\sim \frac{V_{1/2}^{(s)\alpha\beta}(w)}{z-w}\,,\\
G^{\alpha\beta}(z)V_{1/2}^{(s)\gamma\delta}(w) &\sim \frac{g_{1/2,1}\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}V_0^{(s)}(w)}{(z-w)^2} + \frac{1}{z-w}\Big[g_{1/2,2}\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\partial V_0^{(s)}(w) + \\
&\quad + \epsilon_{\beta\delta}\ell_{i,\alpha\gamma}V_1^{(s)+i}(w) + \epsilon_{\alpha\gamma}\ell_{i,\beta\delta}V_1^{(s)-i}(w)\Big]\,,\\
G^{\alpha\beta}(z)V_1^{(s)+i}(w) &\sim \frac{g^+_{1,1}\rho^i_{\gamma\alpha}V_{1/2}^{(s)\gamma\beta}(w)}{(z-w)^2} + \frac{1}{z-w}\Big\{\rho^i_{\gamma\alpha}\big[g^+_{1,2}\partial V_{1/2}^{(s)\gamma\beta}(w) + \\
&\quad + g^+_{1,3}\ell_{j,\delta\beta}(A^{-j}V_{1/2}^{(s)\gamma\delta})(w) + V_{3/2}^{(s)\gamma\beta}(w)\big] + g^+_{1,4}(A^{+i}V_{1/2}^{(s)\alpha\beta})(w)\Big\}\,,\\
G^{\alpha\beta}(z)V_1^{(s)-i}(w) &\sim \frac{g^-_{1,1}\rho^i_{\gamma\beta}V_{1/2}^{(s)\alpha\gamma}(w)}{(z-w)^2} + \frac{1}{z-w}\Big\{\rho^i_{\gamma\beta}\big[g^-_{1,2}\partial V_{1/2}^{(s)\alpha\gamma}(w) + \\
&\quad + g^-_{1,3}\ell_{j,\delta\alpha}(A^{+j}V_{1/2}^{(s)\delta\gamma})(w) + g_{1,5}V_{3/2}^{(s)\alpha\gamma}(w)\big] + g^-_{1,4}(A^{-i}V_{1/2}^{(s)\alpha\beta})(w)\Big\}\,,
\end{aligned}$$

$$
\begin{aligned}
G^{\alpha\beta}(z)V_{3/2}^{(s)\gamma\delta}(w) \sim{}& \frac{g_{3/2,1}\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}V_0^{(s)}(w)}{(z-w)^3} + \frac{1}{(z-w)^2}\Big[\epsilon_{\beta\delta}\ell_{i,\alpha\gamma}g^+_{3/2,2}V_1^{(s)+i}(w) + \\
&+ \epsilon_{\alpha\gamma}\ell_{i,\beta\delta}g^-_{3/2,2}V_1^{(s)-i}(w)\Big] + \frac{1}{z-w}\Big\{\epsilon_{\beta\delta}\ell_{i,\alpha\gamma}\big[g^+_{3/2,3}(\partial A^{+i}V_0^{(s)})(w) + \\
&+ g^+_{3/2,4}(A^{+i}\partial V_0^{(s)})(w) + g^+_{3/2,5}\partial V_1^{(s)+i}(w)\big] + \\
&+ \epsilon_{\alpha\gamma}\ell_{i,\beta\delta}\big[g^-_{3/2,3}(\partial A^{-i}V_0^{(s)})(w) + g^-_{3/2,4}(A^{-i}\partial V_0^{(s)})(w) + \\
&+ g^-_{3/2,5}\partial V_1^{(s)-i}(w)\big] + f^i{}_{jl}\big[g^+_{3/2,6}\epsilon_{\beta\delta}\ell_{i,\alpha\gamma}(A^{+j}V_1^{(s)+l})(w) + \\
&+ g^-_{3/2,6}\epsilon_{\alpha\gamma}\ell_{i,\beta\delta}(A^{-j}V_1^{(s)-l})(w)\big] + \epsilon_{\alpha\beta}\epsilon_{\gamma\delta}V_2^{(s)}(w) + \\
&+ \big(g^+_{3/2,7}\epsilon_{\beta\delta}\epsilon_{\nu\sigma}\ell_{i,\alpha\gamma}r^i_{\mu\rho} + g^-_{3/2,7}\epsilon_{\alpha\gamma}\epsilon_{\mu\rho}\ell_{i,\beta\delta}r^i_{\nu\sigma}\big)(G^{\mu\nu}V_{1/2}^{(s)\rho\sigma})(w)\Big\}\,, \\
G^{\alpha\beta}(z)V_2^{(s)}(w) \sim{}& \frac{g_{2,1}V_{1/2}^{(s)\alpha\beta}(w)}{(z-w)^3} + \frac{1}{(z-w)^2}\Big[g_{2,2}V_{3/2}^{(s)\alpha\beta}(w) + g^+_{2,3}\rho_{i,\gamma\alpha}(A^{+i}V_{1/2}^{(s)\gamma\beta})(w) + \\
&+ g^-_{2,3}\rho_{i,\gamma\beta}(A^{-i}V_{1/2}^{(s)\alpha\gamma})(w)\Big] + \frac{1}{z-w}\Big[g_{2,4}\partial V_{3/2}^{(s)\alpha\beta}(w) + \\
&+ \rho_{i,\gamma\alpha}g^+_{2,5}(\partial A^{+i}V_{1/2}^{(s)\gamma\beta})(w) + g^-_{2,5}(\partial A^{-i}V_{1/2}^{(s)\alpha\gamma})(w)\Big]\,.
\end{aligned}
$$

Here (AB) denotes the minimal normal ordering of 2 operators A and B, i.e. the regular term in the OPE between A and B, and the matrices r^i are defined via

$$
r^i_{\alpha\beta} = \rho^i_{\alpha\gamma}\epsilon_{\gamma\beta}\,, \tag{A.1}
$$

i.e., r^i it is the matrix ρ^i with one of the (α,β) indices raised. With this ansatz, the Jacobi identities with the $\mathcal{N}=4$ superconformal algebra fix the undetermined structure constants uniquely, and we find in addition to eqs. (2.12) and (2.13) the values

$$
\begin{aligned}
&g_{1/2,1} = -4s\,, \qquad g_{1/2,2} = -2\,, \qquad g^\pm_{1,1} = -\frac{4[-1+k_\pm+s(2+k_++k_-)]}{2+k_++k_-}\,, \\
&g^\pm_{1,2} = -\frac{4[1+k_\pm+s(2+k_++k_-)]}{(1+2s)(2+k_++k_-)}\,, \quad g^\pm_{1,3} = -2g^\pm_{1,4} = \frac{8}{2+k_++k_-}\,, \quad g_{1,5} = -1\,, \\
&g_{3/2,1} = -\frac{32s(1+s)(k_+-k_-)}{(1+2s)(2+k_++k_-)}\,, \qquad g^\pm_{3/2,2} = \mp\frac{8(1+s)(1+k_\mp+s(2+k_++k_-)]}{(1+2s)(2+k_++k_-)}\,, \\
&g^\pm_{3/2,3} = \frac{\pm 16s}{2+k_++k_-}\,, \quad g^\pm_{3/2,4} = \frac{\mp 16}{2+k_++k_-}\,, \quad g^\pm_{3/2,5} = \mp\frac{4[1+k_\mp+s(2+k_++k_-)]}{(1+2s)(2+k_++k_-)}\,, \\
&g^\pm_{3/2,6} = -g^\pm_{3/2,7} = \frac{\mp 4}{2+k_++k_-}\,, \quad g_{2,1} = \frac{32s(1+s)(k_+-k_-)}{(1+2s)(2+k_++k_-)}\,, \quad g_{2,2} = -2(3+2s)\,, \\
&g^\pm_{2,3} = \pm\frac{32(1+s)}{2+k_++k_-}\,, \qquad g_{2,4} = -2\,, \quad g^\pm_{2,5} = \pm\frac{16(1+s)}{2+k_++k_-}\,.
\end{aligned} \tag{A.2}
$$

A.2 The linear case

In this subsection we explain our conventions for the OPEs of the fields of the linear A_γ algebra with the component fields of the supermultiplet. For the component fields we use

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the conventions explained in eq. (4.14).

$$
\begin{aligned}
A_\gamma \times V_0^{(s)}: \quad & U\, V_0^{(s)} \sim 0\,, \quad Q^a\, V_0^{(s)} \sim 0\,, \quad A^{\pm,i}\, V_0^{(s)} \sim 0\,, \quad G^a\, V_0^{(s)} \sim \frac{1}{z}\, V_{1/2}^{(s),a}\,, \\
A_\gamma \times V_{1/2}^{(s),a}: \quad & U\, V_{1/2}^{(s),a} \sim 0\,, \quad Q^a\, V_{1/2}^{(s),b} \sim 0\,, \quad A^{\pm,i}\, V_{1/2}^{(s),a} \sim \frac{1}{z}\, \alpha^{\pm,i}_{ab}\, V_{1/2}^{(s),b}\,, \\
& G^a\, V_{1/2}^{(s),b} \sim \frac{2s\,\delta^{ab}}{z^2}\, V_0^{(s)} + \frac{1}{z}\left(\alpha^{+,i}_{ab}\, V_1^{(s),+,i} + \alpha^{-,i}_{ab}\, V_1^{(s),-,i} + \delta^{ab}\,\partial V_0^{(s)}\right)\,, \\
A_\gamma \times V_1^{(s),\pm,i}: \quad & U\, V_1^{(s),\pm,i} \sim 0\,, \quad Q^a\, V_1^{(s),\pm,i} \sim \pm\frac{2}{z}\,\alpha^{\pm,i}_{ab}\, V_{1/2}^{(s),a}\,, \\
& A^{\pm,i}\, V_1^{(s),\pm,j} \sim \frac{2s}{z^2}\,\delta^{ij}\, V_0^{(s)} + \frac{1}{z}\,\epsilon^{ijk}\, V_1^{(s),\pm,k}\,, \quad A^{\pm,i}\, V_1^{(s),\mp,j} \sim 0\,, \\
& G^a\, V_1^{(s),\pm,i} \sim 4(s+\gamma_\mp)\left(\frac{1}{z^2} + \frac{1}{z(2s+1)}\,\partial\right)\alpha^{\pm,i}_{ab}\, V_{1/2}^{(s),b} \mp \frac{1}{z}\,\alpha^{\pm,i}_{ab} V_{3/2}^{(s),b}\,, \\
A_\gamma \times V_{3/2}^{(s),a}: \quad & U\, V_{3/2}^{(s),a} \sim -\frac{1}{z^2}\, V_{1/2}^{(s),a}\,, \\
& Q^a\, V_{3/2}^{(s),b} \sim \frac{4s\delta^{ab}}{z^2}\, V_0^{(s)} + \frac{2}{z}\,\left(\alpha^{+,i}_{ab}\, V_1^{(s),+,i} + \alpha^{-,i}_{ab}\, V_1^{(s),-,i} - \delta^{ab}\,\partial V_0^{(s)}\right)\,, \\
& A^{\pm,i}\, V_{3/2}^{(s),a} \sim \pm\frac{8s(s+1)+\gamma_\mp}{z^2(2s+1)}\alpha^{\pm,i}_{ab}\, V_{1/2}^{(s),b} + \frac{1}{z}\,\alpha^{\pm,i}_{ab}\, V_{3/2}^{(s),b}\,, \\
& G^a\, V_{3/2}^{(s),b} \sim -\frac{16s(s{+}1)(2\gamma{-}1)}{z^3(2s+1)}\delta^{ab} V_0^{(s)} - \frac{8(s{+}1)}{(2s{+}1)}\left(\frac{1}{z^2} + \frac{1}{2(s{+}1)z}\partial\right) \times \\
& \qquad \times \left[(s+\gamma_+)\alpha^{+,i}_{ab}\, V_1^{(s),+,i} - (s+\gamma_-)\alpha^{-,i}_{ab}\, V_1^{(s),-,i}\right]\,. \\
A_\gamma \times V_2^{(s)}: \quad & U\, V_2^{(s)} \sim \frac{8\,s}{z^3}\, V_0^{(s)} - \frac{4}{z^2}\,\partial V_0^{(s)}\,, \\
& Q^a\, V_2^{(s)} \sim -\frac{2(2s+1)}{z^2}\, V_{1/2}^{(s),a} + \frac{2}{z}\,\partial V_{1/2}^{(s),a}\,, \\
& A^{\pm,i}\, V_2^{(s)} \sim \frac{\pm 2(s+1)}{z^2}\, V_1^{(s),\pm,i}\,, \\
& G^a\, V_2^{(s)} \sim \frac{16(2\gamma-1)\,s(s+1)}{z^3(2s+1)}\, V_{1/2}^{(s),a} + \frac{2s+3}{z^2}\, V_{3/2}^{(s),a} + \frac{1}{z}\,\partial V_{3/2}^{(s),a}\,, \\
& T\, V_2^{(s)} \sim -\frac{24(2\gamma-1)s(s+1)}{z^4(2s+1)}\, V_0^{(s)} + \frac{s+2}{z^2}\, V_2^{(s)} + \frac{1}{z}\,\partial V_2^{(s)}\,.
\end{aligned}
$$

Here $\gamma_+ = \gamma$ and $\gamma_- = \bar\gamma = 1-\gamma$. Only the component field $V_2^{(s)}$ is quasi-primary (but not primary).

B The spin 4 OPEs and the structure constants

B.1 The OPEs

The general ansatz for the OPEs of total spin 4 is

$$
\begin{aligned}
V_0^{(1)} \times V_0^{(3)} \sim\; & w_{80} T + w_{81} V_0^{(2)} + w_{82}[V_0^{(1)} V_0^{(1)}] + w_{83}[A^{+i} A^{+}{}_i] + w_{84}[A^{-i} A^{-}{}_i] + \\
& + w_{85} V_0^{(3)} + w_{86}[V_0^{(1)} V_0^{(2)}] + w_{87} V_2^{(1)} + w_{88}[V_0^{(1)}[V_0^{(1)} V_0^{(1)}]] +
\end{aligned}
$$

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$$
\begin{aligned}
&\quad + w_{89}[TV_0^{(1)}] + \epsilon_{\alpha\gamma}\epsilon_{\beta\delta} w_{90}[G^{\alpha\beta}V_{1/2}^{(1)\gamma\delta}] + w_{91}[A^{+}{}_{i}V_1^{(1)+i}] + \\
&\quad + w_{92}[A^{-}{}_{i}V_1^{(1)-i}] + w_{93}[[A^{+}{}_{i}A^{+i}]V_0^{(1)}] + w_{94}[[A^{-}{}_{i}A^{-i}]V_0^{(1)}]\,, \\
V_0^{(1)} \times V_2^{(1)} &\sim w_{95}T + \cdots + w_{108}[[A^{+}{}_{i}A^{+i}]V_0^{(1)}] + w_{109}[[A^{-}{}_{i}A^{-i}]V_0^{(1)}]\,, \\
V_0^{(1)} \times V_1^{(2)+i} &\sim w_{110}A^{+i} + w_{111}V_1^{(1)+i} + w_{112}[A^{+i}V_0^{(1)}] + w_{113}V_1^{(2)+i} + \\
&\quad + w_{114}[V_0^{(1)}V_1^{(1)+i}] + w_{115}[A^{+i}V_0^{(2)}] + w_{116}[A^{+i}V_0^{(1)}]_{-1} + \\
&\quad + w_{117}[A^{+i}[V_0^{(1)}V_0^{(1)}]] + w_{118}[TA^{+i}] + \epsilon_{\beta\delta} r^{i}_{\alpha\gamma}\big(w_{119}[V_{1/2}^{(1)\alpha\beta}V_{1/2}^{(1)\gamma\delta}] + \\
&\quad + w_{120}[G^{\alpha\beta}V_{1/2}^{(1)\gamma\delta}] + w_{121}[G^{\alpha\beta}G^{\gamma\delta}]\big) + f^{i}{}_{jl}\big(w_{122}[A^{+j}V_1^{(1)+l}] + \\
&\quad + w_{123}[A^{+j}A^{+l}]_{-1}\big) + w_{124}[[A^{+}{}_{j}A^{+j}]A^{+i}] + w_{125}[[A^{-}{}_{j}A^{-j}]A^{+i}]\,, \\
V_1^{(1)+i} \times V_0^{(2)} &\sim w_{126}A^{+i} + \cdots + w_{140}[[A^{+}{}_{j}A^{+j}]A^{+i}] + w_{141}[[A^{-}{}_{j}A^{-j}]A^{+i}]\,, \\
V_0^{(1)} \times V_1^{(2)-i} &\sim w_{142}A^{-i} + w_{143}V_1^{(1)-i} + w_{144}[A^{-i}V_0^{(1)}] + w_{145}V_1^{(2)-i} + \\
&\quad + w_{146}[V_0^{(1)}V_1^{(1)-i}] + w_{147}[A^{-i}V_0^{(2)}] + w_{148}[A^{-i}V_0^{(1)}]_{-1} + \\
&\quad + w_{149}[A^{-i}[V_0^{(1)}V_0^{(1)}]] + w_{150}[TA^{-i}] + \epsilon_{\alpha\gamma} r^{i}_{\beta\delta}\big(w_{151}[V_{1/2}^{(1)\alpha\beta}V_{1/2}^{(1)\gamma\delta}] + \\
&\quad + w_{152}[G^{\alpha\beta}V_{1/2}^{(1)\gamma\delta}] + w_{153}[G^{\alpha\beta}G^{\gamma\delta}]\big) + f^{i}{}_{jl}\big(w_{154}[A^{-j}V_1^{(1)-l}] + \\
&\quad + w_{155}[A^{-j}A^{-l}]_{-1}\big) + w_{156}[[A^{-}{}_{j}A^{-j}]A^{-i}] + w_{157}[[A^{+}{}_{j}A^{+j}]A^{-i}]\,, \\
V_1^{(1)-i} \times V_0^{(2)} &\sim w_{158}A^{-i} + \cdots + w_{172}[[A^{-}{}_{j}A^{-j}]A^{-i}] + w_{173}[[A^{+}{}_{j}A^{+j}]A^{-i}]\,, \\
V_1^{(1)+i} \times V_1^{(1)+j} &\sim \eta^{ij}\big(w_{174}I + w_{175}T + w_{176}V_0^{(2)} + w_{177}[V_0^{(1)}V_0^{(1)}] + w_{178}[A^{+l}A^{+}{}_{l}] + \\
&\quad + w_{179}[A^{-l}A^{-}{}_{l}]\big) + w_{180}[A^{+i}A^{+j}] + f^{ij}{}_{l}\big\{w_{181}A^{+l} + w_{182}V_1^{(2)+l} + \\
&\quad + w_{183}[V_0^{(1)}V_1^{(1)+l}] + w_{184}[A^{+l}V_0^{(2)}] + w_{185}[A^{+l}V_0^{(1)}]_{-1} + \\
&\quad + w_{186}[A^{+l}[V_0^{(1)}V_0^{(1)}]] + w_{187}[TA^{+l}] + \epsilon_{\beta\delta} r^{l}_{\alpha\gamma}\big(w_{188}[V_{1/2}^{(1)\alpha\beta}V_{1/2}^{(1)\gamma\delta}] + \\
&\quad + w_{189}[G^{\alpha\beta}V_{1/2}^{(1)\gamma\delta}] + w_{190}[G^{\alpha\beta}G^{\gamma\delta}]\big) + f^{l}{}_{pq}\big(w_{191}[A^{+p}V_1^{(1)+q}] + \\
&\quad + w_{192}[A^{+p}A^{+q}]_{-1}\big) + w_{193}[[A^{+}{}_{p}A^{+p}]A^{+l}] + w_{194}[[A^{-}{}_{p}A^{-p}]A^{-l}]\big\}\,, \\
V_1^{(1)-i} \times V_1^{(1)-j} &\sim \eta^{ij}\big(w_{195}I + w_{196}T + w_{197}V_0^{(2)} + w_{198}[V_0^{(1)}V_0^{(1)}] + w_{199}[A^{+l}A^{+}{}_{l}] + \\
&\quad + w_{200}[A^{-l}A^{-}{}_{l}]\big) + w_{201}[A^{-i}A^{-j}] + f^{ij}{}_{l}\big\{w_{202}A^{-l} + w_{203}V_1^{(2)-l} + \\
&\quad + w_{204}[V_0^{(1)}V_1^{(1)-l}] + w_{205}[A^{-l}V_0^{(2)}] + w_{206}[A^{-l}V_0^{(1)}]_{-1} + \\
&\quad + w_{207}[A^{-l}[V_0^{(1)}V_0^{(1)}]] + w_{208}[TA^{-l}] + \epsilon_{\alpha\gamma} r^{l}_{\beta\delta}\big(w_{209}[V_{1/2}^{(1)\alpha\beta}V_{1/2}^{(1)\gamma\delta}] + \\
&\quad + w_{210}[G^{\alpha\beta}V_{1/2}^{(1)\gamma\delta}] + w_{211}[G^{\alpha\beta}G^{\gamma\delta}]\big) + f^{l}{}_{pq}\big(w_{212}[A^{-p}V_1^{(1)-q}] + \\
&\quad + w_{213}[A^{-p}A^{-q}]_{-1}\big) + w_{214}[[A^{-}{}_{p}A^{-p}]A^{-l}] + w_{215}[[A^{+}{}_{p}A^{+p}]A^{-l}]\big\}\,, \\
V_1^{(1)+i} \times V_1^{(1)-j} &\sim w_{216}[A^{+i}A^{-j}] + r^{i}_{\alpha\gamma}r^{j}_{\beta\delta} w_{217}[G^{\alpha\beta}V_{1/2}^{(1)\gamma\delta}] + w_{218}[A^{+i}V^{(1)-j}] + \\
&\quad + w_{219}[A^{-j}V^{(1)+i}] + w_{220}[A^{+i}[A^{-j}V_0^{(1)}]] + w_{221}[A^{+i}A^{-j}]_{-1}\,, \\
V_{1/2}^{(1)\alpha\beta} \times V_{3/2}^{(1)\gamma\delta} &\sim \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\big\{w_{222}V_0^{(1)} + w_{223}T + w_{224}V_0^{(2)} + w_{225}[V_0^{(1)}V_0^{(1)}] + \\
&\quad + w_{226}[A^{+i}A^{+}{}_{i}] + w_{227}[A^{-i}A^{-}{}_{i}] + w_{228}V_0^{(3)} + w_{229}[V_0^{(1)}V_0^{(2)}] +
\end{aligned}
$$

$$
\begin{aligned}
&+w_{230}V_2^{(1)}+w_{231}[V_0^{(1)}[V_0^{(1)}V_0^{(1)}]]+w_{232}[TV_0^{(1)}]+\\
&+\epsilon_{\rho\mu}\epsilon_{\sigma\nu}w_{233}[G^{\rho\sigma}V_{1/2}^{(1)\mu\nu}]+w_{234}[A^+{}_iV^{(1)+i}]+w_{235}[A^-{}_iV^{(1)-i}]+\\
&+w_{236}[[A^+{}_iA^{+i}]V_0^{(1)}]+w_{237}[[A^-{}_iA^{-i}]V_0^{(1)}]\}+\epsilon_{\beta\delta}\ell_{i,\alpha\gamma}\{w_{238}A^{+i}+\\
&+w_{239}V_1^{(1)+i}+w_{240}[A^{+i}V_0^{(1)}]+w_{241}V_1^{(2)+i}+w_{242}[V_0^{(1)}V_1^{(1)+i}]+\\
&+w_{243}[A^{+i}V_0^{(2)}]+w_{244}[A^{+i}V_0^{(1)}]_{-1}+w_{245}[A^{+i}[V_0^{(1)}V_0^{(1)}]]+\\
&+w_{246}[TA^{+i}]+\epsilon_{\sigma\nu}r^i_{\rho\mu}\big(w_{247}[V_{1/2}^{(1)\rho\sigma}V_{1/2}^{(1)\mu\nu}]+w_{248}[G^{\rho\sigma}V_{1/2}^{(1)\mu\nu}]+\\
&+w_{249}[G^{\rho\sigma}G^{\mu\nu}]\big)+f^i{}_{jl}\big(w_{250}[A^{+j}V_1^{(1)+l}]+w_{251}[A^{+j}A^{+l}]_{-1}\big)+\\
&+w_{252}[[A^+{}_jA^{+j}]A^{+i}]+w_{253}[[A^-{}_jA^{-j}]A^{+i}]\}+\epsilon_{\alpha\gamma}\ell_{i,\beta\delta}\{w_{254}A^{-i}+\\
&+w_{255}V_1^{(1)-i}+w_{256}[A^{-i}V_0^{(1)}]+w_{257}V_1^{(2)-i}+w_{258}[V_0^{(1)}V_1^{(1)-i}]+\\
&+w_{259}[A^{-i}V_0^{(2)}]+w_{260}[A^{-i}V_0^{(1)}]_{-1}+w_{261}[A^{-i}[V_0^{(1)}V_0^{(1)}]]+\\
&+w_{262}[TA^{-i}]+\epsilon_{\rho\mu}r^i_{\sigma\nu}\big(w_{263}[V_{1/2}^{(1)\rho\sigma}V_{1/2}^{(1)\mu\nu}]+w_{264}[G^{\rho\sigma}V_{1/2}^{(1)\mu\nu}]+\\
&+w_{265}[G^{\rho\sigma}G^{\mu\nu}]\big)+f^i{}_{jl}\big(w_{266}[A^{-j}V_1^{(1)-l}]+w_{267}[A^{-j}A^{-l}]_{-1}\big)+\\
&+w_{268}[[A^-{}_jA^{-j}]A^{-i}]+w_{269}[[A^+{}_jA^{+j}]A^{-i}]\}+\\
&+\ell_{i,\alpha\gamma}\ell_{i,\beta\delta}\{w_{270}[A^{+i}A^{-j}]++r^i_{\rho\mu}r^j_{\sigma\nu}w_{271}[G^{\rho\sigma}V_{1/2}^{(1)\mu\nu}]+\\
&+w_{272}[A^{+i}V_1^{(1)-j}]+w_{273}[A^{-j}V_1^{(1)+i}]+w_{274}[A^{+i}[A^{-j}V_0^{(1)}]]+\\
&+w_{275}[A^{+i}A^{-j}]_{-1}\}\,,\\
V_{1/2}^{(1)\alpha\beta}\times V_{1/2}^{(2)\gamma\delta}\sim{}&\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\{w_{276}V_0^{(1)}+w_{277}T+\cdots+w_{291}[[A^-{}_iA^{-i}]V_0^{(1)}]\}+\cdots+\\
&+\ell_{i,\alpha\gamma}\ell_{i,\beta\delta}\{w_{324}[A^{+i}A^{-j}]+\cdots+w_{329}[A^{+i}A^{-j}]_{-1}\}\,,\\
V_0^{(2)}\times V_0^{(2)}\sim{}&n_2I+w_{330}T+w_{331}V_0^{(2)}+w_{332}[V_0^{(1)}V_0^{(1)}]+w_{333}[A^+{}_iA^{+i}]+\\
&+w_{334}[A^-{}_iA^{-i}]\,. \qquad\qquad (B.1)
\end{aligned}
$$

As in the main part of the paper, we have labelled the structure constants in the OPE $V_0^{(1)}\times V_2^{(1)}$ in the same order as in the OPE $V_0^{(1)}\times V_0^{(3)}$ given above it, which is of the same form; the structure constants in the OPE $V_1^{(1)+i}\times V_0^{(2)}$ in the same order as in the OPE $V_0^{(1)}\times V_1^{(2)+i}$ given above it, etc.

B.2 The structure constants

In this section we list the structure constants in the OPEs (B.1). We have fixed the redefinition freedom of $V_0^{(2)}$ and $V_0^{(3)}$ with the conditions (3.18), (3.25), and we have assumed the parity symmetry (3.26) so that eq. (3.27) holds. Under these assumptions the structure constants take the following form:

$$
\begin{aligned}
w_{95}&=-\frac{64(k_--k_+)(-2-k_--k_++k_-k_+)(2+k_-+k_++k_-k_+)}{(2+k_-+k_+)\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,\\
w_{96}&=8\,,\\
w_{97}&=\frac{16(k_--k_+)(5+4k_-+4k_++2k_-k_+)}{-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2}\,,\\
w_{98}&=-\frac{32(-1+k_-)(1+k_-)k_+(2+2k_-+k_+)}{(2+k_-+k_+)\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,
\end{aligned}
$$

$$w_{99}=\frac{32k_-(-1+k_+)(1+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{111}=-\frac{16(-1+k_-)(1+k_-)k_+(1+k_+)(2+2k_-+k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{112}=\frac{64(-1+k_-)(1+k_-)(1+k_+)(2+2k_-+k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{120}=\frac{8(-1+k_-)(1+k_+)(2+2k_-+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{127}=-\frac{16(-1+k_-)(1+k_-)k_+(1+k_+)(2+2k_-+k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{128}=\frac{64(-1+k_-)(1+k_-)(1+k_+)(2+2k_-+k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{132}=\frac{64(-1+k_-)(1+k_-)(2+2k_-+k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{138}=\frac{8(-1+k_-)(1+k_-)k_+(2+2k_-+k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{143}=\frac{16k_-(1+k_-)(-1+k_+)(1+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{144}=-\frac{64(1+k_-)(-1+k_+)(1+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{152}=-\frac{8(1+k_-)(-1+k_+)(2+2k_-+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{159}=\frac{16k_-(1+k_-)(-1+k_+)(1+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{160}=-\frac{64(1+k_-)(-1+k_+)(1+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{164}=-\frac{64(-1+k_+)(1+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{170}=-\frac{8k_-(-1+k_+)(1+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{174}=-\frac{32k_-k_+(1+k_-+2k_+)}{(2+k_-+k_+)^2}\,,$$

$$w_{175}=-\frac{64k_-(2+k_-)(-1+k_+)(1+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{176}=8\,,$$

$$w_{177}=\frac{16(k_--k_+)(5+4k_-+4k_++2k_-k_+)}{-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2}\,,$$

$$w_{178}=-\frac{32k_-k_+(2+k_-+2k_+)\left(-1+2k_-+2k_-^2-2k_+-k_-k_+\right)}{(2+k_-+k_+)^2\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{179}=\frac{32k_-(-1+k_+)(1+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{180}=\frac{32k_-}{(2+k_-+k_+)^2}\,,$$

$$w_{181}=-\frac{32k_-(1+k_-+2k_+)}{(2+k_-+k_+)^2}\,,$$

$$w_{182}=1\,,$$

$$w_{183}=\frac{4(k_--k_+)(5+4k_-+4k_++2k_-k_+)}{-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2}\,,$$

$$w_{187}=-\frac{32(2+k_-+2k_+)\left(-2-k_--k_++2k_-k_++2k_-^2k_+\right)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{188}=\frac{2(2+k_+)(2+k_-+2k_+)\left(1-2k_--2k_-^2+2k_++k_-k_+\right)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{190}=\frac{2(2+k_-+2k_+)\big(-2-4k_--k_++2k_-k_++2k_-^2k_++3k_-k_+^2\big)}{(2+k_-+k_+)\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,$$

$$w_{192}=\frac{8k_-\big(-4-4k_--k_-^2-8k_+-k_-k_++8k_-^2k_++4k_-^3k_+-3k_+^2+4k_-k_+^2+5k_-^2k_+^2\big)}{(2+k_-+k_+)^2\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,$$

$$w_{193}=\frac{16(2+k_-+2k_+)\big(-2-k_--k_++2k_-k_++2k_-^2k_+\big)}{(2+k_-+k_+)^2\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,$$

$$w_{194}=\frac{16(2+k_-+2k_+)\big(-2-3k_--k_++2k_-k_++2k_-^2k_++2k_-k_+^2\big)}{(2+k_-+k_+)^2\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,$$

$$w_{195}=-\frac{32k_-k_+(1+2k_-+k_+)}{(2+k_-+k_+)^2}\,,$$

$$w_{196}=-\frac{64(-1+k_-)(1+k_-)k_+(2+k_+)(2+2k_-+k_+)}{(2+k_-+k_+)\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,$$

$$w_{197}=-8\,,$$

$$w_{198}=-\frac{16(k_--k_+)(5+4k_-+4k_++2k_-k_+)}{-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2}\,,$$

$$w_{199}=\frac{32(-1+k_-)(1+k_-)k_+(2+2k_-+k_+)}{(2+k_-+k_+)\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,$$

$$w_{200}=\frac{32k_-k_+(2+2k_-+k_+)\big(1+2k_--2k_++k_-k_+-2k_+^2\big)}{(2+k_-+k_+)^2\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,$$

$$w_{201}=\frac{32k_+}{(2+k_-+k_+)^2}\,,$$

$$w_{202}=-\frac{32k_+(1+2k_-+k_+)}{(2+k_-+k_+)^2}\,,$$

$$w_{203}=-1\,,$$

$$w_{204}=-\frac{4(k_--k_+)(5+4k_-+4k_++2k_-k_+)}{-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2}\,,$$

$$w_{208}=-\frac{32(2+2k_-+k_+)\big(-2-k_--k_++2k_-k_++2k_-k_+^2\big)}{(2+k_-+k_+)\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,$$

$$w_{209}=\frac{2(2+k_-)(2+2k_-+k_+)\big(1+2k_--2k_++k_-k_+-2k_+^2\big)}{(2+k_-+k_+)\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,$$

$$w_{211}=\frac{2(2+2k_-+k_+)\big(-2-k_--4k_++2k_-k_++3k_-^2k_++2k_-k_+^2\big)}{(2+k_-+k_+)\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,$$

$$w_{213}=\frac{8k_+\big(-4-8k_--3k_-^2-4k_+-k_-k_++4k_-^2k_+-k_+^2+8k_-k_+^2+5k_-^2k_+^2+4k_-k_+^3\big)}{(2+k_-+k_+)^2\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,$$

$$w_{214}=\frac{16(2+2k_-+k_+)\big(-2-k_--k_++2k_-k_++2k_-k_+^2\big)}{(2+k_-+k_+)^2\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,$$

$$w_{215}=\frac{16(2+2k_-+k_+)\big(-2-k_--3k_++2k_-k_++2k_-^2k_++2k_-k_+^2\big)}{(2+k_-+k_+)^2\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,$$

$$w_{216}=-\frac{32}{2+k_-+k_+}\,,$$

$$w_{221}=-\frac{32}{2+k_-+k_+}\,,$$

$$w_{223}=-\frac{64(k_--k_+)\big(8+8k_-+2k_-^2+8k_++9k_-k_++4k_-^2k_++2k_+^2+4k_-k_+^2\big)}{3(2+k_-+k_+)\big(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\big)}\,,$$

$$w_{224}=-8\,,$$

$$w_{225}=-\frac{16(k_--k_+)(5+4k_-+4k_++2k_-k_+)}{-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2}\,,$$

$$w_{226}=\frac{32\big[3k_-k_+(2+k_-+2k_+)\big(-1+2k_-+2k_-^2-2k_+-k_-k_+\big)+(k_-+2k_+)K\big]}{3(2+k_-+k_+)^2K}\,,$$

$$w_{227} = \frac{32\left[3k_-k_+(2+2k_-+k_+)\left(1+2k_--2k_++k_-k_+-2k_+^2\right)-(2k_-+k_+)K\right]}{3(2+k_-+k_+)^2K}\,,$$

$$w_{238} = -\frac{128k_-(3+k_-+2k_+)}{3(2+k_-+k_+)^2}\,,$$

$$w_{241} = 1\,,$$

$$w_{242} = \frac{4(k_--k_+)(5+4k_-+4k_++2k_-k_+)}{-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2}\,,$$

$$w_{246} = -\frac{32(2+k_-+2k_+)\left(-2-k_--k_++2k_-k_++2k_-^2k_+\right)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{247} = -\frac{2\left(8+18k_-+15k_-^2+4k_-^3-2k_+-6k_-k_++2k_-^3k_+-11k_+^2-16k_-k_+^2-6k_-^2k_+^2-4k_+^3-2k_-k_+^3\right)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{249} = \frac{2\left(-8-14k_--5k_-^2-10k_+-2k_-k_++10k_-^2k_++2k_-^3k_+-3k_+^2+14k_-k_+^2+10k_-^2k_+^2+6k_-k_+^3\right)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{251} = \frac{16k_-(2+k_-+2k_+)\left(-2-k_--k_++2k_-k_++2k_-^2k_+\right)}{(2+k_-+k_+)^2\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{252} = \frac{16(2+k_-+2k_+)\left(-2-k_--k_++2k_-k_++2k_-^2k_+\right)}{(2+k_-+k_+)^2\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{253} = \frac{16\left(-12-16k_--5k_-^2-14k_++3k_-k_++14k_-^2k_++2k_-^3k_+-4k_+^2+16k_-k_+^2+12k_-^2k_+^2+4k_-k_+^3\right)}{(2+k_-+k_+)^2\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{254} = \frac{128k_+(3+2k_-+k_+)}{3(2+k_-+k_+)^2}\,,$$

$$w_{257} = 1\,,$$

$$w_{258} = \frac{4(k_--k_+)(5+4k_-+4k_++2k_-k_+)}{-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2}\,,$$

$$w_{262} = \frac{32(2+2k_-+k_+)\left(-2-k_--k_++2k_-k_++2k_-k_+^2\right)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{263} = -\frac{2\left(-8+2k_-+11k_-^2+4k_-^3-18k_++6k_-k_++16k_-^2k_++2k_-^3k_+-15k_+^2+6k_-^2k_+^2-4k_+^3-2k_-k_+^3\right)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{265} = -\frac{2\left(-8-10k_--3k_-^2-14k_+-2k_-k_++14k_-^2k_++6k_-^3k_+-5k_+^2+10k_-k_+^2+10k_-^2k_+^2+2k_-k_+^3\right)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{267} = -\frac{16k_+(2+2k_-+k_+)\left(-2-k_--k_++2k_-k_++2k_-k_+^2\right)}{(2+k_-+k_+)^2\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{268} = -\frac{16(2+2k_-+k_+)\left(-2-k_--k_++2k_-k_++2k_-k_+^2\right)}{(2+k_-+k_+)^2\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{269} = -\frac{16\left(-12-14k_--4k_-^2-16k_++3k_-k_++16k_-^2k_++4k_-^3k_+-5k_+^2+14k_-k_+^2+12k_-^2k_+^2+2k_-k_+^3\right)}{(2+k_-+k_+)^2\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{270} = -\frac{128(k_--k_+)}{3(2+k_-+k_+)^2}\,,$$

$$w_{275} = -\frac{64}{2+k_-+k_+}\,,$$

$$w_{284} = -1\,,$$

$$w_{286} = \frac{16(k_--k_+)(5+4k_-+4k_++2k_-k_+)}{-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2}\,,$$

$$w_{287} = \frac{2(k_--k_+)\left(8+8k_-+2k_-^2+8k_++9k_-k_++4k_-^2k_++2k_+^2+4k_-k_+^2\right)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{288} = \frac{4(2+k_-+2k_+)\left(-2-k_--k_++2k_-k_++2k_-^2k_+\right)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{289} = -\frac{4(2+2k_-+k_+)\left(-2-k_--k_++2k_-k_++2k_-k_+^2\right)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

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$$w_{290}=-\frac{8(k_--k_+)(5+4k_-+4k_++2k_-k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{291}=-\frac{8(k_--k_+)(5+4k_-+4k_++2k_-k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{293}=\frac{16(-1+k_-)(1+k_-)k_+(1+k_+)(2+2k_-+k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{294}=-\frac{64(-1+k_-)(1+k_-)(1+k_+)(2+2k_-+k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{298}=-\frac{32(-1+k_-)(1+k_-)(2+2k_-+k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{302}=-\frac{4(-1+k_-)(1+k_+)(2+2k_-+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{304}=-\frac{4(-1+k_-)(1+k_-)k_+(2+2k_-+k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{309}=-\frac{16k_-(1+k_-)(-1+k_+)(1+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{310}=\frac{64(1+k_-)(-1+k_+)(1+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{314}=\frac{32(-1+k_+)(1+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{318}=\frac{4(1+k_-)(-1+k_+)(2+2k_-+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{320}=\frac{4k_-(-1+k_+)(1+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{325}=\frac{8(k_--k_+)(2+2k_-+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{326}=\frac{8(2+k_-+2k_+)\left(-2-k_--k_++2k_-k_++2k_-^2k_+\right)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{327}=-\frac{8(2+2k_-+k_+)\left(-2-k_--k_++2k_-k_++2k_-k_+^2\right)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{328}=\frac{32(k_--k_+)(5+4k_-+4k_++2k_-k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$n_2=\frac{64(-1+k_-)k_-(1+k_-)(-1+k_+)k_+(1+k_+)(2+2k_-+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)^3\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{330}=\frac{128(-1+k_-)k_-(1+k_-)(2+k_-)(-1+k_+)k_+(1+k_+)(2+k_+)(2+2k_-+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)^2\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)^2}\,,$$

$$w_{331}=-\frac{16(k_--k_+)\left(4+4k_-+k_-^2+4k_++7k_-k_++4k_-^2k_++k_+^2+4k_-k_+^2+k_-^2k_+^2\right)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)}\,,$$

$$w_{332}=\frac{32(-1+k_-)(1+k_-)(2+k_-)(-1+k_+)(1+k_+)(2+k_+)(2+2k_-+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)^2}\,,$$

$$w_{333}=-\frac{64(-1+k_-)k_-(1+k_-)(2+k_-)(-1+k_+)k_+(1+k_+)(2+2k_-+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)^2\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)^2}\,,$$

$$w_{334}=-\frac{64(-1+k_-)k_-(1+k_-)(-1+k_+)k_+(1+k_+)(2+k_+)(2+2k_-+k_+)(2+k_-+2k_+)}{(2+k_-+k_+)^2\left(-4-4k_--k_-^2-4k_++3k_-k_++4k_-^2k_+-k_+^2+4k_-k_+^2+3k_-^2k_+^2\right)^2}\,.$$

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C Structure constants for the OPE $V^{(1)} \times V^{(1)}$

In this appendix we list the values of the 39 coefficients appearing in (4.31).

$$z_1 = -\frac{(k_- + k_+)\, n_1}{2\,(k_- - 1)\,(k_+ - 1)}\,, \qquad z_2 = -\frac{n_1}{(k_- - 1)\,(k_+ - 1)}\,,$$
$$z_3 = \frac{2n_1}{(k_- - 1)\,(k_+ - 1)}\,, \qquad z_4 = -\frac{2n_1}{(k_- - 1)\,(k_+ - 1)}\,,$$
$$z_5 = -\frac{2n_1}{3\,(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,, \qquad z_6 = -2n_1\,,$$
$$z_7 = \frac{4n_1}{k_- - 1}\,, \qquad z_8 = \frac{4n_1}{k_+ - 1}\,,$$
$$z_9 = -\frac{2\,(k_- + k_+ - 2)\, n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}, \qquad z_{10} = \frac{(k_+ - k_-)\, n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,,$$
$$z_{11} = -\frac{n_1}{(k_- - 1)\,(k_+ - 1)}\,, \qquad z_{12} = -\frac{8n_1}{(k_- - 1)\,(k_+ - 1)}\,,$$
$$z_{13} = -\frac{n_1}{(k_- - 1)\,(k_+ - 1)}\,, \qquad z_{14} = \frac{2n_1}{k_- - 1}\,,$$
$$z_{15} = \frac{4n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,, \qquad z_{16} = \frac{2n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,,$$
$$z_{17} = \frac{2n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,, \qquad z_{18} = \frac{4n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,,$$
$$z_{19} = -\frac{2n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,, \qquad z_{20} = \frac{2n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,,$$
$$z_{21} = \frac{2n_1}{k_+ - 1}\,, \qquad z_{22} = -\frac{n_1}{(k_- - 1)\,(k_+ - 1)}\,,$$
$$z_{23} = -\frac{(k_- + k_+ - 4)\, n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,, \qquad z_{24} = \frac{(k_- + k_+ + 4)\, n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,,$$
$$z_{25} = \frac{(k_+ - k_-)\, n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,, \qquad z_{26} = -\frac{(k_- + k_+)\, n_1}{(k_- - 1)\,(k_+ - 1)}\,,$$
$$z_{27} = -\frac{n_1}{(k_- - 1)\,(k_+ - 1)}\,, \qquad z_{28} = -\frac{4n_1}{k_+ - 1}\,,$$
$$z_{29} = \frac{4n_1}{(k_+ - 1)\,(k_- + k_+)}\,, \qquad z_{30} = -\frac{4n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,,$$
$$z_{31} = -\frac{8n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,, \qquad z_{32} = \frac{2n_1}{(k_- - 1)\,(k_+ - 1)}\,,$$
$$z_{33} = \frac{4n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,, \qquad z_{34} = -\frac{4n_1}{k_- - 1}\,,$$
$$z_{35} = \frac{4n_1}{(k_- - 1)\,(k_- + k_+)}\,, \qquad z_{36} = -\frac{4n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,,$$
$$z_{37} = -\frac{8n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,, \qquad z_{38} = -\frac{2n_1}{(k_- - 1)\,(k_+ - 1)}\,,$$
$$z_{39} = -\frac{4n_1}{(k_- - 1)\,(k_+ - 1)\,(k_- + k_+)}\,.$$

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D Structure constants for the OPE $V^{(1)} \times V^{(2)}$

In this appendix we list the values of the 16 coefficients in eq. (4.39). We have omitted a common factor n_1/n_2, where n_1 is defined in (4.31), and n_2 is the coefficient of $V^{(2)}_{1/2}$ in the OPE $V^{(1)}_0 \times V^{(1)}_{3/2}$.

$$
\begin{aligned}
w_1 &= -\frac{k_- + k_+}{2(k_- - 1)(k_+ - 1)}\,, \\
w_2 &= -\frac{2(k_- - k_+)\big(2k_+k_-^2 - 10k_-^2 + 2k_+^2k_- - 26k_+k_- + 17k_- - 10k_+^2 + 17k_+ - 6\big)}{3(k_- - 1)(k_+ - 1)\big(3k_+^2k_-^2 - 2k_+k_-^2 - 2k_-^2 - 2k_+^2k_- - k_+k_- + k_- - 2k_+^2 + k_+\big)}\,, \\
w_3 &= -\frac{4}{(k_- - 1)(k_+ - 1)}\,, \\
w_4 &= \frac{2}{(k_- - 1)(k_+ - 1)}\,, \\
w_5 &= -\frac{2(k_- - k_+)(k_- + k_+)(2k_+k_- + 2k_- + 2k_+ - 1)}{(k_- - 1)(k_+ - 1)\big(3k_+^2k_-^2 - 2k_+k_-^2 - 2k_-^2 - 2k_+^2k_- - k_+k_- + k_- - 2k_+^2 + k_+\big)}\,, \\
w_6 &= \frac{2}{(k_- - 1)(k_+ - 1)}\,, \\
w_7 &= -\frac{4(k_- - k_+)(2k_+k_- + 2k_- + 2k_+ - 1)}{(k_- - 1)(k_+ - 1)\big(3k_+^2k_-^2 - 2k_+k_-^2 - 2k_-^2 - 2k_+^2k_- - k_+k_- + k_- - 2k_+^2 + k_+\big)}\,, \\
w_8 &= -\frac{4\big(4k_+k_-^3 - 4k_-^3 + 5k_+^2k_-^2 - 14k_+k_-^2 + 8k_-^2 - 6k_+^2k_- + 7k_+k_- - 3k_- - 2k_+^2 + k_+\big)}{(k_- - 1)(k_+ - 1)\big(3k_+^2k_-^2 - 2k_+k_-^2 - 2k_-^2 - 2k_+^2k_- - k_+k_- + k_- - 2k_+^2 + k_+\big)}\,, \\
w_9 &= \frac{4\big(4k_-k_+^3 - 4k_+^3 + 5k_-^2k_+^2 - 14k_-k_+^2 + 8k_+^2 - 6k_-^2k_+ + 7k_-k_+ - 3k_+ - 2k_-^2 + k_-\big)}{(k_- - 1)(k_+ - 1)\big(3k_+^2k_-^2 - 2k_+k_-^2 - 2k_-^2 - 2k_+^2k_- - k_+k_- + k_- - 2k_+^2 + k_+\big)}\,, \\
w_{10} &= \frac{8(k_- - k_+)(2k_+k_- + 2k_- + 2k_+ - 1)}{(k_- - 1)(k_+ - 1)\big(3k_+^2k_-^2 - 2k_+k_-^2 - 2k_-^2 - 2k_+^2k_- - k_+k_- + k_- - 2k_+^2 + k_+\big)}\,, \\
w_{11} &= -\frac{8(k_- - k_+)(2k_+k_- + 2k_- + 2k_+ - 1)}{(k_- - 1)(k_+ - 1)\big(3k_+^2k_-^2 - 2k_+k_-^2 - 2k_-^2 - 2k_+^2k_- - k_+k_- + k_- - 2k_+^2 + k_+\big)}\,, \\
w_{12} &= \frac{2}{(k_- - 1)(k_+ - 1)}\,, \\
w_{13} &= \frac{2}{(k_- - 1)(k_+ - 1)}\,, \\
w_{14} &= -\frac{8(k_- - k_+)(2k_+k_- + 2k_- + 2k_+ - 1)}{3(k_- - 1)(k_+ - 1)(k_- + k_+)\big(3k_+^2k_-^2 - 2k_+k_-^2 - 2k_-^2 - 2k_+^2k_- - k_+k_- + k_- - 2k_+^2 + k_+\big)}\,, \\
w_{15} &= \frac{8(k_- + 2k_+ - 1)\big(2k_+k_-^2 - 2k_-^2 - 2k_+k_- + k_- - k_+\big)}{(k_- - 1)(k_+ - 1)(k_- + k_+)\big(3k_+^2k_-^2 - 2k_+k_-^2 - 2k_-^2 - 2k_+^2k_- - k_+k_- + k_- - 2k_+^2 + k_+\big)}\,, \\
w_{16} &= -\frac{8(2k_- + k_+ - 1)\big(2k_-k_+^2 - 2k_+^2 - 2k_-k_+ + k_+ - k_-\big)}{(k_- - 1)(k_+ - 1)(k_- + k_+)\big(3k_+^2k_-^2 - 2k_+k_-^2 - 2k_-^2 - 2k_+^2k_- - k_+k_- + k_- - 2k_+^2 + k_+\big)}\,.
\end{aligned}
$$

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