The BPS spectrum of the 4d $\mathcal{N}=2$ SCFT's H_1, H_2, D_4, E_6, E_7, E_8

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ABSTRACT: Extending results of 1112.3984, we show that all rank 1 $\mathcal{N}=2$ SCFT's in the sequence H_1, H_2, D_4 E_6, E_7, E_8 have canonical finite BPS chambers containing precisely $2h(F) = 12(\Delta - 1)$ hypermultiplets. The BPS spectrum of the canonical BPS chambers saturates the conformal central charge c, and satisfies some intriguing numerology.

Contents

1 Introduction

Consider the seven rank 1 4d $\mathcal{N} = 2$ SCFT's which may be engineered in F-theory using the Kodaira singular fibers [1–9]

$$H_0, \ H_1, \ H_2, \ D_4, \ E_6, \ E_7, \ E_8. \tag{1.1}$$

H_0 has trivial global symmetry and will be neglected in the following. The other six theories have flavor group F equal, respectively, to

$$\mathrm{SU}(2), \ \mathrm{SU}(3), \ \mathrm{SO}(8), \ E_6, \ E_7 \text{ and } E_8. \tag{1.2}$$

We note that (1.2) is precisely the list of *all* simply-laced simple Lie groups F with the property

$$h(F) = 6 \, \frac{r(F) + 2}{10 - r(F)}, \tag{1.3}$$

where $r(F)$ and $h(G)$ are, respectively, the rank and Coxeter number of F. Physically, the relation (1.3) is needed for consistency with the $2d/4d$ correspondence of [10], and is an example of the restrictions on the flavor group F of a 4d $\mathcal{N} = 2$ SCFT following from that principle.

Neglecting H_0, let us list the numbers $2\,h(F)$ for the other six models

$$4, \ 6, \ 12, \ 24, \ 36, \ 60. \tag{1.4}$$

The first four numbers in this list have appeared before in the non-perturbative analysis of the corresponding SCFT's: it is known [11, 12] that the (mass-deformed) SCFT's H_1, H_2, D_4 and E_6 have a finite BPS chamber in which the BPS spectrum consists precisely of (respectively) 4, 6, 12 and 24 hypermultiplets. The H_1 SCFT is the $D_3(\mathrm{SU}(2))$ model of [12, 13], while the H_2, D_4 and E_6 SCFT's coincide, respectively, with the models

$D_2(\mathrm{SU}(3))$, $D_2(\mathrm{SU}(4))$, and $D_2(\mathrm{SO}(8))$ of those papers; then the above statement is a special instance of the general fact that, for all simply-laced Lie groups $G = ADE$, the $D_2(G)$ SCFT has a finite chamber with $r(G)\,h(G)$ hypermultiplets [12], while, for all $p \in \mathbb{N}$, the model $D_p(\mathrm{SU}(2))$ has a special BPS chamber with $2(p-1)$ hypermultiplets.[1]

For the four SCFT's H_1, H_2, D_4, E_6, the number of hypermultiplets in the above preferred chamber, n_h, may be written in a number of intriguing ways: we list just a few

$$n_h = 2\,h(F) = \frac{12\,r(F) + 24}{10 - r(F)} = 12(\Delta - 1) = n_7\,\Delta, \tag{1.5}$$

where Δ is the dimension of the field parametrizing the Coulomb branch of the rank 1 SCFT, and n_7 is the number of parallel 7-branes needed to engineer the SCFT in F-theory [1–9]; see table 1.

The special finite BPS chambers with $n_h = 2\,h(F)$ hypers have the particular property of *saturating* the conformal central charge c of the strongly-coupled SCFT. By this we mean that, for these theories, the exact c is equal to the value for n_h *free* hypermultiplets plus the contribution from the massless photon vector multiplet

$$c = \frac{1}{12}\,n_h + \frac{1}{6}, \tag{1.6}$$

that is, c has the same value as the system of free fields with the same particle content as the BPS spectrum in the *special* chamber. In fact, the c-saturating property holds in general for the standard BPS chamber of all $D_2(G)$ SCFT's [12], and also for all $D_p(\mathrm{SU}(2))$. It was conjectured by Xie and Zhao [15] that a finite BPS chamber with this property exists for a large class of $\mathcal{N} = 2$ models (their examples are close relatives of the present ones). At the level of numerology, for the four SCFT's H_1, H_2, D_4, E_6 we also have a simple relation between the number of hypers in our special chamber, n_h, and the a, k_F conformal central charges: in facts, for all the above SCFT's the central charge a is given by the photon contribution, $5/24$, plus *three-halves* the contribution of n_h free hypers

$$a = \frac{1}{24}\frac{3\,n_h}{2} + \frac{5}{24} \tag{1.7}$$

$$k_F = \frac{n_h + 12}{6}. \tag{1.8}$$

In view of all this impressive numerology involving n_h, it is tempting to *conjecture* that the last two SCFT's in the sequence (1.1), E_7 and E_8, also have canonical finite BPS chambers with, respectively, 36 and 60 hypermultiplets. This will extend our observations, eq. (1.5)–(1.8), to the *full* SCFT sequence (1.1), suggesting that the numerology encodes deep physical properties of rank 1 SCFTs.

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[1]Note that H_1 is the Argyres-Douglas (AD) model of type A_3 [14] which has BPS chambers with any number n_h of BPS hypers in the range $3 \leq n_h \leq 6$; likewise H_2 is the Argyres-Douglas model of type D_4. In both cases it is neither the AD minimal (3 resp. 4 hypers) nor the AD maximal (6 resp. 12 hypers) BPS chamber which is singled out by the property of being c-saturating, but rather their canonical chamber as a $D_p(\mathrm{SU}(2))$ resp. a $D_2(G)$ theory [12] (for D_4 AD these two chambers are equivalent).

SCFT	H_1	H_2	D_4	E_6	E_7	E_8
Δ	4/3	3/2	2	3	4	6
n_7	3	4	6	8	9	10
c	1/2	2/3	7/6	13/6	19/6	31/6
a	11/24	7/12	23/24	41/24	59/24	95/24
k_F	8/3	3	4	6	8	12

Table 1. Numerical invariants for the six SCFT's H_1, H_2, D_4, E_6, E_7 and E_8. The rank of the flavor group, $r(F)$, is equal to the index in the SCFT symbol.

The purpose of the present short note is to *prove* the above *conjecture*, by constructing explicitly the canonical chambers with $2\,h(F)$ hypers. To get the result we use the BPS quivers for the E_7 and E_8 Minahan-Nemeshanski theories identified in [12] together with the mutation algorithm of [11].

The rest of the note is organized as follows. In section 2 we recall the basics of the mutation algorithm. In section 3 we describe the relevant (class of) quivers and apply the mutation method to get the BPS spectra of the six SCFT's, giving full details for the E_6, E_7 and E_8 theories.

2 Basics of the mutation algorithm

We recall the basics of the mutation algorithm referring the reader to [10, 11, 16–18] for the details. Suppose we have a 4d $\mathcal{N}=2$ model which has the BPS quiver property [19], [11, 17]. Let Γ be the lattice of its quantized charges (electric, magnetic, and flavor). The SUSY central charge defines an additive function $Z(\cdot)\colon \Gamma \to \mathbb{C}$, so that a BPS particle of charge $v \in \Gamma$ has central charge $Z(v)$. Fix a half-plane $H_\theta = \{z \in \mathbb{C}\colon \mathrm{Im}(e^{-i\theta}z) > 0\}$ such that no BPS particle has central charge laying on its boundary. We say (conventionally) that the BPS states with central charges in H_θ are *particles*, while those with central charges in $-H_\theta$ are their PCT-conjugate *anti–particles*. The charges of particles span a *strict* convex cone $\Gamma_\theta \subset \Gamma$ which, in a model with the BPS property [19], has the form $\Gamma_\theta \simeq \bigoplus_{i=1}^{r} \mathbb{Z}_{\geq 0}\, e_i^{(\theta)}$, where r is the rank of Γ. The quiver Q_θ of the $\mathcal{N} = 2$ theory is obtained by picking one node per each generator $e_i^{(\theta)}$ of the positive cone Γ_θ and connecting nodes $e_i^{(\theta)}$, $e_j^{(\theta)}$ with $\langle e_i^{(\theta)}, e_j^{(\theta)} \rangle_{\mathrm{Dirac}} \equiv B_{ij}^{(\theta)}$ oriented arrows, where $\langle \cdot, \cdot \rangle_{\mathrm{Dirac}}$ is the Dirac electro-magnetic pairing of the charges.[2] The integral skew-symmetric matrix $B_{ij}^{(\theta)}$ is called the *exchange matrix* of the quiver Q_θ. Q_θ is supplemented with a superpotential $\mathcal{W}_\theta$ (a sum, with complex coefficients, of cycles on Q_θ) [11].

The BPS particles (as contrasted with antiparticles) correspond to *stable* representations X of the quiver Q_θ subjected to the relations $\partial \mathcal{W}_\theta = 0$. A representation X is *stable* iff, for all non-zero proper subrepresentation Y, one has $\arg(e^{-i\theta}Z(Y)) < \arg(e^{-i\theta}Z(X))$,

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[2]Strictly speaking $\langle e_i^{(\theta)}, e_j^{(\theta)} \rangle_{\mathrm{Dirac}}$ is only the number of *net* arrows (i.e. the number of $e_i^{(\theta)} \to e_j^{(\theta)}$ arrows minus the number of the $e_i^{(\theta)} \leftarrow e_j^{(\theta)}$ ones). For *generic* superpotentials pairs of opposite arrows $e_i^{(\theta)} \leftrightarrows e_j^{(\theta)}$ get massive and may be integrated out, leaving the 2-acyclic quiver in the text [11].

where we take $\arg(e^{-i\theta} H_\theta) = [0, \pi]$. The charge $v \in \Gamma_\theta$ of the BPS particle is given by the dimension vector $\sum_i \dim X_i \, e_i^{(\theta)}$ of the corresponding stable representation X.

In particular, the representations S_i with dimension vector equal to a generator $e_i^{(\theta)}$ of Γ_θ are *simple*, and hence automatically stable for all choices of the function $Z(\cdot)$ (consistent with the given positive cone $\Gamma_\theta \subset \Gamma$). Therefore, r BPS states are determined for free; they are necessarily hypermultiplets, since Q_θ has no loop [11, 20].

The above construction depends on an arbitrary choice, the angle θ. Choosing a different angle θ', we get a different convex cone $\Gamma_{\theta'}$ with a different set of generators $e_i^{(\theta')}$. Since the physics does not depend on the conventional choice of the half plane H_θ, the $e_i^{(\theta')}$'s should also be charge vectors of stable BPS hypermultiplets. The idea of the mutation algorithm is to get the full BPS spectrum by collecting all states with charges of the form $e_i^{(\theta)}$'s for all θ. It is easy to see that this gives the full BPS spectrum provided it consists only of hypermultiplets (i.e. particles of spin $\leq 1/2$) and their number n_h is finite.

More concretely, we notice that the BPS particle of larger $\arg(e^{-i\theta} Z(v))$, has a charge v which is a generator $e_{i_1}^{(\theta)}$ of Γ_θ (associated to some node i_1 of Q_θ). We may tilt clockwise the boundary line of H_θ just past the point $Z(e_{i_1}^{(\theta)})$, producing a new half-plane $H_{\theta'}$. In the new frame the state with charge $e_{i_1}^{(\theta)}$ is an *anti*–particle, while its PCT-conjugate of charge $-e_{i_1}^{(\theta)}$ becomes a particle, and in facts a generator of the new positive cone $\Gamma_{\theta'}$. The generators $e_i^{(\theta')}$ of $\Gamma_{\theta'}$ are linear combinations with integral coefficients of the old ones $e_i^{(\theta)}$. The explicit expression of the $e_i^{(\theta')}$'s in terms of the $e_i^{(\theta)}$ is known as the Seiberg duality in physics [21, 22], while in mathematics [23] it is called the basic quiver mutation of Q_θ at the i_1 node, written μ_{i_1},

$$e_i^{(\theta')} = \mu_{i_1}\big(e_i^{(\theta)}\big) = \begin{cases} -e_{i_1}^{(\theta)} & \text{if } i = i_1 \\ e_i^{(\theta)} + \max\{B_{i_1\,i}^{(\theta)}, 0\}\, e_{i_1}^{(\theta)} & \text{otherwise.} \end{cases} \tag{2.1}$$

The mutated quiver $Q_{\theta'} = \mu_{i_1}(Q_\theta)$ is specified by the exchange matrix $B_{ij}^{(\theta')} \equiv \langle e_i^{(\theta')}, e_i^{(\theta')} \rangle_{\text{Dirac}}$. Under the quiver mutation μ_{i_1}, the superpotential $\mathcal{W}_\theta$ changes according to the rules of Seiberg duality [21, 22] (which is equivalent to the the DWZ rule [23]).

The new generators $e_i^{(\theta')}$ are also charge vectors of stable hypers. We may reiterate the procedure by mutating $Q_{\theta'}$ at the node i_2 corresponding to the hypermultiplet with maximal $\arg(e^{-i\theta'} Z(v))$. Again we conclude that the BPS spectrum also contains stable hypers with charges $e_i^{(\theta'')}$. Now suppose that after m mutations we end up with the positive cone $\Gamma_{\theta^{(m)}} \equiv -\Gamma_\theta$; we conclude that $\theta^{(m)} = \theta + \pi$ and hence, with our sequence of m half-plane tiltings, we have scanned the full complex half-plane H_θ, picking up *all* the BPS *particles*, one at each step, according to their (decreasing) phase order in the central charge plane. Thus, whenever this happens, we conclude that we have a BPS chamber in which the BPS spectrum consists of precisely m hypermultiplets.

This happens iff there is a sequence of m basic quiver mutation such that [11]

$$\mu_{i_m} \circ \mu_{i_{m-1}} \circ \cdots \circ \mu_{i_2} \circ \mu_{i_1}\big(e_i^{(\theta)}\big) = -\pi\big(e_i^{(\theta)}\big) \qquad \forall\, i, \tag{2.2}$$

where π is a permutation of the r generators $e_i^{(\theta)}$. If, for the given quiver Q_θ, we are able to find a sequence of quiver mutations satisfying equation (2.2) (for some $\pi \in \mathfrak{S}_r$) we may

claim to have found a finite BPS chamber consisting of m hypermultiplets only, and list the quantum numbers $v_\ell \in \Gamma$ of all BPS particles

$$v_\ell = \mu_{i_{\ell-1}} \circ \mu_{i_{\ell-2}} \circ \cdots \circ \mu_{i_1}\big(e_{i_\ell}^{(\theta)}\big) \qquad \ell = 1, 2, \ldots, m. \tag{2.3}$$

In the the rest of the paper we shall work at fixed θ, and write the positive cone generators simply as e_i, omitting the angle.

There are a few strategies to find particular solutions to eq. (2.2). An elegant one is the complete families of sink/source factorized subquivers of Q_θ introduced in [16] and reviewed in [12]; as explained in these references, this is particular convenient when the factorized subquivers are Dynkin ones endowed with the standard Coxeter sink/source sequences.

For general quivers Q, we may perform a systematic search for solutions on a computer; Keller's quiver mutation applet [24] is quite helpful for both procedures. In doing this, it is convenient to rephrase eq. (2.2) in terms of *tropical y*-seed mutations [25–27]. We recall that the *tropical semifield* $\mathsf{Trop}(u_1, u_2, \ldots, u_r)$ is the free multiplicative Abelian group generated by the indeterminates u_i endowed with the operation $\oplus$ defined by

$$\left(\prod u_i^{l_i}\right) \oplus \left(\prod u_i^{m_i}\right) = \prod u_i^{\min(l_i, m_i)}. \tag{2.4}$$

To a BPS state of charge $\sum_i n_i e_i$ we associate the tropical y-variable $\prod u_i^{n_i} \in \mathsf{Trop}(u_1, u_2, \ldots, u_r)$. We start with the initial y-seed in which we assign to the i-th node of Q the variable associated to the generator e_i of the positive cone, namely $y_i(0) \equiv u_i$, and we perform the sequence of mutations in eq. (2.2) on the y-seed using the Fomin-Zelevinski rules [25–27]

$$y_j(s) = \begin{cases} y_{i_s}(s-1)^{-1} & \text{if } j = i_s \\ y_j(s-1)\, y_{i_s}(s-1)^{[B_{i_s j}(s-1)]_+}\Big(1 \oplus y_{i_s}(s-1)\Big)^{-B_{i_s j}} & \text{otherwise.} \end{cases} \tag{2.5}$$

(here $s = 1, 2, \ldots, m$, and $[x]_+ = \max(x, 0)$). Since the tropical variables $y_{i_s}(s-1)$ correspond to BPS particles with charges in the positive cone Γ_θ, one has $1 \oplus y_{i_s}(s-1) \equiv 1$, and eq. (2.5) reduces to the transformation rule (2.1). In terms of tropical y-variables, then eq. (2.2) becomes

$$y_j(m) = y_{\pi(j)}(0)^{-1}, \tag{2.6}$$

supplemented by the condition that the tropical quantities $y_{i_s}(s-1)$ are monomials in the u_i's. This is the equation we actually solve in the next section.

For many purposes, it is convenient to rephrase the algorithm in the language of [10]. If the sequence of basic quiver mutations μ_{i_a} satisfies eq. (2.2), the associated composition of basic quantum cluster mutations satisfies

$$\mathcal{Q}_{i_m} \circ \mathcal{Q}_{i_{m-1}} \circ \cdots \mathcal{Q}_{i_2} \circ \mathcal{Q}_{i_1} = I_\pi\, \mathbb{K}(q) \tag{2.7}$$

where $\mathbb{K}(q)$ is the quantum half-monodromy [10] and I_π is the unitary operator acting on the generators Y_i of the quantum torus algebra of Q_θ as

$$I_\pi\, Y_i\, I_\pi^{-1} = Y_{\pi(i)}^{-1}. \tag{2.8}$$

The (finite) BPS spectrum may be read directly from the factorization of $\mathbb{K}(q)$ in quantum dilogaritms [10], which is explicit in the LHS of eq. (2.7).

In conclusion: given a solution to eq. (2.2) we have determined a BPS chamber $\mathcal{C}_{\text{fin}}$ (which may be formal in the sense of [19]) containing finitely many hypers, as well as the quantum numbers $v \in \Gamma$ of all these hypers.

In addition, the algorithm specifies the (cyclic) phase order of the central charges $Z(v)$ of the BPS states. From this last information we may read the domain $\mathcal{D}_{\text{fin}} \subset \mathbb{C}^r \equiv (\Gamma \otimes \mathbb{C})^\vee$ of central charges $Z(\cdot) \in (\Gamma \otimes \mathbb{C})^\vee$ for which $\mathcal{C}_{\text{fin}}$ is the *actual* BPS chamber, that is, we may determine the region in the space of the 'physical' parameters of the theory which corresponds to the finite chamber $\mathcal{C}_{\text{fin}}$.

At a generic point in $\mathcal{D}_{\text{fin}}$ the unbroken flavor symmetry is just $\mathrm{U}(1)^{\mathrm{rank}\,F}$. At particular points in parameter space the flavor symmetry may have a non-Abelian enhancement. Let F_{fin} be the flavor symmetry group at a point of maximal enhancement in the domain $\mathcal{D}_{\text{fin}}$. Clearly, the BPS hypers of $\mathcal{C}_{\text{fin}}$ should form representations of F_{fin}. The fact that they do is a non-trivial check of the procedure.

3 Computing the BPS spectra

3.1 The quivers $Q(r,s)$

We begin by fixing uniform and convenient representatives of the quiver mutation-classes for the six $\mathcal{N} = 2$ models in eq. (1.1) with $F \neq 1$. We define $Q(r,s)$ to be the quiver with $(r + s + 2)$ nodes

$$(3.1)$$
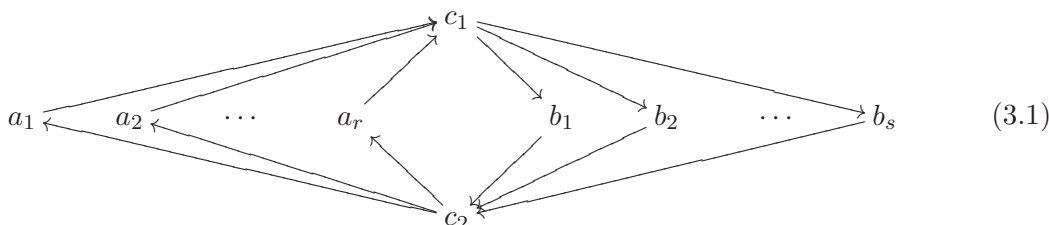
Then the (representative) quivers for our six SCFT's are

SCFT	H_1	H_2	D_4	E_6	E_7	E_8
quiver	$Q(0,1)$	$Q(1,1)$	$Q(2,2)$	$Q(3,3)$	$Q(3,4)$	$Q(3,5)$

$$(3.2)$$

(cfr. ref. [19] for H_1, H_2 and D_4, ref. [11] for E_6, and ref. [12] for E_7 and E_8).

The simplest way to get the table (3.2) is by implementing the flavor groups F in eq. (1.2) directly on the quiver. Indeed, given a $Q(r,s)$ quiver the flavor group F of the corresponding $\mathcal{N} = 2$ QFT is canonically identified by the property that its Dynkin graph is the star with three branches of lengths[3] $[r, s, 2]$. Indeed, a simple computation shows

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[3] As always, in the length of each branch we count the node at the origin of the star; in particular, a branch of length one is no branch at all, while a branch of length zero means that we delete the origin of the star itself. Note that, for all s, the quiver $Q(2,s)$ is mutation equivalent to the quiver of SU(2) SQCD with $N_f = s + 2$ fundamental flavors which has flavor symmetry group SO($2s + 4$), whose Dynkin graphs is the star with three branches of lengths $[2, s, 2]$.

that, if a quiver of the form $Q(r, s)$ is consistent with the 2d/4d correspondence [10] — that is, if $Q(r, s)$ is the BPS quiver of a 2d (2,2) theory with $\hat{c} < 2$ — then the star graph $[r, s, 2]$ is a Dynkin diagram (while 2d (2,2) models whose BPS quivers has the form $Q(r, s)$, with $[r, s, 2]$ an *affine* Dynkin graph, necessarily have $\hat{c} = 2$).

3.2 The c-saturating chamber for H_1, H_2, D_4 and E_6

The first four quivers in (3.2) may be decomposed into Dynkin subquivers in the sense of [16]

$$Q(1, 0) = A_2 \amalg A_1, \qquad\qquad Q(1, 1) = A_2 \amalg A_2,$$
$$Q(2, 2) = A_3 \amalg A_3, \qquad\qquad Q(3, 3) = D_4 \amalg D_4. \qquad (3.3)$$

For a quiver $G \amalg G'$ the charge lattice is $\Gamma = \Gamma_G \oplus \Gamma_{G'}$, where Γ_G is the root lattice of the Lie algebra G. Since the decomposition has the Coxeter property [12, 16], there is a canonical chamber in which the BPS spectrum consists of one hypermultiplet per each of the following charge vectors [16]

$$\Big\{ \alpha \oplus 0 \in \Gamma_G \oplus \Gamma_{G'}, \ \alpha \in \Delta^+(G) \Big\} \bigcup \Big\{ 0 \oplus \beta \in \Gamma_G \oplus \Gamma_{G'}, \ \beta \in \Delta^+(G') \Big\}, \qquad (3.4)$$

where $\Delta^+(G)$ is the set of the positive roots of G. Then the number of hypermultiplets in this canonical finite chamber is

$$n_h = \frac{1}{2} \big(r(G)\, h(G) + r(G')\, h(G') \big), \qquad (3.5)$$

which for the four cases in eq. (3.3) gives (respectively)

$$4, \ 6, \ 12, \ 24, \qquad (3.6)$$

i.e. $n_h = 2\, h(F)$ as expected for a c-saturating chamber.

For sake of comparison with the E_7, E_8 cases in the next subsection, we give more details on the computation of the above spectrum for the E_6 Minahan-Nemeshanski theory [7] using the mutation algorithm. The cases H_1, H_2 and D_4 are similar and simpler.

The two D_4 subquivers of $Q(3, 3)$ are the full subquivers over the nodes $\{a_1, a_2, a_3, c_1\}$ and, respectively, $\{b_1, b_2, b_3, c_1\}$. The quiver $Q(3, 3)$ has an automorphism group $\mathbb{Z}_2 \ltimes (\mathfrak{G}_3 \times \mathfrak{G}_3)$, where the two $\mathfrak{G}_3$ are the triality groups of the D_4 subgraphs, while $\mathbb{Z}_2$ interchanges the two D_4 subquivers (and hence the two $\mathfrak{G}_3$'s). The quiver embedding $D_4 \oplus D_4 \to Q(3, 3)$ induces an embedding of flavor groups

$$\mathrm{SU}(3) \times \mathrm{SU}(3) \to F \equiv E_6, \qquad (3.7)$$

where $\mathrm{SU}(3)$ is the flavor group of the Argyres-Douglas theory of type D_4 characterized by the fact that $\mathrm{Weyl}(\mathrm{SU}(3)) \equiv \mathfrak{G}_3 \equiv$ the triality group of D_4.

The two D_4 subquiver have the 'subspace' orientation; in both bi-partite quivers $Q(3, 3)$ and D_4 we call *even* the nodes a_i and b_i and *odd* the c_i ones. Then, by standard properties of the Weyl group, the quiver mutation 'first all even then all odd'

$$\mu_{c_2} \mu_{c_1} \prod_{i=1}^{3} \mu_{b_i} \prod_{i=1}^{3} \mu_{a_i} \qquad (3.8)$$

transforms the quiver $Q(3,3)$ into itself while acting on $\Gamma_{D_4} \oplus \Gamma_{D_4}$ as $\mathrm{Cox} \oplus \mathrm{Cox}$, where $\mathrm{Cox} \in \mathrm{Weyl}(D_4)$ is the Coxeter element of D_4. Since $(\mathrm{Cox})^3 = -1$, the quiver mutation

$$\left(\mu_{c_2} \mu_{c_1} \prod_{i=1}^{3} \mu_{b_i} \prod_{i=1}^{3} \mu_{a_i} \right)^3 \tag{3.9}$$

is a solution to eq. (2.2) with $\pi = \mathrm{Id}$. Since there are 24 μ's in eq. (3.9), we have found a finite BPS chamber with 24 hypers. eq. (3.9) is invariant under the automorphism group $\mathbb{Z}_2 \ltimes \big(\mathrm{Weyl}(SU(3)) \times \mathrm{Weyl}(SU(3))\big)$ so that there are points in the parameter domain $\mathcal{D}_{\mathrm{fin}}$ corresponding to the above chamber which preserve a flavor group

$$F_{\mathrm{fin}} \supseteq \mathbb{Z}_2 \ltimes \Big(SU(3) \times SU(3) \times U(1)^2 \Big), \tag{3.10}$$

where $\mathbb{Z}_2$ acts by interchanging the two $SU(3)$'s and inverting the sign of the first $U(1)$ charge. The 24 BPS states may be classified in a collection of irrepresentations of the group in the large parenthesis of eq. (3.10) which form $\mathbb{Z}_2$ orbits. From eq. (3.9) we read the phase ordering of the *particles* in the 24 BPS hypers (in addition we have, of course, the PCT conjugate *anti*–particles). Ordered in decreasing phase order, we have

$$\overbrace{(\mathbf{3}, \mathbf{1})_{1,0}, \ (\mathbf{1}, \mathbf{3})_{-1,0},}^{} \quad \overbrace{(\mathbf{1}, \mathbf{1})_{3,1}, \ (\mathbf{1}, \mathbf{1})_{-3,1},}^{} \quad \overbrace{(\overline{\mathbf{3}}, \mathbf{1})_{2,1}, \ (\mathbf{1}, \overline{\mathbf{3}})_{-2,1},}^{}$$
$$\overbrace{(\mathbf{1}, \mathbf{1})_{3,2}, \ (\mathbf{1}, \mathbf{1})_{-3,2},}^{} \quad \overbrace{(\mathbf{3}, \mathbf{1})_{1,1}, \ (\mathbf{1}, \mathbf{3})_{-1,1},}^{} \quad \overbrace{(\mathbf{1}, \mathbf{1})_{0,1}, \ (\mathbf{1}, \mathbf{1})_{0,1},}^{} \tag{3.11}$$

where overbraces collect representations forming a $\mathbb{Z}_2$-orbit. In terms of dimension vectors of the corresponding quiver representations, the quantum numbers of the 24 BPS particles (in decreasing phase order) is

$$a_1, \ a_2, \ a_3; \quad b_1, \ b_2, \ b_3; \quad a_1 + a_2 + a_3 + c_1; \quad b_1 + b_2 + b_3 + c_2;$$
$$a_2 + a_3 + c_1, \ a_1 + a_3 + c_1, \ a_1 + a_2 + c_1; \quad b_2 + b_3 + c_2, \ b_1 + b_3 + c_2, \ b_1 + b_2 + c_2;$$
$$a_1 + a_2 + a_3 + 2\,c_1; \quad b_1 + b_2 + b_3 + 2\,c_2;$$
$$a_1 + c_1, \ a_2 + c_1, \ a_3 + c_1; \quad b_1 + c_2, \ b_2 + c_2, \ b_3 + c_2; \quad c_1; \quad c_2, \tag{3.12}$$

where, for notational convenience, the positive cone generators $e_{a_i}, e_{b_j}, e_{c_k}$ are written simply as a_i, b_j, c_k, respectively.

3.3 The 36-hyper BPS chamber of E_7 MN

The quiver $Q(3,4)$ has no obvious *useful* decomposition into Dynkin subquivers. However, with the help of Keller's quiver mutation applet it is easy to check that the composition of the 36 basic quiver mutations at the sequence of nodes

$$a_1 \ a_2 \ a_3 \ b_1 \ b_2 \ b_3 \ c_1 \ c_2$$
$$a_1 \ a_2 \ b_4 \ b_1 \ b_2 \ b_3 \ c_1 \ c_2 \ a_1 \ a_2 \ a_3 \ b_2 \ c_1 \ c_2 \ b_4 \ b_1 \ b_2 \ b_3 \ c_1 \ c_2 \tag{3.13}$$
$$a_1 \ a_2 \ a_3 \ b_1 \ b_2 \ b_3 \ c_1 \ c_2$$

$$
\begin{array}{l}
a_1,\ \ a_2,\ \ a_3,\ \ b_1,\ \ b_2,\ \ b_3,\ \ a_1 + a_2 + a_3 + c_1,\ \ b_1 + b_2 + b_3 + c_2,\ \ a_2 + a_3 + c_1,\ \ a_1 + a_3 + c_1,\\
a_1 + a_2 + a_3 + b_4 + c_1,\ \ b_2 + b_3 + c_2,\ \ b_1 + b_3 + c_2,\ \ b_1 + b_2 + c_2,\ \ a_1 + a_2 + 2\, a_3 + b_4 + 2\, c_1,\\
b_1 + b_2 + b_3 + 2\, c_2,\ \ a_1 + a_3 + b_4 + c_1,\ \ a_2 + a_3 + b_4 + c_1,\ \ a_1 + a_2 + b_1 + b_2 + b_3 + c_1 + 2\, c_2,\\
b_2 + c_2,\ \ b_4,\ \ a_1 + a_2 + b_2 + c_1 + c_2,\ \ a_1 + a_2 + a_3 + b_2 + 2\, c_1 + c_2,\ \ b_1 + b_4 + c_2,\ \ a_1 + a_2 + c_1,\\
b_3 + b_4 + c_2,\ \ b_1 + b_3 + b_4 + 2\, c_2,\ \ a_1 + a_2 + a_3 + 2\, c_1,\ \ a_2 + c_1,\ \ a_1 + c_1,\ \ b_4 + c_2,\ \ b_3 + c_2,\\
a_3 + c_1,\ \ b_1 + c_2,\ \ c_2,\ \ c_1
\end{array}
$$

Table 2. The charge vectors of the 36 BPS particles in the chamber $\mathcal{C}_{\text{fin}}$ of the E_7 MN theory. To simplify the notation, the positive cone generators $e_{a_i}, e_{b_j}, e_{c_k}$ are written simply as a_i, b_j, c_k, respectively. The particles are listed in decreasing BPS phase order. To get the full BPS spectrum, add the PCT conjugate *anti*–particles.

is a solution to eq. (2.2) for $Q(3,4)$ with[4]

$$
\pi = (a_1\ a_2)(a_3\ b_1\ b_4)(b_2\ b_3)(c_1\ c_2). \tag{3.14}
$$

Moreover no proper subsequence of mutations is a solution to eq. (2.2). Note the similarity with the sequence for E_6 which is a three fold repetition of the first line of (3.13) (the Coxeter sequence of $D_4 \amalg D_4$). Passing from E_6 to E_7 we simply replace the second repetition of the Coxeter sequence for $D_4 \amalg D_4$ with the second line of (3.13) which may also be interpreted as a chain of Coxeter sequences (see remark after eq. (3.17)).

The solution (3.13) corresponds to the finite BPS chamber for the E_7 Minahan Nemeshanski theory [8] with 36 hypermultiplets we were looking for. The (manifest) automorphism of this finite chamber is given by the centralizer of π in the $Q(3,4)$ automorphism group $\mathfrak{S}_3 \times \mathfrak{S}_4$, which is the subgroup $\mathfrak{G}_2 \times \mathfrak{G}_2$ generated by the involutions $(a_1\ a_2)$ and $(b_2\ b_3)$. Then the BPS hypers in this finite chamber form representations of $F_{\text{fin}} = \text{SU}(2) \times \text{SU}(2) \times \text{U}(1)^5$. From the list of charge vectors of the 36 hypers in table 2 we see that this is indeed true.

We stress that the 36-hyper chamber above *is far from being unique*; a part for the other $m = 36$ solutions to eq. (2.2) obtained from (3.13) by applying an automorphism of the quiver $Q(3,4)$, there are other ones; for instance, the sequence of 36 mutations at the nodes

$$
\begin{array}{l}
c_1\ c_2\ a_1\ a_2\ b_1\ b_2\ c_2\ c_1\ a_3\ b_3\ b_4\ b_1\ c_2\ c_1\ b_4\ b_3\ a_1\ a_2\\
b_1\ b_2\ c_2\ c_1\ a_2\ a_1\ b_3\ b_4\ a_3\ b_1\ c_2\ c_1\ a_3\ b_4\ b_2\ a_1\ a_2\ b_3.
\end{array} \tag{3.15}
$$

is a solution to (2.2) with $\pi = (a_1\ a_2)(a_3\ b_1\ b_2)(b_3\ b_4)(c_1\ c_2)$. The properties of all these chambers look very similar, in particular they are expected to have isomorphic F_{fin}.

---

[4]The fact that π is *not* an involution implies that this mutation cannot arise from Coxeter-factorized subquivers as in the previous examples.

3.4 The 60-hyper BPS chamber of E_8 MN

For the quiver $Q(3,5)$ one checks that the composition of the 60 basic quiver mutations at the sequence of nodes

$$a_1 \; a_2 \; a_3 \; b_1 \; b_2 \; b_3 \; c_1 \; c_2 \parallel a_1 \; a_2 \; b_4 \; b_1 \; b_2 \; b_3 \; c_1 \; c_2 \parallel$$
$$a_1 \; a_2 \; a_3 \; b_5 \; b_1 \; b_4 \; c_1 \; c_2 \parallel b_1 \; b_2 \; b_3 \; a_3 \; c_1 \; c_2 \parallel$$
$$a_1 \; a_2 \; b_2 \; b_3 \; b_4 \; b_5 \; c_1 \; c_2 \parallel a_1 \; a_2 \; a_3 \; b_2 \; b_3 \; b_4 \; c_1 \; c_2 \parallel$$
$$a_3 \; b_1 \; b_2 \; b_3 \; b_4 \; b_5 \; c_1 \; c_2 \parallel a_1 \; a_2 \; b_4 \; b_5 \; c_1 \; c_2 \tag{3.16}$$

is a solution to eq. (2.2) with

$$\pi = (a_1 \; a_2)(a_3 \; b_1)(b_2 \; b_3)(b_4 \; b_5)(c_1)(c_2), \tag{3.17}$$

while no proper subsequence solves it. In eq. (3.16) the sequence of mutation is divided into pieces by the dividing symbol $\parallel$; again each piece may be seen as a Coxeter sequence for a suitable $G \amalg G$ (sub)quiver with G either A_3 or D_4.

Thus we have constructed a 60-hyper c-saturating chamber for the E_8 MN theory [8]. The manifest unbroken flavor symmetry in this finite chamber is $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)^6$ whose Weyl group is realized as permutations of the charge vector sets $\{e_{a_1}, e_{a_2}\}$ and $\{e_{b_2}, e_{b_3}\}$.

Again the 60-hyper chamber is *not* unique; for instance, another 60-mutation solution is given by the node sequence

$$c_2 \; c_1 \; b_1 \; b_4 \; b_2 \; a_1 \; a_3 \; a_2 \; c_1 \; c_2 \; b_5 \; b_3 \; b_2 \; b_4 \; c_2 \; c_1 \; a_1 \; a_3 \; a_2 \; b_1 \; b_5 \; b_3 \; c_1 \; c_2 \; a_1 \; a_3 \; a_2 \; b_2 \; b_5 \; b_3$$
$$c_2 \; c_1 \; b_2 \; b_5 \; b_3 \; b_4 \; b_1 \; a_1 \; c_1 \; c_2 \; a_3 \; a_2 \; a_1 \; b_1 \; c_2 \; c_1 \; a_3 \; a_2 \; b_2 \; b_5 \; b_4 \; b_3 \; c_1 \; c_2 \; a_3 \; a_2 \; b_1 \; b_2 \; b_5 \; b_3.$$
$$\tag{3.18}$$

3.5 Decoupling and other finite chambers

Sending the mass parameter dual to a charge e_{b_j} to infinity, $|Z(e_{b_j})| \to +\infty$, all BPS states with non-zero e_{b_j}-charge get infinitely massive and decouple. The surviving states correspond to representations X of the $Q(r,s)$ quiver with $\dim X_{b_j} = 0$, which are stable representations of the quiver $Q(r, s-1)$.

Assuming the decoupling limit may be taken while remaining in the domain $\mathcal{D}_{\mathrm{fin}}$, consistency requires that if we delete from the list of states in table 3 all those which contain the given b_j ($j = 1, 2, \ldots, 5$) with non-zero coefficient, what remains should be the BPS spectrum of the E_7 Minahan-Nemeshanski model in some (not necessarily canonical) finite chamber. In the same vein, a similar truncation of the list in table 2 should produce a finite BPS spectrum of E_6 MN. The fact the all the BPS spectra so obtained are related by the Wall Crossing Formula [28–32] to the canonical chamber determined above is a highly non-trivial check on the procedure.

This decoupling procedure applied to E_7 produces BPS chambers with 27 hypermultiplets, which are easily shown to be equivalent to the canonical 24-hypers one. In the E_8 case we get a chamber of E_7 with either 42 or 43 hypers, depending of which e_{b_j} charge we make infinitely heavy.

$a_1, \ a_2, \ a_3, \ b_1, \ b_2, \ b_3, \ a_1 + a_2 + a_3 + c_1, \ b_1 + b_2 + b_3 + c_2, \ a_2 + a_3 + c_1, \ a_1 + a_3 + c_1,$
$a_1 + a_2 + a_3 + b_4 + c_1, \ b_2 + b_3 + c_2, \ b_1 + b_3 + c_2, \ b_1 + b_2 + c_2, \ a_1 + a_2 + 2\,a_3 + b_4 + 2\,c_1,$
$b_1 + b_2 + b_3 + 2\,c_2, \ a_1 + a_3 + b_4 + c_1, \ a_2 + a_3 + b_4 + c_1, \ a_1 + a_2 + b_1 + b_2 + b_3 + c_1 + 2\,c_2,$
$a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + b_5 + c_1 + 2\,c_2, \ b_1 + c_2, \ a_3 + c_1, \ a_3 + b_4 + c_1,$
$2\,a_1 + 2\,a_2 + a_3 + 2\,b_1 + b_2 + b_3 + b_5 + 2\,c_1 + 3\,c_2,$
$2\,a_1 + 2\,a_2 + a_3 + b_1 + b_2 + b_3 + b_5 + 2\,c_1 + 2\,c_2, \ a_3 + b_2 + b_4 + c_1 + c_2,$
$a_3 + b_3 + b_4 + c_1 + c_2, \ a_1 + a_2 + a_3 + b_1 + b_5 + c_1 + c_2, \ a_3 + b_2 + b_3 + b_4 + c_1 + 2\,c_2,$
$a_1 + a_2 + a_3 + b_5 + c_1, \ a_2 + a_3 + b_5 + c_1, \ a_1 + a_3 + b_5 + c_1, \ b_3 + c_2, \ b_2 + c_2,$
$a_1 + a_2 + a_3 + b_4 + b_5 + c_1, \ a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + b_4 + 2\,c_1 + 3\,c_2,$
$a_1 + a_2 + b_1 + b_2 + b_3 + c_1 + 3\,c_2, \ a_1 + a_2 + 2\,a_3 + b_4 + 2\,b_5 + 2\,c_1,$
$a_1 + a_3 + b_4 + b_5 + c_1, \ a_2 + a_3 + b_4 + b_5 + c_1, \ a_1 + a_2 + b_2 + b_3 + c_1 + 2\,c_2,$
$a_1 + a_2 + b_1 + b_2 + c_1 + 2\,c_2, \ a_1 + a_2 + b_1 + b_3 + c_1 + 2\,c_2, \ a_3 + b_5 + c_1,$
$2\,a_1 + 2\,a_2 + b_1 + b_2 + b_3 + 2\,c_1 + 3\,c_2, \ a_3 + b_4 + b_5 + c_1, \ a_1 + a_2 + b_1 + c_1 + c_2,$
$a_3 + b_4 + b_5 + c_1 + c_2, \ a_1 + a_2 + b_3 + c_1 + c_2, \ a_1 + a_2 + b_2 + c_1 + c_2,$
$b_4, \ b_5, \ a_1 + a_2 + c_1, \ b_4 + b_5 + c_2, \ a_2 + c_1, \ a_1 + c_1, \ b_5 + c_2, \ b_4 + c_2$

Table 3. Charge vectors of the 60 BPS particles in the chamber $\mathcal{C}_{\text{fin}}$ of the E_8 MN theory (in decreasing phase order). One has to add the PCT conjugate *anti*–particles.

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