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Borcherds and Kac-Moody extensions of simple finite-dimensional Lie algebras

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Abstract: We study the Borcherds superalgebra obtained by adding an odd (fermionic) null root to the set of simple roots of a simple finite-dimensional Lie algebra. We compare it to the Kac-Moody algebra obtained by replacing the odd null root by an ordinary simple root, and then adding more simple roots, such that each node that we add to the Dynkin diagram is connected to the previous one with a single line. This generalizes the situation in maximal supergravity, where the E_n symmetry algebra can be extended either to a Borcherds superalgebra or to the Kac-Moody algebra E_{11}, and both extensions can be used to derive the spectrum of p-form potentials in the theory. We show that also in the general case, the Borcherds and Kac-Moody extensions lead to the same 'p-form spectrum' of representations of the simple finite-dimensional Lie algebra.

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Contents

1 Introduction

Maximal supergravity in D dimensions contains p-form potentials that transform in representations of a global symmetry group. Including also the non-dynamical $(D-1)$- and D-forms that are possible to add to the theory, all these representations can be derived from the infinite-dimensional Kac-Moody algebra E_{11} [1–6]. Considering E_{11} as the 'very extension' of (the split real form of) the exceptional Lie algebra E_8, the corresponding derivation also works for half-maximal supergravity theories, where the role of E_8 is played by B_7, B_8 or D_8 [7]. The spectrum of p-form representations for maximal supergravity in D dimensions can alternatively be derived from a Borcherds algebra which depends on D [8–12]. The fact that these Borcherds algebras lead to the same p-form spectrum as E_{11} was explained in [10], and an alternative explanation was given in [13]. Since Borcherds algebras arise from Bianchi identities in half-maximal supergravity [14] as well as in maximal theories [11, 12] it raises the question whether also these Borcherds algebras lead to the same p-form spectra as the corresponding very extended Kac-Moody algebras. The present paper gives an affirmative answer to that question, by generalizing the result in [13].

The Borcherds algebra associated to maximal supergravity in $3 \leq D \leq 7$ dimensions, with a Lie algebra $\mathfrak{g}$ corresponding to the global symmetry group, can be constructed from $\mathfrak{g}$ by adding an extra simple root in a certain way (or equivalently, an extra node to the Dynkin diagram of $\mathfrak{g}$). It is in fact not a Lie algebra but a Lie *superalgebra*, where the eigenvectors corresponding to the extra simple root are odd elements, and furthermore the eigenvalues are zero. If we instead add $N \geq 1$ ordinary simple roots (such that each node that we add to the Dynkin diagram is connected to the previous one with a single line), then we obtain a Kac-Moody algebra, which for $N = D$ is the very extended Kac-Moody algebra E_{11}. Up to level $p = N$, the level decomposition of this Kac-Moody algebra under $\mathfrak{g} \oplus \mathfrak{sl}_N$, restricted to antisymmetric $\mathfrak{sl}_N$ tensors, gives the same 'p-form spectrum' of $\mathfrak{g}$ representations as the level decomposition of the Borcherds algebra under $\mathfrak{g}$. We will show that the corresponding result holds for any such Borcherds and Kac-Moody extensions of a simple finite-dimensional Lie algebra $\mathfrak{g}$. More precisely, for *any* such $\mathfrak{g}$, *any* way of adding the first extra node (the only extra node for the Borcherds algebra), and *any* total number

N of extra nodes for the Kac-Moody algebra, the two algebras lead to the same p-form spectrum (up to level $p = N$). The fact that N can be larger than the spacetime dimension D is important for applications to the superspace approach that has been employed recently in [11, 12, 14], since p-form superfields with $p > D$ need not be zero.

In this paper we denote the Borcherds and Kac-Moody extensions by U and W, and we let V be an intermediate Kac-Moody algebra. The algebras U, V and W are described in section 2, 3 and 4, respectively, and in the end we show that W gives the same p-form spectrum as U. The reader who finds the paper difficult to follow is invited to read [13] first, where U, V and W corresponds U_{n+1}, E_{n+1} and E_{11}.

2 The Borcherds algebra U

As indicated in the introduction, our definition of Borcherds algebras include also the generalization [15] of the original Borcherds algebras [16] to superalgebras. However, we will here only define a very special case of such superalgebras, and refer to [17] for the full definition. As noted in [10], footnote 8, there is an error in the definition in [17], but this has no importance for the special cases we consider here.

A Borcherds algebra is given by a (generalized) Cartan matrix a_{IJ}, which is a non-degenerate symmetric real matrix, where the rows and columns are labelled by some index set. This set can in general be infinite, but here we restrict it to be finite and write $I, J, \ldots = 0, 1, \ldots, r$ for some r. For each value I of the indices we associate two Chevalley generators e_I and f_I which are both either odd (fermionic) or even (bosonic) elements of the Borcherds algebra. We assume e_0 and f_0 to be odd, and use the indices $i, j, \ldots = 1, 2, \ldots, r$ for the even generators. Furthermore, we assume that $a_{00} = 0$ and $a_{ii} > 0$. With these restrictions, the conditions that define a_{IJ} to be the Cartan matrix of a Borcherds algebra are

$$I \neq J \Rightarrow a_{IJ} \leq 0, \qquad 2\frac{a_{iJ}}{a_{ii}} \in \mathbb{Z}. \tag{2.1}$$

Note that this matrix is symmetric, unlike general Cartan matrices of Kac-Moody algebras with the standard definition (see for example [18, 19]). However, we can 'de-symmetrize' a_{IJ} and define an in general non-symmetric matrix A_{IJ} by

$$A_{iJ} = 2\frac{a_{iJ}}{a_{ii}}, \qquad A_{0i} = a_{0i}, \qquad A_{00} = a_{00} = 0. \tag{2.2}$$

Any multiple of a_{IJ} gives the same Borcherds algebra as a_{IJ}. Together with the second condition in (2.1), this implies that we can assume all the diagonal entries a_{ii} to be even integers. It then follows from the same condition that all the entries in a_{IJ} are integers, in particular a_{0i}. We conclude that A_{IJ} is an integer-valued matrix, with $A_{00} = 0$ and $A_{ii} = 2$. The off-diagonal entries are non-positive integers, in general with $A_{IJ} \neq A_{JI}$, but if $A_{IJ} = 0$, then $A_{JI} = 0$ as well. Thus A_{ij} satisfies the definition of a Cartan matrix of a Kac-Moody algebra, and, as a last restriction, we require this Kac-Moody algebra to be finite, that is, a simple finite-dimensional Lie algebra $\mathfrak{g}$.

The Borcherds algebra U associated to a_{IJ} (or A_{IJ}) is now defined as the Lie superalgebra generated by the Chevalley generators e_I, f_I and $h_I = [\![e_I, f_I]\!]$ modulo the relations

$$[\![h_I, e_J]\!] = A_{IJ}e_J, \qquad [\![h_I, f_J]\!] = -A_{IJ}f_J, \qquad [\![e_I, f_J]\!] = \delta_{IJ}h_J,$$
$$[\![e_0, e_0]\!] = [\![f_0, f_0]\!] = (\text{ad } e_i)^{1-A_{iJ}}(e_J) = (\text{ad } f_i)^{1-A_{iJ}}(f_J) = 0, \tag{2.3}$$

where $i \neq J$, and $[\![x, y]\!]$ denotes the supercommutator of two elements x and y. This is a symmetric anticommutator $[\![x, y]\!] \equiv \{x, y\} = \{y, x\}$ if both x and y are odd elements, and an ordinary antisymmetric commutator $[\![x, y]\!] \equiv [x, y] = -[y, x]$ if at least one of the elements is even.

The Borcherds algebra U has a bilinear form, which we write as $\langle x|y\rangle$ for two elements x and y, and define by

$$\langle h_I|h_J\rangle = A_{IJ}, \qquad \langle e_I|f_J\rangle = \delta_{IJ}, \qquad \langle e_I|e_J\rangle = \langle f_I|f_J\rangle = \langle h_I|e_J\rangle = \langle h_I|f_J\rangle = 0. \tag{2.4}$$

The definition can then be extended to the full algebra U in such a way that the bilinear form is invariant,

$$\langle [\![x, y]\!]|z\rangle = \langle x|[\![y, z]\!]\rangle, \tag{2.5}$$

and supersymmetric, which means that $\langle x|y\rangle = -\langle y|x\rangle$ if both elements are odd, and $\langle x|y\rangle = \langle y|x\rangle$ if at least one of them is even.

The odd generators e_0 and f_0 give rise to a $\mathbb{Z}$-grading of U which is consistent with the $\mathbb{Z}_2$-grading that U naturally is equipped with as a superalgebra. This means that it can be written as a direct sum of subspaces U_p for all integers p, such that

$$[\![U_p, U_q]\!] \subseteq U_{p+q}, \tag{2.6}$$

where U_p consists of odd elements if p is odd, and of even elements if p is even. (These subspaces should not be confused with U_{n+1} in [13], which is simply U here.) Among the Chevalley generators e_0 belongs to U_{-1}, whereas f_0 belongs to U_1, and all the others belong to U_0.

It follows from the grading (2.6) that each subspace U_p constitute a representation $\mathbf{r}_p$ of $\mathfrak{g}$ (called $\mathbf{s}_p$ in [13]). One can easily see that there is an isomorphism between the subspaces U_1 and U_{-1}, such that elements mapped to each other have eigenvalues with opposite signs under the adjoint action of h_i, and therefore the representations $\mathbf{r}_1$ and $\mathbf{r}_{-1}$ are conjugate to each other. Accordingly, we introduce indices

$$\mathcal{M}, \mathcal{N}, \ldots = 1, 2, \ldots, \dim \mathbf{r}_1, \tag{2.7}$$

and write the basis elements of U_{-1} and U_1 as $E_\mathcal{M}$ and $F^\mathcal{N}$, respectively, chosen such that $\langle E_\mathcal{M}|F^\mathcal{N}\rangle = \delta_\mathcal{M}{}^\mathcal{N}$. For $p \geq 2$ the subspace U_{-p} is then spanned by the elements

$$E_{\mathcal{M}_1\cdots\mathcal{M}_p} \equiv [\![E_{\mathcal{M}_1}, [\![E_{\mathcal{M}_2}, \ldots, [\![E_{\mathcal{M}_{p-1}}, E_{\mathcal{M}_p}]\!]\cdots]\!]]\!] \tag{2.8}$$

and U_p by the elements

$$F^{\mathcal{N}_1\cdots\mathcal{N}_p} \equiv [\![F^{\mathcal{N}_1}, [\![F^{\mathcal{N}_2}, \ldots, [\![F^{\mathcal{N}_{p-1}}, F^{\mathcal{N}_p}]\!]\cdots]\!]]\!]. \tag{2.9}$$

As explained in [13], each representation $\mathbf{r}_p$ is determined by the lower (or upper) indices in the tensor

$$f_{\mathcal{N}_1\cdots\mathcal{N}_p}{}^{\mathcal{P}_1\cdots\mathcal{P}_p} = \langle E_{\mathcal{N}_1\cdots\mathcal{N}_p}|F^{\mathcal{P}_1\cdots\mathcal{P}_p}\rangle, \tag{2.10}$$

and all such tensors can be computed recursively, starting from the constants

$$f_{\mathcal{M}}{}^{\mathcal{N}}{}_{\mathcal{P}}{}^{\mathcal{Q}} = \langle[\{E_{\mathcal{M}}, F^{\mathcal{N}}\}, E_{\mathcal{P}}]|F^{\mathcal{Q}}\rangle, \tag{2.11}$$

which are the structure constants of U_{-1} considered as a (generalized Jordan) triple system with the triple product $[\{E_{\mathcal{M}}, F^{\mathcal{N}}\}, E_{\mathcal{P}}]$. To find the recursion formula, we first use the Jacobi identity to compute

$$\begin{aligned}
[\![F^{\mathcal{N}}, E_{\mathcal{M}_1\cdots\mathcal{M}_p}]\!] &= [\![\{F^{\mathcal{N}}, E_{\mathcal{M}_1}\}, E_{\mathcal{M}_2\cdots\mathcal{M}_p}]\!] \\
&\qquad - [\![E_{\mathcal{M}_1}, [\![F^{\mathcal{N}}, E_{\mathcal{M}_2\cdots\mathcal{M}_p}]\!]]\!] \\
&= [\![\{F^{\mathcal{N}}, E_{\mathcal{M}_1}\}, E_{\mathcal{M}_2\cdots\mathcal{M}_p}]\!] \\
&\qquad - [\![E_{\mathcal{M}_1}, [\![\{F^{\mathcal{N}}, E_{\mathcal{M}_2}\}, E_{\mathcal{M}_3\cdots\mathcal{M}_p}]\!]]\!] \\
&\qquad + [\![E_{\mathcal{M}_1}, [\![E_{\mathcal{M}_2}, [\![F^{\mathcal{N}}, E_{\mathcal{M}_3\cdots\mathcal{M}_p}]\!]]\!]]\!] \\
&= [\![\{F^{\mathcal{N}}, E_{\mathcal{M}_1}\}, E_{\mathcal{M}_2\cdots\mathcal{M}_p}]\!] \\
&\qquad - [\![E_{\mathcal{M}_1}, [\![\{F^{\mathcal{N}}, E_{\mathcal{M}_2}\}, E_{\mathcal{M}_3\cdots\mathcal{M}_p}]\!]]\!] \\
&\qquad + [\![E_{\mathcal{M}_1}, [\![E_{\mathcal{M}_2}, [\![\{F^{\mathcal{N}}, E_{\mathcal{M}_3}\}, E_{\mathcal{M}_4\cdots\mathcal{M}_p}]\!]]\!]]\!] \\
&\qquad\ \cdots \\
&\qquad + (-1)^{p+1}[\![E_{\mathcal{M}_1}, [\![E_{\mathcal{M}_2}, \ldots, [\![E_{\mathcal{M}_{p-1}}, \{F^{\mathcal{N}}, E_{\mathcal{M}_p}\}]\!]\cdots]\!]]\!] \\
&= \sum_{i=1}^{p-1}\sum_{j=i+1}^{p} (-1)^{i+1} f_{\mathcal{M}_i}{}^{\mathcal{N}}{}_{\mathcal{M}_j}{}^{\mathcal{P}} E_{\mathcal{M}_1\cdots\mathcal{M}_{i-1}\mathcal{M}_{i+1}\cdots\mathcal{M}_{j-1}\mathcal{P}\mathcal{M}_{j+1}\cdots\mathcal{M}_p} \\
&\qquad\quad + (-1)^p f_{\mathcal{M}_p}{}^{\mathcal{N}}{}_{\mathcal{M}_{p-1}}{}^{\mathcal{P}} E_{\mathcal{M}_1\cdots\mathcal{M}_{p-2}\mathcal{P}},
\end{aligned} \tag{2.12}$$

and then, using the invariance of the bilinear form, we obtain

$$\begin{aligned}
f_{\mathcal{M}_1\cdots\mathcal{M}_p}{}^{\mathcal{N}_1\cdots\mathcal{N}_p} &= \langle E_{\mathcal{M}_1\cdots\mathcal{M}_p}|F^{\mathcal{N}_1\cdots\mathcal{N}_p}\rangle \\
&= (-1)^{p+1}\langle[\![F^{\mathcal{N}_1}, E_{\mathcal{M}_1\cdots\mathcal{M}_p}]\!]|F^{\mathcal{N}_2\cdots\mathcal{N}_p}\rangle \\
&= \sum_{i=1}^{p-1}\sum_{j=i+1}^{p} (-1)^{i+p} f_{\mathcal{M}_i}{}^{\mathcal{N}_1}{}_{\mathcal{M}_j}{}^{\mathcal{P}} f_{\mathcal{M}_1\cdots\mathcal{M}_{i-1}\mathcal{M}_{i+1}\cdots\mathcal{M}_{j-1}\mathcal{P}\mathcal{M}_{j+1}\cdots\mathcal{M}_p}{}^{\mathcal{N}_2\cdots\mathcal{N}_p} \\
&\qquad\quad - f_{\mathcal{M}_p}{}^{\mathcal{N}_1}{}_{\mathcal{M}_{p-1}}{}^{\mathcal{P}} f_{\mathcal{M}_1\cdots\mathcal{M}_{p-2}\mathcal{P}}{}^{\mathcal{N}_2\cdots\mathcal{N}_p}.
\end{aligned} \tag{2.13}$$

The subspace U_0 is spanned by $\mathfrak{g}$ and h_0. Since U_0 is a finite-dimensional representation of $\mathfrak{g}$ it must be fully reducible, and since its dimension is $(\dim \mathfrak{g} + 1)$ it must (as a Lie algebra) be the direct sum of $\mathfrak{g}$ and a one-dimensional abelian subalgebra, spanned by an element c. It then follows from the invariance of the bilinear form that the commutation relations between the elements in U_0 and $U_{\pm 1}$ are

$$\begin{aligned}
\{E_{\mathcal{M}}, F^{\mathcal{N}}\} &= (t_\alpha)_{\mathcal{M}}{}^{\mathcal{N}} t^\alpha + \delta_{\mathcal{M}}{}^{\mathcal{N}} c, & [t^\alpha, c] &= 0, \\
[t^\alpha, E_{\mathcal{M}}] &= (t^\alpha)_{\mathcal{M}}{}^{\mathcal{N}} E_{\mathcal{N}}, & [c, E_{\mathcal{M}}] &= \langle c|c\rangle E_{\mathcal{M}}, \\
[t^\alpha, F^{\mathcal{N}}] &= -(t^\alpha)_{\mathcal{M}}{}^{\mathcal{N}} F^{\mathcal{M}}, & [c, F^{\mathcal{N}}] &= -\langle c|c\rangle F^{\mathcal{N}},
\end{aligned} \tag{2.14}$$

where t^α are the basis elements of $\mathfrak{g}$, and $(t_\alpha)_\mathcal{M}{}^\mathcal{N}$ are the components of t_α in the representation $\mathbf{r}_1$. The adjoint index α has been lowered with the restriction of the invariant bilinear form to $\mathfrak{g}$ (the Killing form), so that $\langle t^\alpha | t_\beta \rangle = \delta^\alpha{}_\beta$, and the normalization of c has been fixed by the first equation in (2.14) as we will see in the next section. Thus we end up with the expression

$$f_\mathcal{M}{}^\mathcal{N}{}_\mathcal{P}{}^\mathcal{Q} = \langle [\{E_\mathcal{M}, F^\mathcal{N}\}, E_\mathcal{P}] | F^\mathcal{Q} \rangle = (t_\alpha)_\mathcal{M}{}^\mathcal{N} (t^\alpha)_\mathcal{P}{}^\mathcal{Q} + \langle c|c \rangle \, \delta_\mathcal{M}{}^\mathcal{N} \delta_\mathcal{P}{}^\mathcal{Q} \tag{2.15}$$

for the structure constants $f_\mathcal{M}{}^\mathcal{N}{}_\mathcal{P}{}^\mathcal{Q}$, which can then be inserted in (2.13).

3 The Kac-Moody algebra V

Let B_{IJ} be the matrix obtained from A_{IJ} by replacing the entry $A_{00} = 0$ by $B_{00} = 2$. Thus we have

$$B_{00} = 2, \qquad B_{Ii} = A_{Ii}, \qquad B_{iI} = A_{iI}. \tag{3.1}$$

We then define the Kac-Moody algebra V associated to the Cartan matrix B_{IJ} as the Lie algebra generated by e_I, f_I and $h_I = [e_I, f_I]$ modulo the relations (2.3), but now with A_{IJ} replaced by B_{IJ}, and all Chevalley generators being even elements, so that the supercommutators are ordinary antisymmetric commutators. The relations corresponding to (2.4) define a bilinear form on V which is invariant and, unlike the one on U, fully symmetric. We write it as $(x|y)$ for two elements x and y to distinguish it from the invariant bilinear form on U. Note that we have $(h_0|h_0) = 2$, whereas $\langle h_0 | h_0 \rangle = 0$.

In the same way as for U, the generators e_0 and f_0 give rise to a $\mathbb{Z}$-grading of V, where each subspace V_p constitutes a representation $\mathbf{s}_p$ of $\mathfrak{g}$. The difference between A_{IJ} and B_{IJ} does not affect the commutation relations between $\mathfrak{g}$ and e_I or f_I, and therefore we have $\mathbf{s}_{\pm 1} = \mathbf{r}_{\pm 1}$. Furthermore, the Lie algebra V_0 is, in the same way as U_0, the direct sum of $\mathfrak{g}$ and a one-dimensional abelian subalgebra spanned by an element d. Using the same notation for the basis elements of $V_{\pm 1}$ as for $U_{\pm 1}$, the commutation relations between the elements in V_0 and $V_{\pm 1}$ are then

$$\begin{aligned}
[E_\mathcal{M}, F^\mathcal{N}] &= (t_\alpha)_\mathcal{M}{}^\mathcal{N} t^\alpha + \delta_\mathcal{M}{}^\mathcal{N} d, & [t^\alpha, d] &= 0, \\
[t^\alpha, E_\mathcal{M}] &= (t^\alpha)_\mathcal{M}{}^\mathcal{N} E_\mathcal{N}, & [d, E_\mathcal{M}] &= (d|d) E_\mathcal{M}, \\
[t^\alpha, F^\mathcal{N}] &= -(t^\alpha)_\mathcal{M}{}^\mathcal{N} F^\mathcal{M}, & [d, F^\mathcal{N}] &= -(d|d) F^\mathcal{N}.
\end{aligned} \tag{3.2}$$

Let us compare d in V_0 with the corresponding element c in U_0. From the invariance of the bilinear form it follows that c and d are determined up to normalization by the conditions $\langle c|\mathfrak{g} \rangle = 0$ and $(d|\mathfrak{g}) = 0$, respectively. This implies in turn that both c and d are linear combinations

$$\begin{aligned}
c &= c_0 h_0 + c_1 h_1 + \cdots + c_r h_r, \\
d &= d_0 h_0 + d_1 h_1 + \cdots + d_r h_r
\end{aligned} \tag{3.3}$$

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(identifying the Chevalley generators of U and V with each other). The first equations in (2.14) and (3.2) fix the coefficients c_0 and d_0 to $c_0 = d_0 = 1$. Furthermore, the conditions $\langle c|\mathfrak{g}\rangle = 0$ and $(d|\mathfrak{g}) = 0$ do not involve A_{00} or B_{00}, which are the only entries that differ between A_{IJ} or B_{IJ}, so they are in fact equivalent, and we conclude that $c = d$. Now we have

$$\begin{aligned}[c, e_0] &= c_0[h_0, e_0] + c_1[h_1, e_0] + \cdots + c_r[h_r, e_0] \\ &= (c_0A_{00} + c_1A_{10} + \cdots + c_rA_{r0})e_0 \\ &= (c_1A_{10} + \cdots + c_rA_{r0})e_0 \end{aligned} \tag{3.4}$$

in U, and

$$\begin{aligned}[d, e_0] &= d_0[h_0, e_0] + d_1[h_1, e_0] + \cdots + d_r[h_r, e_0] \\ &= (d_0B_{00} + d_1B_{10} + \cdots + d_rB_{r0})e_0 \\ &= (2 + c_1A_{10} + \cdots + c_rA_{r0})e_0 \end{aligned} \tag{3.5}$$

in V. On the other hand, from (2.14) and (3.2) we have $[c, e_0] = \langle c|c\rangle e_0$ in U, and $[d, e_0] = (d|d)e_0$ in V, so we conclude that $(d|d) = \langle c|c\rangle + 2$. It follows that the structure constants of V_{-1} considered as a triple system are

$$\begin{aligned} g_{\mathcal{M}}{}^{\mathcal{N}}{}_{\mathcal{P}}{}^{\mathcal{Q}} = ([[E_{\mathcal{M}}, F^{\mathcal{N}}], E_{\mathcal{P}}]|F^{\mathcal{Q}}) &= (t_\alpha)_{\mathcal{M}}{}^{\mathcal{N}}(t^\alpha)_{\mathcal{P}}{}^{\mathcal{Q}} + (\langle c|c\rangle + 2)\delta_{\mathcal{M}}{}^{\mathcal{N}}\delta_{\mathcal{P}}{}^{\mathcal{Q}} \\ &= f_{\mathcal{M}}{}^{\mathcal{N}}{}_{\mathcal{P}}{}^{\mathcal{Q}} + 2\delta_{\mathcal{M}}{}^{\mathcal{N}}\delta_{\mathcal{P}}{}^{\mathcal{Q}}. \end{aligned} \tag{3.6}$$

4 The extended Kac-Moody algebra W

Let C be the matrix obtained from B by adding $N-1$ more rows and columns, labelled by $m, n, \ldots = -N+1, -N+2, \ldots, -1$, so that

$$C_{IJ} = B_{IJ}, \qquad\qquad C_{mI} = C_{Im} = 0, \tag{4.1}$$

and C_{mn} is the well known Cartan matrix of $A_{N-1} = \mathfrak{sl}_N$. Let W be the Kac-Moody algebra given by the Cartan matrix C. This corresponds to adding $N-1$ more nodes to the Dynkin diagram of V, each connected to the previous one by a single line.

In the same way as for U and V, the generators e_0 and f_0 give rise to a $\mathbb{Z}$-grading of W, where each subspace W_p constitutes a representation $\mathbf{t}_p$ of $\mathfrak{g}$, but also a representation of $\mathfrak{sl}_N$. Considering V as a subalgebra of W we can write the basis elements of W_1 and W_{-1} as $E_{\mathcal{M}a}$ and $F^{\mathcal{M}b}$, respectively, where $a, b, \ldots = 0, 1, \ldots, N-1$, and

$$\begin{aligned} E_{\mathcal{M}0} &= E_{\mathcal{M}}, \qquad & E_{\mathcal{M}(-m)} &= [[\cdots[[e_m, e_{m+1}], e_{m+2}], \ldots, e_{-1}], E_{\mathcal{M}}], \\ F^{\mathcal{M}0} &= F^{\mathcal{M}}, \qquad & F^{\mathcal{M}(-m)} &= (-1)^m[[\cdots[[f_m, f_{m+1}], f_{m+2}], \ldots, f_{-1}], F^{\mathcal{M}}]. \end{aligned} \tag{4.2}$$

For $p \geq 2$, the subspace W_p is then spanned by the elements

$$E_{\mathcal{M}_1\cdots\mathcal{M}_p\ a_1\cdots a_p} = \big[E_{\mathcal{M}_1\, a_1}, \big[E_{\mathcal{M}_2\, a_2}, \ldots, \big[E_{\mathcal{M}_{p-1}\, a_{p-1}}, E_{\mathcal{M}_p\, a_p}\big]\cdots\big]\big], \tag{4.3}$$

and W_{-p} by the elements

$$F^{\mathcal{M}_1\cdots\mathcal{M}_p\;a_1\cdots a_p} = \left[F^{\mathcal{M}_1\,a_1}, \left[F^{\mathcal{M}_2\,a_2}, \ldots, \left[F^{\mathcal{M}_{p-1}\,a_{p-1}}, F^{\mathcal{M}_p\,a_p}\right]\cdots\right]\right]. \tag{4.4}$$

Following the steps in [20] it is straightforward to show that the structure constants of the triple system W_{-1} are related to those of V_{-1} as

$$\begin{aligned} h_{\mathcal{M}}{}^{\mathcal{N}}{}_{\mathcal{P}}{}^{\mathcal{Q}}{}_a{}^b{}_c{}^d &= ([[E_{\mathcal{M}a}, F^{\mathcal{N}b}], E_{\mathcal{P}c}]|F^{\mathcal{Q}d}) \\ &= g_{\mathcal{M}}{}^{\mathcal{N}}{}_{\mathcal{P}}{}^{\mathcal{Q}}\,\delta_a{}^b\delta_c{}^d - \delta_{\mathcal{M}}{}^{\mathcal{N}}\delta_{\mathcal{P}}{}^{\mathcal{Q}}\,\delta_a{}^b\delta_c{}^d + \delta_{\mathcal{M}}{}^{\mathcal{N}}\delta_{\mathcal{P}}{}^{\mathcal{Q}}\,\delta_c{}^b\delta_a{}^d, \end{aligned} \tag{4.5}$$

and if we antisymmetrize in a and c we obtain

$$h_{\mathcal{M}}{}^{\mathcal{N}}{}_{\mathcal{P}}{}^{\mathcal{Q}}{}_{[a}{}^b{}_{c]}{}^d = (g_{\mathcal{M}}{}^{\mathcal{N}}{}_{\mathcal{P}}{}^{\mathcal{Q}} - 2\,\delta_{\mathcal{M}}{}^{\mathcal{N}}\delta_{\mathcal{P}}{}^{\mathcal{Q}})\delta_{[a}{}^b\delta_{c]}{}^d = f_{\mathcal{M}}{}^{\mathcal{N}}{}_{\mathcal{P}}{}^{\mathcal{Q}}\,\delta_{[ac]}{}^{bd}. \tag{4.6}$$

Thus we get back the structure constants (2.15) for the triple system U_{-1}, times $\delta_{[a}{}^b\delta_{c]}{}^d$. As we will see next, this relation between the two triple systems can be viewed as the reason why U and W lead to the same p-form spectrum, or to be precise, why $\mathbf{t}_p = \mathbf{r}_p$ for $1 \le p \le N$, which is the main result of this paper.

As for U, each representation $\mathbf{t}_p$ is determined by the lower indices in the tensor

$$h_{\mathcal{M}_1\cdots\mathcal{M}_p}{}^{\mathcal{N}_1\cdots\mathcal{N}_p}{}_{[a_1\cdots a_p]}{}^{b_1\cdots b_p} = (E_{\mathcal{M}_1\cdots\mathcal{M}_p\;a_1\cdots a_p}|F^{\mathcal{N}_1\cdots\mathcal{N}_p\;b_1\cdots b_p}). \tag{4.7}$$

In the same way as we obtained (2.13) for U, we now obtain

$$\begin{aligned} h_{\mathcal{M}_1\cdots\mathcal{M}_p}{}^{\mathcal{N}_1\cdots\mathcal{N}_p}{}_{[a_1\cdots a_p]}{}^{b_1\cdots b_p} &= (E_{\mathcal{M}_1\cdots\mathcal{M}_p\,[a_1\cdots a_p]}|F^{\mathcal{N}_1\cdots\mathcal{N}_p\;b_1\cdots b_p}) \\ &= \sum_{i=1}^{p-1}\sum_{j=i+1}^{p} h_{\mathcal{M}_i}{}^{\mathcal{N}_1}{}_{\mathcal{M}_j}{}^{\mathcal{P}}{}_{[a_i}{}^{b_1}{}_{a_j}{}^{c}\times \\ &\qquad \times h_{\mathcal{M}_1\cdots\mathcal{P}\cdots\mathcal{M}_p}{}^{\mathcal{N}_2\cdots\mathcal{N}_p}{}_{a_1\cdots a_{i-1}a_{i+1}\cdots a_{j-1}|c|a_{j+1}\cdots a_p]}{}^{b_2\cdots b_p} \\ &\qquad - h_{\mathcal{M}_p}{}^{\mathcal{N}_1}{}_{\mathcal{M}_{p-1}}{}^{\mathcal{P}}{}_{[a_p}{}^{b_1}{}_{a_{p-1}}{}^{c} h_{\mathcal{M}_1\cdots\mathcal{M}_{p-2}\mathcal{P}}{}^{\mathcal{N}_2\cdots\mathcal{N}_p}{}_{a_1\cdots a_{p-2}]c}{}^{b_2\cdots b_p} \end{aligned} \tag{4.8}$$

for W, where we have simplified the notation by writing

$$\mathcal{M}_1\cdots\mathcal{P}\cdots\mathcal{M}_p = \mathcal{M}_1\cdots\mathcal{M}_{i-1}\mathcal{M}_{i+1}\cdots\mathcal{M}_{j-1}\mathcal{P}\mathcal{M}_{j+1}\cdots\mathcal{M}_p. \tag{4.9}$$

The difference compared to (2.13) is that f is replaced by h, that each $\mathbf{r}_1$ index is accompanied by an $\mathfrak{sl}_N$ index and, most important, that the prefactor $(-1)^{i+p}$ is replaced by 1. We will now show, by induction over p, that

$$h_{\mathcal{M}_1\cdots\mathcal{M}_p}{}^{\mathcal{N}_1\cdots\mathcal{N}_p}{}_{[a_1\cdots a_p]}{}^{b_1\cdots b_p} = (-1)^{\sigma(p)}\delta_{[a_1\cdots a_p]}{}^{b_1\cdots b_p}\,f_{\mathcal{M}_1\cdots\mathcal{M}_p}{}^{\mathcal{N}_1\cdots\mathcal{N}_p}, \tag{4.10}$$

for all integers $p \ge 1$, where $\sigma(p) = p(p-1)/2$. For $p = 1$ we have

$$h_{\mathcal{M}}{}^{\mathcal{N}}{}_a{}^b = (E_{\mathcal{M}a}|F^{\mathcal{N}b}) = \delta_a{}^b\delta_{\mathcal{M}}{}^{\mathcal{N}} = \delta_a{}^b\langle E_{\mathcal{M}}|F^{\mathcal{N}}\rangle = \delta_a{}^b f_{\mathcal{M}}{}^{\mathcal{N}}. \tag{4.11}$$

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Assume now that (4.10) holds for $p = q - 1$, where q is some integer $q \geq 2$. Then

$$\begin{aligned}
&h_{\mathcal{M}_1\cdots\mathcal{M}_q}{}^{\mathcal{N}_1\cdots\mathcal{N}_q}{}_{[a_1\cdots a_q]}{}^{b_1\cdots b_q}\\
&\qquad = \langle E_{\mathcal{M}_1\cdots\mathcal{M}_q\,[a_1\cdots a_q]}|F^{\mathcal{N}_1\cdots\mathcal{N}_q\,b_1\cdots b_q}\rangle\\
&\qquad = \sum_{i=1}^{q-1}\sum_{j=i+1}^{q} h_{\mathcal{M}_i}{}^{\mathcal{N}_1}{}_{\mathcal{M}_j}{}^{\mathcal{P}}{}_{[a_i}{}^{b_1}{}_{a_j}{}^{c}\times\\
&\qquad\qquad\qquad \times h_{\mathcal{M}_1\cdots\mathcal{P}\cdots\mathcal{M}_q}{}^{\mathcal{N}_2\cdots\mathcal{N}_q}{}_{a_1\cdots a_{i-1}a_{i+1}\cdots a_{j-1}|c|a_{j+1}\cdots a_q]}{}^{b_2\cdots b_q}\\
&\qquad\qquad - h_{\mathcal{M}_q}{}^{\mathcal{N}_1}{}_{\mathcal{M}_{q-1}}{}^{\mathcal{P}}{}_{[a_q}{}^{b_1}{}_{a_{q-1}}{}^{c} h_{\mathcal{M}_1\cdots\mathcal{M}_{q-2}\mathcal{P}}{}^{\mathcal{N}_2\cdots\mathcal{N}_q}{}_{a_1\cdots a_{q-2}]c}{}^{b_2\cdots b_q}\\
&\qquad = \sum_{i=1}^{q-1}\sum_{j=i+1}^{q} f_{\mathcal{M}_i}{}^{\mathcal{N}_1}{}_{\mathcal{M}_j}{}^{\mathcal{P}}\delta_{[a_i a_j}{}^{b_1 c}\times\\
&\qquad\qquad\qquad \times (-1)^{\sigma(q-1)} f_{\mathcal{M}_1\cdots\mathcal{P}\cdots\mathcal{M}_q}{}^{\mathcal{N}_2\cdots\mathcal{N}_q}\delta_{a_1\cdots a_{i-1}a_{i+1}\cdots a_{j-1}|c|a_{j+1}\cdots a_q]}{}^{[b_2\cdots b_q]}\\
&\qquad\qquad - (-1)^{\sigma(q-1)} f_{\mathcal{M}_q}{}^{\mathcal{N}_1}{}_{\mathcal{M}_{q-1}}{}^{\mathcal{P}}\delta_{[a_q a_{q-1}}{}^{b_1 c} f_{\mathcal{M}_1\cdots\mathcal{M}_{q-2}\mathcal{P}}{}^{\mathcal{N}_2\cdots\mathcal{N}_q}\delta_{a_1\cdots a_{q-2}]c}{}^{[b_2\cdots b_q]}\\
&\qquad = \sum_{i=1}^{q-1}\sum_{j=i+1}^{q} (-1)^{\sigma(q-1)} f_{\mathcal{M}_i}{}^{\mathcal{N}_1}{}_{\mathcal{M}_j}{}^{\mathcal{P}} f_{\mathcal{M}_1\cdots\mathcal{P}\cdots\mathcal{M}_q}{}^{\mathcal{N}_2\cdots\mathcal{N}_q}\delta_{[a_i a_1\cdots a_{i-1}a_{i+1}\cdots a_q]}{}^{b_1 b_2\cdots b_q}\\
&\qquad\qquad - (-1)^{\sigma(q-1)} f_{\mathcal{M}_q}{}^{\mathcal{N}_1}{}_{\mathcal{M}_{q-1}}{}^{\mathcal{P}} f_{\mathcal{M}_1\cdots\mathcal{M}_{q-2}\mathcal{P}}{}^{\mathcal{N}_2\cdots\mathcal{N}_q}\delta_{[a_q a_1\cdots a_{q-1}]}{}^{b_1 b_2\cdots b_q}\\
&\qquad = \sum_{i=1}^{q-1}\sum_{j=i+1}^{q} (-1)^{\sigma(q-1)+i+1} f_{\mathcal{M}_i}{}^{\mathcal{N}_1}{}_{\mathcal{M}_j}{}^{\mathcal{P}} f_{\mathcal{M}_1\cdots\mathcal{P}\cdots\mathcal{M}_q}{}^{\mathcal{N}_2\cdots\mathcal{N}_q}\delta_{[a_1\cdots a_q]}{}^{b_1\cdots b_q}\\
&\qquad\qquad + (-1)^{\sigma(q-1)+q} f_{\mathcal{M}_q}{}^{\mathcal{N}_1}{}_{\mathcal{M}_{q-1}}{}^{\mathcal{P}} f_{\mathcal{M}_1\cdots\mathcal{M}_{q-2}\mathcal{P}}{}^{\mathcal{N}_2\cdots\mathcal{N}_q}\delta_{[a_1\cdots a_q]}{}^{b_1\cdots b_q}\\
&\qquad = (-1)^{\sigma(q-1)+q-1}\delta_{[a_1\cdots a_q]}{}^{b_1\cdots b_q} f_{\mathcal{M}_1\cdots\mathcal{M}_q}{}^{\mathcal{N}_1\cdots\mathcal{N}_q}\\
&\qquad = (-1)^{\sigma(q)}\delta_{[a_1\cdots a_q]}{}^{b_1\cdots b_q} f_{\mathcal{M}_1\cdots\mathcal{M}_q}{}^{\mathcal{N}_1\cdots\mathcal{N}_q},
\end{aligned} \tag{4.12}$$

where we first have inserted the assumption of the induction, and then used (2.13). By the principle of induction, it follows that (4.10) holds for all integers $p \geq 1$. Since the lower $\mathbf{r}_1$ indices on the left hand side of (4.10) determine $\mathbf{r}_p$, and those on the right hand side determine $\mathbf{t}_p$, we conclude that $\mathbf{r}_p = \mathbf{t}_p$ as long as the delta factor does not vanish, that is, for $1 \leq p \leq N$.

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