Published for SISSA by Springer

Received: *April 15, 2011*
Accepted: *May 25, 2011*
Published: *June 27, 2011*

From correlators to Wilson loops in Chern-Simons matter theories

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Abstract: We study n-point correlation functions for chiral primary operators in three dimensional supersymmetric Chern-Simons matter theories. Our analysis is carried on in $\mathcal{N} = 2$ superspace and covers $\mathcal{N} = 2, 3$ supersymmetric CFT's, the $\mathcal{N} = 6$ ABJM and the $\mathcal{N} = 8$ BLG models. In the limit where the positions of adjacent operators become light-like, we find that the one-loop n-point correlator divided by its tree level expression coincides with a light-like n-polygon Wilson loop. Remarkably, the result can be simply expressed as a linear combination of five dimensional two-mass easy boxes. We manage to evaluate the integrals analytically and find a vanishing result, in agreement with previous findings for Wilson loops.

Keywords: Supersymmetry and Duality, Chern-Simons Theories

ArXiv ePrint: 1103.3675

doi:10.1007/JHEP06(2011)118

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1 Introduction

In the last few years, AdS/CFT correspondence and stringy-inspired technologies for computing scattering amplitudes have led to the discovery of new remarkable properties of supersymmetric Yang-Mills theories in four dimensions.

For planar $\mathcal{N} = 4$ SYM theory, a duality between MHV scattering amplitudes and light-like polygon Wilson loops has been found first at strong coupling [1, 2] and successively tested at weak coupling by a perturbative field theory approach [3]–[9]. On the field theory side this duality is related to the emergence of a new hidden symmetry, the dual superconformal symmetry [10, 11], which corresponds at strong coupling to the invariance of the type IIB string theory on $AdS_5 \times S_5$ under a suitable combination of bosonic and fermionic T-dualities [12]. The dual superconformal generators are part of the infinite set of generators of the Yangian symmetry of the theory [13], thus being intimately related to its integrability [14, 15]–[17].

More recently, a novel duality has been discovered [18] which involves correlation functions of gauge invariant BPS scalar operators of $\mathcal{N} = 4$ SYM theory. An n-point correlation function $\mathcal{C}_n$ in the limit where adjacent points become light-like separated should be equal

to a light-like n-polygon Wilson loop in the adjoint representation of the gauge group. The precise identification

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\mathcal{C}_n}{\mathcal{C}_n^{tree}} = \langle Tr_{adj} P \exp\left(ig \int_{\Gamma_n} dz^\mu A_\mu(z)\right)\rangle \tag{1.1}$$

has been tested perturbatively up to two loops in a number of cases [18].

While this new duality is still lacking a proof in the string theory regime, in field theory a physical explanation can be given in terms of an infinitely fast moving scalar particle interacting with a low energy gluon. In the light-like limit, in fact, the scalar particle flowing around the loop becomes infinitely energetic compared to the gluon it interacts with. As a consequence, its propagator becomes an almost free propagator, except for an eikonal phase $P\exp\left(ig\int_{\Gamma_n} dz^\mu A_\mu(z)\right)$ which arises as the result of a path integral saddle point approximation. According to this explanation, the connection between correlators and polygonal Wilson loops should be true not only for $\mathcal{N}=4$ SYM but also for general conformal gauge theories in any dimensions [18].

Since Wilson loops are dual to planar scattering amplitudes, a direct duality between n-point correlation functions and n-point scattering amplitudes must exist. This has been investigated in [19, 20], where this duality has been established at one-loop for generic n and proved at two loops for $n = 4, 5, 6$.

It is interesting to investigate whether the amplitudes/WL/correlators dualities and the existence of underlying hidden symmetries extend to class of theories in different dimensions for which a string dual description is known.

In this paper we concentrate on the class of $\mathcal{N}=2$, $U(N)_{k_1} \times U(M)_{k_2}$ Chern-Simons matter theories in three dimensions with generic (k_1, k_2) CS levels and generic superpotential. It includes, as particular cases, the $\mathcal{N}=6$ ABJM theory [21, 22] dual to a IIA string theory on $AdS_4 \times CP_3$, the $\mathcal{N}=8$ BLG theory [23–25] describing the low-energy dynamics of M2-branes in M-theory and $\mathcal{N}=2,3$ superconformal theories [26, 27] for which a dual description in terms of deformed backgrounds has been conjectured [28, 29].

For the ABJM theory, preliminary results are already available in the literature. At tree level, scattering amplitudes exhibit dual superconformal symmetry [30, 31] whose generators are the level-one generators of a Yangian symmetry. Invariance under Yangian symmetry has been explicitly proved at tree level for four and six-point amplitudes [32] and argued in general through the construction of a generating function [33]. These symmetries suggest that string theory on $AdS_4 \times CP_3$ should be self-dual under a suitable combination of bosonic and fermionic T-dualities. Efforts in this direction are complicated by the emergence of singularities [34]–[38].

A first indication of a duality between scattering amplitudes and Wilson loops comes from the fact that both the one-loop four-point amplitude [39] and the light-like square WL [40] vanish.

In this paper we move one step further in the direction of establishing amplitudes/WL/correlators dualities, by studying correlation functions of gauge invariant BPS scalar operators.

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For generic $\mathcal{N}=2$, two-level Chern-Simons matter theories we compute the n-point correlator at one-loop. We prove that in the light-like limit its expression, divided by the corresponding tree level correlator, coincides with the one-loop expression for a light-like n-polygon Wilson loop, once the Feynman combining parameters of the correlator integral are identified with the affine parameters which parametrize the light-like edges of the WL polygon. Remarkably, we find that both quantities can be expressed in terms of a five dimensional two-mass-easy box integral.

While in the ABJM case, and whenever $K_2 = -K_1$ and $M = N$, the identification gets trivialized by the fact that both the correlator and the Wilson loop are proportional to a vanishing color factor, in the more general cases the color factor in front is not zero and a non-trivial identification emerges.

We manage to compute the five dimensional box integral analytically and prove that, once plugged back into the correlation function/WL, it gives a vanishing result.

Our final statement is then

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\mathcal{C}_n^{\,1-loop}}{\mathcal{C}_n^{\,tree}} = \langle W_n \rangle^{1-loop} \;=\; 0 \qquad , \qquad \text{for any } n \tag{1.2}$$

This identity is true for any value of the CS levels and for N, M finite (no planar limit is required). It holds for the whole class of theories under study, independently of their degree of supersymmetry. This is a consequence of the fact that at the order we are working, the superpotential does not enter the calculation. Note that at one loop they are all superconformal theories, being the beta-functions trivially zero [26, 27]. We expect that theories with different number of supersymmetries and with or without superconformal invariance will undergo a different destiny starting from two-loops [45].

We stress that in the general case the identification between correlators and Wilson loops is valid independently of the fact that they both eventually vanish. Therefore, our result is a first non-trivial indication of a correlator/WL duality at work and supports the conjecture of [18] which states that this duality should hold for generic conformal gauge theories in any dimensions.

For $\mathcal{N} > 4$, four-point scattering amplitudes have been proved to vanish at one loop [39]. Therefore, for the special case $n = 4$, our result completes the amplitudes/WL/correlators duality for theories with a number of supersymmetries greater than four.

We prove that the n-polygon Wilson loop is zero at first order for any value of the CS levels and independently of the chiral couplings. Thus, the proof is true also for pure Chern-Simons theories, just set matter fields and one of the two gauge fields to zero. Therefore, our result provides the analytical proof of the conjecture made in [40] according to which one-loop light-like Wilson loops should vanish in pure Chern-Simons theories.

The paper is organized as follows. In section 2 we introduce the class of CS matter theories we are interested in and list the corresponding gauge invariant chiral operators. Working in $\mathcal{N}=2$ superspace, we classify different theories according to the particular choice of the coupling constants in the superpotential. In section 3 we focus on the evaluation of n-point correlators for dimension-one chiral operators. In particular, we evaluate

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the building block which enters one-loop calculations and discuss its representation in terms of a 5d two-mass easy box integral. In section 4 we prove that in the light-like limit, the expression for the one-loop correlator divided by its tree level counterpart is identical to the first order contribution to a light-like n-polygon Wilson loop. This identification holds independently of the value of the couplings and even before computing the actual Feynman integrals. In section 5, equipped with the exact result for the 5d box integral, we give the analytical proof that correlators and Wilson loops vanish at this order. In section 6 we prove that our results for dimension-one BPS operators extend to correlation functions of operators with arbitrary dimension. Conclusions with a discussion of future perspectives follow, plus appendix A which contains our conventions and appendix B where we present a detailed discussion of the unexpected emergence of a 5d box integral.

2 $\mathcal{N} = 2$ Chern-Simons matter theories

In three dimensions, we consider a $\mathcal{N} = 2$ supersymmetric $\mathrm{U}(N) \times \mathrm{U}(M)$ Chern-Simons theory for vector multiplets $(V, \hat{V})$ coupled to chiral multiplets A^i and B_i, $i = 1, 2$ in the fundamental representation of a global $\mathrm{SU}(2)_A \times \mathrm{SU}(2)_B$. The vector multiplets $V, \hat{V}$ are in the adjoint representation of the gauge groups $\mathrm{U}(N)$ and $\mathrm{U}(M)$ respectively, while A^i are in the $(N, \bar{M})$ and B_i in the $(\bar{N}, M)$ bifundamental representations.

In $\mathcal{N} = 2$ superspace the action reads (for superspace conventions see appendix A)

$$\mathcal{S} = \mathcal{S}_{\mathrm{CS}} + \mathcal{S}_{\mathrm{mat}} \tag{2.1}$$

with

$$\begin{aligned}
\mathcal{S}_{\mathrm{CS}} &= \int d^3x\, d^4\theta \int_0^1 dt \left\{ K_1 Tr\left[V \overline{D}^\alpha \left(e^{-tV} D_\alpha e^{tV} \right) \right] + K_2 Tr\left[\hat{V} \overline{D}^\alpha \left(e^{-t\hat{V}} D_\alpha e^{t\hat{V}} \right) \right] \right\} \\
\mathcal{S}_{\mathrm{mat}} &= \int d^3x\, d^4\theta\, Tr\left(\bar{A}_i e^V A^i e^{-\hat{V}} + \bar{B}^i e^{\hat{V}} B_i e^{-V} \right) \\
&\quad + \int d^3x d^2\theta\, Tr\left[h_1 (A^1 B_1)^2 + h_2 (A^2 B_2)^2 + h_3 (A^1 B_1 A^2 B_2) + h_4 (A^2 B_1 A^1 B_2) \right] + h.c.
\end{aligned} \tag{2.2}$$

Here $4\pi K_1$, $4\pi K_2$ are two independent integers, as required by gauge invariance of the effective action. In the perturbative regime we take $K_1, K_2 \gg N, M$.

For generic values of the couplings, the action (2.1) is invariant under the following gauge transformations

$$e^V \to e^{i\bar{\Lambda}_1} e^V e^{-i\Lambda_1} \qquad e^{\hat{V}} \to e^{i\bar{\Lambda}_2} e^{\hat{V}} e^{-i\Lambda_2} \tag{2.3}$$

$$A^i \to e^{i\Lambda_1} A^i e^{-i\Lambda_2} \qquad B_i \to e^{i\Lambda_2} B_i e^{-i\Lambda_1} \tag{2.4}$$

where Λ_1, Λ_2 are two chiral superfields parametrizing $\mathrm{U}(N)$ and $\mathrm{U}(M)$ gauge transformations, respectively. Antichiral superfields transform according to the conjugate of (2.4). The action is also invariant under the $\mathrm{U}(1)_R$ R-symmetry group under which the A^i and B_i fields have charges $\frac{1}{2}$.

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For special values of the couplings we can have enhancement of global symmetries and/or R-symmetry with consequent enhancement of supersymmetry.

For $K_1 = -K_2$ and $h_1 = h_2 = 0$ we have $\mathcal{N} = 2$ ABJM/ABJ-like CFT's [26]. In this case the theory is invariant under two global U(1)'s

$$\begin{aligned} \mathrm{U}(1)_A: \quad & A^1 \to e^{i\alpha} A^1\,, \qquad \mathrm{U}(1)_B: \quad B_1 \to e^{i\beta} B_1 \\ & A^2 \to e^{-i\alpha} A^2\,, \qquad\qquad\quad B_2 \to e^{-i\beta} B_2 \end{aligned} \tag{2.5}$$

If, in addition, we choose $h_3 = -h_4 \equiv h$, the global symmetry becomes $\mathrm{U}(1)_R \times \mathrm{SU}(2)_A \times \mathrm{SU}(2)_B$ and gets enhanced to $\mathrm{SU}(4)_R$ for $h = 1/K$ [21, 41]. For this particular values of the couplings we recover the $\mathcal{N} = 6$ superconformal ABJ theory [22] and for $N = M$ the ABJM theory [21].

In the more general case $K_1 \neq -K_2$, setting $h_1 = h_2 = \frac{1}{2}(h_3 + h_4)$ the corresponding superpotential reads

$$\mathcal{S}_{\rm pot} = \frac{1}{2} \int d^3x\, d^2\theta\, Tr\left[h_3 (A^i B_i)^2 + h_4 (B_i A^i)^2\right] + h.c. \tag{2.6}$$

This is the class of $\mathcal{N} = 2$ theories studied in [28, 29] with SU(2) invariant superpotential, where SU(2) rotates simultaneously A^i and B_i.

When $h_3 = -h_4$, that is $h_1 = h_2 = 0$, we have the particular set of $\mathcal{N} = 2$ theories with global $\mathrm{SU}(2)_A \times \mathrm{SU}(2)_B$ symmetry [28, 29]. This is the generalization of ABJ/ABJM-like theories to $K_1 \neq -K_2$. For particular values of the couplings [27] the theory is superconformal invariant.

Finally, another interesting fixed point corresponds to $h_3 = \frac{1}{K_1}$ and $h_4 = \frac{1}{K_2}$. The $\mathrm{U}(1)_R$ R-symmetry is enhanced to $\mathrm{SU}(2)_R$ and the theory is $\mathcal{N} = 3$ superconformal [27–29].

The quantization of the theory can be easily carried on in superspace after performing gauge fixing (for details, see for instance [26, 27]). In coordinate space and using Landau gauge, this leads to gauge propagators

$$\begin{aligned} \langle V^A(1)\, V^B(2)\rangle &= \frac{1}{4\pi K_1}\, \overline{D}^\alpha D_\alpha \frac{\delta^4(\theta_1 - \theta_2)}{|x_1 - x_2|}\, \delta^{AB} \\ \langle \hat{V}^A(1)\, \hat{V}^B(2)\rangle &= \frac{1}{4\pi K_2}\, \overline{D}^\alpha D_\alpha \frac{\delta^4(\theta_1 - \theta_2)}{|x_1 - x_2|}\, \delta^{AB} \end{aligned} \tag{2.7}$$

whereas the scalar propagators are

$$\begin{aligned} \langle \bar{A}^{\hat{a}}{}_a(1)\, A^b{}_{\hat{b}}(2)\rangle &= \frac{1}{4\pi}\, D^2 \bar{D}^2 \frac{\delta^4(\theta_1 - \theta_2)}{|x_1 - x_2|}\, \delta^{\hat{a}}{}_{\hat{b}}\, \delta_a{}^b \\ \langle \bar{B}^a{}_{\hat{a}}(1)\, B^{\hat{b}}{}_b(2)\rangle &= \frac{1}{4\pi}\, D^2 \bar{D}^2 \frac{\delta^4(\theta_1 - \theta_2)}{|x_1 - x_2|}\, \delta^a{}_b\, \delta_{\hat{a}}{}^{\hat{b}} \end{aligned} \tag{2.8}$$

The vertices needed for one-loop calculations can be easily read from the action (2.2)

$$\int d^3x\, d^4\theta\, \left[Tr(\bar{A}_i V A^i) - Tr(B_i V \bar{B}^i) + Tr(\bar{B}^i \hat{V} B_i) - Tr(A^i \hat{V} \bar{A}_i)\right] \tag{2.9}$$

We note that A and B vertices always carry an opposite sign.

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In $\mathcal{N}=2$ superspace language, the most general gauge invariant, BPS scalar operator is

$$\mathcal{O}^{i_1,\cdots,i_l}_{j_1,\cdots,j_l} = Tr(A^{i_1}B_{j_1}\cdots A^{i_l}B_{j_l}) \tag{2.10}$$

It has classical dimension $\Delta = l$ and belongs to a suitable representation of $\mathrm{SU}(2)_A \times \mathrm{SU}(2)_B$. Indeed, according to the particular theory we are considering, the sequence of indices may be constrained by the request for the operator to be a chiral primary ($\mathcal{O}^{i_1,\cdots,i_l}_{j_1,\cdots,j_l} \neq \bar{D}^2\mathcal{X}^{i_1,\cdots,i_l}_{j_1,\cdots,j_l}$). In the ABJ/ABJM case, this amounts to require the indices to be completely symmetrized, as follows from the observation that the equations of motion set antisymmetric products equal to $\bar{D}^2$(something).

For theories with $\mathrm{U}(1)_A \times \mathrm{U}(1)_B$ invariance (2.5), the composite operator $\mathcal{O}^{i_1,\cdots,i_l}_{j_1,\cdots,j_l}$ is not in general invariant, unless it contains the same number of A^1 and A^2 and the same number of B^1 and B^2 as well.

3 The n-point correlation functions

We are interested in computing correlation functions of the scalar composite operators in (2.10). We begin by considering the simplest case of a dimension-one operator

$$\mathcal{O}^i_j(Z) = Tr(A^i(Z)B_j(Z)) \qquad , \qquad \bar{\mathcal{O}}^j_i(Z) = Tr(\bar{A}_i(Z)\bar{B}^j(Z)) \tag{3.1}$$

Here $Z = (x^\mu, \theta^\alpha, \overline{\theta}^{\dot{\alpha}})$ and i, j are flavor indices that we omit in what follows. The generalization to higher dimensional operators is discussed in section 6, where we prove that one-loop correlation functions for BPS operators of arbitrary dimension can be expressed in terms of one-loop correlation functions of dimension-one operators.

We focus on the evaluation of the expression

$$\mathcal{C}_n = \left\langle \mathcal{O}(Z_1)\,\bar{\mathcal{O}}(Z_2)\cdots\mathcal{O}(Z_{n-1})\,\bar{\mathcal{O}}(Z_n)\right\rangle\big|_{\theta_i=\overline{\theta}_i=0} \tag{3.2}$$

which corresponds to the correlator for the lowest scalar component of (3.1).

At tree level, the correlation function is given by the product of free chiral propagators (2.8) which, evaluated at $\theta = \overline{\theta} = 0$, are simply $\frac{1}{4\pi}\frac{1}{|x_i - x_j|}$. Taking into account all the possibilities of contracting the fields, the expression (3.2) will be a linear combination of connected and disconnected diagrams. We concentrate only on the connected part. Using the simplified notation $x_{i,\,j} = |x_i - x_j|$, the tree level connected correlator reads

$$\mathcal{C}_n^{\,tree} = \frac{MN}{(4\pi)^n}\sum_{\{i_1,\cdots,i_n\}}\frac{1}{x_{i_1,\,i_2}}\frac{1}{x_{i_2,\,i_3}}\cdots\frac{1}{x_{i_n,\,i_1}} \tag{3.3}$$

where the sum is over all non-cyclic permutations compatible with the constraint that contractions are allowed only between chirals and antichirals. Since we will be eventually interested in the behavior of the correlator in the light-cone limit $x^2_{i,\,i+1} \to 0$, in (3.3) we select the most singular term which corresponds to the cyclic order $\{1, 2, \cdots, n\}$

$$\mathcal{C}_n^{\,tree} \to \frac{MN}{(4\pi)^n}\prod_{i=1}^{n}\frac{1}{x_{i,\,i+1}} \tag{3.4}$$

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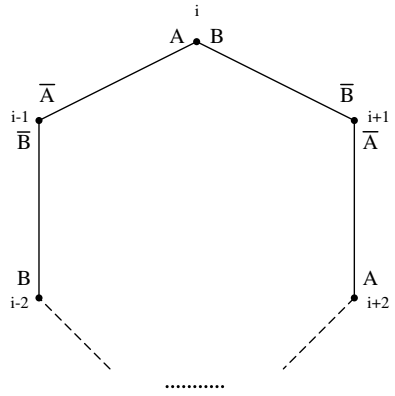

Figure 1. The correlation function of dimension-one operators in the light-like limit.

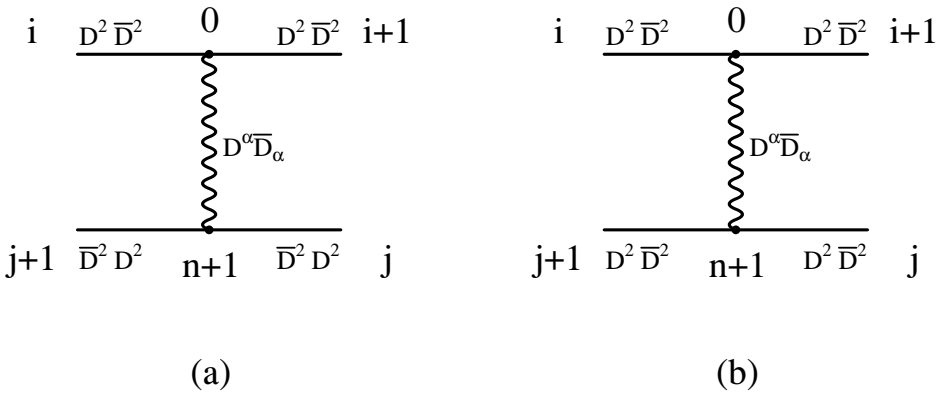

Figure 2. Building blocks for one-loop corrections.

where $x_{n+1} = x_1$.

Diagrammatically, this is given by a planar n-polygon with the operators at the vertices (See figure 1).

First order corrections in the $\frac{N}{K_1}$, $\frac{M}{K_2}$ couplings are obtained by attaching a V or $\hat{V}$ gauge propagator at the edges of the polygon in all possible ways. At this order, chiral interaction vertices from the superpotential do not contribute, so the results are valid for any $\mathcal{N} = 2$ theory.

When the gauge propagator has both ends attached to a single chiral line the result is zero. In fact, one loop corrections to chiral propagators vanish because it is possible to perform D-algebra in such a way that no enough spinorial derivatives survive inside the loop.

We are then left with contributions where the gauge propagator joins two different edges. It is useful to compute the generic building blocks $\mathcal{B}_{ij}$ depicted in figure 2, where the edges $x_{i,\,i+1}$ and $x_{j,\,j+1}$ are connected by a wave line representing either a V or a $\hat{V}$ propagator.

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3.1 One-loop building block

As shown in figure 2, there are two different configurations for the one-loop building block, depending on the chirality of the external vertices. Diagram 2a) corresponds to the case where vertices i and j are antichirals and $i+1$ and $j+1$ chirals. Diagram 2b) corresponds to the case where vertices i and $j+1$ are antichirals, while the other two are chirals.

In order to evaluate the building blocks $\mathcal{B}_{ij}$ we need perform D-algebra to end up with a non-vanishing result when evaluated at $\theta_k = \overline{\theta}_k = 0$, $k = i, i+1, j, j+1$. Starting with the configurations of figure 2 for the spinorial derivatives, we free the gauge and one of the chiral lines from derivatives by integrating by parts at one of the vertices. Among different terms which get produced, the only non-trivial contribution in the $\theta_k = \overline{\theta}_k = 0$ limit is the one where a $D^2\bar{D}^2$ structure survives on three chiral propagators. Together with the derivatives coming out from the spinorial integrations, these derivatives are sufficient to kill the fermionic delta functions, leading to a non-vanishing result. As a result of the D-algebra, the ordinary Feynman diagram we are left with has three space-time derivatives acting on chiral propagators.

Summing the contributions from the V and $\hat{V}$ insertions, the final result for the two configurations is

$$\begin{aligned}\mathcal{B}^{(a)}_{ij} &= -\frac{2}{(4\pi)^5}\left(\frac{1}{K_1}+\frac{1}{K_2}\right)\epsilon_{\mu\nu\rho}\,\partial_i^\mu\,\partial_{i+1}^\nu\,\partial_{j+1}^\rho\,\mathcal{I}(i,j)\\ \mathcal{B}^{(b)}_{ij} &= \frac{2}{(4\pi)^5}\left(\frac{1}{K_1}+\frac{1}{K_2}\right)\epsilon_{\mu\nu\rho}\,\partial_i^\mu\,\partial_{i+1}^\nu\,\partial_{j+1}^\rho\,\mathcal{I}(i,j)\end{aligned} \tag{3.5}$$

in terms of the integral

$$\mathcal{I}(i,j) = \int \frac{d^3x_0\,d^3x_{n+1}}{x_{0,\,i}\,x_{0,\,i+1}\,x_{0,\,n+1}\,x_{j,\,n+1}\,x_{j+1,\,n+1}} \tag{3.6}$$

The remarkable fact is that the expression $\epsilon_{\mu\nu\rho}\,\partial_i^\mu\,\partial_{i+1}^\nu\,\partial_{j+1}^\rho\,\mathcal{I}(i,j)$ can be manipulated by using Feynman combining and Mellin-Barnes representation and reduced to a single integral in five dimensions. Precisely, as proved in details in appendix B, the following identity holds

$$\epsilon_{\mu\nu\rho}\,\partial_i^\mu\,\partial_{i+1}^\nu\,\partial_{j+1}^\rho\,\mathcal{I}(i,j) = \frac{8}{\pi^2}\,\frac{\epsilon_{\mu\nu\rho}\,x^\mu_{i,\,i+1}\,x^\nu_{i+1,\,j}\,x^\rho_{j,\,j+1}}{x_{i,\,i+1}\,x_{j,\,j+1}} \times \int d^5x_0\,\frac{1}{x^2_{0,i}\,x^2_{0,i+1}\,x^2_{0,j}\,x^2_{0,j+1}} \tag{3.7}$$

Therefore, the building block which describes the gauge correction to the tree level expression $\frac{1}{x_{i,\,i+1}\,x_{j,\,j+1}}$ can be still written as the product of the two free propagators times a five dimensional scalar integral. Interpreting the x_j variables as the dual coordinates of a set of 5d momenta $p_j = x_{j+1} - x_j$, this can be seen as a box integral in five dimensions.

3.2 One-loop results and their light-like limit

Given the results (3.5), (3.7), we are now ready to evaluate the one-loop n-point correlator. The generic contribution will be the product of the blocks (3.5) times $(n-2)$ free propagators.

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By antisymmetry of the ϵ tensor we can ascertain that contributions coming from the gauge propagator connecting two adjacent edges vanish identically. In fact, setting $x_i = x_{j+1}$ or $x_j = x_{i+1}$, it is immediate to see that the structure $\epsilon_{\mu\nu\rho}\, x^{\mu}_{i,\,i+1}\, x^{\nu}_{i+1,\,j}\, x^{\rho}_{j,\,j+1}$ is zero. Therefore, we are left only with contributions where the gauge propagator connects two non-adjacent edges.

When the two lines are separated by an odd number of free propagators the block $\mathcal{B}^{(a)}_{ij}$ has to be used. In this case, given the particular structure of the operator and the fact that only the $\langle A\bar{A}\rangle$, $\langle B\bar{B}\rangle$ propagators are non-vanishing, the vertices employed to construct the block are necessarily of the same type: If one is a A-vertex, the second one is a A-vertex as well. These carry the same sign so that this contribution is given by $\mathcal{B}^{(a)}_{ij}$, without any sign change. On the other hand, when the two non-adjacent lines are separated by an even number of free lines we need use the block $\mathcal{B}^{(b)}_{ij}$. In this case the two employed vertices are of different kind and since these carry opposite sign (see eq. (2.9)) we obtain an extra minus which compensates the sign difference between the blocks, so that both kinds of contributions end up having the same sign.

In conclusion, taking into account color factors, the leading term of the correlation function at one-loop is

$$\mathcal{C}_n^{\,1-loop} \to \mathcal{C}_n^{\rm tree} \times \frac{-1}{4\,\pi^5}\left[\frac{N}{K_1}+\frac{M}{K_2}\right]\sum_{i=1}^{n-2}\sum_{j=i+2}^{n-\delta_{i,1}} \epsilon_{\mu\nu\rho}\, x^{\mu}_{i,\,i+1}\, x^{\nu}_{i+1,\,j}\, x^{\rho}_{j,\,j+1}\, \mathcal{J}(i,j) \tag{3.8}$$

where the sum extends to the $n(n-3)/2$ ways to connect two non-adjacent edges, and $\mathcal{J}(i,j)$ is

$$\mathcal{J}(i,j) = \int d^5x_0\, \frac{1}{x^2_{0,i}\, x^2_{0,i+1}\, x^2_{0,j}\, x^2_{0,j+1}} \tag{3.9}$$

In the ABJM case and for all theories with $K_2 = -K_1$ and $M = N$ the color factor in front vanishes, so that correlation functions are trivially zero at one loop. The same happens for the BLG theory, as it can be easily checked by computing the color factor for gauge group $\mathrm{SU}(N)\times\mathrm{SU}(M)$ which turns out to be $(N-1/N)/K_1 + (M-1/M)/K_2$.

We concentrate on more general theories for which the color factor does not vanish. The first non-trivial expression in (3.8) is the four point correlation function. Setting $n=4$, the sum reduces to two contributions having the same integral factor

$$\mathcal{C}_4^{\,1-loop} \propto \epsilon_{\mu\nu\rho}\left(x^{\mu}_{1,\,2}\, x^{\nu}_{2,\,3}\, x^{\rho}_{3,\,4} + x^{\mu}_{2,\,3}\, x^{\nu}_{3,\,4}\, x^{\rho}_{4,\,1}\right)\int d^5x_0\, \frac{1}{x^2_{0,1}\, x^2_{0,2}\, x^2_{0,3}\, x^2_{0,4}} \tag{3.10}$$

It is immediate to see that the structure in front of the integral vanishes, due to the contraction with the ϵ tensor. Hence, the connected four point correlation function is identically zero, no need to perform the integral.

This trivial result is no longer true for higher point correlation functions, so that in general we really have to work out the $\mathcal{J}(i,j)$ integral. We do it in the light-like limit $x^2_{i,\,i+1} \to 0$, which greatly simplifies the computation and, as shown in [18], is the correct prescription to test a correspondence to light-like Wilson loops.

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Since the prefactor $\mathcal{C}_n^{tree}$ in (3.8) is divergent in this limit, we consider the ratio of the one-loop correlator to the tree level result. Moreover, in order to get a real output, we require the $n(n-3)/2$ diagonals of the n-polygon to be space-like ($x_{i,j}^2 > 0$, $j \neq i+1$).

To evaluate the integral (3.9) we first shift the integration variable $x_0 \to x_0 + x_i$, and reduce it to a Feynman scalar box integral in five dimensions with external momenta $x_{i,i+1}$, $x_{i+1,j}$, $x_{j,j+1}$ and $x_{j+1,i}$. In the light-like limit the integral is recognized to be the two mass easy box, with two of the momenta massless by construction and the other two massive. In the $j = i+2$ case, i.e. when the two edges are separated by a single free line, one more external leg becomes massless and the integral simplifies further.

Feynman parametrizing the scalar five dimensional box and performing the loop integration yields

$$\mathcal{J}(i,j) = \frac{\pi^3}{2}\int_0^1 [d\alpha]_4 \frac{1}{\Big(\alpha_1\,\alpha_3\,x_{i,j}^2 + \alpha_2\,\alpha_4\,x_{i+1,j+1}^2 + \alpha_1\,\alpha_4\,x_{i,j+1}^2 + \alpha_2\,\alpha_3\,x_{i+1,j}^2\Big)^{\frac{3}{2}}} \tag{3.11}$$

where $[d\alpha]_4 = \delta(1-\sum_{k=1}^4 \alpha_k)\prod_{k=1}^4 d\alpha_k$.

The delta-function constraint can be solved by performing the following change of variables

$$\alpha_1 = (1-\beta_1)(1-\beta_3) \quad , \quad \alpha_2 = \beta_1(1-\beta_3) \quad , \quad \alpha_3 = (1-\beta_2)\beta_3 \quad , \quad \alpha_4 = \beta_2\beta_3 \tag{3.12}$$

Consequently, the integral reduces to

$$\mathcal{J}(i,j) = \frac{\pi^3}{2}\int_0^1 \prod_{i=1}^3 d\beta_i \;\times \tag{3.13}$$

$$\frac{\beta_3^{-\frac{1}{2}}(1-\beta_3)^{-\frac{1}{2}}}{\Big[(1-\beta_2)\,(1-\beta_1)\,x_{i,j}^2 + \beta_1\,\beta_2\,x_{i+1,j+1}^2 + \beta_2\,(1-\beta_1)\,x_{i,j+1}^2 + \beta_1\,(1-\beta_2)\,x_{i+1,j}^2\Big]^{\frac{3}{2}}}$$

where the β_3-integration can be trivially performed, leading to

$$\mathcal{J}(i,j) = \frac{\pi^4}{2}\int_0^1 d\beta_1 d\beta_2 \;\times \tag{3.14}$$

$$\frac{1}{\Big[(1-\beta_2)\,(1-\beta_1)\,x_{i,j}^2 + \beta_1\,\beta_2\,x_{i+1,j+1}^2 + \beta_2\,(1-\beta_1)\,x_{i,j+1}^2 + \beta_1\,(1-\beta_2)\,x_{i+1,j}^2\Big]^{\frac{3}{2}}}$$

Finally, the last two integrations can be performed with the help of *Mathematica*.

In conclusion, the general one-loop contribution to the n-point correlator corresponding to a Feynman diagram where a vector line connects the $x_{i,i+1}$ and $x_{j,j+1}$ free propagators, in the light-cone limit reads

$$\epsilon_{\mu\nu\rho}\,x_{i,i+1}^{\mu}\,x_{i+1,j}^{\nu}\,x_{j,j+1}^{\rho}\,\mathcal{J}(i,j) \;= \tag{3.15}$$

$$\pi^4\,\mathcal{S}_{i,j}\,\log\left[\frac{(1+x_{i+1,j}\,\mathcal{L}_{i,j})\,(1+x_{i,j+1}\,\mathcal{L}_{i,j})}{(1-x_{i+1,j}\,\mathcal{L}_{i,j})\,(1-x_{i,j+1}\,\mathcal{L}_{i,j})}\frac{(1-x_{i,j}\,\mathcal{L}_{i,j})\,(1-x_{i+1,j+1}\,\mathcal{L}_{i,j})}{(1+x_{i,j}\,\mathcal{L}_{i,j})\,(1+x_{i+1,j+1}\,\mathcal{L}_{i,j})}\right]$$

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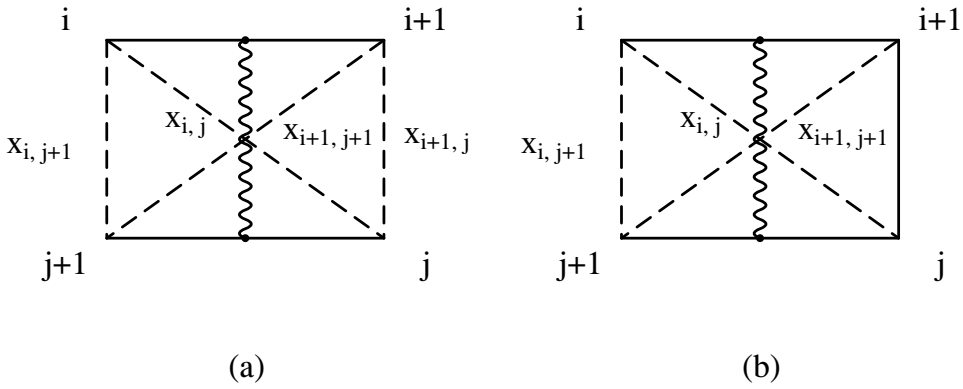

Figure 3. The building blocks for the correlation functions only depend on the diagonals of the polygon, which are drawn with dashed lines. Case (a) corresponds to the $n(n-5)/2$ blocks where all the involved diagonals are long. Case (b) depicts one of the n blocks with short diagonals.

where we have defined

$$\mathcal{S}_{i,\,j} = \frac{2\,\epsilon_{\mu\nu\rho}\,x^{\mu}_{i,\,i+1}\,x^{\nu}_{i+1,\,j}\,x^{\rho}_{j,\,j+1}}{\sqrt{x^2_{i,j}+x^2_{i+1,j+1}-x^2_{i+1,j}-x^2_{i,j+1}}\sqrt{x^2_{i,j}x^2_{i+1,j+1}-x^2_{i+1,j}x^2_{i,j+1}}} \tag{3.16}$$

and

$$\mathcal{L}_{i,\,j} = \frac{\sqrt{x^2_{i,j}+x^2_{i+1,j+1}-x^2_{i+1,j}-x^2_{i,j+1}}}{\sqrt{x^2_{i,j}x^2_{i+1,j+1}-x^2_{i+1,j}x^2_{i,j+1}}} \tag{3.17}$$

Focusing on the argument of the logarithm in (3.15) we note that it depends only on the diagonals connecting the four vertices of the block x_i, x_{i+1}, x_j and x_{j+1}, as depicted in figure 3(a). This is due to the fact that the correlator, being Poincaré invariant, has to be a function of the only invariants that we can construct. In the light-like limit these are the $n(n-3)/2$ space-like diagonals.[1]

We distinguish two sets of diagonals. We call "short" diagonals those connecting two vertices separated by a pair of light-like edges, whereas we call "long" diagonals the remaining $n(n-5)/2$ ones.

An example of the appearance of short diagonals is depicted in figure 3(b), where the vertices x_{i+1} and x_j are connected by a null edge, so the space-like segments $x_{i,\,j}$ and $x_{i+1,\,j+1}$ are short diagonals. In this case, the corresponding contribution can be obtained from the general expression (3.15) by collapsing $x_{i+1,\,j} \to 0$, and as a result the logarithm contains just three factors instead of four.

Going back to (3.15), by straightforward algebra we can rewrite the argument of the

[1] Actually not all diagonals are independent and their number could in principle be reduced by the Gram constraints. Since these constraints are difficult to implement we will not pursue this technique.

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logarithm as

$$\frac{(1+x_{i+1,j}\,\mathcal{L}_{i,j})\,(1+x_{i,j+1}\,\mathcal{L}_{i,j})}{(1-x_{i+1,j}\,\mathcal{L}_{i,j})\,(1-x_{i,j+1}\,\mathcal{L}_{i,j})}\frac{(1-x_{i,j}\,\mathcal{L}_{i,j})\,(1-x_{i+1,j+1}\,\mathcal{L}_{i,j})}{(1+x_{i,j}\,\mathcal{L}_{i,j})\,(1+x_{i+1,j+1}\,\mathcal{L}_{i,j})} = \frac{(1+x_{i+1,j}\,\mathcal{L}_{i,j})^2\,(1+x_{i,j+1}\,\mathcal{L}_{i,j})^2}{(1+x_{i,j}\,\mathcal{L}_{i,j})^2\,(1+x_{i+1,j+1}\,\mathcal{L}_{i,j})^2} \tag{3.18}$$

As proven in section 5, $\mathcal{L}_{i,j}$'s are real functions as long as all the diagonals are space-like. Under this assumption, eq. (3.18) is the square of a real expression and the logarithm in (3.15) is well defined. A similar argument applies also to the case of short diagonals, leading to the same conclusions.

Finally, inserting the result (3.15) back into eq. (3.8) and summing over all possible contractions, we obtain the complete analytical result for the ratio $\mathcal{C}_n^{1-loop}/\mathcal{C}_n^{\text{tree}}$ in the light-like limit. The positiveness of the arguments of all logarithms allows us to safely rewrite the sum as

$$\frac{\mathcal{C}_n^{1-loop}}{\mathcal{C}_n^{\text{tree}}} = -\frac{1}{4\,\pi}\left[\frac{N}{K_1}+\frac{M}{K_2}\right]\log\left\{\prod_{i=1}^{n-2}\prod_{j=i+2}^{n-\delta_{i,1}}\left[\frac{(1+x_{i+1,j}\,\mathcal{L}_{i,j})\,(1+x_{i,j+1}\,\mathcal{L}_{i,j})}{(1-x_{i+1,j}\,\mathcal{L}_{i,j})\left(1-x_{i,j+1}^2\,\mathcal{L}_{i,j}\right)}\frac{(1-x_{i,j}\,\mathcal{L}_{i,j})\,(1-x_{i+1,j+1}\,\mathcal{L}_{i,j})}{(1+x_{i,j}\,\mathcal{L}_{i,j})\,(1+x_{i+1,j+1}\,\mathcal{L}_{i,j})}\right]^{\mathcal{S}_{i,j}}\right\} \tag{3.19}$$

In general, this expression is not zero as long as the distances $x_{i,j}$ are arbitrary. However they are not all independent, being the diagonals of a polygon in three spacetime dimensions. In section 5 we come back to this result and prove that it is actually zero when implementing an explicit parametrization which constrains the $x_{i,j}$ segments to be the diagonals of a three dimensional polygon.

4 Connection with light-like Wilson loops

In this section we discuss the relation between the n-point correlation function just computed and light-like polygonal Wilson loops.

For the set of theories described by the action (2.2) we consider the Wilson loop operator

$$\langle W_n(A,\hat{A})\rangle = \left\langle \frac{1}{2N}\,Tr\,\mathcal{P}\text{exp}\left(i\oint_{\Gamma_n}A_\mu dz^\mu\right)+\frac{1}{2M}\,Tr\,\mathcal{P}\text{exp}\left(i\oint_{\Gamma_n}\hat{A}_\mu dz^\mu\right)\right\rangle \tag{4.1}$$

where Γ_n is a n-polygon with vertices x_i, $i=1,\cdots,n$, and light-like edges, $x_{i,i+1}^2=0$. We require all the diagonals to be strictly positive in order to get real results. The edges can be parametrized as

$$z_i^\mu(\tau_i) = x_i^\mu - x_{i,i+1}^\mu\,\tau_i \qquad , \qquad 0\le\tau_i\le 1 \tag{4.2}$$

The perturbative evaluation of these operators up to two loops has been carried on in [40]. Here, we briefly summarize their findings by pointing out what is needed for a comparison with correlation functions.

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The one-loop contribution to a WL is obtained by expanding the path-ordered exponential at second order in the gauge fields. Concentrating on one of the gauge fields, let's say A_μ, it is given by

$$\langle W(A)\rangle^{1-loop} = \frac{i^2}{N}\sum_{i\geq j}\int d\tau_i\, d\tau_j\, \dot{z}_i^\mu\, \dot{z}_j^\nu\, \langle Tr(A_\mu(z_i)\, A_\nu(z_j))\rangle \tag{4.3}$$

where $\tau_i, \tau_j, i \neq j$ run independently between 0 and 1, whereas for $i = j$ the integration domain is meant to be $0 \leq \tau_i \leq 1$ and $0 \leq \tau_j \leq \tau_i$. Dots indicate derivatives with respect to the affine parameters.

Plugging in the explicit expression for the gauge propagator, which in Landau gauge reads

$$\langle (A_\mu)^a_b(z_i)\, (A_\nu)^c_d(z_j)\rangle = -\frac{1}{8\pi K_1}\,\epsilon_{\mu\nu\rho}\,\frac{(z_i - z_j)^\rho}{|z_i - z_j|^3}\,\delta^a_d\,\delta^c_b \tag{4.4}$$

the contribution from a diagram where the gauge vector connects the (x_i, x_{i+1}) and (x_j, x_{j+1}) edges is proportional to $\epsilon_{\mu\nu\rho}\, x^\mu_{i,i+1}\, x^\nu_{i+1,j}\, x^\rho_{j,j+1}\, \mathcal{K}(i,j)$, where

$$\mathcal{K}(i,j) = \frac{\pi^4}{2}\int_0^1 d\tau_i d\tau_j\ \times \tag{4.5}$$
$$\frac{1}{\Big[(1-\tau_i)\,(1-\tau_j)\,x^2_{i,j} + \tau_i\,\tau_j\,x^2_{i+1,j+1} + \tau_j\,(1-\tau_i)\,x^2_{i,j+1} + \tau_i\,(1-\tau_j)\,x^2_{i+1,j}\Big]^{\frac{3}{2}}}$$

where we have taken into account that the contributions for $j = i$ and $j = i + 1$ vanish, due to the antisymmetry of the ϵ tensor.

Now, including all the coefficients and summing the analogous contribution coming from $\hat{A}$, the one-loop WL can be written as

$$\langle W(A,\hat{A})\rangle^{1-loop} = -\frac{1}{4\pi^5}\left(\frac{N}{K_1} + \frac{M}{K_2}\right)\sum_{i=1}^{n-2}\sum_{j=i+2}^{n-\delta_{i,1}}\epsilon_{\mu\nu\rho}\, x^\mu_{i,i+1}\, x^\nu_{i+1,j}\, x^\rho_{j,j+1}\, \mathcal{K}(i,j) \tag{4.6}$$

where the sum runs over all possible ways to connect two non-adjacent lines. We note that at this order matter fields do not enter the calculation. Therefore, this result is valid also for pure Chern-Simons theories.

As for the correlation functions, the overall color factor in (4.6) vanishes for all the theories with $K_2 = -K_1$ and $M = N$, ABJM case included. For this set of theories the correlation functions/WL duality is then trivial at the first perturbative order.

Interesting non-trivial results can be found, instead, for theories where the color factor does not vanish. In fact, the main observation is that, identifying the affine parameters τ_i, τ_j with the Feynman parameters β_1, β_2 in (3.14), the $\mathcal{K}(i,j)$ integral is precisely the same as the integral $\mathcal{J}(i,j)$ arising in the computation of an n-point correlation function in the light-like limit. Since the integral (3.14) is the Feynman parametrization of a 5d box integral, we can claim that also the one-loop WL can be formally expressed in terms of a 5d scalar integral.

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The exact relation between correlation functions and WL, at one-loop reads

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\mathcal{C}_n^{\,1-loop}}{\mathcal{C}_n^{\rm tree}} \;=\; \langle W(A,\hat{A})\rangle^{1-loop} \tag{4.7}$$

all in terms of the 5d integral (3.9).

We note that the two expressions coincide, independently of the values of the couplings K_1, K_2 and for any value of the gauge ranks (N, M), as no planar limit is required.

5 One-loop vanishing of correlators and Wilson loops

In this section we give an analytical proof that the expression (3.19) vanishes for any value of n. In other words, the light-like limit of n-point correlation functions of dimension-one BPS operators is zero at one loop.

Given the identification (4.7), as a by-product we also prove that light-like n-polygon Wilson loops vanish at first order. This result generalizes the one in [40] valid only for $n = 4, 6$ and proves the conjecture made there that WL should be one-loop vanishing for any n.

As we read in (3.19), the one-loop correction to a correlation function is proportional to the logarithm of a product of factors with schematic form $\left(\frac{1\pm x\,\mathcal{L}_{i,j}}{1\mp x\,\mathcal{L}_{i,j}}\right)^{\mathcal{S}_{i,j}}$. We prove that this product always evaluates to 1.

In (3.19) the factors are grouped according to the pair of edges involved in a given gauge vector exchange (see blocks in figure 2). The basic idea of the proof is to reorganize them by group together all the factors which depend on the same diagonal $x_{i,j}$. It is easy to ascertain that each long diagonal is involved in four contributions, coming from the four possible interactions connecting the edges which are adjacent to the diagonal itself (See figure 4 (a)). In the case of a short diagonal, one of these contributions vanishes (it would be a correction to the vertex), thus we are left with just three pieces (See figure 4(b)).

Once this reshuffling of factors has been performed in (3.19), we prove that the product of contributions involving the same reference diagonal evaluates to $+1$ for long diagonals and to -1 for short ones. We consider a generic diagonal and parametrize all distances in full generality, so that once we establish this property for one diagonal, we can apply it to all the contributions to the correlator.

Let us focus on one particular diagonal $x_{i,j}$, and suppose it is long. The corresponding block of factors then depends only on the nearest neighbours of the vertices x_i and x_j, which are x_{i-1}, x_{i+1}, x_{j-1}, x_{j+1}. These six points are parametrized by 18 coordinates. However, four of them can be eliminated by light-likeness of the edges $x_{i,\,i+1}$, $x_{i,\,i-1}$, $x_{j,\,j+1}$, $x_{j,\,j-1}$. By using translation invariance, we choose a convenient reference frame where $x_i^\mu = (0,0,0)$, so removing three more coordinates. Using rotational invariance, we eliminate two further parameters by choosing $x_j^\mu = (0, b, 0)$ where $b > 0$. In this way, the reference diagonal lies in the $t = 0$ plane. We parametrize the rest of the block in terms of the nine remaining variables as follows

$$\begin{aligned} x_{i-1}^\mu &= r_1\,(1, \cos\phi_1, \sin\phi_1)\,, & x_{i+1}^\mu &= r_3\,(1, \cos\phi_3, \sin\phi_3) \\ x_{j-1}^\mu &= x_j^\mu + r_2\,(1, \cos\phi_2, \sin\phi_2)\,, & x_{j+1}^\mu &= x_j^\mu + r_4\,(1, \cos\phi_4, \sin\phi_4) \end{aligned} \tag{5.1}$$

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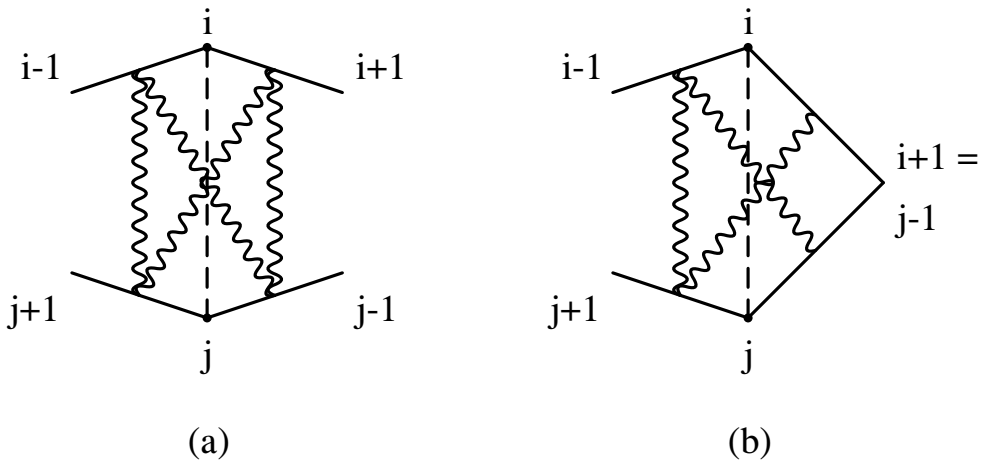

Figure 4. In picture (a) the four blocks in which the reference diagonal $x_{i,j}$ is involved are depicted. In picture (b) the case of a short diagonal and its three blocks is shown. Each wiggled line has to be interpreted separately.

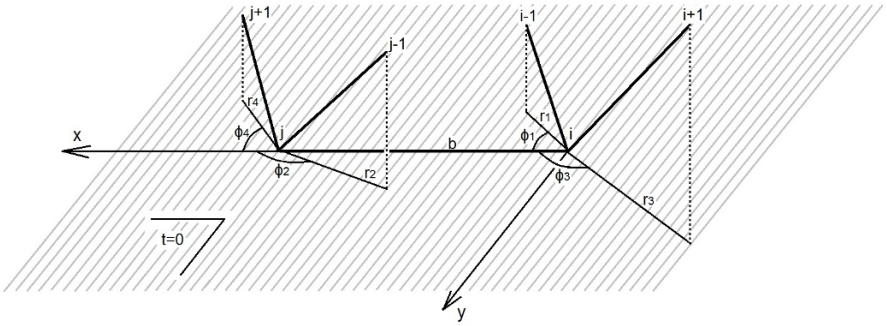

Figure 5. Parametrization of the block of contributions involving the same reference diagonal $x_{i,j}$.

This parametrization is sketched in figure 5: the ϕ_i's are the angles held by the projections of the light-like lines on the $t = 0$ plane, while the moduli of the r_i's measure the lengths of these same projections. It is obvious that the edges are light-like and the reference diagonal $x_{i,j}$ is space-like by construction. At this stage, the other diagonals are not necessarily space-like. The quest for them to be space-like implies that r_1, r_3 and r_2, r_4 should have separately the same sign, in order for adjacent segments to point alternatively to the future and to the past. In the following we will assume that they are all positive, but the final statement can be exhaustively shown to be valid for any choice of these signs.

Let us now evaluate the product of the four contributions for the reference diagonal $x_{i,j}$, namely

$$\left(\frac{1+x_{i,j}\ \mathcal{L}_{i,j}}{1-x_{i,j}\ \mathcal{L}_{i,j}}\right)^{\mathcal{S}_{i,j}} \quad \left(\frac{1+x_{i,j}\ \mathcal{L}_{i-1,j-1}}{1-x_{i,j}\ \mathcal{L}_{i-1,j-1}}\right)^{\mathcal{S}_{i-1,j-1}}$$
$$\left(\frac{1-x_{i,j}\ \mathcal{L}_{i-1,j}}{1+x_{i,j}\ \mathcal{L}_{i-1,j}}\right)^{\mathcal{S}_{i-1,j}} \quad \left(\frac{1-x_{i,j}\ \mathcal{L}_{i,j-1}}{1+x_{i,j}\ \mathcal{L}_{i,j-1}}\right)^{\mathcal{S}_{i,j-1}} \tag{5.2}$$

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By plugging in the parametrization (5.1) we obtain a nice symmetric expression

$$\left(\frac{1+\left|\frac{\sin\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)}\right|}{1-\left|\frac{\sin\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)}\right|}\right)^{\mathrm{Sign}\left[\frac{\sin\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)}\right]}\left(\frac{1-\left|\frac{\sin\left(\frac{\phi_3-\phi_2}{2}\right)}{\cos\left(\frac{\phi_3+\phi_2}{2}\right)}\right|}{1+\left|\frac{\sin\left(\frac{\phi_3-\phi_2}{2}\right)}{\cos\left(\frac{\phi_3+\phi_2}{2}\right)}\right|}\right)^{\mathrm{Sign}\left[\frac{\sin\left(\frac{\phi_3-\phi_2}{2}\right)}{\cos\left(\frac{\phi_3+\phi_2}{2}\right)}\right]}$$
$$\left(\frac{1-\left|\frac{\sin\left(\frac{\phi_1-\phi_4}{2}\right)}{\cos\left(\frac{\phi_1+\phi_4}{2}\right)}\right|}{1+\left|\frac{\sin\left(\frac{\phi_1-\phi_4}{2}\right)}{\cos\left(\frac{\phi_1+\phi_4}{2}\right)}\right|}\right)^{\mathrm{Sign}\left[\frac{\sin\left(\frac{\phi_1-\phi_4}{2}\right)}{\cos\left(\frac{\phi_1+\phi_4}{2}\right)}\right]}\left(\frac{1+\left|\frac{\sin\left(\frac{\phi_3-\phi_4}{2}\right)}{\cos\left(\frac{\phi_3+\phi_4}{2}\right)}\right|}{1-\left|\frac{\sin\left(\frac{\phi_3-\phi_4}{2}\right)}{\cos\left(\frac{\phi_3+\phi_4}{2}\right)}\right|}\right)^{\mathrm{Sign}\left[\frac{\sin\left(\frac{\phi_3-\phi_4}{2}\right)}{\cos\left(\frac{\phi_3+\phi_4}{2}\right)}\right]} \tag{5.3}$$

where $\mathrm{Sign}(x)$ is the sign function. We notice that the explicit parametrization allows us to fix a loose end from section 3.2, namely we have ascertained that the terms $x_{i,j}\,\mathcal{L}_{\dots}$ are real positive functions. Furthermore we observe that the apparently awkward exponents $\mathcal{S}_{i,j}$ (3.16) are surprisingly just $\pm$ signs.

Expression (5.3) can be written in a compact fashion (here and in the following $\phi_5 = \phi_1$ is understood)

$$\prod_{i=1}^{4}\left(\frac{1+\left|\frac{\sin\left(\frac{\phi_i-\phi_{i+1}}{2}\right)}{\cos\left(\frac{\phi_i+\phi_{i+1}}{2}\right)}\right|}{1-\left|\frac{\sin\left(\frac{\phi_i-\phi_{i+1}}{2}\right)}{\cos\left(\frac{\phi_i+\phi_{i+1}}{2}\right)}\right|}\right)^{\mathrm{Sign}\left[\frac{\sin\left(\frac{\phi_i-\phi_{i+1}}{2}\right)}{\cos\left(\frac{\phi_i+\phi_{i+1}}{2}\right)}\right]} \tag{5.4}$$

We observe that the expression depends exclusively on the four angles of the parametrization but not on any of the five dimensionful parameters. We also note that in each contribution the arguments of absolute values and Sign functions are the same. Because of that and using the fact that $\left(\frac{1\pm x}{1\mp x}\right)^{\pm} = \frac{1+x}{1-x}$ we may simplify expression (5.4) to obtain

$$\prod_{i=1}^{4}\frac{\cos\left(\frac{\phi_i+\phi_{i+1}}{2}\right)+\sin\left(\frac{\phi_i-\phi_{i+1}}{2}\right)}{\cos\left(\frac{\phi_i+\phi_{i+1}}{2}\right)-\sin\left(\frac{\phi_i-\phi_{i+1}}{2}\right)} \tag{5.5}$$

This is equivalent to

$$\prod_{i=1}^{4}\cot\left(\frac{\phi_i}{2}-\frac{\pi}{4}\right)\,\tan\left(\frac{\phi_{i+1}}{2}-\frac{\pi}{4}\right)=1 \tag{5.6}$$

Therefore, this completes the proof for long diagonals.

For a short diagonal $x_{i,j}$, it suffices to take the result above and set e.g., $x_{i+1} = x_{j-1}$. Then the contribution involving $\mathcal{L}_{i,j-1}$ vanishes by construction leaving

$$\frac{\cos\left(\frac{\phi_1+\phi_2}{2}\right)+\sin\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)-\sin\left(\frac{\phi_1-\phi_2}{2}\right)}\,\frac{\cos\left(\frac{\phi_1+\phi_4}{2}\right)-\sin\left(\frac{\phi_1-\phi_4}{2}\right)}{\cos\left(\frac{\phi_1+\phi_4}{2}\right)+\sin\left(\frac{\phi_1-\phi_4}{2}\right)}\,\frac{\cos\left(\frac{\phi_3+\phi_4}{2}\right)+\sin\left(\frac{\phi_3-\phi_4}{2}\right)}{\cos\left(\frac{\phi_3+\phi_4}{2}\right)-\sin\left(\frac{\phi_3-\phi_4}{2}\right)} \tag{5.7}$$

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When parametrizing as in eq. (5.1) the condition $x_{i+1} = x_{j-1}$ is forced by choosing

$$r_2 = r_3, \quad r_3 \cos(\phi_3) = b + r_2 \cos(\phi_2), \quad \sin(\phi_3) = \sin(\phi_2) \tag{5.8}$$

These equations are solved by $\phi_3 = \pi - \phi_2$ and some function $r_2 = r_2(a, \phi_3)$ which is irrelevant. Plugging it into (5.7) finally simplifies the expression to -1, in a completely analogous way as in the long diagonal case. Since there are n contributions of the short type, and since n is even, the overall contribution of short diagonals is equal to $+1$.

Summarizing, we have shown that the combined collection of all short and long diagonal contributions to the argument of the logarithm is equal to $+1$. Therefore, the logarithm is equal to zero, thus proving the vanishing of the n-point correlator and Wilson loops at one loop.

6 Generalization to higher dimensional operators

So far we have considered one-loop corrections $\mathcal{C}_n^{1-loop}$ to correlators of dimension-one operators $\mathcal{O}_j^i = Tr(A^i B_j)$. In this section, we show that one-loop corrections to correlators $\mathcal{C}_{n,2l}$ of higher dimensional operators (2.10) can be simply computed once $\mathcal{C}_n^{1-loop}$ is known. In particular, since $\mathcal{C}_n^{1-loop}$ is zero, the same holds for any correlator $\mathcal{C}_{n,2l}$ with $l > 1$. We emphasize that the derivation of this result is valid for any value of the gauge group parameters (N, M).

The most divergent part of connected correlators of higher dimensional operators in the light-like limit $x_{i,i+1}^2 \to 0$ at tree level reads

$$\begin{aligned} \mathcal{C}_{n,2l}^{\text{tree}} &\propto \sum_{s=1}^{2l-1} \mathcal{T}_s^{\text{tree}} \\ \mathcal{T}_s^{\text{tree}} &= \prod_{j=1}^{n/2} \left(\frac{1}{x_{2j-1,2j}}\right)^s \left(\frac{1}{x_{2j,2j+1}}\right)^{2l-s} \end{aligned} \tag{6.1}$$

Eq. (6.1) extends eq. (3.4) to the $l > 1$ case. The general contribution in the sum (6.1) is a polygon with edges alternately made by s and $2l - s$ propagators (see figure 6(a)). Each value of s defines a different topology $\mathcal{T}_s$. In the rest of the discussion, it is useful to divide each topology $\mathcal{T}_s$ into classes $\mathcal{T}_{s,a}$ where the parameter a counts the number of $\langle A\bar{A}\rangle$ propagators inside a block of s lines (see figure 6(b)).

One-loop corrections to $\mathcal{C}_{n,2l}$ are given by inserting a gauge propagator V or $\hat{V}$ in all possible ways between the edges of the polygon $\mathcal{C}_{n,2l}^{\text{tree}}$.

As in the $l = 1$ case, the only non-trivial insertions occur when the gauge propagator connects two non-consecutive edges in the polygon. All other possible insertions are zero due to D-algebra constraints or to the antisymmetry of the ϵ tensor.

The non-trivial corrections have the form (3.5). However, since now we have more than one chiral propagator in each edge, we have more than one possibility to insert a gauge line between the same two edges of a correlator.

The combinatorial factor is in principle different for corrections involving different pairs of edges in each class $\mathcal{T}_{s,a}$. However, a careful computation which takes into account the

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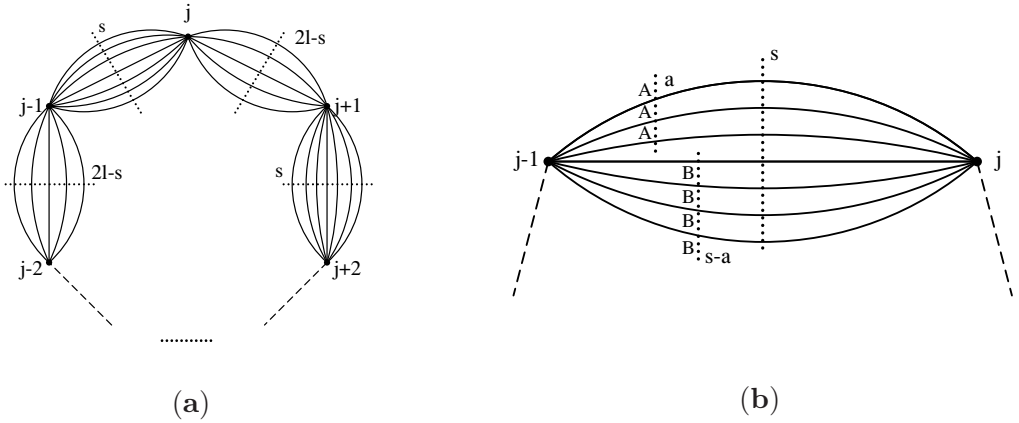

Figure 6. General form of the contributions to $\mathcal{C}^{\text{tree}}_{n,2l}$. In figure (a), structure of the leading divergent terms in the limit $x^2_{i,i+1} \to 0$. In figure (b), the parameter a counts the number of $\langle A\bar{A}\rangle$ propagators in a set of s lines.

relative signs between A and B vertices (2.9) and between the two building blocks (3.5) shows that the combinatorial factor depends only on the (a, s) parameters and it is thus a common factor for all corrections inside each $\mathcal{T}_{s,a}$ class. Precisely, the one-loop correction to the generic $\mathcal{T}_{s,a}$ class reads

$$\mathcal{T}^{1-loop}_{s,a} \propto \mathcal{T}^{\text{tree}}_{s,a} \times (s-2a)^2 \sum_{i=1}^{n-2} \sum_{j=i+2}^{n-\delta_{i,1}} \epsilon_{\mu\nu\rho}\, x^{\mu}_{i,\,i+1}\, x^{\nu}_{i+1,\,j}\, x^{\rho}_{j,\,j+1}\, \mathcal{J}(i,j) \tag{6.2}$$

where $\mathcal{J}(i,j)$ is the five dimensional box integral (3.9).

This formula closely resembles eq. (3.8). In particular, the sums in these two expressions are the same. Thus, having computed the one-loop corrections to the n-point function for dimension-one operators, we immediately have the result for any $\mathcal{T}^{1-loop}_{s,a}$. The complete one-loop correction to the correlator $\mathcal{C}_{n,2l}$ can be then recovered through (6.1). In particular, since we have proved that $\mathcal{C}^{1-loop}_{n}/\mathcal{C}^{\text{tree}}_{n}$ vanishes in the light-like limit, so $\mathcal{C}^{1-loop}_{n,2l}/\mathcal{C}^{\text{tree}}_{n,2l}$ does.

7 Conclusions

In this paper we have focused on the novel proposal that a multiple light-like limit of the correlation function of n protected operators reproduces a Wilson loop evaluated on a null n-polygon. This statement has been argued and verified perturbatively in $\mathcal{N} = 4$ SYM, and it has been claimed to be valid for any conformal gauge theory in any dimensions [18].

We have confirmed this expectation for a class of supersymmetric Chern-Simons matter theories in three dimensions at first order in perturbation theory. Our check goes as follows: We have computed one-loop corrections to the correlation function of n BPS scalar operators in a manifestly $\mathcal{N} = 2$ supersymmetric formalism. Remarkably, they can be expressed in terms of five dimensional box integrals. Then we have performed the light-like limit of the correlator as prescribed in [18] in order to compare it to the Wilson loop

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expectation values on n-cusped light-like polygons. These were found explicitly in [40] in the cases of $n = 4$ and $n = 6$ cusps. The former was shown to vanish analytically, whereas numerical evidence hinted at the vanishing of the latter. This suggested that all light-like Wilson loop should not receive first order quantum corrections in Chern-Simons theories. We have managed to show analytically that both one-loop corrections to the correlators and to the Wilson loops vanish, thus confirming the correlator/Wilson loop duality for a class of three dimensional theories, and proving the claim on the vanishing of light-like Wilson loops at first order. We point out that our check is not just a mere identity between two vanishing contributions, since the equality between correlators and Wilson loops in the light-like limit already holds when expressing them in terms of integrals, before showing that these expressions actually vanish.

In our computation this equivalence seems to apply to any $\mathcal{N} = 2$ Chern-Simons matter theories, but this is just an artifact of the low perturbative order. Indeed quantum corrections arise purely from the Chern-Simons sector both for the correlation functions and for Wilson loops and the matter sector is not involved. On one hand, this confirms the idea that the relation should be valid in any conformal field theory: Indeed, all Chern-Simons matter theories are naively conformal invariant at one loop. On the other hand, this shadows any difference between the gauge theories spanned by our $\mathcal{N} = 2$ Lagrangian (2.2), both as concerns supersymmetry and conformality. Models with different amounts of symmetry should be discriminated starting from two-loop order, where we expect that the equivalence between correlators and Wilson loops may hold for the subset of conformal field theories only. The Wilson loop on a four cusped null contour is available at two loops in literature [40]; the computation of correlation functions at the same order is then highly desirable and is planned for a future investigation [45].

The correlators/Wilson loop equivalence is just a corner of the chain of dualities conjectured in [18–20]. Dualities involving scattering amplitudes are of great interest, the hope being to eventually extract information on those from the knowledge of simpler objects such as Wilson loops and correlation functions.

In three dimensions results on loop amplitudes are limited to the scattering of four external particles at first order [39]. When the theory possesses enough supersymmetry these amplitudes have been shown to vanish, completing the test of dualities in the one-loop $n = 4$ case. Differently from the correlator/Wilson loop equivalence, dualities involving scattering amplitudes seem to require supersymmetry already at one loop, indicating that their origin should be different from the former. Indeed the duality between MHV scattering amplitudes and Wilson loops is intimately connected to dual superconformal invariance [10] on the field theory side and to T-duality in the AdS dual description [12]. Results on dual superconformal invariance have been extended to tree level scattering amplitudes in three dimensional theories in [30, 31], whereas fermionic T-dualities seem to be ill-defined for the σ-model in the dual picture [34]–[38].

In order to shed more light on the role of superconformal invariance and dualities for Chern-Simons matter theories the knowledge of a larger sample of scattering amplitudes is mandatory. In particular it would be highly desirable to compute the six point amplitude at one-loop order and the four point amplitude at two loops, which should not be trivial.

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This task represents another challenging line of research [46].

Acknowledgments

This work has been supported in part by INFN and MIUR.

A Notations and conventions

For three dimensional $\mathcal{N}=2$ superspace we follow the conventions of [43].
The metric for the fermionic coordinates θ^α ($\alpha = 1,\, 2$) of $\mathcal{N}=2$ superspace is

$$C^{\alpha\beta} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad C_{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{A.1}$$

which is used to rise and lower spinorial indices as

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta \qquad \psi_\alpha = \psi^\beta C_{\beta\alpha} \tag{A.2}$$

and obeys the relation

$$C^{\alpha\beta}\, C_{\gamma\delta} = \delta^\alpha{}_\gamma\, \delta^\beta{}_\delta - \delta^\alpha{}_\delta\, \delta^\beta{}_\gamma \tag{A.3}$$

Spinors are contracted according to

$$\psi\chi = \psi^\alpha\, \chi_\alpha = \chi^\alpha\, \psi_\beta = \chi\psi \qquad \psi^2 = \frac{1}{2}\, \psi^\alpha\, \psi_\alpha \tag{A.4}$$

We consider a three dimensional Minkowski spacetime with mostly plus signature $g^{\mu\nu} = \mathrm{diag}\,(-1,1,1)$. The corresponding Dirac $(\gamma^\mu)^\alpha{}_\beta$ matrices satisfy the algebra

$$(\gamma^\mu)^\alpha{}_\gamma\, (\gamma^\nu)^\gamma{}_\beta = -g^{\mu\nu}\delta^\alpha{}_\beta + i\,\epsilon^{\mu\nu\rho}\,(\gamma_\rho)^\alpha{}_\beta \tag{A.5}$$

The following identities for traces of Dirac matrices can be read from the above algebra

$$tr(\gamma^\mu\,\gamma^\nu) = (\gamma^\mu)^\alpha{}_\beta\,(\gamma^\nu)^\beta{}_\alpha = -2\,g^{\mu\nu} \tag{A.6}$$

$$tr(\gamma^\mu\,\gamma^\nu\,\gamma^\rho) = -(\gamma^\mu)^\alpha{}_\beta\,(\gamma^\nu)^\beta{}_\gamma\,(\gamma^\rho)^\gamma{}_\alpha = 2\,i\,\epsilon^{\mu\nu\rho} \tag{A.7}$$

$$\begin{aligned} tr(\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma) &= (\gamma^\mu)^\alpha{}_\beta\,(\gamma^\nu)^\beta{}_\gamma\,(\gamma^\rho)^\gamma{}_\delta\,(\gamma^\sigma)^\delta{}_\alpha = \\ &= 2\,(g^{\mu\nu}\,g^{\rho\sigma} - g^{\mu\rho}\,g^{\nu\sigma} + g^{\mu\sigma}\,g^{\nu\rho}) \end{aligned} \tag{A.8}$$

The scalar product of two bispinors follows

$$p^{\alpha\beta}\,k_{\alpha\beta} \;=\; 2\,p\cdot k \tag{A.9}$$

Vectors and bispinors are exchanged according to

$$\begin{array}{lll} \text{for coordinates} & x^{\alpha\beta} = \frac{1}{2}\,(\gamma_\mu)^{\alpha\beta}\,x^\mu & x^\mu = (\gamma^\mu)_{\alpha\beta}\,x^{\alpha\beta} \\ \text{for derivatives} & \partial_{\alpha\beta} = (\gamma^\mu)_{\alpha\beta}\,\partial_\mu & \partial_\mu = \frac{1}{2}\,(\gamma_\mu)^{\alpha\beta}\,\partial_{\alpha\beta} \\ \text{for fields} & A_{\alpha\beta} = \frac{1}{\sqrt{2}}\,(\gamma^\mu)_{\alpha\beta}\,A_\mu & A_\mu = \frac{1}{\sqrt{2}}\,(\gamma_\mu)^{\alpha\beta}\,A_{\alpha\beta} \end{array} \tag{A.10}$$

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Supercovariant derivatives are defined as

$$D_\alpha = \partial_\alpha + \frac{i}{2}\,\overline{\theta}^\beta\,\partial_{\alpha\beta} \qquad , \qquad \overline{D}_\alpha = \bar{\partial}_\alpha + \frac{i}{2}\,\theta^\beta\,\partial_{\alpha\beta} \tag{A.11}$$

and satisfy the anticommutator

$$\{D_\alpha,\,\overline{D}_\beta\} = i\,\partial_{\alpha\beta} \tag{A.12}$$

The components of a chiral and an anti-chiral superfield, $Z(x_L,\theta)$ and $\bar{Z}(x_R,\bar{\theta})$, are a complex boson ϕ, a complex two-component fermion ψ and a complex auxiliary scalar F. Their component expansions are given by

$$\begin{aligned} Z &= \phi(x_L) + \theta^\alpha\psi_\alpha(x_L) - \theta^2\,F(x_L) \\ \bar{Z} &= \bar{\phi}(x_R) + \overline{\theta}^\alpha\bar{\psi}_\alpha(x_R) - \overline{\theta}^2\,\bar{F}(x_R) \end{aligned} \tag{A.13}$$

where $x_L^\mu = x^\mu + i\theta\gamma^\mu\bar{\theta}$, $x_R^\mu = x^\mu - i\theta\gamma^\mu\bar{\theta}$.

The components of the real vector superfield $V(x,\theta,\bar{\theta})$ in Wess-Zumino gauge ($V| = D_\alpha V| = D^2V| = 0$) are the gauge field $A_{\alpha\beta}$, a complex two-component fermion λ_α, a real scalar σ and an auxiliary scalar D, such that

$$V = i\,\theta^\alpha\overline{\theta}_\alpha\,\sigma(x) + \theta^\alpha\overline{\theta}^\beta\,\sqrt{2}\,A_{\alpha\beta}(x) - \theta^2\,\overline{\theta}^\alpha\bar{\lambda}_\alpha(x) - \overline{\theta}^2\,\theta^\alpha\lambda_\alpha(x) + \theta^2\,\overline{\theta}^2\,D(x) \tag{A.14}$$

The U(N) generators are $T^A = (T^0, T^a)$, where $T^0 = \frac{1}{\sqrt{N}}$ and T^a ($a = 1,\ldots,N^2-1$) are a set of $N\times N$ hermitian matrices. The generators are normalized as $Tr(T^AT^B) = \delta^{AB}$.

B The emergence of a five dimensional integral

In this appendix we give a detailed proof of eq. (3.7) which allows to express a double three dimensional integral as a one-loop five dimensional box integral.

In order to simplify the notation, in the expression $\epsilon_{\mu\nu\rho}\,\partial_i^\mu\,\partial_{i+1}^\nu\,\partial_{j+1}^\rho\,\mathcal{I}(i,j)$ we choose $i=1, j=3$. Applying the derivatives to the integrand, the expression that we need evaluate is then

$$\begin{aligned} &\epsilon^{\mu\nu\rho}\,\partial_{1\mu}\,\partial_{2\nu}\,\partial_{4\rho}\int d^3x_0\,d^3x_5\,\frac{1}{x_{1,0}\,x_{2,0}\,x_{0,5}\,x_{3,5}\,x_{4,5}} \qquad\qquad (B.1)\\ &= -\epsilon_{\mu\nu\rho}\int d^3x_0\,d^3x_5\,\frac{x_{1,0}^\mu\,x_{2,0}^\nu\,x_{4,5}^\rho}{(x_{1,0}^2)^{3/2}\,(x_{2,0}^2)^{3/2}\,(x_{0,5}^2)^{1/2}\,(x_{3,5}^2)^{1/2}\,(x_{4,5}^2)^{3/2}} \equiv \mathcal{I} \end{aligned}$$

We first focus on the x_0-integration which can be performed by introducing Feynman parameters

$$\begin{aligned} &\epsilon_{\mu\nu\rho}\int d^3x_0\,\frac{x_{1,0}^\mu\,x_{2,0}^\nu}{(x_{1,0}^2)^{3/2}\,(x_{2,0}^2)^{3/2}\,(x_{0,5}^2)^{1/2}} = \\ &\frac{4}{\pi^{3/2}}\,\Gamma\left(\frac{7}{2}\right)\int\prod_{i=1}^3 dy_i\,\delta(\textstyle\sum y_i - 1)\,y_1^{1/2}y_2^{1/2}y_3^{-1/2}\int d^3x_0\,\frac{\epsilon_{\mu\nu\rho}\,x_{1,0}^\mu\,x_{2,0}^\nu}{[(x_0-\rho_1)^2+\Omega_1]^{7/2}} \end{aligned} \tag{B.2}$$

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where $\rho_1^\mu = y_1 x_1^\mu + y_2 x_2^\mu + y_3 x_5^\mu$ and $\Omega_1 = y_1 y_2 x_{1,2}^2 + y_1 y_3 x_{1,5}^2 + y_2 y_3 x_{2,5}^2$.

Performing the shift $x_0^\mu \to x_0^\mu + \rho_1^\mu$ and integrating over x_0 we obtain

$$4\,\epsilon_{\mu\nu\rho}\,x_{1,5}^\mu\,x_{2,5}^\nu \int \prod_{i=1}^{3} dy_i\,\delta(\sum_i y_i - 1)\frac{(y_1\,y_2\,y_3)^{1/2}}{(y_1 y_2 x_{1,2}^2 + y_1 y_3 x_{1,5}^2 + y_2 y_3 x_{2,5}^2)^2} \tag{B.3}$$

Now, in order to render the remaining x_5 integration in (B.1) doable, we manipulate the expression (B.3) by using the Mellin-Barnes integral representation. According to the general identity

$$\frac{1}{(k^2+A^2+B^2)^a} = \frac{1}{(k^2)^a\Gamma(a)}\int_{-i\infty}^{+i\infty}\frac{dsdt}{(2\pi i)^2}\Gamma(-s)\Gamma(-t)\Gamma(a+s+t)\left(\frac{A^2}{k^2}\right)^s\left(\frac{B^2}{k^2}\right)^t \tag{B.4}$$

we rewrite (B.3) as

$$\frac{4\,\epsilon_{\mu\nu\rho}\,x_{1,5}^\mu\,x_{2,5}^\nu}{\sqrt{\pi}}\int_{-i\infty}^{i\infty}\frac{du\,dv}{(2\pi i)^2}\frac{\Gamma\left(-u|-\frac{1}{2}-u|-v|-\frac{1}{2}-v|2+u+v|\frac{3}{2}+u+v\right)}{(x_{1,2}^2)^{u+v+2}\,(x_{1,5}^2)^{-u}\,(x_{2,5}^2)^{-v}} \tag{B.5}$$

where we have introduced the short notation $\Gamma(z_1|...|z_n) \equiv \Gamma(z_1)...\Gamma(z_n)$.

We insert this expression back into eq. (B.1) and perform the x_5-integration. Once again, using Feynman combining we can write (we neglect factors which are independent of x_5)

$$-\epsilon_{\mu\nu\rho}\int d^3x_5\,\frac{x_{1,5}^\mu\,x_{2,5}^\nu\,x_{4,5}^\rho}{(x_{1,5}^2)^{-u}\,(x_{2,5}^2)^{-v}\,(x_{3,5}^2)^{1/2}\,(x_{4,5}^2)^{3/2}} = \tag{B.6}$$

$$-\frac{2\Gamma(2-u-v)}{\Gamma(-u|-v)\pi}\int\prod_{i=1}^{4}dy_i\,\delta(\sum y_i-1)y_1^{-u-1}y_2^{-v-1}y_3^{-1/2}y_4^{1/2}\int\frac{d^3x_5\,\epsilon_{\mu\nu\rho}\,x_{1,5}^\mu\,x_{2,5}^\nu\,x_{4,5}^\rho}{[(x_5-\rho_2)^2+\Omega_2]^{2-u-v}}$$

where we have defined

$$\begin{aligned}\rho_2^\mu &= y_1x_1^\mu + y_2x_2^\mu + y_3x_3^\mu + y_4x_4^\mu \\ \Omega_2 &= y_1y_2x_{1,2}^2 + y_2y_3x_{2,3}^2 + y_3y_4x_{3,4}^2 + y_4y_1x_{4,1}^2 + y_1y_3x_{1,3}^2 + y_2y_4x_{2,4}^2\end{aligned} \tag{B.7}$$

After shifting $x_5^\mu \to x_5^\mu + \rho_2^\mu$, we may integrate over x_5 and obtain

$$\epsilon_{\mu\nu\rho}\,x_{3,1}^\mu\,x_{3,2}^\nu\,x_{3,4}^\rho\,\frac{2\sqrt{\pi}\,\Gamma(\frac{1}{2}-u-v)}{\Gamma(-u|-v)}\int\prod_{i=1}^{4}dy_i\,\frac{\delta(\sum y_i-1)\,y_1^{-u-1}y_2^{-v-1}y_3^{1/2}y_4^{1/2}}{\Omega_2^{1/2-u-v}} \tag{B.8}$$

The first remarkable observation is that this expression is exactly the Feynman parametrization of a five dimensional scalar square integral with indices $(-u,-v,3/2,3/2)$. Precisely, we have

$$\text{(B.8)} \quad = \epsilon_{\mu\nu\rho}\,x_{3,1}^\mu\,x_{3,2}^\nu\,x_{3,4}^\rho\,\frac{1}{2\pi}\int d^5x_5\frac{1}{(x_{1,5}^2)^{-u}\,(x_{2,5}^2)^{-v}\,(x_{3,5}^2)^{3/2}\,(x_{4,5}^2)^{3/2}} \tag{B.9}$$

The identification with a higher dimensional integral is strictly formal, and should be intended at the level of its Feynman-parametrized form. In any case, we are dealing with

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a scalar integral which depends only on the Lorentz invariants $x_{i,j}^2$ and these invariants are unambiguously well-defined both in three and five dimensions.

Collecting all the factors from (B.5), (B.9), we are left with the following expression for the initial integral

$$\mathcal{I} = \frac{2}{\pi^{\frac{3}{2}}}\,\epsilon_{\mu\nu\rho}\,x_{1,3}^{\mu}\,x_{2,3}^{\nu}\,x_{3,4}^{\rho}\int\frac{d^5x_5}{(x_{3,5}^2)^{3/2}\,(x_{4,5}^2)^{3/2}}\times \int_{-i\infty}^{+i\infty}\frac{du\,dv}{(2\pi i)^2}\,\frac{\Gamma\left(-u|-v|-\frac{1}{2}-u|-\frac{1}{2}-v|2+u+v|\frac{3}{2}+u+v\right)}{(x_{1,2}^2)^{u+v+2}\,(x_{1,5}^2)^{-u}\,(x_{2,5}^2)^{-v}} \tag{B.10}$$

The second remarkable observation is that the MB integral in this expression can be identified with the MB-representation of a five dimensional scalar triangle with exponents $(3/2, 3/2, 3/2)$. Therefore, we can write

$$\mathcal{I} = \frac{1}{4\pi^2}\,\epsilon_{\mu\nu\rho}\,x_{1,3}^{\mu}\,x_{2,3}^{\nu}\,x_{3,4}^{\rho}\int d^5x_0\,d^5x_5\,\frac{1}{(x_{0,1}^2)^{3/2}\,(x_{0,2}^2)^{3/2}\,(x_{0,5}^2)^{3/2}\,(x_{3,5}^2)^{3/2}\,(x_{4,5}^2)^{3/2}} \tag{B.11}$$

At this point it might seem that we have traded a complicated two-loop tensor integral in three dimensions with a complicated two-loop scalar integral in five dimensions. But here comes the magic: We can use the uniqueness relations applied to the x_5-triangle integral.

We recall that for a generic triangle integral in D dimensions with arbitrary exponents

$$\mathcal{T}[D;\alpha_1,\alpha_2,\alpha_3;x_{0,3}^2,x_{0,4}^2,x_{3,4}^2] = \int\frac{d^Dx_5}{(x_{0,5}^2)^{\alpha_1}\,(x_{3,5}^2)^{\alpha_2}\,(x_{4,5}^2)^{\alpha_3}}, \tag{B.12}$$

the following identity holds [42]

$$\mathcal{T}[D;\alpha_1,\alpha_2,\alpha_3;x_{0,3}^2,x_{0,4}^2,x_{3,4}^2] = \frac{\Gamma(\sum_i\alpha_i-\frac{D}{2})}{\Gamma(D-\sum_i\alpha_i)}\prod_i\frac{\Gamma(\frac{D}{2}-\alpha_i)}{\Gamma(\alpha_i)}\times\frac{1}{(x_{3,4}^2)^{\alpha_2+\alpha_3-D/2}}\times \mathcal{T}\left[D;\sum\alpha_i-\frac{D}{2},\frac{D}{2}-\alpha_3,\frac{D}{2}-\alpha_2;x_{0,3}^2,x_{0,4}^2,x_{3,4}^2\right] \tag{B.13}$$

Applying this identity to the x_5-triangle in (B.11) where we identify $D=5$ and $\alpha_1=\alpha_2=\alpha_3=3/2$, we obtain

$$\mathcal{I} = \frac{2}{\pi^4}\,\frac{\epsilon_{\mu\nu\rho}\,x_{1,3}^{\mu}\,x_{2,3}^{\nu}\,x_{3,4}^{\rho}}{x_{3,4}}\int d^5x_0\,d^5x_5\,\frac{1}{(x_{0,1}^2)^{3/2}\,(x_{0,2}^2)^{3/2}\,(x_{0,5}^2)^2\,x_{3,5}^2\,x_{4,5}^2} \tag{B.14}$$

The advantage of doing it is that the exponents of the x_0 triangle are now $(3/2, 3/2, 2)$ and satisfy the uniqueness condition $\alpha_1+\alpha_2+\alpha_3=D$ in five dimensions. Therefore, we can use the general result for *unique* triangles [42]

$$\int\frac{d^Dx_0}{(x_{0,1}^2)^{\alpha_1}\,(x_{0,2}^2)^{\alpha_2}\,(x_{0,5}^2)^{\alpha_3}}\Big|_{\alpha_1+\alpha_2+\alpha_3=D} = \pi^{D/2}\prod_i\frac{\Gamma(D/2-a_i)}{\Gamma(a_i)}\,\frac{1}{(x_{1,2}^2)^{D/2-\alpha_3}\,(x_{1,5}^2)^{D/2-\alpha_2}\,(x_{2,5}^2)^{D/2-\alpha_1}} \tag{B.15}$$

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and finally write

$$\mathcal{I} = \frac{8}{\pi^2} \frac{\epsilon_{\mu\nu\rho}\, x_{1,3}^{\mu}\, x_{2,3}^{\nu}\, x_{3,4}^{\rho}}{x_{1,2}\, x_{3,4}} \int d^5 x_5 \frac{1}{x_{5,1}^2\, x_{5,2}^2\, x_{5,3}^2\, x_{5,4}^2} \tag{B.16}$$

This concludes the proof of eq. (3.7) for $i = 1, j = 3$. The generalization of the formula to any i, j is trivial.

References

[1] L.F. Alday and J.M. Maldacena, *Gluon scattering amplitudes at strong coupling*, *JHEP* **06** (2007) 064 [arXiv:0705.0303] [SPIRES].

[2] L.F. Alday and J. Maldacena, *Comments on gluon scattering amplitudes via AdS/CFT*, *JHEP* **11** (2007) 068 [arXiv:0710.1060] [SPIRES].

[3] G.P. Korchemsky, J.M. Drummond and E. Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops*, *Nucl. Phys.* **B 795** (2008) 385 [arXiv:0707.0243] [SPIRES].

[4] A. Brandhuber, P. Heslop and G. Travaglini, *MHV Amplitudes in $N = 4$ Super Yang-Mills and Wilson Loops*, *Nucl. Phys.* **B 794** (2008) 231 [arXiv:0707.1153] [SPIRES].

[5] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *On planar gluon amplitudes/Wilson loops duality*, *Nucl. Phys.* **B 795** (2008) 52 [arXiv:0709.2368] [SPIRES].

[6] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *Conformal Ward identities for Wilson loops and a test of the duality with gluon amplitudes*, *Nucl. Phys.* **B 826** (2010) 337 [arXiv:0712.1223] [SPIRES].

[7] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *The hexagon Wilson loop and the BDS ansatz for the six- gluon amplitude*, *Phys. Lett.* **B 662** (2008) 456 [arXiv:0712.4138] [SPIRES].

[8] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *Hexagon Wilson loop = six-gluon MHV amplitude*, *Nucl. Phys.* **B 815** (2009) 142 [arXiv:0803.1466] [SPIRES].

[9] C. Anastasiou et al., *Two-Loop Polygon Wilson Loops in $N = 4$ SYM*, *JHEP* **05** (2009) 115 [arXiv:0902.2245] [SPIRES].

[10] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *Dual superconformal symmetry of scattering amplitudes in $N = 4$ super-Yang-Mills theory*, *Nucl. Phys.* **B 828** (2010) 317 [arXiv:0807.1095] [SPIRES].

[11] A. Brandhuber, P. Heslop and G. Travaglini, *A note on dual superconformal symmetry of the $N = 4$ super Yang-Mills S-matrix*, *Phys. Rev.* **D 78** (2008) 125005 [arXiv:0807.4097] [SPIRES].

[12] N. Berkovits and J. Maldacena, *Fermionic T-duality, Dual Superconformal Symmetry and the Amplitude/Wilson Loop Connection*, *JHEP* **09** (2008) 062 [arXiv:0807.3196] [SPIRES].

[13] J.M. Drummond, J.M. Henn and J. Plefka, *Yangian symmetry of scattering amplitudes in $N = 4$ super Yang-Mills theory*, *JHEP* **05** (2009) 046 [arXiv:0902.2987] [SPIRES].

[14] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208] [SPIRES].

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[15] N. Beisert, *The dilatation operator of $N = 4$ super Yang-Mills theory and integrability*, *Phys. Rept.* **405** (2005) 1 [hep-th/0407277] [SPIRES].

[16] N. Beisert, R. Ricci, A.A. Tseytlin and M. Wolf, *Dual Superconformal Symmetry from $AdS_5 \times S^5$ Superstring Integrability*, *Phys. Rev.* **D 78** (2008) 126004 [arXiv:0807.3228] [SPIRES].

[17] L.F. Alday, J. Maldacena, A. Sever and P. Vieira, *Y-system for Scattering Amplitudes*, *J. Phys.* **A 43** (2010) 485401 [arXiv:1002.2459] [SPIRES].

[18] L.F. Alday, B. Eden, G.P. Korchemsky, J. Maldacena and E. Sokatchev, *From correlation functions to Wilson loops*, arXiv:1007.3243 [SPIRES].

[19] B. Eden, G.P. Korchemsky and E. Sokatchev, *From correlation functions to scattering amplitudes*, arXiv:1007.3246 [SPIRES].

[20] B. Eden, G.P. Korchemsky and E. Sokatchev, *More on the duality correlators/amplitudes*, arXiv:1009.2488 [SPIRES].

[21] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [arXiv:0806.1218] [SPIRES].

[22] O. Aharony, O. Bergman and D.L. Jafferis, *Fractional M2-branes*, *JHEP* **11** (2008) 043 [arXiv:0807.4924] [SPIRES].

[23] J. Bagger and N. Lambert, *Modeling multiple M2's*, *Phys. Rev.* **D 75** (2007) 045020 [hep-th/0611108] [SPIRES].

[24] J. Bagger and N. Lambert, *Gauge Symmetry and Supersymmetry of Multiple M2-Branes*, *Phys. Rev.* **D 77** (2008) 065008 [arXiv:0711.0955] [SPIRES].

[25] A. Gustavsson, *Algebraic structures on parallel M2-branes*, *Nucl. Phys.* **B 811** (2009) 66 [arXiv:0709.1260] [SPIRES].

[26] M.S. Bianchi, S. Penati and M. Siani, *Infrared stability of ABJ-like theories*, *JHEP* **01** (2010) 080 [arXiv:0910.5200] [SPIRES].

[27] M.S. Bianchi, S. Penati and M. Siani, *Infrared Stability of $N = 2$ Chern-Simons Matter Theories*, *JHEP* **05** (2010) 106 [arXiv:0912.4282] [SPIRES].

[28] D. Gaiotto and A. Tomasiello, *The gauge dual of Romans mass*, *JHEP* **01** (2010) 015 [arXiv:0901.0969] [SPIRES].

[29] D. Gaiotto and A. Tomasiello, *Perturbing gauge/gravity duals by a Romans mass*, *J. Phys.* **A 42** (2009) 465205 [arXiv:0904.3959] [SPIRES].

[30] Y.-t. Huang and A.E. Lipstein, *Dual Superconformal Symmetry of $N = 6$ Chern-Simons Theory*, *JHEP* **11** (2010) 076 [arXiv:1008.0041] [SPIRES].

[31] D. Gang, Y.-t. Huang, E. Koh, S. Lee and A.E. Lipstein, *Tree-level Recursion Relation and Dual Superconformal Symmetry of the ABJM Theory*, *JHEP* **03** (2011) 116 [arXiv:1012.5032] [SPIRES].

[32] T. Bargheer, F. Loebbert and C. Meneghelli, *Symmetries of Tree-level Scattering Amplitudes in $N = 6$ Superconformal Chern-Simons Theory*, *Phys. Rev.* **D 82** (2010) 045016 [arXiv:1003.6120] [SPIRES].

[33] S. Lee, *Yangian Invariant Scattering Amplitudes in Supersymmetric Chern-Simons Theory*, *Phys. Rev. Lett.* **105** (2010) 151603 [arXiv:1007.4772] [SPIRES].

JHEP06(2011)118

[34] I. Adam, A. Dekel and Y. Oz, *On Integrable Backgrounds Self-dual under Fermionic T-duality*, *JHEP* **04** (2009) 120 [arXiv:0902.3805] [SPIRES].

[35] P.A. Grassi, D. Sorokin and L. Wulff, *Simplifying superstring and D-brane actions in AdS_4xCP^3 superbackground*, *JHEP* **08** (2009) 060 [arXiv:0903.5407] [SPIRES].

[36] I. Adam, A. Dekel and Y. Oz, *On the fermionic T-duality of the $AdS_4 \times CP^3\sigma$-model*, *JHEP* **10** (2010) 110 [arXiv:1008.0649] [SPIRES].

[37] I. Bakhmatov, *On AdS_4xCP^3 T-duality*, *Nucl. Phys.* **B 847** (2011) 38 [arXiv:1011.0985] [SPIRES].

[38] A. Dekel and Y. Oz, *Self-Duality of Green-Schwarz σ-models*, *JHEP* **03** (2011) 117 [arXiv:1101.0400] [SPIRES].

[39] A. Agarwal, N. Beisert and T. McLoughlin, *Scattering in Mass-Deformed $N >= 4$ Chern-Simons Models*, *JHEP* **06** (2009) 045 [arXiv:0812.3367] [SPIRES].

[40] J.M. Henn, J. Plefka and K. Wiegandt, *Light-like polygonal Wilson loops in 3d Chern-Simons and ABJM theory*, *JHEP* **08** (2010) 032 [arXiv:1004.0226] [SPIRES].

[41] M. Benna, I. Klebanov, T. Klose and M. Smedback, *Superconformal Chern-Simons Theories and AdS_4/CFT_3 Correspondence*, *JHEP* **09** (2008) 072 [arXiv:0806.1519] [SPIRES].

[42] N.I. Usyukina and A.I. Davydychev, *New results for two loop off-shell three point diagrams*, *Phys. Lett.* **B 332** (1994) 159 [hep-ph/9402223] [SPIRES].

[43] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, *Superspace, or one thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1 [hep-th/0108200] [SPIRES].

[44] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, S. Caron-Huot and J. Trnka, *The All-Loop Integrand For Scattering Amplitudes in Planar $N = 4$ SYM*, *JHEP* **01** (2011) 041 [arXiv:1008.2958] [SPIRES].

[45] M.S. Bianchi, M. Leoni, A. Mauri, S. Penati and A. Santambrogio, in preparation.

[46] M.S. Bianchi, M. Leoni, A. Mauri, S. Penati and A. Santambrogio, *Scattering Amplitudes/Wilson Loops Duality in ABJM Theory*, in preparation.

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