

# The real radiation antenna function for $S \rightarrow Q\bar{Q}q\bar{q}$ at NNLO QCD

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ABSTRACT: As a first step towards the application of the antenna subtraction formalism to NNLO QCD reactions with massive quarks, we determine the real radiation antenna function and its integrated counterpart for reactions of the type  $S \rightarrow Q\bar{Q}q\bar{q}$ , where  $S$  denotes an uncolored initial state and  $Q, q$  a massive and massless quark, respectively. We compute the corresponding integrated antenna function in terms of harmonic polylogarithms. As an application and check of our results we calculate the contribution proportional to  $\alpha_s^2 e_Q^2 N_f$  to the inclusive heavy-quark pair production cross section in  $e^+e^-$  annihilation.

KEYWORDS: NLO Computations, QCD

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**1 Introduction**

The calculation of differential cross sections and distributions in perturbative QCD beyond the leading order requires, apart from the renormalization of the QCD coupling and the quark masses, methods to regularize and handle the infrared (IR) divergences that appear in the intermediate steps of such computations. One general approach is to construct subtraction terms such that, after adding/subtracting these terms, the IR singularities are regulated and cancelled in tree-amplitudes involving the radiation of real massless partons and in associated loop amplitudes in the computation of IR safe observables (up to factorization of collinear initial-state singularities).

For calculations at NLO QCD a widely used version of this approach is the dipole subtraction method for massless QCD [1] and for QCD with massive quarks and other colored massive particles [2–4], which was slightly modified in [5–9] and has found a number of computer implementations [7, 9–12]. Other NLO subtraction methods were constructed and applied, too, including those of [13–16], and the antenna method (see below).

Computations of differential cross sections at NNLO QCD involve three types of contributions: squares of tree-level double real emission amplitudes (with  $n + 2$  final-state partons), interferences of one-loop and tree-level amplitudes ( $n + 1$  final-state partons) and of Born, one-loop, and two-loop amplitudes ( $n$  final-state partons). For general discussions of the IR structure, see [17–24]. Various techniques have been devised to handle the IR divergences of these individual contributions. These include the sector decomposition algorithm [25–29], the antenna formalism [30–35], and the subtraction methods [36–49]. Application to reactions at NNLO QCD include  $pp \rightarrow H + X$  [47, 50],  $pp \rightarrow W + X$  [51–53],  $e^+e^- \rightarrow 2$  jets [54, 55], and  $e^+e^- \rightarrow 3$  jets [56–61], where the jet calculations just mentioned were made for massless partons.

The antenna method [30–32], which we employ in this paper, was first worked out fully to NNLO QCD for  $e^+e^-$  annihilation into massless final-state partons. The general set-up of this approach applies also to colored initial states and/or massive colored particles in the final state. The extension to processes with initial-state and massless final-state partons at NLO QCD was made in [30, 31, 62]. For reactions with initial-state and massless final-state partons at NNLO QCD, results were presented in [63–65]. For processes with massive quarks  $Q$  in the final state, the antenna subtraction terms at NLO QCD were explicitly worked out for colorless initial states and for the hadronic reactions  $h_1 h_2 \rightarrow Q\bar{Q}$ ,  $Q\bar{Q} + \text{jet}$  in [66] and in [67], respectively. As is well-known, the IR singularity structure of the matrix elements for reactions with massive colored particles is less entangled than that of their massless counterparts but, on the other hand, the (analytical) computation of the integrated subtraction terms is more difficult.

In this paper we are concerned with the production of a heavy-quark pair  $Q\bar{Q}$  by an uncolored initial state  $S$  at NNLO QCD, i.e., we consider reactions of the type

$$S \rightarrow Q \bar{Q} + X, \tag{1.1}$$

at order  $\alpha_s^2$ . This includes the production of a pair of heavy quarks by electron-positron annihilation,  $e^+e^- \rightarrow \gamma^*, Z^* \rightarrow Q\bar{Q}X$ , and the decay of a color and electrically neutral massive boson of any spin into  $Q\bar{Q}X$ . The following ingredients are necessary for the computation of arbitrary differential distributions to order  $\alpha_s^2$ : i) The amplitudes  $S \rightarrow Q\bar{Q}$  to order  $\alpha_s^2$ . They are known in analytical form for  $S = \text{vector}$  [68, 69], axial vector [70, 71], scalar and pseudoscalar [72]. ii) The tree- and one-loop amplitudes for  $S \rightarrow Q\bar{Q}g$ . The one-loop amplitudes can be computed with standard methods and are known for  $e^+e^-$  annihilation, i.e.  $S = \gamma^*, Z^*$  [73–75]. iii) The tree-level amplitudes  $S \rightarrow Q\bar{Q}Q\bar{Q}$ ,  $Q\bar{Q}gg$ , and  $Q\bar{Q}q\bar{q}$ , where  $q$  denotes a massless quark. Apart from the tree-level amplitudes  $S \rightarrow Q\bar{Q}$  and  $S \rightarrow Q\bar{Q}Q\bar{Q}$ , the matrix elements give rise to IR singularities, which are regulated within the above-mentioned subtraction methods by appropriate counterterms. The calculation of the differential cross section  $d\sigma_{\text{NNLO}}$  for  $S$  decaying into two massive quark jets is standard. The contribution of order  $\alpha_s^2$  to the two-jet cross section is given schematically, using the notation of [55], by

$$d\sigma_{\text{NNLO}} = \int_{d\Phi_4} (d\sigma_{\text{NNLO}}^R - d\sigma_{\text{NNLO}}^S) + \int_{d\Phi_4} d\sigma_{\text{NNLO}}^S + \int_{d\Phi_3} (d\sigma_{\text{NNLO}}^{V,1} - d\sigma_{\text{NNLO}}^{VS,1}) + \int_{d\Phi_3} d\sigma_{\text{NNLO}}^{VS,1} + \int_{d\Phi_2} d\sigma_{\text{NNLO}}^{V,2}. \tag{1.2}$$

Here  $d\sigma_{\text{NNLO}}^R$ ,  $d\sigma_{\text{NNLO}}^{V,1}$ , and  $d\sigma_{\text{NNLO}}^{V,2}$  denote the contributions from the tree-level amplitudes  $Q\bar{Q}gg$  and  $Q\bar{Q}q\bar{q}$  (and  $Q\bar{Q}Q\bar{Q}$ , which does not require a subtraction term), the one-loop  $Q\bar{Q}g$  amplitude, and the two-loop  $Q\bar{Q}$  amplitude, respectively. The term  $d\sigma_{\text{NNLO}}^S$  ( $d\sigma_{\text{NNLO}}^{VS,1}$ ) is a subtraction term that coincides with  $d\sigma_{\text{NNLO}}^R$  ( $d\sigma_{\text{NNLO}}^{V,1}$ ) in all singular limits.

The IR singularities of the two-loop term  $d\sigma_{\text{NNLO}}^{V,2}$  are explicitly known within dimensional regularization [68–72]. The construction of  $d\sigma_{\text{NNLO}}^S$  and  $d\sigma_{\text{NNLO}}^{VS,1}$  depends on the subtraction method used — the integration of these terms over the four- and three-parton

phase spaces  $d\Phi_4$  and  $d\Phi_3$ , respectively, is in any case a difficult task. To our knowledge this has not yet been done for massive quarks in analytical form. As mentioned above, we shall use the antenna framework. As a first step in the computation of  $d\sigma(Q\bar{Q})$  to NNLO QCD within this approach, we determine in this paper the subtraction term for the  $Q\bar{Q}q\bar{q}$  final state and its integral over the four-parton phase space. The new aspect of this computation is the analytic integration of the massive tree-level antenna function associated with the process

$$\gamma^*(q) \rightarrow Q(p_1) \bar{Q}(p_2) + q(p_3) \bar{q}(p_4). \quad (1.3)$$

over the full four-particle phase space. This (integrated) antenna function is not only of relevance for the specific process at hand, but serves also as a building block for constructing subtraction terms for other processes (1.1) within the antenna formalism.

The paper is organized as follows. In section 2 we determine the antenna function for (1.3) and in section 3 we integrate this function analytically over the four-parton phase space. As an application and check of our results, we compute in section 4 the cross section for the inclusive production of a massive  $Q\bar{Q}$  pair plus  $N_f$  massless quarks by  $e^+e^-$  annihilation via a virtual photon — more precisely, the contribution of order  $e_Q^2\alpha_s^2N_f$  to this cross section. Section 5 contains a summary and outlook.

## 2 Antenna subtraction at NNLO QCD

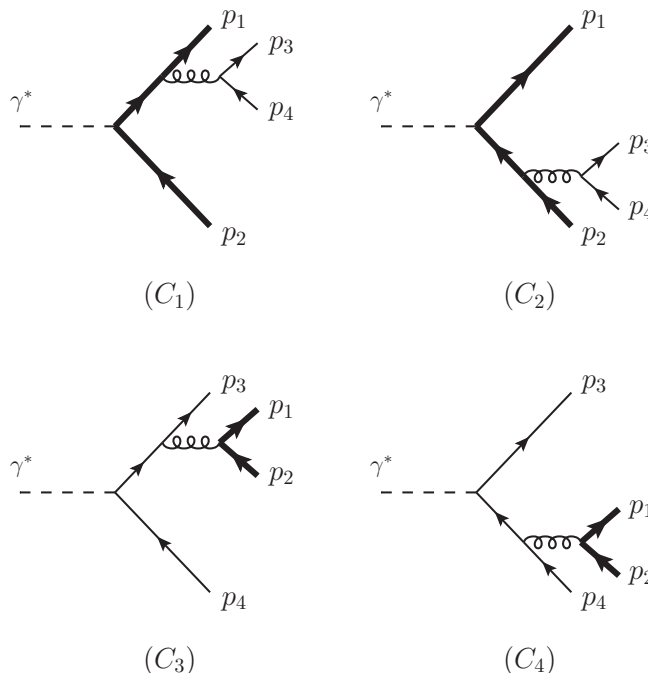
In the following, we restrict our attention to the case of reaction (1.3) where, for definiteness, we consider one massless quark flavor  $q$  in the final state. As mentioned above, we focus on constructing a subtraction term which coincides with the squared matrix element of  $\gamma^* \rightarrow Q\bar{Q}q\bar{q}$  in all single and double unresolved limits. In figure 1 the Feynman diagrams are shown that are associated with this process at order  $\alpha_s^2$ . The corresponding contribution to the cross section for 2-jet production may be written as follows:

$$d\sigma_{\text{NNLO}}^{R,Q\bar{Q}q\bar{q}} = 4\pi\alpha(4\pi\alpha_s)^2(N_c^2 - 1) d\Phi_4(p_1, p_2, p_3, p_4; q) J_2^{(4)}(p_1, p_2, p_3, p_4) \\ \times \left\{ e_Q^2 \left| \mathcal{M}_{Qq\bar{q}\bar{Q}}^0 \right|^2 + e_q^2 \left| \mathcal{M}_{qQ\bar{Q}\bar{q}}^0 \right|^2 + 2e_Q e_q \text{Re} \left( \mathcal{M}_{Qq\bar{q}\bar{Q}}^0 \mathcal{M}_{qQ\bar{Q}\bar{q}}^{0,\dagger} \right) \right\}, \quad (2.1)$$

where the matrix elements  $\mathcal{M}_{Qq\bar{q}\bar{Q}}^0$  and  $\mathcal{M}_{qQ\bar{Q}\bar{q}}^0$  correspond to the diagrams  $C_1$ ,  $C_2$  and  $C_3$ ,  $C_4$ , respectively. The dependence on the electromagnetic and strong coupling and the dependence on the number of colors  $N_c$  are extracted from the matrix elements;  $e_Q$  ( $e_q$ ) is the electric charge of the massive (massless) quark in units of the positron charge. Summation over all spins is understood. In the formulae below, the polarizations of  $\gamma^*$  are summed, but not averaged.

The phase space measure  $d\Phi_4$  in  $d = 4 - 2\epsilon$  dimensions is

$$d\Phi_4(p_1, p_2, p_3, p_4; q) = \mu^{12-3d} \prod_{i=1}^4 \frac{d^{d-1}p_i}{(2\pi)^{d-1}2p_i^0} (2\pi)^d \delta^{(d)} \left( q - \sum_{i=1}^4 p_i \right), \quad (2.2)$$



**Figure 1.** Feynman diagrams contributing to  $\gamma^* \rightarrow Q\bar{Q}q\bar{q}$  at tree-level. Bold (thin) lines refer to massive (massless) quarks.

where  $\mu$  is a mass scale. The jet function  $J_m^{(n)}$  in (2.1) ensures that only configurations are taken into account where  $n$  outgoing partons form  $m$  jets.

Because the squared matrix element  $|\mathcal{M}_{qQ\bar{Q}\bar{q}}^0|^2$  does not involve infrared singular configurations, no subtraction is required. Its contribution to the  $Q\bar{Q}$  production cross section is given in [76]. The same holds for the interference terms between  $\mathcal{M}_{Qq\bar{q}\bar{Q}}^0$  and  $\mathcal{M}_{qQ\bar{Q}\bar{q}}^0$ . Moreover, due to Furry’s theorem these terms yield a vanishing contribution to the cross section if the observable under consideration does not distinguish between quarks and antiquarks.

In the framework of antenna subtraction the main building blocks for constructing NNLO subtraction terms are the antenna functions, which can be derived from physical color-ordered squared matrix elements for tree-level  $1 \rightarrow 3$  and  $1 \rightarrow 4$  processes and one-loop  $1 \rightarrow 3$  processes. A detailed and completely general analysis of how the subtraction terms are constructed from the various antenna functions is given in [32] for massless final state partons. This procedure applies also to the case of massive quarks. Its application to the specific process  $\gamma^* \rightarrow Q\bar{Q}q\bar{q}$  is outlined below. The three-parton tree-level antenna functions with massive quarks have been calculated in [66, 67], whereas the four-parton tree-level and three-parton one-loop antenna functions that involve massive quarks are still missing.

In the case of  $\gamma^* \rightarrow Q\bar{Q}q\bar{q}$  the construction of the subtraction terms may be divided into two parts. In a first step, the single unresolved configurations are subtracted. Within

the antenna method, the corresponding subtraction term reads

$$\begin{aligned}
 d\sigma_{\text{NNLO}}^{S,a} &= 4\pi\alpha (4\pi\alpha_s)^2 e_Q^2 (N_c^2 - 1) d\Phi_4(p_1, p_2, p_3, p_4; q) \left| \mathcal{M}_{Q\bar{Q}}^0 \right|^2 \\
 &\times \left[ E_3^0(1_Q, 3_q, 4_{\bar{q}}) A_3^0\left(\widetilde{(13)}_Q, \widetilde{(43)}_g, 2_{\bar{Q}}\right) J_2^{(3)}(\widetilde{p}_{13}, \widetilde{p}_{43}, p_2) \right. \\
 &\quad \left. + E_3^0(2_{\bar{Q}}3_q, 4_{\bar{q}}) A_3^0\left(1_Q, \widetilde{(34)}_g, \widetilde{(24)}_{\bar{Q}}\right) J_2^{(3)}(p_1, \widetilde{p}_{34}, \widetilde{p}_{24}) \right]. \quad (2.3)
 \end{aligned}$$

The massive quark-antiquark antenna function  $A_3^0(i_Q, k_g, j_{\bar{Q}})$  and the quark-gluon antenna  $E_3^0(i_Q, j_q, k_{\bar{q}})$  with a massive radiator quark are given in [66]. The momenta  $\widetilde{p}_{ik}$  and  $\widetilde{p}_{jk}$  are redefined on-shell momenta, constructed from linear combinations of the momenta  $p_i$ ,  $p_j$  and  $p_k$ . The tree-level two-parton matrix element squared (summed over colors and spins, with the photon coupling and  $N_c$  factored out) is

$$\left| \mathcal{M}_{Q\bar{Q}}^0(\gamma^* \rightarrow Q\bar{Q}) \right|^2 = 4 [(1 - \epsilon) q^2 + 2m^2], \quad (2.4)$$

where  $m$  denotes the mass of  $Q$ . In a second step, the double unresolved configuration, where both  $q$  and  $\bar{q}$  become soft, has to be subtracted. In the case at hand, the appropriately normalized squared matrix element  $\left| \mathcal{M}_{Qq\bar{q}\bar{Q}}^0 \right|^2$  can be used as subtraction term. In the terminology of [32] this is the antenna function  $B_4^0(1_Q, 3_q, 4_{\bar{q}}, 2_{\bar{Q}})$  associated with the color-ordering  $Qq\bar{q}\bar{Q}$ , where a color-connected massless quark-antiquark pair is radiated between a pair of massive quarks. More precisely, this antenna function is defined by

$$B_4^0(1_Q, 3_q, 4_{\bar{q}}, 2_{\bar{Q}}) = \frac{\left| \mathcal{M}_{Qq\bar{q}\bar{Q}}^0 \right|^2}{\left| \mathcal{M}_{Q\bar{Q}}^0 \right|^2}, \quad (2.5)$$

where the normalization factor is given in (2.4). Obviously (2.5) has the appropriate behaviour in the singular double unresolved limit where  $q$  and  $\bar{q}$  become simultaneously soft. However (2.5) contains also singularities due to single unresolved limits, which have to be subtracted from the antenna function. In the end, the corresponding subtraction term for the double unresolved configuration reads

$$\begin{aligned}
 d\sigma_{\text{NNLO}}^{S,b} &= 4\pi\alpha (4\pi\alpha_s)^2 e_Q^2 (N_c^2 - 1) d\Phi_4(p_1, p_2, p_3, p_4; q) \left| \mathcal{M}_{Q\bar{Q}}^0 \right|^2 \\
 &\times \left[ B_4^0(1_Q, 3_q, 4_{\bar{q}}, 4_Q) - E_3^0(1_Q, 3_q, 4_{\bar{q}}) A_3^0\left(\widetilde{(13)}_Q, \widetilde{(43)}_g, 2_{\bar{Q}}\right) \right. \\
 &\quad \left. - E_3^0(2_{\bar{Q}}3_q, 4_{\bar{q}}) A_3^0\left(1_Q, \widetilde{(34)}_g, \widetilde{(24)}_{\bar{Q}}\right) \right] J_2^{(2)}(\widetilde{p}_{134}, \widetilde{p}_{243}), \quad (2.6)
 \end{aligned}$$

where  $\widetilde{p}_{ikl}$  and  $\widetilde{p}_{jkl}$  are linear combinations of the momenta  $p_i$ ,  $p_j$ ,  $p_k$  and  $p_l$ .

The subtracted differential cross section

$$d\sigma_{\text{NNLO}}^{R,Q\bar{Q}q\bar{q}} - d\sigma_{\text{NNLO}}^{S,a} - d\sigma_{\text{NNLO}}^{S,b} \quad (2.7)$$

is free of IR divergences and can be integrated over the four-parton phase space numerically in  $d = 4$  dimensions.

For the antenna function  $B_4^0$  we find

$$\begin{aligned}
B_4^0(1_Q, 3_q, 4_{\bar{q}}, 2_{\bar{Q}}) = & \frac{1}{(q^2 + 2m^2)} \left\{ \frac{1}{s_{34}s_{134}^2} [s_{12}s_{13} + s_{12}s_{14} + s_{13}s_{23} + s_{14}s_{24}] \right. \\
& + \frac{1}{s_{34}s_{234}^2} [s_{12}s_{23} + s_{12}s_{24} + s_{13}s_{23} + s_{14}s_{24}] \\
& + \frac{1}{s_{34}^2 s_{134}^2} [2s_{12}s_{13}s_{14} + s_{13}s_{14}s_{24} + s_{13}s_{14}s_{23} \\
& - s_{13}^2 s_{24} - s_{14}^2 s_{23}] \\
& + \frac{1}{s_{34}^2 s_{234}^2} [2s_{12}s_{23}s_{24} + s_{13}s_{23}s_{24} + s_{14}s_{23}s_{24} \\
& - s_{13}s_{24}^2 - s_{14}s_{23}^2] \\
& + \frac{1}{s_{34}s_{134}s_{234}} [2s_{12}^2 + s_{12}s_{23} + s_{12}s_{24} + s_{12}s_{13} + s_{12}s_{14}] \\
& + \frac{1}{s_{34}^2 s_{134}s_{234}} [-s_{13}s_{24}^2 - s_{14}s_{23}^2 - s_{13}^2 s_{24} - s_{14}^2 s_{23} \\
& + s_{13}s_{14}s_{23} + s_{13}s_{14}s_{24} + s_{13}s_{23}s_{24} + s_{14}s_{23}s_{24} \\
& - 2s_{12}s_{13}s_{24} - 2s_{12}s_{14}s_{23}] + \frac{2s_{12}}{s_{134}s_{234}} \\
& + m^2 \left( \frac{8s_{13}s_{14}}{s_{34}^2 s_{134}^2} + \frac{8s_{23}s_{24}}{s_{34}^2 s_{234}^2} - \frac{4}{s_{134}^2} - \frac{4}{s_{234}^2} \right. \\
& - \frac{2}{s_{34}s_{134}^2} [s_{12} + s_{23} + s_{24}] - \frac{2}{s_{34}s_{234}^2} [s_{12} + s_{13} + s_{14}] \\
& + \frac{2}{s_{34}s_{134}s_{234}} [4s_{12} - s_{13} - s_{14} - s_{24} - s_{23}] \\
& \left. - \frac{8}{s_{34}^2 s_{134}s_{234}} [s_{14}s_{23} + s_{13}s_{24}] \right) \\
& \left. - m^4 \left( \frac{8}{s_{34} s_{134}^2} + \frac{8}{s_{34} s_{234}^2} \right) \right\} + \mathcal{O}(\varepsilon), \tag{2.8}
\end{aligned}$$

where  $s_{ij} = 2p_i \cdot p_j$  and  $s_{ijk} = s_{ij} + s_{ik} + s_{jk}$ . For the sake of brevity we have not written down in (2.8) the terms of order  $\varepsilon$ . For the numerical computation of (2.7) in  $d = 4$  dimensions, only the four-dimensional antenna function  $B_4^0$  is required. However, the integrated antenna function, which we compute in the next section, must be determined with  $B_4^0$  in  $d$  dimensions.

For completeness, we derive also the behaviour of the antenna function  $B_4^0$  in the double soft limit  $p_3 \rightarrow \lambda p_3$ ,  $p_4 \rightarrow \lambda p_4$  with  $\lambda \rightarrow 0$ . We obtain  $B_4^0 = \lambda^{-4} S_{12} + \mathcal{O}(\lambda^{-3})$  with

$$\begin{aligned}
S_{12}(3_q, 4_{\bar{q}}) = & \frac{2}{s_{34}^2 (s_{13} + s_{14})(s_{23} + s_{24})} (s_{12}s_{34} - s_{13}s_{24} - s_{14}s_{23}) \\
& + \frac{2}{s_{34}^2} \left( \frac{s_{13}s_{14} - s_{34}m^2}{(s_{13} + s_{14})^2} + \frac{s_{23}s_{24} - s_{34}m^2}{(s_{23} + s_{24})^2} \right). \tag{2.9}
\end{aligned}$$

The double-soft factor (2.9) is not identical to the respective factor  $S_{12}^{m=0}$  for a massless quark  $Q$  given in [32], but agrees with this factor in the limit  $m \rightarrow 0$ . A respective difference between the  $m \neq 0$  versus  $m = 0$  case was shown in [49] for  $Q\bar{Q}gg$  final states in the double-soft gluon limit.

### 3 The integrated $Qq\bar{q}\bar{Q}$ antenna function

The introduction of subtraction terms must be counterbalanced by adding their integrated counterparts, cf. (1.2). In the context of the antenna method this implies the analytic integration of the antenna function  $B_4^0(1_Q, 3_q, 4_{\bar{q}}, 2_{\bar{Q}})$  over the four-parton antenna phase space  $d\Phi_{X_{Qq\bar{q}\bar{Q}}}$  associated with a massive and a massless quark-antiquark pair. The corresponding integrated antenna function  $\mathcal{B}_{Qq\bar{q}\bar{Q}}^0$  is defined by

$$\mathcal{B}_{Qq\bar{q}\bar{Q}}^0(q^2, m, \mu, \varepsilon) = (8\pi^2 (4\pi)^{-\varepsilon} e^{\varepsilon\gamma_E})^2 \int d\Phi_{X_{Qq\bar{q}\bar{Q}}} B_4^0(1_Q, 3_q, 4_{\bar{q}}, 2_{\bar{Q}}). \quad (3.1)$$

The antenna phase space  $d\Phi_{X_{Qq\bar{q}\bar{Q}}}$  is closely related to the full four-particle phase space (2.2):

$$d\Phi_4(p_1, p_2, p_3, p_4, q) = P_2(q^2, m) d\Phi_{X_{Qq\bar{q}\bar{Q}}}, \quad (3.2)$$

where  $P_2(q^2, m)$  denotes the integrated phase space for two particles of equal mass  $m$  in  $d = 4 - 2\varepsilon$  space-time dimensions:

$$P_2(q^2, m) = 2^{-3+2\varepsilon} \pi^{-1+\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)} \left(\frac{\mu^2}{q^2}\right)^\varepsilon \left(1 - \frac{4m^2}{q^2}\right)^{\frac{1}{2}-\varepsilon}. \quad (3.3)$$

The integration of the antenna function over the antenna phase space has to be performed analytically in  $d = 4 - 2\varepsilon$  dimensions.

#### 3.1 Reduction to master integrals

In order to calculate the integrated antenna function (3.1), we first reduce the number of integrals to be computed by exploiting linear dependences between phase-space integrals with the help of integration-by-parts (IBP) identities [77]. These identities were originally derived for loop integrals, but are also very useful in computing phase space integrals [78, 79].

Using the Cutkosky rules [80] we introduce two massive and two massless cut-propagators

$$\frac{1}{D_i} = 2\pi i \delta^+(p_i^2 - m^2) = \frac{1}{p_i^2 - m^2 + i0} - \frac{1}{p_i^2 - m^2 - i0}, \quad i = 1, 2, \quad (3.4)$$

$$\frac{1}{D_i} = 2\pi i \delta^+(p_i^2) = \frac{1}{p_i^2 + i0} - \frac{1}{p_i^2 - i0}, \quad i = 3, 4. \quad (3.5)$$

Then we can write:

$$d\phi_4(p_1, p_2, p_3, p_4, q) = \frac{\mu^{12-3d}}{i^4 (2\pi)^{3d}} \delta^{(d)}\left(q - \sum_{i=1}^4 p_i\right) \prod_{i=1}^4 \frac{d^d p_i}{D_i}. \quad (3.6)$$



Next we express each of the invariants  $s_{ij}$ ,  $s_{klm}$  in the  $d$ -dimensional version of (2.8) in terms of the four cut-propagators and five further appropriately chosen propagators and scalar products, such that each term of  $\mathcal{B}_{Qq\bar{q}\bar{Q}}^0$  can be viewed as a four-particle cut through a three-loop vacuum polarization diagram, which may contain irreducible scalar products in the numerator. It was shown in [78] that IBP reduction of such integrals can be carried out in the same way as for loop integrals, using that integrals where at least one of the four cut-propagators is missing in the integrand do not contribute to the original cut-integral. Using the implementation AIR [81] of the Laporta reduction algorithm [82] we decompose the integral (3.1) along these lines. As a result we can express the integrated antenna function in terms of five independent integrals (master integrals):

$$\begin{aligned}
\mathcal{B}_{Qq\bar{q}\bar{Q}}^0(q^2, m, \mu, \epsilon) = & \frac{(8\pi^2 (4\pi)^{-\epsilon} e^{\epsilon\gamma_E})^2}{P_2(q^2, m) |\mathcal{M}_{Q\bar{Q}}|^2} \left\{ \left[ \frac{64 + 208z + 48z^2 - 10z^3 + 5z^4}{z(1-z)^3} \frac{1}{\epsilon^2} \right. \right. \\
& + \frac{2(32 - 448z - 112z^2 + 133z^3 - 49z^4)}{3z(1-z)^3} \frac{1}{\epsilon} \\
& + \left. \frac{2(1720 + 5656z - 164z^2 - 163z^3 + 214z^4)}{9z(1-z)^3} + \mathcal{O}(\epsilon) \right] \\
& \times (q^2)^{-1} T_1(q^2, m^2, \epsilon) \\
& + \left[ - \frac{4(48 + 56z + 6z^2 - 5z^3)}{z(1-z)^3} \frac{1}{\epsilon^2} \right. \\
& + \frac{4(72 + 56z + 234z^2 - 101z^3)}{3z(1-z)^3} \frac{1}{\epsilon} \\
& - \left. \frac{8(1296 + 1222z + 153z^2 - 241z^3)}{9z(1-z)^3} + \mathcal{O}(\epsilon) \right] \\
& \times (q^2)^{-2} T_2(q^2, m^2, \epsilon) \\
& + \left[ - \frac{4(16 + 64z + 26z^2 - z^3)}{z(1-z)^3} \frac{1}{\epsilon^2} \right. \\
& - \frac{4(8 - 232z - 50z^2 + 13z^3)}{3z(1-z)^3} \frac{1}{\epsilon} \\
& - \left. \frac{8(440 + 1730z + 247z^2 + 13z^3)}{9z(1-z)^3} + \mathcal{O}(\epsilon) \right] \\
& \times (q^2)^{-2} T_3(q^2, m^2, \epsilon) \\
& + \left[ - \frac{8(4 - z^2)}{3(1-z)z} \frac{1}{\epsilon} + \frac{4(20 + 12z + 49z^2)}{9(1-z)z} + \mathcal{O}(\epsilon) \right] \\
& \times T_4(q^2, m^2, \epsilon) \\
& + \left[ \frac{2(4 - z^2)}{3(1-z)} \frac{1}{\epsilon} - \frac{4(8 + 9z + z^2)}{9(1-z)} + \mathcal{O}(\epsilon) \right] \\
& \times q^2 T_5(q^2, m^2, \epsilon) \left. \right\}, \tag{3.7}
\end{aligned}$$



$$\begin{aligned}
& \times {}_3F_2\left(-\frac{1}{2} + \epsilon, -2 + 3\epsilon, -3 + 4\epsilon; \epsilon, -1 + 2\epsilon; z\right) \\
& + 2^{-12+8\epsilon} \pi^{-5+3\epsilon} \frac{\Gamma(1-\epsilon)^3 \Gamma(-1+\epsilon)}{\Gamma(3-3\epsilon) \Gamma(2-2\epsilon)} \\
& \quad \times z^{1-\epsilon} {}_3F_2\left(\frac{1}{2}, -1 + 2\epsilon, -2 + 3\epsilon; 2-\epsilon, \epsilon; z\right) \\
& + 2^{-15+10\epsilon} \pi^{-5+3\epsilon} \frac{\Gamma(1-\epsilon) \Gamma(-1+\epsilon)^2}{\Gamma(2-2\epsilon)} \\
& \quad \times z^{2-2\epsilon} {}_3F_2\left(\frac{3}{2} - \epsilon, \epsilon, -1 + 2\epsilon; 3-2\epsilon, 2-\epsilon; z\right) \Big\}, \tag{3.14}
\end{aligned}$$

$$\begin{aligned}
T_2 = (q^2)^3 \left(\frac{\mu^2}{q^2}\right)^{3\epsilon} & \left\{ 2^{-12+6\epsilon} \pi^{-5+3\epsilon} \frac{\Gamma(1-\epsilon)^4}{3\Gamma(3-3\epsilon) \Gamma(4-4\epsilon)} \right. \\
& \times {}_3F_2\left(-\frac{1}{2} + \epsilon, -3 + 3\epsilon, -4 + 4\epsilon; \epsilon, -2 + 2\epsilon; z\right) \\
& + 2^{-13+8\epsilon} \pi^{-5+3\epsilon} \frac{\Gamma(1-\epsilon)^3 \Gamma(-1+\epsilon)}{3\Gamma(3-3\epsilon) \Gamma(2-2\epsilon)} \\
& \quad \times z^{1-\epsilon} {}_3F_2\left(\frac{1}{2}, -2 + 2\epsilon, -3 + 3\epsilon; 2-\epsilon, -1 + \epsilon; z\right) \\
& + 2^{-16+10\epsilon} \pi^{-5+3\epsilon} \frac{\Gamma(2-\epsilon) \Gamma(-2+\epsilon) \Gamma(-1+\epsilon)}{\Gamma(2-2\epsilon)} \\
& \quad \times z^{3-2\epsilon} {}_3F_2\left(\frac{5}{2} - \epsilon, \epsilon, -1 + 2\epsilon; 4-2\epsilon, 3-\epsilon; z\right) \Big\}, \tag{3.15}
\end{aligned}$$

$$\begin{aligned}
T_3 = (q^2)^3 \left(\frac{\mu^2}{q^2}\right)^{3\epsilon} & \left\{ 2^{-12+6\epsilon} \pi^{-5+3\epsilon} \frac{\Gamma(1-\epsilon)^4}{\Gamma(3-3\epsilon) \Gamma(4-4\epsilon)} \right. \\
& \times {}_3F_2\left(-\frac{1}{2} + \epsilon, -2 + 3\epsilon, -4 + 4\epsilon; \epsilon, -1 + 2\epsilon; z\right) \\
& + 2^{-11+8\epsilon} \pi^{-5+3\epsilon} \frac{\Gamma(1-\epsilon)^3 \Gamma(-1+\epsilon)}{3\Gamma(3-3\epsilon) \Gamma(2-2\epsilon)} \\
& \quad \times z^{1-\epsilon} {}_3F_2\left(\frac{1}{2}, -1 + 2\epsilon, -3 + 3\epsilon; 2-\epsilon, \epsilon; z\right) \\
& + 2^{-15+10\epsilon} \pi^{-5+3\epsilon} \frac{\Gamma(1-\epsilon) \Gamma(-1+\epsilon)^2}{\Gamma(2-2\epsilon)} \\
& \quad \times z^{2-2\epsilon} {}_3F_2\left(\frac{3}{2} - \epsilon, \epsilon, -2 + 2\epsilon; 3-2\epsilon, 2-\epsilon; z\right) \Big\}. \tag{3.16}
\end{aligned}$$

We use the package `HypExp2` [85] for expanding the hypergeometric functions in  $\epsilon$  to the required orders. For the above three master integrals the result of these expansions is given in appendix A in terms of harmonic polylogarithms (HPL) [86, 87]. In [49] the phase space volume  $T_1$  was also computed by expansion in  $\epsilon$  in terms of harmonic polylogarithms.

The remaining two master integrals  $T_4, T_5$  are computed using the differential equations method [88–90]. (For a review see [91].) By differentiating  $T_4, T_5$  with respect to  $z = 4m^2/q^2$  we obtain linear combinations of integrals which in turn can be reduced by IBP identities to the above five master integrals under consideration. In this way we derive

inhomogeneous first order differential equations in  $z$  for  $T_4$  and  $T_5$ .

The inhomogeneous part of the differential equation for  $T_4$  only depends on the three master integrals  $T_1, T_2, T_3$  whose series expansions in  $\epsilon$  were already derived above in terms of HPL. We expand this equation for  $T_4$  in  $\epsilon$ :

$$T_4 = T_4^{(0)} + T_4^{(1)}\epsilon + \mathcal{O}(\epsilon^2). \tag{3.17}$$

In this way we obtain first order differential equations for the coefficients  $T_4^{(0)}$  and  $T_4^{(1)}$ , whose inhomogeneous parts are determined in terms of HPL. In each case the general solution is composed of the general solution of the homogeneous equation, which contains a constant of integration, and an integral over the inhomogeneous part. The differential equations for  $T_5$  are obtained and solved in a similar way. Here the inhomogeneous part depends on the other four master integrals.

The integration over the inhomogeneous parts is carried out using the package HPL [92]. After partial fraction decomposition and expansion of shuffle products we can write the respective integrand such that all terms containing HPL are of the form  $k f(z)^j H(\dots, z)$ , where  $k$  is a constant,  $f(z)$  is one of the functions  $\frac{1}{1-z}, \frac{1}{z}, \frac{1}{1+z}$  appearing to a positive integer power  $j$ , and where  $H(\dots, z)$  is a HPL of weight  $w$  and argument  $z$ . In the case of  $j = 1$  the primitive is  $k$  times an HPL of weight  $w + 1$ , which follows directly from the definition of the HPL [86]. In the remaining cases where  $j \neq 1$  we can perform partial integration and partial fractioning in sequence until each term is either of the above form with  $j = 1$  or is just an algebraic function of  $z$ .

In order to fix the constants of integration of the integral  $T_5$ , one can consider the massless limit  $z = 0$  where  $T_5$  remains finite. In this limit the integral was computed in [79]. We use that result as a boundary condition in order to determine the constants of integration for  $T_5$ .

In the case of the differential equations obtained for  $T_4^{(0)}$  and  $T_4^{(1)}$ , the massless limit  $z = 0$  and the threshold limit  $z = 1$  can not be used as boundary conditions for the determination of the integration constants. Instead we choose an appropriate integral which is known to vanish in the limit  $z = 1$  and which can be expressed in terms of the five master integrals by IBP reduction. Via this reduction we use the latter limit as boundary condition for fixing these integration constants.

For  $T_4$  and  $T_5$  the result of their expansion to order  $\epsilon$  in terms of harmonic polylogarithms is also given in appendix A.

For the integrals  $T_4$  and  $T_5$  we performed numerical cross checks using VEGAS [93]. A strong analytical check of all five master integrals is provided by the analysis described in the subsequent section.

In the following we use the variable

$$y \equiv \frac{1 - \sqrt{1-z}}{1 + \sqrt{1-z}}. \tag{3.18}$$

Inserting our results for the master integrals into (3.7) we obtain as our main result the

integrated antenna function expanded in  $\epsilon$ :

$$\begin{aligned}
\mathcal{B}_{Qq\bar{q}\bar{Q}}^0(q^2, y, \mu, \epsilon) = & \left(\frac{\mu^2}{q^2}\right)^{2\epsilon} \left[ \frac{1}{\epsilon^2} \left\{ -\frac{1}{6} + \left( \frac{1}{6} - \frac{1}{6(1-y)} - \frac{1}{6(1+y)} \right) H(0; y) \right\} \right. \\
& + \frac{1}{\epsilon} \left\{ \left( -\frac{8}{9} + \frac{23}{36(1-y)} - \frac{13}{36(1+y)} + \frac{1+2y}{6(1+4y+y^2)} \right) H(0; y) \right. \\
& - \frac{4}{3} H(1; y) + \left( \frac{4}{3} - \frac{4}{3(1-y)} - \frac{4}{3(1+y)} \right) H(2; y) \\
& + \left( \frac{4}{3} - \frac{4}{3(1-y)} - \frac{4}{3(1+y)} \right) H(-1, 0; y) \\
& - \left( \frac{1}{3} - \frac{1}{3(1-y)} - \frac{1}{3(1+y)} \right) H(0, 0; y) \\
& - \left( \frac{2}{3} - \frac{2}{3(1-y)} - \frac{2}{3(1+y)} \right) H(1, 0; y) \\
& \left. - \frac{43}{36} - \frac{2\pi^2}{9} + \frac{2\pi^2}{9(1-y)} + \frac{2\pi^2}{9(1+y)} - \frac{y}{1+4y+y^2} \right\} \\
& - \left( \frac{883-15\pi^2}{108} - \frac{757-15\pi^2}{108(1-y)} - \frac{271-15\pi^2}{108(1+y)} - \frac{2+7y}{3(1+4y+y^2)^2} \right. \\
& \left. + \frac{61+167y}{18(1+4y+y^2)} \right) H(0; y) - \left( \frac{14}{9} + \frac{2}{(1-y)^2} - \frac{95}{36(1-y)} \right. \\
& \left. - \frac{35}{36(1+y)} - \frac{5+22y}{6(1+4y+y^2)} \right) H(0, 0; y) \\
& - \left( \frac{5\pi^2}{3} - \frac{5\pi^2}{3(1-y)} - \frac{5\pi^2}{3(1+y)} \right) H(-1; y) \\
& - \left( \frac{86-5\pi^2}{9} + \frac{5\pi^2}{9(1-y)} + \frac{5\pi^2}{9(1+y)} + \frac{8y}{1+4y+y^2} \right) H(1; y) \\
& - \left( \frac{64}{9} - \frac{46}{9(1-y)} + \frac{26}{9(1+y)} - \frac{4(1+2y)}{3(1+4y+y^2)} \right) H(2; y) \\
& - \left( \frac{8}{3} - \frac{8}{3(1-y)} - \frac{8}{3(1+y)} \right) H(3; y) - \frac{32}{3} H(1, 1; y) \\
& - \left( \frac{20}{3} - \frac{20}{3(1-y)} - \frac{20}{3(1+y)} \right) H(-2, 0; y) \\
& - \left( \frac{4}{9} - \frac{28}{9(1-y)} + \frac{20}{9(1+y)} \right) H(-1, 0; y) \\
& + \left( \frac{32}{3} - \frac{32}{3(1-y)} - \frac{32}{3(1+y)} \right) H(-1, 2; y) \\
& - \left( \frac{58}{9} + \frac{13}{18(1-y)} - \frac{47}{18(1+y)} - \frac{1+2y}{3(1+4y+y^2)} \right) H(1, 0; y) \\
& - \left( \frac{16}{3} - \frac{16}{3(1-y)} - \frac{16}{3(1+y)} \right) H(1, 2; y) \\
& + \left( \frac{22}{3} - \frac{22}{3(1-y)} - \frac{22}{3(1+y)} \right) H(2, 0; y)
\end{aligned}$$

$$\begin{aligned}
& + \left( \frac{32}{3} - \frac{32}{3(1-y)} - \frac{32}{3(1+y)} \right) H(2, 1; y) \\
& + \left( \frac{28}{3} - \frac{28}{3(1-y)} - \frac{28}{3(1+y)} \right) H(-1, -1, 0; y) \\
& - \left( \frac{4}{3} - \frac{4}{3(1-y)} - \frac{4}{3(1+y)} \right) H(-1, 0, 0; y) \\
& - \left( 4 - \frac{4}{1-y} - \frac{4}{1+y} \right) H(-1, 1, 0; y) \\
& + \left( \frac{2}{3} - \frac{2}{3(1-y)} - \frac{2}{3(1+y)} \right) H(0, 0, 0; y) \\
& - \left( 4 - \frac{4}{1-y} - \frac{4}{1+y} \right) H(1, -1, 0; y) \\
& - \left( \frac{2}{3} - \frac{2}{3(1-y)} - \frac{2}{3(1+y)} \right) H(1, 0, 0; y) \\
& - \frac{707 - 113\pi^2 + 468 \zeta(3)}{108} + \frac{79\pi^2 + 468\zeta(3)}{108(1+y)} - \frac{77\pi^2 - 468 \zeta(3)}{108(1-y)} \\
& - \frac{2(1+4y)}{(1+4y+y^2)^2} + \frac{12 - \pi^2 - 36y - 2\pi^2 y}{6(1+4y+y^2)} + \mathcal{O}(\epsilon) \Big]. \tag{3.19}
\end{aligned}$$

#### 4 The correction of $\alpha_s^2 e_Q^2 N_f$ to the ratio $R$

As an application and check of our results of section 3, we consider the ratio

$$R = \frac{\sigma(e^+e^- \rightarrow \gamma^* \rightarrow Q\bar{Q} + X)}{\sigma_{\text{pt}}}, \tag{4.1}$$

to order  $\alpha_s^2$  and to lowest order in  $\alpha$ . Here  $\sigma_{\text{pt}} = e^4/(12\pi q^2)$  is the massless Born cross section for  $e^+e^- \rightarrow \gamma^* \rightarrow \mu^+\mu^-$ . In the following, we consider one heavy quark, carrying the electric charge  $e_Q$ , and  $N_f$  massless quark flavors. Here we are only interested in the contribution proportional to  $\alpha_s^2 e_Q^2 N_f$  to the ratio (4.1). This contribution is gauge-invariant and IR finite. Apart from the tree-level contributions, which are closely related to the integrated antenna function of section 3, this term receives a two-loop contribution which was computed within dimensional regularization first in [68]. Using this result and our result of section 3, we can check the IR poles of  $\mathcal{B}_{Qq\bar{q}\bar{Q}}^0$ . Furthermore we compare this contribution to  $R$  with the previous result of [94], which was obtained in  $d = 4$  using different methods.

Throughout this section we use the subscripts  $\alpha_s^2 e_Q^2 N_f$  or  $\alpha_s^2 N_f$  when referring to the contribution of these terms to a given quantity. We have

$$\sigma(e^+e^- \rightarrow \gamma^* \rightarrow Q\bar{Q} + X)_{\alpha_s^2 e_Q^2 N_f} = \frac{1}{2q^2} \frac{1}{4} \frac{e^4 e_Q^2}{(q^2)^2} L^{\mu\nu} \sum_X H_{\mu\nu, \alpha_s^2 N_f}^{Q\bar{Q}X}, \tag{4.2}$$

with the lepton tensor

$$L_{\mu\nu} = 4(k_{1\mu}k_{2\nu} + k_{1\nu}k_{2\mu} - g_{\mu\nu} k_1 \cdot k_2), \tag{4.3}$$

where  $k_{1\mu}$  and  $k_{2\mu}$  denote the momenta of the incoming electron and positron,  $q_\mu = k_{1\mu} + k_{2\mu}$ , and the contributions

$$H_{\mu\nu, \alpha_s^2 N_f}^X = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi_{\alpha_s^2 N_f}^X(q^2, m, \mu, \epsilon) \quad (4.4)$$

to the hadron tensor. They will be given below. Performing the tensor contractions in  $d = 4 - 2\epsilon$  dimensions, one obtains

$$R_{\alpha_s^2 e_Q^2 N_f} = 6\pi e_Q^2 (1 - \epsilon) \sum_X \Pi_{\alpha_s^2 N_f}^{Q\bar{Q}X}. \quad (4.5)$$

The contribution to (4.5) from the  $Q\bar{Q}q\bar{q}$  final state is closely related to the integrated antenna function  $\mathcal{B}_{Qq\bar{q}\bar{Q}}^0$  given in eq. (3.19). After restoring all couplings and color factors we find

$$\Pi_{\alpha_s^2 N_f}^{Q\bar{Q}q\bar{q}} = \frac{(4\pi\alpha_s)^2 4C_F N_c T_R N_f}{q^2(3 - 2\epsilon)} \frac{P_2(q^2, m) |\mathcal{M}_{Q\bar{Q}}|^2}{(8\pi^2(4\pi)^{-\epsilon} e^{\gamma_E \epsilon})^2} \mathcal{B}_{Qq\bar{q}\bar{Q}}^0(q^2, y, \mu, \epsilon), \quad (4.6)$$

where  $C_F = (N_c^2 - 1)/(2N_c)$  and  $T_R = \frac{1}{2}$ . The expressions for  $|\mathcal{M}_{Q\bar{Q}}|^2$  and  $P_2(q^2, m)$  are given in (2.4) and (3.3), respectively. The second normalization factor in (4.6) is obtained in straightforward fashion; it reflects the relation between the decay rate of a virtual photon and the integrated antenna function, see (2.5), (3.1) and (3.2). In our calculation of the antenna function the hadronic tensor (4.4) was contracted with  $-g_{\mu\nu}$  instead of  $L_{\mu\nu}$ . This is corrected by the additional factor  $1/(q^2(3 - 2\epsilon))$ .

The two-particle contribution  $\Pi^{Q\bar{Q}}$  can be expressed by the Dirac and Pauli heavy quark form factors  $F_1$  and  $F_2$ .

$$\begin{aligned} \Pi_{\alpha_s^2 N_f}^{Q\bar{Q}} = & \frac{N_c P_2(q^2, m)}{3 - 2\epsilon} \left[ 4 \left( 1 + \frac{2y}{(1+y)^2} - \epsilon \right) |F_1|_{\alpha_s^2 N_f}^2 \right. \\ & \left. + \frac{(1+y^2 + y(10 - 8\epsilon))}{2y} |F_2|_{\alpha_s^2 N_f}^2 + 4(3 - 2\epsilon) \text{Re}(F_1 F_2^*)_{\alpha_s^2 N_f} \right]. \end{aligned} \quad (4.7)$$

The contributions required in (4.7) can be read off from the expressions for the UV-renormalized form factors above threshold given in [68].

In [68] the renormalization constants for the heavy quark mass and wave function were defined in the on-shell scheme, whereas the renormalization of the strong coupling constant and the gluon wave-function was performed in the  $\overline{\text{MS}}$  scheme. In order to obtain an LSZ residue equal to one, we apply on-shell renormalization for the external gluon, too. (Nominally, this avoids contributions from three-particle cuts.) This change of the renormalization scheme as compared to [68] leaves the two-loop contributions  $\propto \alpha_s^2 N_f$  unchanged. However, it changes the  $QQg$  renormalization constant  $Z_{1F}(\epsilon, \mu^2/m^2)$  as compared to the one of [68] at the one-loop level by the additional term

$$\delta Z_{1F, \alpha_s N_f} = \frac{\alpha_s N_f T_R (4\pi)^\epsilon}{6\pi\epsilon} \Gamma(1 + \epsilon). \quad (4.8)$$

This change induces a counterterm contribution proportional to  $\alpha_s^2 N_f$  from the three particle  $Q\bar{Q}g$  final state, which reads:

$$\Pi_{\alpha_s^2 N_f}^{Q\bar{Q}g} = \frac{(4\pi\alpha_s)\mu^{2\epsilon} 2C_F N_c}{q^2(3 - 2\epsilon)} P_2(q^2, m) |\mathcal{M}_{Q\bar{Q}}|^2 \left( 2\delta Z_{1F, \alpha_s N_f} \mathcal{A}_{Q\bar{Q}g}^0 \right), \quad (4.9)$$

where  $\mathcal{A}_{Qg\bar{Q}}^0$  is the integrated massive tree-level three parton quark-antiquark antenna as given, e.g., in [66].

Adding the contributions (4.6), (4.7), and (4.9) to (4.5), all IR poles cancel. This provides a strong check for the IR divergent part of the integrated antenna function given in (3.19).

Next we compare our result for  $R_{\alpha_s^2 e_Q^2 N_f}$  with the result of [94], which was obtained in  $d = 4$  using different techniques. Introducing the QCD coupling  $\alpha_s$  and the appropriate color factor in eq. (42) of ref. [94] the relevant part of this equation becomes

$$R_{\alpha_s^2 e_Q^2 N_f} = \left(\frac{\alpha_s}{\pi}\right)^2 e_Q^2 C_F T_R N_c N_f \times \left[ -\frac{1}{3} W \ln\left(\frac{\mu^2}{q^2}\right) + f_R^{(0)} + w(3-w^2)(f_1^{(0)} + f_2^{(0)}) + w^3 f_2^{(0)} \right], \quad (4.10)$$

where

$$w = \frac{1-y}{1+y} = \sqrt{1-z}, \quad (4.11)$$

and the explicit expressions of the functions  $W$ ,  $f_R^{(0)}$ ,  $f_1^{(0)}$ , and  $f_2^{(0)}$  can be found in reference [94]. In [94] the result for  $f_1^{(0)}$  is expressed in terms of the integrals

$$\begin{aligned} T_2(\eta, \xi) &= \int_0^1 dx \frac{\arctan(\xi x)}{x^2 + \eta^2}, \\ T_2^*(\eta, \xi) &= \int_0^1 dx \frac{\ln(x^2 + \xi^2)}{x^2 + \eta^2}, \\ T_3(\eta, \xi, \chi) &= \int_0^1 dx \frac{\ln(x^2 + \xi^2) \arctan(\chi x)}{x^2 + \eta^2}. \end{aligned} \quad (4.12)$$

(We have used here the notation of [94] for these integrals; they should not be confused with the master integrals of section 3.) In order to be able to compare our result with (4.10) in analytic fashion, we have computed the particular combination of these integrals, which appears in the function  $f_1^{(0)}$ , in terms of polylogarithms. We find

$$\begin{aligned} & -T_3(1, 0, w) + T_3\left(1, \frac{1}{w}, w\right) - T_3\left(1, w, \frac{1}{w}\right) + 2 \ln(w) T_2(1, w) - \frac{\pi}{2} T_2^*\left(1, \frac{1}{w}\right) \\ &= \text{Li}_3\left(-\frac{w}{1-w}\right) - \text{Li}_3\left(\frac{w}{1+w}\right) + \ln(w) \left( \text{Li}_2\left(\frac{w}{1+w}\right) - \text{Li}_2\left(-\frac{w}{1-w}\right) \right) \\ & \quad + \frac{1}{6} \ln^3(1+w) - \frac{1}{6} \ln^3(1-w) - \frac{\pi^2}{3} \ln(1+w) - \frac{\pi^2}{6} \ln(1-w) \\ & \quad + \frac{\pi^2}{4} \ln(w) + \frac{1}{2} \ln^2(w) (\ln(1-w) - \ln(1+w)) + \pi G, \end{aligned} \quad (4.13)$$

where  $G$  is Catalan's constant. With this formula we find agreement<sup>1</sup> between our result (4.5) and the result (4.10) of [94].

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<sup>1</sup>In the course of this comparison, we found that eq. (23) of [94] contains a typographical error. In the fifth line of this equation,  $\ln p^2$  must be replaced by  $\ln^2 p$ .



## 5 Summary and outlook

As a first step towards extending the antenna subtraction method to NNLO QCD reactions with massive quarks, we have determined the real radiation antenna function and its integrated counterpart for reactions of the type  $S \rightarrow Q\bar{Q}q\bar{q}$ , where  $S$  denotes an uncolored initial state. We were able to determine the integrated antenna function in completely analytic fashion, namely in terms of harmonic polylogarithms, for which efficient evaluation codes are available. We checked our results by computing the contribution proportional to  $\alpha_s^2 e_Q^2 N_f$  to the inclusive heavy-quark pair production cross section in  $e^+e^-$  annihilation via a virtual photon and by comparison with results in the literature.

An obvious next step in this line of investigation is the determination of the antenna function and its integrated version for  $S \rightarrow Q\bar{Q}gg$ . The results of this paper indicate that for the  $Q\bar{Q}gg$  final state, the integrated antenna function can also be obtained analytically in a relatively compact form.

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## A The master integrals

In this appendix we give analytic results for the five master integrals  $T_1, T_2, T_3, T_4, T_5$  to the required orders in  $\epsilon$ . The integrals are given in terms of the variable

$$y \equiv \frac{1 - \sqrt{1 - \frac{4m^2}{q^2}}}{1 + \sqrt{1 - \frac{4m^2}{q^2}}}. \quad (\text{A.1})$$

The harmonic polylogarithms are given in the notation of [86]. The  $\epsilon$ -expansion is needed to order  $\epsilon^2$  for  $T_1, T_2$  and  $T_3$  and to order  $\epsilon$  for  $T_4$  and  $T_5$ . For convenience, we define

$$T_1(q^2, y, \mu^2, \epsilon) = C^3(\epsilon) \left(\frac{\mu^2}{q^2}\right)^{3\epsilon} \frac{(q^2)^2}{2^{11}\pi^5(1+y)^6} \mathcal{T}_1(y, \epsilon), \quad (\text{A.2})$$

with

$$C(\epsilon) = \left(\frac{4\pi}{e^{\gamma_E}}\right)^\epsilon. \quad (\text{A.3})$$

We find

$$\mathcal{T}_1(y, \epsilon) = \frac{1}{12}(1+y) (-1 - 23y + (-34 + 4\pi^2)y^2 + (34 + 4\pi^2)y^3 + 23y^4 + y^5)$$

$$\begin{aligned}
& +y(1+5y+6y^2+5y^3+y^4)H(0;y)-4y^2(1+y)^2H(-1,0;y) \\
& +2y^2(1+y)^2H(0,0;y)\} \\
& +\epsilon\left\{4y^2(1+y)^2H(-3;y)+2y(1+5y+6y^2+5y^3+y^4)H(-2;y)\right. \\
& -\frac{1}{6}(1+y)(-1-23y-2(17+8\pi^2)y^2+(34-16\pi^2)y^3+23y^4+y^5)H(-1;y) \\
& -\frac{1}{6}y(-45+(-183+8\pi^2)y+2(-35+8\pi^2)y^2+(45+8\pi^2)y^3 \\
& +51y^4+4y^5)H(0;y)-\frac{5}{6}(-1-24y-57y^2+57y^4+24y^5+y^6)H(1;y) \\
& +10y(1+5y+6y^2+5y^3+y^4)H(2;y)+20y^2(1+y)^2H(3;y) \\
& +28y^2(1+y)^2H(-2,0;y)-8y^2(1+y)^2H(-1,-2;y) \\
& +2y(5+17y+16y^2+17y^3+5y^4)H(-1,0;y)-40y^2(1+y)^2H(-1,2;y) \\
& -y(1+y)^2(1-5y+y^2)H(0,0;y)-56y^2(1+y)^2H(-1,-1,0;y) \\
& +12y^2(1+y)^2H(-1,0,0;y)-6y^2(1+y)^2H(0,0,0;y) \\
& +\frac{1}{72}(71-24(-71+3\pi^2)y-24(71+3\pi^2)y^5-71y^6 \\
& +y^2(4047-456\pi^2-720\zeta(3))-120y^3(5\pi^2+12\zeta(3)) \\
& \left.-3y^4(1349+152\pi^2+240\zeta(3)))\right\} \\
& +\epsilon^2\left\{-\frac{7}{432}(-445+9\pi^2)+\frac{5}{432}(-623+75\pi^2)y^6+(-12y^2-24y^3\right. \\
& -12y^4)H(-4;y)+(-2y+6y^2+16y^3+6y^4-2y^5)H(-3;y) \\
& +\left(15y-\frac{1}{3}(-183+80\pi^2)y^2-\frac{10}{3}(-7+16\pi^2)y^3-\frac{5}{3}(9+16\pi^2)y^4\right. \\
& \left.-17y^5-\frac{4y^6}{3}\right)H(-2;y) \\
& +\left(\frac{355}{36}+\frac{710y}{3}+\frac{6745y^2}{12}-\frac{6745y^4}{12}-\frac{710y^5}{3}-\frac{355y^6}{36}\right)H(1;y) \\
& +\left(75y+305y^2+\frac{350y^3}{3}-75y^4-85y^5-\frac{20y^6}{3}\right)H(2;y) \\
& +(-10y+30y^2+80y^3+30y^4-10y^5)H(3;y) \\
& +(-60y^2-120y^3-60y^4)H(4;y) \\
& \left.+(8y^2+16y^3+8y^4)H(-3,-1;y)\right\}
\end{aligned}$$

$$\begin{aligned}
&+(-132y^2-264y^3-132y^4) H(-3, 0; y) \\
&+(40y^2+80y^3+40y^4) H(-3, 1; y) \\
&+(56y^2+112y^3+56y^4) H(-2, -2; y) \\
&+(4y+20y^2+24y^3+20y^4+4y^5) H(-2, -1; y) \\
&+(-38y-78y^2-32y^3-78y^4-38y^5) H(-2, 0; y) \\
&+(20y+100y^2+120y^3+100y^4+20y^5) H(-2, 1; y) \\
&+(280y^2+560y^3+280y^4) H(-2, 2; y) \\
&+(24y^2+48y^3+24y^4) H(-1, -3; y) \\
&+(20y+68y^2+64y^3+68y^4+20y^5) H(-1, -2; y) \\
&+\left(\frac{1}{3}+8y+\frac{1}{3}(57+160\pi^2)y^2+\frac{320\pi^2y^3}{3}+\frac{1}{3}(-57+160\pi^2)y^4\right. \\
&\quad \left.-8y^5-\frac{y^6}{3}\right) H(-1, -1; y) \\
&+\left(\frac{7}{3}+19y-\frac{1}{3}(-408+7\pi^2)y^2-\frac{2}{3}(-136+7\pi^2)y^3-\frac{1}{3}(-180+7\pi^2)y^4\right. \\
&\quad \left.-13y^5-\frac{11y^6}{3}\right) H(-1, 0; y) \\
&+\left(\frac{5}{3}+40y+95y^2-95y^4-40y^5-\frac{5y^6}{3}\right) H(-1, 1; y) \\
&+(100y+340y^2+320y^3+340y^4+100y^5) H(-1, 2; y) \\
&+(120y^2+240y^3+120y^4) H(-1, 3; y) \\
&+\left(-\frac{15y}{2}+\frac{1}{6}(-75+7\pi^2)y^2+\frac{1}{3}(4+7\pi^2)y^3+\frac{1}{6}(39+7\pi^2)y^4\right. \\
&\quad \left.+\frac{y^5}{2}+\frac{y^6}{3}\right) H(0, 0; y) \\
&+\left(\frac{5}{3}+40y+95y^2-95y^4-40y^5-\frac{5y^6}{3}\right) H(1, -1; y) \\
&+\left(\frac{10}{3}+80y+190y^2-190y^4-80y^5-\frac{10y^6}{3}\right) H(1, 0; y) \\
&+\left(\frac{25}{3}+200y+475y^2-475y^4-200y^5-\frac{25y^6}{3}\right) H(1, 1; y) \\
&+(20y+100y^2+120y^3+100y^4+20y^5) H(2, -1; y) \\
&+(40y+200y^2+240y^3+200y^4+40y^5) H(2, 0; y)
\end{aligned}$$

$$\begin{aligned}
& + (100y + 500y^2 + 600y^3 + 500y^4 + 100y^5) H(2, 1; y) \\
& + (40y^2 + 80y^3 + 40y^4) H(3, -1; y) \\
& + (80y^2 + 160y^3 + 80y^4) H(3, 0; y) + (200y^2 + 400y^3 + 200y^4) H(3, 1; y) \\
& + (296y^2 + 592y^3 + 296y^4) H(-2, -1, 0; y) \\
& + (-36y^2 - 72y^3 - 36y^4) H(-2, 0, 0; y) \\
& + (-16y^2 - 32y^3 - 16y^4) H(-1, -2, -1; y) \\
& + (264y^2 + 528y^3 + 264y^4) H(-1, -2, 0; y) \\
& + (-80y^2 - 160y^3 - 80y^4) H(-1, -2, 1; y) \\
& + (-112y^2 - 224y^3 - 112y^4) H(-1, -1, -2; y) \\
& + (92y + 236y^2 + 160y^3 + 236y^4 + 92y^5) H(-1, -1, 0; y) \\
& + (-560y^2 - 1120y^3 - 560y^4) H(-1, -1, 2; y) \\
& + (-6y + 18y^2 + 48y^3 + 18y^4 - 6y^5) H(-1, 0, 0; y) \\
& + (-80y^2 - 160y^3 - 80y^4) H(-1, 2, -1; y) \\
& + (-160y^2 - 320y^3 - 160y^4) H(-1, 2, 0; y) \\
& + (-400y^2 - 800y^3 - 400y^4) H(-1, 2, 1; y) \\
& + (y - 19y^2 - 36y^3 - 19y^4 + y^5) H(0, 0, 0; y) \\
& + (-592y^2 - 1184y^3 - 592y^4) H(-1, -1, -1, 0; y) \\
& + (72y^2 + 144y^3 + 72y^4) H(-1, -1, 0, 0; y) \\
& + (-28y^2 - 56y^3 - 28y^4) H(-1, 0, 0, 0; y) \\
& + (14y^2 + 28y^3 + 14y^4) H(0, 0, 0, 0; y) \\
& + \left( \frac{71}{36} - \frac{2}{3} (-71 + 16\pi^2) y - \frac{2}{3} (71 + 16\pi^2) y^5 - \frac{71y^6}{36} \right. \\
& - \frac{1}{12} y^2 (-1349 + 512\pi^2 - 384 \zeta(3)) \\
& \left. - \frac{1}{12} y^4 (1349 + 512\pi^2 - 384 \zeta(3)) - \frac{8}{3} y^3 (17\pi^2 - 24 \zeta(3)) \right) H(-1; y) \\
& + \left( -\frac{3}{4} (-47 + \pi^2) y - \frac{1}{12} (1849 + 9\pi^2) y^5 - \frac{71y^6}{9} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{12}y^2(-1121+109\pi^2+192\zeta(3)) \\
& -\frac{1}{12}y^4(4275+109\pi^2+192\zeta(3))-\frac{1}{18}y^3(3293+249\pi^2+576\zeta(3)) \Big) H(0; y) \\
& +\frac{1}{144}y^2(59185-6021\pi^2+76\pi^4-18720\zeta(3)) \\
& +\frac{1}{144}y^4(-59185+2301\pi^2+76\pi^4-18720\zeta(3)) \\
& +\frac{1}{18}y^3(-249\pi^2+19\pi^4-3204\zeta(3)) \\
& +\frac{1}{18}y^5(-3115+240\pi^2-324\zeta(3)) \\
& -\frac{1}{18}y(-3115+198\pi^2+324\zeta(3)) \Big\}. \tag{A.4}
\end{aligned}$$

$$T_2(q^2, y, \mu, \epsilon) = C^3(\epsilon) \left(\frac{\mu^2}{q^2}\right)^{3\epsilon} \frac{(q^2)^3}{2^{11}\pi^5(1+y)^8} T_2(y, \epsilon), \tag{A.5}$$

with

$$\begin{aligned}
T_2(y, \epsilon) = & -\frac{1}{72}(1+y)(-1-23y+67y^2+(149-24\pi^2)y^3-(149+24\pi^2)y^4-67y^5 \\
& +23y^6+y^7)+\frac{1}{6}(y-13y^3-14y^4-13y^5+y^7) H(0; y) \\
& +4y^3(1+y)^2 H(-1, 0; y)-2y^3(1+y)^2 H(0, 0; y) \\
& +\epsilon \left\{ -4y^3(1+y)^2 H(-3; y)+\frac{1}{3}(y-13y^3-14y^4-13y^5+y^7) H(-2; y) \right. \\
& -\frac{1}{36}(1+y)(-1-23y+67y^2+(149+96\pi^2)y^3+(-149+96\pi^2)y^4-67y^5 \\
& +23y^6+y^7) H(-1; y) \\
& +\frac{1}{36}y(45-30y+(-545+48\pi^2)y^2+(-6+96\pi^2)y^3+(319+48\pi^2)y^4 \\
& +146y^5-51y^6-4y^7) H(0; y) \\
& -\frac{5}{36}(-1-24y+44y^2+216y^3-216y^5-44y^6+24y^7+y^8) H(1; y) \\
& +\frac{5}{3}(y-13y^3-14y^4-13y^5+y^7) H(2; y)-20y^3(1+y)^2 H(3; y) \\
& -28y^3(1+y)^2 H(-2, 0; y)+8y^3(1+y)^2 H(-1, -2; y) \\
& +\frac{1}{3}y(5-23y^2+4y^3-23y^4+5y^6) H(-1, 0; y)+40y^3(1+y)^2 H(-1, 2; y) \\
& -\frac{1}{6}y(1+y)^2(1-2y+32y^2-2y^3+y^4) H(0, 0; y)+56y^3(1+y)^2 H(-1, -1, 0; y) \\
& -12y^3(1+y)^2 H(-1, 0, 0; y)+6y^3(1+y)^2 H(0, 0, 0; y) \\
& \left. +\frac{1}{432}(71-24(-71+3\pi^2)y-2704y^2+2704y^6-24(71+3\pi^2)y^7-71y^8) \right\}
\end{aligned}$$

$$\begin{aligned}
 & +24y^3 (-601+60\pi^2+180\zeta(3)) +24y^5 (601+60\pi^2+180\zeta(3)) \\
 & +24y^4 (79\pi^2+360\zeta(3)) \} \\
 & +\epsilon^2 \left\{ 12y^3(1+y)^2 H(-4; y) - \frac{1}{3}y(1+y)^2 (1-2y+32y^2-2y^3+y^4) H(-3; y) \right. \\
 & + \frac{1}{18}y (45-30y+5(-109+96\pi^2))y^2 + (-6+960\pi^2)y^3 + (319+480\pi^2)y^4 \\
 & +146y^5-51y^6-4y^7) H(-2; y) \\
 & - \frac{5}{216}(-71-1704y+2704y^2 \\
 & +14424y^3-14424y^5-2704y^6+1704y^7+71y^8) H(1; y) \\
 & - \frac{5}{18}y (-45+30y+545y^2+6y^3-319y^4-146y^5+51y^6+4y^7) H(2; y) \\
 & - \frac{5}{3}(y+29y^3+60y^4+29y^5+y^7) H(3; y) +60y^3(1+y)^2 H(4; y) \\
 & -8y^3(1+y)^2 H(-3, -1; y) +132y^3(1+y)^2 H(-3, 0; y) \\
 & -40y^3(1+y)^2 H(-3, 1; y) -56y^3(1+y)^2 H(-2, -2; y) \\
 & + \frac{2}{3}(y-13y^3-14y^4-13y^5+y^7) H(-2, -1; y) \\
 & - \frac{1}{3}y (19+47y^2+252y^3+47y^4+19y^6) H(-2, 0; y) \\
 & + \frac{10}{3}(y-13y^3-14y^4-13y^5+y^7) H(-2, 1; y) \\
 & -280y^3(1+y)^2 H(-2, 2; y) -24y^3(1+y)^2 H(-1, -3; y) \\
 & + \frac{2}{3}y (5-23y^2+4y^3-23y^4+5y^6) H(-1, -2; y) \\
 & - \frac{1}{18}(1+y) (-1-23y+67y^2+(149+960\pi^2)y^3+(-149+960\pi^2)y^4 \\
 & -67y^5+23y^6+y^7) H(-1, -1; y) \\
 & + \frac{1}{18}(-7+57y+182y^2+7(-85+6\pi^2)y^3+6(87+14\pi^2)y^4 \\
 & +(269+42\pi^2)y^5+358y^6-39y^7-11y^8) H(-1, 0; y) \\
 & - \frac{5}{18}(-1-24y+44y^2+216y^3-216y^5-44y^6+24y^7+y^8) H(-1, 1; y) \\
 & + \frac{10}{3}y (5-23y^2+4y^3-23y^4+5y^6) H(-1, 2; y) -120y^3(1+y)^2 H(-1, 3; y) \\
 & + \frac{1}{36}y(1+y) (-45+51y-2(62+21\pi^2)y^2-2(211+21\pi^2)y^3 \\
 & -83y^4+y^5+2y^6) H(0, 0; y)
 \end{aligned}$$

$$\begin{aligned}
& -\frac{5}{18}(-1-24y+44y^2+216y^3-216y^5-44y^6+24y^7+y^8) H(1,-1;y) \\
& -\frac{5}{9}(-1-24y+44y^2+216y^3-216y^5-44y^6+24y^7+y^8) H(1,0;y) \\
& -\frac{25}{18}(-1-24y+44y^2+216y^3-216y^5-44y^6+24y^7+y^8) H(1,1;y) \\
& +\frac{10}{3}(y-13y^3-14y^4-13y^5+y^7) H(2,-1;y) \\
& +\frac{20}{3}(y-13y^3-14y^4-13y^5+y^7) H(2,0;y) \\
& +\frac{50}{3}(y-13y^3-14y^4-13y^5+y^7) H(2,1;y)-40y^3(1+y)^2 H(3,-1;y) \\
& -80y^3(1+y)^2 H(3,0;y)-200y^3(1+y)^2 H(3,1;y) \\
& -296y^3(1+y)^2 H(-2,-1,0;y)+36y^3(1+y)^2 H(-2,0,0;y) \\
& +16y^3(1+y)^2 H(-1,-2,-1;y)-264y^3(1+y)^2 H(-1,-2,0;y) \\
& +80y^3(1+y)^2 H(-1,-2,1;y)+112y^3(1+y)^2 H(-1,-1,-2;y) \\
& +\frac{2}{3}y(23-5y^2+196y^3-5y^4+23y^6) H(-1,-1,0;y)+560y^3(1+y)^2 H(-1,-1,2;y) \\
& -y(1+y)^2(1-2y+32y^2-2y^3+y^4) H(-1,0,0;y)+80y^3(1+y)^2 H(-1,2,-1;y) \\
& +160y^3(1+y)^2 H(-1,2,0;y) \\
& +400y^3(1+y)^2 H(-1,2,1;y)+\frac{1}{6}(y+113y^3+208y^4+113y^5+y^7) H(0,0,0;y) \\
& +592y^3(1+y)^2 H(-1,-1,-1,0;y)-72y^3(1+y)^2 H(-1,-1,0,0;y) \\
& +28y^3(1+y)^2 H(-1,0,0,0;y)-14y^3(1+y)^2 H(0,0,0,0;y) \\
& +\frac{1}{216}(71-24(-71+16\pi^2)y-2704y^2+2704y^6-24(71+16\pi^2)y^7-71y^8 \\
& +24y^3(-601+124\pi^2-288\zeta(3))+24y^5(601+124\pi^2-288\zeta(3)) \\
& +96y^4(19\pi^2-144\zeta(3))) H(-1;y)+\frac{1}{216}y(1269-2550y+8266y^5 \\
& -5547y^6-284y^7-3\pi^2(9-453y^2-718y^3-453y^4+9y^6) \\
& +6y^3(3535+1152\zeta(3))+y^2(-15937+3456\zeta(3)) \\
& +y^4(41759+3456\zeta(3))) H(0;y) \\
& +\frac{1}{2592}(3115-105152y^2+105152y^6-3115y^8-1368\pi^4y^3(1+y)^2 \\
& +3\pi^2(-21-1584y+1740y^2+20088y^3+2352y^4-11448y^5-4684y^6)
\end{aligned}$$

$$\begin{aligned}
 &+1920y^7+125y^8)+y(74760-7776\zeta(3))+268704y^4\zeta(3) \\
 &-24y^7(3115+324\zeta(3))+24y^3(-25145+7992\zeta(3)) \\
 &+24y^5(25145+7992\zeta(3)) \left. \vphantom{+1920y^7} \right\}. \tag{A.6}
 \end{aligned}$$

$$T_3(q^2, y, \mu, \epsilon) = C^3(\epsilon) \left( \frac{\mu^2}{q^2} \right)^{3\epsilon} \frac{(q^2)^3}{2^{11}\pi^5(1+y)^8} \bar{T}_3(y, \epsilon), \tag{A.7}$$

with

$$\begin{aligned}
 \bar{T}_3(y, \epsilon) = &-\frac{1}{72}(1+y)(-3-109y+(-443+24\pi^2)y^2+(-277+72\pi^2)y^3 \\
 &+(277+72\pi^2)y^4+(443+24\pi^2)y^5+109y^6+3y^7) \\
 &+\frac{1}{3}y(2+18y+42y^2+57y^3+42y^4+18y^5+2y^6)H(0; y) \\
 &-4y^2(1+y)^4H(-1, 0; y)+2y^2(1+y)^4H(0, 0; y) \\
 &+\epsilon \left\{ 4y^2(1+y)^4H(-3; y) \right. \\
 &+\frac{2}{3}y(2+18y+42y^2+57y^3+42y^4+18y^5+2y^6)H(-2; y) \\
 &+\frac{1}{36}(3+112y+24(23+4\pi^2)y^2+48(15+8\pi^2)y^3+576\pi^2y^4 \\
 &+48(-15+8\pi^2)y^5+24(-23+4\pi^2)y^6-112y^7-3y^8)H(-1; y) \\
 &-\frac{1}{18}y(-90+3(-235+8\pi^2)y+6(-189+16\pi^2)y^2+(-653+144\pi^2)y^3 \\
 &+6(51+16\pi^2)y^4+3(133+8\pi^2)y^5+134y^6+6y^7)H(0; y) \\
 &-\frac{5}{36}(-3-112y-552y^2-720y^3+720y^5+552y^6+112y^7+3y^8)H(1; y) \\
 &+\frac{10}{3}y(2+18y+42y^2+57y^3+42y^4+18y^5+2y^6)H(2; y) \\
 &+20y^2(1+y)^4H(3; y)+28y^2(1+y)^4H(-2, 0; y) \\
 &-8y^2(1+y)^4H(-1, -2; y) \\
 &+\frac{4}{3}y(5+33y+63y^2+80y^3+63y^4+33y^5+5y^6)H(-1, 0; y) \\
 &-40y^2(1+y)^4H(-1, 2; y)-\frac{2}{3}y(1+y)^4(1-7y+y^2)H(0, 0; y) \\
 &-56y^2(1+y)^4H(-1, -1, 0; y)+12y^2(1+y)^4H(-1, 0, 0; y) \\
 &-6y^2(1+y)^4H(0, 0, 0; y) \\
 &+\frac{1}{432}(213-8(-985+36\pi^2)y-8(985+36\pi^2)y^7-213y^8
 \end{aligned}$$



$$\begin{aligned}
& -36y^2 (-1067+88\pi^2+120\zeta(3)) - 36y^6 (1067+88\pi^2+120\zeta(3)) \\
& - 24y^3 (-2077+336\pi^2+720\zeta(3)) - 24y^5 (2077+336\pi^2+720\zeta(3)) \\
& - 24y^4 (467\pi^2+1080\zeta(3)) \} \\
& + \epsilon^2 \left\{ -12y^2(1+y)^4 H(-4; y) - \frac{4}{3}y(1+y)^4 (1-7y+y^2) H(-3; y) \right. \\
& - \frac{1}{9}y (-90+15(-47+16\pi^2)y+6(-189+160\pi^2)y^2+(-653+1440\pi^2)y^3 \\
& + 6(51+160\pi^2)y^4+3(133+80\pi^2)y^5+134y^6+6y^7) H(-2; y) \\
& - \frac{5}{216} (-213-7880y-38412y^2-49848y^3+49848y^5+38412y^6 \\
& + 7880y^7+213y^8) H(1; y) \\
& - \frac{5}{9}y (-90-705y-1134y^2-653y^3+306y^4+399y^5+134y^6+6y^7) H(2; y) \\
& - \frac{20}{3}y(1+y)^4 (1-7y+y^2) H(3; y) - 60y^2(1+y)^4 H(4; y) \\
& + 8y^2(1+y)^4 H(-3, -1; y) - 132y^2(1+y)^4 H(-3, 0; y) \\
& + 40y^2(1+y)^4 H(-3, 1; y) + 56y^2(1+y)^4 H(-2, -2; y) \\
& + \frac{4}{3}y (2+18y+42y^2+57y^3+42y^4+18y^5+2y^6) H(-2, -1; y) \\
& - \frac{4}{3}y (19+87y+105y^2+104y^3+105y^4+87y^5+19y^6) H(-2, 0; y) \\
& + \frac{20}{3}y (2+18y+42y^2+57y^3+42y^4+18y^5+2y^6) H(-2, 1; y) \\
& + 280y^2(1+y)^4 H(-2, 2; y) + 24y^2(1+y)^4 H(-1, -3; y) \\
& + \frac{8}{3}y (5+33y+63y^2+80y^3+63y^4+33y^5+5y^6) H(-1, -2; y) \\
& - \frac{1}{18}(1+y) (-3-109y-(443+960\pi^2)y^2-(277+2880\pi^2)y^3 \\
& + (277-2880\pi^2)y^4+(443-960\pi^2)y^5+109y^6+3y^7) H(-1, -1; y) \\
& + \frac{1}{18} (-21+116y-6(-415+7\pi^2)y^2+(4740-168\pi^2)y^3 \\
& + (5018-252\pi^2)y^4+(1860-168\pi^2)y^5-6(-47+7\pi^2)y^6 \\
& - 332y^7-33y^8) H(-1, 0; y) \\
& - \frac{5}{18} (-3-112y-552y^2-720y^3+720y^5+552y^6+112y^7+3y^8) H(-1, 1; y) \\
& + \frac{40}{3}y (5+33y+63y^2+80y^3+63y^4+33y^5+5y^6) H(-1, 2; y)
\end{aligned}$$

$$\begin{aligned}
& +120y^2(1+y)^4 H(-1, 3; y) \\
& + \frac{1}{18}y(1+y) (-90+3(-89+7\pi^2)y+(-87+63\pi^2)y^2+(190+63\pi^2)y^3 \\
& + (176+21\pi^2)y^4+19y^5+3y^6) H(0, 0; y) \\
& - \frac{5}{18}(-3-112y-552y^2-720y^3+720y^5+552y^6+112y^7+3y^8) H(1, -1; y) \\
& - \frac{5}{9}(-3-112y-552y^2-720y^3+720y^5+552y^6+112y^7+3y^8) H(1, 0; y) \\
& - \frac{25}{18}(-3-112y-552y^2-720y^3+720y^5+552y^6+112y^7+3y^8) H(1, 1; y) \\
& + \frac{20}{3}y(2+18y+42y^2+57y^3+42y^4+18y^5+2y^6) H(2, -1; y) \\
& + \frac{40}{3}y(2+18y+42y^2+57y^3+42y^4+18y^5+2y^6) H(2, 0; y) \\
& + \frac{100}{3}y(2+18y+42y^2+57y^3+42y^4+18y^5+2y^6) H(2, 1; y) \\
& + 40y^2(1+y)^4 H(3, -1; y) + 80y^2(1+y)^4 H(3, 0; y) \\
& + 200y^2(1+y)^4 H(3, 1; y) + 296y^2(1+y)^4 H(-2, -1, 0; y) \\
& - 36y^2(1+y)^4 H(-2, 0, 0; y) - 16y^2(1+y)^4 H(-1, -2, -1; y) \\
& + 264y^2(1+y)^4 H(-1, -2, 0; y) - 80y^2(1+y)^4 H(-1, -2, 1; y) \\
& - 112y^2(1+y)^4 H(-1, -1, -2; y) \\
& + \frac{8}{3}y(23+123y+189y^2+218y^3+189y^4+123y^5+23y^6) H(-1, -1, 0; y) \\
& - 560y^2(1+y)^4 H(-1, -1, 2; y) \\
& - 4y(1+y)^4(1-7y+y^2) H(-1, 0, 0; y) \\
& - 80y^2(1+y)^4 H(-1, 2, -1; y) - 160y^2(1+y)^4 H(-1, 2, 0; y) \\
& - 400y^2(1+y)^4 H(-1, 2, 1; y) \\
& + \frac{2}{3}y(1-27y-105y^2-159y^3-105y^4-27y^5+y^6) H(0, 0, 0; y) \\
& - 592y^2(1+y)^4 H(-1, -1, -1, 0; y) + 72y^2(1+y)^4 H(-1, -1, 0, 0; y) \\
& - 28y^2(1+y)^4 H(-1, 0, 0, 0; y) + 14y^2(1+y)^4 H(0, 0, 0, 0; y) \\
& - \frac{1}{108}y(-2556+13204y^6+426y^7+3\pi^2(18+354y+1050y^2+1513y^3 \\
& +1050y^4+354y^5+18y^6)+27y^5(2257+64\zeta(3))+96y^4(1021+72\zeta(3)))
\end{aligned}$$

$$\begin{aligned}
& +48y^2(-35+144\zeta(3))+9y(-1765+192\zeta(3))+y^3(58955+10368\zeta(3)) H(0; y) \\
& +\frac{1}{216}(213-8(-985+192\pi^2)y-8(985+192\pi^2)y^7-213y^8 \\
& -24y^5(2077+1008\pi^2-1152\zeta(3))-96y^4(331\pi^2-432\zeta(3)) \\
& -36y^2(-1067+320\pi^2-192\zeta(3))-36y^6(1067+320\pi^2-192\zeta(3)) \\
& +y^3(49848-24192\pi^2+27648\zeta(3))) H(-1; y) \\
& +\frac{1}{2592}(9345-9345y^8+1368\pi^4y^2(1+y)^4+3\pi^2(-63-6672y \\
& -48216y^2-75792y^3-37392y^4+29328y^5+32376y^6+9680y^7+375y^8) \\
& -1426464y^4\zeta(3)-324y^2(-5123+1184\zeta(3))-324y^6(5123+1184\zeta(3)) \\
& -8y(-42917+3888\zeta(3))-8y^7(42917+3888\zeta(3)) \\
& -24y^3(-89405+42336\zeta(3))-24y^5(89405+42336\zeta(3))) \}. \tag{A.8}
\end{aligned}$$

$$T_4(q^2, y, \mu, \epsilon) = C^3(\epsilon) \left( \frac{\mu^2}{q^2} \right)^{3\epsilon} \frac{q^2}{2^{11}\pi^5} T_4(y, \epsilon), \tag{A.9}$$

with

$$\begin{aligned}
T_4(y, \epsilon) = & \left[ \frac{3}{(1+y)^4} - \frac{6}{(1+y)^3} + \frac{4}{(1+y)^2} - \frac{1}{1+y} \right] H(0; y) \\
& + \left[ \frac{4}{(1+y)^3} - \frac{7}{(1+y)^2} + \frac{3}{1+y} \right] H(0, 0; y) \\
& + \left[ \frac{2}{(1+y)^2} - \frac{2}{1+y} \right] H(-1, 0; y) \\
& - \frac{1}{4} + \frac{3}{(1+y)^3} + \frac{-27+\pi^2}{6(1+y)^2} + \frac{12-\pi^2}{6(1+y)} \\
& + \epsilon \left\{ \left[ \frac{8}{(1+y)^3} - \frac{14}{(1+y)^2} + \frac{6}{1+y} \right] H(-3; y) \right. \\
& + \left[ \frac{6}{(1+y)^4} - \frac{12}{(1+y)^3} + \frac{8}{(1+y)^2} - \frac{2}{1+y} \right] H(-2; y) \\
& + \left[ \frac{1}{2} + \frac{6}{(1+y)^3} + \frac{-27-4\pi^2}{3(1+y)^2} + \frac{4(3+\pi^2)}{3(1+y)} \right] H(-1; y) \\
& + \left[ -2 + \frac{39}{2(1+y)^4} + \frac{-27-4\pi^2}{(1+y)^3} + \frac{9+20\pi^2}{3(1+y)^2} + \frac{39-16\pi^2}{6(1+y)} \right] H(0; y) \\
& + \left[ \frac{5}{2} + \frac{30}{(1+y)^3} - \frac{45}{(1+y)^2} + \frac{20}{1+y} \right] H(1; y) \\
& + \left[ \frac{30}{(1+y)^4} - \frac{60}{(1+y)^3} + \frac{40}{(1+y)^2} - \frac{10}{1+y} \right] H(2; y)
\end{aligned}$$

$$\begin{aligned}
 & + \left[ \frac{40}{(1+y)^3} - \frac{70}{(1+y)^2} + \frac{30}{1+y} \right] H(3; y) \\
 & + \left[ \frac{40}{(1+y)^3} - \frac{74}{(1+y)^2} + \frac{34}{1+y} \right] H(-2, 0; y) \\
 & + \left[ \frac{4}{(1+y)^2} - \frac{4}{1+y} \right] H(-1, -2; y) \\
 & + \left[ \frac{24}{(1+y)^4} - \frac{48}{(1+y)^3} + \frac{43}{(1+y)^2} - \frac{19}{1+y} \right] H(-1, 0; y) \\
 & + \left[ \frac{20}{(1+y)^2} - \frac{20}{1+y} \right] H(-1, 2; y) \\
 & + \left[ \frac{18}{(1+y)^3} - \frac{65}{2(1+y)^2} + \frac{29}{2(1+y)} \right] H(0, 0; y) \\
 & + \left[ \frac{28}{(1+y)^2} - \frac{28}{1+y} \right] H(-1, -1, 0; y) \\
 & + \left[ \frac{24}{(1+y)^3} - \frac{42}{(1+y)^2} + \frac{18}{1+y} \right] H(-1, 0, 0; y) \\
 & + \left[ -\frac{12}{(1+y)^3} + \frac{21}{(1+y)^2} - \frac{9}{1+y} \right] H(0, 0, 0; y) \\
 & + \left[ \frac{8}{(1+y)^3} - \frac{12}{(1+y)^2} + \frac{4}{1+y} \right] H(1, 0, 0; y) \\
 & - \frac{25}{8} - \frac{7\pi^2}{2(1+y)^4} + \frac{100 + \pi^2 - 44\zeta(3)}{4(1+y)} \\
 & + \left. \frac{75 + 14\pi^2 - 24\zeta(3)}{2(1+y)^3} + \frac{-225 - 15\pi^2 + 92\zeta(3)}{4(1+y)^2} \right\}. \tag{A.10}
 \end{aligned}$$

$$T_5(q^2, y, \mu, \epsilon) = C^3(\epsilon) \left( \frac{\mu^2}{q^2} \right)^{3\epsilon} \frac{1}{2^{11}\pi^5} \mathcal{T}_5(y, \epsilon), \tag{A.11}$$

with

$$\begin{aligned}
 \mathcal{T}_5(y, \epsilon) = & \left( \frac{2}{(1+y)^2} - \frac{2}{1+y} \right) H(0; y) - \left( 2 + \frac{4}{(1+y)^2} - \frac{4}{1+y} \right) H(-1, 0; y) \\
 & + \left( 4 + \frac{6}{(1+y)^2} - \frac{10}{1+y} \right) H(0, 0; y) + \left( 2 + \frac{8}{(1+y)^2} - \frac{8}{1+y} \right) H(1, 0; y) \\
 & + 1 + \frac{\pi^2}{6} + \frac{\pi^2}{(1+y)^2} - \frac{2 + \pi^2}{1+y} \\
 & + \epsilon \left\{ \left( 8 + \frac{12}{(1+y)^2} - \frac{20}{1+y} \right) H(-3; y) + \left( \frac{4}{(1+y)^2} - \frac{4}{1+y} \right) H(-2; y) \right. \\
 & + \left( 2 + \frac{10\pi^2}{3} + \frac{32\pi^2}{3(1+y)^2} - \frac{4(3+8\pi^2)}{3(1+y)} \right) H(-1; y) \\
 & + \left( 8 - 4\pi^2 + \frac{4(18-5\pi^2)}{3(1+y)^2} - \frac{32(3-\pi^2)}{3(1+y)} \right) H(0; y) \\
 & \left. + \left( 10 - \frac{2\pi^2}{3} - \frac{8\pi^2}{3(1+y)^2} - \frac{4(15-2\pi^2)}{3(1+y)} \right) H(1; y) \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \left( \frac{20}{(1+y)^2} - \frac{20}{1+y} \right) H(2; y) + \left( 40 + \frac{60}{(1+y)^2} - \frac{100}{1+y} \right) H(3; y) \\
& + \left( 40 + \frac{52}{(1+y)^2} - \frac{92}{1+y} \right) H(-2, 0; y) - \left( 4 + \frac{8}{(1+y)^2} - \frac{8}{1+y} \right) H(-1, -2; y) \\
& - \left( 14 + \frac{8}{(1+y)^2} - \frac{8}{1+y} \right) H(-1, 0; y) - \left( 20 + \frac{40}{(1+y)^2} - \frac{40}{1+y} \right) H(-1, 2; y) \\
& + \left( 28 + \frac{40}{(1+y)^2} - \frac{68}{1+y} \right) H(0, 0; y) + \left( 4 + \frac{16}{(1+y)^2} - \frac{16}{1+y} \right) H(1, -2; y) \\
& + \left( 14 + \frac{56}{(1+y)^2} - \frac{56}{1+y} \right) H(1, 0; y) + \left( 20 + \frac{80}{(1+y)^2} - \frac{80}{1+y} \right) H(1, 2; y) \\
& - \left( 28 + \frac{56}{(1+y)^2} - \frac{56}{1+y} \right) H(-1, -1, 0; y) \\
& + \left( 24 + \frac{36}{(1+y)^2} - \frac{60}{1+y} \right) H(-1, 0, 0; y) \\
& + \left( 12 + \frac{48}{(1+y)^2} - \frac{48}{1+y} \right) H(-1, 1, 0; y) \\
& - \left( 12 + \frac{18}{(1+y)^2} - \frac{30}{1+y} \right) H(0, 0, 0; y) \\
& + \left( 12 + \frac{48}{(1+y)^2} - \frac{48}{1+y} \right) H(1, -1, 0; y) \\
& + \left( 12 + \frac{32}{(1+y)^2} - \frac{40}{1+y} \right) H(1, 0, 0; y) \\
& + \left( 12 + \frac{48}{(1+y)^2} - \frac{48}{1+y} \right) H(1, 1, 0; y) \\
& + \frac{1}{6} (84 + 7\pi^2 - 18\zeta(3)) + \frac{-28 - 5\pi^2 - 10\zeta(3)}{1+y} + \frac{5\pi^2 + 22\zeta(3)}{(1+y)^2} \Big\} + \mathcal{O}(\epsilon^2). \quad (\text{A.12})
\end{aligned}$$

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