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5-loop Konishi from linearized TBA and the XXX magnet

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ABSTRACT: Using the linearized TBA equations recently obtained in arXiv:1002.1711 we show analytically that the 5-loop anomalous dimension of the Konishi operator agrees with the result obtained previously from the generalized Lüscher formulae. The proof is based on the relation between this linear system and the XXX model TBA equations.

KEYWORDS: AdS-CFT Correspondence, Lattice Integrable Models

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1 Introduction

One of the most important problems in testing the AdS/CFT correspondence [1–3] is to understand the finite size spectrum of the $AdS_5 \times S^5$ superstring. For large volumes the asymptotic Bethe Ansatz (ABA) describes the spectrum of the model [4]. It takes into account all power like corrections in the size, but neglects the exponentially small wrapping corrections [5].

In [6] it was shown that the leading order wrapping corrections can also be expressed by the infinite volume scattering data through the generalized Lüscher formulae [7]. In [6] the 4-loop anomalous dimension of the Konishi operator was obtained by means of the generalized Lüscher formulae in perfect agreement with direct field theoretic computations [8, 9]. Subsequently wrapping interactions computed from Lüscher corrections were found to be crucial for the agreement of some structural properties of twist two operators [10] with LO and NLO BFKL expectations [11, 12].

More recently [13] the 5-loop wrapping correction to the anomalous dimension of the Konishi operator was also computed from the generalized Lüscher approach yielding the result:

$$\Delta^{(10)} = \Delta_{\text{asympt}}^{(10)} + g^{10} \left\{ -\frac{81\zeta(3)^2}{16} + \frac{81\zeta(3)}{32} - \frac{45\zeta(5)}{4} + \frac{945\zeta(7)}{32} - \frac{2835}{256} \right\},$$

with g being the coupling constant related to the 't Hooft coupling λ through $\lambda = 4\pi^2 g^2$. Later this result has been extended to the class of twist two operators as well [14].

Due to the integrability of the string worldsheet theory, the Thermodynamic Bethe Ansatz (TBA) approach for the mirror model [5, 15] offers a tool to investigate the spectrum of the string theory. The TBA equations were derived first for the ground state [16–20]. Later using an analytic continuation trick [21] they were extended to excited states lying in the $sl(2)$ sector of the theory [20, 22] as well. The TBA equations passed some tests both in the weak and in the strong coupling limit. In the strong coupling limit it was

shown [23] that the TBA equations reproduce correctly the 1-loop string energies in the quasi-classical limit. In the weak coupling regime they give (by construction) the same leading order wrapping corrections in g as predicted by the generalized Lüscher formulae.

However, to extract the next to leading order wrapping correction in g from TBA is more difficult as at this order the modification of the ABA equations must be taken into account. In the TBA approach the modified ABA equations depend also on the asymptotically non-vanishing Y -functions (which satisfy non-trivial coupled equations even in the small g limit), making the next to leading order calculation of wrapping interactions a non-trivial task.

In the TBA formulation of the finite size problem the energy of an N -particle state takes the form:

$$E = J + \sum_{i=1}^N \mathcal{E}(p_i) - \frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} du \frac{d\tilde{p}^Q}{du} \log(1 + Y_Q), \quad (1.1)$$

where J is the angular momentum carried by the string rotating around the equator of S^5 , \tilde{p}^Q is the mirror momentum and the functions Y_Q are the unknown functions (Y -functions) associated to the mirror Q -particles, furthermore

$$\mathcal{E}(p) = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}} \quad (1.2)$$

is the dispersion relation of the string theory particles.

In this paper we will focus on the g^{10} order computation of the anomalous dimension (energy) of the Konishi operator $\text{Tr}(D^2 Z^2 - (DZ)^2)$ and expanding the considerations of [24] we prove analytically that the TBA equations and the generalized Lüscher formulae of [13] give the same result for the 5-loop anomalous dimension of the Konishi operator. This operator corresponds to the $N = J = 2$ choice in (1.1). For its TBA equations see [22, 25]. In this paper we will use the notations and conventions used in [22].

It is known for the Konishi operator that in the weak coupling regime the wrapping corrections start at the order of g^8 thus the ABA equations for the momenta get corrections from wrapping at g^8 order, i.e. $\delta p_k \sim g^8$, where δp_k is the wrapping correction to the asymptotic value of the k th momentum.

The energy formula (1.1) can be expanded around the asymptotic solution if the Y -functions are small. This happens either for large J as the Y_Q -functions are exponentially small in this limit or at fixed J for small g . From the asymptotic solution of the TBA equations [26] it is known that $Y_Q \sim g^8$, this is why up to $O(g^{10})$ it is enough to take into account only the first term, linear in Y_Q , in the series expansion of the integral term of (1.1).

$$E \simeq J + \sum_{i=1}^2 \mathcal{E}(p_i^{ABA}) + \sum_{i=1}^2 \left. \frac{\partial \mathcal{E}}{\partial p_i} \right|_{ABA} \delta p_i - \frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} du \frac{d\tilde{p}^Q}{du} Y_Q + O(g^{12}), \quad (1.3)$$

where p_i^{ABA} denotes the solution of the ABA and the derivative of \mathcal{E} must be taken at the asymptotic values of the momenta. As the function $\frac{d\tilde{p}^Q}{du}$ starts at $O(1)$ in g^2 and

$\left. \frac{\partial \mathcal{E}}{\partial p_i} \right|_{ABA} \sim g^2$ it can be seen that only the asymptotic form of the Y_Q functions contribute to the wrapping correction in leading order and the momentum perturbations start to play a role only at $O(g^{10})$.

Taking the asymptotic form of the Y_Q functions given in [26] it is easy to see that all terms in the above energy expression identically agree with those of ref. [13] except the one containing the momentum correction. Thus only the momentum quantization equations should be compared to see whether both approaches give the same result for δp_i .

In a recent publication [24] this agreement was verified by numerically solving the linearized TBA equations. We will use the results (and notations) of this paper.

Let $u_k = u_k^o + \delta u_k$, where u_k^o is the asymptotic value of the u_k and $\delta u_k \sim g^8$ is its wrapping correction. Then δu_k satisfies the equation:

$$\sum_{j=1}^2 \left. \frac{\delta ABA(u_k, \{u_i\})}{\delta u_j} \right|_{u_i=u_i^o} \delta u_j + \Phi_k^{(8)} = 0$$

where $\Phi_k^{(8)}$ is the $O(g^8)$ correction to the ABA. For small g all Y_Q functions are small and the TBA equations can be linearized around the asymptotic solution.

In [24] it has been shown that at $O(g^8)$ the linear problem for the functions associated to the vw -strings decouples from the other type of variables and takes the form

$$\begin{aligned} \mathcal{Y}_{M|vw} &= (A_{M-1|vw} \mathcal{Y}_{M-1|vw} + A_{M+1|vw} \mathcal{Y}_{M+1|vw}) \star s - Y_{M+1}^o \star s, \\ \mathcal{Y}_{0|vw} &= 0, \quad M = 1, 2, \dots, \end{aligned} \tag{1.4}$$

where $\mathcal{Y}_{M|vw}$ is the $O(g^8)$ perturbation of the asymptotic $Y_{M|vw}^o$ function defined by the formula $Y_{M|vw} = Y_{M|vw}^o (1 + \mathcal{Y}_{M|vw})$, $A_{M|vw} = \lim_{g \rightarrow 0} \frac{Y_{M|vw}^o}{1 + Y_{M|vw}^o}$ and it is given explicitly by¹

$$A_{M|vw}(u) = \frac{M(M+2)}{(M+1)^2} \frac{(u^2 - w^2 + M^2 - 1)(u^2 - w^2 + (M+2)^2 - 1)}{[u^2 - w^2 + (M+1)^2]^2 + 4w^2 - 4M(M+2)}, \tag{1.5}$$

where

$$u_1 = -u_2 = w = \frac{1}{\sqrt{3}} \tag{1.6}$$

is the $O(1)$ solution of the ABA for the Konishi state. Furthermore \star denotes convolution, s is the TBA kernel $s(x) = \frac{1}{4 \cosh(\frac{\pi}{2}x)}$, and finally in the source term of the linear problem Y_Q^o is the leading, $O(g^8)$, asymptotic expression of the Y_Q functions:

$$\begin{aligned} Y_Q^o(u) &= g^8 \frac{64 Q^2 [-1 + Q^2 + u^2 - w^2]^2}{(Q^2 + u^2)^4 [(Q-1)^2 + (u-w)^2][(Q+1)^2 + (u-w)^2]} \times \\ &\quad \times \frac{1}{[(Q-1)^2 + (u+w)^2][(Q+1)^2 + (u+w)^2]}. \end{aligned} \tag{1.7}$$

¹We simplified the formula (2.15) of ref. [24] to make its relation to the XXX magnet apparent.

It turns out [24] that apart from the Y_Q^o functions and $\mathcal{Y}_{1|vw}$ no perturbations of the other Y -functions enter the final formula for $\delta\mathcal{R}_k = -\Phi_k^{(8)}$:

$$\begin{aligned} \delta\mathcal{R}_k &= \frac{1}{\pi} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} du Y_m^o(u) \frac{u - u_k}{(m+1)^2 + (u - u_k)^2} + \rho_k \\ &+ \frac{1}{\pi} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} du Y_{m+1}^o(u) \left\{ \mathcal{F}_m(u - u_k) - \frac{u - u_k}{m^2 + (u - u_k)^2} \right\}, \end{aligned} \tag{1.8}$$

where

$$\mathcal{F}_m(u) = \frac{-i}{4} \left\{ \psi\left(\frac{m+iu}{4}\right) - \psi\left(\frac{m-iu}{4}\right) - \psi\left(\frac{m+2+iu}{4}\right) + \psi\left(\frac{m+2-iu}{4}\right) \right\} \tag{1.9}$$

with the usual ψ function $\psi(z) = \Gamma'(z)/\Gamma(z)$ and ρ_k is the contribution coming from the $Y_{M|vw}$ -functions:

$$\rho_k = \int_{-\infty}^{\infty} du A_{1|vw}(u) \mathcal{Y}_{1|vw}(u) \frac{1}{2 \sinh \frac{\pi}{2}(u - u_k)}. \tag{1.10}$$

On the other hand the generalized Lüscher approach provides [13] the following expression for $\Phi_k^{(8)}$:

$$\begin{aligned} \Phi^{(8)}(u_k) &= \sum_{M=1}^{\infty} \int_{-\infty}^{\infty} du Y_M^o(u) \times \\ &\times \frac{1}{\pi} \left[-\frac{u - u_k}{(M+1)^2 + (u - u_k)^2} - \frac{u - u_k}{(M-1)^2 + (u - u_k)^2} + \frac{u_k}{-1 + M^2 + u^2 - u_k^2} \right]. \end{aligned} \tag{1.11}$$

In [24] it has been numerically verified that $\Phi_k^{(8)}$ given by (1.8) and (1.11) agrees. In this paper we will show this fact analytically. The key point of the proof is to recognize that the coefficient functions $A_{M|vw}$ of the linear problem (1.4) are related to the Y -functions of the inhomogeneous spin- $\frac{1}{2}$ XXX chain (for a review, see [27]) and that (with a different source term) the linear problem (1.4) is identical to the variation of the TBA equations² of the XXX magnet with respect to the inhomogeneity parameters. Exploiting these facts we can express the quantity $\Phi_k^{(8)}$ by the Y -functions of the XXX magnet and show that the formulae (1.8) and (1.11) are identical (up to a sign).

2 Linearized TBA equations

Let us rewrite the linearized AdS TBA system (1.4) as follows:

$$D_m \delta L_m - s \star (\delta L_{m+1} + \delta L_{m-1}) = -s \star Y_{m+1}^o, \quad m = 1, 2, \dots \tag{2.1}$$

Note that our unknown functions $\delta L_m(u)$ are rescaled (by $A_{m|vw}(u)$) with respect to the ones used in ref. [24] and the coefficient functions $D_m(u)$ are the inverses of the functions $A_{m|vw}(u)$ given by (1.5). In this note we will only use the fact that the $Y_m^o(u)$ functions

²In this case the term TBA is used in the sense of finite size effects.

are regular and even in u , but their explicit form (1.7) is not needed. In (2.1) $\delta L_0 = 0$ by convention and we also note that $\delta L_1(u_k) = 0$ because of the rescaling by $A_{1|vw}(u)$, since the latter function vanishes at $u = u_k$. We first have to solve (2.1) and then the relevant quantity to be calculated is

$$\rho_k = \int_{-\infty}^{\infty} du \frac{\delta L_1(u)}{2 \sinh \frac{\pi}{2}(u - u_k)}. \quad (2.2)$$

No principal value prescription is needed since the integrand is regular at $u = u_k$. If we can calculate ρ_k then the leading correction to the Bethe-Yang quantization is given by (1.8).

To avoid the singularities coming from $D_1(u)$ at u_k we shift the integration contour in the imaginary direction by a small amount $i\gamma$:

$$D_m^\gamma \delta L_m^\gamma - s \star (\delta L_{m+1}^\gamma + \delta L_{m-1}^\gamma) = -s^\gamma \star Y_{m+1}^o, \quad m = 1, 2, \dots \quad (2.3)$$

Here we use the notation $f^\gamma(u) = f(u + i\gamma)$ for any function $f(u)$. Although we need to solve (2.3) in a particular case only, it turns out to be useful to study the corresponding general linear problem, for a general (infinite) vector of unknowns ξ and arbitrary (infinite) source vector j :

$$\mathbf{M} \xi = j, \quad (2.4)$$

where the operator matrix is given by

$$\mathbf{M} = \begin{pmatrix} D_1^\gamma & -s\star & 0 & \dots \\ -s\star & D_2^\gamma & -s\star & \dots \\ 0 & -s\star & D_3^\gamma & \dots \\ \vdots & & & \ddots \end{pmatrix}. \quad (2.5)$$

In our case the unknowns are

$$\xi = \begin{pmatrix} \delta L_1^\gamma \\ \delta L_2^\gamma \\ \vdots \end{pmatrix} \quad (2.6)$$

and the source term is of the form

$$j = I = \begin{pmatrix} -s^\gamma \star Y_2^o \\ -s^\gamma \star Y_3^o \\ \vdots \end{pmatrix}. \quad (2.7)$$

The operator matrix \mathbf{M} is symmetric, $\mathbf{M}^T = \mathbf{M}$. Therefore, assuming that the inverse of \mathbf{M} exists uniquely³ we can formally solve (2.4) as

$$\xi = \mathbf{R} j, \quad \mathbf{R} = \mathbf{M}^{-1}, \quad (2.8)$$

such that the inverse operator \mathbf{R} is also symmetric: $\mathbf{R}^T = \mathbf{R}$. Writing the solution (2.8) in components we have

$$\xi_m(x) = \sum_{m'=1}^{\infty} \int_{-\infty}^{\infty} dy R_{mm'}(x, y) j_{m'}(y), \quad (2.9)$$

³See appendix A about the existence and unicity of the inverse of \mathbf{M} .

where, due to the symmetry of the operator, the kernels satisfy

$$R_{mm'}(x, y) = R_{m'm}(y, x). \tag{2.10}$$

Using this notation, we have

$$\begin{aligned} \rho_k &= \int_{-\infty}^{\infty} du \frac{\delta L_1^\gamma(u)}{2 \sinh \frac{\pi}{2}(u + i\gamma - u_k)} \\ &= \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \frac{1}{2 \sinh \frac{\pi}{2}(u + i\gamma - u_k)} R_{1m}(u, v) I_m(v). \end{aligned} \tag{2.11}$$

In this paper we will compute the relevant quantity ρ_k given by (2.11) without solving explicitly the linearized TBA equations (2.1). This can be done by recognizing that an explicitly solvable auxiliary linear problem can be defined via the XXX model which is of the form (2.4) with a special right hand side j . This linear problem is the linearization of the TBA system corresponding to the XXX model such that the coefficient functions D_m are related to the XXX model Y-functions. The construction and the solution of this linear problem is given in the next section.

3 XXX model TBA equations

The XXX model transfer matrix eigenvalue relevant for our considerations is

$$t_m(u) = (m + 1) \{(u - u_1)(u - u_2) + m(m + 2)\}, \quad m = -1, 0, 1, 2, \dots \tag{3.1}$$

This is a zero isospin solution of the T-system equations for the inhomogeneous XXX spin chain⁴ of length 2 (the corresponding Baxter Q-operator has one real Bethe root):

$$t_m(u+i) t_m(u-i) = t_{m+1}(u) t_{m-1}(u) + t_0(u+(m+1)i) t_0(u-(m+1)i), \quad m = 0, 1, 2, \dots \tag{3.2}$$

The Y-system elements are given by the usual definitions

$$y_m(u) = \frac{t_{m+1}(u) t_{m-1}(u)}{t_0(u + (m + 1)i) t_0(u - (m + 1)i)}, \tag{3.3}$$

$$1 + y_m(u) = \frac{t_m(u + i) t_m(u - i)}{t_0(u + (m + 1)i) t_0(u - (m + 1)i)} \tag{3.4}$$

and satisfy the Y-system equations

$$y_m(u + i) y_m(u - i) = [1 + y_{m+1}(u)] [1 + y_{m-1}(u)]. \tag{3.5}$$

Now the crucial observation is that with this solution

$$D_m(u) = \frac{1}{A_{m|vw}(u)} = \frac{1 + y_m(u)}{y_m(u)}, \tag{3.6}$$

where the functions $A_{m|vw}(u)$ are given by (1.5). More precisely, (3.6) holds for the symmetric case (1.6).

⁴Here we consider the case when the inhomogeneities u_1 and u_2 are real.

Our T-functions (except t_0) have no physical roots (zeroes with imaginary parts less than unity) if

$$\left| \frac{u_1 - u_2}{2} \right| < \sqrt{2} \tag{3.7}$$

and therefore (for $m \geq 1$) only $y_1(u)$ has physical roots. The corresponding TBA equations are of the form

$$y_m(u) = \{t(u - u_1) t(u - u_2)\}^{\delta_{m1}} \exp \{s \star (L_{m+1} + L_{m-1})(u)\}, \tag{3.8}$$

where

$$t(u) = \tanh \frac{\pi}{4} u \quad \text{and} \quad L_m(u) = \ln(1 + y_m(u)). \tag{3.9}$$

Taking the derivative (∂_k) of (3.8) with respect to u_k gives

$$\frac{\partial_k y_m(u)}{y_m(u)} = -\frac{\pi \delta_{m1}}{2 \sinh \frac{\pi}{2}(u - u_k)} + (s \star \partial_k L_{m+1})(u) + (s \star \partial_k L_{m-1})(u). \tag{3.10}$$

After shifting the u variable by $i\gamma$ and making the specialization⁵ (1.6) we get the auxiliary linear problem which is precisely of the form (2.4) with

$$\xi_m(u) = \partial_k L_m^\gamma(u) \tag{3.11}$$

and

$$j_m(u) = -\frac{\pi \delta_{m1}}{2 \sinh \frac{\pi}{2}(u + i\gamma - u_k)}. \tag{3.12}$$

Substituting (3.11) and (3.12) into (2.9) we get a relation between the solution (3.11) and certain matrix elements of the inverse operator

$$\partial_k L_m^\gamma(u) = -\frac{\pi}{2} \int_{-\infty}^{\infty} dv \frac{R_{m1}(u, v)}{\sinh \frac{\pi}{2}(v + i\gamma - u_k)}. \tag{3.13}$$

4 Calculation of ρ_k

From (2.11) and the symmetry property of the inverse operator \mathbf{R} it can be seen that the knowledge of the right hand side of (3.13) is enough to compute ρ_k without solving explicitly the complicated linearized TBA equations (2.1) of the AdS/CFT. Making use of (3.13) we get

$$\begin{aligned} \rho_k &= -\frac{1}{\pi} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} dv I_m(v) \partial_k L_m^\gamma(v) \\ &= \frac{1}{\pi} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du s(v + i\gamma - u) Y_{m+1}^o(u) \partial_k L_m^\gamma(v) \\ &= \frac{1}{\pi} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} du Y_{m+1}^o(u) (s \star \partial_k L_m)(u). \end{aligned} \tag{4.1}$$

⁵In the rest of the paper ∂_k is understood as first taking the derivative with respect to u_k and then taking the specialization corresponding to (1.6).

This can be simplified further if we introduce the gauge transformed T-system functions

$$\hat{t}_m(u) = \left\{ \prod_k r_m(u - u_k) \right\} t_m(u), \tag{4.2}$$

where

$$r_m(u) = \frac{1}{4} \frac{\gamma(2 + m + iu) \gamma(2 + m - iu)}{\gamma(4 + m + iu) \gamma(4 + m - iu)} \tag{4.3}$$

with $\gamma(z) = \Gamma(z/4)$. It is easy to check that in this gauge we have

$$\hat{t}_m(u + i) \hat{t}_m(u - i) = 1 + y_m(u). \tag{4.4}$$

Since \hat{t}_m ($m \geq 1$) has no roots in the physical strip we can write

$$\hat{t}_m(u) = \exp \{ (s \star L_m)(u) \} \tag{4.5}$$

and by taking the ∂_k derivative we obtain

$$\partial_k \ln \hat{t}_m(u) = s \star \partial_k L_m(u). \tag{4.6}$$

ρ_k can now be written as

$$\rho_k = \frac{1}{\pi} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} du Y_{m+1}^o(u) \partial_k \ln \hat{t}_m(u). \tag{4.7}$$

Calculating the derivative we find

$$\partial_k \ln \hat{t}_m(u) = -\mathcal{F}_m(u - u_k) + \frac{2(u - u_k)}{m^2 + (u - u_k)^2} - \frac{u + u_k}{u^2 - u_k^2 + m(m + 2)}. \tag{4.8}$$

Putting everything together, we find the result

$$\begin{aligned} \delta \mathcal{R}_k = & \frac{1}{\pi} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} du Y_m^o(u) \frac{u - u_k}{(m + 1)^2 + (u - u_k)^2} \\ & + \frac{1}{\pi} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} du Y_{m+1}^o(u) \left\{ \frac{u - u_k}{m^2 + (u - u_k)^2} - \frac{u_k}{u^2 - u_k^2 + m(m + 2)} \right\}. \end{aligned} \tag{4.9}$$

This is precisely the same (up to a sign) as (1.11), the result obtained by using the generalized Lüscher formalism [13]. Thus we have shown that up to 5-loop order the TBA equations and the generalized Lüscher formulae give the same result for the anomalous dimension of the Konishi operator.

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A Existence and uniqueness of the inverse matrix

The problem of finding the solution of the linearized TBA equations (2.3) is essentially equivalent to finding the inverse of the infinite matrix (2.5). In this appendix we show the existence and uniqueness of this matrix inversion problem. Uniqueness, which is essentially equivalent to the absence of zero modes, is important because this enables us to calculate ρ_k unambiguously from (2.2). The infinite matrix (2.5) can be written as

$$\mathbf{M} = \mathbf{D} - \mathbf{P} s \star, \quad (\text{A.1})$$

where $\mathbf{D} = \langle D_1^\gamma, D_2^\gamma, \dots \rangle$ is diagonal and \mathbf{P} is a constant tridiagonal matrix given by $P_{ij} = \delta_{i+1j} + \delta_{i-1j}$. We can rewrite \mathbf{M} as

$$\mathbf{M} = (\mathbf{1} - \mathbf{A})\mathbf{D}, \quad \text{where} \quad \mathbf{A} = \mathbf{P} s \star \mathbf{D}^{-1}. \quad (\text{A.2})$$

The action of the operator \mathbf{A} on a vector with components $f_i(x)$ can be written as

$$(\mathbf{A}f)_i(x) = \sum_j P_{ij} \int_{-\infty}^{\infty} dy s(x-y) d_j(y) f_j(y), \quad (\text{A.3})$$

where $d_j(y) = 1/D_j^\gamma(y)$. The crucial observation is that the absolute value of this function is always smaller than its asymptotic value, Δ_j :

$$|d_j(y)| < \Delta_j = \frac{j(j+2)}{(j+1)^2} \quad j = 1, 2, \dots, \quad (\text{A.4})$$

at least for small enough γ . For later use we now define the operator \mathbf{B} , which is obtained from \mathbf{A} by replacing $d_j(y)$ with its asymptotic value:

$$(\mathbf{B}f)_i(x) = \sum_j P_{ij} \int_{-\infty}^{\infty} dy s(x-y) \Delta_j f_j(y). \quad (\text{A.5})$$

We also define analogously

$$\mathbf{M}_\infty = (\mathbf{1} - \mathbf{B})\mathbf{D}_\infty = \mathbf{D}_\infty - \mathbf{P} s \star. \quad (\text{A.6})$$

The vectors of our linear space are given as infinite vectors

$$f \sim \{f_1(x), f_2(x), \dots\}, \quad (\text{A.7})$$

or, equivalently, in Fourier space as

$$f \sim \{\tilde{f}_1(\omega), \tilde{f}_2(\omega), \dots\}, \quad (\text{A.8})$$

where, as usual,

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dx e^{ix\omega} f(x). \quad (\text{A.9})$$

We now equip our space with the hermitean scalar product

$$(g|f) = \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} dx g_i^*(x) f_i(x) = \frac{1}{2\pi} \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} d\omega \tilde{g}_i^*(\omega) \tilde{f}_i(\omega) \quad (\text{A.10})$$

and the corresponding norm $\|f\|^2 = (f|f)$. With this norm our vector space becomes a Hilbert space. We assume throughout this paper that both the vector variables ξ and the source terms j in equations of the form (2.4) belong to this Hilbert space. This is a natural assumption since it is easy to see that both source terms (2.7) and (3.12) and, more importantly, the vector on the left hand side of (3.13) are elements of this Hilbert space.

For later purpose we note that the action of the operator \mathbf{B} on the elements of the Hilbert space is simple in terms of the Fourier transformed components. Using the notation $\mathbf{B}f = h$, we have

$$\tilde{h}_i(\omega) = \sum_j P_{ij} \tilde{s}(\omega) \Delta_j \tilde{f}_j(\omega), \quad (\text{A.11})$$

where

$$\tilde{s}(\omega) = \frac{1}{2 \cosh \omega} = \frac{1}{q + \frac{1}{q}}, \quad q = e^{|\omega|} \geq 1. \quad (\text{A.12})$$

We now observe that

$$\begin{aligned} |(\mathbf{A}f)_i(x)| &\leq \sum_j P_{ij} \int_{-\infty}^{\infty} dy s(x-y) |d_j(y)| |f_j(y)| \\ &< \sum_j P_{ij} \int_{-\infty}^{\infty} dy s(x-y) \Delta_j \hat{f}_j(y) = (\mathbf{B}\hat{f})_i(x), \end{aligned} \quad (\text{A.13})$$

where⁶

$$\hat{f}_i(x) = |f_i(x)|, \quad (\hat{f}|\hat{f}) = (f|f). \quad (\text{A.14})$$

This inequality implies that \mathbf{A} is “smaller” than \mathbf{B} , in the sense that

$$\|\mathbf{A}f\| < \|\mathbf{B}\hat{f}\| \quad \text{and} \quad |(g|\mathbf{A}f)| < (\hat{g}|\mathbf{B}\hat{f}). \quad (\text{A.15})$$

On the other hand, \mathbf{B} is smaller than unity, in the following sense. We first write

$$(g|\mathbf{B}f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{s}(\omega) \sum_{j=1}^{\infty} \Delta_j \tilde{f}_j(\omega) \{ \tilde{g}_{j+1}^*(\omega) + \tilde{g}_{j-1}^*(\omega) \} \quad (\text{A.16})$$

and after using the simple inequality $2|ab| \leq |a|^2 + |b|^2$ we have

$$|(g|\mathbf{B}f)| < \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{s}(\omega) \sum_{j=1}^{\infty} \left\{ |\tilde{f}_j^*(\omega)|^2 + |\tilde{g}_j^*(\omega)|^2 \right\} < \frac{1}{2} \|f\|^2 + \frac{1}{2} \|g\|^2. \quad (\text{A.17})$$

Thus the norm of \mathbf{B} is not exceeding unity since from the above inequality it follows that

$$|(f|\mathbf{B}f)| < (f|f) \quad (\text{A.18})$$

and similarly

$$|(f|\mathbf{A}f)| < (\hat{f}|\mathbf{B}\hat{f}) < (\hat{f}|\hat{f}) = (f|f). \quad (\text{A.19})$$

⁶All strict inequalities in this appendix are valid for nonzero Hilbert space vectors f, g .

The inequalities (A.18) and (A.19) imply *uniqueness* of the inverse of the operators $\mathbf{1} - \mathbf{B}$ and $\mathbf{1} - \mathbf{A}$ since by multiplying the equations

$$f = \mathbf{B}f, \quad f = \mathbf{A}f \tag{A.20}$$

by f we arrive at a contradiction.

More precisely, the solution of $(\mathbf{1} - \mathbf{A})f = 0$ as an infinite component vector $\{f_1(x), f_2(x), \dots\}$ may formally exist, but the above considerations show that f cannot be a vector of the Hilbert space. The analogous $\mathbf{M}_\infty \xi = 0$ equation can be explicitly solved in Fourier space. This corresponds to the recursion relation

$$\left(q + \frac{1}{q}\right) \frac{(k+1)^2}{k(k+2)} \tilde{\xi}_k = \tilde{\xi}_{k+1} + \tilde{\xi}_{k-1}, \quad k = 1, 2, \dots \tag{A.21}$$

with the boundary condition $\tilde{\xi}_0 = 0$. The formal solution is easily found:

$$\tilde{\xi}_k(\omega) = C_1(\omega)a(k), \quad a(k) = \frac{k}{k+1} \left(q^{k+2} - q^{-k-2}\right) - \frac{k+2}{k+1} \left(q^k - q^{-k}\right). \tag{A.22}$$

Here $C_1(\omega)$ is an arbitrary (ω -dependent) normalization constant. Of course, this ξ cannot be an element of the Hilbert space, since its components are exploding in k . This shows why the Hilbert space requirement is natural: linearization only makes sense as long as the linearized variable remains small.

The general solution of the recursion relation (A.21) is

$$\tilde{\xi}_k(\omega) = C_1(\omega)a(k) + C_2(\omega)b(k), \quad b(k) = \frac{k+2}{k+1} q^{-k} - \frac{k}{k+1} q^{-k-2}, \tag{A.23}$$

where $C_2(\omega)$ is a second normalization constant. Using the building blocks $a(k)$ and $b(k)$ the inverse of \mathbf{M}_∞ in Fourier space can be written as

$$\tilde{R}_{\infty kl}(\omega) = \begin{cases} \lambda(\omega) a(k)b(l) & k \leq l, \\ \lambda(\omega) a(l)b(k) & k \geq l, \end{cases} \tag{A.24}$$

where

$$\lambda(\omega) = \frac{\cosh \omega}{4 \sinh^3 |\omega|}. \tag{A.25}$$

One can show that

$$\mathbf{R}_\infty = \mathbf{D}_\infty^{-1}(\mathbf{1} - \mathbf{B})^{-1} = \mathbf{D}_\infty^{-1} \mathbf{b}, \tag{A.26}$$

where \mathbf{b} is the sum of the Neumann series

$$\mathbf{b} = (\mathbf{1} - \mathbf{B})^{-1} = \mathbf{1} + \mathbf{B} + \mathbf{B}^2 + \dots \tag{A.27}$$

Since \mathbf{A} is “smaller” than \mathbf{B} , the inverse of $\mathbf{1} - \mathbf{A}$ also exists in the form of the Neumann series

$$\mathbf{a} = (\mathbf{1} - \mathbf{A})^{-1} = \mathbf{1} + \mathbf{A} + \mathbf{A}^2 + \dots \tag{A.28}$$

since it can be shown easily that

$$\|(\mathbf{1} + \mathbf{A} + \mathbf{A}^2 + \dots + \mathbf{A}^k)f\| < \|(\mathbf{1} + \mathbf{B} + \mathbf{B}^2 + \dots + \mathbf{B}^k)\hat{f}\| < \|\mathbf{b}\hat{f}\|. \tag{A.29}$$

It is also evident that the inverse of \mathbf{M} , which can be written as

$$\mathbf{R} = \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{P}_S \star \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{P}_S \star \mathbf{D}^{-1}\mathbf{P}_S \star \mathbf{D}^{-1} + \dots \quad (\text{A.30})$$

is manifestly symmetric.

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