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Proving the AGT relation for $N_f = 0, 1, 2$ antifundamentals

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Abstract: Using recursive relations satisfied by Nekrasov partition functions and by irregular conformal blocks we prove the AGT correspondence in the case of $\mathcal{N} = 2$ superconformal SU(2) quiver gauge theories with $N_f = 0, 1, 2$ antifundamental hypermultiplets.

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Contents

1 Introduction

Last year Alday, Gaiotto and Tachikawa conjectured [1] that partition functions of $\mathcal{N} = 2$ superconformal SU(2) quivers are directly related to correlation functions of the two dimensional Liouville field theory. This relation has been soon extended to similar relations between the general SU(N) quiver theories and A_{N-1} Toda theories [2] and to other objects like surface and loop operators and their Liouville counterparts [3–8]. Other extensions concern non-conformal limit of the AGT [9–13] correspondence and its 5-dimensional version [14, 15]. Yet another generalization has been recently found in [16].

An explanation of the AGT relation was given by Dijkgraaf and Vafa [17]. The idea was to relate both sides of the correspondence to a certain class of matrix models. These relations were further analyzed in a number of papers [18–24]. Another M-theory explanation was presented in [25, 26].

An essential part of the AGT conjecture is an exact correspondence between instanton parts of Nekrasov partition functions in 4-dimensional $\mathcal{N} = 2$ SCFT [27] and conformal blocks of the 2-dimensional CFT [28]. This relation has passed many checks [29–35] and lead to many interesting results both on the conformal blocks and on the $\mathcal{N} = 2$ quivers [36–52].

In spite of all these developments only in the case of $\mathcal{N} = 2$ SU(2) SYM with a single adjoint matter multiplet an analytic proof of the AGT relation is known. The main idea of the proof given by Fateev and Litvinov [52] is to show that the corresponding Nekrasov function and the 1-point conformal block on the torus satisfy exactly the same recursion relations. These relations were first conjectured by Poghossian [38] and then proven on the CFT side in [41] and on the $\mathcal{N} = 2$ SU(2) SYM side in [52]. The aim of the present paper is to extend this proof to the case of $N_f = 0, 1, 2$ antifundamental hypermultiplets.

In section 2 we use the results of Marshakov, Mironov and Morozov [10] and Poghossian [38] to derive the recursive relations for all irregular blocks by analyzing appropriate decoupling limits of the Zamolodchikov elliptic recursive relation for the 4-point conformal block on the sphere [53–55]. This allows in particular to prove the relation between two representations of the irregular block with two μ parameters conjectured in [9]. As a side

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topic we clarify the relations of polynomials appearing in the recursive relations to the fusion polynomials and to the null states in the degenerate Verma modules. These new, intriguing result deserves further investigations.

In section 3 we follow the method of Fateev and Litvinov [52] to analyze singularities of the Nekrasov functions and the factorization of the residues in the case of an arbitrary number of antifundamentals. The complete derivation of recursive formulae along this line requires the large $\mathsf{p} = \frac{a}{\hbar}$ asymptotic of the Nekrasov functions. This is simple in the cases $N_f = 0, 1$. We were also able to calculate the asymptotic in the $N_f = 2$ case. In all three cases the recursions obtained are identical on both sides of the correspondence. Calculating the asymptotics in the cases $N_f = 3, 4$ turned out to be difficult and is still a challenging open problem.

2 Recursive relations for irregular blocks

The Gaiotto states [9] can be defined by the conditions

$$\begin{aligned} L_1 \left|\Delta, \Lambda^2\right\rangle &= -\Lambda^2 \left|\Delta, \Lambda^2\right\rangle, & L_n \left|\Delta, \Lambda^2\right\rangle &= 0 \quad \text{for} \quad n \geq 2. \\ L_2 \left|\Delta, \mu, \Lambda\right\rangle &= -\Lambda^2 \left|\Delta, \mu, \Lambda\right\rangle, & & \\ L_1 \left|\Delta, \mu, \Lambda\right\rangle &= -2\mu\Lambda \left|\Delta, \mu, \Lambda\right\rangle, & L_n \left|\Delta, \mu, \Lambda\right\rangle &= 0 \quad \text{for} \quad n \geq 3. \end{aligned}$$

An explicit form of these states was found in [10]

$$\left|\Delta, \Lambda^2\right\rangle = \sum_{n=0} \Lambda^{2n} \left|\Delta, n\right\rangle = \sum_{n} \Lambda^{2n} (-1)^n \sum_{|J|=n} \left[B^n_{c,\Delta}\right]^{[1^n],J} L_{-J} \left|\Delta\right\rangle, \tag{2.1}$$

$$\begin{aligned} \left|\Delta, \mu, \Lambda\right\rangle &= \sum_{n=0} \Lambda^{n} \left|\Delta, \mu, n\right\rangle \\ &= \sum_{n} \Lambda^{n} \sum_{|J|=n} \sum_{p=0}^{n/2} (-1)^{n-p} (2\mu)^{n-2p} \left[B^n_{c,\Delta}\right]^{[1^{n-2p},2^p],J} L_{-J} \left|\Delta\right\rangle. \end{aligned} \tag{2.2}$$

In the formulae above $\left[B^n_{c,\Delta}\right]^{I,J}$ denotes the inverse of the Gram matrix

$$\left[B^n_{c,\Delta}\right]_{I,J} = \left\langle\Delta\right| L_I L_{-J} \left|\Delta\right\rangle$$

in the standard basis

$$L_{-J} \left|\Delta\right\rangle = L_{-j_1} \ldots L_{-j_k} \left|\Delta\right\rangle, \quad j_1 \leqslant \cdots \leqslant j_k, \quad |J| = \sum_{i=1}^{k} j_i$$

of the Verma module $\mathcal{V}_{c,\Delta}$ of the central charge c and the highest weight Δ.

The irregular blocks [9] are defined as scalar products of the Gaiotto states:

$$\begin{aligned} \left\langle\Delta, \Lambda^2 \middle| \Delta, \Lambda^2\right\rangle &= \sum_{n} \Lambda^{4n} \left\langle\Delta, n \,\middle|\, \Delta, n\right\rangle, \\ \left\langle\Delta, \mu, \tfrac{1}{2}\Lambda \,\middle|\, \Delta, \Lambda^2\right\rangle &= \sum_{n} \Lambda^{3n} 2^{-n} \left\langle\Delta, \mu, n \,\middle|\, \Delta, n\right\rangle, \\ \left\langle\Delta, \mu_1, \tfrac{1}{2}\Lambda \,\middle|\, \Delta, \mu_2, \tfrac{1}{2}\Lambda\right\rangle &= \sum_{n} \Lambda^{2n} 2^{-2n} \left\langle\Delta, \mu_1, n \,\middle|\, \Delta, \mu_2, n\right\rangle, \end{aligned}$$

or in terms of the 3-point conformal block:[1]

$$\left\langle \Delta, \Lambda^2 \right| V_{\Delta_2}(1) \left|\Delta_1\right\rangle = \sum_n \Lambda^{2n} \left\langle \Delta, n \right| V_{\Delta_2}(1) \left|\Delta_1\right\rangle , \tag{2.3}$$

$$\left\langle \Delta, \mu_3, \tfrac{1}{2}\Lambda \right| V_{\Delta_2}(1) \left|\Delta_1\right\rangle = \sum_n \Lambda^{2n} 2^{-2n} \left\langle \Delta, \mu_3, n \right| V_{\Delta_2}(1) \left|\Delta_1\right\rangle ,$$

where the conformal weights are related to μ_i parameters by

$$\Delta_i = \frac{1}{4}\left(Q^2 - \lambda_i^2\right) , \qquad 2\mu_1 = \lambda_2 + \lambda_1 , \qquad 2\mu_2 = \lambda_2 - \lambda_1 . \tag{2.4}$$

It was shown in [10] that all irregular blocks above can be obtained by appropriate decoupling limits of the 4-point conformal block on the sphere:

$$\mathcal{B}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x) = \sum_{n=0} x^n \sum_{|J|=|K|=n} \left\langle \Delta_4 \right| V_{\Delta_3}(1) L_{-J} \left|\Delta\right\rangle \left[B^n_{c,\Delta}\right]^{J,K} \left\langle \Delta \right| L_K V_{\Delta_2}(1) \left|\Delta_1\right\rangle .$$

If

$$2\mu_3 = \lambda_3 - \lambda_4 , \quad 2\mu_4 = \lambda_3 + \lambda_4 , \tag{2.5}$$

then [10]:

$$\mathcal{B}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x) \quad \underset{\mu_4 x = \Lambda}{\xrightarrow{\mu_4 \to \infty}} \quad \left\langle \Delta, \mu_3, \tfrac{1}{2}\Lambda \right| V_{\Delta_2}(1) \left|\Delta_1\right\rangle , \tag{2.6}$$

$$\mathcal{B}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x) \quad \underset{\mu_3\mu_4 x = \Lambda^2}{\xrightarrow{\mu_3, \mu_4 \to \infty}} \quad \left\langle \Delta, \Lambda^2 \right| V_{\Delta_2}(1) \left|\Delta_1\right\rangle , \tag{2.7}$$

$$\mathcal{B}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x) \quad \underset{\mu_1\mu_4 x = \Lambda^2}{\xrightarrow{\mu_1, \mu_4 \to \infty}} \quad \left\langle \Delta, \mu_2, \tfrac{1}{2}\Lambda \,\middle|\, \Delta, \mu_3, \tfrac{1}{2}\Lambda \right\rangle , \tag{2.8}$$

$$\mathcal{B}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x) \quad \underset{\mu_1\mu_2\mu_4 x = \Lambda^3}{\xrightarrow{\mu_1, \mu_2, \mu_4 \to \infty}} \quad \left\langle \Delta, \mu_3, \tfrac{1}{2}\Lambda \,\middle|\, \Delta, \Lambda^2 \right\rangle , \tag{2.9}$$

$$\mathcal{B}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x) \quad \underset{\mu_1\mu_2\mu_3\mu_4 x = \Lambda^4}{\xrightarrow{\mu_1, \mu_2, \mu_3, \mu_4 \to \infty}} \quad \left\langle \Delta, \Lambda^2 \,\middle|\, \Delta, \Lambda^2 \right\rangle . \tag{2.10}$$

As it was demonstrated in [38] the recursive relations for the irregular blocks can be derived by analyzing decoupling limits of Zamolodchikov's recursive formula for the elliptic 4-point block [53–55]

$$\mathcal{H}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](\,q) = 1 + \sum_{n=1}^{\infty} (16q)^{\,n} H^n_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] \tag{2.11}$$

defined by

$$\mathcal{B}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x) = \left(\frac{x}{16q}\right)^{\frac{\lambda^2}{4}} (1-x)^{\frac{Q^2}{4}-\Delta_1-\Delta_3} [\theta_3(q)]^{3Q^2-4(\Delta_1+\Delta_2+\Delta_3+\Delta_4)} \mathcal{H}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](q) \tag{2.12}$$

where

$$\theta_3(q) = \sum_{-\infty}^{\infty} q^{n^2} , \qquad q(x) = \mathrm{e}^{-\pi \frac{K(1-x)}{K(x)}} , \qquad K(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}} .$$

[1] The present notation for 3-point conformal block is related to that used in [41, 56] by $\langle \xi' | V_\Delta(1) | \xi'' \rangle = \rho(\xi', \nu_\Delta, \xi'')$. The normalization condition takes the form $\langle \Delta' | V_\Delta(1) | \Delta'' \rangle = 1$.

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The coefficients in (2.11) are uniquely determined by Zomolodchikov's recursive formula:

$$H^n_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] = \delta_{n,0} + \sum_{1\leqslant rs\leqslant n} \frac{A_{rs}\prod\limits_{i=1}^{4} Y_{rs}(\mu_i)}{\Delta-\Delta_{rs}}\, H^{n-rs}_{\Delta_{rs}+rs}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right], \tag{2.13}$$

where μ_i are defined by (2.4), (2.5) and

$$\Delta_{rs} = \frac{Q^2}{4} - \frac{1}{4}\left(rb+sb^{-1}\right)^2, \tag{2.14}$$

$$A_{rs} = \frac{1}{2}\prod_{\substack{p=1-r\\(p,q)\neq}}^{r}\ \prod_{\substack{q=1-s\\(0,0),(r,s)}}^{s} \frac{1}{pb+qb^{-1}}, \tag{2.15}$$

$$Y_{rs}(\mu) = \prod_{\substack{p=1-r\\ p+r=1 \bmod 2}}^{r-1}\ \prod_{\substack{q=1-s\\ q+s=1 \bmod 2}}^{s-1}\left(\mu - \frac{pb+qb^{-1}}{2}\right). \tag{2.16}$$

In the limits $\mu_4\to\infty,\ \mu_4 x=\Lambda$ and $\mu_3\mu_4\to\infty,\ \mu_3\mu_4 x=\Lambda^2$ one has

$$16\mu_4 q(x) \longrightarrow \mu_4 x = \Lambda\,, \qquad 16\mu_3\mu_4 q(x) \longrightarrow \mu_3\mu_4 x = \Lambda^2\,,$$

and the limits of (2.12) take the form

$$\left\langle \Delta,\mu_3,\tfrac{1}{2}\Lambda\right| V_{\Delta_2}(1)\left|\Delta_1\right\rangle = \exp\left(-\tfrac{1}{64}\Lambda^2 - \tfrac{1}{2}\mu_3\Lambda\right)\left(1+\sum_{n=1}^{\infty} H_n(\Delta,\mu_1,\mu_2,\mu_3)\Lambda^n\right),$$

$$\left\langle \Delta,\Lambda^2\right| V_{\Delta_2}(1)\left|\Delta_1\right\rangle = \exp\left(-\tfrac{1}{2}\Lambda^2\right)\left(1+\sum_{n=1}^{\infty} H_n(\Delta,\mu_1,\mu_2)\Lambda^{2n}\right).$$

Since $\lim\limits_{\mu\to\infty}\mu^{-rs}Y_{rs}(\mu) = 1$ the coefficients $H_n(\Delta,\mu_1,\mu_2,\mu_3)$ and $H_n(\Delta,\mu_1,\mu_2)$ satisfy the recursive relations:

$$H_n(\Delta,\mu_1,\mu_2,\mu_3) = \delta^n_0 + \sum_{1\leqslant rs\leqslant n}\frac{A_{rs}Y_{rs}(\mu_1)Y_{rs}(\mu_2)Y_{rs}(\mu_3)}{\Delta-\Delta_{rs}} H_{n-rs}(\Delta_{rs}+rs,\mu_1,\mu_2,\mu_3)\,,$$

$$H_n(\Delta,\mu_1,\mu_2) = \delta^n_0 + \sum_{1\leqslant rs\leqslant n}\frac{A_{rs}Y_{rs}(\mu_1)Y_{rs}(\mu_2)}{\Delta-\Delta_{rs}}\, H_{n-rs}(\Delta_{rs}+rs,\mu_1,\mu_2)\,. \tag{2.17}$$

In the other cases (2.8), (2.9), (2.10) the limit of the prefactor in (2.12) is simply 1 and one gets

$$\langle \Delta,\mu_1,n\,|\,\Delta,\mu_2,n\rangle = \delta^n_0 + \sum_{1\leqslant rs\leqslant n}\frac{A_{rs}Y_{rs}(\mu_1)Y_{rs}(\mu_2)}{\Delta-\Delta_{rs}}\,\langle \Delta_{rs}+rs,\mu_1,n-rs\,|\,\Delta_{rs}+rs,\mu_2,n-rs\rangle\,, \tag{2.18}$$

$$\langle \Delta,\mu,n\,|\,\Delta,n\rangle = \delta^n_0 + \sum_{1\leqslant rs\leqslant n}\frac{A_{rs}Y_{rs}(\mu)}{\Delta-\Delta_{rs}}\,\langle \Delta_{rs}+rs,\mu,n-rs\,|\,\Delta_{rs}+rs,n-rs\rangle\,, \tag{2.19}$$

$$\langle \Delta,n\,|\,\Delta,n\rangle = \delta^n_0 + \sum_{1\leqslant rs\leqslant n}\frac{A_{rs}}{\Delta-\Delta_{rs}}\,\langle \Delta_{rs}+rs,n-rs\,|\,\Delta_{rs}+rs,n-rs\rangle\,. \tag{2.20}$$

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Comparing (2.17) and (2.18) one obtains the equivalence of two different realizations of the irregular block with two μ parameters proposed in [9]:

$$\langle \Delta, \Lambda^2 \big| V_{\Delta_2}(1) \, |\Delta_1\rangle = \mathrm{e}^{-\frac{\Lambda^2}{2}} \left\langle \Delta, \mu_1, \tfrac{1}{2}\Lambda \,\middle|\, \Delta, \mu_2, \tfrac{1}{2}\Lambda \right\rangle .$$

This completes the derivation of the recursive relations required for the proof of the AGT conjecture.

We close this section by some remarks on Y_{rs} polynomials. They show up in the derivation of the Zamolodchikov recursive relation in the factorization formula for the fusion polynomial [56][2]

$$\langle \Delta_{rs} | \, O^{\dagger}_{rs} V_{\Delta_2}(1) \, |\Delta_1\rangle = (-1)^{rs} Y_{rs}(\mu_1) Y_{rs}(\mu_2) \tag{2.21}$$

where as before the relation between parameters is given by (2.4) and O_{rs} denotes the combination of the Virasoro algebra generators creating the singular state of level rs out of the degenerate vacuum $|\Delta_{rs}\rangle$ normalized by the condition that the coefficient in front of $L_{-1}^{rs} \, |\Delta_{rs}\rangle$ is equal 1. Let us note that factorization formula (2.21) is a direct consequence of the null vector decoupling theorem [57].

Another interpretation of Y_{rs} can be obtained by the derivation of the recursive formulae for irregular blocks directly from their expressions in terms of the inverse Gram matrix. Let us consider the block

$$\left\langle \Delta, \mu, \tfrac{1}{2}\Lambda \,|\, \Delta, \Lambda^2 \right\rangle = \sum_n \Lambda^{3n} 2^{-n} \sum_{p=0}^{n/2} (-1)^p (2\mu)^{n-2p} \left[B^n_{c,\Delta} \right]^{[1^{n-2p},2^p],[1^n]} .$$

In the generic case the only singularities of $\left[B^n_{c,\Delta} \right]^{M,N}$ as a function of Δ are simple poles at zeros of the Kac determinant $\Delta_{rs} = \frac{Q^2}{4} - \frac{1}{4}\left(rb + sb^{-1}\right)^2$, $r \geqslant 1, \ s \geqslant 1, n \geqslant rs \geqslant 1$. Since the degree of each minor of the Gram matrix as a function of Δ is strictly lower than the degree of the Kac determinant itself there are no regular terms in the expansion:

$$\langle \Delta, \mu, n | \Delta, n\rangle = \delta_{n,0} + \sum_{1 \leq rs \leqslant n} \frac{\mathcal{R}^{\mu}_{rs,n}}{\Delta - \Delta_{rs}} \tag{2.22}$$

except $n = 0$. For the residue calculation it is convenient to choose a specific basis in each subspace $\mathcal{V}^n_{c,\Delta} \subset \mathcal{V}_{c,\Delta}$ of level $n \geqslant rs$ formed by vectors:

$$L_{-K} O_{rs} \, |\Delta\rangle \, , \quad |K| = n - rs \, ,$$

[2]In ref. [56] the case of $N = 1$ SCFT is considered but the reasoning is the CFT case is essentialy the same.

and by an arbitrary basis in the orthogonal complement of Span$\{L_{-K}O_{rs}\,|\Delta\rangle\}$. Due to the singular behavior of the Gram matrix in the limit $\Delta \to \Delta_{rs}$ [41, 56] one has at the residue

$$
\begin{aligned}
&\lim_{\Delta\to\Delta_{rs}} (\Delta - \Delta_{rs}) \left\langle \Delta, \mu, n \,\middle|\, \Delta, n \right\rangle = \\
&= \lim_{\Delta\to\Delta_{rs}} (\Delta - \Delta_{rs}) \sum_{|K|=|M|=n-rs} \langle \Delta, \mu, n|\, L_{-K}O_{rs}\,|\Delta\rangle \left[G^{n-rs}_{c,\Delta}\right]^{K,M} \langle\Delta|\, O^{\dagger}_{rs}L_M\,|\Delta, n\rangle \\
&= \lim_{\Delta\to\Delta_{rs}} (\Delta - \Delta_{rs}) \sum_{|K|=n-rs} \sum_{|J|=n} \sum_{p=0}^{\frac{n}{2}} \\
&\qquad (2\mu)^{n-2p}\,(-1)^p 2^{-n} \left[G^{n}_{c,\Delta}\right]^{J,[1^{n-2p},2^p]} \langle\Delta|\, L_J L_{-K} O_{rs}\,|\Delta\rangle \left[G^{n-rs}_{c,\Delta}\right]^{K,[1^{n-rs}]} \\
&= A_{rs} \sum_{p=0}^{\frac{n}{2}} \sum_{q=0}^{\frac{n-rs}{2}} (2\mu)^{n-2p}\,(-1)^p 2^{-n}\, e^{[1^{n-2p},2^p]}_{[1^{n-rs-2q},2^q]} \left[G^{n-rs}_{c,\Delta_{rs}+rs}\right]^{[1^{n-rs-2q},2^q],[1^{n-rs}]}
\end{aligned}
\tag{2.23}
$$

where

$$
A_{rs} \;=\; \lim_{\Delta\to\Delta_{rs}} \left(\frac{\langle\Delta|\, O^{\dagger}_{rs} O_{rs}\,|\Delta\rangle}{\Delta - \Delta_{rs}(c)} \right)^{-1}
$$

and e^I_K are the coefficients of the state $L_{-K}O_{rs}\,|\Delta_{rs}\rangle$ in the standard basis of $\mathcal{V}^{|K|+rs}_{\Delta_{rs}}$:

$$
L_{-K}O_{rs}\,|\Delta_{rs}\rangle = \sum_{|I|=|K|+rs} e^I_K L_{-I}\,|\Delta_{rs}\rangle\,.
$$

For our normalization of $O_{rs}\,|\Delta_{rs}\rangle$ the exact form (2.15) of the coefficient A_{rs} was first proposed by Al. Zamolodchikov in [53] and then justified in [58]. In formula (2.23) only the coefficients corresponding to the states generated from $|\Delta_{rs}\rangle$ by the operators L_{-2}, L_{-1} are present. Let us define such coefficients for the singular state:

$$
O_{rs}\,|\Delta_{rs}\rangle = \sum_{k=0}^{\frac{rs}{2}} c^{(rs)}_k\, L^{rs-2k}_{-1} L^k_{-2}\,|\Delta_{rs}\rangle + \ldots
$$

Then for an arbitrary q, the sum over p in (2.23) is given in terms of $c^{(rs)}_p$ and μ:

$$
\begin{aligned}
X_{rs}(\mu) \;&=\; 2^{-rs} \sum_{p=q}^{\frac{rs}{2}+q} (2\mu)^{rs-2(p-q)}\,(-1)^{p-q}\, e^{[1^{n-2p},2^p]}_{[1^{n-rs-2q},2^q]} \\
&=\; 2^{-rs} \sum_{p=0}^{\frac{rs}{2}} (2\mu)^{rs-2p}\,(-1)^p\, c^{(rs)}_p.
\end{aligned}
$$

This yields the factorization formula for the residues $\mathcal{R}^{\mu}_{rs,n}$ in (2.22):

$$
\lim_{\Delta\to\Delta_{rs}} (\Delta - \Delta_{rs}) \left\langle \Delta, \mu, n \,\middle|\, \Delta, n \right\rangle \;=\; A_{rs} X_{rs}(\mu) \left\langle \Delta_{rs} + rs, \mu, n - rs \,\middle|\, \Delta_{rs} + rs, n - rs \right\rangle.
$$

Expansion (2.22) and the formula above yield the recursive relation:

$$\left\langle \Delta,\mu,n \middle| \Delta,n \right\rangle = \delta_0^n \sum_{1\leqslant rs\leqslant n} \frac{A_{rs}X_{rs}(\mu)}{\Delta-\Delta_{rs}} \left\langle \Delta_{rs}+rs,\mu,n-rs \middle| \Delta_{rs}+rs,n-rs \right\rangle .$$

Comparing with (2.19) one gets $X_{rs}(\mu)=Y_{rs}(\mu)$ which provides the factorization formula for the X_{rs} polynomial:

$$2^{-rs}\sum_{p=0}^{\frac{rs}{2}}(2\mu)^{rs-2p}\,(-1)^p\,c_p^{(rs)} \;=\; 2^{-rs}\prod_{\substack{k=1-r\\ k+r=1\,\mathrm{mod}\,2}}^{r-1}\;\prod_{\substack{l=1-s\\ l+s=1\,\mathrm{mod}\,2}}^{s-1}(2\mu-kb-lb^{-1}) \tag{2.24}$$

and implies an unexpected (from the point of view of the original definition of X_{rs}) relation with the fusion polynomial (2.21):

$$\langle\Delta_{rs}|\,O_{rs}^\dagger V_{\Delta_2}(1)\,|\Delta_1\rangle = (-1)^{rs}X_{rs}(\mu_1)X_{rs}(\mu_2)\ .$$

3 Recursive relations for the Nekrasov partition functions

We shall discuss the instanton contribution to the Nekrasov partition function of the $\mathcal{N}=2$ supersymmetric, U(2) gauge theory with N_f hypermultiplets in the antifundamental representation [27]. It can be written as a sum over pairs of Young diagrams,

$$Z^{N_f}(\mathsf{p}_\alpha,\mu_f,b,\hbar;q) \;=\; 1+\sum_{N=1}^{\infty} Z_N^{N_f}(\mathsf{p}_1,\mathsf{p}_2,\mu_f,b)\left(q\hbar^{N_f-4}\right)^N , \tag{3.1}$$

$$Z_N^{N_f}(\mathsf{p}_1,\mathsf{p}_2,\mu_f,b) \;=\; \sum_{|\vec{Y}|=N} Z^{N_f}(\mathsf{p}_\alpha,\mu_f,b;\vec{Y}) , \tag{3.2}$$

where[3] $\hbar\mathsf{p}_\alpha = a_\alpha,\ \alpha=1,2$ are the vev-s of the scalar component of the $\mathcal{N}=2$ gauge supermultiplet, $\hbar b=\epsilon_1,\ \hbar b^{-1}=\epsilon_2$ are the parameters of the Ω background, $Q=b+b^{-1}$, $\hbar\left(\mu_f+\frac{Q}{2}\right)=m_f,\ f=1,\dots N_f$ are the mass parameters of the hypermultiplets and $|\vec{Y}|$ denotes the total number of boxes in the pair of Young diagrams $\vec{Y}=(Y_1,Y_2)$.

The contribution to the partition function parameterized by a specific pair of Young diagrams is of the form [59]

$$Z^{N_f}(\mathsf{p}_\alpha,\mu_f,b;\vec{Y})=\left(\prod_{\alpha=1}^{2}\prod_{\langle m,n\rangle\in Y_\alpha} S_\alpha(\langle m,n\rangle)\right)\left(\prod_{\alpha,\beta=1}^{2}\prod_{\langle m,n\rangle\in Y_\alpha}\frac{1}{E_{\alpha\beta}(\langle m,n\rangle)\big(Q-E_{\alpha\beta}(\langle m,n\rangle)\big)}\right) \tag{3.3}$$

where

$$S_\alpha(\langle m,n\rangle) \;=\; \prod_{f=1}^{N_f}\left(\mathsf{p}_\alpha+(m-1)b+(n-1)b^{-1}+\mu_f+\tfrac{Q}{2}\right),$$

$$E_{\alpha\beta}(\langle m,n\rangle) \;=\; \mathsf{p}_\alpha-\mathsf{p}_\beta-bH_{Y_\beta}(\langle m,n\rangle)+b^{-1}\left(V_{Y_\alpha}(\langle m,n\rangle)+1\right).$$

[3] Our notation is close to the one in [52].

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The $N-$box diagram Y can be described by an ordered sequence of natural numbers $k_1 \geq k_2 \ldots \geq k_l > k_{l+1} = 0$ corresponding to the heights of columns of Y. The vertical distance from the edge of the diagram of the box $\langle m, n\rangle$ situated in the $n-$th row (counted from the lowest one) of the $m-$th column (counted from the left) is equal to

$$V_Y(\langle m, n\rangle) = k_m(Y) - n,$$

while the horizontal distance reads

$$H_Y(\langle m, n\rangle) = k_n(Y^{\mathrm{T}}) - m$$

where Y^{T} denotes the transposed diagram.

The contribution to the partition functions from instantons of topological charge N can be expressed as a contour integral [27]

$$Z_N^{N_f}(\mathsf{p}_1, \mathsf{p}_2, \mu_f, b) = \frac{Q^N}{N!} \oint_{\mathbb{R}} \frac{d\phi_N}{2\pi i} \cdots \oint_{\mathbb{R}} \frac{d\phi_1}{2\pi i} \left(\prod_{k=1}^{N} \frac{\mathcal{Q}_f(\phi_k)}{\mathcal{P}(\phi_k - i0)\mathcal{P}(\phi_k + Q + i0)} \right. \\ \left. \times \prod_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\phi_{ij}(\phi_{ij} - Q)}{(\phi_{ij} - b - i0)(\phi_{ij} - b^{-1} - i0)} \right) \tag{3.4}$$

where $Q = b + b^{-1}$, $\phi_{ij} = \phi_i - \phi_j$ and

$$\mathcal{P}(\phi) = (\phi - \mathsf{p}_1)(\phi - \mathsf{p}_2), \qquad \mathcal{Q}_f(\phi) = \prod_{f=1}^{N_f} \left(\phi + \mu_f + \tfrac{Q}{2}\right).$$

A contribution to the Nekrasov partition function parameterized by a pair of Young diagrams (Y_1, Y_2) corresponds to a specific choice of the integration contours in (3.4) [52] (see also the appendix). If the box $\langle r, s\rangle$ belongs to the diagram Y_α, then for some k the contour of integration over ϕ_k surrounds only the pole at $\mathsf{p}_\alpha + (r-1)b + (s-1)b^{-1}$, yielding the corresponding residue. This pole is present in the integrand if and only if at least $rs - 1$ integrals were already computed and the contributions from all the poles at $\mathsf{p}_\alpha + (m-1)b + (n-1)b^{-1}$ with $m \leq r,\ n \leq s, (m, n) \neq (r, s)$ (one pole per one integral) were taken into account. If we visualize the computation of the contribution corresponding to a pair (Y_1, Y_2) as "building" the Young diagrams by adding subsequent boxes one by one, than one can add a box $\langle r, s\rangle$ only if all the boxes in the rectangular $1 \leq m \leq r,\ 1 \leq n \leq s$ save the right upper corner $\langle m, n\rangle = \langle r, s\rangle$ are already present.

It was demonstrated in [52] that the poles of $Z_N^{N_f}(\mathsf{p}_1, \mathsf{p}_2, \mu_f, b)$ appear solely at $\mathsf{p}_{12} \equiv \mathsf{p}_1 - \mathsf{p}_2 = \mp(rb + sb^{-1})$, with $1 \leq r, s \leq N,\ rs \leq N$. Moreover, these and only these pairs of diagrams which include as a subset of Y_1 the rectangle $1 \leq m \leq r,\ 1 \leq n \leq s$ contribute to the pole at $\mathsf{p}_{12} = -rb - sb^{-1}$ and only those which include the rectangle $1 \leq m \leq r,\ 1 \leq n \leq s$ as a subset of Y_2 contribute to the pole at $\mathsf{p}_{21} = -rb - sb^{-1}$.

Let us calculate the residue of the pole at $\mathsf{p}_{12} = -rb - sb^{-1}$. One has to take into account only those contributions in (3.4) for which rs out of N integrals are evaluated by calculating the residues at $\mathsf{p}_1, \mathsf{p}_1+b, \ldots, \mathsf{p}_1+(r-1)b+(s-1)b^{-1}$. All the other contributions are finite in the limit $\mathsf{p}_{12} = -rb - sb^{-1}$. By a suitable re-labeling of indices (which yields a combinatorial factor $\binom{N}{rs}$) we may denote by ϕ_{mn} the variable of integration in the integral evaluated by taking a residue at $\mathsf{p}_1 + (m-1)b + (n-1)b^{-1}$. If we declare the indices i, j and k, l to satisfy

$$1 \leq i, j \leq rs, \qquad rs < k, l \leq N,$$

then the residue of $Z_N(\mathsf{p}_1, \mathsf{p}_2, \mu_f, b)$ at $\mathsf{p}_{12} = -rb - sb^{-1}$ is equal to

$$\begin{aligned}\text{Res}\, Z_N^{N_f}(\mathsf{p}_1, \mathsf{p}_2, \mu_f, b) = \frac{Q^{N-rs}}{(N-rs)!} \int_{\mathbb{R}} \frac{d\phi_N}{2\pi i} \cdots \int_{\mathbb{R}} \frac{d\phi_{rs+1}}{2\pi i} \prod_{k \neq l} \frac{\phi_{kl}(\phi_{kl} - Q)}{(\phi_{kl} - b - i0)(\phi_{kl} - b^{-1} - i0)} \\ \times \prod_{k=rs+1}^{N} \frac{\mathcal{Q}_f(\phi_k)}{\mathcal{P}(\phi_k)\mathcal{P}(\phi_k + Q)} \frac{Q^{rs}}{(rs)!} K_{rs}\end{aligned}$$

where

$$\begin{aligned}K_{rs} &= \text{Res} \int_{\mathbb{R}} \frac{d\phi_{rs}}{2\pi i} \cdots \int_{\mathbb{R}} \frac{d\phi_1}{2\pi i} \prod_{k,i} \frac{\phi_{ki}^2(\phi_{ki}^2 - Q^2)}{(\phi_{ki}^2 - (b-i0)^2)(\phi_{ki}^2 - (b-i0)^{-2})} \\ &\quad \times \prod_{i=1}^{rs} \frac{\mathcal{Q}_f(\phi_i)}{\mathcal{P}(\phi_i - i0)\mathcal{P}(\phi_i + Q + i0)} \prod_{i \neq j} \frac{\phi_{ij}(\phi_{ij} - Q)}{(\phi_{ij} - b - i0)(\phi_{ij} - b^{-1} - i0)} \\ &= \text{Res}\, Z_N^{N_f}(\mathsf{p}_1, \mathsf{p}_2, \mu_f, b, \vec{Y}_{r,s}) \prod_{k=rs+1}^{N} \prod_{m=1}^{r} \prod_{n=1}^{s} \frac{(\phi_k - x_{mn})^2((\phi_k - x_{ms})^2 - Q^2)}{((\phi_k - x_{ms})^2 - b^2)((\phi_k - x_{ms})^2 - b^{-2})}.\end{aligned}$$

In the last line, $Z_N^{N_f}(\mathsf{p}_1, \mathsf{p}_2, \mu_f, b, \vec{Y}_{rs})$ is a contribution to the partition function corresponding to the pair $\vec{Y}_{rs}$ such that Y_1 is a $r \times s$ rectangle and $Y_2 = \emptyset$, while $x_{mn} = \mathsf{p}_1 + (m-1)b + (n-1)b^{-1}$.

After some simple algebra we get

$$\begin{aligned}&\frac{1}{\mathcal{P}(\phi_k)\mathcal{P}(\phi_k + Q)} \prod_{m=1}^{r} \prod_{n=1}^{s} \frac{(\phi_k - x_{mn})^2((\phi_k - x_{mn})^2 - Q^2)}{((\phi_k - x_{mn})^2 - b^2)((\phi_k - x_{mn})^2 - b^{-2})} \\ &\quad = \frac{1}{(\phi_k - \mathsf{p}_1 - rb)(\phi_k + Q - \mathsf{p}_1 - rb)(\phi_k - \mathsf{p}_1 - sb^{-1})(\phi_k + Q - \mathsf{p}_1 - sb^{-1})} \\ &\quad\quad \times \frac{(\phi_k - \mathsf{p}_1 - rb - sb^{-1})(\phi_k + Q - \mathsf{p}_1 - rb - sb^{-1})}{(\phi_k - \mathsf{p}_2)(\phi_k + Q - \mathsf{p}_2)}.\end{aligned} \tag{3.5}$$

For $\mathsf{p}_{12} = -rb - sb^{-1}$ the factor in the last line of (3.5) is equal to 1 and

$$\begin{aligned}&\frac{1}{\mathcal{P}(\phi_k)\mathcal{P}(\phi_k + Q)} \prod_{m=1}^{r} \prod_{n=1}^{s} \frac{(\phi_k - x_{mn})^2((\phi_k - x_{mn})^2 - Q^2)}{((\phi_k - x_{mn})^2 - b^2)((\phi_k - x_{mn})^2 - b^{-2})}\Big|_{\mathsf{p}_{12} = -rb - sb^{-1}} \\ &\quad = \frac{1}{\widetilde{\mathcal{P}}(\phi_k)\widetilde{\mathcal{P}}(\phi_k + Q)}\end{aligned} \tag{3.6}$$

with

$$\widetilde{\mathcal{P}}(\phi) = (\phi - \mathsf{p}_1 - rb)(\phi - \mathsf{p}_1 - sb^{-1}). \tag{3.7}$$

It leads to the relation

$$\operatorname{Res} Z_N^{N_f}(\mathsf{p}_1, \mathsf{p}_2, \mu_f, b) = \operatorname{Res} Z^{N_f}(\mathsf{p}_1, \mathsf{p}_2, \mu_f, b; \vec{Y}_{rs})\, Z_{N-rs}^{N_f}(\mathsf{p}_1 + rb, \mathsf{p}_1 + sb^{-1}, \mu_f, b). \tag{3.8}$$

For the pair of diagrams $\vec{Y}_{rs} = (r \times s, \emptyset)$ one has

$$\begin{aligned} H_{Y_1}(\langle m,n\rangle) &= r - m, \qquad & H_{Y_2}(\langle m,n\rangle) &= -m, \\ V_{Y_1}(\langle m,n\rangle) &= s - n, \qquad & V_{Y_2}(\langle m,n\rangle) &= -n, \end{aligned}$$

and

$$\begin{aligned} E_{11}(\langle m,n\rangle) &= (m-r)b + (s-n+1)b^{-1}, \\ E_{12}(\langle m,n\rangle) &= \mathsf{p}_{12} + mb + (s-n+1)b^{-1}. \end{aligned}$$

The formula (3.3) thus gives

$$\begin{aligned} \operatorname{Res} \prod_{\alpha,\beta=1}^{2} \prod_{\langle m,n\rangle \in Y_i} \frac{1}{E_{\alpha\beta}(\langle m,n\rangle)(Q - E_\alpha(\langle m,n\rangle))}\Bigg|_{\mathsf{p}_{12}=-rb-sb^{-1}} &= \\ = \prod_{m=1-r}^{r} \prod_{n=1-s}^{s} \frac{1}{mb + nb^{-1}}, \qquad \langle m,n\rangle \neq \langle 0,0\rangle. \end{aligned} \tag{3.9}$$

Let us now assume

$$\mathsf{p}_1 = -\mathsf{p}_2 = \mathsf{p}$$

which corresponds to SU(2) rather than U(2) gauge group. The pole at $\mathsf{p}_{12} = -rb - sb^{-1}$ then corresponds to

$$\mathsf{p} = -\frac{1}{2}\left(rb + sb^{-1}\right).$$

For the pair of diagrams $\vec{Y}_{rs} = (r \times s, \emptyset)$ we thus have

$$\begin{aligned} &\prod_{\alpha=1}^{2} \prod_{\langle m,n\rangle \in Y_\alpha} S_\alpha(\langle m,n\rangle)\Big|_{\mathsf{p}=-\frac{rb+sb^{-1}}{2}} \\ &\qquad = \prod_{f=1}^{N_f} \prod_{m=1}^{r} \prod_{n=1}^{s} \left[\left(m - 1 - \tfrac{1}{2}r\right) b + \left(n - 1 - \tfrac{1}{2}s\right) b^{-1} + \mu_f + \tfrac{Q}{2}\right] \\ &\qquad = \prod_{f=1}^{N_f} Y_{rs}(\mu_f) \end{aligned}$$

where Y_{rs} are defined by (2.16). Since $Z_N(\mathsf{p}_1, \mathsf{p}_2, \mu_f, b)$ is a symmetric function of $\mathsf{p}_1 - \mathsf{p}_2 = 2\mathsf{p}$

$$Z_N(\mathsf{p}, \mu_f, b) \equiv Z_N(\mathsf{p}, -\mathsf{p}, \mu_f, b) = Z_N(-\mathsf{p}, \mu_f, b),$$

the residues at the poles at $\mathsf{p}_{12} = -rb - sb^{-1}$ and $\mathsf{p}_{21} = -rb - sb^{-1}$ differ only by a sign. Equations (3.8) and (3.9) then yield for $4\mathsf{p}^2 \to \left(rb + sb^{-1}\right)^2$

$$Z_N^{N_f}(\mathsf{p}, \mu_f, b) \;=\; -\frac{4A_{rs} \prod_{f=1}^{N_f} Y_{rs}(\mu_f)}{4\mathsf{p}^2 - \left(rb + sb^{-1}\right)^2}\; Z_{N-rs}^{N_f}\big(\tfrac{1}{2}(rb - sb^{-1}), \mu_f, b\big) + \mathcal{O}(1) \tag{3.10}$$

where A_{rs} is given by formula (2.15). For $\mathsf{p} \to \infty$ relation (3.3) implies

$$Z_N^{N_f}(\mathsf{p}, \mu_f, b) = \mathcal{O}\left(\mathsf{p}^{2N(N_f-2)}\right),$$

hence

$$\lim_{p\to\infty} Z_N^{N_f}(\mathsf{p}, \mu_f, b; q) = \delta_{0,N} \quad \text{for} \quad N_f = 0, 1.$$

Together with (3.10) this gives the recursion relations:

$$\begin{aligned}
Z_N^0(\mathsf{p}, b) \;&=\; \delta_{0,N} - \sum_{1\leqslant rs\leqslant N} \frac{4A_{rs}}{4\mathsf{p}^2 - \left(rb + sb^{-1}\right)^2}\; Z_{N-rs}^0\left(\tfrac{1}{2}(rb - sb^{-1}), b\right),\\
Z_N^1(\mathsf{p}, \mu, b) \;&=\; \delta_{0,N} - \sum_{1\leqslant rs\leqslant N} \frac{4A_{rs}\, Y_{rs}(\mu)}{4\mathsf{p}^2 - \left(rb + sb^{-1}\right)^2}\; Z_{N-rs}^1\left(\tfrac{1}{2}(rb - sb^{-1}), b\right).
\end{aligned}$$

Regarding the partition function as a function of the conformal dimension $\Delta = \frac{1}{4}Q^2 - \mathsf{p}^2$ rather than the function of p, one gets the recursion formulae of the form:

$$Z_N^0(\Delta, b) \;=\; \delta_{0,N} + \sum_{1\leqslant rs\leqslant N} \frac{A_{rs}}{\Delta - \Delta_{rs}} Z_{N-rs}^0\left(\Delta_{rs} + rs, b\right), \tag{3.11}$$

$$Z_N^1(\Delta, \mu, b) \;=\; \delta_{0,N} + \sum_{1\leqslant rs\leqslant N} \frac{A_{rs}\, Y_{rs}(\mu)}{\Delta - \Delta_{rs}} Z_{N-rs}^1\left(\Delta_{rs} + rs, \mu, b\right), \tag{3.12}$$

where Δ_{rs} is given by relation (2.14). Comparing with (2.20) and (2.19) one gets the AGT relation for $N_f = 0, 1$:

$$\begin{aligned}
Z^0(\Delta, b, \hbar;\, \hbar^4\Lambda^4) \;&=\; \left\langle \Delta, \Lambda^2 \middle| \Delta, \Lambda^2 \right\rangle,\\
Z^1(\Delta, \mu, b, \hbar;\, \hbar^3\Lambda^3) \;&=\; \left\langle \Delta, \mu, \tfrac{1}{2}\Lambda \,\middle|\, \Delta, \Lambda^2 \right\rangle.
\end{aligned}$$

Calculating the large p asymptotic in the case $N_f = 2$ is more involved. Let us first note that for $\alpha \neq \beta$ and $\mathsf{p} = \mathsf{p}_\alpha = -\mathsf{p}_\beta \to \infty$:

$$\frac{S_\alpha(\langle m, n\rangle)}{E_{\alpha\beta}(\langle m, n\rangle)\big(Q - E_{\alpha\beta}(\langle m, n\rangle)\big)} \;=\; \frac{\mathsf{p}_\alpha^2}{-(\mathsf{p}_\alpha - \mathsf{p}_\beta)^2} + \mathcal{O}(\mathsf{p}^{-1}) \qquad \to -\frac{1}{4},$$

hence, for $|\vec{Y}| = N$

$$\prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{2} \prod_{\langle m,n\rangle \in Y_1} \frac{S_\alpha(\langle m, n\rangle)}{E_{\alpha\beta}(\langle m, n\rangle)\big(Q - E_{\alpha\beta}(\langle m, n\rangle)\big)} \qquad \to \left(-\frac{1}{4}\right)^N$$

and

$$\lim_{\mathsf{p}\to\infty} Z^2(\mathsf{p},\mu_1,\mu_2,b,\hbar;q)=\sum_{N=0}^{\infty}\Big(-\frac{q}{4\hbar^2}\Big)^N\sum_{|\vec{Y}|=N}\prod_{\alpha=1}^{2}\prod_{\langle m,n\rangle\in Y_\alpha}\frac{1}{E_{\alpha\alpha}(\langle m,n\rangle)\big(Q-E_{\alpha\alpha}(\langle m,n\rangle)\big)}$$

$$=\left(\sum_{N=0}^{\infty}\Big(-\frac{q}{4\hbar^2}\Big)^N\sum_{|Y|=N}\prod_{\langle m,n\rangle\in Y}\frac{1}{E_Y(\langle m,n\rangle)\big(Q-E_Y(\langle m,n\rangle)\big)}\right)^2,$$

where

$$E_Y(\langle m,n\rangle)=-bH_Y(\langle m,n\rangle)+b^{-1}(1+V_Y(\langle m,n\rangle)).$$

In order to calculate the sum

$$\mathcal{Z}_N(b)=\sum_{|Y|=N}\prod_{\langle m,n\rangle\in Y}\frac{1}{E_Y(\langle m,n\rangle)\big(Q-E_Y(\langle m,n\rangle)\big)} \tag{3.13}$$

we shall use the integral representation

$$\mathcal{Z}_N(b)=\frac{Q^N}{N!}\oint_{\mathbb{R}}\frac{d\phi_N}{2\pi i}\dots\oint_{\mathbb{R}}\frac{d\phi_1}{2\pi i}\prod_{k=1}^{N}\frac{1}{(\phi_k-i0)(\phi_k+Q+i0)}\prod_{\substack{i,j=1\\ i\neq j}}^{N}\frac{\phi_{ij}\big(\phi_{ij}-Q\big)}{\big(\phi_{ij}-b-i0\big)\big(\phi_{ij}-b^{-1}-i0\big)}. \tag{3.14}$$

The relation (3.14) can be obtained using the results of [61] and [62] (formulae 6.4-6.12 in [61]). Applying the reasoning presented in appendix to the integral in (3.14) one can show that the only singularities of $\mathcal{Z}_N(b)$ result from the collision of the poles at $\phi_k=(r-1)b+(s-1)b^{-1}$ (above the real axis) with the pole at $\phi_k=-Q$ (below the real axis), i.e. for

$$rb+sb^{-1}=0.$$

Since r and s are positive integers the only singularities of $\mathcal{Z}_N(b)$ can occur for purely imaginary b. However, for $b=i\beta,\ \beta\in\mathbb{R}$,

$$\begin{aligned}&E_Y(\langle m,n\rangle)(Q-E_Y(\langle m,n\rangle)\\ &\qquad=\Big(1+V_Y(\langle m,n\rangle)+\beta^2H_Y(\langle m,n\rangle)\Big)\Big(1+H_Y(\langle m,n\rangle)+\beta^{-2}V_Y(\langle m,n\rangle)\Big)\end{aligned}$$

is strictly positive. Thus the residue at the would-be pole at $rb+sb^{-1}=0$ actually vanishes and $\mathcal{Z}_N(b)$ is a holomorphic function of b. Now, for $b\to\infty$

$$\frac{1}{E_Y(\langle m,n\rangle)\big(Q-E_Y(\langle m,n\rangle)\big)}\to 0$$

unless $H_Y(\langle m,n\rangle)=0$, i.e. Y consists solely of the first column and therefore

$$\lim_{b\to\infty}\sum_{|Y|=N}\prod_{\langle m,n\rangle\in Y}\frac{1}{E_Y(\langle m,n\rangle)\big(Q-E_Y(\langle m,n\rangle)\big)}=\prod_{m=1}^{N}\frac{1}{1+V_Y(\langle m,1\rangle)}=\frac{1}{N!}.$$

The Liouville's boundedness theorem then implies that $\mathcal{Z}_N(b)$ is a b independent constant:[4]

$$\sum_{|Y|=N}\prod_{\langle m,n\rangle\in Y}\frac{1}{E_Y(\langle m,n\rangle)\big(Q-E_Y(\langle m,n\rangle)\big)}=\frac{1}{N!}\,.$$

One thus gets

$$\lim_{\mathsf{p}\to\infty} Z^2(\mathsf{p},\mu_1,\mu_2,b,\hbar;q) = \mathrm{e}^{-\frac{1}{2}\frac{q}{\hbar^2}}$$

which along with (3.10) implies the recursive relation

$$\begin{aligned} Z^2_N(\Delta,\mu_1,\mu_2,b) &= \frac{(-1)^N}{2^N N!} \\ &+ \sum_{1\leqslant rs\leqslant N}\frac{A_{rs}\,Y_{rs}(\mu_1)Y_{rs}(\mu_2)}{\Delta-\Delta_{rs}}Z^2_{N-rs}\left(\Delta_{rs}+rs,\mu_1,\mu_2,b\right). \end{aligned} \tag{3.15}$$

Comparing with recursive relations (2.17) and (2.18) one gets the AGT correspondence for $N_f=2$

$$Z^2(\Delta,\mu_1,\mu_2,b,\hbar;\,\hbar^2\Lambda^2) = \left\langle\Delta,\Lambda^2\right| V_{\Delta_2}(1)\left|\Delta_1\right\rangle = \mathrm{e}^{-\frac{\Lambda^2}{2}}\left\langle\Delta,\mu_1,\tfrac{1}{2}\Lambda\,\middle|\,\Delta,\mu_2,\tfrac{1}{2}\Lambda\right\rangle.$$

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A Singularities of the partition function

In this appendix we shall repeat (a slightly elaborated version of) a proof of the following lemma due to Fateev and Litvinov [52]:

Lemma 1 *Let $f(\phi_1,\ldots,\phi_n)$ be a holomorphic function of all its arguments which grows for $\phi_i\to\infty$ slow enough to ensure the convergence of the integral*

$$F^f_N(\{a_\alpha\};\{\epsilon_a\})=\int\limits_{\mathbb{R}}\frac{d\phi_1}{2\pi i}\cdots\int\limits_{\mathbb{R}}\frac{d\phi_N}{2\pi i}\;f(\phi_1,\ldots,\phi_N)\prod_\alpha\prod_{i=1}^N\frac{1}{\phi_i-a_\alpha-0^+}\;\prod_a\prod_{\substack{i,j=1\\ i\neq j}}^N\frac{1}{\phi_{ij}-\epsilon_a-0^+}\,. \tag{A.1}$$

Then $F^f_N(\{a_\alpha\};\{\epsilon_a\})$ is a holomorphic function of all a_α.

Proof. Let us perform the integral over ϕ_N by closing the contour in the upper half-plane. With this choice the poles at $\phi_N=a_\alpha$ and $\phi_N=\phi_j+\epsilon_a$ are inside the contour, while the poles at $\phi_N=\phi_j-\epsilon_a$ stay outside. When we vary a_α, some of these poles move and, when the positions of two poles coincide, a higher order singularity of the integrand may occur.

If this happens for the poles on the same side of the integration contour, then we can move the contour away from such a colliding pair and the integral stays finite. On the

[4] An independent check of this property can be made by evaluating with a help of [60], Theorem 2.7, the value of $\mathcal{Z}_N(i)$.

other hand, if the colliding poles “squeeze” the integration contour in between then we can deform the contour of integration away only at a price of evaluating a residue (which can become singular in the limit) at one of the poles.

More precisely: take a pole at $\phi_N = \phi_j - \epsilon_b$ and deform the integration contour to the sum of $\mathcal{C}$ and $-\mathcal{C}_{k,b}$, where $\mathcal{C}$ encloses all the poles of ϕ_N in the upper half-plane **and** the pole at $\phi_N = \phi_j - \epsilon_b$, while $\mathcal{C}_{j,b}$ surrounds just $\phi_N = \phi_j - \epsilon_b$. By the argument above a singularity of the integral over ϕ_N may appear only from the integral over $\mathcal{C}_{j,b}$. Adding such contributions for all $j < N$ and all b (and neglecting for the moment all the factors which do not depend on ϕ_N) we get:

$$
\begin{aligned}
\mathcal{I}_N &= \sum_{j,b} \oint_{-\mathcal{C}_{k,b}} \frac{d\phi_N}{2\pi i}\, f(\phi_1,\ldots,\phi_N) \prod_\alpha \frac{1}{\phi_N - a_\alpha - 0^+} \prod_a \prod_{i=1}^{N-1} \frac{1}{\phi_N - \phi_i - \epsilon_a - 0^+}\, \frac{1}{\phi_N - \phi_i - \epsilon_a + 0^+} \\
&= \sum_{j,b} f(\phi_1,\ldots,\phi_j - \epsilon_b) \prod_\alpha \frac{1}{\phi_j - a_\alpha - \epsilon_b - 0^+}\, \frac{1}{2\epsilon_b} \prod_{\substack{a,\, i<N \\ (a,i)\neq(b,j)}} \frac{1}{\phi_{ji} - \epsilon_a - \epsilon_b - 0^+}\, \frac{1}{\phi_{ji} + \epsilon_a - \epsilon_b - 0^+}.
\end{aligned} \tag{A.2}
$$

There are indeed several new poles on the right hand side of this equation. The first type is at $\phi_j = a_\alpha + \epsilon_b$ above the real axis. In order to expose the others let us note that (A.2) is symmetric under $(a,i) \leftrightarrow (b,j)$ so that symmetrizing its right hand side we get

$$
\begin{aligned}
2\mathcal{I}_N = &\sum_{j,b} f(\phi_1,\ldots,\phi_j - \epsilon_b) \prod_\alpha \frac{1}{\phi_j - a_\alpha - \epsilon_b - 0^+}\, \frac{1}{2\epsilon_b} \prod_{\substack{a,i<N \\ (a,i)\neq(b,j)}} \frac{1}{\phi_{ij} + \epsilon_a + \epsilon_b + 0^+}\, \frac{1}{\phi_{ij} - \epsilon_a + \epsilon_b + 0^+} \\
&+ \sum_{i,a} f(\phi_1,\ldots,\phi_i - \epsilon_a) \prod_\alpha \frac{1}{\phi_i - a_\alpha - \epsilon_a - 0^+}\, \frac{1}{2\epsilon_a} \prod_{\substack{b,j<N \\ (b,j)\neq(a,i)}} \frac{1}{\phi_{ij} - \epsilon_a - \epsilon_b - 0^+}\, \frac{1}{\phi_{ij} - \epsilon_a + \epsilon_b - 0^+}.
\end{aligned}
$$

It follows from the expression above that $\mathcal{I}_N$ has poles above the real axis at $\phi_i = \phi_j + \epsilon_a + \epsilon_b$ and poles below the real axis at $\phi_i = \phi_j - \epsilon_a - \epsilon_b$. It seems that there is also a pole at $\phi_i = \phi_j + \epsilon_a - \epsilon_b$. However, the residue of such a “would be” pole contains a factor

$$
\frac{1}{2\epsilon_b}\, \frac{1}{\phi_{ij} + \epsilon_a + \epsilon_b} + \frac{1}{2\epsilon_a}\, \frac{1}{\phi_{ij} - \epsilon_a - \epsilon_b} = \frac{\epsilon_a + \epsilon_b}{2\epsilon_a \epsilon_b}\, \frac{\phi_{ij} - \epsilon_a + \epsilon_b}{\phi_{ij}^2 - (\epsilon_a + \epsilon_b)^2}
$$

and $\mathcal{I}_N$ is actually finite for $\phi_i \to \phi_j + \epsilon_a - \epsilon_b$.

After performing the integral over ϕ_N we arrive at the expression of the same structure as (A.1) with $N-1$ instead of N integrals, the function f replaced with some other holomorphic function of $\phi_1, \ldots \phi_{N-1}$, and the sets $\{a_\alpha\}, \{\epsilon_a\}$ enlarged by adding points $\{a_\alpha + \epsilon_b\}, \{\epsilon_a + \epsilon_b\}$ with all possible ϵ_b. After performing all but the last integral we arrive at the formula

$$
F_N^f\left(\{a_\alpha\};\{\epsilon_a\}\right) = \int_{\mathbb{R}} \frac{d\phi_1}{2\pi i}\, \tilde{f}(\phi_1) \prod_\alpha \prod_A \frac{1}{\phi_1 - a_\alpha - e_A - i0^+} = \prod_\alpha \prod_A \tilde{f}(a_\alpha + e_A),
$$

with $\{e_A\} = \{0, \epsilon_a, \epsilon_a + \epsilon_b, \ldots\}$ and some holomorphic function $\tilde{f}$. The thesis of the lemma follows.

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Let us now suppose that the function f appearing in (A.1) contains a factor

$$\prod_{\nu}\prod_{i=1}^{N}\frac{1}{\phi_i-\xi_\nu+i0^+},$$

with simple poles below the real axis. Adapting the reasoning given in the proof above we see that F_N^f is no longer holomorphic. Its singularities may only come from the collision of poles at $\phi_i = a_\alpha + e_A$ and $\phi_i = \xi_\nu$ with the ϕ_i integration contour squeezed in between and are thus located at

$$a_\alpha = \xi_\nu + e_A.$$

In the situation of interest in section 3 we have $\alpha = 1, 2,\ \ \xi_1 = a_1 - \epsilon$ and $\xi_2 = a_2 - \epsilon$. The poles in the variables ϕ_i can appear only at

$$\phi_i = a_\alpha + (r-1)\epsilon_1 + (s-1)\epsilon_2, \qquad r, s \in \mathbb{Z}_+$$

and the only singularities of the partition function as a function of a_α are the (simple in the generic case) poles at

$$a_1 - a_2 = \mp(m\epsilon_1 + n\epsilon_2).$$

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