

Augmented recursion for one-loop gravity amplitudes

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ABSTRACT: We present a semi-recursive method for calculating the rational parts of one-loop gravity amplitudes which utilises axial gauge diagrams to determine the non-factorising pieces of the amplitude. This method is used to compute the amplitudes $M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$ and $M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+, 6^+)$.

KEYWORDS: Models of Quantum Gravity, NLO Computations

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1 Introduction

On-shell recursive techniques have proven very successful in the computation of scattering amplitudes in gauge theories and in theories of gravity [1–4]. The recursive techniques for tree scattering amplitudes make use of both the rationality of the amplitudes and their complex factorisation properties. Specifically, in a theory with massless states, if we use a spinor helicity representation for the polarisation vectors it is possible to write the amplitude entirely in terms of spinorial variables $A(\lambda_\alpha^i, \bar{\lambda}_{\dot{\alpha}}^i)$ where the massless momentum of the i^{th} particle is $\lambda_\alpha^i \bar{\lambda}_{\dot{\alpha}}^i = (\sigma_\mu)_{\alpha\dot{\alpha}} k_i^\mu$.¹

Within this formalism it is possible to probe the analytic structure of the amplitude by choosing a pair a, b of external momenta and shifting these according to

$$\bar{\lambda}^a \longrightarrow \bar{\lambda}^a - z\bar{\lambda}^b, \quad \lambda^b \longrightarrow \lambda^b + z\lambda^a \tag{1.1}$$

where we suppress the spinor indices. The analytic behaviour of the shifted amplitude $A(z)$ can then be studied.

If $A(z)$

1. is a rational function,
2. has simple poles at points z_i , and
3. vanishes as $z \longrightarrow \infty$

¹We use the usual spinor products $\langle ab \rangle = \epsilon^{\alpha\beta} \lambda_\alpha^a \lambda_\beta^b$, $[ab] = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\lambda}_{\dot{\alpha}}^a \bar{\lambda}_{\dot{\beta}}^b$, which satisfy $\langle ab \rangle [ba] = (k_a + k_b)^2 \equiv s_{ab}$, chains of spinor products such as $[a|b|c] \equiv [ab] \langle bc \rangle$ and $[a|P_{ef\dots}|b] = [a|e|b] + [a|f|b] + \dots$ etc and $t_{abc} \equiv (k_a + k_b + k_c)^2$.

then applying Cauchy's theorem to $A(z)/z$ with a contour at infinity yields

$$A(0) = - \sum_{z_i} \text{Res} \left(\frac{A(z)}{z}, z_i \right). \tag{1.2}$$

This technique has proven very effective in computing tree amplitudes and has been extended from the purely gluonic case to a variety of other applications including that of gravity [3, 4]. Alternate shifts exist [5] which can be used to re-derive the CSW formulation for Yang-Mills [6] and gravity [7].

The result (1.2) holds even if condition 2 above is relaxed to poles of finite order; however this condition allows us to use the factorisation theorems to determine the residues in terms of lower point amplitudes. At tree level the factorisation is relatively simple: amplitudes must factorise on multi-particle and collinear poles. For a partition of the external momenta (S_L, S_R) with at least two momenta on either side, and defining $K^\mu \equiv \sum_{i \in S_L} k_i^\mu$, the n -point tree amplitude A_n^{tree} factorises as K becomes on shell as

$$A_n^{\text{tree}} \xrightarrow{K^2 \rightarrow 0} \sum_{\sigma} \left[A_{r+1}^{\text{tree}}(k_i \in S_L, K^\sigma) \frac{i}{K^2} A_{n-r+1}^{\text{tree}}((-K)^{-\sigma}, k_i \in S_R) \right] \tag{1.3}$$

where σ denotes the internal state of the intermediate particle and r is the length of S_L . Consequently, simple poles in the shifted amplitude $A(z)$ occur at values of z where $K^2(z) = 0$. Only those K 's containing precisely one of k_a or k_b will be z dependent. When the corresponding $K^2(z)$ vanishes the residue will be the product of the tree amplitudes defined at $z = z_i$. Thus we can express the n -point tree amplitude in terms of lower point amplitudes,

$$A_n^{\text{tree}}(0) = \sum_{i, \sigma} A_{r_i+1}^{\text{tree}, \sigma}(z_i) \frac{i}{K^2} A_{n-r_i+1}^{\text{tree}, -\sigma}(z_i), \tag{1.4}$$

where the summation over i is only over factorisations where the a and b legs are on opposite sides of the pole.

Beyond tree level there are three potential barriers to using recursion. Firstly the amplitudes, in general, contain non-rational functions such as logarithms and dilogarithms; secondly, the amplitudes may contain higher-order poles for complex momenta and, finally, the amplitudes may not vanish asymptotically with z . Nonetheless a variety of techniques based upon recursion and unitarity have been developed. A one-loop amplitude for massless particles may be expressed as

$$A^{\text{1-loop}} = \sum_{n=2,3,4; i} c_i I_n^i + R + O(\epsilon) \tag{1.5}$$

where the scalar integral functions I_n^i are the various scalar box, triangle and bubble functions. The function R contains the remaining rational terms. The one-loop amplitude can then be specified by computing the coefficients, c_i , and the purely rational term R . The c_i are rational coefficients which can be computed by various applications of the four-dimensional unitarity technique [8–11] or indeed recursively [12].

There are a variety of strategies for evaluating the rational terms. They may be evaluated using D -dimensional unitarity, by recursion or by specialised Feynman diagram techniques [13–31]. In general, the rational term R does not simply satisfy the previously-stated requisites for recursion 2 and 3. If the amplitude has only simple poles but does not vanish as $z \rightarrow \infty$ then it can be possible to formulate recursion by the use of an auxiliary recursion relation [32, 33]. However there are rational amplitudes for which one cannot find a shift which only generates simple poles such as the one-minus amplitude $A^{1\text{-loop}}(1^-, 2^+, \dots, n^+)$. These amplitudes vanish at tree level and consequently are purely rational at one-loop. A shift on these amplitudes yields double and single poles

$$A \sim \frac{a}{(z - z_i)^2} + \frac{b}{(z - z_i)} + \dots = \frac{a}{(z - z_i)^2} \left(1 + \frac{b}{a}(z - z_i) + \dots \right) \quad (1.6)$$

The double pole is *not* in itself a barrier to using recursion with the double pole contributing

$$- \text{Res} \left(\frac{1}{z(z - z_i)^2}, z_i \right) = \frac{1}{z_i^2}. \quad (1.7)$$

However to obtain the full residue in a recursive construction one must have specific formulae for this double pole and for the coincident single pole, or the ‘pole under the double pole’.

In ref. [34] the form of the pole in Yang-Mills was postulated to be

$$\frac{1}{(K^2)^2} \left(1 + \sum_{a_i, b_i} S(a_1, \hat{K}^+, a_2) K^2 S(b_1, \hat{K}^-, b_2) \right) \quad (1.8)$$

where the ‘soft’ factors are

$$S(a, s^+, b) = \frac{\langle ab \rangle}{\langle as \rangle \langle sb \rangle}, \quad S(a, s^-, b) = \frac{[ab]}{[as] [sb]} \quad (1.9)$$

With this ansatz recursion correctly reproduces the known one-minus one-loop amplitudes. In ref. [35] it was shown that the consistency requirements for recursion in QCD are sufficient to determine these soft factors.

The above postulate, or variations thereof, however does not work for gravity amplitudes [36]. In this article we will demonstrate how to apply recursion techniques in gravity scattering amplitudes by determining the ‘pole under the pole’ using an axial gauge formalism. By only keeping the pole terms it is relatively simple to extract these from the diagrammatic approach. We demonstrate this by calculating the previously-unknown amplitudes $M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$ and $M(1^-, 2^+, 3^+, 4^+, 5^+, 6^+)$. We assume that the shifted amplitudes have vanishing behaviour as $z \rightarrow \infty$. The expressions we derive have the correct symmetries and soft limits, providing strong evidence for the validity of this assumption. Further, we compare the result numerically with a completely independent computation of $M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$ from ‘string-based rules’ for gravity [37–40].

2 Recursion

The factorisation of one-loop massless amplitudes is described in ref. [41],

$$\begin{aligned}
 A_n^{1\text{-loop}} \xrightarrow{K^2 \rightarrow 0} & \sum_{\lambda=\pm} \left[A_{r+1}^{1\text{-loop}}(k_i, \dots, k_{i+r-1}, K^\lambda) \frac{i}{K^2} A_{n-r+1}^{\text{tree}}((-K)^{-\lambda}, k_{i+r}, \dots, k_{i-1}) \right. \\
 & + A_{r+1}^{\text{tree}}(k_i, \dots, k_{i+r-1}, K^\lambda) \frac{i}{K^2} A_{n-r+1}^{1\text{-loop}}((-K)^{-\lambda}, k_{i+r}, \dots, k_{i-1}) \\
 & \left. + A_{r+1}^{\text{tree}}(k_i, \dots, k_{i+r-1}, K^\lambda) \frac{i}{K^2} A_{n-r+1}^{\text{tree}}((-K)^{-\lambda}, k_{i+r}, \dots, k_{i-1}) F_n(K^2; k_1, \dots, k_n) \right],
 \end{aligned} \tag{2.1}$$

where the one-loop ‘factorisation function’ F_n is helicity-independent. Naïvely this only contains single poles, however for complex momenta there are double poles. These can be interpreted as due to the three-point all-plus (or all-minus) one-loop amplitude also containing a pole

$$A_3^{1\text{-loop}}(K^+, a^+, b^+) = \frac{1}{K^2} V^{1\text{-loop}}(K^+, a^+, b^+) \tag{2.2}$$

where, for pure Yang-Mills,

$$V^{1\text{-loop}}(K^+, a^+, b^+) = -\frac{i}{48\pi^2} [K a] [a b] [b K]. \tag{2.3}$$

To see this explicitly, let us consider the five-point Yang-Mills amplitude $A_5^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$ [42]:

$$\begin{aligned}
 A_5^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+) = & \\
 & \frac{i}{48\pi^2} \frac{1}{\langle 34 \rangle^2} \left[-\frac{[25]^3}{[12][51]} + \frac{\langle 14 \rangle^3 [45] \langle 35 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 45 \rangle^2} - \frac{\langle 13 \rangle^3 [32] \langle 42 \rangle}{\langle 15 \rangle \langle 54 \rangle \langle 32 \rangle^2} \right].
 \end{aligned} \tag{2.4}$$

If we carry out a complex shift on legs 1 and 5,

$$\lambda^5 \longrightarrow \lambda^5 + z\lambda^1, \quad \bar{\lambda}^1 \longrightarrow \bar{\lambda}^1 - z\bar{\lambda}^5, \tag{2.5}$$

then $\langle 45 \rangle \longrightarrow \langle 45 \rangle + z\langle 41 \rangle$ which vanishes at $z = -\langle 45 \rangle / \langle 41 \rangle$ and the amplitude has a double pole at this point.²

A recursive approach suggests drawing the diagrams shown in figure 1. The third of these involves the one-loop vertex $V^{1\text{-loop}}(K^+, 4^+, \hat{5}^+)$. Computing with this does correctly generate the double pole in the amplitude [34, 36], however it needs augmentation to give an expression with the correct single pole. By trial and error, adding the second term in (1.8) gives the correct single pole and completes the computation of the amplitude.

When calculating the gravity amplitude $M_5^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$ we must consider the same class of diagrams as in figure 1 together with permutations over the external legs. For gravity the vertex

$$V^{1\text{-loop}}(K^+, a^+, b^+) = -\frac{i\kappa^3}{1440\pi^2} ([K a] [a b] [b K])^2 \tag{2.6}$$

²The term which gives rise to the double pole $[45] / \langle 45 \rangle^2$ is the one-loop splitting function [8, 9] which only gives a linear collinear pole for real momenta.

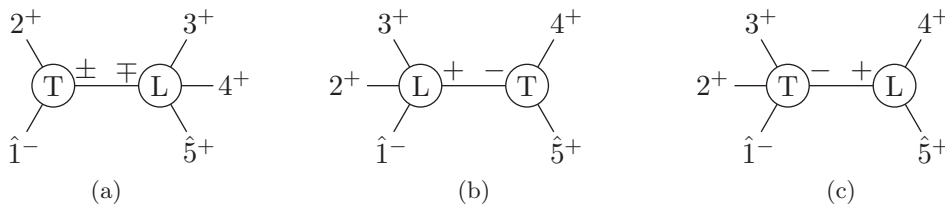


Figure 1: Diagrams contributing to the recursive construction of $A(1^-, 2^+, 3^+, 4^+, 5^+)$ with legs 1 and 5 shifted in the manner of (2.5). The diagram (c) contains the one-loop vertex $V^{1\text{-loop}}(\hat{K}^+, 4^+, \hat{5}^+)$ that contributes the double-pole.

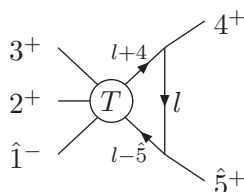


Figure 2: The form of the tree insertion that augments the recursion in order to construct the double pole and its underlying single pole.

can be used to generate a double pole term which has the correct soft and collinear limit, but attempts [36] to implement a universal correction for the single pole analogous to that of (1.8) have failed.

We find that the resolution is to replace the factorisation term of figure 1c with a tree insertion diagram of the form shown in figure 2 and compute this using axial gauge diagrammatics. In section 3 we present the axial gauge rules, in section 4 the computation of the five point one-minus gravity amplitude and in appendix A the result of the computation of the six point one-minus amplitude.

3 Axial gauge diagrammatics

We use axial gauge diagrammatic methods to determine the singular structure necessary to augment the recursion. We identify and compute the singularities arising when we shift a negative-helicity leg a and a positive-helicity leg b as in (1.1). These singularities arise from propagators involving just external momenta and from the loop momentum integration.

Following ref. [43] we use a set of Feynman rules for Yang-Mills amplitudes based on scalar propagators connecting three and four point vertices. The starting point is the expansion of the axial gauge propagator in terms of polarisation vectors,

$$\begin{aligned}
 i \frac{d_{\mu\nu}}{k^2} &= \frac{i}{k^2} \left(-g_{\mu\nu} + 2 \frac{k_\mu q_\nu + q_\mu k_\nu}{2k \cdot q} \right) \\
 &= \frac{i}{k^2} [\epsilon_\mu^+(k) \epsilon_\nu^-(k) + \epsilon_\mu^-(k) \epsilon_\nu^+(k) + \epsilon_\mu^0(k) \epsilon_\nu^0(k)],
 \end{aligned}
 \tag{3.1}$$

where

$$\epsilon_{\mu}^{+}(k) = \frac{[k^{\flat}|\gamma_{\mu}|q\rangle}{\sqrt{2}\langle k^{\flat}q\rangle}, \quad \epsilon_{\mu}^{-}(k) = \frac{[q|\gamma_{\mu}|k^{\flat}\rangle}{\sqrt{2}[k^{\flat}q]}, \quad \epsilon_{\mu}^0(k) = 2\frac{\sqrt{k^2}}{2k \cdot q}q_{\mu}. \quad (3.2)$$

Here q is a null reference momentum which may be complex. For any momentum k we define its q -nullified form

$$k^{\flat} := k - \frac{k^2}{2k \cdot q}q. \quad (3.3)$$

Contracting the polarisation vectors into the usual Yang-Mills three-point vertex yields the familiar three-point MHV and $\overline{\text{MHV}}$ vertices,

$$\begin{aligned} \frac{1}{i\sqrt{2}}V_3(1^-, 2^-, 3^+) &= \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle} = \frac{\langle 12 \rangle [3q]^2}{[1q][2q]}, \\ \frac{1}{i\sqrt{2}}V_3(1^+, 2^+, 3^-) &= -\frac{[12]^3}{[23][31]} = \frac{[21]\langle 3q \rangle^2}{\langle 1q \rangle \langle 2q \rangle}, \end{aligned} \quad (3.4)$$

along with a $V_3(1^+, 2^-, 3^0)$ vertex. In the formula above, all momenta are q -nullified. As vertices of this last type must be attached together in pairs, it is natural to absorb the resulting four-point configurations into an effective four-point vertex along with the Yang-Mills four-point vertex. These effective four-point vertices contain prefactors

$$\frac{[pq]}{\langle pq \rangle} \quad \text{and} \quad \frac{\langle mq \rangle}{[mq]}, \quad (3.5)$$

for each positive-helicity leg p , and each negative-helicity leg m , respectively.

When adopting a recursive approach which involves shifting a negative-helicity leg a and a positive-helicity leg b , the recursion-optimised choice for the reference momentum q is

$$\lambda_q = \lambda_a, \quad \bar{\lambda}_q = \bar{\lambda}_b. \quad (3.6)$$

With this choice of q the prefactors of four-point vertices (3.5) involving a shifted leg vanish. Furthermore from (3.4) the legs a and b can only enter a diagram on an MHV or $\overline{\text{MHV}}$ three-point vertex respectively.

Thus for the single-minus amplitudes, at tree and one-loop level, the external negative-helicity leg must enter the diagram via an MHV three-point vertex and this must have a negative-helicity internal leg. This leaves insufficient negative helicities to have a four-point vertex anywhere in the diagram. At tree level there are no non-vanishing diagrams whilst at one-loop we have a single MHV three-point vertex and several $\overline{\text{MHV}}$ three-point vertices in each diagram. These rules apply to both Yang-Mills and gravity calculations. For gravity we define the tree amplitudes using the Kawai-Lewellen-Tye (KLT) expressions [44].

We now wish to characterise the singularities when either s_{bc} or s_{ay} vanish. Singularities arise in the integration from the region of loop momentum where the denominators of three adjacent propagators vanish simultaneously, as the two null legs to which they

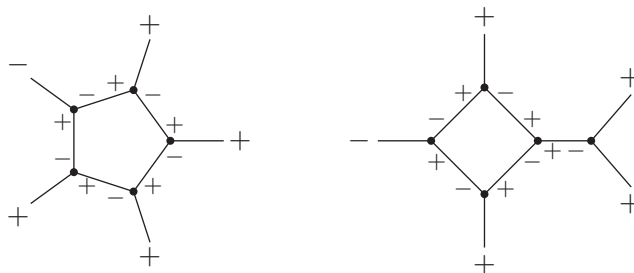


Figure 3: With the constraints that (1) the negative-helicity leg enters via an MHV three-point vertex and (2) the four-point vertices vanish, we only have non-vanishing diagrams with a single three-point MHV vertex with the remaining vertices three-point $\overline{\text{MHV}}$ with internal helicities organised as shown in these sample diagrams.

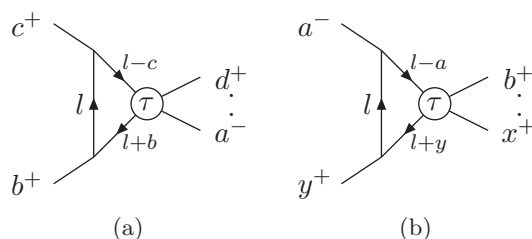


Figure 4: Singularities in s_{bc} and s_{ay} arise in integrations over the terms shown.

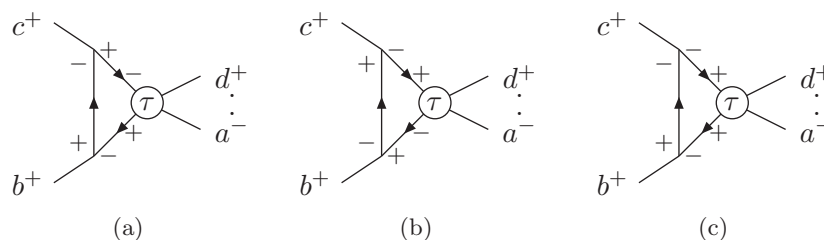


Figure 5: The three possible helicity structures of figure 4a.

connect become collinear. The diagrams of interest for any single-minus amplitude can then be collected into the forms shown in figure 4. Note that we evaluate these diagrams for real momenta and only carry out analytic shifts on the final expressions. The circles in these diagrams represent the sums of all possible tree diagrams with two internal legs and the given external legs. We denote these by $\tau(a, b, \dots)$. In the integration region of interest all the legs of τ are close to null and the internal legs are close to collinear.

In each case there are three options for the helicities within the loop, as illustrated in figure 5. Let us consider figure 4a. With the configuration of figure 5c, τ vanishes at the integration singularity because it is a one-minus tree amplitude in this limit and so the diagram has vanishing residue.

The diagram 5b evaluates to

$$\int d^4l \frac{[b|l|a]\langle c|l|a\rangle}{\langle b a \rangle \langle c a \rangle} \times \frac{\langle l - c, a \rangle^2}{\langle l + b, a \rangle^2} \frac{\tau((l - c)^+, d^+, \dots, a^-, (l + b)^-)}{l^2(l + k_b)^2(l - k_c)^2} \quad (3.7)$$

where the momenta in the spinor products are q -nullified as in (3.3). We construct a basis for the loop momentum built on b and c via

$$l = \alpha_1(k_b + k_c) + \alpha_2(k_b - k_c) + (\alpha_3 + i\alpha_4) \frac{\langle c a \rangle}{\langle b a \rangle} \lambda_b \bar{\lambda}_c + (\alpha_3 - i\alpha_4) \frac{\langle b a \rangle}{\langle c a \rangle} \lambda_c \bar{\lambda}_b \quad (3.8)$$

Under this parametrisation,

$$\int \frac{d^4l}{l^2(l + k_b)^2(l - k_c)^2} f(l) = \frac{1}{s_{bc}} \int d\alpha_i F(\alpha_i) f(l(\alpha_i)) \quad (3.9)$$

where $F(\alpha_i)$ has no dependence on s_{bc} .³

Also,

$$\begin{aligned} [b|l|a] &= (\alpha_1 - \alpha_2 + \alpha_3 + i\alpha_4) [b|c|a], \\ [c|l|a] &= (\alpha_1 + \alpha_2 + \alpha_3 - i\alpha_4) [c|b|a]. \end{aligned} \quad (3.10)$$

After these manipulations the integrand from figure 4a becomes

$$\frac{[bc]}{\langle bc \rangle} \frac{\langle l - c, a \rangle^2}{\langle l + b, a \rangle^2} \tau((l - c)^+, d^+, \dots, a^-, (l + b)^-) \times F'(\alpha_i). \quad (3.11)$$

When l , b and c become collinear, τ approaches the collinear limit of an MHV tree amplitude. Within τ there are diagrams with an explicit s_{bc} pole. The singular factor from the integration around the collinear limit combines with the explicit pole factor to give the double pole discussed previously. In addition to the leading behaviour of τ in the collinear limit, we need to know its finite piece in order to determine the residue at this pole. The diagram 5a gives the same contribution.

We can apply a similar analysis to the contributions from diagrams of the type shown in figure 4b. In this case τ approaches either a one-minus or an all-plus tree amplitude in the region of interest and so vanishes. Thus diagrams of the type shown in figure 4b give no contribution.

In order to evaluate the contribution from (3.11) we must evaluate the tree structures to order $\langle bc \rangle^0$. For diagrams within τ involving $1/s_{bc}$ this means going beyond leading order. The loop part of these diagrams is a triangle and the calculation is readily done exactly. The diagrams without this propagator need only be calculated to leading order. In this regard, not only is the recursive approach selecting a subset of diagrams for calculation, it is also allowing us to calculate these diagrams in a very convenient limit.

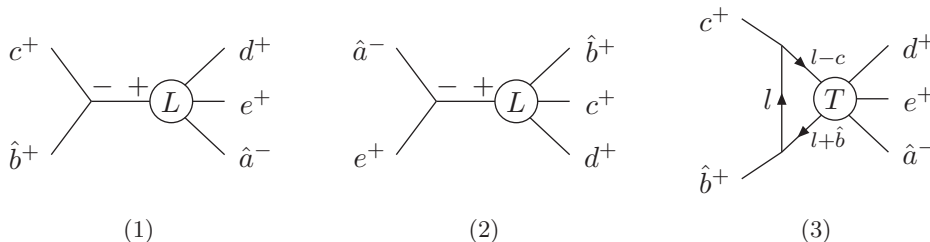
³ We have not explicitly introduced a regulator, although even for the finite amplitudes under consideration individual diagrams may diverge. If we were to use a Pauli-Villars regulator with mass M_{PV} for instance, we could still extract the same momentum-dependent prefactor but the remaining integral would depend on the α_i and M_{PV}^2/s_{bc} . Knowing that any divergent pieces cancel in the full amplitude allows us to consider only the finite pieces, which are independent of M_{PV} .

In the following section we apply an augmented recursive analysis to the calculation of the amplitude $M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$. For gravity, using the KLT relations for tree amplitudes [44] the equivalent expression to (3.11) is

$$\frac{[bc]^3 \langle l-c, a \rangle^4}{\langle bc \rangle \langle l+b, a \rangle^4} \tau_{\text{grav}}((l-c)^+, d^+, \dots, a^-, (l+b)^-) \times F'(\alpha_i). \quad (3.12)$$

4 The graviton scattering amplitude $M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$

To compute this amplitude recursively, as discussed in the previous section, we must compute three types of contribution:



together with the contributions obtained by summing over the distinct permutations of c , d and e .

The first two of these involve single poles only, so we only need the loop structures to leading order and we can use the corresponding four-pt one-loop amplitudes. The final structure contains a double pole so we must evaluate both the tree structure on the right and the loop pieces more carefully. The first diagram uses the four-point one-minus amplitude whereas the second requires the four-point all-plus amplitude, both are given in (B.4). We obtain,

$$R_1(a, b, c, d, e) = \frac{1}{5760} \frac{\langle ad \rangle^2 \langle ae \rangle^2 [bc][de]^4 (\langle cd \rangle^2 \langle ae \rangle^2 + \langle ac \rangle \langle cd \rangle \langle de \rangle \langle ae \rangle + \langle ac \rangle^2 \langle de \rangle^2)}{\langle ab \rangle^2 \langle bc \rangle \langle ce \rangle^2 \langle cd \rangle^2 \langle de \rangle^2},$$

$$R_2(a, b, c, d, e) = -\frac{3}{5760} \frac{\langle ae \rangle [be]^4}{\langle cd \rangle^2 [ab]^2 [ae]} ([bc]^2 [de]^2 + [bc][cd][de][be] + [cd]^2 [be]^2). \quad (4.1)$$

For the third diagram, we need the tree diagrams which constitute τ_{grav} of (3.12). Mindful of the recursive analysis that we will ultimately perform, we calculate these diagrams as Laurent series in $\langle bc \rangle$, dropping terms that will not contribute to the residues.

We require the five-point contributions with two off-shell legs B^- and C^+ , carrying momenta $B \equiv l+b$ and $C \equiv c-l$, respectively.

The KLT relation [44] between Yang-Mills amplitudes and gravity amplitudes at five points is

$$M(a^-, B^-, C^+, d^+, e^+) = s_{BC} s_{de} A(a^-, B^-, C^+, d^+, e^+) A(a^-, C^+, B^-, e^+, d^+) + s_{Bd} s_{Ce} A(a^-, B^-, d^+, C^+, e^+) A(a^-, d^+, B^-, e^+, C^+), \quad (4.2)$$

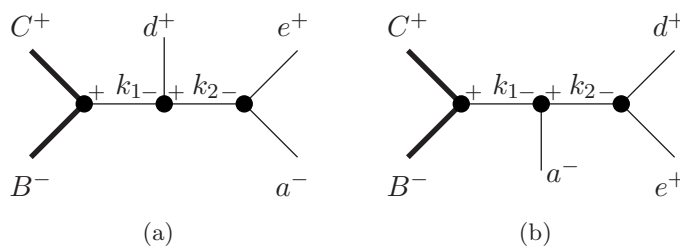
where we have chosen a form of the KLT relations that restricts the $\langle bc \rangle$ pole to the first term. The KLT relations are only valid for on-shell momenta, although these momenta

may be in higher dimensions. If we assume the deviation from eq. (4.2) may be neglected in the region around $B^2 = C^2 = 0$, we see that the gravity tree structure has the form,

$$\langle bc \rangle \left(\frac{T^{\text{leading}}}{\langle bc \rangle} + T^{\text{sub-leading}} \right) \left(\frac{T^{\text{leading}}}{\langle bc \rangle} + T^{\text{sub-leading}} \right), \quad (4.3)$$

where all diagrams contribute to the sub-leading pieces but only diagrams involving a $V_3(B^-, C^+, x)$ vertex contribute to the leading pieces. The second term in eq. (4.2) is only needed to leading order and its contribution to the residue will be directly determined by the on-shell Yang-Mills MHV amplitudes. The amplitude generated using the leading and sub-leading singularity terms from (4.2) has the correct symmetries and collinear limits. Additionally, the five-point amplitude has been verified by a completely independent string-based rules computation. The general case is worthy of further study [45].

First we establish the double pole term. This arises from the poles in each of the Yang-Mills tree amplitudes in the first term of (4.2). We evaluate this diagrammatically. The Yang-Mills amplitude, $A(a^-, B^-, C^+, d^+, e^+)$ receives contributions from five diagrams. The two which contribute to the pole are:



with contributions

$$\begin{aligned} D_a &= \frac{\langle Ba \rangle^2}{\langle Ca \rangle^2} \frac{\langle a|bc|a \rangle [de][eb]}{s_{bc} \langle da \rangle \langle ea \rangle [ae][ab]} f_a(\alpha_i), \\ D_b &= \frac{\langle Ba \rangle^2}{\langle Ca \rangle^2} \frac{\langle a|bc|a \rangle \langle ca \rangle}{\langle bc \rangle [ab] \langle de \rangle \langle da \rangle \langle ea \rangle} f_b(\alpha_i), \end{aligned} \quad (4.4)$$

where we have used $\langle a|BC|a \rangle = \langle a|(b+l)(c-l)|a \rangle = \langle a|bc|a \rangle f(\alpha_i)$, etc.. The parameters contained in f_a and f_b are the same for both diagrams and the sum of the two contributions is

$$D_a + D_b = \frac{\langle Ba \rangle^2}{\langle Ca \rangle^2} \frac{\langle a|bc|a \rangle}{s_{bc} [ab] \langle da \rangle \langle ea \rangle} \left(\frac{[bad|e] - [cb|e]}{[ae] \langle de \rangle} \right) f_a(\alpha_i), \quad (4.5)$$

where the second term is sub-leading in the $\langle bc \rangle$ pole.

The leading pole in the other Yang-Mills factor is obtained analogously and, combining, we obtain the leading pole in (4.2),

$$\frac{\langle Ba \rangle^4}{\langle Ca \rangle^4} s_{bc} s_{de} \frac{\langle ab \rangle \langle ac \rangle [de]}{\langle bc \rangle \langle ea \rangle [ae] \langle de \rangle} \frac{\langle ab \rangle \langle ac \rangle [de]}{\langle bc \rangle \langle da \rangle [ad] \langle de \rangle} f'_a(\alpha_i). \quad (4.6)$$

Combining this with the factors arising from the left hand part of the full diagram and integrating over the α_i the leading term in the Laurent series is proportional to

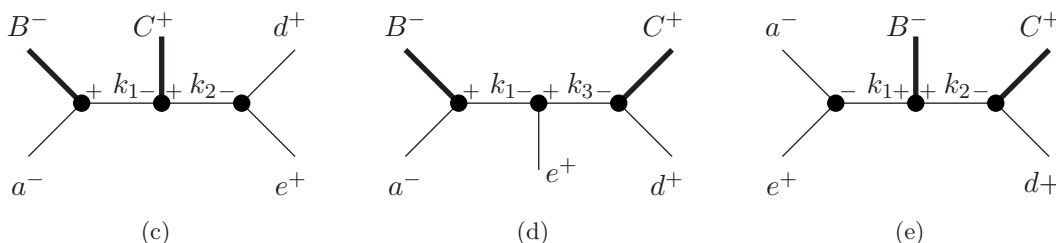
$$\frac{[bc]^3}{\langle bc \rangle} [bc][de] \frac{\langle ab \rangle \langle ac \rangle [de]}{\langle ea \rangle [ae] \langle de \rangle} \frac{\langle ab \rangle \langle ac \rangle [de]}{\langle bc \rangle \langle da \rangle [ad]}, \quad (4.7)$$

which clearly displays the double pole factor. The constant of proportionality is most readily fixed by looking at collinear limits.

We must now enumerate the contributions that are sub-leading in the s_{bc} pole. These come from a variety of sources. We express these single-pole terms as the double-pole factor of (4.7), multiplied by a factor δ . Firstly, we have the sub-leading contribution of (4.5) together with the corresponding contribution from the other Yang-Mills factor,

$$\delta_1 = \frac{s_{bc}[be]}{[b|ad|e]} + \frac{s_{bc}[bd]}{[b|ae|d]}. \quad (4.8)$$

Next we have the sub-leading diagrams for the Yang-Mills amplitudes in the first term of (4.2) shown below:



We note that in the first two of these the k_1 propagator feeds into the two diagrams that would make a one-minus four-point tree if k_1 and C were both null. As we know that this vanishes when all the legs are null, the sum of the first two diagrams must be of the form: $C^2 X + k_1^2 Y$. We can drop the terms containing a C^2 factor as we are already at sub-leading order, leaving something proportional k_1^2 . Thus taking both terms together leads to the cancellation of the s_{aB} propagator:

$$D_c + D_d = \frac{\langle Ba \rangle^2}{\langle Ca \rangle^2} \frac{[b|B|a]}{[ab] \langle ea \rangle \langle de \rangle} \frac{\langle ca \rangle}{\langle cd \rangle} f_c(\alpha_i) + \mathcal{O}(\langle bc \rangle). \quad (4.9)$$

Pulling out a factor of (4.7) leaves

$$\delta_2 = \frac{s_{bc}[e|a|c]}{s_{ab}[e|d|c]}. \quad (4.10)$$

We can apply the same procedure to the final diagram giving

$$\delta_3 = \frac{\langle bc \rangle \langle de \rangle}{s_{ab}[de]} \left(\frac{[e|B|a][eb]}{\langle da \rangle \langle cd \rangle} + \frac{[d|B|a][db]}{\langle ea \rangle \langle ce \rangle} \right) \quad (4.11)$$

Finally we need the second term in (4.2), $s_{Bd} s_{Ce} A(a, B, d, C, e) A(a, d, B, e, C)$, which we evaluate using MHV tree amplitudes. After extracting the double-pole factor we obtain

$$\delta_4 = \frac{\langle bc \rangle \langle de \rangle [d|B|a][e|C|a]}{[bc][de] \langle ab \rangle^2 \langle cd \rangle \langle ce \rangle}. \quad (4.12)$$

We thus have the leading and sub-leading poles expressed as

$$\frac{[bc]^3}{\langle bc \rangle} [bc][de] \frac{\langle ab \rangle \langle ac \rangle [de]}{\langle ea \rangle [ae] \langle de \rangle} \frac{\langle ab \rangle \langle ac \rangle [de]}{\langle bc \rangle \langle da \rangle [ad]} \times \left(1 + \sum_i \delta_i \right) \quad (4.13)$$

We can now use the pole to determine the amplitude recursively. This involves applying the shift (1.1) and evaluating at $z = -\langle bc \rangle / \langle ac \rangle$. The coefficient of the double pole in (4.7) has a z dependence under this shift which generates a further contribution to the single pole since

$$\text{Res} \left(\frac{f(z)}{z(z-z_i)^2}, z_i \right) = -\frac{f(z_i)}{z_i^2} + \frac{1}{z_i} \left. \frac{df}{dz} \right|_{z=z_i}. \quad (4.14)$$

Carrying this out and combining with the contributions of the δ s gives

$$\Delta(a, b, c, d, e) = -\frac{1}{2} \frac{\langle ad \rangle \langle bc \rangle}{\langle ab \rangle \langle cd \rangle} - \frac{1}{2} \frac{\langle ae \rangle \langle bc \rangle}{\langle ab \rangle \langle ce \rangle} \quad (4.15)$$

$$-3 \frac{[db][eb]}{\langle dc \rangle \langle ec \rangle} \frac{\langle bc \rangle \langle de \rangle}{[bc][de]} - 3 \frac{[dc][ec]}{\langle dc \rangle \langle ec \rangle} \frac{\langle bc \rangle \langle de \rangle \langle ca \rangle^2}{[bc][de] \langle ba \rangle^2} \quad (4.16)$$

$$- \frac{7}{2} \frac{[dc][eb]}{\langle dc \rangle \langle ec \rangle} \frac{\langle bc \rangle \langle de \rangle \langle ca \rangle}{[bc][de] \langle ba \rangle} - \frac{7}{2} \frac{[db][ec]}{\langle dc \rangle \langle ec \rangle} \frac{\langle bc \rangle \langle de \rangle \langle ca \rangle}{[bc][de] \langle ba \rangle}. \quad (4.17)$$

The full one-minus amplitude can now be written as the sum over recursive contributions arising from three orderings of the external legs,

$$M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+) = R(1, 2, 3, 4, 5) + R(1, 2, 4, 5, 3) + R(1, 2, 5, 3, 4). \quad (4.18)$$

with the full amplitude having a factor of $i\kappa^5/16\pi^2$ as in (B.1).

Each recursive term is a sum over the three classes of recursive diagram,

$$R(a, b, c, d, e) = R_1(a, b, c, d, e) + R_2(a, b, c, d, e) + R_3(a, b, c, d, e), \quad (4.19)$$

where R_1 and R_2 are given by (4.1), and

$$R_3(a, b, c, d, e) = \frac{1}{5760} \frac{\langle ab \rangle^2 \langle ac \rangle^4 [bc]^4 [de]}{\langle ad \rangle \langle ae \rangle \langle bc \rangle^2 \langle cd \rangle \langle ce \rangle \langle de \rangle} (1 + \Delta(a, b, c, d, e)). \quad (4.20)$$

The overall normalisation can be obtained by evaluating the parameter integrals or, more easily, fixed by factorising the known four-point amplitude. The individual factors on the terms in Δ are also obtainable by parameter integration or more conveniently by the normalisation of the collinear limits.

This form for the amplitude has the correct collinear limits and is symmetric under interchange of any pair of positive-helicity legs. We have also checked that the amplitude agrees with that calculated by string-based rules. This calculation can readily be extended to the six-point case, $M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+, 6^+)$. We have constructed the amplitude and again checked that it has the correct symmetries and collinear limits. This result is presented in appendix A. *Mathematica* code for both the five- and six-point amplitudes may be found at <http://pyweb.swan.ac.uk/~dunbar/graviton.html>.

5 Conclusions and remarks

In this article we have demonstrated how to augment recursion in order to determine the rational terms in amplitudes with double poles under a complex shift. Double poles are generic in amplitudes, however it is often possible to carry out a recursion which avoids them. However, double poles are unavoidable in the case of the one-minus Yang-Mills amplitudes $A^{1\text{-loop}}(1^-, 2^+, 3^+, \dots, n^+)$ and the gravity amplitudes $M^{1\text{-loop}}(1^-, 2^+, 3^+, \dots, n^+)$. In the absence of a universal soft factor analogous to (1.8), in order to perform the augmented recursion the sub-leading poles must be determined on a case-by-case basis. While we have done this for both the five- and six-point one-minus gravity amplitudes, this procedure could be used to calculate the seven-point or indeed any higher-point one-minus amplitude.

A Six-point single-minus amplitude

The six-point one-loop single-minus graviton scattering amplitude can also be calculated using augmented recursion. The calculation follows that of the five-point amplitude with the addition of factorisations involving a four-point tree amplitude and a four-point loop amplitude. The shift employed is once again $\bar{\lambda}_1 \rightarrow \bar{\lambda}_1 - z\bar{\lambda}_2$, $\lambda_2 \rightarrow \lambda_2 + z\lambda_1$. The amplitude is given by

$$\begin{aligned}
 M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+, 6^+) = & \tag{A.1} \\
 & \sum_{\substack{x \in \{3,4,5,6\} \\ \{y_1, y_2, y_3\} \cup \{x\} = \{3,4,5,6\}}} \left\{ R_1^{(6)}(1, x|2, y_1, y_2, y_3; z_x) + R_2^{(6)}(2, x|1, y_1, y_2, y_3; z_x) \right. \\
 & \left. + R_3^{(6)}(2, x|1, y_1, y_2, y_3; z_x) \right\} \\
 + & \sum_{\substack{\{x_1, x_2\} \subset \{3,4,5,6\} \\ \{y_1, y_2\} \cup \{x_1, x_2\} = \{3,4,5,6\}}} R_4^{(6)}(1, x_1, x_2|2, y_1, y_2; z_{x_1, x_2}).
 \end{aligned}$$

In each of these terms the vertical bar denotes a split of the external momenta with the relevant pole arising when the shifted total momentum to the right of bar is null.

The $R_1^{(6)}$ terms are the factorisations involving a three-point $\overline{\text{MHV}}$ tree and a five-point all-plus one-loop amplitude:

$$R_1^{(6)}(1, x|2, y_1, y_2, y_3; z_x) = \frac{[x 2]^2 \langle 1 x \rangle}{[1 2]^2 [1 x]} M^{1\text{-loop}}(p^+, \hat{2}^+, y_1^+, y_2^+, y_3^+), \tag{A.2}$$

with $z_x = [1 x] / [2 x]$ and $p = (\lambda^x + \lambda^1 [1 2] / [x 2]) \bar{\lambda}^x$. Similarly, the $R_2^{(6)}$ terms are the factorisations involving a three-point MHV tree and a five-point one-minus one-loop amplitude:

$$R_2^{(6)}(2, x|1, y_1, y_2, y_3; z_x) = \frac{\langle 1 x \rangle^2 [2 x]}{[1 2]^2 \langle 2 x \rangle} M^{1\text{-loop}}(\hat{1}^-, y_1^+, y_2^+, y_3^+, p^+), \tag{A.3}$$

with $z_x = -\langle x 2 \rangle / \langle x 1 \rangle$ and $p = \lambda^x (\bar{\lambda}^x + \langle 1 2 \rangle \bar{\lambda}^2 / \langle 1 x \rangle)$. The $R_4^{(6)}$ terms are the factorisations involving a four-point MHV tree and a four-point all-plus one-loop amplitudes:

$$R_4^{(6)}(1, x_1, x_2|2, y_1, y_2; z_{x_1, x_2}) = \frac{M^{\text{tree}}(\hat{1}^-, -p^-, x_1^+, x_2^+) M^{1\text{-loop}}(\hat{2}^+, y_1^+, y_2^+, p^+)}{t_{1x_1x_2}}, \tag{A.4}$$

with $z_{x_1x_2} = t_{1x_1x_2}/[2|P_{x_1x_2}|1\rangle$ and $p = k_{\hat{1}} + k_{x_1} + k_{x_2}$.

The $R_3^{(6)}$ terms are the augmented pieces arising from the $\langle 2x \rangle$ poles. Here, $z_x = -\langle x2 \rangle / \langle x1 \rangle$.

$$\begin{aligned}
 R_3^{(6)}(2, x|1, y_1, y_2, y_3; z_x) = & \\
 & \frac{[2x]^3}{\langle 2x \rangle} \left\{ -\text{KLT}_F(\hat{1}, \hat{2}, x, y_1, y_2, y_3) \right. \\
 & + \frac{[2x]}{\langle 2x \rangle} s_{y_2y_3} \text{YM}_L(\hat{1}, \hat{2}, x, y_1, y_2, y_3) \\
 & \quad \times [s_{y_1y_3} \text{YM}_L(\hat{1}, \hat{2}, x, y_3, y_1, y_2) + (s_{y_1y_2} + s_{y_1y_3}) \text{YM}_L(\hat{1}, \hat{2}, x, y_3, y_2, y_1)] \\
 & + [2x] s_{y_2y_3} \text{YM}_L(\hat{1}, \hat{2}, x, y_1, y_2, y_3) \\
 & \quad \times [s_{y_1y_3} \text{YM}_S(\hat{1}, \hat{2}, x, y_3, y_1, y_2) + (s_{y_1y_2} + s_{y_1y_3}) \text{YM}_S(\hat{1}, \hat{2}, x, y_3, y_2, y_1)] \\
 & + [2x] s_{y_2y_3} \text{YM}_S(\hat{1}, \hat{2}, x, y_1, y_2, y_3) \\
 & \quad \times [s_{y_1y_3} \text{YM}_L(\hat{1}, \hat{2}, x, y_3, y_1, y_2) + (s_{y_1y_2} + s_{y_1y_3}) \text{YM}_L(\hat{1}, \hat{2}, x, y_3, y_2, y_1)] \\
 & + \frac{[2x]}{\langle 2x \rangle} s_{y_1y_3} \text{YM}_L(\hat{1}, \hat{2}, x, y_2, y_1, y_3) \\
 & \quad \times [s_{y_2y_3} \text{YM}_L(\hat{1}, \hat{2}, x, y_3, y_2, y_1) + (s_{y_2y_1} + s_{y_2y_3}) \text{YM}_L(\hat{1}, \hat{2}, x, y_3, y_1, y_2)] \\
 & + [2x] s_{y_1y_3} \text{YM}_L(\hat{1}, \hat{2}, x, y_2, y_1, y_3) \\
 & \quad \times [s_{y_2y_3} \text{YM}_S(\hat{1}, \hat{2}, x, y_3, y_2, y_1) + (s_{y_2y_1} + s_{y_2y_3}) \text{YM}_S(\hat{1}, \hat{2}, x, y_3, y_1, y_2)] \\
 & + [2x] s_{y_1y_3} \text{YM}_S(\hat{1}, \hat{2}, x, y_2, y_1, y_3) \\
 & \quad \times [s_{y_2y_3} \text{YM}_L(\hat{1}, \hat{2}, x, y_3, y_2, y_1) + (s_{y_2y_1} + s_{y_2y_3}) \text{YM}_L(\hat{1}, \hat{2}, x, y_3, y_1, y_2)] \left. \right\}, \tag{A.5}
 \end{aligned}$$

where the KLT terms contributing to the double pole have leading Yang-Mills factors:

$$\begin{aligned}
 \text{YM}_L(a, b, c, d, e, f) = & \frac{\langle ac \rangle \langle ba \rangle}{[ab] \langle da \rangle \langle ea \rangle \langle fa \rangle} \left\{ \frac{[bc] \langle ca \rangle}{t_{def}} \left(\frac{[f|P_{de}|a\rangle}{\langle de \rangle} + \frac{[d|P_{ef}|a\rangle}{\langle fe \rangle} \right) \right. \\
 & \left. + \frac{[d|P_{ef}|a\rangle [b|P_{ef}|a\rangle]}{\langle ef \rangle t_{efa}} + \frac{[d|P_{ef}|a\rangle [ef] [fb]}{[af] t_{efa}} + \frac{[f|P_{de}|a\rangle [fb]}{[af] \langle de \rangle} \right\}, \tag{A.6}
 \end{aligned}$$

and sub-leading factors:

$$\begin{aligned}
 \text{YM}_S(a, b, c, d, e, f) = & \frac{\langle ba \rangle}{[ab] \langle da \rangle \langle ea \rangle \langle fa \rangle} \left\{ \frac{[d|P_{ef}|a\rangle [b|P_{ef}|a\rangle^2}{\langle ef \rangle t_{efa}^2} \right. \\
 & + \frac{[d|P_{ef}|a\rangle [ef] [fb]}{[af] t_{efa}} \left(\frac{[b|P_{ef}|a\rangle}{t_{efa}} + \frac{[bf]}{[af]} \right) - \frac{[f|P_{de}|a\rangle [bf]^2}{[af]^2 \langle de \rangle} \left. \right\} \\
 & + \frac{1}{2} \frac{[bc] \langle ac \rangle^2}{[ab] \langle cd \rangle \langle de \rangle \langle ef \rangle \langle fa \rangle} - \frac{1}{2} \frac{[f|k_b - k_c|a\rangle \langle ac \rangle [fb]}{[ab] \langle cd \rangle \langle de \rangle \langle ea \rangle s_{fa}} \\
 & - \frac{1}{2} \frac{\langle a|(k_b - k_c)P_{ef}|a\rangle \langle ac \rangle}{[ab] \langle cd \rangle \langle da \rangle \langle ea \rangle t_{efa}} \left(\frac{[b|P_{ef}|a\rangle}{\langle fe \rangle \langle fa \rangle} + \frac{[bf][ef]}{s_{fa}} \right). \tag{A.7}
 \end{aligned}$$

Finally the finite terms in the KLT sum are:

$$\begin{aligned}
 \text{KLT}_F(a, b, c, d, e, f) = & \\
 & \frac{s_{ef} \langle ab \rangle^4}{2 \langle ad \rangle \langle ae \rangle \langle af \rangle \langle bd \rangle^2 \langle be \rangle^2 \langle bf \rangle \langle ef \rangle^2} \\
 & \times \left\{ \langle ae \rangle (6 \langle a|bP_{ef}|b \rangle [d|b|a] + 7 \langle a|bP_{ef}|b \rangle [d|c|a] + 7 \langle a|cP_{ef}|b \rangle [d|b|a] + 6 \langle a|cP_{ef}|b \rangle [d|c|a]) \right. \\
 & \left. + \langle ab \rangle (6 \langle e|fb|a \rangle [d|b|a] + 7 \langle e|fb|a \rangle [d|c|a] + 7 \langle e|fc|a \rangle [d|b|a] + 6 \langle e|fc|a \rangle [d|c|a]) \right\} \\
 & + \frac{s_{df} \langle ab \rangle^4}{2 \langle ad \rangle \langle ae \rangle \langle bd \rangle \langle be \rangle \langle bf \rangle^2 \langle de \rangle \langle df \rangle} \\
 & \times \{ 6 [e|b|a] [f|b|a] + 7 [e|b|a] [f|c|a] + 7 [e|c|a] [f|b|a] + 6 [e|c|a] [f|c|a] \} \\
 & + \frac{\langle ab \rangle^4}{2 \langle ad \rangle \langle ae \rangle \langle af \rangle \langle bd \rangle \langle be \rangle \langle bf \rangle^2 \langle de \rangle} \{ [d|c|a] [e|b|a] [f|c|a] + [d|b|a] [e|c|a] [f|b|a] \} \\
 & + \{ d \leftrightarrow e \} \tag{A.8}
 \end{aligned}$$

Expressed naïvely, without attempting optimisation, as a rational polynomial of the λ_α^i it has a `LeafCount` of 355,053. For comparison, the `LeafCount` of the five-point one-minus gravity amplitude is 4,549, and for the six-point one-minus Yang-Mills amplitude is 1,541.

B Graviton scattering amplitudes

We define tree and one-loop amplitudes in gravity for which all field couplings have been removed, i.e.,

$$\begin{aligned}
 \mathcal{M}_n^{\text{tree}}(1, 2, \dots, n) &= i\kappa^{(n-2)} M_n^{\text{tree}}(1, 2, \dots, n), \\
 \mathcal{M}_n^{1\text{-loop}}(1, 2, \dots, n) &= \frac{i\kappa^n}{(4\pi)^2} M_n^{1\text{-loop}}(1, 2, \dots, n). \tag{B.1}
 \end{aligned}$$

As for Yang-Mills amplitudes we express amplitudes using the spinor helicity formalism. For the four dimensional case there are only two graviton helicities and their polarisation tensors can be constructed from direct products of Yang-Mills polarisations vectors,

$$\varepsilon_{\mu\nu}^+ = \varepsilon_\mu^+ \varepsilon_\nu^+, \quad \varepsilon_{\mu\nu}^- = \varepsilon_\mu^- \varepsilon_\nu^-. \tag{B.2}$$

If we consider the Feynman diagrams for a gravity one-loop scattering amplitude, performing a Passarino-Veltman reduction [46–48] allows us to reduce any one-loop amplitude to the form

$$M_n^{1\text{-loop}}(1, \dots, n) = \sum_i c_i I_4^i + \sum_j d_j I_3^j + \sum_k e_k I_2^k + R + O(\epsilon). \tag{B.3}$$

Relatively few graviton scattering amplitudes have been computed. In fact, only the four-point amplitudes have been computed for all helicity configurations [39, 40, 49–51] and

all possible matter types circulating in the loop. For four points there are three independent helicity configurations for the external gravitons: $M(1^+, 2^+, 3^+, 4^+)$, $M(1^-, 2^+, 3^+, 4^+)$ and $M(1^-, 2^-, 3^+, 4^+)$. The all-plus and one-minus vanish at tree level and have one-loop amplitudes which are purely rational (to order ϵ^0). These amplitudes, for any matter content, are

$$\begin{aligned}
 M^{1\text{-loop}}(1^+, 2^+, 3^+, 4^+) &= -N_s \left(\frac{st}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \right)^2 \frac{(s^2 + st + t^2)}{1920}, \\
 M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+) &= N_s \left(\frac{st}{u} \right)^2 \left(\frac{[24]^2}{[12] \langle 23 \rangle \langle 34 \rangle [41]} \right)^2 \frac{(s^2 + st + t^2)}{5760}, \tag{B.4}
 \end{aligned}$$

where $s = (k_1 + k_2)^2$, $t = (k_1 + k_4)^2$, $u = (k_1 + k_3)^2$ and $N_s = N_B - N_F$ is the number of bosonic states in the loop minus the number of fermionic states. The amplitudes for pure gravity are found by putting $N_s = 2$ in the above expressions since a graviton has two helicity states. These amplitudes vanish in any supersymmetric theory. The n -point all-plus and one-minus amplitudes are also particle-type independent up to a prefactor of N_s , as can be seen from the vanishing of these amplitudes in any supersymmetric theory as a consequence of supersymmetric Ward identities [52, 53]. It is therefore sufficient, for these configurations, to compute the amplitude with a scalar particle circulating in the loop.

Beyond four points most of the explicit graviton amplitudes are for scattering in supersymmetric theories. For $\mathcal{N} = 8$ supergravity the n -point MHV is known [54], as are the NMHV six- [55, 56] and seven-point amplitudes [57]. In ref. [54] a ‘dimension shift’ relation [31] allowed the conjecture of an ansatz for the all-plus n -point amplitudes. This amplitude is an ingredient in the recursion of the one-minus amplitude.

C String-based rules calculation of $M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$

The string-based rules were introduced in refs. [37, 38, 58] as a method of calculating (one-loop) gauge theory amplitudes. Their extension to gravity, in the form we use here, was given in refs. [39, 40]. In this appendix we summarise these rules and then describe how they are applied to compute $M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$. Our presentation treats the method as something of a ‘black box’ for obtaining field-theory results and we refer those interested in the details of its string-theory origins and derivation to the literature.

C.1 Summary of the string-based rules for gravity amplitudes

String-based rules use one-loop ϕ^3 -like graphs to compute the one-loop corrections to a field theory amplitude. The terms produced take the form of a rational function of the kinematic variables, within a Feynman parametrisation of a tensor loop integral. This approach has the advantage of significant computational savings over the traditional Feynman graph method: far fewer graphs are involved and early application of simplifications from the spinor-helicity formalism reduce the complexity of the associated expressions.

We begin by drawing all one-loop ϕ^3 graphs excluding massless bubbles (which vanish in dimensional regularisation) and tadpoles. We label the outermost legs of the graphs with the particles’ momenta, k_1, \dots, k_n . An internal line bears the same label as the first

line or leg found going anti-clockwise about its outermost vertex. (For examples of such graphs and labellings, see figure 6.) All independent labellings of external legs contribute for gravity amplitudes. The one-loop correction to the amplitude is then given by

$$M^{1\text{-loop}}(1, 2, \dots, n) = \sum_{\text{graphs } \gamma} \mathcal{D}(\gamma),$$

where the contribution from a graph γ with an n_ℓ -propagator loop is

$$\mathcal{D}(\gamma) = \Gamma(n_\ell - 2 + \epsilon) \left(\prod_{m=1}^{n_\ell-1} \int_0^{x_{i_{m+1}}} dx_{i_m} \right) \frac{K^{(\gamma)}(x_{i_1}, \dots, x_{i_{n_\ell-1}})}{\left\{ \sum_{1 \leq k < l \leq n_\ell} P_{i_k} \cdot P_{i_l} x_{i_k i_l} (1 - x_{i_k i_l}) \right\}^{n_\ell - 2 + \epsilon}}. \quad (\text{C.1})$$

In this formula, i_1, \dots, i_{n_ℓ} are the labels of the lines adjoining the loop going clockwise. $x_{ij} \equiv x_i - x_j$, with $x_{i_{n_\ell}}$ fixed at 1, and P_{i_k} is the momentum entering the loop along the line with innermost label i_k . The ‘reduced kinematic factor’ for γ , $K^{(\gamma)}(x_{i_1}, \dots, x_{i_{n_\ell-1}})$, is a polynomial in the x_{i_k} which for a gravity theory with no supersymmetries is of order $2n_\ell$.

We compute $K^{(\gamma)}(x_{i_1}, \dots, x_{i_{n_\ell-1}})$ as follows: the starting point is the overall graviton kinematic factor

$$\mathcal{K} = \int \prod_{i=1}^n dx_i d\bar{x}_i \prod_{1 \leq i < j \leq n} \exp \left\{ k_i \cdot k_j G_B^{ij} + (k_i \cdot \varepsilon_j - k_j \cdot \varepsilon_i) \dot{G}_B^{ij} - \varepsilon_i \cdot \varepsilon_j \ddot{G}_B^{ij} \right. \\ \left. + (k_i \cdot \bar{\varepsilon}_j - k_j \cdot \bar{\varepsilon}_i) \dot{\bar{G}}_B^{ij} - \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j \ddot{\bar{G}}_B^{ij} - (\varepsilon_i \cdot \bar{\varepsilon}_j - \varepsilon_j \cdot \bar{\varepsilon}_i) H_B^{ij} \right\} \Big|_{\text{multi-linear}}, \quad (\text{C.2})$$

where ‘multi-linear’ indicates that we retain only the coefficient of $\prod_{i=1}^n \varepsilon_i \bar{\varepsilon}_i$ in the expansion of the exponentials. The graviton polarisation tensor are then reconstructed using (B.2). \mathcal{K} contains much structure from the string theory perspective: $G_B^{ij} \equiv G_B(x_{ij})$ is the bosonic Green’s function on the string world-sheet, and the $x_i(\bar{x}_i)$ are closed-string left(right)-moving co-ordinates. Other objects present are the derivatives of G_B^{ij} : $\dot{G}_B^{ij} = \partial G_B^{ij} / \partial x_i$, which is antisymmetric in i, j , and $\ddot{G}_B^{ij} = \partial^2 G_B^{ij} / \partial x_i^2$ (with similar expressions for the right-moving $\dot{\bar{G}}_B^{ij}$ and $\ddot{\bar{G}}_B^{ij}$); and $H_B^{ij} = \partial^2 G_B^{ij} / \partial x_i \partial \bar{x}_i$. However, for our purposes we will simply treat (C.2) as an object for obtaining reduced kinematic factors by the application of some substitution rules that implement the field theory limit.

For the helicity configuration under consideration and a judicious choice of reference momenta for the polarisation vectors, we shall see that the coefficients of the second-order derivatives of the Green’s functions vanish. Nevertheless, in general this is not so and we should eliminate the \dot{G}_B and \ddot{G}_B from \mathcal{K} using integration by parts, which may lead to additional factors of the H_B appearing. Each H_B factor should then be eliminated by replacing it with the Feynman denominator relevant to the diagram under consideration (i.e. the expression found within the curly braces in (C.1)). At this point we simply drop the integration over the world-sheet co-ordinates and the leading factors of $\exp(k_i \cdot k_j G_B^{ij})$ from \mathcal{K} (their contributions are built into the rules).

Now consider a consecutive pair of lines joining on to the loop in a ϕ^3 graph, labelled (i, j) going clockwise. We can ‘pinch off’ this pair of lines by attaching them instead to

a new vertex and then drawing a new line carrying the label j from this vertex back to the loop. Any of the graphs drawn for string-based rules may be obtained this way (for example, the graph of figure 6d is obtained from figure 6a first by pinching off (2,3), and then (3,4)). For each such pinch used to reach a graph we: (1) discard all terms in the expression obtained above *except* those containing *exactly* one power of $\dot{G}_B^{ij} \dot{G}_B^{ij}$; (2) replace i with j in all remaining \dot{G}_B, \dot{G}_B ; and (3) multiply by $-1/k_{ij}^2$, where k_{ij} is the momentum carried by the new line formed by the pinch.

Next we apply the substitution rules. These act on the derivatives of the Green's functions in a left/right-independent manner, replacing them with polynomials in the x_{ij} in a way that depends on the particle content of the loop. In particular, the rule for a single scalar degree of freedom running around the loop is the simple substitution

$$\dot{G}_B^{ij}, \dot{G}_B^{ij} \rightarrow x_{ij} - \frac{1}{2} \text{sign } x_{ij}. \tag{C.3}$$

There are other rules for particles of higher spin in the loop (including the graviton), but by the discussion in appendix B, (C.3) is all we need for a one-minus amplitude. Therefore we compute the amplitude by applying (C.3) to the reduced kinematic factors and multiplying by $N_s = 2$. Finally we make change of the integration variables in (C.1) to a_k using $x_{i_k} = \sum_{l=1}^k a_l$, which yields an integral in the usual Feynman parametrisation.

C.2 Application to $M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$

The (topologies of the) graphs that have a non-vanishing contribution to $M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$ are shown in figure 6. There are 117 such labelled graphs in total: 12 massless pentagons (figure 6a), 30 one-mass boxes (figure 6b), 15 two-mass triangles (figure 6c) and 30 one-mass triangles (figure 6d). There are also 30 massive bubbles, but these vanish by the pinching process when using the spinor helicity choice (C.4) below.

In order to define the ε_i and $\bar{\varepsilon}_i$ (which are set to the same values after multi-linearisation in (C.2)), we choose k_5 as the reference momentum for the first graviton and k_1 for the rest. In the spinor-helicity formalism the polarisation vectors are

$$\varepsilon_1^\mu = \frac{[5|\gamma^\mu|1\rangle}{\sqrt{2}[15]} \quad \text{and} \quad \varepsilon_i^\mu = \frac{[i|\gamma^\mu|1\rangle}{\sqrt{2}\langle 1i\rangle} \quad \text{for } i \neq 1. \tag{C.4}$$

We have the standard spinor-helicity results that $k_i \cdot \varepsilon_i = k_5 \cdot \varepsilon_1 = k_1 \cdot \varepsilon_i = 0$ for all i , and furthermore for this choice $\varepsilon_i \cdot \varepsilon_j$ vanishes for all i, j , so there are no second derivatives of Green's functions to handle. After dropping the $\exp(k_i \cdot k_j \dot{G}_B^{ij})$, (C.2) becomes,

$$\begin{aligned} & (k_2 \cdot \varepsilon_1 \dot{G}_B^{12} + k_3 \cdot \varepsilon_1 \dot{G}_B^{13} + k_4 \cdot \varepsilon_1 \dot{G}_B^{14})(k_3 \cdot \varepsilon_2 \dot{G}_B^{23} + k_4 \cdot \varepsilon_2 \dot{G}_B^{24} + k_5 \cdot \varepsilon_2 \dot{G}_B^{25}) \\ & \times (k_2 \cdot \varepsilon_3 \dot{G}_B^{32} + k_4 \cdot \varepsilon_3 \dot{G}_B^{34} + k_5 \cdot \varepsilon_3 \dot{G}_B^{35})(k_2 \cdot \varepsilon_4 \dot{G}_B^{42} + k_3 \cdot \varepsilon_4 \dot{G}_B^{43} + k_5 \cdot \varepsilon_4 \dot{G}_B^{45}) \\ & \times (k_2 \cdot \varepsilon_5 \dot{G}_B^{52} + k_3 \cdot \varepsilon_5 \dot{G}_B^{53} + k_4 \cdot \varepsilon_5 \dot{G}_B^{54}) \times (l \rightarrow r). \end{aligned} \tag{C.5}$$

Here, '(l \rightarrow r)' denotes taking the expression to the left and replacing all \dot{G}_B^{ij} with \dot{G}_B^{ij} . We can now see why there are no bubble graphs in the problem. They come from three pinches that form two independent trees, but any such sequence of pinches will either pull

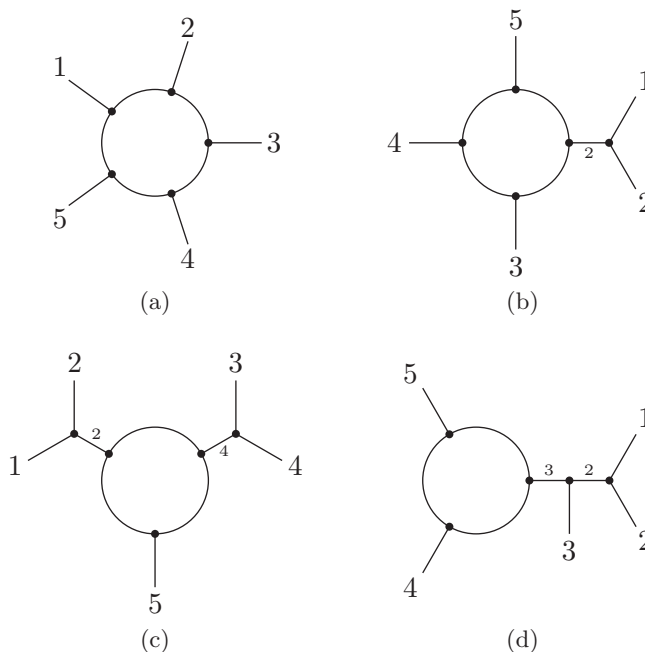


Figure 6: Topologies for ϕ^3 -like Feynman diagrams that have a non-vanishing contribution to the string-based rules calculation of $M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$. (The labellings shown are non-vanishing examples; other orderings also contribute.)

out a factor of the form $(k_1 + k_i) \cdot \varepsilon_i$, which vanishes by conservation of momentum and the remarks below (C.4), or simply run out of pinchable G_B s.

Since the same substitution rule (C.3) is applied independently to both the left- and right-moving sectors, and at each step in the pinching we pull out terms containing exactly one power of both \dot{G}_B^{ij} and \ddot{G}_B^{ij} , we can in fact proceed in a rather more straightforward manner by applying the pinching and substituting for just the left-moving factors of (C.5), then taking the square of the result as $K^{(\gamma)}(x_{i_1}, \dots, x_{i_{n_\ell-1}})$, taking care *not* to square the kinematic factors that arise from the trees during pinching.

We do this for all 117 graphs and substitute back into (C.1), changing the variables to the usual Feynman parameters. Each graph thereby yields an expression of the form

$$\mathcal{D}(\gamma) = \sum_{\{p_i\}} X^{(\gamma)}(r_1, \dots, r_{n_\ell}) I_{n_\ell}^{(\gamma)}[a_1^{r_1} \dots a_{n_\ell}^{r_{n_\ell}}], \tag{C.6}$$

where $X^{(\gamma)}(r_1, \dots, r_{n_\ell})$ is a rational coefficient. The n_ℓ -gonal tensor Feynman integral with momentum configuration relevant to the graph γ is defined as

$$I_{n_\ell}^{(\gamma)}[a_1^{r_1} \dots a_{n_\ell}^{r_{n_\ell}}] = \Gamma(n_\ell - 2 + \epsilon) \int_0^1 d^{n_\ell} a \frac{a_1^{r_1} \dots a_{n_\ell}^{r_{n_\ell}} \delta(1 - \sum_i a_i)}{\left\{ \sum_{k,l=1}^{n_\ell} S_{kl}^{(\gamma)} a_k a_l - i\varepsilon \right\}^{n_\ell-2+\epsilon}}, \tag{C.7}$$

with the array $S_{kl}^{(\gamma)}$ given in terms of the momenta P_{i_k} entering γ 's loop by

$$S_{kl}^{(\gamma)} = \begin{cases} 0 & \text{for } k = l, \\ -\frac{1}{2}(P_{i_k} + \dots + P_{i_{l-1}})^2 & \text{otherwise.} \end{cases}$$

These integrals may be evaluated by the recursive approach detailed in refs. [42, 59]. In this way we have constructed an integral database using computer algebra, indexed by the tuples $\{(r_1, \dots, r_{n_\ell}) | \sum_i r_i \leq 2n_\ell\}$ for integrals with $3 \leq n_\ell \leq 5$ and up to $5 - n_\ell$ massive legs. The table contains both the rational pieces of the integrals and the rational coefficients of their (di)logarithms. Since $M^{1\text{-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$ is entirely rational, we can use the vanishing of the logarithmic terms as a consistency check. Computationally the most complicated integrals are the pentagons where we must evaluate integrals with Feynman parameter polynomials of order ten.

The coefficients produced in this manner are far too large and cumbersome to present here (or even form a compact analytic expression for the amplitude at present); nevertheless, they are amenable to exact numeric arithmetic at a kinematic point. The results have been compared with, and agree with, the recursion-derived expression (4.18).

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