

Actions on the quiver: discrete quotients on the Coulomb branch

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ABSTRACT: This paper introduces two operations in quiver gauge theories. The first operation, *collapse*, takes a quiver with a permutation symmetry S_n and gives a quiver with adjoint loops. The corresponding 3d $\mathcal{N} = 4$ Coulomb branches are related by an orbifold of S_n . The second operation, *multi-lacing*, takes a quiver with n nodes connected by edges of multiplicity k and replaces them by n nodes of multiplicity qk . The corresponding Coulomb branch moduli spaces are related by an orbifold of type \mathbb{Z}_q^{n-1} . Collapse generalises known cases that appeared in the literature [1–3]. These two operations can be combined to generate new relations between moduli spaces that are constructed using the magnetic construction.

KEYWORDS: Supersymmetric Gauge Theory, Discrete Symmetries, Brane Dynamics in Gauge Theories, Supersymmetry and Duality

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1 Introduction

In recent years diagrammatic techniques on $3d \mathcal{N} = 4$ quiver gauge theories have been developed and provide insight and simple tools to study their moduli spaces of vacua. Examples of diagrammatic techniques on $3d \mathcal{N} = 4$ quiver gauge theories include *quiver subtraction* [4] which can be used to study the stratification of the Coulomb branch into a finite number of symplectic leaves [5–7] or to study hyper-Kähler quotients of Coulomb branches [8]. The former also has a realisation in string theory [9, 10].

More pertinent to this work is the study of discrete actions on the Coulomb branch via diagrammatic techniques. Examples include *folding* [11, 12] and *discrete gauging* [1–3, 12]. These realise discrete actions on the Coulomb branch.

One result of this work is a generalisation of the discrete gauging of a bouquet of $U(1)$ in $3d \mathcal{N} = 4$ unitary quiver gauge theories introduced in [1–3]. The discrete gauging of such a bouquet has a realisation in M-theory as a stack of M5 branes probing an A_k singularity and making these M5 branes coincident. There is a natural S_n action permuting the n M5. The reduction of the M-theory set-up to Type IIA corresponds to the same S_n action on NS5 branes. The magnetic quiver [13] for the theory on the worldvolume of the D6 branes has a bouquet of the $U(1)$ gauge nodes when the NS5 are not coincident and becomes a $U(n)$ gauge node with an adjoint hypermultiplet when the NS5 are coincident. This is illustrated below with the $U(1)$ in the bouquet shown in orange:

$$\mathcal{C} \left(\begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \\ \text{1} \quad \text{2} \quad \dots \quad \text{k} \quad \dots \quad \text{2} \quad \text{1} \\ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \\ \text{1} \quad \dots \quad \text{n} \quad \dots \quad \text{1} \end{array} \right) \xrightarrow{S_n} \mathcal{C} \left(\begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \\ \text{1} \quad \text{2} \quad \dots \quad \text{k} \quad \dots \quad \text{2} \quad \text{1} \\ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \\ \text{1} \quad \dots \quad \text{n} \quad \dots \quad \text{1} \end{array} \right) \quad (1.1)$$

An application of the discrete quotients of Coulomb branches was to physically realise the results of Kostant-Brylinski [14], which relate certain nilpotent orbit closures under discrete quotients.

In this work the notion of a bouquet of $U(1)$ gauge nodes in a unitary $3d \mathcal{N} = 4$ quiver gauge theory is encompassed and extended to a *complete graph* of $U(1)$ gauge nodes. There is a natural S_n permutation of these $U(1)$ gauge nodes. This outer automorphism symmetry can be gauged and consequently there is a discrete S_n action on the Coulomb branch.

A second result of this work is to realise orbifold actions on the Coulomb branch by turning simply laced edges into non-simply laced edges [15]. The relationships between quivers with simply laced and non-simply laced edges were studied in [16, 17] and are studied further here.

These two new diagrammatic techniques, named *collapse* and *multi-lacing* respectively, are applied to various quivers and examples are presented for each technique applied on its own and also in combination.

There is a deep and rich connection between Coulomb branches and the mathematics of symplectic singularities, and in particular to slices in the nilpotent cone of semi-simple Lie algebras. The techniques of collapse and multi-lacing may be applied to quivers to study these moduli spaces. For example, the S_2 quotient of the Kleinian A-type singularity to the D-type singularity is constructed as a Coulomb branch. In addition many new discrete

quotients of known moduli spaces such as $\overline{\text{min.}A_2} = a_2$, $\overline{\text{min.}A_3} = a_3$, and $\overline{\text{min}D_4} = d_4$ are constructed. Another outcome of this work is the Coulomb branch construction of a_2/S_4 and d_4/S_4 , which appear respectively as a Słodowy slice in the nilpotent cone of \mathfrak{e}_8 between the nilpotent orbits with Bala-Carta labels $E_8(a_6)$ and $E_8(b_6)$ and as a Słodowy slice in the nilpotent cone of \mathfrak{f}_4 between the nilpotent orbits with Bala-Carter labels $F_4(a_3)$ and A_2 [18, 19]. One way of constructing d_4/S_4 as a Coulomb branch is through the discrete gauging of a bouquet. The discrete gauging of a bouquet is not amenable for the case of a_2/S_4 . Instead, the isomorphism $S_4 \cong \mathbb{Z}_2^2 \rtimes S_3$ is employed. On the quiver, a combination of the two new techniques, orbifold and complete graph discrete quotients, are used to achieve the desired result. Specifically, we have the first construction of a_2/S_4 as a Coulomb branch and now two ways of constructing d_4/S_4 .

Another outcome of this work is the discrete quotient on the Coulomb branch by a semi-direct product of discrete groups rather than a direct product of discrete groups.

The quivers for the $3d$ mirrors of certain Argyres-Douglas theories contain a complete graph [20, 21]. The discrete quotient on the Coulomb branch of the $3d$ mirrors are computed.

Outline of the paper. The paper is organised as follows. In section 2 the complete graph quiver is introduced and the notation that is used throughout this paper is defined. The claim that there is an S_n quotient between the Coulomb branches of quivers with complete graphs and a $U(n)$ gauge node with some number of adjoint hypermultiplets is stated. In section 3 the monopole formula is introduced as well as the Molien sum on the monopole formula. The claim made in section 2 is proved. The Molien sum on the monopole formula realising the orbifold relation between simply and non-simply laced quivers is also presented. In section 4 the Abelianisation procedure is introduced. In section 5 many examples of the S_n quotient are presented. These include the S_2 gauging of the A-type singularity to the D-type singularity, and, additionally, the $S_2 \times S_2$ gauging of the A_3 affine quiver. In section 6 examples of \mathbb{Z}_q quotient are presented. In section 7 a combination of the S_n and \mathbb{Z}_q quotients on the Coulomb branch are combined. In section 8 the S_n quotient is generalised to other types of gauge group. In section 9 conclusion are drawn and future work is discussed.

2 Discrete quotient on the quiver

Loops and complete graphs. The first family of quivers studied in this paper is $\mathcal{CG}_{n,k}$, the family of complete graphs shown in figure 1.

The quiver $\mathcal{CG}_{n,k}$ consists of n nodes of $U(1)$ with each pair of $U(1)$ s connected by k hypermultiplets transforming in the bi-fundamental representation. An overall $U(1)$ acts freely and the actual gauge group is $U(1)^n/U(1)$. This family received some attention as the $3d$ mirror of the Argyres Douglas theories of type (A_{n-1}, A_{nk-1}) [20, 22], i.e. the Coulomb/Higgs branch of the $\mathcal{CG}_{n,k}$ theory is the Higgs/Coulomb branch of the $3d$ (A_{n-1}, A_{nk-1}) theory, respectively. For most quivers studied in this paper only even k is needed, and furthermore a convenient parametrisation is $k = 2g - 2$, where $g \in \mathbb{Z}^+$ is suggestive of the genus of a Riemann surface, as becomes clearer in section 5.5. Note that there is an S_n permutation symmetry of the $U(1)$ gauge nodes which comes from the outer automorphism symmetry of

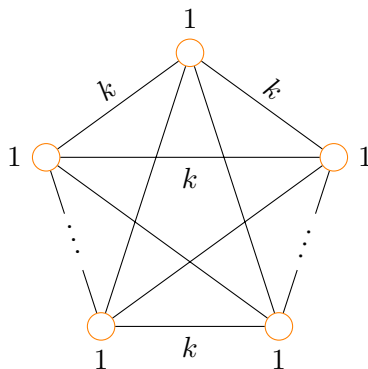


Figure 1. The complete graph quiver $\mathcal{CG}_{n,k}$ with n nodes of $U(1)$ and edge multiplicity k . Henceforth $k = 2g - 2$.



Figure 2. Multi-loop quiver $\mathcal{ML}_{n,g}$ of $U(n)$ gauge group with g adjoints.

$\mathcal{CG}_{n,2g-2}$. Correspondingly, the Coulomb branch has this S_n symmetry, and one can construct many orbifolds by gauging a subgroup of this symmetry.

The second family of quivers studied here is the multi-loop quiver $\mathcal{ML}_{n,g}$ shown in figure 2. This quiver consists of a $U(n)$ gauge group with g hypermultiplets in the adjoint representation. As the centre $U(1)$ is acting trivially on the matter content, the actual gauge group is $PSU(n)$.

The two main results of this paper are

1. The Coulomb branches of $\mathcal{ML}_{n,g}$ and $\mathcal{CG}_{n,2g-2}$ are related by a quotient of the S_n symmetry:

$$\mathcal{C}(\mathcal{ML}_{n,g}) = \mathcal{C}(\mathcal{CG}_{n,2g-2})/S_n. \tag{2.1}$$

Here the edge multiplicity k is required to be even. Odd multiplicity gives non-integer g which makes the quiver of figure 2 ill defined. This S_n action represented on quivers is the *collapse* of the $\mathcal{CG}_{n,2g-2}$ quiver to $\mathcal{ML}_{n,g}$.

2. Given a complete graph quiver $\mathcal{CG}_{n,k}$, if the edge multiplicity is increased to be q -fold, the quiver becomes $\mathcal{CG}_{n,qk}$, and the Coulomb branches are related by a quotient of \mathbb{Z}_q^{n-1} :

$$\mathcal{C}(\mathcal{CG}_{n,qk}) = \mathcal{C}(\mathcal{CG}_{n,k})/\mathbb{Z}_q^{n-1}. \tag{2.2}$$

This \mathbb{Z}_q^{n-1} action represented on quivers is the *multi-lacing* of the $\mathcal{CG}_{n,k}$ quiver to $\mathcal{CG}_{n,qk}$.

There are some corollaries of these two results.

Combined action. Notice that the S_n symmetry is preserved after the \mathbb{Z}_q^{n-1} quotient: $\mathcal{CG}_{n,1}$ and $\mathcal{CG}_{n,q}$ have the same outer-automorphism symmetry, so one can combine the two

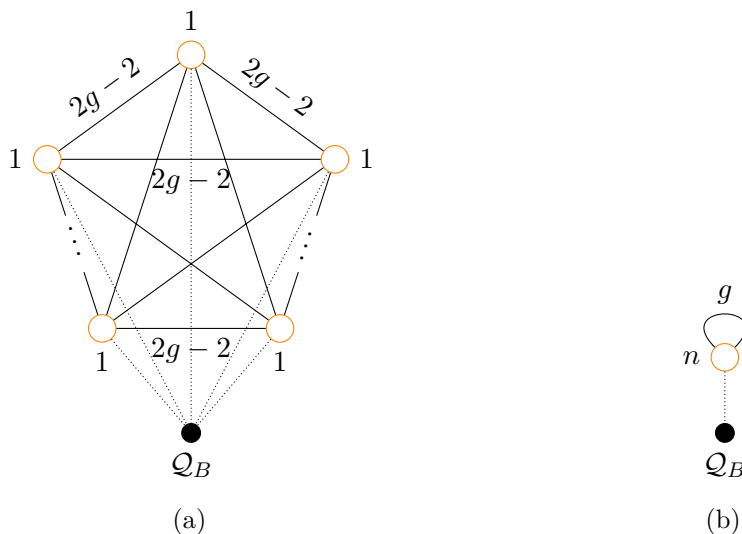


Figure 3. (a): Quiver $\mathcal{Q}_{3(a)}$ which has the complete graph subquiver $\mathcal{CG}_{n,2g-2}$, connecting to background \mathcal{Q}_B . (b): Quiver $\mathcal{Q}_{3(b)}$ for $U(n)$ gauge group with g adjoints, connecting to background \mathcal{Q}_B .

relations (2.1) and (2.2) above, and result in the following quotient relation:

$$\mathcal{C}(\mathcal{ML}_{n,g}) = \mathcal{C}(\mathcal{CG}_{n,1}) / \mathbb{Z}_{2g-2}^{n-1} \rtimes S_n. \tag{2.3}$$

The composition is a semi-direct product, since the S_n symmetry also permutes these $n-1$ \mathbb{Z}_{qS} .

Connecting to a background quiver. The two families of quivers $\mathcal{CG}_{n,2g-2}$ and $\mathcal{ML}_{n,g}$ can be connected to a *pivot* node of a quiver \mathcal{Q}_B .

Then there are two families of quivers $\mathcal{Q}_{3(a)}$ and $\mathcal{Q}_{3(b)}$, shown in figure 3(a) and figure 3(b) respectively. The dotted lines in the quivers $\mathcal{Q}_{3(a)}$ and $\mathcal{Q}_{3(b)}$ refer to a link of any type to the same pivot node in \mathcal{Q}_B . For example, the dotted links could be a single bi-fundamental hypermultiplet, multiple bi-fundamental hypermultiplets, or directed non-simply laced edges. They can also be connected to multiple pivot nodes in \mathcal{Q}_B as long as the links to that pivot are all the same for the $U(1)$ nodes in the complete graph.

By adding this pivot node, the quiver outer-automorphism still contains the same S_n , hence the same quotient relation on the Coulomb branch holds:

$$\mathcal{C}(\mathcal{Q}_{3(b)}) = \mathcal{C}(\mathcal{Q}_{3(a)}) / S_n. \tag{2.4}$$

For the special case of $g = 1$ and each connection to some gauge node in \mathcal{Q}_B is a single bi-fundamental, claim (2.4) reproduces the S_n quotient result for a bouquet of $U(1)$ in [1–3].

Moreover, the complete graph part of $\mathcal{Q}_{3(a)}$ contains subgraphs which themselves are complete graphs of m nodes ($m \leq n$). So Claim (2.4) implies that any S_m with $m \leq n$ can be quotiented from $\mathcal{Q}_{3(a)}$. An important consequence of this relation is that every unitary quiver with a loop can be replaced by a quiver that has no loops, but instead admits a discrete permutation symmetry.

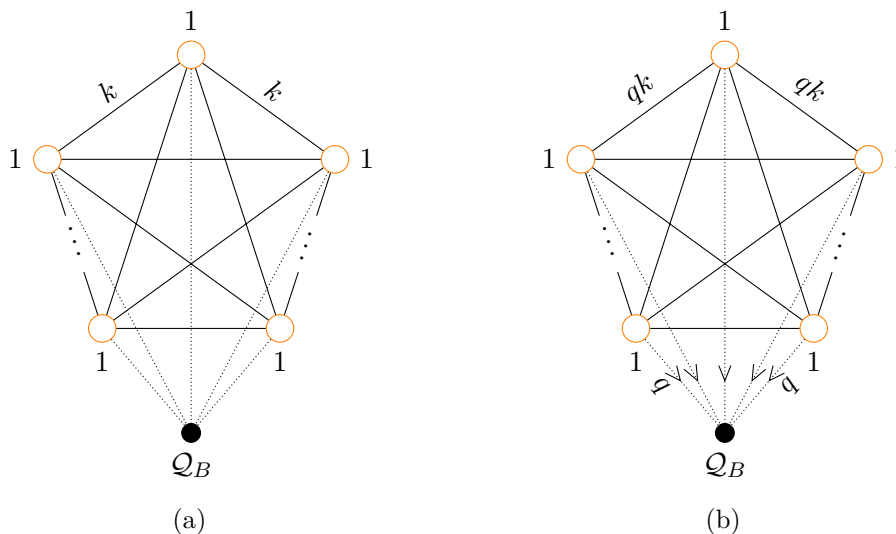


Figure 4. (a): $\mathcal{CG}_{n,k}$ with simply laced edges connecting a simply laced \mathcal{Q}_B . (b): $\mathcal{CG}_{n,qk}$ with q laced edges connecting a simply laced \mathcal{Q}_B .

The \mathbb{Z}_q quotient relation also holds when connecting to a background quiver. The following claims are made:¹

$$c(\mathcal{Q}_{4(b)}) = \begin{cases} c(\mathcal{Q}_{4(a)}/\mathbb{Z}_q^{n-1}), & \text{when } \mathcal{Q}_B \text{ is empty or unframed,} \\ c(\mathcal{Q}_{4(a)}/\mathbb{Z}_q^n), & \text{when } \mathcal{Q}_B \text{ is framed,} \end{cases} \quad (2.5)$$

the reason for the different action for the framed and unframed \mathcal{Q}_B is discussed in section 3.3.

Notice that the S_n symmetry is preserved after \mathbb{Z}_q^n quotient, and the S_n symmetry permutes these n \mathbb{Z}_q s. The following claims are made:

$$c(\mathcal{Q}_{4(b)}/S_n) = \begin{cases} c(\mathcal{Q}_{4(a)}/\mathbb{Z}_q^{n-1} \rtimes S_n), & \text{when } \mathcal{Q}_B \text{ is empty or unframed,} \\ c(\mathcal{Q}_{4(a)}/\mathbb{Z}_q^n \rtimes S_n), & \text{when } \mathcal{Q}_B \text{ is framed.} \end{cases} \quad (2.6)$$

3 Generalised Molien sum and Coulomb branch Hilbert series

The Molien sum is a method to calculate the Hilbert series of an orbifold from a known Hilbert series, more details are in appendix B. In this section the generalised Molien sum on the monopole formula is introduced and (2.1) and (2.4) are proved using the Hilbert series.

3.1 Conformal dimension and monopole formula

The monopole formula, introduced in [23], is a method to calculate the Coulomb branch Hilbert series of a 3d $\mathcal{N} = 4$ gauge theory. To evaluate the monopole formula, firstly the conformal dimension of the bare monopole operator [24] is determined. Further details about the monopole formula are in appendix A.

¹This claim does not rely on the complete graph structure. In fact, one can replace the complete graph with an arbitrary simply-laced Abelian quiver, and the quotient relation on Coulomb branch still holds.

The conformal dimension for the $\mathcal{CG}_{n,2g-2}$ theory is computed as

$$\Delta(m_1, \dots, m_n) = (g-1) \sum_{1 \leq i < j \leq n} |m_i - m_j|, \quad (3.1)$$

where m_i labels the magnetic charges and $(m_1, \dots, m_n) \in \text{Hom}(U(1), U(n)) = \Lambda_\omega^{U(n)^\vee} = \mathbb{Z}^n$. The key point is that the $\mathcal{ML}_{n,g}$ theory has exactly the same conformal dimension.

Given this fact for the standalone quivers $\mathcal{CG}_{n,2g-2}$ and $\mathcal{ML}_{n,g}$ it is a simple extension to show that the conformal dimension when these quivers are attached to a background quiver \mathcal{Q}_B , as in quivers $\mathcal{Q}_{3(b)}$ and $\mathcal{Q}_{3(a)}$, is also the same,

$$\Delta(m_1, \dots, m_n; \vec{B}) = (g-1) \sum_{1 \leq i < j \leq n} |m_i - m_j| + \sum_{1 \leq j \leq l} \sum_{1 \leq i \leq n} \Delta_j(m_i; \vec{B}) + \Delta[\mathcal{Q}_B]. \quad (3.2)$$

Here, $\Delta[\mathcal{Q}_B]$ is the contribution from \mathcal{Q}_B . There could be links to each of the l gauge nodes in \mathcal{Q}_B , the contribution from these links to the j^{th} node in \mathcal{Q}_B is denoted $\Delta_j(m_i; \vec{B})$ for $1 \leq j \leq l$. The contribution $\Delta_j(m_i; \vec{B})$ takes the same functional form for all m_i . This is the only requirement for the links to \mathcal{Q}_B in the quivers. So those links can be a single bi-fundamental hypermultiplet, multiple bi-fundamental hypermultiplets, and also directed non-simply laced edges, as long as they are all the same. Note that if a non-simply laced edge is used then these must all be directed in the same way for all connections to \mathcal{Q}_B .

The conformal dimensions for the $\mathcal{CG}_{n,2g-2}$ and the $\mathcal{ML}_{n,g}$ are the same, but the difference in Hilbert series for the Coulomb branch of $\mathcal{Q}_{3(b)}$ and $\mathcal{Q}_{3(a)}$ only comes from the magnetic lattice and the dressing factors.

3.2 Generalised Molien sum of S_n on the monopole formula

We take the specific action of S_n on the monopole formula and explain how the generalised Molien sum works.

Let us start with the quiver $\mathcal{Q}_{3(a)}$, which contains n identical $U(1)$ nodes and a background \mathcal{Q}_B . Quiver $\mathcal{Q}_{3(a)}$ is unchanged under permutation of those n $U(1)$ nodes, i.e. there is a $S_n \subset \text{Out}(\mathcal{Q}_{3(a)})$.

We label the magnetic flux associated to the i -th node $U(1)_i$ as m_i , and the collective magnetic fluxes associated to \mathcal{Q}_B are denoted by \vec{B} . We label the dressing factor as $P_{U(1)^n} = \frac{1}{(1-t^2)^n}$ and $P_B(\vec{B})$. The monopole formula of $\mathcal{Q}_{3(a)}$ can be written as:

$$\text{HS}(\mathcal{C}(\mathcal{Q}_{3(a)})) = \sum_{m_1, \dots, m_n, \vec{B}} P_{U(1)^n} P_B(\vec{B}) \cdot z_1^{m_1} \dots z_n^{m_n} Z(\vec{B}) \cdot t^{\Delta(m_1, \dots, m_n; \vec{B})}, \quad (3.3)$$

where the z_i are topological fugacities for the $U(1)_i$ and $Z(\vec{B})$ is a monomial of topological fugacities associated to gauge nodes in \mathcal{Q}_B . The action of S_n permutes magnetic fluxes m_i , topological fugacities z_i and Casimir invariants, however leaves $P_B(\vec{B})$ and $Z(\vec{B})$ invariant by construction. This can be implemented in the monopole formula by restricting it to the fixed loci of $\sigma \in S_n$, which is discussed below in detail.

Elements of S_n can be written as cycles: $\sigma = (i_1 \dots i_{\sigma_1})(i_{\sigma_1+1} \dots i_{\sigma_1+\sigma_2}) \dots (i_{n-\sigma_k+1} \dots i_n)$, where $\sum_{l=1}^k \sigma_l = n$. For example, $(1\ 3)(2\ 5\ 4) \in S_5$ acts on the magnetic fluxes as:

$$(m_1, m_2, m_3, m_4, m_5) \rightarrow (m_3, m_5, m_1, m_2, m_4). \quad (3.4)$$

Under the action of σ , fluxes within the same cycle are identified, i.e. $m_{i_1} = \dots = m_{i_{\sigma_1}}, \dots, m_{i_{n-\sigma_k+1}} = \dots = m_{i_n}$. The same rule applies for the topological fugacities z_i .

The next step is to determine the dressing factors under σ . The Casimir invariants form the polynomial ring of \mathbb{C}^n . The degree of each coordinate is scaled by a factor of 2 and the Hilbert series $\frac{1}{(1-t^2)^n}$ is exactly the dressing factor of $U(1)^n$. The Hilbert series of the invariant ring under S_n follows from the Molien formula:

$$\begin{aligned} \text{HS} \left((\mathbb{C}^n)^{S_n} \right) &= \frac{1}{n!} \sum_{\sigma \in S_n} \frac{1}{\det(1-t^2 \cdot \sigma)} \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \frac{1}{(1-t^{2\sigma_1}) \dots (1-t^{2\sigma_k})} \\ &= \frac{1}{(1-t^2) \dots (1-t^{2n})}. \end{aligned} \tag{3.5}$$

which is the dressing factor for the unbroken $U(n)$. Hence, on the fixed loci of σ , the dressing factors turns into:

$$P_{U(1)^n} = \frac{1}{(1-t^2)^n} \xrightarrow{\sigma} \frac{1}{\det(1-t^2 \cdot \sigma)} = \frac{1}{(1-t^{2\sigma_1}) \dots (1-t^{2\sigma_k})} \tag{3.6}$$

Now the monopole formula under the action of σ is expressed. The fugacities z_i and $Z(\vec{B})$ are set to 1 for brevity, however recall that the z_i have the same behaviour as $t^{\Delta(m_1, \dots, m_n; \vec{B})}$ under S_n and $Z(\vec{B})$ is invariant. The monopole formula under the element σ becomes:

$$\text{HS} \left(\mathcal{C}(\mathcal{Q}_{3(a)}) \right)^\sigma = \sum_{\vec{B}} \sum_{m_{i_1} = \dots = m_{i_{\sigma_1}}} \dots \sum_{m_{i_{n-\sigma_k+1}} = \dots = m_{i_n}} \left(\prod_{l=1}^k \frac{1}{(1-t^{2\sigma_l})} \right) P_B(\vec{B}) \cdot t^{\Delta(m_1, \dots, m_n; \vec{B})}. \tag{3.7}$$

According to Molien formula, the monopole formula applied to $\mathcal{C}(\mathcal{Q}_{3(a)})/S_n$ is the average over the S_n action:

$$\text{HS} \left(\mathcal{C}(\mathcal{Q}_{3(a)}/S_n) \right) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{HS} \left(\mathcal{C}(\mathcal{Q}_{3(a)}) \right)^\sigma. \tag{3.8}$$

This generalised Molien sum is realised through cycle index for special cases in [2] and also agrees with the Hilbert series of the wreathed quiver [11].

Proof of the S_n relation. In this paragraph, the claim (2.4) is proved at the level of the Hilbert series. For claim (2.4) to be true, the following must hold:

$$\text{HS} \left(\mathcal{C}(\mathcal{Q}_{3(a)}/S_n) \right) = \text{HS} \left(\mathcal{C}(\mathcal{Q}_{3(b)}) \right). \tag{3.9}$$

To begin with, the monopole formula of $\mathcal{Q}_{3(b)}$ can be written as:

$$\text{HS} \left(\mathcal{C}(\mathcal{Q}_{3(b)}) \right) = \sum_{\vec{B}} \sum_{m_1 \geq \dots \geq m_n} P_{U(n)}(m_1, \dots, m_n) P_B(\vec{B}) \cdot t^{\Delta(m_1, \dots, m_n; \vec{B})}, \tag{3.10}$$

where the magnetic fluxes are restricted to the principle Weyl chamber, and $P_{U(n)}(m_1, \dots, m_n)$ are the dressing factors of $U(n)$. The value of $P_{U(n)}$ is determined by the number of identical

magnetic fluxes [23]. If the values of the fluxes (m_1, \dots, m_n) satisfy $m_{i_1} = \dots = m_{i_{\sigma_1}} > \dots > m_{i_{n-\sigma_k+1}} = \dots = m_{i_n}$, then $P_{U(n)}(m_1, \dots, m_n) = \frac{1}{(1-t^2)\dots(1-t^{2\sigma_1})} \dots \frac{1}{(1-t^2)\dots(1-t^{2\sigma_k})}$. A short-hand notation for the fixed loci of vector of magnetic fluxes under σ is introduced as:

$$(m, \sigma) : \quad m_{i_1} = \dots = m_{i_{\sigma_1}}, \dots, m_{i_{n-\sigma_k+1}} = \dots = m_{i_n}, \quad (3.11a)$$

$$(m; \sigma) : \quad m_{i_1} = \dots = m_{i_{\sigma_1}} > \dots > m_{i_{n-\sigma_k+1}} = \dots = m_{i_n}, \quad \begin{array}{l} \text{with all possible orderings} \\ \text{of the cycles in } \sigma \end{array} \quad (3.11b)$$

For example, $(m, (1\ 3)(2\ 5\ 4))$ means $m_1 = m_3, m_2 = m_5 = m_4$, and $(m; (1\ 3)(2\ 5\ 4))$ means $m_1 = m_3 > m_2 = m_5 = m_4 \cup m_2 = m_5 = m_4 > m_1 = m_3$. In addition $\sum_{(m, \sigma)}$ and $\sum_{(m; \sigma)}$ denotes the sum of the magnetic flux with respect to these conditions.

Without restricting the monopole formula (3.10) to the principal Weyl chamber, it can be expressed as:

$$\text{HS}(\mathcal{C}(\mathcal{Q}_{3(b)})) = \sum_{\vec{B}} \sum_{\sigma \in S_n} \frac{\sigma_1! \dots \sigma_k!}{n!} \sum_{(m; \sigma)} \frac{1}{(1-t^2)\dots(1-t^{2\sigma_1})} \dots \frac{1}{(1-t^2)\dots(1-t^{2\sigma_k})} P_B(\vec{B}) \cdot t^{\Delta(m_1, \dots, m_n; \vec{B})}. \quad (3.12)$$

With (3.5), the monopole formula (3.12) is rewritten as:

$$\begin{aligned} \text{HS}(\mathcal{C}(\mathcal{Q}_{3(b)})) &= \sum_{\vec{B}} \sum_{\sigma \in S_n} \frac{\sigma_1! \dots \sigma_k!}{n!} \sum_{(m; \sigma)} \left(\prod_{\sigma_i \in \sigma} \left(\frac{1}{\sigma_i!} \sum_{\rho' \in S_{\sigma_i}} \frac{1}{(1-t^{2\rho'_1}) \dots (1-t^{2\rho'_l})} \right) \right) P_B(\vec{B}) \cdot t^{\Delta(m_1, \dots, m_n; \vec{B})} \\ &= \frac{1}{n!} \sum_{\vec{B}} \sum_{\sigma \in S_n} \sum_{\rho \in S_{\sigma_1} \times \dots \times S_{\sigma_k}} \sum_{(m; \sigma)} \frac{1}{(1-t^{2\rho_1}) \dots (1-t^{2\rho_l})} P_B(\vec{B}) \cdot t^{\Delta(m_1, \dots, m_n; \vec{B})} \\ &= \frac{1}{n!} \sum_{\vec{B}} \sum_{\rho \in S_n} \sum_{\sigma \ni \rho} \sum_{(m; \sigma)} \frac{1}{(1-t^{2\rho_1}) \dots (1-t^{2\rho_l})} P_B(\vec{B}) \cdot t^{\Delta(m_1, \dots, m_n; \vec{B})}. \end{aligned} \quad (3.13)$$

Here $(\rho'_1, \dots, \rho'_l)$ are σ_i -partition corresponding to the sizes of cycles of $\rho' \in S_{\sigma_i}$, and (ρ_1, \dots, ρ_l) are n -partition corresponding to the sizes of cycles of $\rho \in S_{\sigma_1} \times \dots \times S_{\sigma_k}$, the minimal permutation subgroup containing σ . A further bit of notation is that the sum over all $\sigma \in S_n$ whose minimal permutation subgroup contains $S_{\rho_1} \times \dots \times S_{\rho_l}$ is denoted $\sum_{\sigma \ni \rho}$. Moreover, it follows that $\sum_{\sigma \in S_n} \sum_{\rho \in S_{\sigma_1} \times \dots \times S_{\sigma_k}}$ is equivalent to $\sum_{\rho \in S_n} \sum_{\sigma \ni \rho}$. Also note that:

$$\sum_{(m, \rho)} t^{\Delta(m_1, \dots, m_n; \vec{B})} = \sum_{\sigma \ni \rho} \sum_{(m; \sigma)} t^{\Delta(m_1, \dots, m_n; \vec{B})}, \quad (3.14)$$

for example, $\sum_{(m, (1\ 3)(2\ 5\ 4))} = \sum_{m_1=m_3, m_2=m_5=m_4} = \sum_{m_1=m_2=m_3=m_4=m_5} + \sum_{m_1=m_3 > m_2=m_5=m_4} + \sum_{m_1=m_3 < m_2=m_5=m_4}$.

Combining (3.7), (3.13) and (3.14), the following HS is obtained:

$$\begin{aligned} \text{HS}(\mathcal{C}(\mathcal{Q}_{3(b)})) &= \frac{1}{n!} \sum_{\vec{B}} \sum_{\rho \in S_n} \sum_{(m, \rho)} \frac{1}{(1-t^{2\rho_1}) \dots (1-t^{2\rho_l})} P_B(\vec{B}) \cdot t^{\Delta(m_1, \dots, m_n; \vec{B})} \\ &= \frac{1}{n!} \sum_{\vec{B}} \sum_{\sigma \in S_n} \sum_{(m, \sigma)} \frac{1}{(1-t^{2\sigma_1})} \dots \frac{1}{(1-t^{2\sigma_k})} P_B(\vec{B}) \cdot t^{\Delta(m_1, \dots, m_n; \vec{B})} \\ &= \text{HS}(\mathcal{C}(\mathcal{Q}_{3(a)}/S_n)), \end{aligned} \quad (3.15)$$

here the first line to the second line is just a relabelling from ρ to σ . Hence, Claim (2.4) is proved at the level of the Hilbert series.

Ungauging. One can always ungauged on \mathcal{Q}_B if it is non-empty. However, if \mathcal{Q}_B is empty, one can set an arbitrary $m_i = 0$ and take a $\frac{1}{1-t^2}$ factor away from the whole dressing factor. The proof proceeds analogous to the above. Hence, the Claim (2.1) is also verified on the level of the Hilbert series.

3.3 Generalised Molien sum of \mathbb{Z}_q^n on the monopole formula

In this section, the assumption that the edges between the complete graph and \mathcal{Q}_B and \mathcal{Q}_B itself are simply laced is made. Apart from the outer-automorphism symmetry on the complete graph quiver, there is another discrete action \mathbb{Z}_q^n acting on the Cartan of the gauge group. After the \mathbb{Z}_q^n quotient, the edge multiplicity of edges in the complete graph increase q -fold, and the edge connecting complete graph and \mathcal{Q}_B pick up a lace q pointing to \mathcal{Q}_B . The edges in the complete graph can also be thought of as becoming double q -laced. However, for Abelian nodes, one cannot distinguish between the edge multiplicity and charge multiplicity at the level of the Coulomb branch. If \mathcal{Q}_B is empty or unframed, a $U(1) \subset U(1)^n$ is decoupled from complete graph, then the action reduced to \mathbb{Z}_q^{n-1} .

At the level of Hilbert series:

$$\text{HS}(\mathcal{C}(\mathcal{Q}_{4(b)})) = \begin{cases} \text{HS}(\mathcal{C}(\mathcal{Q}_{4(a)}/\mathbb{Z}_q^{n-1})) , & \text{when } \mathcal{Q}_B \text{ is empty or unframed ,} \\ \text{HS}(\mathcal{C}(\mathcal{Q}_{4(a)}/\mathbb{Z}_q^n)) , & \text{when } \mathcal{Q}_B \text{ is framed .} \end{cases} \quad (3.16)$$

To show this, the refinement fugacities are turned back on. The \mathbb{Z}_q^n acts on these fugacities in the following way:

$$\mathbb{Z}_q^n : (z_1, \dots, z_n) \rightarrow (\omega_q^{k_1} z_1, \dots, \omega_q^{k_n} z_n) , \quad (3.17)$$

where ω_q is the primitive q -th root of unity and $(k_1, \dots, k_n) \in \mathbb{Z}_q^n$. Similar to [17, 25], the Molien formula yields:

$$\begin{aligned} \text{HS}(\mathcal{C}(\mathcal{Q}_{4(a)}/\mathbb{Z}_q^n)) &= \frac{1}{q^n} \sum_{k_1, \dots, k_n} \sum_{m_1, \dots, m_n} P(m_1, \dots, m_n) \cdot \omega_q^{m_1 \cdot k_1} z_1^{m_1} \dots \omega_q^{m_n \cdot k_n} z_n^{m_n} \cdot t^{\Delta(m_1, \dots, m_n)} \\ &= \sum_{m_1, \dots, m_n} P(m_1, \dots, m_n) \cdot z_1^{q \cdot m_1} \dots z_n^{q \cdot m_n} \cdot t^{\Delta(q \cdot m_1, \dots, q \cdot m_n)} \\ &= \text{HS}(\mathcal{C}(\mathcal{Q}_{4(b)})) . \end{aligned} \quad (3.18)$$

Considering the ungauging process: if \mathcal{Q}_B is framed, there is no need to ungauged; if \mathcal{Q}_B is empty or unframed, one can set one $m_i = 0$ (or other choices so long as it cancels the same shift invariance), as a result, the \mathbb{Z}_q action on z_i is trivial and the overall action becomes \mathbb{Z}_q^{n-1} :

$$\text{HS}(\mathcal{C}(\mathcal{Q}_{4(a)}/\mathbb{Z}_q^{n-1}))|_{m_i=0} = \text{HS}(\mathcal{C}(\mathcal{Q}_{4(b)}))|_{m_i=0} . \quad (3.19)$$

Hence, Claims (2.2) and (2.5) are verified on the level of the Hilbert series.

As we can see from the proof, this \mathbb{Z}_q quotient has two origins: one is to increase the edge multiplicity q -times, the other is to increase the charge of hypermultiplets q -times. On the Coulomb branch geometry these two constructions have no difference, but can be distinguished by deformation and the corresponding Higgs branch.

4 Abelianisation

In this section the discrete action on the Coulomb branch of a 3d $\mathcal{N} = 4$ theory with unitary gauge groups via the Abelianisation method introduced in [26] is discussed. From the monopole formula it is easy to read the representation of the monopole operators under continuous global symmetry, but it is hard to find the exact representation under the discrete action. Abelianisation provides a systemic way to find out the discrete action induced by quiver outer-automorphism.

Start with an Abelian theory of rank n , i.e, the gauge group is $U(1)^n$, with hypermultiplets with charge $\rho \oplus \bar{\rho}$. The monopole operators transform under the topological $U(1)_{\text{top}}^n$. For each topological charge $A \in \mathbb{Z}^n$, there exists a (bare) monopole operator v_A . The Coulomb branch chiral ring is generated by the set of bare monopole operators v_A and the set of Casimir invariants ϕ_i of gauge group, which are subject to relations determined by the charge matrix ρ .

The simplest example is the theory of a single $U(1)$ with no hypermultiplet. The (bare) monopole operator can be written as a holomorphic function of v_1 and v_{-1} : for a positive integer k , $v_k = v_1^k$ and $v_{-k} = v_{-1}^k$. Hence, the chiral ring generators are v_1, v_{-1}, ϕ . The relation between generators is $v_1 v_{-1} = 1$. The ring $\mathbb{C}[v_1, v_{-1}, \phi]/\langle v_1 v_{-1} = 1 \rangle$ is exactly $T^*\mathbb{C}^\times$.

For a non-Abelian theory with unitary gauge group G of rank n , if the representation of hypermultiplets contains at least all the roots of G , one can always find a corresponding Abelian theory whose gauge group is the Cartan subgroup of the non-Abelian gauge group [27]. The charges of the hypermultiplets in this Abelian theory should add up to the charges of the representations of hypermultiplets under the Cartan subgroup, i.e. the bare monopole operators in both theories have the same conformal dimension contribution. The difference between the two Coulomb branch chiral rings is a quotient of the Weyl group W_G .

For the most general case, the Weyl group action on the Coulomb branch is:

$$\phi_i \longleftrightarrow \phi_{\sigma(i)}, \quad v_{(n_1, \dots, n_k)} \longleftrightarrow v_{(n_{\sigma(1)}, \dots, n_{\sigma(k)})}. \quad (4.1)$$

5 Examples of S_n quotients

The S_n gauging on quivers which are or contain certain complete graphs are computed and the corresponding S_n on their Coulomb branches are studied. This action is studied through the Hilbert series computed with the monopole formula and through Abelianisation for each example.

5.1 From Kleinian A to Kleinian D — An S_2 gauging

Consider the Abelian quiver $\mathcal{Q}_{5(a)} = \mathcal{CG}_{2,2g-2}$ of figure 5(a), whose Coulomb branch is the A -type singularity $A_{2g-3} \cong \mathbb{C}^2/\mathbb{Z}_{2g-2}$. The Coulomb branch of the quiver $\mathcal{Q}_{5(b)}$ in figure 5(b) is the D -type singularity $D_{g+1} \cong \mathbb{C}^2/\text{Dic}_{g-1}$. Embracing (2.4), one is led to an interesting relation between the two quivers of figure 5: as a complete graph, $\mathcal{Q}_{5(a)}$ has an S_2 outer-automorphism. By gauging this S_2 , in view of (2.4), $\mathcal{Q}_{5(b)}$ is obtained, a quiver with a single $U(2)$ gauge node and with g hypermultiplets in the adjoint representation. Hence, the following relationship between the Coulomb branches of these two quivers is found:

$$\mathcal{C}(\mathcal{Q}_{5(b)}) = \mathcal{C}(\mathcal{Q}_{5(a)})/S_2. \quad (5.1)$$

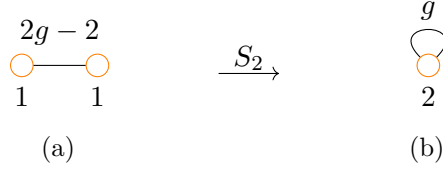


Figure 5. (a): the Abelian quiver $\mathcal{CG}_{2,2g-2}$, whose Coulomb branch is the A_{2g-3} surface singularity. (b): the quiver $\mathcal{ML}_{2,g}$, whose Coulomb branch is the D_{g+1} Kleinian singularity.

This, in fact, reproduces the well-known relation between the two Kleinian singularities:

$$D_{g+1} \cong A_{2g-3}/S_2, \quad (5.2)$$

but now it is phrased as an operation on quivers.

Monopole formula. Let us examine the relationship (5.1) using the Hilbert series, which for these singularities are:

$$\text{HS}(A_{2g-3}) = \text{PE} \left[t^2 + \left(q + \frac{1}{q} \right) t^{2g-2} - t^{4g-4} \right], \quad (5.3a)$$

$$\text{HS}(D_{g+1}) = \text{PE} \left[t^4 + t^{2g-2} + t^{2g} - t^{4g} \right], \quad (5.3b)$$

where q is the fugacity of the topological $U(1)_J$. The global symmetry on A_{2g-3} is $U(1)_J$, except for the special case of $g = 2$, where the global symmetry is enhanced to $SU(2)$. There is no continuous global symmetry on D_{g+1} , since the S_2 action breaks the $U(1)_J$ symmetry.

Now the result is verified further from the monopole formula and generalised Molien sum. The magnetic fluxes of $\mathcal{Q}_{5(a)}$ are denoted as m_1 and m_2 , and the overall $U(1)$ is ungauged by taking $m_2 = 0$. The monopole formula is:

$$\text{HS}(\mathcal{C}(\mathcal{Q}_{5(a)})) = \frac{1}{1-t^2} \sum_{m_1=-\infty}^{\infty} q^{m_1+m_2} \cdot t^{(2g-2)|m_1-m_2}|_{m_2=0} = \text{PE} \left[t^2 + \left(q + \frac{1}{q} \right) t^{2g-2} - t^{4g-4} \right], \quad (5.4)$$

reproducing the result in (5.3a). Using the same notation, the monopole formula of $\mathcal{Q}_{5(b)}$ is:

$$\text{HS}(\mathcal{C}(\mathcal{Q}_{5(b)})) = \frac{1}{1-t^4} + \frac{1}{1-t^2} \sum_{m_1 > m_2}^{\infty} t^{(2g-2)|m_1-m_2}|_{m_2=0} = \text{PE} \left[t^4 + t^{2g-2} + t^{2g} - t^{4g} \right], \quad (5.5)$$

reproducing the result in (5.3b). Let the non-trivial action of S_2 be denoted as (1 2). Using (3.7), the monopole formula on the fixed loci $m_1 = m_2 = 0$ of (1 2) is:

$$\begin{aligned} \text{HS}(\mathcal{C}(\mathcal{Q}_{5(a)}))^{(1\ 2)} &= \frac{1-t^2}{1-t^4} \left(t^{(2g-2)|m_1-m_2}|_{m_1=m_2=0} \right) \\ &= \frac{1}{1+t^2}. \end{aligned} \quad (5.6)$$

Generators	U(1) _J Charge	Degree	Generators	Degree
ϕ	0	2	x	4
u	1	$2g - 2$	w	$2g - 2$
v	-1	$2g - 2$	z	$2g$
Relation			Relation	
$\phi^{2g-2} + uv = 0$	0	$4g - 4$	$z^2 - xw^2 + x^g = 0$	$4g$

(a) A_{2g-3} singularity.

(b) D_{g+1} singularity.

Table 1. Generators and relations: (a) for Kleinian A_{2g-3} ; and (b) for Kleinian D_{g+1} .

By applying the generalised Molien sum (3.8), there is agreement that:

$$\begin{aligned}
 \frac{1}{2} \left(\text{HS} \left(\mathcal{C}(\mathcal{Q}_{5(a)}) \right) |_{q=1} + \text{HS} \left(\mathcal{C}(\mathcal{Q}_{5(a)}) \right)^{(1 \ 2)} \right) &= \frac{1}{2} \left(\frac{1 - t^{4g-4}}{(1 - t^2)(1 - t^{2g-2})^2} + \frac{1}{1 + t^2} \right) \\
 &= \text{PE} \left[t^4 + t^{2g-2} + t^{2g} - t^{4g} \right] \\
 &= \text{HS} \left(\mathcal{C}(\mathcal{Q}_{5(b)}) \right). \tag{5.7}
 \end{aligned}$$

From the PE, the numbers and representations/charges of the generators and relations at each degree are read off. From (5.3a), A_{2g-3} has one generator at degree 2 with charge 0, two generators at degree $2g - 2$ with charge +1 and -1, and one relation between them at degree $4g - 4$ with charge 0. From (5.3b), D_{g+1} has three generators at degree 4, $2g - 2$, and $2g$ respectively; and one relation between them at degree $4g$. The explicit generators and relations for the A_{2g-3} and D_{g+1} singularities are given in table 1(a) and table 1(b), respectively.

Now let us construct the D_{g+1} surface singularity starting from the Kleinian A_{2g-3} singularity. The action of S_2 on the generators of A_{2g-3} is given by

$$\phi \rightarrow -\phi, \quad u \rightarrow v, \quad v \rightarrow u. \tag{5.8}$$

There are four fundamental invariants under this S_2 action: ϕ^2 , $u + v$, $\phi(u - v)$, and uv . Together with the relations there are three independent generators: $\phi^2 = x$, $u + v = 2w$, and $\phi(u - v) = 2z$. These have the same degrees as the generators of D_{g+1} . Moreover, the relation between x , w , and z is indeed $z^2 - xw^2 + x^g = 0$, which coincides with the relation of the D_{g+1} singularity, cf. table 1(b).

Abelianisation. Now let us verify the generators and relations of the monopole operators via Abelianisation, as described in section 4. For the Abelian theory $\mathcal{Q}_{5(a)} = \mathcal{CG}_{2,2g-2}$, the Casimir operators of the gauge group $U(1)^2$ is labelled as ϕ_1 and ϕ_2 , and the general monopole operators as $v_{(n_1, n_2)}$, where $(n_1, n_2) \in \mathbb{Z}^2$. The chiral ring relations between these operators are:

$$v_{(n_1, n_2)} v_{(m_1, m_2)} = v_{(n_1+m_1, n_2+m_2)} (\phi_1 - \phi_2)^{(2g-2)\text{ABS}_+(n_1-n_2, m_1-m_2)}, \tag{5.9a}$$

$$\text{with } \text{ABS}_+(n, m) := \max(n, 0) + \max(m, 0) - \max(n + m, 0). \tag{5.9b}$$

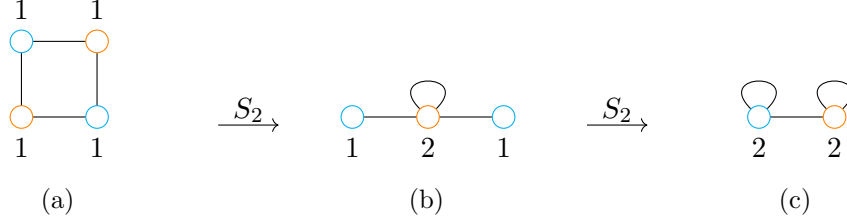


Figure 6. (a): Quiver $\mathcal{Q}_{6(a)}$ is the affine A_3 quiver. The Coulomb branch is the closure of minimal nilpotent orbit a_3 . The orange and cyan parts are two $\mathcal{CG}_{2,0}$ sub-graphs. (b): non-Abelian quiver $\mathcal{Q}_{6(b)}$, whose Coulomb branch is the closure of next-to-minimal nilpotent orbit $\overline{n.min.B_2}$. (c): non-Abelian quiver $\mathcal{Q}_{6(c)}$.

The defining Coulomb branch relations are:

$$v_{(1,1)}v_{(-1,-1)} = 1, \quad (5.10a)$$

$$v_{(1,0)}v_{(-1,0)} = (\phi_1 - \phi_2)^{2g-2}, \quad (5.10b)$$

$$v_{(0,1)}v_{(0,-1)} = (\phi_1 - \phi_2)^{2g-2}, \quad (5.10c)$$

$$v_{(1,0)}v_{(0,1)} = v_{(1,1)}(\phi_1 - \phi_2)^{2g-2}, \quad (5.10d)$$

$$v_{(-1,0)}v_{(0,-1)} = v_{(-1,-1)}(\phi_1 - \phi_2)^{2g-2}, \quad (5.10e)$$

$$v_{(kn_1, kn_2)} = v_{(n_1, n_2)}^k. \quad (5.10f)$$

Here, (5.10f) reflects the holomorphicity of the chiral ring. From (5.10a) it is clear that the freely acting diagonal $U(1)$ contributes to the relation of $T^*\mathbb{C}^\times$: $\mathbb{C}/[v_{(1,1)}, v_{(-1,-1)}, \phi_1 + \phi_2]/\langle v_{(1,1)}v_{(-1,-1)} = 1 \rangle$. Next, (5.10b) is the relation of the A_{2g-3} singularity: $\mathbb{C}/[v_{(1,0)}, v_{(-1,0)}, \phi_1 - \phi_2]/\langle v_{(1,0)}v_{(-1,0)} = (\phi_1 - \phi_2)^{2g-2} \rangle$. The generators $v_{(1,0)}$, $v_{(-1,0)}$, $\phi_1 - \phi_2$ can be identified with u , v , ϕ , respectively, as in table 1(a). The relation (5.10c) is then derived from (5.10a) and (5.10b)

$$v_{(0,1)}v_{(0,-1)} = v_{(1,1)}v_{(-1,0)}v_{(-1,-1)}v_{(1,0)} = (\phi_1 - \phi_2)^{2g-2}. \quad (5.11)$$

Hence, the overall Coulomb branch of $\mathcal{Q}_{5(a)}$ is $T^*\mathbb{C}^\times \times A_{2g-3}$, in which $T^*\mathbb{C}^\times$ can be decoupled via the ungauging process. After the ungauging there is an identification $v_{n_1+k, n_2+k} \sim v_{n_1, n_2}$.

According to the statement above, the Coulomb branch of non-Abelian theory $\mathcal{Q}_{5(b)}$ is a quotient of the Coulomb branch of $\mathcal{Q}_{5(a)}$ by the Weyl group S_2 . The nontrivial action of S_2 is simply permuting the two $U(1)$ s in the Cartan subgroup:

$$v_{(n_1, n_2)} \longleftrightarrow v_{(n_2, n_1)}, \quad \phi_1 \longleftrightarrow \phi_2, \quad (5.12)$$

which agrees with (5.8). The free part $T^*\mathbb{C}^\times$ is invariant under this action, and the singular part A_{2g-3} becomes D_{g+1} . Therefore, the overall Coulomb branch of $\mathcal{Q}_{5(b)}$ is $T^*\mathbb{C}^\times \times D_{g+1}$.

5.2 A_3 affine quiver with S_2 gauging

A similar analysis is repeated for the affine A_3 quiver. The Coulomb branch is the closure of the minimal nilpotent orbit of \mathfrak{sl}_4 which is denoted a_3 .

An S_2 quotient can be taken on this quiver if a pair of $U(1)$ s diagonally opposite are treated as a sub-complete graph of $\mathcal{CG}_{2,0}$ on which the S_2 gauging occurs, as shown in

figure 6(a) and figure 6(b). Note that this is the only possibility, as the alternative, which takes a pair of U(1) on the same side as $\mathcal{CG}_{2,1}$ for the gauging, has odd edge multiplicity. Importantly, the magnetic quiver for $\overline{n.min.B_2}$ is found, which is consistent with the result of Kostant and Brylinski [14] and also consistent with previous studies on discrete actions on Coulomb branches of $3d \mathcal{N} = 4$ quiver gauge theories [11]. Furthermore, since there are two pairs of diagonally opposite U(1)s as two sub-complete graphs of $\mathcal{CG}_{2,0}$, an $S_2 \times S_2$ quotient can also be taken, the quiver is shown in figure 6(c).

Monopole formula. Let us derive the chiral ring relations of these three quivers via Hilbert series and show how the S_2 and $S_2 \times S_2$ quotients act on the generators. The unrefined Hilbert series

read:

$$\text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(a)} \right) \right) = \frac{(1+t^2)(1+8t^2+t^4)}{(1-t^2)^6}, \quad (5.13a)$$

$$\text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(b)} \right) \right) = \frac{(1+t^2)(1+3t^2+t^4)}{(1-t^2)^6}, \quad (5.13b)$$

$$\text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(c)} \right) \right) = \frac{1+3t^2+11t^4+10t^6+11t^8+3t^{10}+t^{12}}{(1-t^2)^3(1-t^4)^3}. \quad (5.13c)$$

The coefficient of t^2 term indicates global symmetry on the Coulomb branches are SU(4), $\text{SO}(5) \cong \text{USp}(4)$ and $\text{SU}(2) \times \text{SU}(2)$ respectively.

It is straightforward to show that these Hilbert series follow from the Molien sum relation (3.8):

$$\text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(a)} \right) \right)^{(1\ 3)} = \text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(a)} \right) \right)^{(2\ 4)} = \frac{1+t^2}{(1-t^2)^4}, \quad (5.14a)$$

$$\text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(a)} \right) \right)^{(1\ 3)(2\ 4)} = \frac{1+t^4}{(1+t^2)(1-t^4)^2}, \quad (5.14b)$$

$$\begin{aligned} \frac{1}{2} \left(\text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(a)} \right) \right) + \text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(a)} \right) \right)^{(1\ 3)} \right) &= \frac{1}{2} \left(\text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(a)} \right) \right) + \text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(a)} \right) \right)^{(2\ 4)} \right) \\ &= \frac{1}{2} \left(\frac{(1+t^2)(1+8t^2+t^4)}{(1-t^2)^6} + \frac{1+t^2}{(1-t^2)^4} \right) \\ &= \frac{(1+t^2)(1+3t^2+t^4)}{(1-t^2)^6} \\ &= \text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(b)} \right) \right), \end{aligned} \quad (5.14c)$$

$$\begin{aligned} \frac{1}{4} \left(\text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(a)} \right) \right) + \text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(a)} \right) \right)^{(1\ 3)} \right) &+ \text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(a)} \right) \right)^{(2\ 4)} + \text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(a)} \right) \right)^{(1\ 3)(2\ 4)} \\ &= \frac{1}{4} \left(\frac{(1+t^2)(1+8t^2+t^4)}{(1-t^2)^6} + 2 \frac{1+t^2}{(1-t^2)^4} + \frac{1+t^4}{(1+t^2)(1-t^4)^2} \right) \\ &= \frac{1+3t^2+11t^4+10t^6+11t^8+3t^{10}+t^{12}}{(1-t^2)^3(1-t^4)^3} \\ &= \text{HS} \left(\mathcal{C} \left(\mathcal{Q}_{6(c)} \right) \right), \end{aligned} \quad (5.14d)$$

where (1 3) represents the permutation action between the orange U(1)² and (2 4) represents the permutation action between the cyan U(1)².

Generators	Reps of SU(4)	Degree
M_i^j	$[1, 0, 1] + [0, 0, 0]$	2
Relations		
$\text{Tr}(M) = 0$	$[0, 0, 0] + [0, 2, 0] + [1, 0, 1]$	2
$M_i^j M_k^l - M_i^l M_k^j = 0$ ($\text{rk}(M) \leq 1$)	$[0, 1, 0]$	4

Table 2. Generators and relations of Coulomb branch of $\mathcal{Q}_{6(a)}$, which is $a_3 \cong \mathbb{C}[M_i^j]/\langle \text{tr}(M) = 0, \text{rk}(M) \leq 1 \rangle$, where $i, j = 1, 2, 3, 4$.

Generators	Reps of USp(4)	Degree
S_{ij}	$[2, 0]$	2
Relations		
$\Omega^{jk} S_{ij} S_{kl} = 0$	$[0, 0] + [0, 1]$	4

Table 3. Generators and relations of Coulomb branch of $\mathcal{Q}_{6(b)}$, which is a_3/S_2 . The global symmetry is $\text{SO}(5) \cong \text{USp}(4)$. Ω is the standard skew-symmetric matrix.

Now let us extract the information of the independent generators and relations of the first two quivers via the refined PL:

$$\text{PL}((5.13a)) = [1, 0, 1]_{\text{su}4} t^2 - [0, 1, 0]_{\text{su}4} t^4 + O(t^5), \tag{5.15a}$$

$$\text{PL}((5.13b)) = [2, 0]_{\text{usp}4} t^2 - ([0, 0]_{\text{usp}4} + [0, 1]_{\text{usp}4}) t^4 + O(t^5). \tag{5.15b}$$

As a result, the explicit generators and relations are derived as shown in tables 2 and 3.

In this case, it is hard to guess the orbifold action on the generators from the Hilbert series alone.

Abelianisation. Alternatively, Abelianisation can be used to inspect the quotients in figure 6. For the Abelian theory $\mathcal{Q}_{6(a)}$, the Casimir operators of the orange $U(1)^2$ are labelled as ϕ_1, ϕ_3 , and for the cyan $U(1)^2$ as ϕ_2, ϕ_4 . A general monopole operator is denoted by $v_{(n_1, n_2, n_3, n_4)}$, where $(n_1, n_3) \in \mathbb{Z}^2$ are the magnetic fluxes for the orange $U(1)^2$ gauge nodes and $(n_2, n_4) \in \mathbb{Z}^2$ are the magnetic fluxes for the cyan $U(1)^2$ gauge nodes. The chiral ring relations are:

$$v_{(1,1,1,1)} v_{(-1,-1,-1,-1)} = 1, \tag{5.16a}$$

$$v_{(1,0,0,0)} v_{(-1,0,0,0)} = (\phi_4 - \phi_1)(\phi_1 - \phi_2), \tag{5.16b}$$

$$v_{(0,1,0,0)} v_{(0,-1,0,0)} = (\phi_1 - \phi_2)(\phi_2 - \phi_3), \tag{5.16c}$$

$$v_{(0,0,1,0)} v_{(0,0,-1,0)} = (\phi_2 - \phi_3)(\phi_3 - \phi_4), \tag{5.16d}$$

$$v_{(0,0,0,1)} v_{(0,0,0,-1)} = (\phi_3 - \phi_4)(\phi_4 - \phi_1), \tag{5.16e}$$

$$v_{(1,0,0,0)} v_{(0,1,0,0)} = v_{(1,1,0,0)}(\phi_1 - \phi_2), \tag{5.16f}$$

$$v_{(0,1,0,0)} v_{(0,0,1,0)} = v_{(0,1,1,0)}(\phi_2 - \phi_3), \tag{5.16g}$$

$$v_{(0,0,1,0)}v_{(0,0,0,1)} = v_{(0,0,1,1)}(\phi_3 - \phi_4), \quad (5.16h)$$

$$v_{(0,0,0,1)}v_{(1,0,0,0)} = v_{(1,0,0,1)}(\phi_4 - \phi_1), \quad (5.16i)$$

$$v_{(1,0,0,0)}v_{(0,1,0,0)}v_{(0,0,1,0)} = v_{(1,1,1,0)}(\phi_1 - \phi_2)(\phi_2 - \phi_3), \quad (5.16j)$$

$$v_{(0,1,0,0)}v_{(0,0,1,0)}v_{(0,0,0,1)} = v_{(0,1,1,1)}(\phi_2 - \phi_3)(\phi_3 - \phi_4), \quad (5.16k)$$

$$v_{(0,0,1,0)}v_{(0,0,0,1)}v_{(1,0,0,0)} = v_{(1,0,1,1)}(\phi_3 - \phi_4)(\phi_4 - \phi_1), \quad (5.16l)$$

$$v_{(0,0,0,1)}v_{(1,0,0,0)}v_{(0,1,0,0)} = v_{(1,1,0,1)}(\phi_4 - \phi_1)(\phi_1 - \phi_2), \quad (5.16m)$$

$$v_{(1,0,0,0)}v_{(0,1,0,0)}v_{(0,0,1,0)}v_{(0,0,0,1)} = v_{(1,1,1,1)}(\phi_1 - \phi_2)(\phi_2 - \phi_3)(\phi_3 - \phi_4)(\phi_4 - \phi_1), \quad (5.16n)$$

$$v_{(kn_1, kn_2)} = v_{(n_1, n_2)}^k. \quad (5.16o)$$

From (5.16a) it is clear that the freely acting diagonal $U(1)$ contributes to the relation of $T^*\mathbb{C}^\times$: $\mathbb{C}[v_{(1,1,1,1)}, v_{(-1,-1,-1,-1)}, \phi_1 + \phi_2 + \phi_3 + \phi_4] / \langle v_{(1,1,1,1)}v_{(-1,-1,-1,-1)} = 1 \rangle$. After decoupling this free $U(1)$, there is the identification $v_{(n_1+k, n_2+k, n_3+k, n_4+k)} \sim v_{(n_1, n_2, n_3, n_4)}$. By rearranging the remaining generators into matrix form:

$$\begin{pmatrix} \phi_1 - \phi_2 & v_{(0,1,0,0)} & v_{(0,1,1,0)} & v_{(1,0,0,0)} \\ v_{(0,-1,0,0)} & \phi_2 - \phi_3 & v_{(0,0,1,0)} & v_{(0,0,1,1)} \\ v_{(0,-1,-1,0)} & v_{(0,0,-1,0)} & \phi_3 - \phi_4 & v_{(0,0,0,1)} \\ v_{(-1,0,0,0)} & v_{(0,0,-1,-1)} & v_{(0,0,0,-1)} & \phi_4 - \phi_1 \end{pmatrix}. \quad (5.17)$$

The chiral ring relations (5.16b)–(5.16n) hold by requiring the rank of the matrix is no greater than 1. Hence the generators and relations can be identified with M_i^j in table 2.

The non-trivial S_2 action which permutes the orange $U(1)^2$ is given by

$$v_{(n_1, n_2, n_3, n_4)} \longleftrightarrow v_{(n_3, n_2, n_1, n_4)}, \quad \phi_1 \longleftrightarrow \phi_3, \quad (5.18)$$

and the non-trivial S_2 action permuting the cyan $U(1)^2$ reads

$$v_{(n_1, n_2, n_3, n_4)} \longleftrightarrow v_{(n_1, n_4, n_3, n_2)}, \quad \phi_2 \longleftrightarrow \phi_4. \quad (5.19)$$

Having studied these simple examples, let us turn to another nice consequence of the statements in section 2, where in the following example a known hypersurface is studied, but in a different light.

5.3 Symplectic hypersurfaces in 2 quaternionic dimensions and S_2 gauging

Symplectic singularities which are hypersurfaces are very rare. According to a conjecture by [28, 29] these are

1. Klein singularities (quaternionic dimension 1),
2. Intersections of Slodowy slices in the $\text{Sp}(g)$ nilpotent cone (dimension 2),
3. The transverse slice between the regular and minimal orbits in G_2 (dimension 3).

It is easy to construct quivers such that their moduli spaces are symplectic hypersurfaces [30].

For case 2, the family of these symplectic hypersurfaces are the Coulomb branches of the multiloop quiver $\mathcal{Q}_{7(b)}$ on the r.h.s. of figure 7. This family of hypersurfaces was studied

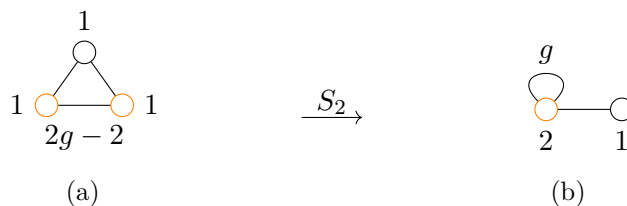


Figure 7. (a): Quiver $\mathcal{Q}_{7(a)}$ is composed of two parts: the complete graph $\mathcal{CG}_{2,2g-2}$ in orange, and the background \mathcal{Q}_B , given by the uncoloured U(1) node. They are connected by edges of multiplicity 1. (b): Quiver $\mathcal{Q}_{7(b)}$ is composed of $\mathcal{Q}_{5(b)}$, shown in orange, connecting to the same background quiver \mathcal{Q}_B .

in [31] as the Higgs branch of a certain $4d \mathcal{N} = 2 A_1$ Class \mathcal{S} theory on a Riemann surface of genus g and 1 puncture.

The Coulomb branch of this quiver was identified in [32] as $\mathcal{S}_{[2g-2,1,1]}^{[2g]}$, the intersection of the nilpotent cone of $\mathrm{Sp}(g)$ with the Słodowy slice to the orbit $[2g-2, 1, 1]$. Incidentally, this moduli space has a Higgs branch realisation by the S_2 quotient of the affine D_{g+1} quiver:

$$\mathcal{S}_{[2g-2,1,1]}^{[2g]} = \mathcal{H} \left(\begin{array}{c} \text{graph with } g-2 \text{ nodes and } 2 \text{ edges} \\ \text{orange node with } 2 \text{ loops} \end{array} \right) = \mathcal{C} \left(\begin{array}{c} \text{graph with } g \text{ loops} \\ \text{orange node with } 2 \text{ loops} \end{array} \right). \quad (5.20)$$

Notice that the orange U(2) node of $\mathcal{Q}_{7(b)}$ has loops. According to the claim (2.4), the Coulomb branch of $\mathcal{Q}_{7(b)}$ admits a double-cover, whose quiver $\mathcal{Q}_{7(a)}$ is shown on the l.h.s. of figure 7:

$$\mathcal{C}(\mathcal{Q}_{7(b)}) \cong \mathcal{C}(\mathcal{Q}_{7(a)})/S_2. \quad (5.21)$$

In $\mathcal{Q}_{7(a)}$, the orange U(1) nodes on the bottom together with the $2g-2$ edges can be treated as a complete graph. There is a S_2 symmetry permuting these two nodes. By gauging this S_2 symmetry, it becomes the orange U(2) node with g adjoints in quiver $\mathcal{Q}_{7(b)}$.

Monopole formula. Let us derive the chiral ring relations of these two quivers from Hilbert series and show how this S_2 quotient acts on the generators. Firstly, let us look at the refined Hilbert series of these two quivers:

$$\begin{aligned} \mathrm{HS}(\mathcal{C}(\mathcal{Q}_{7(a)})) &= \left(1+t^2 - \left(q+\frac{1}{q}\right)[1]t^{2g+1} - ([2]+1)t^{4g-2} + ([2]+1)t^{4g}\right. \\ &\quad \left.+ \left(q+\frac{1}{q}\right)[1]t^{6g-3} - t^{8g-4} - t^{8g-2}\right) \cdot \mathrm{PE} \left[([2]+1)t^2 + \left(q+\frac{1}{q}\right)[1]t^{2g-1} \right], \end{aligned} \quad (5.22a)$$

$$\mathrm{HS}(\mathcal{C}(\mathcal{Q}_{7(b)})) = \mathrm{PE} \left[[2]t^2 + [1]t^{2g-1} - t^{4g} \right], \quad (5.22b)$$

where q is the fugacity for U(1), and $[n]$ refers to the character for the $n+1$ dimensional representation² of SU(2). The Coulomb branch global symmetry of $\mathcal{Q}_{7(a)}$ is $\mathrm{U}(2) \cong \mathrm{SU}(2) \times \mathrm{U}(1)$; while the Coulomb branch global symmetry of $\mathcal{Q}_{7(b)}$ is SU(2). The S_2 quotient breaks the U(1) global symmetry and leaves the SU(2) intact.

²Here, for a rank k Lie group, the Dynkin label $[n_1, \dots, n_k]$ is used to represent both the irrep of corresponding highest weight and its character.

Generators	Reps of (U(1), SU(2))	Degree
Φ	(0, [0])	2
M_{ij}	(0, [2])	2
X_i	(1, [1])	$2g - 1$
Y_i	(-1, [1])	$2g - 1$
Relations		
$\epsilon^{ik} \epsilon^{jl} M_{ij} M_{kl} - f_1 \Phi^2 = 0$	(0, [0])	4
$\epsilon^{ik} M_{ij} X_k - f_2 \Phi X_j = 0$	(1, [1])	$2g + 1$
$\epsilon^{ik} M_{ij} Y_k - f_2 \Phi Y_j = 0$	(-1, [1])	$2g + 1$
$X_i Y_j - f_3 \Phi^{2g-2} M_{ij} = 0$	(0, [2])	$4g - 2$
$\epsilon^{ij} X_i Y_j - f_4 \Phi^{2g-1} = 0$	(0, [0])	$4g - 2$

Table 4. Generators and relations of Coulomb branch of $\mathcal{Q}_{7(a)}$, with $f_{1,2,3,4}$ constants that depend on g .

Generators	Reps of SU(2)	Degree
M_{ij}	[2]	2
B_i	[1]	$2g - 1$
Relation		
$(\epsilon^{ik} \epsilon^{jl} M_{ij} M_{kl})^g - f \epsilon^{ik} \epsilon^{jl} M_{ij} B_k B_l = 0$	[0]	$4g$

Table 5. Generators and relations of Coulomb branch of $\mathcal{Q}_{7(b)}$, where f is a constant depending on g .

It is straightforward to verify that the monopole formula satisfies the generalised Molien sum (3.8):

$$\text{HS}(\mathcal{C}(\mathcal{Q}_{7(b)})) = \frac{1}{2} \left(\text{HS}(\mathcal{C}(\mathcal{Q}_{7(a)})) + \text{HS}(\mathcal{C}(\mathcal{Q}_{7(a)}))^{(1\ 2)} \right) |_{q=1}. \quad (5.23)$$

To extract the information about the independent generators and relations for each variety, the PL of the HS is taken:

$$\begin{aligned} \text{PL}((5.22a)) &= ([2] + 1)t^2 - t^4 + \left(q + \frac{1}{q}\right) [1] t^{2g-1} - \left(q + \frac{1}{q}\right) [1] t^{2g+1} \\ &\quad - ([2] + 1)t^{4g-2} + \mathcal{O}(t^{4g-1}), \end{aligned} \quad (5.24a)$$

$$\text{PL}((5.22b)) = [2] t^2 + [1] t^{2g-1} - t^{4g}. \quad (5.24b)$$

Building on this, the explicit generators and relations are derived as shown in tables 4 and 5.

Next, let us construct $\mathcal{C}(\mathcal{Q}_{7(b)})$ from $\mathcal{C}(\mathcal{Q}_{7(a)})$. The S_2 action on the generators of $\mathcal{C}(\mathcal{Q}_{7(a)})$ is given by:

$$\Phi \rightarrow -\Phi, \quad X_i \rightarrow Y_i, \quad Y_i \rightarrow X_i. \quad (5.25)$$

There are five fundamental invariants under the S_2 action: M_{ij} , Φ^2 , $X_i + Y_i$, $\Phi(X_i - Y_i)$, and $X_i Y_j$. Due to the relations, there are only two independent generators: M_{ij} and $X_i + Y_i = B_i$. Moreover, the relation between M_{ij} and B_i is indeed given by $(\epsilon^{ik} \epsilon^{jl} M_{ij} M_{kl})^g - f \epsilon^{ik} \epsilon^{jl} M_{ij} B_k B_l = 0$.

Abelianisation. For the Abelian theory $\mathcal{Q}_{7(a)}$, the Casimir operators of the bottom $U(1)^2$ are denoted as ϕ_1 , ϕ_2 , and for the top $U(1)$ as ϕ_3 . Likewise, a generic monopole operator is denoted by $v_{(n_1, n_2, n_3)}$, with $(n_1, n_2, n_3) \in \mathbb{Z}^3$. The chiral ring relations for these operators are:

$$v_{(n_1, n_2, n_3)} v_{(m_1, m_2, m_3)} = v_{(n_1+m_1, n_2+m_2, n_3+m_3)} (\phi_1 - \phi_2)^{(2g-2)\text{ABS}_+(n_1-n_2, m_1-m_2)} \cdot (\phi_2 - \phi_3)^{\text{ABS}_+(n_2-n_3, m_2-m_3)} (\phi_3 - \phi_1)^{\text{ABS}_+(n_3-n_1, m_3-m_1)}. \quad (5.26)$$

The defining Coulomb branch relations are thus given by:

$$v_{(1,1,1)} v_{(-1,-1,-1)} = 1, \quad (5.27a)$$

$$v_{(1,0,0)} v_{(-1,0,0)} = (\phi_1 - \phi_2)^{2g-2} (\phi_3 - \phi_1), \quad (5.27b)$$

$$v_{(0,1,0)} v_{(0,-1,0)} = (\phi_1 - \phi_2)^{2g-2} (\phi_2 - \phi_3), \quad (5.27c)$$

$$v_{(0,0,1)} v_{(0,0,-1)} = (\phi_2 - \phi_3) (\phi_3 - \phi_1), \quad (5.27d)$$

$$v_{(1,0,0)} v_{(0,1,0)} = v_{(1,1,0)} (\phi_1 - \phi_2)^{2g-2}, \quad (5.27e)$$

$$v_{(-1,0,0)} v_{(0,-1,0)} = v_{(-1,-1,0)} (\phi_1 - \phi_2)^{2g-2}, \quad (5.27f)$$

$$v_{(1,0,0)} v_{(0,0,1)} = v_{(1,0,1)} (\phi_3 - \phi_1), \quad (5.27g)$$

$$v_{(-1,0,0)} v_{(0,0,-1)} = v_{(-1,0,-1)} (\phi_3 - \phi_1), \quad (5.27h)$$

$$v_{(0,1,0)} v_{(0,0,1)} = v_{(0,1,1)} (\phi_2 - \phi_3), \quad (5.27i)$$

$$v_{(0,-1,0)} v_{(0,0,-1)} = v_{(0,-1,-1)} (\phi_2 - \phi_3), \quad (5.27j)$$

$$v_{(1,0,0)} v_{(0,1,0)} v_{(0,0,1)} = v_{(1,1,1)} (\phi_1 - \phi_2)^{2g-2} (\phi_2 - \phi_3) (\phi_3 - \phi_1), \quad (5.27k)$$

$$v_{(-1,0,0)} v_{(0,-1,0)} v_{(0,0,-1)} = v_{(-1,-1,-1)} (\phi_1 - \phi_2)^{2g-2} (\phi_2 - \phi_3) (\phi_3 - \phi_1), \quad (5.27l)$$

$$v_{(kn_1, kn_2, kn_3)} = v_{(n_1, n_2, n_3)}^k. \quad (5.27m)$$

Here (5.27m) reflects the holomorphicity of the chiral ring. The freely acting diagonal $U(1)$ is seen from (5.27a), which contributes to the relation of $T^*\mathbb{C}^\times: \mathbb{C}/[v_{(1,1,1)}, v_{(-1,-1,-1)}, \phi_1 + \phi_2 + \phi_3]/\langle v_{(1,1,1)} v_{(-1,-1,-1)} = 1 \rangle$. After decoupling this $T^*\mathbb{C}^\times$, there is the identification $v_{(n_1+k, n_2+k, n_3+k)} \sim v_{(n_1, n_2, n_3)}$. Here $\phi_1 - \phi_2$, $v_{(0,0,1)}$, $v_{(0,0,-1)}$, $\phi_1 + \phi_2 - 2\phi_3$ can be rearranged and identified with Φ , M_{ij} in table 4. Also there is an arrangement of $v_{(0,1,0)}$, $v_{(0,-1,0)}$, $v_{(1,0,0)}$, $v_{(-1,0,0)}$ into X_i , Y_i . Under this identification, the relations in table 4 from (5.27b)–(5.27j) are recovered.

According to the statement above, the Coulomb branch of the non-Abelian theory $\mathcal{Q}_{7(b)}$ is a quotient of the Coulomb branch of $\mathcal{Q}_{7(a)}$ by the Weyl group S_2 . The non-trivial S_2 action is simply permuting the two bottom $U(1)$ s:

$$v_{(n_1, n_2, n_3)} \longleftrightarrow v_{(n_2, n_1, n_3)}, \quad \phi_1 \longleftrightarrow \phi_2, \quad (5.28)$$

which agrees with (5.25). This action leads to the Coulomb branch of $\mathcal{Q}_{7(b)}$.

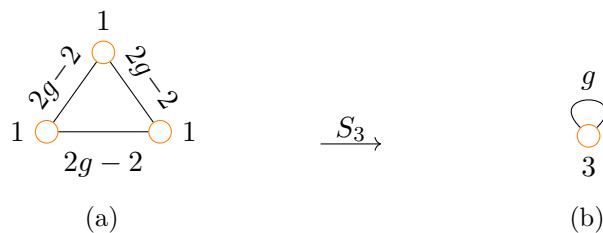


Figure 8. (a): Abelian quiver $\mathcal{CG}_{3,2g-2}$. (b): Quiver $\mathcal{ML}_{3,g}$.

5.4 Complete graph of three nodes — $\mathcal{CG}_{3,2g-2}$ quiver with S_3 gauging

The quiver $\mathcal{Q}_{8(a)} = \mathcal{CG}_{3,2g-2}$ is the magnetic quiver of the AD theory (A_2, A_{6g-7}) .

There is a clear S_3 automorphism acting on $\mathcal{Q}_{8(a)}$. From (2.4), by gauging this S_3 , the quiver $\mathcal{Q}_{8(b)}$ is found which is a single $U(3)$ node with g adjoints, shown in figure 8.

Consequently, there exists the following relation between the Coulomb branches of the two quivers:

$$\mathcal{C}(\mathcal{Q}_{8(b)}) = \mathcal{C}(\mathcal{Q}_{8(a)})/S_3. \quad (5.29)$$

Again, Hilbert series and Abelianisation techniques are use in turn to shed light on the fine details.

Monopole formula. Let us examine the generators and relations via their Coulomb branch Hilbert series:

$$\begin{aligned} \text{HS}(\mathcal{C}(\mathcal{Q}_{8(a)})) &= \left(1 + 2t^{4g-4} - \left(z_1 + z_2 + z_1 z_2 + \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_1 z_2} \right) t^{8g-8} + 2t^{12g-12} + t^{16g-16} \right) \\ &\cdot \text{PE} \left[2t^2 + \left(z_1 + z_2 + z_1 z_2 + \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_1 z_2} \right) t^{4g-4} - 2t^{4g-4} \right], \end{aligned} \quad (5.30a)$$

$$\text{HS}(\mathcal{C}(\mathcal{Q}_{8(b)})) = \frac{1 + 2t^{4g-2} + 2t^{4g} + t^{8g-2}}{(1-t^4)(1-t^6)(1-t^{4g-4})^2}, \quad (5.30b)$$

here z_1 and z_2 are the fugacities for the topological $U(1)_1 \times U(1)_2$. The Coulomb branch global symmetry of $\mathcal{Q}_{8(a)}$ is $U(1)_1 \times U(1)_2$. The S_3 quotient breaks the $U(1)_1 \times U(1)_2$ global symmetry, leaving behind no continuous Coulomb branch global symmetry for $\mathcal{Q}_{8(b)}$.

Again, it is straightforward to verify that the generalised Molien sum (3.8) is satisfied:

$$\begin{aligned} \text{HS}(\mathcal{C}(\mathcal{Q}_{8(b)})) &= \frac{1}{2} \left(\text{HS}(\mathcal{C}(\mathcal{Q}_{8(a)})) + \text{HS}(\mathcal{C}(\mathcal{Q}_{8(a)}))^{(1\ 2)} + \text{HS}(\mathcal{C}(\mathcal{Q}_{8(a)}))^{(2\ 3)} + \text{HS}(\mathcal{C}(\mathcal{Q}_{8(a)}))^{(3\ 1)} \right. \\ &\quad \left. + \text{HS}(\mathcal{C}(\mathcal{Q}_{8(a)}))^{(1\ 2\ 3)} + \text{HS}(\mathcal{C}(\mathcal{Q}_{8(a)}))^{(1\ 3\ 2)} \right) \Big|_{z_1=z_2=1}. \end{aligned} \quad (5.31)$$

As before, the PL enables us to extract the information on the independent generators and relations:

$$\begin{aligned} \text{PL}((5.30a)) &= 2t^2 + \left(z_1 + z_2 + z_1 z_2 + \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_1 z_2} \right) t^{4g-4} \\ &\quad - \left(3 + z_1 + z_2 + z_1 z_2 + \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_1 z_2} \right) t^{8g-8} + \mathcal{O}(t^{8g-7}), \end{aligned} \quad (5.32a)$$

$$\text{PL}((5.30b)) = t^4 + t^6 + 2t^{4g-4} + 2t^{4g-2} + 2t^{4g} - 3t^{8g-4} - 3t^{8g-2} - 3t^{8g} + \mathcal{O}(t^{8g+1}). \quad (5.32b)$$

The explicit generators and relations are summarised in tables 6 and 7.

Generators	Charge of $U(1)_1 \times U(1)_2$	Degree
Φ_1, Φ_2, Φ_3	$(0, 0)$	2
X_1	$(1, 0)$	$4g - 4$
X_2	$(0, 1)$	$4g - 4$
X_3	$(-1, -1)$	$4g - 4$
Y_1	$(-1, 0)$	$4g - 4$
Y_2	$(0, -1)$	$4g - 4$
Y_3	$(1, 1)$	$4g - 4$
Relations		
$\Phi_1 + \Phi_2 + \Phi_3 = 0$	$(0, 0)$	2
$X_1 Y_1 - (\Phi_2 \Phi_3)^{2g-2} = 0$	$(0, 0)$	$8g - 8$
$X_2 Y_2 - (\Phi_3 \Phi_1)^{2g-2} = 0$	$(0, 0)$	$8g - 8$
$X_3 Y_3 - (\Phi_1 \Phi_2)^{2g-2} = 0$	$(0, 0)$	$8g - 8$
$Y_2 Y_3 - X_1 \Phi_1^{2g-2} = 0$	$(1, 0)$	$8g - 8$
$X_2 X_3 - Y_1 \Phi_1^{2g-2} = 0$	$(-1, 0)$	$8g - 8$
$Y_1 Y_3 - X_2 \Phi_2^{2g-2} = 0$	$(0, 1)$	$8g - 8$
$X_3 X_1 - Y_2 \Phi_2^{2g-2} = 0$	$(0, -1)$	$8g - 8$
$Y_2 Y_1 - X_3 \Phi_3^{2g-2} = 0$	$(-1, -1)$	$8g - 8$
$X_1 X_2 - Y_3 \Phi_3^{2g-2} = 0$	$(1, 1)$	$8g - 8$

Table 6. Generators and relations of Coulomb branch of $\mathcal{Q}_{8(a)}$. At degree 2, instead of two independent generators, three generators are used together with 1 relation to demonstrate the S_3 symmetry. When $g = 2$, it reduces to the generators and relations of $a_2 \cong \mathbb{C}[M_i^j] / \langle \text{tr}(M) = 0, \text{rk}(M) \leq 1 \rangle$, where $i, j = 1, 2, 3$.

Generators	Degree
p_2	4
p_3	6
u_1, u_2	$4g - 4$
v_1, v_2	$4g - 2$
w_1, w_2	$4g$
Relations	
r_1, r_2, r_3	$8g - 4$
r'_1, r'_2, r'_3	$8g - 2$
r''_1, r''_2, r''_3	$8g$

Table 7. Generators and relations of Coulomb branch of $\mathcal{Q}_{8(b)}$.

Next, one can construct $\mathcal{C}(\mathcal{Q}_{8(b)})$ from $\mathcal{C}(\mathcal{Q}_{8(a)})$ explicitly. The S_3 action on the generators of $\mathcal{C}(\mathcal{Q}_{8(a)})$ is given by:

$$\Phi_i \rightarrow -\Phi_{\sigma(i)}, \quad X_i \rightarrow X_{\sigma(i)}, \quad Y_i \rightarrow Y_{\sigma(i)}. \quad (5.33)$$

This action is induced from the S_3 subgroup of outer-automorphism of $\text{SL}(3, \mathbb{C})$, for more details see appendix C. The generators are given by: $p_2 = \Phi_1^2 + \Phi_2^2 + \Phi_3^2$, $p_3 = \Phi_1\Phi_2(\Phi_1 - \Phi_2) + \Phi_2\Phi_3(\Phi_2 - \Phi_3) + \Phi_3\Phi_1(\Phi_3 - \Phi_1)$, $u_1 = X_1 + X_2 + X_3$, $u_2 = Y_1 + Y_2 + Y_3$, $v_1 = X_1\Phi_1 + X_2\Phi_2 + X_3\Phi_3$, $v_2 = Y_1\Phi_1 + Y_2\Phi_2 + Y_3\Phi_3$, $w_1 = X_1(\Phi_2 - \Phi_3) + X_2(\Phi_3 - \Phi_1) + X_3(\Phi_1 - \Phi_2)$ and $w_2 = Y_1(\Phi_2 - \Phi_3) + Y_2(\Phi_3 - \Phi_1) + Y_3(\Phi_1 - \Phi_2)$. The 9 relations $r_1, r_2, r_3, r'_1, r'_2, r'_3, r''_1, r''_2$ and r''_3 after the quotient can be worked out explicitly for small g .

Abelianisation. For the Coulomb branch of $\mathcal{Q}_{8(a)} \cong \mathcal{CG}_{3,2g-2}$, the Casimir operators of gauge group $\text{U}(1)^3$ are labelled as ϕ_1, ϕ_2 and ϕ_3 , and the generic monopole operator as $v_{(n_1, n_2, n_3)}$, with $(n_1, n_2, n_3) \in \mathbb{Z}^3$. The chiral ring relations between these operators are:

$$\begin{aligned} v_{(n_1, n_2, n_3)} v_{(m_1, m_2, m_3)} &= v_{(n_1+m_1, n_2+m_2, n_3+m_3)} (\phi_1 - \phi_2)^{(2g-2)\text{ABS}_+(n_1-n_2, m_1-m_2)} \\ &\cdot (\phi_2 - \phi_3)^{(2g-2)\text{ABS}_+(n_2-n_3, m_2-m_3)} (\phi_3 - \phi_1)^{(2g-2)\text{ABS}_+(n_3-n_1, m_3-m_1)} \end{aligned} \quad (5.34)$$

which allows for the extraction of the determining Coulomb branch relations:

$$v_{(1,1,1)} v_{(-1,-1,-1)} = 1, \quad (5.35a)$$

$$v_{(1,0,0)} v_{(-1,0,0)} = (\phi_1 - \phi_2)^{2g-2} (\phi_3 - \phi_1)^{2g-2}, \quad (5.35b)$$

$$v_{(0,1,0)} v_{(0,-1,0)} = (\phi_1 - \phi_2)^{2g-2} (\phi_2 - \phi_3)^{2g-2}, \quad (5.35c)$$

$$v_{(0,0,1)} v_{(0,0,-1)} = (\phi_2 - \phi_3)^{2g-2} (\phi_3 - \phi_1)^{2g-2}, \quad (5.35d)$$

$$v_{(1,0,0)} v_{(0,1,0)} = v_{(1,1,0)} (\phi_1 - \phi_2)^{2g-2}, \quad (5.35e)$$

$$v_{(0,1,0)} v_{(0,0,1)} = v_{(0,1,1)} (\phi_2 - \phi_3)^{2g-2}, \quad (5.35f)$$

$$v_{(0,0,1)} v_{(1,0,0)} = v_{(1,0,1)} (\phi_3 - \phi_1)^{2g-2}, \quad (5.35g)$$

$$v_{(-1,0,0)} v_{(0,-1,0)} = v_{(-1,-1,0)} (\phi_1 - \phi_2)^{2g-2}, \quad (5.35h)$$

$$v_{(0,-1,0)} v_{(0,0,-1)} = v_{(0,-1,-1)} (\phi_2 - \phi_3)^{2g-2}, \quad (5.35i)$$

$$v_{(0,0,-1)} v_{(-1,0,0)} = v_{(-1,0,-1)} (\phi_3 - \phi_1)^{2g-2}, \quad (5.35j)$$

$$v_{(1,0,0)} v_{(0,1,0)} v_{(0,0,1)} = v_{(1,1,1)} (\phi_1 - \phi_2)^{2g-2} (\phi_2 - \phi_3)^{2g-2} (\phi_3 - \phi_1)^{2g-2}, \quad (5.35k)$$

$$v_{(-1,0,0)} v_{(0,-1,0)} v_{(0,0,-1)} = v_{(-1,-1,-1)} (\phi_1 - \phi_2)^{2g-2} (\phi_2 - \phi_3)^{2g-2} (\phi_3 - \phi_1)^{2g-2}, \quad (5.35l)$$

$$v_{(kn_1, kn_2, kn_3)} = v_{(n_1, n_2, n_3)}^k. \quad (5.35m)$$

The freely acting $\text{U}(1)$ is the diagonal $\text{U}(1)$ as seen from (5.35a), which contributes to $T^*\mathbb{C}^\times = \mathbb{C}[v_{(1,1,1)}, v_{(-1,-1,-1)}, \phi_1 + \phi_2 + \phi_3] / \langle v_{(1,1,1)} v_{(-1,-1,-1)} = 1 \rangle$. After decoupling this $T^*\mathbb{C}^\times$ there is the identification $v_{(n_1+k, n_2+k, n_3+k)} \sim v_{(n_1, n_2, n_3)}$. The generators $v_{(1,0,0)}, v_{(0,1,0)}, v_{(-1,-1,0)}, v_{(-1,0,0)}, v_{(0,-1,0)}, v_{(1,1,0)}, \phi_1 - \phi_2, \phi_2 - \phi_3, \phi_3 - \phi_1$ can be identified with $X_1, X_2, X_3, Y_1, Y_2, Y_3, \Phi_3, \Phi_1, \Phi_2$ respectively in table 6. The relations (5.35b)–(5.35j) can be repackaged into matrix form with constraint of rank no larger than 1:

$$\begin{pmatrix} (\phi_1 - \phi_2)^2 & v_{(0,1,0)} & v_{(-1,0,0)} \\ v_{(0,-1,0)} & (\phi_2 - \phi_3)^2 & v_{(0,0,1)} \\ v_{(1,0,0)} & v_{(0,0,-1)} & (\phi_3 - \phi_1)^2 \end{pmatrix}, \quad (5.36)$$

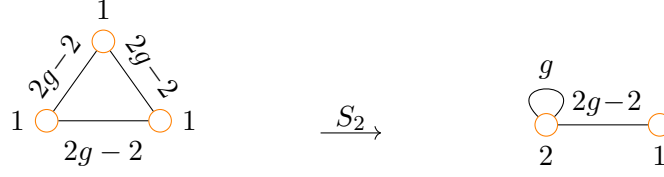


Figure 9. Magnetic quivers of the $S_2 \in S_3$ quotient on the Coulomb branch of $\mathcal{CG}_{3,2g-2}$.

they are exactly the generators and relations of a_2/\mathbb{Z}_{2g-2}^2 , as in (C.3).

The S_3 Weyl group action on the Abelian Coulomb branch is:

$$\phi_i \longleftrightarrow \phi_{\sigma(i)}, \quad v_{(n_1, n_2, n_3)} \longleftrightarrow v_{(n_{\sigma(1)}, n_{\sigma(2)}, n_{\sigma(3)})}, \quad (5.37)$$

which agrees with (5.33). This action leads to the Coulomb branch of $\mathcal{Q}_{8(b)}$.

Quotient by an S_2 subgroup. A quotient by an $S_2 \subset S_3$ symmetry can also be applied on the Coulomb branch of $\mathcal{Q}_{8(a)} \cong \mathcal{CG}_{3,2g-2}$. The quivers are shown in figure 9.

The S_2 Weyl group action on the Abelian Coulomb branch is a subgroup action of S_3 in (5.37):

$$\phi_i \longleftrightarrow \phi_{\sigma(i)}, \quad v_{(n_1, n_2, n_3)} \longleftrightarrow v_{(n_{\sigma(1)}, n_{\sigma(2)}, n_3)}, \quad (5.38)$$

where $\sigma \in S_2 \cong \{\text{Id}, (1\ 2)\}$. The Hilbert series are detailed in section 7.2.

5.5 3d mirror of $D_p(\text{SU}(n))$ theories with S_m gauging

The complete graph can be found as a subquiver in the 3d mirror of $D_p(\text{SU}(n))$. The $D_p(\text{SU}(n))$ theories can be realised as $6d \mathcal{N} = (2, 0)$ theories compactified on a sphere with one Type I irregular puncture and one regular (1^n) maximal puncture [33]. The procedure to derive magnetic quivers for these theories are given in [21] and the resulting magnetic quiver takes the form of $\mathcal{Q}_{3(a)}$. In particular, the irregular puncture gives the complete graph part, and the full puncture gives the form of Q_B which is a linear quiver of $(1) - \dots - (n-1)$ in this case.

Let us restrict to the case where $p \geq n$, here in the complete graph part of the magnetic quivers of $D_p(\text{SU}(n))$ theories there are $m = \text{GCD}(p, n)$ nodes with edge multiplicity $\frac{n(p-n)}{m^2}$. The Q_B is given, as before, by a linear quiver $(1) - \dots - (n-1)$. Each $U(1)$ node in the complete graph is connected to the $U(n-1)$ in Q_B by $\frac{n}{m}$ bi-fundamentals. There are also $\frac{(n-m)(p-n-m)}{2m}$ free hypermultiplets together.

Then equation (2.4) implies one can gauge any $S_{\sigma_1} \times \dots \times S_{\sigma_k}$ permutation subgroup of S_n of the 3d mirror of the $D_p(\text{SU}(n))$ theories. For $D_{n(2g-1)}(\text{SU}(n))$ theories, after gauging S_n , a magnetic quiver for Class \mathcal{S} theories with genus g and one maximal regular (1^n) puncture is found.

Some examples are listed in the following.

- $\mathcal{Q}_{10(a)}$ and $\mathcal{Q}_{10(b)}$ give the magnetic quivers for $D_9(\text{SU}(6))$ and $D_9(\text{SU}(3))$ theories. Both of them can be quotiented by S_2 or S_3 . For $D_9(\text{SU}(3))$ theory, the results after the quotient are shown in figure 11. The same operation can be done for $D_9(\text{SU}(6))$.

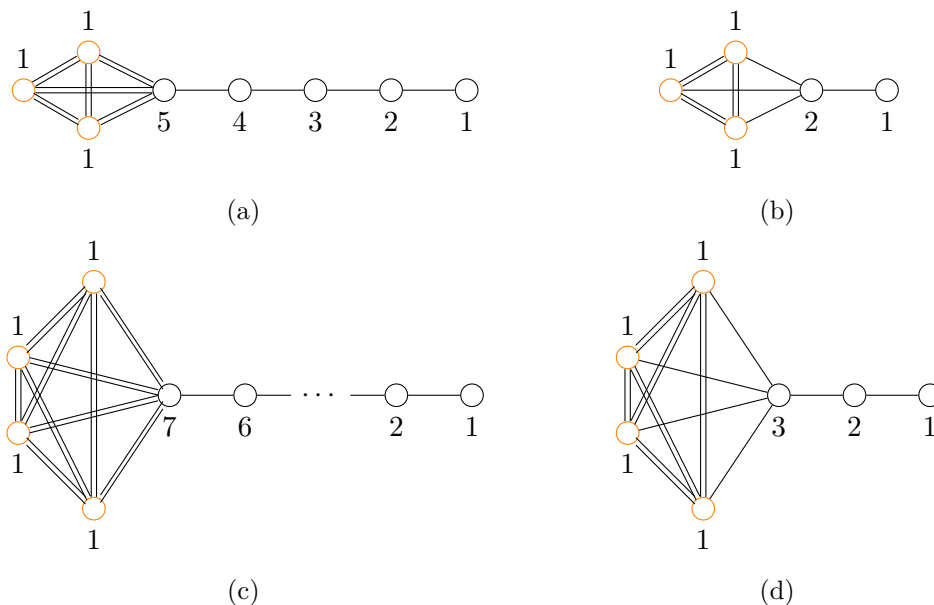


Figure 10. (a): magnetic Quiver for $D_9(\text{SU}(6))$. (b): magnetic Quiver for $D_9(\text{SU}(3))$. (c): magnetic Quiver for $D_{12}(\text{SU}(8))$. (d): magnetic Quiver for $D_{12}(\text{SU}(4))$.

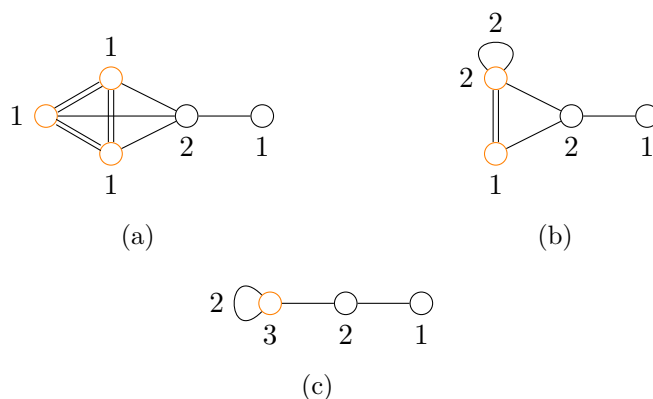


Figure 11. (a): magnetic Quiver for $D_9(\text{SU}(3))$. (b): S_2 quotient on Coulomb branch of $\mathcal{Q}_{11(a)}$. (c): S_3 quotient on Coulomb branch of $\mathcal{Q}_{11(a)}$.

- $\mathcal{Q}_{10(c)}$ and $\mathcal{Q}_{10(d)}$ gives the magnetic quivers for $D_{12}(\text{SU}(8))$ and $D_{12}(\text{SU}(4))$ theories. Both of them can be quotiented by S_2 , $S_2 \times S_2$, S_3 , or S_4 . For $D_{12}(\text{SU}(4))$ theory, the results after the quotient are shown in figure 12. The same operation can be done for $D_{12}(\text{SU}(8))$.

The unrefined Coulomb branch Hilbert series for $\mathcal{Q}_{11(a)}$ and the S_2 and S_3 quotients $\mathcal{Q}_{11(b)}$ and $\mathcal{Q}_{11(c)}$ are computed using monopole formula and Molien sums. Note that in the S_3 quotient, the element (1 2), (1 3), and (2 3) give the same contribution, as do the pair (1 2 3) and (1 3 2). The Hilbert Series after the action of an element in S_n only depends on the conjugacy class of the element.

$$\text{HS}(\mathcal{C}(\mathcal{Q}_{11(a)})) = \frac{\left(1+4t^2+10t^4+32t^6+65t^8+104t^{10}+153t^{12}+150t^{14}+153t^{16}+104t^{18}+65t^{20}+32t^{22}+10t^{24}+4t^{26}+t^{28}\right)}{(1-t^4)^{-1}(1-t^2)^6(1-t^6)^5}, \tag{5.39a}$$

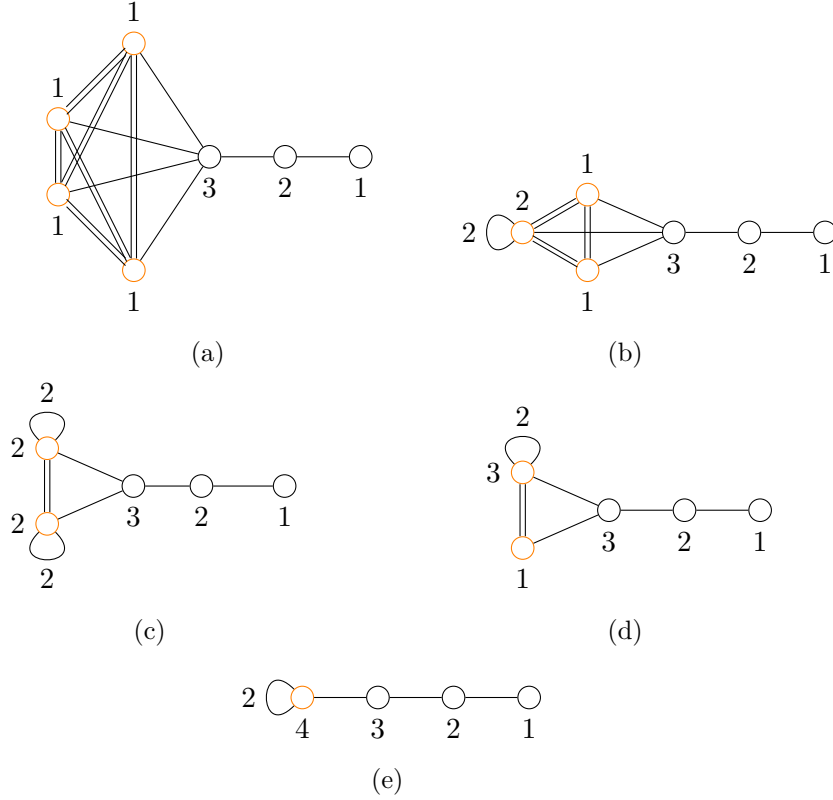


Figure 12. (a): magnetic Quiver for $D_{12}(SU(4))$. (b): S_2 quotient on Coulomb branch of $\mathcal{Q}_{12(a)}$. (c): $S_2 \times S_2$ quotient on Coulomb branch of $\mathcal{Q}_{12(a)}$. (d): S_3 quotient on Coulomb branch of $\mathcal{Q}_{12(a)}$. (e): S_4 quotient on Coulomb branch of $\mathcal{Q}_{12(a)}$.

$$\begin{aligned}
 \text{HS}(\mathcal{C}(\mathcal{Q}_{11(b)})/S_2) &= \frac{1}{2} \left(\text{HS}(\mathcal{C}(\mathcal{Q}_{11(a)})) + \text{HS}(\mathcal{C}(\mathcal{Q}_{11(a)}))^{(1\ 2)} \right) \\
 &= \frac{1}{2} \left(\text{HS}(\mathcal{C}(\mathcal{Q}_{11(a)})) + \frac{(1-t^4)(1+2t^2+4t^4+8t^6+7t^8+8t^{10}+4t^{12}+2t^{14}+t^{16})}{(1-t^2)^6(1-t^6)^3} \right) \\
 &= \frac{\left(1+3t^2+7t^4+19t^6+34t^8+52t^{10}+71t^{12}+70t^{14} \right. \\
 &\quad \left. +71t^{16}+52t^{18}+34t^{20}+19t^{22}+7t^{24}+3t^{26}+t^{28} \right)}{(1-t^4)^{-1}(1-t^2)^6(1-t^6)^5}, \tag{5.39b}
 \end{aligned}$$

$$\begin{aligned}
 \text{HS}(\mathcal{C}(\mathcal{Q}_{11(c)})/S_3) &= \frac{1}{6} \left(\text{HS}(\mathcal{C}(\mathcal{Q}_{11(a)})) + \text{HS}(\mathcal{C}(\mathcal{Q}_{11(a)}))^{(1\ 2)} + \text{HS}(\mathcal{C}(\mathcal{Q}_{11(a)}))^{(2\ 3)} \right. \\
 &\quad \left. + \text{HS}(\mathcal{C}(\mathcal{Q}_{11(a)}))^{(1\ 3)} + \text{HS}(\mathcal{C}(\mathcal{Q}_{11(a)}))^{(1\ 2\ 3)} + \text{HS}(\mathcal{C}(\mathcal{Q}_{11(a)}))^{(1\ 3\ 2)} \right) \\
 &= \frac{1}{6} \left(\text{HS}(\mathcal{C}(\mathcal{Q}_{11(a)})) + 3 \frac{(1-t^4)(1+2t^2+4t^4+8t^6+7t^8+8t^{10}+4t^{12}+2t^{14}+t^{16})}{(1-t^2)^6(1-t^6)^3} + 2 \frac{1-t^4}{(1-t^2)^7} \right) \\
 &= \frac{(1-t^8)(1+2t^2+3t^4+5t^6+8t^8+11t^{10}+14t^{12}+11t^{14}+8t^{16}+5t^{18}+3t^{20}+2t^{22}+t^{24})}{(1-t^2)^6(1-t^6)^5}. \tag{5.39c}
 \end{aligned}$$

Both $\mathcal{Q}_{11(a)}$ and $\mathcal{Q}_{11(c)}$ admit a well-defined 4d construction, in the sense that they are magnetic quivers for such 4d $\mathcal{N} = 2$ SCFTs. In contrast, it remains to be explored if $\mathcal{Q}_{11(b)}$ also is a magnetic quiver of some 4d theory (compactified from 6d). An intuitive guess is that all these theories can be compactified on a genus g Riemann surface with certain

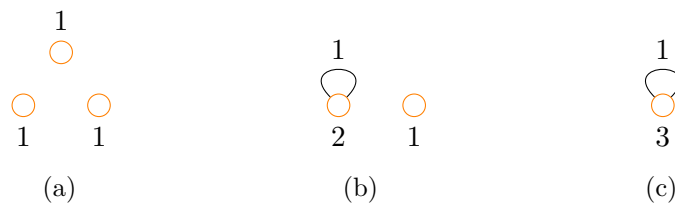


Figure 13. (a): $\mathcal{CG}_{3,0}$. (b): $\mathcal{ML}_{2,1}$ with $\mathcal{CG}_{1,0}$. (c): $\mathcal{ML}_{3,1}$.

number of punctures, the genus g contributes to the number of loops, and the punctures should encode the partition data of the subgroups of S_3 . The investigation of the discrete quotient in the 4d point of view is left to future work.

5.6 Pure $U(1)^3$ with S_3 gauging — D-brane physics

In this section a well-studied case is used to show the claim and Molien sum technique also work for bad quivers. The quiver before discrete gauging consists of three disconnected $U(1)$ nodes. After S_2 gauging is one $U(2)$ node with one adjoint and one $U(1)$ node. After S_3 gauging is $U(3)$ node with one adjoint.

In this case, the supersymmetry is enhanced to $\mathcal{N} = 8$, the Coulomb branch receives no quantum correction. The discrete gauging between each quiver simply reflects the classical description: $(\mathbb{R}^3 \times S^1)^r / \mathcal{W}_G$, where \mathcal{W}_G is the Weyl group of the gauge group. In [34], it is expressed as the cotangent bundle of the maximal torus of the Langlands (GNO) dual group of the gauge group: T^*T^\vee / \mathcal{W} . Applying this description to the quivers above, the following is found:

$$\mathcal{C}(\mathcal{CG}_{3,0}) = T^*(\mathbb{C}^\times)^3 \quad (5.40a)$$

$$\mathcal{C}(\mathcal{CG}_{3,0}/S_2) = \mathcal{C}(\mathcal{ML}_{2,1} \times \mathcal{CG}_{1,0}) = T^*(\mathbb{C}^\times)^2/S_2 \times T^*\mathbb{C}^\times \quad (5.40b)$$

$$\mathcal{C}(\mathcal{CG}_{3,0}/S_3) = \mathcal{C}(\mathcal{ML}_{3,1}) = T^*(\mathbb{C}^\times)^3/S_3 \quad (5.40c)$$

The monopole formula diverges when applied to these quivers:

$$\text{HS}(\mathcal{C}(\mathcal{CG}_{3,0})) = \frac{1}{(1-t^2)^3} \sum_{m_1, m_2, m_3} z_1^{m_1} z_2^{m_2} z_3^{m_3} \quad (5.41a)$$

$$\text{HS}(\mathcal{C}(\mathcal{CG}_{3,0}/S_2)) = \frac{1}{1-t^2} \sum_{m_3} z_3^{m_3} \left(\frac{1}{(1-t^2)(1-t^4)} \sum_{m_1=m_2} z^{m_1+m_2} + \frac{1}{(1-t^2)^2} \sum_{m_1>m_2} z^{m_1+m_2} \right) \quad (5.41b)$$

$$\begin{aligned} \text{HS}(\mathcal{C}(\mathcal{CG}_{3,0}/S_3)) &= \frac{1}{(1-t^2)(1-t^4)(1-t^6)} \sum_{m_1=m_2=m_3} z^{m_1+m_2+m_3} \\ &+ \frac{1}{(1-t^2)^2(1-t^4)} \left(\sum_{m_1>m_2=m_3} + \sum_{m_1=m_2>m_3} \right) z^{m_1+m_2+m_3} \\ &+ \frac{1}{(1-t^2)^3} \sum_{m_1>m_2>m_3} z^{m_1+m_2+m_3} \end{aligned} \quad (5.41c)$$

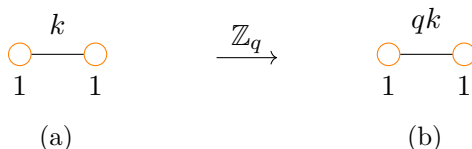


Figure 14. (a): Quiver $\mathcal{CG}_{2,k}$, whose Coulomb branch is the A_{k-1} singularity. (b): Quiver $\mathcal{CG}_{2,kq}$ of (2.1), whose Coulomb branch is the A_{kq-1} singularity.

The reason is that there are no less than one freely acting $U(1)$ s in each quiver. However, the divergent monopole formula is still a valid tool. For example, the Coulomb branch of pure $U(1)$ theory is $\mathbb{C}[\phi, u_-, u_+]/\langle u_- u_+ - 1 \rangle$. The monopole formula for this theory is $\frac{1}{1-t^2} \sum_{m_1} z^{m_1}$. This can be interpreted as: for any fixed monopole charge m_1 , there is one monopole operator $u_+^{m_1} \phi^n = u_-^{-m_1} \phi^n$ at degree $2n$. From the geometry point of view, there is a \mathbb{C} fibring over \mathbb{C}^\times .

After demonstrating the S_n quotient, now let us demonstrate the \mathbb{Z}_q quotient.

6 Examples of \mathbb{Z}_q quotient

The \mathbb{Z}_q gauging on quivers which are or contain certain complete graphs are computed and the corresponding \mathbb{Z}_q on their Coulomb branches are studied. This action is studied through the Hilbert series computed with the monopole formula for each example.

6.1 Kleinian A with \mathbb{Z}_q quotient

Now let us look at the simplest example of Claim (2.2). The $A_{k-1} \cong \mathbb{C}^2/\mathbb{Z}_k$ singularity can be realised through the Coulomb branch of an Abelian quiver $\mathcal{CG}_{2,k}$, as $\mathcal{Q}_{14(a)}$ in figure 14. Since A_{kq-1} is a \mathbb{Z}_q quotient of A_{k-1} , the Coulomb branch of $\mathcal{CG}_{2,kq}$ is a \mathbb{Z}_q quotient of the Coulomb branch of $\mathcal{CG}_{2,k}$:

$$\mathcal{C}(\mathcal{CG}_{2,kq}) = \mathcal{C}(\mathcal{CG}_{2,k})/\mathbb{Z}_q. \quad (6.1)$$

The \mathbb{Z}_q action on $\mathcal{C}(\mathcal{CG}_{2,k}) = A_{k-1} \cong \mathbb{C}[\phi, u, v]/\langle \phi^k - uv \rangle$ is given by:

$$u \rightarrow \omega_q \cdot u, \quad v \rightarrow \omega_q^{-1} \cdot v, \quad (6.2)$$

and the fundamental invariants are $u^q = u', v^q = v'$, subject to $uv = \phi^k$. This agrees with $\mathcal{C}(\mathcal{CG}_{2,kq}) = A_{kq-1} \cong \mathbb{C}[\phi, u', v']/\langle \phi^{kq} - u'v' \rangle$. The \mathbb{Z}_q action can also be implemented in the Hilbert series: it acts on the $U(1)_J$ fugacity as in (3.17):

$$z \rightarrow \omega_q \cdot z. \quad (6.3)$$

Combined with the Molien sum (3.18), the Hilbert Series of $\mathcal{C}(\mathcal{CG}_{2,kq})$ is obtained from the Hilbert Series of $\mathcal{C}(\mathcal{CG}_{2,k})$:

$$\begin{aligned} \text{HS}(A_{kq-1}) &= \frac{1}{q} \sum_{i=1}^q \text{HS}(A_{k-1})|_{z \rightarrow \omega_q^i \cdot z} \\ &= \frac{1}{q} \sum_{i=1}^q \text{PE} \left[t^2 + \left(\omega_q^i \cdot z + \frac{1}{\omega_q^i \cdot z} \right) t^k - t^{2k} \right] \\ &= \text{PE} \left[t^2 - \left(z^q + \frac{1}{z^q} \right) t^{kq} - t^{2kq} \right]. \end{aligned} \quad (6.4)$$

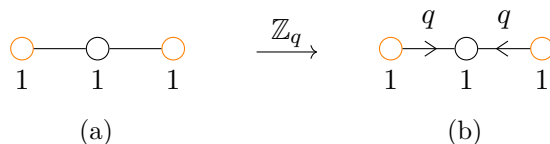


Figure 15. (a): Quiver $\mathcal{Q}_{15(a)}$ is an Abelian linear quiver of three nodes. The Coulomb branch is $\mathbb{H}^2 \cong \mathbb{C}^4$. (b): Quiver $\mathcal{Q}_{15(b)}$ has two q laced edges pointing to the middle U(1). The Coulomb branch is $\mathbb{C}^4/\mathbb{Z}_q$.

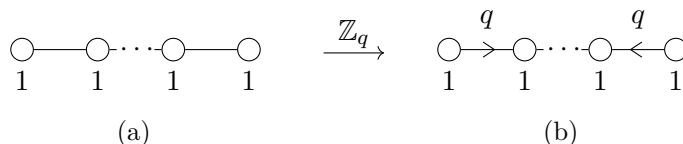


Figure 16. (a): Quiver $\mathcal{Q}_{16(a)}$ is a linear Abelian quiver composed of $n + 1$ nodes. The Coulomb branch is $h_{n,1} = \mathbb{H}^n \cong \mathbb{C}^{2n}$. (b): Quiver $\mathcal{Q}_{16(b)}$ has two q laced edges pointing to the middle, as leftmost and rightmost. The Coulomb branch is $h_{n,q} = \mathbb{C}^{2n}/\mathbb{Z}_q$. The $h_{n,q}$ appears an elementary slice in the affine Grassmannian, and in more general symplectic singularities.

From the quivers in figure 14, the \mathbb{Z}_q quotient can be implemented through the edge multiplicity k . On the Coulomb branch level, instead of q -times the edge multiplicity, the action (6.3) can also be interpreted as q -times the charge (double q lace) of hypermultiplets under the gauge $U(1)^2$. The edge multiplicity and double lace q on Coulomb branch cannot be distinguished; the difference is manifest on the Higgs branch, which is left to future work. Moreover, the two potential theory also differ in their higher symmetries, as recently analysed in [17, 25, 35].

6.2 \mathbb{H}^2 with \mathbb{Z}_q quotient

As an extension of the A -type singularity to 4 complex dimensions, there is $\mathbb{C}^4/\mathbb{Z}_q$ with the action of \mathbb{Z}_q : $\text{diag}(\omega_q, \omega_q, \omega_q^{-1}, \omega_q^{-1})$. The magnetic quiver of this singularity is $\mathcal{Q}_{15(b)}$. The Coulomb branch of $\mathcal{Q}_{15(b)}$ can be seen as a \mathbb{Z}_q quotient on the Coulomb branch of $\mathcal{Q}_{15(a)}$ by the Claim (2.5):

$$\mathcal{C}(\mathcal{Q}_{15(b)}) = \mathcal{C}(\mathcal{Q}_{15(a)})/\mathbb{Z}_q \tag{6.5}$$

here the middle node is treated as the background quiver.

This construction can be further extended to $2n$ complex dimensions: i.e. $\mathbb{C}^{2n}/\mathbb{Z}_q$ with the action $\text{diag}(\omega_q, \dots, \omega_q, \omega_q^{-1}, \dots, \omega_q^{-1})$. The Coulomb branch construction is shown in figure 16 and the Coulomb branches obey the relation:

$$\mathcal{C}(\mathcal{Q}_{16(b)}) = \mathcal{C}(\mathcal{Q}_{16(a)})/\mathbb{Z}_q. \tag{6.6}$$

In this case, however, the leftmost and rightmost nodes cannot be seen as identical, so the quiver does not fit the Claim (2.5). But, since the proof of Claim (2.5) only assumes the subquiver to be an Abelian instead of a complete graph, it can be applied to any simply-laced Abelian subquiver.

6.3 a_2 with \mathbb{Z}_q^2 gauging

Another interesting example of (2.2) is the quotient on $\mathcal{CG}_{3,1}$. The Coulomb branch of $\mathcal{CG}_{3,1}$ is a_2 , the closure of minimal nilpotent orbit of \mathfrak{sl}_3 . The coordinate ring of this Coulomb branch is written as $\mathbb{C}[M_i^j]/\langle \text{Tr}(M) = 0, \text{rk}(M) \leq 1 \rangle$, where $i, j = 1, 2, 3$. According to Claim (2.2), there is the following relationship:

$$c \left(\begin{array}{ccc} & 1 & \\ q & \circ & q \\ \circ & & \circ \\ 1 & q & 1 \end{array} \right) = c \left(\begin{array}{ccc} & 1 & \\ \circ & & \circ \\ \circ & & \circ \\ 1 & & 1 \end{array} \right) / \mathbb{Z}_q^2. \quad (6.7)$$

The Coulomb branch ring of $\mathcal{CG}_{3,q}$ is the ring of invariants $\mathbb{C}[M_i^j]^{\mathbb{Z}_q \times \mathbb{Z}_q} / \langle \text{Tr}(M) = 0, \text{rk}(M) \leq 1 \rangle$. The $\mathbb{Z}_q \times \mathbb{Z}_q$ action is given by: $M_i^j \rightarrow \omega_q^{k_i - k_j} M_i^j$, with $(k_i, k_j) \in \mathbb{Z}_q^2$.

7 Examples of combined S_n and \mathbb{Z}_q quotients

After studying the S_n and \mathbb{Z}_q quotients individually, it is time to explore the combined action.

7.1 Kleinian review: Dih or Dic

In section 5.1, we showed the S_2 quotient relation between A_{2g-3} and D_{g+1} through quivers. The Claim (2.3) indicates that $\mathcal{C}(\mathcal{CG}_{2,k}) \cong A_{k-1}$ and $\mathcal{C}(\mathcal{ML}_{2,qk+1}) \cong D_{qk+2}$ are related by a $\mathbb{Z}_{2q} \rtimes S_2 \cong Dih_{2q}$ quotient. However, this claim might be confusing without specifying the module of the group action. For example, when $k = 1$, the two Coulomb branches $A_0 \cong \mathbb{C}^2$ and $D_{q+2} \cong \mathbb{C}^2 / Dic_q$ are related by the non-split extension $\mathbb{Z}_{2q} \cdot S_2 = Dic_q$ instead of the split extension $\mathbb{Z}_{2q} \rtimes S_2 = Dih_{2q}$. To fit the claim, we should take the module of the generators span $[u, v, \phi]$ instead of \mathbb{C}^2 , then the actions are:

$$\sigma = \begin{pmatrix} \omega_{2q} & 0 & 0 \\ 0 & \omega_{2q}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (7.1)$$

which exactly generate the group $Dih_{2q} = \langle \sigma, \tau | \sigma^{2q} = \tau^2 = \sigma\tau\sigma\tau = \text{Id} \rangle$.

7.2 A_2 affine quiver with S_4 gauging

In this section the combined discrete quotient on the Coulomb branch of the affine A_2 quiver $\mathcal{CG}_{3,1}$ is studied. As in section 6.3, a \mathbb{Z}_q^2 quotient on a_2 is performed, which takes the $\mathcal{CG}_{3,1}$ quiver to the $\mathcal{CG}_{3,q}$ quiver. Recalling section 5.4, a further S_3 quotient on the Coulomb branch of $\mathcal{CG}_{3,2g-2}$ is performed, which results in the $\mathcal{ML}_{3,2}$ quiver — a single $U(3)$ gauge node with 2 adjoint hypermultiplets.

Let us consider the simplest example of the combined action. A S_2 or S_3 symmetry can be quotiented from $\mathcal{CG}_{3,2}$, the corresponding quivers are shown in table 8. According to Claim (2.6), the Coulomb branches are $a_2 / \mathbb{Z}_2^2 \rtimes S_2$ and $a_2 / \mathbb{Z}_2^2 \rtimes S_3$ respectively. There are surprising examples due to the coincidences of the group action: $\mathbb{Z}_2^2 \rtimes S_2 \cong Dih_4$ and $\mathbb{Z}_2^2 \rtimes S_3 \cong S_4$. The latter leads to a recently studied symplectic singularity [18]. The details are discussed below.

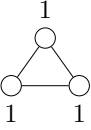
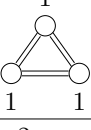
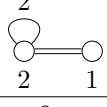

Label	Quiver	Discrete Quotient	HS	Volume
$\mathcal{C}(\mathcal{CG}_{3,1}) = a_2$			$\frac{1+4t^2+t^4}{(1-t^2)^4}$	$\frac{3}{8}$
$\mathcal{C}(\mathcal{CG}_{3,2}) = a_2/\mathbb{Z}_2^2$		$\mathbb{Z}_2^2 \cong Klein_4$	$\frac{1+4t^4+t^8}{(1-t^2)^2(1-t^4)^2}$	$\frac{3}{32}$
$a_2/\mathbb{Z}_2^2 \rtimes \mathfrak{S}_2$		$\mathbb{Z}_2^2 \rtimes \mathfrak{S}_2 \cong Dih_4$	$\frac{1+t^2+2t^4+4t^6+2t^8+t^{10}+t^{12}}{(1-t^4)^4}$	$\frac{3}{64}$
$a_2/\mathbb{Z}_2^2 \rtimes \mathfrak{S}_3 \cong a_2/\mathfrak{S}_4$		$\mathbb{Z}_2^2 \rtimes \mathfrak{S}_3 \cong \mathfrak{S}_4$	$\frac{1+2t^6+2t^8+t^{14}}{(1-t^4)^3(1-t^6)}$	$\frac{1}{64}$

Table 8. Quivers from discrete quotient of a_2 .

In terms of gauging outer automorphism symmetries both \mathfrak{S}_2 and \mathfrak{S}_3 could be gauged on the Coulomb branch of $\mathcal{CG}_{3,2}$, but not on $\mathcal{CG}_{3,1}$ due to its odd edge multiplicity.

The Hilbert series of all possible combinations of \mathbb{Z}_2^2 and \mathfrak{S}_2 and \mathfrak{S}_3 gaugings on a_2 are computed. The quivers, unrefined HS, and the volume of the Coulomb branches are provided in table 8.

Note that the ratios of the volumes³ of the moduli spaces is also consistent with the gaugings performed. Indeed, $\frac{\text{vol}^{-1}(a_2/\mathbb{Z}_2^2)}{\text{vol}^{-1}(a_2)} = 4$, $\frac{\text{vol}^{-1}(a_2/\mathbb{Z}_2^2 \rtimes S_2)}{\text{vol}^{-1}(a_2)} = 8$, and $\frac{\text{vol}^{-1}(a_2/\mathbb{Z}_2^2 \rtimes S_3)}{\text{vol}^{-1}(a_2)} = 24$.

In addition, full details of the computation of the Hilbert Series of these quotients using the Molien sum (3.8) and (3.18) are provided. Here, the fugacities (z'_1, z'_2, z'_3) for $\mathcal{CG}_{3,1}$, and rescaled fugacities $(z_1, z_2, z_3) = (z_1'^2, z_2'^2, z_3'^2)$ for $\mathcal{CG}_{3,2}$ are turned on. The quiver is ungauged by setting $m_3 = 0$ and $z'_3 = z_3 = 1$:

$$\begin{aligned}
 \text{HS}(a_2) &= \frac{1}{(1-t^2)^2} \sum_{m_1, m_2} z_1'^{m_1} z_2'^{m_2} t^{|m_1|+|m_2|+|m_1-m_2|} \\
 &= \frac{1+2t^2 - \left(z_1' + z_2' + z_1' z_2' + \frac{1}{z_1'} + \frac{1}{z_2'} + \frac{1}{z_1' z_2'} \right) t^4 + 2t^6 + t^8}{(1-z_1'^2 t^2)(1-z_2'^2 t^2)(1-z_1' z_2' t^2) \left(1 - \frac{t^2}{z_1'} \right) \left(1 - \frac{t^2}{z_2'} \right) \left(1 - \frac{t^2}{z_1' z_2'} \right)} \quad (7.2a)
 \end{aligned}$$

$$\begin{aligned}
 \text{HS}(a_2/\mathbb{Z}_2^2) &= \frac{1}{4} \sum_{i=1}^2 \sum_{j=1}^2 \text{HS}(a_2)|_{z_1' \rightarrow \omega^i, z_2' \rightarrow \omega^j} \\
 &= \frac{1}{(1-t^2)^2} \sum_{m_1, m_2} z_1'^{2m_1} z_2'^{2m_2} t^{2|m_1|+2|m_2|+2|m_1-m_2|} |_{z_1'^2=z_1, z_2'^2=z_2} \\
 &= \frac{1+2t^4 - \left(z_1 + z_2 + z_1 z_2 + \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_1 z_2} \right) t^8 + 2t^{12} + t^{16}}{(1-z_1 t^4)(1-z_2 t^4)(1-z_1 z_2 t^4) \left(1 - \frac{t^4}{z_1} \right) \left(1 - \frac{t^4}{z_2} \right) \left(1 - \frac{t^4}{z_1 z_2} \right) (1+t^2)^{-2}} \quad (7.2b)
 \end{aligned}$$

³The (inverse) volume of the Coulomb branches can be calculated from Hilbert series as $\text{vol}^{-1} = \lim_{t \rightarrow 1} (1-t)^d \text{HS}$, where d is the complex dimension of the Coulomb branch. If two space are related by a G quotient, then the ratio between the volumes should be the order $|G|$.

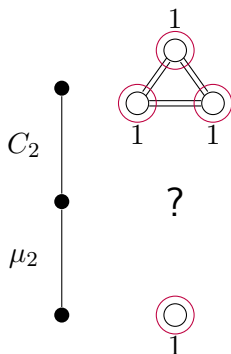


Figure 17. Hasse diagram of a_2/S_4 .

$$\begin{aligned}
 \text{HS}(a_2/\mathbb{Z}_2^2 \rtimes S_2) &= \frac{1}{2} \left(\text{HS}(a_2/\mathbb{Z}_2^2) + \text{HS}(a_2/\mathbb{Z}_2^2)^{(1\ 2)} \right) \\
 &= \frac{1}{2} \left(\text{HS}(a_2/\mathbb{Z}_2^2) + \frac{1}{1-t^4} \sum_{m_1=m_2} z_1^{m_1} z_2^{m_2} t^{2|m_1|+2|m_2|+2|m_1-m_2|} \right) \Big|_{z_1=z_2=z} \\
 &= \frac{1+t^2+2t^4 + \left(2+z+\frac{1}{z}\right)t^6 + 2t^8 + t^{10} + t^{12}}{(1-zt^4)(1-z^2t^4) \left(1-\frac{1}{z}t^4\right) \left(1-\frac{1}{z^2}t^4\right)} \tag{7.2c}
 \end{aligned}$$

$$\begin{aligned}
 \text{HS}(a_2/\mathbb{Z}_2^2 \rtimes S_3) &= \frac{1}{6} \left(\text{HS}(a_2/\mathbb{Z}_2^2) + \text{HS}(a_2/\mathbb{Z}_2^2)^{(1\ 2)} + \text{HS}(a_2/\mathbb{Z}_2^2)^{(1\ 3)} + \text{HS}(a_2/\mathbb{Z}_2^2)^{(2\ 3)} \right. \\
 &\quad \left. + \text{HS}(a_2/\mathbb{Z}_2^2)^{(1\ 2\ 3)} + \text{HS}(a_2/\mathbb{Z}_2^2)^{(1\ 3\ 2)} \right) \\
 &= \frac{1}{6} \left(\text{HS}(a_2/\mathbb{Z}_2^2) + \frac{1}{1-t^4} \left(\sum_{m_1=m_2} + \sum_{m_1=0, m_2} + \sum_{m_1, m_2=0} \right) z_1^{m_1} z_2^{m_2} t^{2|m_1|+2|m_2|+2|m_1-m_2|} \right. \\
 &\quad \left. + \frac{1-t^2}{1-t^6} + \frac{1-t^2}{1-t^6} \right) \Big|_{z_1=z_2=1} \\
 &= \frac{1+2t^6+2t^8+t^{14}}{(1-t^4)^3(1-t^6)}. \tag{7.2d}
 \end{aligned}$$

Note that in (7.2d), the elements (1 2), (1 3) and (2 3) give the same contribution, so do the elements (1 2 3) and (1 3 2). In general, the elements in the same conjugacy of S_n class give the same contribution.

Finally, a comment on the $a_2/\mathbb{Z}_2^2 \rtimes S_3$. This case raises a special interest due to the accidental identification of the discrete symmetry $\mathbb{Z}_2^2 \rtimes S_3$ with S_4 . This symplectic singularity is also found and constructed explicitly as part of the nilpotent cone of \mathfrak{e}_8 in [18], as the slice between $E_8(a_6)$ and $E_8(b_6)$. The original construction is given in appendix C. By comparing the Hilbert series with the generators and relations, the claim that the quiver and the Hilbert Series presented in the last row of table 8 are indeed the magnetic quiver and Hilbert Series for a_2/S_4 , respectively is made.

In addition, the Hasse diagram and decorated quiver description of a_2/S_4 is given in figure 17. In the diagram, C_2 is the Kleinian singularity A_3 with an intrinsic symmetry S_2 , and μ_2 is a non-normal slice with normalisation A_3 .⁴ The decorated quiver with a simply-laced

⁴The slice C_2 can be generalised to C_p . The ring of C_p is locally the same as D_{p+1} . Geometrically there is a monodromy map changing the canonical set of generators of D_{p+1} . It appears as the top slice of the $a_2/\mathbb{Z}_p^2 \rtimes S_3$ Hasse diagram. The non-normal slice μ_2 can be generalised to μ_p , whose normalisation is A_{2p-1} . The ring of μ_p is $\mathbb{C}[s^2t^2, s^3t^3, s^{2p}, t^{2p}, s^{2p+1}t, st^{2p+1}]$. It appears as the bottom slice of the $a_2/\mathbb{Z}_p^2 \rtimes S_3$ Hasse diagram.

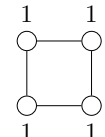
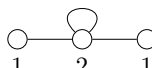
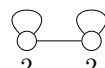
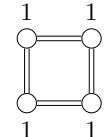
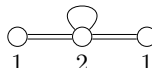
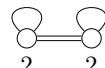
Label	Quiver	Discrete Quotient	HS	Volume
a_3			$\frac{(1+t^2)(1+8t^2+t^4)}{(1-t^2)^6}$	$\frac{5}{8}$
$a_3/\mathfrak{S}_2 \cong \overline{\text{n.min } B_2}$		\mathfrak{S}_2	$\frac{(1+t^2)(1+3t^2+t^4)}{(1-t^2)^6}$	$\frac{5}{16}$
$a_3/\mathfrak{S}_2 \times \mathfrak{S}_2$		$\mathfrak{S}_2 \times \mathfrak{S}_2 \cong \text{Klein}_4$	$\frac{1+3t^2+11t^4+10t^6+11t^8+3t^{10}+t^{12}}{(1-t^2)^3(1-t^4)^3}$	$\frac{5}{32}$
a_3/\mathbb{Z}_2^3		\mathbb{Z}_2^3	$\frac{(1+t^4)(1+8t^4+t^8)}{(1-t^2)^3(1-t^4)^3}$	$\frac{5}{64}$
$a_3/\mathbb{Z}_2^3 \rtimes \mathfrak{S}_2$		$\mathbb{Z}_2^3 \rtimes \mathfrak{S}_2 \cong \mathbb{Z}_2 \times \text{Dih}_4$	$\frac{(1+t^4)(1-t^2+5t^4-t^6+t^8)}{(1-t^2)^3(1-t^4)^3}$	$\frac{5}{128}$
$a_3/\mathbb{Z}_2^3 \rtimes \mathfrak{S}_2 \times \mathfrak{S}_2$		$\mathbb{Z}_2^3 \rtimes \mathfrak{S}_2 \times \mathfrak{S}_2$	$\frac{1-2t^2+6t^4-4t^6+8t^8-4t^{10}+6t^{12}-2t^{14}+t^{16}}{(1-t^2)^3(1-t^4)^2(1-t^8)}$	$\frac{5}{256}$

Table 9. Quivers from discrete quotient of a_3 .

bouquet is introduced and discussed in [36, 37]. The investigation of the decorated quiver with a non-simply-laced bouquet or complete graphs is left for future work.

7.3 A_3 affine quiver with S_2 and \mathbb{Z}_2 gauging

A similar analysis for the affine A_3 quiver: $\mathcal{CG}_{4,1}$ is performed. The Coulomb branch is the closure of the minimal nilpotent orbit of A_3 , which is denoted a_3 . Hence, for a_3 , quotients by S_2 , S_2^2 , \mathbb{Z}_2^3 , and products thereof can be taken. The quotient of S_2 and S_2^2 are discussed above in section 5.2. The resulting quivers, unrefined HS, and volume of the resulting moduli space are summarised in table 9.

Again, one verifies that the (inverse) volume ratios agree with the order of the quotient group: $\frac{\text{vol}^{-1}(a_3/S_2)}{\text{vol}^{-1}(a_3)} = 2$, $\frac{\text{vol}^{-1}(a_3/S_2 \times S_2)}{\text{vol}^{-1}(a_3)} = 4$, $\frac{\text{vol}^{-1}(a_3/\mathbb{Z}_2^3)}{\text{vol}^{-1}(a_3)} = 8$, $\frac{\text{vol}^{-1}(a_3/\mathbb{Z}_2^3 \rtimes S_2)}{\text{vol}^{-1}(a_3)} = 16$, and $\frac{\text{vol}^{-1}(a_3/\mathbb{Z}_2^3 \rtimes S_2 \times S_2)}{\text{vol}^{-1}(a_3)} = 32$.

7.4 D_4 affine quiver with S_4 gauging

The Coulomb branch of the D_4 affine quiver is d_4 , the closure of the minimal nilpotent orbit of \mathfrak{so}_8 . The quiver has a natural S_4 action that permutes the 4 $U(1)$ nodes. By gauging this S_4 and its subgroups, the orbifolds of d_4 by the subgroup that is gauged can be obtained, the quivers are labelled by (1) in table 10. This construction is well-studied in [1–3, 11, 38]. Here a different realisation of the orbifold of d_4 by S_4 and its subgroups is provided by using both the \mathbb{Z}_q and the S_n quotients, the quivers are labelled by (2) in the table 10. The symplectic singularity in the last row of this table is also found and constructed explicitly as part of the nilpotent cone of \mathfrak{f}_4 in [18], as the slice between $E_8(a_6)$ and $E_8(b_6)$.

Label	Quiver	Discrete Quotient	HS	Volume
d_4			$\frac{(1+t^2)(1+17t^2+48t^4+17t^6+t^8)}{(1-t^2)^{10}}$	$\frac{21}{128}$
$d_4/\mathfrak{S}_2(1)$		\mathfrak{S}_2	$\frac{(1+t^2)(1+10t^2+20t^4+10t^6+t^8)}{(1-t^2)^{10}}$	$\frac{21}{256}$
$d_4/\mathfrak{S}_2(2)$		$\mathbb{Z}_2 \cong \mathfrak{S}_2$	$\frac{(1+3t^2+6t^4+3t^6+t^8)(1+8t^2+55t^4+64t^6+55t^8+8t^{10}+t^{12})}{(1-t^2)^5(1-t^4)^5}$	$\frac{21}{256}$
$d_4/\mathfrak{S}_2 \times \mathfrak{S}_2(1)$		$\mathfrak{S}_2 \times \mathfrak{S}_2$	$\frac{1+10t^2+55t^4+150t^6+288t^8+336t^{10}+288t^{12}+150t^{14}+55t^{16}+10t^{18}+t^{20}}{(1-t^2)^5(1-t^4)^5}$	$\frac{21}{512}$
$d_4/\mathfrak{S}_2 \times \mathfrak{S}_2(2)$		$\mathbb{Z}_2^2 \cong \mathfrak{S}_2 \times \mathfrak{S}_2$	$\frac{1+5t^2+45t^4+130t^6+314t^8+354t^{10}+314t^{12}+130t^{14}+45t^{16}+5t^{18}+t^{20}}{(1-t^2)^5(1-t^4)^5}$	$\frac{21}{512}$
$d_4/\mathfrak{S}_3(1)$		\mathfrak{S}_3	$\frac{(1+t^2)(1+3t^2+6t^4+3t^6+t^8)}{(1-t^2)^{10}}$	$\frac{7}{256}$
$d_4/\mathfrak{S}_3(2)$		$\mathbb{Z}_3 \times \mathfrak{S}_2 \cong \mathfrak{S}_3$	$\frac{(1+t^2)(1+9t^2+31t^4+119t^6+234t^8+392t^{10}+645t^{12}+540t^{14}+645t^{16}+392t^{18}+234t^{20}+119t^{22}+31t^{24}+9t^{26}+t^{28})}{(1-t^2)^5(1-t^4)^5}$	$\frac{7}{256}$
$d_4/\mathfrak{S}_4(1)$		\mathfrak{S}_4	$\frac{1+3t^2+13t^4+25t^6+46t^8+48t^{10}+46t^{12}+25t^{14}+13t^{16}+3t^{18}+t^{20}}{(1-t^2)^5(1-t^4)^5}$	$\frac{7}{1024}$
$d_4/\mathfrak{S}_4(2)$		\mathfrak{S}_4	$\frac{1+3t^2+13t^4+25t^6+46t^8+48t^{10}+46t^{12}+25t^{14}+13t^{16}+3t^{18}+t^{20}}{(1-t^2)^5(1-t^4)^5}$	$\frac{7}{1024}$

Table 10. Quivers from discrete quotient of d_4 .

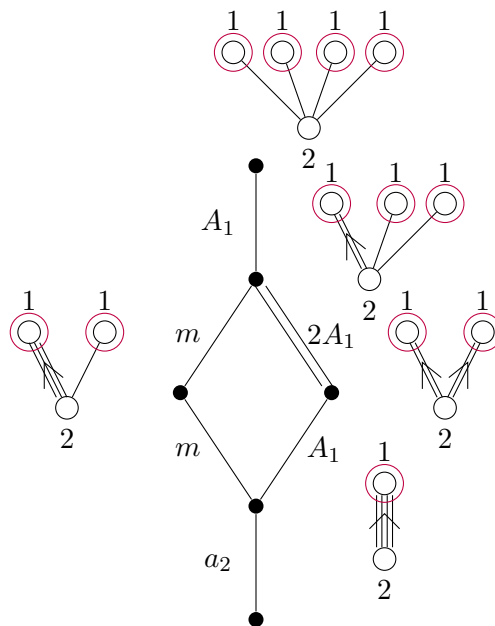


Figure 18. Hasse diagram and decorated quiver of $d_4/S_4(1)$.

A particularly interesting case is the symplectic singularity d_4/S_4 , where the action matches for both cases (1) and (2), shown in the last two rows of table 10. Therefore there are two different constructions of this singularity. One is a unitary quiver with an adjoint loop, and the other is a non-simply laced unitary quiver with adjoint. The Hasse diagrams are shown in figure 18 and figure 19, where one can observe that it is computed in two different ways, starting from each corresponding quiver. The brane system for the quiver in the last row will be discussed in an upcoming paper.

The Hasse diagram can be derived from the decorated quiver technique:

7.5 $\mathcal{CG}_{4,2}$ quiver with S_4 gauging

As a next example, consider the $\mathcal{CG}_{4,2}$ quiver, for which one can quotient S_2 , $S_2 \times S_2$, S_3 , or S_4 . The resulting quivers are collected in table 11.

Again, one verifies straightforwardly that the computed (inverse) volume ratios agree with the orders of the quotient groups: $\frac{\text{vol}^{-1}(\mathcal{CG}_{4,2}/S_2)}{\text{vol}^{-1}(\mathcal{CG}_{4,2})} = 2$, $\frac{\text{vol}^{-1}(\mathcal{CG}_{4,2}/S_2 \times S_2)}{\text{vol}^{-1}(\mathcal{CG}_{4,2})} = 4$, $\frac{\text{vol}^{-1}(\mathcal{CG}_{4,2}/S_3)}{\text{vol}^{-1}(\mathcal{CG}_{4,2})} = 6$, and $\frac{\text{vol}^{-1}(\mathcal{CG}_{4,2}/S_4)}{\text{vol}^{-1}(\mathcal{CG}_{4,2})} = 24$.

7.6 Orbifolds and covers of a hypersurface

Recalling the hypersurface $\mathcal{Q}_{7(b)}$ and its cover $\mathcal{Q}_{7(a)}$, one can study all possible discrete quotients of the latter. Specifically, this is a complete graph with only two nodes and edge multiplicity $2g - 2$, and connected to a background $U(1)$ gauge node. The \mathbb{Z}_2 quotient can be taken by either changing the edge multiplicity and changing the connection into non-simply laced, or quotient S_2 by merging the two edges. See table 12 for a summary.

The most interesting case is the moduli space $\mathcal{C}(\mathcal{Q}_{7(a)})/\mathbb{Z}_q \rtimes S_2$. The Hilbert series indicates that it is a complete intersection: it has 3 generators in adjoint representation

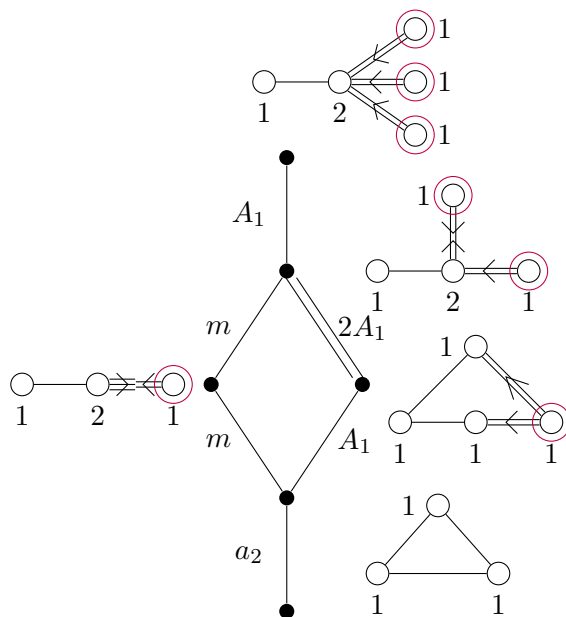


Figure 19. Hasse diagram and decorated quiver of $d_4/S_4(2)$.

Label	Quiver	Discrete Quotient	HS	Volume
$\mathcal{CG}_{4,2}$			$\frac{1+6t^6+5t^8+5t^{12}+6t^{14}+t^{20}}{(1-t^2)^3(1-t^6)^2(1-t^8)}$	$\frac{1}{96}$
$\mathcal{CG}_{4,2}/\mathfrak{S}_2$		\mathfrak{S}_2	$\frac{1+4t^6+5t^8+2t^{10}+2t^{12}+5t^{14}+4t^{16}+t^{22}}{(1-t^2)^2(1-t^4)(1-t^6)^2(1-t^8)}$	$\frac{1}{192}$
$\mathcal{CG}_{4,2}/\mathfrak{S}_2 \times \mathfrak{S}_2$		$\mathfrak{S}_2 \times \mathfrak{S}_2$	$\frac{1+2t^6+6t^8+2t^{10}+2t^{12}+2t^{14}+6t^{16}+2t^{18}+t^{24}}{(1-t^2)(1-t^4)^2(1-t^6)^2(1-t^8)}$	$\frac{1}{384}$
$\mathcal{CG}_{4,2}/\mathfrak{S}_3$		\mathfrak{S}_3	$\frac{1+2t^6+3t^8+4t^{10}+2t^{12}+2t^{14}+4t^{16}+3t^{18}+2t^{20}+t^{26}}{(1-t^2)(1-t^4)(1-t^6)^3(1-t^8)}$	$\frac{1}{576}$
$\mathcal{CG}_{4,2}/\mathfrak{S}_4$		\mathfrak{S}_4	$\frac{1+2t^8+3t^{10}+4t^{12}+t^{14}+2t^{16}+t^{18}+4t^{20}+3t^{22}+2t^{24}+t^{32}}{(1-t^4)(1-t^6)^3(1-t^8)^2}$	$\frac{1}{2304}$

Table 11. Quivers from discrete quotient of $\mathcal{CG}_{4,2}$.

Label	Quiver	Discrete Quotient	HS	Volume
$\mathcal{C}(\mathcal{Q}_{7(a)})$			$\frac{1+t^2+2t^{2g}-1-2t^{2g+1}-t^{4g-2}-t^{4g}}{(1-t^2)^3(1-t^{2g}-1)^2}$	$\frac{g}{(2g-1)^2}$
$\mathcal{C}(\mathcal{Q}_{7(b)}) = \mathcal{C}(\mathcal{Q}_{7(a)})/\mathfrak{S}_2$		\mathfrak{S}_2	$\frac{1-t^{4g}}{(1-t^2)^3(1-t^{2g}-1)^2}$	$\frac{g}{2(2g-1)^2}$
$\mathcal{C}(\mathcal{Q}_{7(a)})/\mathbb{Z}_2$		\mathbb{Z}_2	$\frac{1+t^2+4t^{4g}-2-4t^{4g}-t^{8g-4}-t^{8g-2}}{(1-t^2)^3(1-t^{4g}-2)^2}$	$\frac{g}{2(2g-1)^2}$
$\mathcal{C}(\mathcal{Q}_{7(a)})/\mathbb{Z}_2 \rtimes \mathfrak{S}_2$		$\mathbb{Z}_2 \rtimes \mathfrak{S}_2$	$\frac{(1-t^{4g})(1-t^{8g-4})}{(1-t^2)^3(1-t^{4g}-2)^3}$	$\frac{g}{4(2g-1)^2}$
$\mathcal{C}(\mathcal{Q}_{7(a)})/\mathbb{Z}_q$		\mathbb{Z}_q	$\frac{1+t^2+2qt^{2qg}-q-2qt^{2qg}-q+2-t^{4qg}-2q-t^{4qg}-2q+2}{(1-t^2)^3(1-t^{2qg}-q)^2}$	$\frac{g}{q(2g-1)^2}$
$\mathcal{C}(\mathcal{Q}_{7(a)})/\mathbb{Z}_q \rtimes \mathfrak{S}_2$		$\mathbb{Z}_q \rtimes \mathfrak{S}_2$	$\frac{1+(q-1)t^{2qg}-q-qt^{2qg}-q+2-qt^{4qg}-2q-(q-1)t^{4qg}-2q+2+t^{6qg}-3q+2}{(1-t^2)^3(1-t^{2qg}-q)^3}$	$\frac{g}{2q(2g-1)^2}$

Table 12. Quivers from discrete quotients of a_2 .

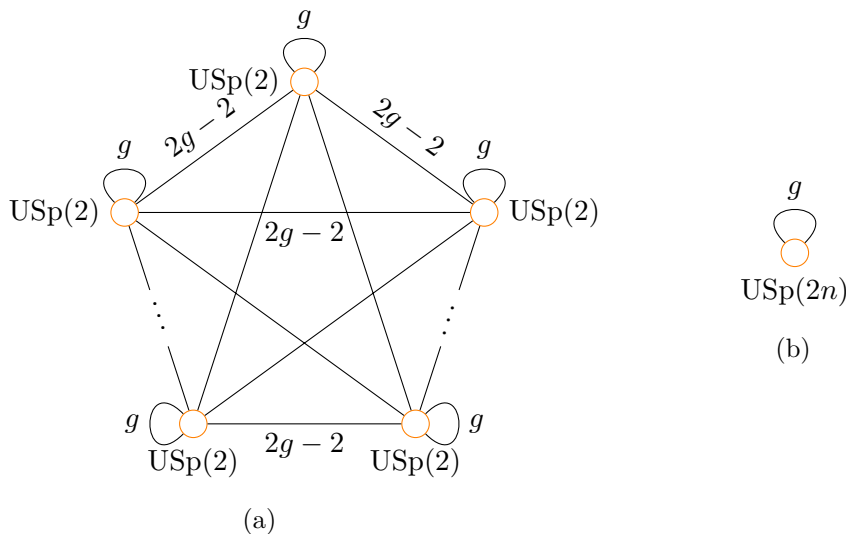


Figure 20. (a): complete graph quiver with n nodes of $USp(2)$ with g adjoints and edges of multiplicity $2g - 2$ connecting all pairs. (b): Quiver for $USp(2n)$ gauge group with g adjoints.

of $SU(2)$ at degree 2, and 3 generators in adjoint representation of $SU(2)$ at degree $4g - 2$. Also there is one relation in the trivial representation at degree $4g$, and another relation in the trivial representation at degree $8g - 4$.

8 Generalisations

It is straightforward to generalise the S_n quotient to quivers with other types of gauge groups with multiple adjoint hypermultiplets;⁵ see table 13. For instance, figures 20, 21, and 22 illustrate such cases. The proof proceeds analogous to that section 3.2. The following results are found:

$$\mathcal{C}(\mathcal{Q}_{20(b)}) = \mathcal{C}(\mathcal{Q}_{20(a)})/S_n \tag{8.1}$$

$$\mathcal{C}(\mathcal{Q}_{21(b)}) = \mathcal{C}(\mathcal{Q}_{21(a)})/S_n \tag{8.2}$$

$$\mathcal{C}(\mathcal{Q}_{22(b)}) = \mathcal{C}(\mathcal{Q}_{22(a)})/S_n. \tag{8.3}$$

$USp(4)$ example. $USp(4)$ is used as an example to demonstrate the generalisation. A S_2 quotient on the Coulomb branch can be performed as follows:

$$\mathcal{C}(\mathcal{Q}_{USp(2,g)}) = \mathcal{C} \left(\begin{array}{c} \begin{array}{cc} \begin{array}{c} g \\ \circlearrowleft \\ USp(2) \end{array} & \begin{array}{c} 2g-2 \\ \text{---} \\ \begin{array}{c} g \\ \circlearrowleft \\ USp(2) \end{array} \end{array} \end{array} \right) \xrightarrow{S_2} \mathcal{C}(\mathcal{Q}_{USp(2,g)}/S_2) = \mathcal{C} \left(\begin{array}{c} \begin{array}{c} g \\ \circlearrowleft \\ USp(4) \end{array} \end{array} \right). \tag{8.4}$$

⁵This is the natural generalisation of the bouquets with different gauge nodes of [2].

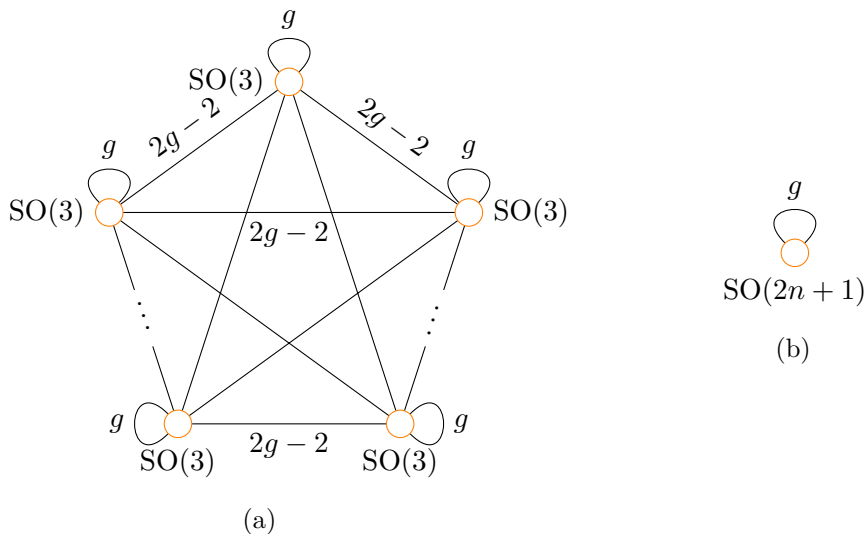


Figure 21. (a): complete graph quiver with n nodes of $\text{SO}(3)$ with g adjoints and edges of multiplicity $2g - 2$ connecting all pairs. (b): Quiver for $\text{SO}(2n + 1)$ gauge group with g adjoints.

The Hilbert series for the two Coulomb branches are calculated by an S_2 Molien sum

$$\begin{aligned}
 & \text{HS}(\mathcal{C}(\mathcal{Q}_{\text{USp}}(2, g))) \\
 &= \frac{1}{(1-t^4)^2} + \left(\frac{1}{(1-t^2)(1-t^4)} \sum_{m_1=0, m_2>0} + \frac{1}{(1-t^2)(1-t^4)} \sum_{m_1>0, m_2=0} \right. \\
 & \quad \left. + \frac{1}{(1-t^2)^2} \sum_{m_1, m_2>0} \right) t^{(4g-4)(|m_1|+|m_2|)+(2g-2)(|m_1-m_2|+|m_1+m_2|)} \\
 &= \frac{1+t^{8g-8}+2t^{8g-6}+2t^{12g-10}+t^{12g-8}+t^{20g-16}}{(1-t^4)^2(1-t^{8g-8})(1-t^{12g-12})}. \tag{8.5a}
 \end{aligned}$$

$$\begin{aligned}
 & \text{HS}(\mathcal{C}(\mathcal{Q}_{\text{USp}}(2, g)/S_2)) \\
 &= \frac{1}{2} \left(\text{HS}(\mathcal{Q}_{\text{USp}}(2, g)) + \frac{1}{1-t^8} + \frac{1}{1-t^4} \sum_{m_1=m_2>0} t^{(4g-4)(|m_1|+|m_2|)+(2g-2)(|m_1-m_2|+|m_1+m_2|)} \right) \\
 &= \frac{1+t^{8g-6}+t^{8g-4}+t^{8g-2}+t^{12g-10}+t^{12g-8}+t^{12g-6}+t^{20g-12}}{(1-t^4)(1-t^8)(1-t^{8g-8})(1-t^{12g-12})}. \tag{8.5b}
 \end{aligned}$$

and the last line agrees with the direct monopole formula evaluation of the $\text{USp}(4)$ theory. Thus, validating the discrete quotient proposal. It then also follows that the (inverse) volume ratio agrees with the order quotient group: $\frac{\text{vol}^{-1}(\mathcal{Q}_{\text{USp}}(2, g)/S_2)}{\text{vol}^{-1}(\mathcal{Q}_{\text{USp}}(2, g))} = 2$.

As in [2], this statement can be further generalised to symplectic gauge groups with g hypermultiplets in the traceless second rank anti-symmetric product Λ^2 , and special orthogonal groups with g hypermultiplets in traceless second rank symmetric product S^2 ; see table 13. Again, the proof proceeds analogous to that in section 3.2. The following results are found:

$$\mathcal{C}(\mathcal{Q}_{23(b)}) = \mathcal{C}(\mathcal{Q}_{23(a)})/S_n \tag{8.6}$$

$$\mathcal{C}(\mathcal{Q}_{24(b)}) = \mathcal{C}(\mathcal{Q}_{24(a)})/S_n \tag{8.7}$$

$$\mathcal{C}(\mathcal{Q}_{25(b)}) = \mathcal{C}(\mathcal{Q}_{25(a)})/S_n. \tag{8.8}$$

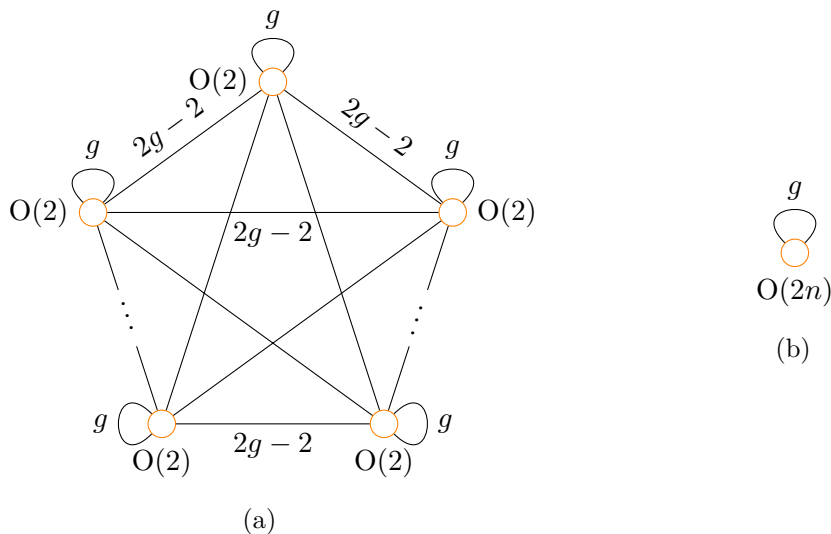


Figure 22. (a): complete graph quiver with n nodes of $O(2)$ with edges of multiplicity $2g - 2$ connecting all pairs. (b): Quiver for $O(2n)$ gauge group with g adjoints.

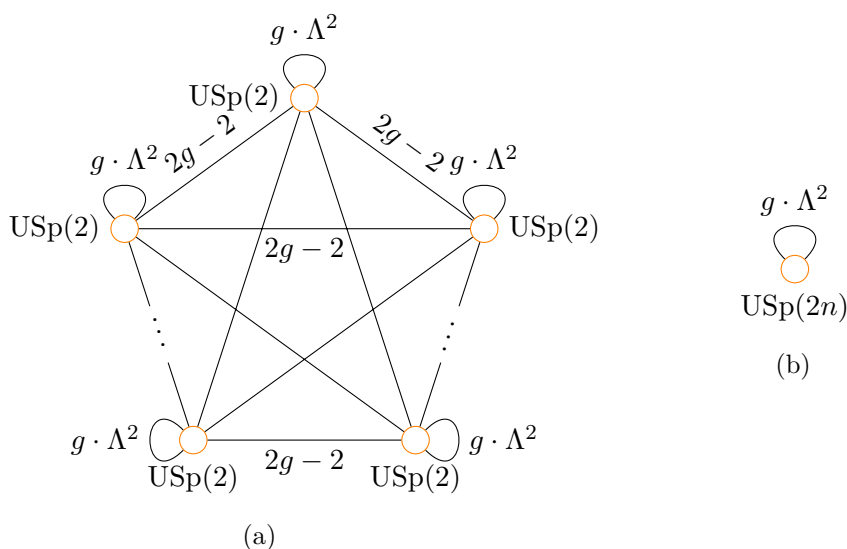


Figure 23. (a): complete graph quiver with n nodes of $USp(2)$ and edges of multiplicity $2g - 2$ connecting all pairs. (b): Quiver for $USp(2n)$ gauge group with $g \cdot \Lambda^2$ hypermultiplets.

Those generalizations of S_n quotient to orthosymplectic quivers are summarised in table 13.

Weyl group quotient. In this section, only connected gauge groups are considered. Note that

$$\mathcal{C}(\mathcal{CG}_{2,4g-4}) = \mathcal{C} \left(\begin{array}{cc} & 4g-4 \\ \circ & \circ \\ 1 & 1 \end{array} \right) \xrightarrow{S_2} \mathcal{C}(\mathcal{ML}_{USp(2),g}) = \mathcal{C} \left(\begin{array}{c} g \\ \circ \\ USp(2) \end{array} \right). \quad (8.9)$$

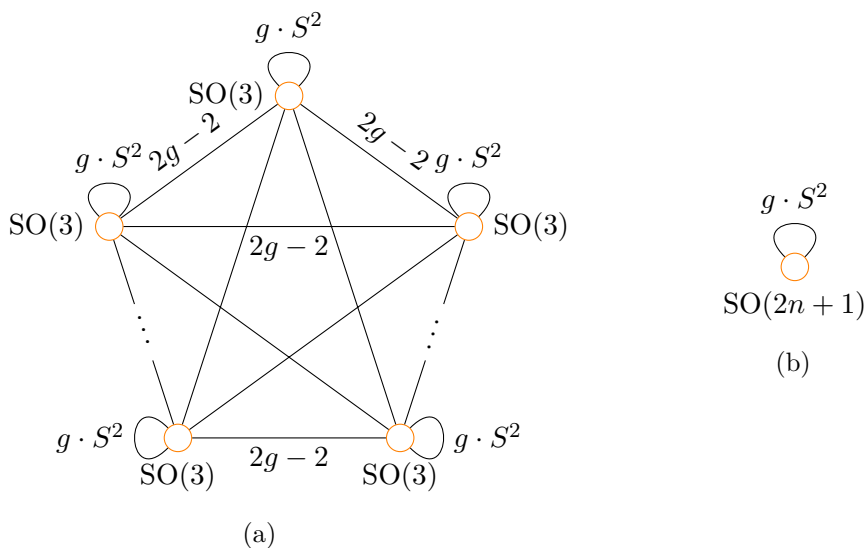


Figure 24. (a): complete graph quiver with n nodes of $SO(3)$ with $g \cdot S^2$ hypermultiplets and edges of multiplicity $2g - 2$ connecting all pairs. (b): Quiver for $SO(2n + 1)$ gauge group with $g \cdot S^2$ hypermultiplets.

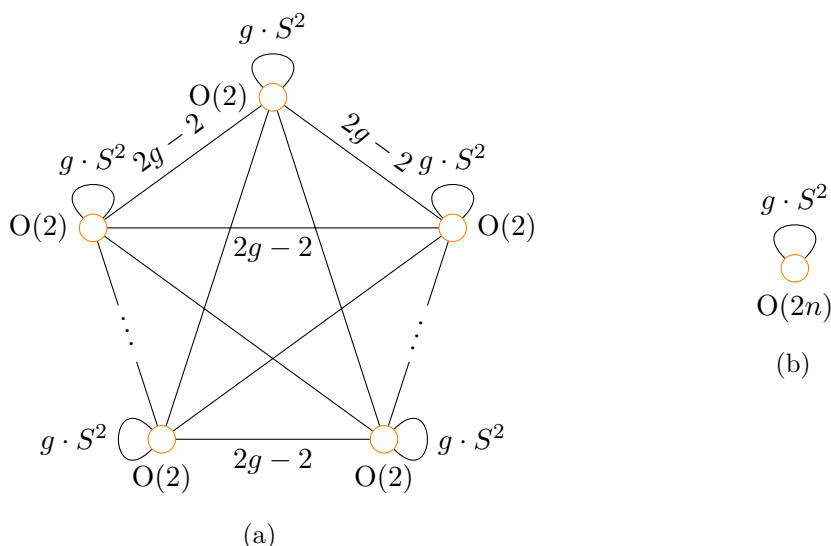


Figure 25. (a): complete graph quiver with n nodes of $O(2)$ with edges of multiplicity $2g - 2$ connecting all pairs. (b): Quiver for $O(2n)$ gauge group with $g \cdot S^2$ hypermultiplets.

G	G'	V
U(1)	U(n)	adj
USp(2)	USp($2n$)	adj, Λ^2
SO(3)	SO($2n + 1$)	adj, S^2
O(2)	O($2n$)	adj, S^2

Table 13. The rank 1 gauge nodes G of the complete graph and the rank n gauge node G' after S_n quotient. V is the allowed representation of the loop attached to G and G' .

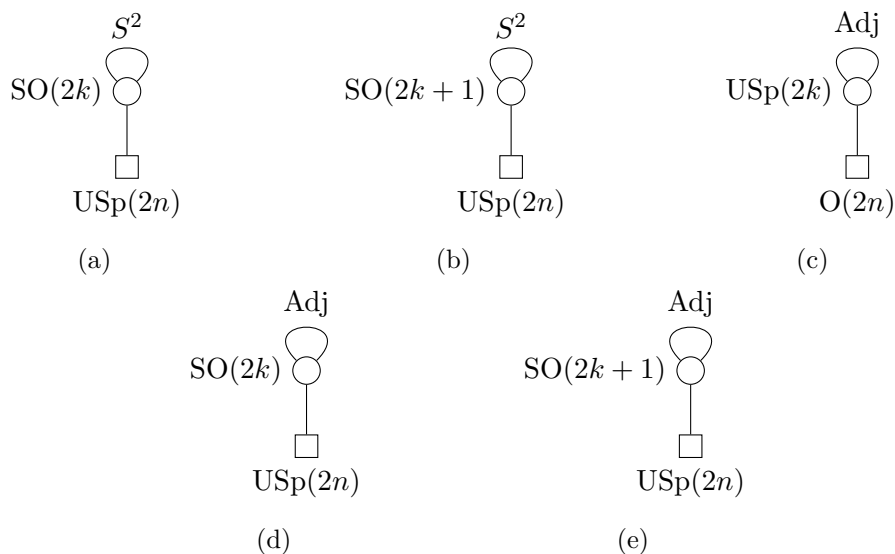


Figure 26. ADHM-like quivers.

This means that every $\text{USp}(2)$ node can be acquired from a S_2 quotient of $\mathcal{CG}_{2,4g-4}$. Together with S_n , one may expect an Abelian theory \mathcal{T}_w (not necessarily a quiver), such that the $S_2^n \rtimes S_n = \mathcal{W}_{\text{USp}(2n)}$ quotient of the Coulomb branch of \mathcal{T}_w is the Coulomb branch of $\mathcal{ML}_{\text{USp}(2n),g}$.

Now this result is generalised as follows: suppose that the non-zero weights of the matter representation \mathcal{R} contains all the non-zero roots of the gauge group G_i , then the Coulomb branch can be written as the Coulomb branch of an Abelian theory quotient the Weyl group of G_i [27]. The Abelian theory \mathcal{T}_w is given by $\text{rank}(G_i)$ copies of $\text{U}(1)$ gauge groups, and the hypermultiplet charges are given by non-zero weights in the original theory, minus one copy of the non-zero weights in the adjoint representation, i.e.

$$\mathcal{C}(\mathcal{Q}_G) = \mathcal{C}(\mathcal{T}_w/\mathcal{W}_{G_i}). \tag{8.10}$$

This generalises (2.4) in two ways. Firstly, in (8.10) any representation “larger” than the adjoint can be included. Secondly, the gauge group G can be any type.

In the orthogonal or symplectic case, from the Abelian theory the \mathbb{Z}_2 factors in the Weyl group of G_i can be understood as the \mathbb{Z}_2 automorphisms between \mathcal{N}_i and $\bar{\mathcal{N}}_i$ of the hypermultiplet representations $\mathcal{R}_i = \mathcal{N}_i \oplus \bar{\mathcal{N}}_i$. Thus, after the quotient half-hypermultiplets appear in the orthosymplectic quivers.

It is straightforward to see that in quivers figure 2 and figure 3(b) the requirement for the hypermultiplet representations is satisfied. The requirement for a “larger” representation can be understood in the following way. For an Abelian theory, there is no negative contribution to the conformal dimension. In the corresponding non-Abelian theory, the negative contribution from the vector multiplet needs to be fully compensated. In general, for quivers with hypermultiplets representation “smaller” than the adjoint, one can only expect the Coulomb branch is birational to the Abelianized Coulomb branch [26].

The simplest example for orthosymplectic quivers are shown in figure 26.

Quiver	Coulomb branch
$\mathcal{Q}_{26(a)}$	$(A_{2n+3})^k/\mathcal{W}_{D_k}$
$\mathcal{Q}_{26(b)}$	$(A_{2n+3})^k/\mathcal{W}_{B_k}$
$\mathcal{Q}_{26(c)}$	$(A_{2n-1})^k/\mathcal{W}_{C_k}$
$\mathcal{Q}_{26(d)}$	$(A_{2n-1})^k/\mathcal{W}_{D_k}$
$\mathcal{Q}_{26(e)}$	$(A_{2n-1})^k/\mathcal{W}_{B_k}$

G	\mathcal{W}_G	$ \mathcal{W}_G $
A_k	S_k	$k!$
B_k	$\mathbb{Z}_2^k \rtimes S_k$	$2^k \cdot k!$
C_k	$\mathbb{Z}_2^k \rtimes S_k$	$2^k \cdot k!$
D_k	$\mathbb{Z}_2^{k-1} \rtimes S_k$	$2^{k-1} \cdot k!$

(a)

(b) The Weyl group of classical Lie algebras.

Table 14. (a): the Coulomb branches of the quivers in figure 26. (b): the Weyl group of classical Lie algebras.

With (8.10), the Coulomb branches of those quivers are summarised in table 14(a). For completeness, the Weyl groups are listed in table 14(b).

9 Conclusions and outlook

In this work, the S_n discrete quotient on the Coulomb branch of $3d \mathcal{N} = 4$ quiver with a bouquet of n $U(1)$ is generalised to any quiver with a complete graph of n $U(1)$. This result is proven on the level of the monopole formula by using the generalised Molien sum. This S_n plays the role of both outer-automorphism of the quiver and Weyl group of gauge group. This result is also generalised to complete graphs of any rank 1 nodes with loops in certain representations.

Another discrete action considered is \mathbb{Z}_q^{n-1} . This can be achieved by changing the edge multiplicity to q -fold or the charge to q -times. These two operations on quivers cannot be distinguished on Coulomb branch level, but can be distinguished by their Higgs branches. The idea behind both actions is to reflect the discrete action on the magnetic lattices into an action on the Coulomb branch geometry. Further, \mathbb{Z}_q^{n-1} and S_n can be combined into $\mathbb{Z}_q^{n-1} \rtimes S_n$ action on the Coulomb branch.

These results are expanding our knowledge of relations between different quivers. It allows us to determine whether a Coulomb branch can be obtained as an orbifold of another, and construct new quivers following the rules. For example, a quiver construction of a_2/S_4 and a new quiver construction of d_4/S_4 were obtained in this paper.

There are plenty of directions to explore in the future. First of all, the Hasse diagrams of Coulomb branches of quivers with g adjoint hypermultiplets is not fully understood yet. Secondly, for the theories which cannot be Abelianised into a well-defined theory in the sense of [27], it is an open challenge to derive the S_n action on the chiral ring generators. Thirdly, it is interesting to classify all quivers whose Coulomb branches are orbifolds of flat space. For example, the orbifold quivers for the complex reflection groups are analysed in an upcoming work. Finally, it remains to be explored what those discrete actions on quivers imply on $4d$ theories. That is to say, the discrete action is well-defined on the magnetic quiver, but to what extent can this be transferred to the $4d \mathcal{N} = 2$ theory (Argyres-Douglas or class \mathcal{S}).

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A Monopole formula

The Coulomb branch chiral ring operators are monopole operators dressed by complex scalars in the vector multiplet. The Hilbert series counts those dressed monopole operators, graded by their conformal dimension. A bare monopole operator is labelled by its magnetic charge $m \in \text{Hom}(U(1), G) = \Lambda_\omega^{G^\vee}$, which is the dual lattice of the weight lattice of the gauge group G . For gauge group G and hypermultiplet representation \mathcal{R} , the conformal dimension is given by [24, 39–41]

$$\Delta(m) = - \sum_{\alpha \in \Phi^+} |\langle \alpha, m \rangle| + \frac{1}{4} \sum_{\omega_i \in \mathcal{R}} |\langle \omega_i, m \rangle|. \quad (\text{A.1})$$

The set of positive roots α of the algebra \mathfrak{g} are denoted by Φ^+ , and the ω_i are the weights for the matter representation \mathcal{R} for the hypermultiplets. If the representation \mathcal{R} is polarizable, $\mathcal{R} = \mathcal{N} \oplus \bar{\mathcal{N}}$, then the conformal dimension simplifies to

$$\Delta(m) = - \sum_{\alpha \in \Phi^+} |\langle \alpha, m \rangle| + \frac{1}{2} \sum_{\omega_i \in \mathcal{N}} |\langle \omega_i, m \rangle|. \quad (\text{A.2})$$

Then the refined Hilbert series is given by:

$$\text{HS}(z, t) = \sum_{m \in \Lambda_\omega^{G^\vee} / \mathcal{W}_G} z^{J(m)} t^{2\Delta(m)} P_G[G_m](t). \quad (\text{A.3})$$

Here t is a fugacity for R-charge, and z is a fugacity for counting topological quantum numbers $J(m)$. There is a dressing factor, $P_G[G_m](t)$, accounting for the dressing of the monopole operators by the complex scalar in the vector multiplet, and it depends on the residual gauge group G_m that is unbroken by the magnetic charges. The dressing factor is counting Casimir invariants of the unbroken gauge group G_m .

Weyl chamber and dressing factor. Here detailed explanation of the role of Weyl chamber and the dressing factor $P_G[G_m](t)$ is given, since this is where the S_n is acting in the monopole formula.

The principal Weyl chamber for $U(n)$ in $\mathcal{Q}_{3(b)}$ can be taken as $\{m_1 \geq \dots \geq m_n\}$ in \mathbb{R}^n . This is S_n quotient on \mathbb{R}^n , the principal Weyl chamber for $U(1)^n$ in $\mathcal{Q}_{3(a)}$. The magnetic lattice that is summed over is the intersection of principal Weyl chamber with \mathbb{Z}^n .

The principal Weyl chamber is viewed as a polyhedral cone, as in [42, 43]. One can decompose the principal Weyl chamber σ into a union of loci,

$$\sigma = \bigcup_i \sigma_i \tag{A.4}$$

The summation over magnetic lattice in (A.3) is a summation over all the loci $\sigma_i \cap \mathbb{Z}^n$. Each locus is fixed by a subgroup of the Weyl group \mathcal{W}_G and gives a pattern of breaking of the gauge group.

For a given magnetic charge m , the unbroken gauge group G_m is a subgroup of G which commute with the magnetic charges m . The whole Weyl group of G is also broken to the Weyl group of G_m , denoted by \mathcal{W}_m . The Weyl group of G_m , \mathcal{W}_m , is just the subgroup of \mathcal{W}_G that fixes the locus σ_i that m lives in.

The complex scalars of the unbroken gauge group G_m combines into Casimirs, giving the ring $\mathbb{C}[\mathfrak{g}_m]^{G_m} = \mathbb{C}[\mathfrak{h}_m]^{\mathcal{W}_m}$, whose Hilbert series gives the dressing factor and the calculation is explained in next appendix. This \mathcal{W}_m action on the $\mathbb{C}[\mathfrak{h}_m]$ justifies the action on the dressing factor that were used in the main context.

Notice that the dressing factors $P_G[G_m](t)$ have the same order of pole at $t = 1$. The residue at $t = 1$ is related by a factor of $1/|\mathcal{W}_m|$, since it only counts \mathcal{W}_m invariant polynomials. It immediately follows that the volumes of the Coulomb branches of quivers $\mathcal{Q}_{3(b)}$ and $\mathcal{Q}_{3(a)}$ are related by an $|S_n|$ quotient, as a consistency check of (2.4).

B Molien sum

The Molien sum is used to compute the Hilbert series of an orbifold V/Γ , where V is a n -dim module of discrete group Γ , i.e. a n -dim complex vector space with Γ acting on with certain representation:

$$\text{HS} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{1}{\det(1 - \gamma \cdot t)}. \tag{B.1}$$

To explain this formula, recall the orthogonality of characters, i.e. the sum of characters of any non-trivial irrep of Γ is zero:

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ in trivial irrep} \\ 0 & \text{if } \gamma \text{ in non-trivial irrep} \end{cases} \tag{B.2}$$

For any $\gamma \in \Gamma$, there exists a basis (x_1, \dots, x_n) of V diagonalise γ , such that $\gamma = \text{diag}(\gamma_1, \dots, \gamma_n)$ under this basis. The Molien sum is independent of basis. The basis of the space of polynomials $\prod_{d=0}^{\infty} \text{Sym}^d(V)$ can be encoded into a formal series:

$$\frac{1}{(1 - x_1) \cdots (1 - x_n)} \tag{B.3}$$

The element γ acts on this series as:

$$\frac{1}{(1 - x_1) \cdots (1 - x_n)} \xrightarrow{\gamma} \frac{1}{(1 - \gamma_1 x_1) \cdots (1 - \gamma_n x_n)}. \tag{B.4}$$

If x_i is set to $x_i = t$, the coefficient of t^d is the character of γ on $Sym^d(V)$:

$$\frac{1}{(1 - \gamma_1 t) \cdots (1 - \gamma_n t)} = \sum_{d=0}^{\infty} \chi_d(\gamma) t^d. \tag{B.5}$$

Now the sum over the group Γ is performed, only the trivial rep will remain and each copy of trivial rep will contribute one to the coefficients. One can see from Molien sum, the result is exactly the Hilbert series:

$$\begin{aligned} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{1}{\det(1 - \gamma \cdot t)} &= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{1}{(1 - \gamma_1 t) \cdots (1 - \gamma_n t)} \\ &= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_{d=0}^{\infty} \chi_d(\gamma) t^d \\ &= \sum_{d=0}^{\infty} \dim \left((Sym^d(V))^{\Gamma} \right) t^d \\ &= \text{HS}. \end{aligned} \tag{B.6}$$

The Molien sum can be generalised for any variety with Γ acting on it:

$$\begin{aligned} \text{HS} &= \sum_{\gamma \in \Gamma} \text{HS}_0^{\gamma} \\ &= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_{d=0}^{\infty} \chi_d(\gamma) t^d, \end{aligned} \tag{B.7}$$

where HS_0 is the Hilbert series of the original variety and $\chi_d(\gamma)$ is the character of γ in the vector space spanned by polynomials of degree d . See [11] for more examples. The key of Molien sum is using the orthogonality property of characters to throw away any terms transforming non-trivially under Γ , so that the remaining terms are invariant under the action of Γ . Note that the Molien sum is representation dependent. The same logic can be extend to continuous group, for example, the Weyl integral for Lie groups.

C Outer S_4 action on a_2

In this section the S_4 action on the ring of a_2 following [18] is discussed. The ring can be expressed as a 3×3 complex matrix $M \in \text{SL}(3, \mathbb{C})$ with conditions traceless and the rank no greater than 1: $\mathbb{C}[M_j^i] / \langle \text{Tr}(M) = 0, \text{rk}(M) \leq 1 \rangle$, with $i, j = 1, 2, 3$. We can also use an alternative choice of coordinates by set $M_j^i = x_i y_j$: $\mathbb{C}[x_i y_j] / \langle x_1 y_1 + x_2 y_2 + x_3 y_3 = 0 \rangle$, with $i, j = 1, 2, 3$. This set of coordinates automatically impose the condition of $\text{rk}(M) \leq 1$. In the matrix form:

$$\begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{pmatrix}. \tag{C.1}$$

The action of $S_4 = S_3 \rtimes \mathbb{Z}_2^2$ are generated by the action of S_3 together with the action of $\text{Klein}_4 = \mathbb{Z}_2^2$. Start with the action of Klein_4 , which is a subgroup of $\text{SL}(3, \mathbb{C})$:

$$(x_1, y_1) \rightarrow (-x_1, -y_1), (x_2, y_2) \rightarrow (-x_2, -y_2), (x_3, y_3) \rightarrow (-x_3, -y_3). \tag{C.2}$$

It is easy to see that there are only two independent \mathbb{Z}_2 actions on the ring of a_2 , for example, the $(x_1, y_1) \rightarrow (-x_1, -y_1)$ together with $(x_2, y_2) \rightarrow (-x_2, -y_2)$ can be composed to $(x_3, y_3) \rightarrow (-x_3, -y_3)$. The fundamental invariants under this action (after eliminating with the relation) are $x_i y_i$ and $x_i^2 y_j^2$ where $i \neq j$. We have the induced relations by requiring the following matrix has rank no greater than 1 (any 2×2 sub-matrix has zero determinant):

$$\begin{pmatrix} x_1^2 y_1^2 & x_1^2 y_2^2 & x_1^2 y_3^2 \\ x_2^2 y_1^2 & x_2^2 y_2^2 & x_2^2 y_3^2 \\ x_3^2 y_1^2 & x_3^2 y_2^2 & x_3^2 y_3^2 \end{pmatrix}. \tag{C.3}$$

Now let us look at the S_3 action, which is not a subgroup of $SL(3, \mathbb{C})$. Here the invariants under Klein₄ are labelled as $\alpha_i = x_{i+1} y_{i+1} - x_{i-1} y_{i-1}$, $\beta_i = x_i^2 y_{i-1}^2$ and $\gamma_i = x_{i-1}^2 y_i^2$. The action of S_3 is:

$$\alpha_i \rightarrow \alpha_{\sigma(i)}, \beta_i \rightarrow \beta_{\sigma(i)}, \gamma_i \rightarrow \gamma_{\sigma(i)}. \tag{C.4}$$

Combine these two actions (C.2) and (C.4), there is the action of S_4 on a_2 . We call it outer S_4 since it is not a subgroup of $SL(3, \mathbb{C})$ but belongs to its outer-automorphism. One can examine it and find consistency with the action in section 5.4.

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