Published for SISSA by Springer

Received: *March 5, 2024*
Accepted: *March 12, 2024*
Published: *May 6, 2024*

Universal predictions of Siegel modular invariant theories near the fixed points

Gui-Jun Ding,[a] **Ferruccio Feruglio**[b] **and Xiang-Gan Liu**[c]

[a] *Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*
[b] *INFN, Sezione di Padova, Via Marzolo 8, I-35131 Padua, Italy*
[c] *Department of Physics and Astronomy, University of California, Irvine, CA 92697-4575 U.S.A.*

E-mail: dinggj@ustc.edu.cn, feruglio@pd.infn.it, xianggal@uci.edu

Abstract: We analyze a general class of locally supersymmetric, CP and modular invariant models of lepton masses depending on two complex moduli taking values in the vicinity of a fixed point, where the theory enjoys a residual symmetry under a finite group. Like in models that depend on a single modulus, we find that all physical quantities exhibit a universal scaling with the distance from the fixed point. There is no dependence on the level of the construction, the weights of matter multiplets and their representations, with the only restriction that electroweak lepton doublets transform as irreducible triplets of the finite modular group. Also the form of the kinetic terms, which here are assumed to be neither minimal nor flavor blind, is irrelevant to the outcome. The result is remarkably simple and the whole class of examined theories gives rise to five independent patterns of neutrino mass matrices. Only in one of them, the predicted scaling agrees with the observed neutrino mass ratios and lepton mixing angles, exactly as in single modulus theories living close to $\tau = i$.

Keywords: Flavour Symmetries, Neutrino Mixing, Theories of Flavour, Discrete Symmetries

ArXiv ePrint: 2402.14915

https://doi.org/10.1007/JHEP05(2024)052

Contents

1 Introduction

The flavour puzzle is one of the big unsolved problems in particle physics. The observed regularities in the fermion mass spectrum and the mixing angles still escape our understanding and their consistent description requires more than twenty free parameters. One of the tools that have been exploited so far to tackle the puzzle relies on flavour symmetries, acting in generation space and supposed to reduce the number of flavour parameters, including those associated with CP violation. In a bottom-up approach, the freedom of the model builder is however disarmingly huge. Not only we can freely choose flavour groups and representations of matter fields but, more importantly, we should add an ad hoc symmetry-breaking sector to reconcile the approximate predictions of the exact symmetry limit with the accurate experimental data. The complicated architecture of such a sector often results in a limited predictive power of the whole approach.

Modular invariance offers in principle some advantage over traditional flavour symmetries of the above type [1]. The starting point is the symmetry-breaking sector itself consisting, in the simplest case, of a single complex field, the modulus, living in the upper half of the complex plane and transforming nontrivially under the modular group. In a supersymmetric realization [2, 3], modular invariance severely restricts the Yukawa couplings, which are required to be modular forms, holomorphic functions of the modulus with appropriate transformation properties under the modular group. Radiative corrections and supersymmetry breaking effects are negligible in a large portion of the parameter space [4]. Invariance under the modular group offers a simple, axion-free solution to the strong CP problem [5]. Moreover, such a framework is intimately related to the basic properties of superstring compactifications [6–27], allowing the bottom-up and top-down approaches to reinforce each other [28]. An intense activity in both directions has been pursued in the last years [29, 30].

These nice features are partially spoiled by the freedom associated with the transformation law of the matter fields under the modular group and with the choice of kinetic terms. The former requires the specification of a level n and, for each matter multiplet $\varphi^{(I)}$, a weight k_I and a representation ρ_I of the adopted finite modular group. The latter affects the fermion mass spectrum [31], except for the case of minimal or flavor universal Kähler potential. Some progress in dealing with this problem has been made with the introduction of eclectic flavour symmetries [32–38], but the prize is the reintroduction of a nonminimal symmetry-breaking sector [39–42]. Even considering the lepton sector alone, by exploiting the existing freedom a large number of models correctly reproducing neutrino masses, lepton mixing angles, and predicting leptonic CP-violating phases have been formulated [29, 30]. The variety of different realizations allowed in a pure bottom-up approach has not allowed so far to identify a unique scenario.

Despite our inability to designate a single successful theory, from all the existent effective models we might have learned something significant about the principle underlying the fermion sector. Indeed, the leptonic models formulated so far suggest a preference for a value of the modulus near the self-dual fixed point $\tau_0 = i$ [43], which preserves the symmetry under $\tau \to -1/\tau$. Such a preference is even more pronounced for the subset of CP-invariant models, where the violation of CP is spontaneous and entirely controlled by the modulus τ. Indeed, if electroweak lepton doublets are assigned to irreducible triplets of the finite modular group, a typical choice allowing to minimize the number of free parameters, the behavior of modular invariant models of lepton masses near the fixed points $\tau_0 = i$ and $\tau_0 = -1/2 + i\,\sqrt{3}/2$ is universal [44]. All physical quantities scale with the distance $|\tau - \tau_0|$ in a way that is largely independent of the level n, the weight k_I, the specific representations ρ_I and even the form of kinetic terms. Only a few patterns of neutrino mass matrices can be realized, depending on the chosen fixed point. In particular, near $\tau_0 = i$ almost all the successful models predict a normal ordering of neutrino masses, with mixing angles and neutrino mass differences that are all accommodated by $|\tau - \tau_0| \approx 0.1$. The key feature under this universal behavior is the residual $\mathbb{Z}_4$ symmetry enjoyed by the theory at the fixed point, spontaneously broken by $(\tau - \tau_0)$. In the bottom-up approach, the modulus τ is treated as a free parameter, optimized to maximize the agreement with the data. Remarkably, fixed points τ_0 are extrema of a modular invariant energy density [45, 46]. Moreover, minima of the energy density close to but distinct from the fixed points have been established in modular invariant theories [47–52]. Cosmological evolution can offer a mechanism for moduli trapping near the points enjoying an enhanced symmetry [53–55]. In general, $(\tau - \tau_0)$ provides a useful expansion parameter to understand the hierarchy among charged fermion masses and mixing angles [56–67].

JHEP05(2024)052

Theories with a single modulus might offer an oversimplified description of the fermion mass spectrum. For example, in superstring compactifications fermion masses typically depend on several moduli. It is thus natural to ask whether a universal behavior persists in multi-moduli theories in the vicinity of the fixed points, where a residual symmetry under a discrete group G_0 applies. Moreover, if such universality is indeed exhibited in these richer theories, there is the chance of finding new realistic patterns for fermion mass matrices. An attractive generalization of the upper half of the complex plane and the modular group $\mathrm{SL}(2,\mathbb{Z})$ is the Siegel upper half plane, where the Siegel modular group $\mathrm{Sp}(2g,\mathbb{Z})$ operates. The Siegel upper half plane has complex dimension $g(g+1)/2$ and coincides with the ordinary upper half complex plane when the genus g is one. Higher genera are realized in string theory compactification [68–72]. Bottom-up realizations have been formulated in ref. [73], and their spontaneous CP breaking has been analyzed in ref. [74].

The purpose of the present paper is to study the simplest possible such generalization, at genus $g = 2$, and its property in the vicinity of the fixed points, which have been fully classified in [75–77]. We focus on the lepton sector, adopting completely generic level n, weights k_I of the involved multiplets and representations ρ_I, with the only assumption that the electroweak lepton doublets L are assigned to an irreducible triplet of the relevant modular group. We also allow the kinetic terms to be the most general ones. These features are implemented in a CP-invariant locally supersymmetric theory. There are strong indications that the four-dimensional CP symmetry is a gauge symmetry [78–80], even starting from a higher-dimensional theory where CP is not conserved. It has been conjectured, as a general property of string theory, that CP is indeed a gauge symmetry of the four-dimensional theory. In this context, CP can only be violated spontaneously, by complex expectation values of fields, in our case the moduli. The moduli will be restricted to a suitable region of the Siegel upper half plane, invariant under a subgroup of $\mathrm{Sp}(4,\mathbb{Z})$ hosting three-dimensional irreducible representations of the finite modular group. The widest such a region has complex dimension two and provides a nontrivial extension of the framework studied in refs. [43, 44].

In section 2 we will review the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$ and the two-dimensional invariant regions of the Siegel upper half plane. We identify the region that allows to assign lepton doublets to irreducible triplets of the relevant finite modular group and we analyze the fixed points belonging to this region. In section 3 we define our CP-invariant locally supersymmetric theory and we describe the requirements for modular invariance. At the fixed points the theory enjoys a residual symmetry under a finite group G_0. An important step is provided by a field redefinition that, linearizing the action of G_0, considerably simplifies our task. In section 4, for each fixed point we identify the group G_0 and we provide the decomposition of all irreducible triplets of the finite modular group under G_0. This information is sufficient to find the pattern of neutrino mass matrices, in the basis where kinetic terms are canonical and charged lepton mass matrices diagonal, expressed as a series expansion in powers of $(\tau - \tau_0)$. Details on the group theory of the residual symmetry G_0 can be found in the appendices.

The result is remarkably simple. Apart from the unrealistic case of a neutrino mass matrix vanishing to all orders of the $(\tau - \tau_0)$ expansion, only five patterns are found. Four of them coincide with those arising in $\mathrm{SL}(2,\mathbb{Z})$-invariant single modulus theories in the vicinity of the fixed points $\tau_0 = i$ and $\tau_0 = -1/2 + i\sqrt{3}/2$. In particular, one of these four patterns is especially effective in accommodating the existing data, with no required tuning of the free parameters. The last pattern predicts neutrino masses of the same order of magnitude and mixing angles of approximately the same size and does not provide any explanation for the smallness of $\Delta m^2_{sol}/\Delta m^2_{atm}$ and $\sin^2\theta_{13}$. It is intriguing that, not only the universal behavior of the theory near the fixed points is confirmed, but essentially no realistic patterns of neutrino mass matrices different from those found in the single modulus case are exhibited in this class of multi-moduli theories.

2 Siegel modular group

One of the most natural generalizations of theories invariant under the modular group $\mathrm{SL}(2,\mathbb{Z})$, where masses and mixing angles depend on a single complex modulus τ, is the class of theories invariant under the Siegel (or symplectic) modular group $\mathrm{Sp}(4,\mathbb{Z})$ [73], where moduli are described by a two-by-two complex symmetric matrix τ belonging to the Siegel upper-half plane $\mathcal{H}_2$:

$$\tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix}, \qquad \det(\mathtt{Im}(\tau)) > 0\,, \qquad \mathrm{Tr}(\mathtt{Im}(\tau)) > 0\,. \tag{2.1}$$

Assuming CP and $\mathcal{N}=1$ supersymmetric invariance, and neglecting gauge interactions, these theories involve a set of chiral supermultiplets, $\Phi = (\tau, \varphi^{(I)})$, where τ is dimensionless and gauge-invariant. The Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$, adopted as flavour symmetry, acts on $\Phi = (\tau, \varphi^{(I)})$ as [73]:

$$\begin{cases} \tau \to \gamma\tau = (A\tau + B)(C\tau + D)^{-1}\,, \\ \varphi^{(I)} \to [\det(C\tau + D)]^{-k_I}\rho_I(\gamma)\varphi^{(I)}\,, \end{cases} \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z})\,, \tag{2.2}$$

with suitable conditions on the submatrices A, B, C and D. The weight k_I is assumed to be an integer and $\rho_I(\gamma)$ denotes a unitary representation of a finite copy Γ_n of $\mathrm{Sp}(4,\mathbb{Z})$.[1]

Such a finite copy, known as finite Siegel modular group, is defined as the quotient group $\Gamma_n = \mathrm{Sp}(4,\mathbb{Z})/\Gamma(n)$ where $\Gamma(n)$ is the principal congruence subgroup $\Gamma(n)$ of level n:

$$\Gamma(n) = \left\{\gamma \in \mathrm{Sp}(4,\mathbb{Z}) \;\middle|\; \gamma \equiv \mathbb{1} \,\mathtt{mod}\, n\right\}, \tag{2.3}$$

n being a generic positive integer. The finite Siegel modular group has finite order [81, 82]:

$$|\Gamma_n| = n^{10}\prod_{p|n}\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p^4}\right), \tag{2.4}$$

where the product is over the prime divisors p of n. $|\Gamma_n|$ rapidly grows with n: for example $|\Gamma_2| = 720$, $|\Gamma_3| = 51840$. The group Γ_2 is isomorphic to S_6, and Γ_3 is $\mathrm{Sp}(4,F_3)$, the double covering of Burkhardt group. These finite groups do not possess three-dimensional irreducible representations suitable to accommodate three fermion generations. The smallest irreducible representation with degree greater than one is 5 for Γ_2 and 4 for Γ_3. If n is a power of an odd prime, the next lowest dimensional (complex, irreducible) representations of Γ_n, after the

[1]For a generic genus g, the Siegel modular group is $\mathrm{Sp}(2g,\mathbb{Z})$ and the finite modular groups are denoted by $\Gamma_{g,n}$. Here we use the concise notation $\Gamma_n = \Gamma_{2,n}$.

trivial representation, are two of dimension $(n^2-1)/2$ and two of dimension $(n^2+1)/2$ [83]. In this series we do not find any three-dimensional irreducible representation. It is quite possible that also when n is a power of an even prime, the smallest nontrivial irreducible representation of Γ_n are quite large. Though we do not have a formal proof, we suspect that none of the finite Siegel modular groups Γ_n possesses three-dimensional irreducible representations. The requirement of three-dimensional irreducible representations is not mandatory, but is a very convenient one since it reduces the number of independent parameters needed to describe the fermion mass spectrum of the theory.

2.1 Invariant regions

In ref. [73] we have shown how to overcome this difficulty, by restricting τ to a convenient region Σ of the Siegel upper-half plane $\mathcal{H}_2$, invariant under a subgroup of $\mathrm{Sp}(4,\mathbb{Z})$. Individual points in Σ are left invariant by a common subgroup H of $\mathrm{Sp}(4,\mathbb{Z})$, the stabilizer

$$H\,\tau = \tau\,. \tag{2.5}$$

The region Σ, as a whole, is left invariant by the normalizer

$$N(H) = \left\{\gamma \in \mathrm{Sp}(4,\mathbb{Z}) | \gamma^{-1} H \gamma = H\right\}\,. \tag{2.6}$$

In general, H is a proper subgroup of $N(H)$. If Σ consists of a single point, the stabilizer and the normalizer coincide. In our theory, we can consistently restrict the domain of moduli to a region Σ of this type, and replace $\mathrm{Sp}(4,\mathbb{Z})$ with $N(H)$ and Γ_n with $N_n(H)$, where the integer n is the level of the representation and $N_n(H)$ is a finite copy of the normalizer, obtained through the same steps leading to the Siegel finite modular groups Γ_n. The group $N(H)$ acts on $\Phi = (\tau, \varphi^{(I)})$ as in eq. (2.2), where γ belongs to $N(H)$ and $\rho_I(\gamma)$ is a unitary representation of $N_n(H)$. Since in general $N_n(H)$ is smaller than Γ_n, there is a chance that it possesses three-dimensional irreducible representations.

Finally, up to Sigel modular transformations, CP transformations on the chiral multiplets read (we use a bar to denote conjugation of fields) [32, 70, 71, 74, 84, 85]:

$$\tau \xrightarrow{\mathcal{CP}} -\bar{\tau}\,, \qquad \varphi^{(I)} \xrightarrow{\mathcal{CP}} X_I\, \bar{\varphi}^{(I)}\,, \tag{2.7}$$

where the CP transformation matrix X_I is fixed by the following consistency conditions [74]

$$X_I\, \rho_I^*(S) X_I^{-1} = \rho_I(S^{-1}), \qquad X_I\, \rho_I^*(T_i) X_I^{-1} = \rho_I(T_i^{-1})\,, \tag{2.8}$$

up to an overall irrelevant phase. Here S and T_i $(i = 1,2,3)$ denote the generators of $\mathrm{Sp}(4,\mathbb{Z})$:

$$S = \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix}\,, \qquad T_i = \begin{pmatrix} \mathbb{1}_2 & B_i \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{2.9}$$

with

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\,, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\,, \quad B_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{2.10}$$

In the basis where the unitary representation matrices $\rho_I(S)$ and $\rho_I(T_i)$ are symmetric, the consistency conditions of eq. (2.8) are solved by $X_I = \mathbb{1}$ which is the canonical CP transformation. In our analysis, it will be convenient to make use of generalized CP transformations, which combine eq. (2.7) with a modular transformation. They will be discussed in section 3.1.

Invariant region τ	Stabilizer H	generators of the Normalizer $N(H)$
$\Sigma_1 = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$	$Z_2 \times Z_2$	eq. (2.13)
$\Sigma_2 = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_1 \end{pmatrix}$	$Z_2 \times Z_2$	eq. (2.18)

Table 1. Invariant regions $\Sigma_{1,2}$ of complex dimension two in the Siegel upper half plane $\mathcal{H}_2$. The generators of the corresponding normalizers are shown in eqs. (2.13) and (2.18).

2.2 Two-dimensional invariant regions

Having to abandon the full Siegel upper-half plane $\mathcal{H}_2$, the widest regions Σ have complex dimension two. In the Siegel upper-half plane $\mathcal{H}_2$, there are two such regions, left invariant by the action of a subgroup $N(H)$ of $\mathrm{Sp}(4,\mathbb{Z})$ [75–77].

2.2.1 Σ_1

One of them is

$$\Sigma_1 = \left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \in \mathcal{H}_2 \right\} . \tag{2.11}$$

The stabilizer H is the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group whose generators are:

$$-\mathbb{1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2.12}$$

The normalizer $N(H)$ is generated by the elements:

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad G_1' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$G_2' = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{2.13}$$

The action of the group generators on the moduli τ_1 and τ_2 is

$$\begin{aligned} G_1 : &\quad \tau_1 \to \tau_1, &\quad \tau_2 \to -\frac{1}{\tau_2}, \\ G_2 : &\quad \tau_1 \to \tau_1, &\quad \tau_2 \to \tau_2 + 1, \\ G_1' : &\quad \tau_1 \to -\frac{1}{\tau_1}, &\quad \tau_2 \to \tau_2, \end{aligned}$$

JHEP05(2024)052

$$\begin{aligned} G'_2 : &\quad \tau_1 \to \tau_1 + 1, \qquad \tau_2 \to \tau_2\,, \\ G_3 : &\quad \tau_1 \to \tau_2 \qquad\quad\;\; \tau_2 \to \tau_1. \end{aligned} \tag{2.14}$$

The generators $G_{1,2}$, $G'_{1,2}$ and G_3 obey the following relations

$$\begin{aligned} &G_1^4 = (G_1G_2)^3 = 1, && G_1^2G_2 = G_2G_1^2, && G_1'^4 = (G_1'G_2')^3 = 1, && G_1'^2G_2' = G_2'G_1'^2\,, \\ &G_1G_1' = G_1'G_1, && G_1G_2' = G_2'G_1, && G_2G_1' = G_1'G_2, && G_2G_2' = G_2'G_2\,, \\ &G_3^2 = 1, && G_3G_1 = G_1'G_3, && G_3G_2 = G_2'G_3\,. \end{aligned} \tag{2.15}$$

Therefore the normalizer $N(H)$ is isomorphic to $(\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})) \rtimes (\mathbb{Z}_2)_M$, where the last factor, generated by G_3 that exchanges τ_1 and τ_2, represents the so-called mirror symmetry in the string theory context. We find that the normalizer $N_n(H)$ corresponding to region Σ_1 has no three-dimensional irreducible representations. The triplet representations of the group $\mathrm{SL}(2,\mathbb{Z}_n) \times \mathrm{SL}(2,\mathbb{Z}_n)$ can be obtained from the direct product of $\mathrm{SL}(2,\mathbb{Z}_n)$ singlets with the irreducible triplets of another $\mathrm{SL}(2,\mathbb{Z}_n)$. The two triplet representations $\mathrm{SL}(2,\mathbb{Z}_n) \times \mathrm{SL}(2,\mathbb{Z}_n)$ related by the mirror symmetry $(\mathbb{Z}_2)_M$ would form a six dimensional representation of the normalizer $N_n(H)$. Hence $N_n(H)$ has no three-dimensional irreducible representations. We tested this argument by varying n from 2 to 18, and indeed we found no three-dimensional irreducible representations for $N_n(H)$, but only irreducible representations of one, two, four, six, eight, nine, and higher dimension. Thus the region Σ_1 is only suitable to describe electroweak lepton doublets transforming in a reducible representation of the finite normalizer. We do not consider such a case here and we proceed by examining the region Σ_2.

2.2.2 Σ_2

The other two-dimensional region is

$$\Sigma_2 = \left\{ \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_1 \end{pmatrix} \in \mathcal{H}_2 \right\}\,. \tag{2.16}$$

The stabilizer H is the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group generated by:

$$-\mathbb{1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad h = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}. \tag{2.17}$$

The normalizer $N(H)$ is generated by:

$$\begin{aligned} G_1 &= \begin{pmatrix} 1&0&1&0 \\ 0&1&0&1 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 1&0&0&1 \\ 0&1&1&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \\ G_3 &= \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ -1&0&0&0 \\ 0&-1&0&0 \end{pmatrix}, \qquad G_4 = \begin{pmatrix} 1&0&0&0 \\ 0&-1&0&0 \\ 0&0&1&0 \\ 0&0&0&-1 \end{pmatrix}. \end{aligned} \tag{2.18}$$

JHEP05(2024)052

level n	order	n. of conjugacy classes	n. of irreducible triplets
2	48	10	4
4	6144	109	24
8	393216	938	56
16	25165824	5572	56
32	1610612736	?	?
...	...	...	...
3	1152	35	0
5	28800	54	0
7	225792	77	0
11	3484800	135	0
13	9539712	170	0
...	...	...	...

Table 2. Orders, number of conjugacy classes and of irreducible triplets of the finite normalizers $N_n(H)$ in the region Σ_2. The results have been obtained with the software GAP.

The action of the group generators on the moduli τ_1 and τ_2 is

$$\begin{aligned} G_1: &\quad \tau_1 \to \tau_1 + 1, &\quad \tau_3 \to \tau_3\,, \\ G_2: &\quad \tau_1 \to \tau_1, &\quad \tau_3 \to \tau_3 + 1\,, \\ G_3: &\quad \tau_1 \to \frac{\tau_1}{\tau_3^2 - \tau_1^2}, &\quad \tau_3 \to -\frac{\tau_3}{\tau_3^2 - \tau_1^2}\,, \\ G_4: &\quad \tau_1 \to \tau_1, &\quad \tau_3 \to -\tau_3\,. \end{aligned} \tag{2.19}$$

The generators G_1, G_2, G_3 and G_4 fulfill the following relations

$$\begin{aligned} &G_3^2 = R, \qquad R^2 = G_4^2 = (G_1G_3)^3 = (G_2G_3)^6 = (G_2G_4)^2 = 1\,, \\ &G_2G_3G_2^2G_3G_2 = G_3^3G_2^{-2}G_3, \qquad G_1G_2 = G_2G_1, \qquad G_1G_4 = G_4G_1, \qquad G_3G_4 = G_4G_3\,, \\ &G_1R = RG_1, \qquad G_2R = RG_2, \qquad G_3R = RG_3, \qquad G_4R = RG_4\,, \end{aligned} \tag{2.20}$$

where $R = -\mathbb{1}$.

In table 2, for n small and equal to a power of two or an odd prime, we show the number of irreducible triplets of $N_n(H)$. We see that when the level n is a prime (or a power of primes) different from two, there are no three-dimensional irreducible representations, at least for the first few values of n. When n is a power of two, there are several three-dimensional irreducible representations and their number grows with n, until n is equal to 8. We find that for $n = 2, 4, 8$ there are $4, 24, 56$ three-dimensional irreducible representations, and for $n = 16$ there are still only 56 three-dimensional irreducible representations. Moreover, there are 24 independent singlet representations in all $N_n(H)$. We conjecture that there are no new three-dimensional irreducible representations for $n > 8$. Moreover, the 56 three-dimensional irreducible representations at level 8 include those al the lower levels $n < 8$. If our conjecture is correct, all irreducible triplets of $N_n(H)$ can be obtained from the product of the 24

	Fixed points τ	Residual symmetry in $\mathrm{Sp}(4,\mathbb{Z})$	G_0 = Residual symmetry in $N(H)$	CP
1.	$\begin{pmatrix} \zeta & \zeta+\zeta^{-2} \\ \zeta+\zeta^{-2} & -\zeta^{-1} \end{pmatrix}$	Z_{10}	—	+
2.	$\begin{pmatrix} \eta & \frac{1}{2}(\eta-1) \\ \frac{1}{2}(\eta-1) & \eta \end{pmatrix}$	$\mathrm{GL}(2,3)$	$D_4 \quad (\tau \in \Sigma_2)$	+
3.	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$(Z_4 \times Z_4) \rtimes Z_2$	$\begin{cases} (Z_4 \times Z_4) \rtimes Z_2 & (\tau \in \Sigma_1) \\ D_4 \circ Z_4 & (\tau \in \Sigma_2) \end{cases}$	+
4.	$\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$	$[72,30] \cong (Z_6 \times Z_6) \rtimes Z_2$	$\begin{cases} (Z_6 \times Z_6) \rtimes Z_2 & (\tau \in \Sigma_1) \\ D_4 \times Z_3 & (\tau \in \Sigma_2) \end{cases}$	+
5.	$\frac{i\sqrt{3}}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	$(Z_6 \times Z_2) \rtimes Z_2$	$D_4 \quad (\tau \in \Sigma_2)$	+
6.	$\begin{pmatrix} \omega & 0 \\ 0 & i \end{pmatrix}$	$Z_{12} \times Z_2$	$Z_{12} \times Z_2 \quad (\tau \in \Sigma_1)$	+

Table 3. Inequivalent fixed points of $\mathrm{Sp}(4,\mathbb{Z})$ in the Siegel upper half plane $\mathcal{H}_2$. We set $\zeta = e^{2\pi i/5}$, $\eta = \frac{1}{3}(1+i2\sqrt{2})$, $\omega = -1/2+i\sqrt{3}/2$. All other fixed points are related to one of these through a $\mathrm{Sp}(4,\mathbb{Z})$ transformation. The group with `GAP` id [72,30] is isomorphic to $(Z_6 \times Z_6) \rtimes Z_2$.

singlets and 56 triplets of $\Gamma_{2,4,8}$. Out of the resulting 1344 representations, we find that the inequivalent ones are 168. In the rest of this paper, we assume that these 168 representations exhaust the number of inequivalent irreducible triplets.

2.3 Fixed points

We analyze a class of theories, where the matrix τ is restricted to Σ_2 and the flavor group is the corresponding normalizer $N(H)$ defined in eq. (2.18). At a generic point of Σ_2, the flavour symmetry is broken down to the stabilizer H, eq. (2.17). An exception is provided by the fixed points τ_0 of Σ_2, where the residual symmetry group G_0, a subgroup of $N(H)$, is bigger than H. The inequivalent fixed points of $\mathrm{Sp}(4,\mathbb{Z})$ in the upper half-plane are displayed in table 3. Fixed points 2 and 5 belong to Σ_2. Fixed points 3 and 4 belong to both Σ_1 and Σ_2. Here we will only consider the latter option. The fixed point 6 belongs to the region Σ_1 and will not be discussed. Finally, the fixed point 1, where CP is preserved, does not belong to either Σ_1 or Σ_2. We have also checked that it is not related by any $\mathrm{Sp}(4,\mathbb{Z})$ transformation to one point in either Σ_1 or Σ_2. This point will be dismissed in our analysis, too.

In modular invariant theories depending on a single modulus τ where lepton doublets are assigned to irreducible triplets of the finite modular group, it has been shown that in the vicinity of the fixed points the predictions are universal: they are independent of the finite modular group acting on the lepton multiplets, the weights of the matter multiplets and even the form of the kinetic terms. We are led to study the behavior of Siegel modular invariant theories when the matrix τ, conveniently restricted to Σ_2, falls in the vicinity of one of fixed points 2, 3, 4 and 5.

3 Modular invariant models of lepton masses

In the rest of this paper, we analyze CP-invariant and $\mathcal{N} = 1$ locally supersymmetric models of lepton masses and mixing angles. Neglecting gauge interactions that are not of interest in our analysis, they are described by an action $\mathscr{S}$ fully specified by a gauge-invariant real function $\mathcal{G}$

$$\mathcal{G} = \frac{K}{M_{Pl}^2} + \log\left|\frac{w}{M_{Pl}^3}\right|^2 , \tag{3.1}$$

where restriction to the scalar components of the supermultiplets is understood, and M_{Pl} is the Planck mass. Here K is the Kähler potential, a real gauge-invariant function of the chiral multiplets and their conjugate of dimensionality $(\text{mass})^2$, and w is the superpotential, an analytic gauge-invariant function of the chiral multiplets of dimensionality $(\text{mass})^3$. We do not discuss supersymmetry breaking effects, which in theories with a single modulus are negligible in a large portion of the parameter space [4]. The chiral superfield content includes the matrix τ of eq. (2.1), restricted to the region Σ_2, and a set of matter fields $\varphi^{(I)}$. Siegel modular invariance requires $\mathscr{S}$ to remain unchanged under the transformations of eq. (2.2), where γ is restricted to the normalizer $N(H)$ of Σ_2. The transformation law of $\varphi^{(I)}$ is specified by a unitary representation $\rho_I(\gamma)$ of the finite modular group $N_n(H)$ and by the integer weight k_I (irreducible components of $\varphi^{(I)}$ admitting independent weights). Invariance under CP requires $\mathscr{S}$ to remain unmodified under the transformations of eq. (2.7).

Notice that for the action $\mathscr{S}$ to be invariant under the transformations of eq. (2.2), the Kähler potential K and the superpotential w do not need to be separately invariant. If modular transformations of eq. (2.2), restricted to the scalar components z of the supermultiplets $(\tau, \varphi^{(I)})$, induce a Kähler transformation:

$$\begin{aligned} K &\xrightarrow{\gamma} K + M_{Pl}^2 F_\gamma(z) + M_{Pl}^2 \bar{F}_\gamma(\bar{z}) , \\ w &\xrightarrow{\gamma} e^{-F_\gamma(z)} w , \end{aligned} \tag{3.2}$$

$\mathcal{G}$ does not change and the theory is invariant, provided the fermionic partners ψ of the scalar multiplets z undergo an extra chiral rotation of the type:[2]

$$\psi \xrightarrow{\gamma} e^{\frac{F_\gamma(z)-\bar{F}_\gamma(z)}{4}} \psi . \tag{3.3}$$

If $F_\gamma(z)$ has a nontrivial field dependence, eq. (3.3) represents a local chiral rotation on the fermion components and anomaly cancellation is required to guarantee invariance. Thus, up to potential anomalies requiring a cancellation mechanism, a Kähler transformation is always a symmetry of the theory. As a particular case, we can consider $F = -2i\alpha_\gamma$, with α_γ a field-independent real constant and eq. (3.2) becomes

$$K \xrightarrow{\gamma} K , \qquad w \xrightarrow{\gamma} e^{2i\alpha_\gamma} w . \tag{3.4}$$

For example, consider the choice:

$$K = -3M_{Pl}^2 \log(-i\tau + i\bar{\tau}) , \qquad w = c\frac{M_{Pl}^3}{\eta(\tau)^6} , \tag{3.5}$$

[2]Gauginos are required to transform with an opposite phase.

where $\eta(\tau)$ is the Dedekind eta function and c a constant. When performing the transformation $T : \tau \to \tau + 1$, we reproduce eq. (3.4) with $\alpha_T = -\pi/4$, thanks to the property

$$\eta(\tau + 1) = e^{i\frac{\pi}{12}}\,\eta(\tau). \tag{3.6}$$

This theory is invariant under T,[3] despite the presence of nontrivial phases in the transformation of the superpotential. We conclude that in the local case, the covariance of w is only required up to a phase $2\alpha_\gamma$. Since this phase depends on the group element γ, consistency with the group law requires[4]

$$\alpha_{\gamma_1} + \alpha_{\gamma_2} = \alpha_{\gamma_1\gamma_2}\,. \tag{3.7}$$

Here we will make use of the possibility in eq. (3.4), which is rarely exploited in the bottom-up approach. Working in the context of local supersymmetry, we consider the general case where K and w satisfy

$$\begin{aligned} K &\xrightarrow{\gamma} K + k_w M_{Pl}^2 \log\det(C\tau + D) + k_w M_{Pl}^2 \log\det(C\bar{\tau} + D)\,, \\ w &\xrightarrow{\gamma} \det(C\tau + D)^{-k_w}\boldsymbol{r}_s(\gamma) w, \end{aligned} \tag{3.8}$$

where $\boldsymbol{r}_s$ is a singlet representation of $N_n(H)$, which can differ from the trivial one by a phase factor. A suitable mechanism of anomaly cancellation, not affecting the analysis of the light fermion masses, is understood in the present analysis.

As a side remark, we stress that, in general, requiring w to transform as a nontrivial singlet $\boldsymbol{r}_s$ is not possible in rigid supersymmetry, where the theory depends separately on K and w. Indeed, if w acquires a phase, the invariance of the theory can be guaranteed by an R-symmetry. The transformation in eq. (3.4) should read:

$$w(\theta) \xrightarrow{\gamma} e^{2i\alpha_\gamma} w(e^{-i\alpha_\gamma}\theta), \tag{3.9}$$

which can be absorbed by a redefinition of the Grassmann measure

$$\begin{aligned} \theta &\xrightarrow{\gamma} e^{+i\alpha_\gamma}\,\theta\,, \\ d^2\theta &\xrightarrow{\gamma} e^{-2i\alpha_\gamma}\,d^2\theta\,. \end{aligned} \tag{3.10}$$

When w is a polynomial, the required R-invariance can be achieved if the R-charges of the chiral multiplets add up to $+2$ in each term of w. In a modular invariant theory, w is not necessarily polynomial since one of these fields is typically a modular form $Y(\tau)$. Due to the non-homogeneous dependence of $Y(\tau)$ on τ, if we assign a nonvanishing R-charge to τ, $Y(\tau)$ will not possess a definite R-charge, in general. Thus, the only possible R-charge assignment

[3] Indeed, it is invariant under the full $\mathrm{SL}(2,\mathbb{Z})$.

[4] We can replace this condition with a less restrictive one, where $v(\gamma) = e^{2i\alpha_\gamma}$ is a multiplier system obeying:

$$v(\gamma_1)v(\gamma_2)j(\gamma_1,\gamma_2\tau)^{-k_w}j(\gamma_2,\tau)^{-k_w} = v(\gamma_1\gamma_2)j(\gamma_1\gamma_2,\tau)^{-k_w}\,, \qquad j(\gamma,\tau) = \det(C\tau + D),$$

where k_w is the weight of w. When k is an integer, this condition becomes equivalent to the one in eq. (3.7).

of both τ and $Y(\tau)$ is $R = 0$. But this amounts to saying that the scalar component of τ does not transform under the considered symmetry. This is what happens when dealing with a traditional flavour symmetry. Specific combinations of modular transformations behave as traditional flavour symmetries, such as S^2 in SL$(2, \mathbb{Z})$-invariant theories or transformations belonging to the center of the metaplectic modular group, arising in the context of fractional weight modular forms [86–89]. In the following, we will not consider these special cases, and we focus on nontrivial transformations of the moduli τ.

We will exploit the above framework to describe masses, mixing angles, and phases in the lepton sector. The matter multiplets $\varphi^{(I)}$ include two Higgs doublets $H_{u,d}$, electroweak epton doublets L, lepton singlets E^c and, when neutrino masses arise from the seesaw mechanism, heavy neutrino singlets N^c. Assuming Majorana neutrino masses, the low-energy superpotential w of the lepton sector reads:

$$w = -\frac{1}{2\Lambda_L}(H_u L)^T \mathcal{Y}(\tau)(H_u L) - E^{cT}\mathcal{Y}_e(\tau)(H_d L), \tag{3.11}$$

where the first term can arise from the seesaw mechanism when heavy singlets N^c are integrated out. To minimize the number of free parameters, L will be assigned to an irreducible triplet of the finite modular group $N_n(H)$. In the class of theories described by the action $\mathscr{S}$, the lepton mass matrices are obtained by combining the holomorphic contribution arising from the superpotential w with the non-holomorphic data coming from the Kähler potential. In general, there is a large freedom related to both the allowed transformation properties of the matter fields (level n, weight k_I and representation ρ_I) and the inefficiency of modular invariance in constraining the Kähler potential. Moreover, the holomorphic contribution is expressed in terms of Siegel modular forms, whose knowledge for generic levels and weights (n, k) is limited. These obstacles can be largely overcome if τ falls in the vicinity of a fixed point τ_0. In this case, even giving up the full power of modular invariance, a considerable amount of information about lepton masses and mixing angles survives from the approximate invariance under CP and the stability group G_0.

3.1 A field redefinition

At the fixed points τ_0, both CP and the stability group G_0 are preserved. To establish CP conservation, it might be convenient to use a nonstandard definition of CP transformation $g\mathcal{CP} \equiv \gamma^{-1} \circ \mathcal{CP}$ on τ and the matter fields:

$$\tau \xrightarrow{g\mathcal{CP}} \tau_{\rm CP} = \gamma^{-1}(-\bar{\tau})\,, \qquad \gamma = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}\,,$$

$$\varphi^{(I)} \xrightarrow{g\mathcal{CP}} \det(C_0^t \bar{\tau} + A_0^t)^{-k_I} \rho_I(\gamma^{-1}) X_I\, \bar{\varphi}^{(I)}\;\;, \tag{3.12}$$

where γ is an element of Sp$(4, \mathbb{Z})$ fulfilling $\gamma\tau_0 = -\bar{\tau}_0$, so that the fixed point τ_0 is invariant under $g\mathcal{CP}$. In general both CP and the stability group G_0 are nonlinearly realized on $(\tau, \varphi^{(I)})$ and it is preferable to move to a field basis $(u, \Phi^{(I)})$ where CP and G_0 act linearly. We will choose a basis where $u(\tau_0) = 0$, so that the group G_0 and CP are unbroken at the origin of the field space, $u = 0$. This allows us to adopt u as an order parameter for the breaking of

G_0 and CP. The new basis can be defined by the transformation:

$$
\begin{aligned}
u =& e^{-i\alpha}(\tau-\tau_0)(\tau-\bar{\tau}_0)^{-1}\,, \qquad \tau=(1-e^{i\alpha}u)^{-1}(\tau_0-e^{i\alpha}u\bar{\tau}_0)\,,\\
\Phi^{(I)} =& [\det(1-e^{i\alpha}u)]^{k_I}\varphi^{(I)}\,,
\end{aligned}
\tag{3.13}
$$

where the phase α can be conveniently adjusted to simplify the action of CP. Denoting by

$$
h_i=\begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}
\tag{3.14}
$$

the generators of the stability group G_0, $h_i\tau_0=\tau_0$,we find[5]

$$
\begin{aligned}
u &\xrightarrow{h_i} h_i u=(A_i-\tau_0 C_i)\; u\; (A_i-\bar{\tau}_0 C_i)^{-1}\,,\\
\Phi^{(I)} &\xrightarrow{h_i} \Omega_I(h_i)\Phi^{(I)}\,, \qquad \Omega_I(h_i)=\det(C_i\tau_0+D_i)^{-k_I}\rho_I(h_i)\,.
\end{aligned}
\tag{3.15}
$$

It is always possible to parametrize u in such a way that this linear transformation is unitary. Moreover, if $\rho_I(h_i)$ is unitary, so is $\Omega_I(h_i)$, since $[\det(C_i\tau_0+D_i)]^q=1$, where q is the order of h_i. The explicit parametrization of u and $\Phi^{(I)}$ will be specified in the next sections. Moreover, under the general CP symmetry $g\mathcal{CP}$, the new fields u and $\Phi^{(I)}$ transform as follows

$$
\begin{aligned}
u &\xrightarrow{g\mathcal{CP}} e^{-2i\alpha}\,(\bar{\tau}_0 C_0+A_0)^{-1}\,\bar{u}\,(\tau_0 C_0+A_0)\;,\\
\Phi^{(I)} &\xrightarrow{g\mathcal{CP}} \det(\tau_0 C_0+A_0)^{k_I} X_I'\bar{\Phi}^{(I)}, \quad X_I'=\rho_I(\gamma^{-1})X_I\,.
\end{aligned}
\tag{3.16}
$$

The phase $\det(\tau_0 C_0+A_0)^{k_I}$ can be absorbed by field redefinition further, thus the CP transformation matrix is essentially X_I'. The explicit form of X_I' can be determined by solving a set of consistency conditions, as shown in the following sections. In the basis $(u,\Phi^{(I)})$, CP and G_0 are linearly realized in the field space, and are broken by the VEV of the flavon $u(\tau)$, which remains small if τ is near τ_0. Assuming that the lepton doublets L transform as an irreducible triplet of $N_n(H)$, we can build the most general set of lepton mass matrices with the correct transformation properties under G_0 and CP from the decomposition of such a triplet under G_0. As we will see, both the moduli u and the irreducible triplets of $N_n(H)$ decompose into the direct sum of G_0-singlets. Thus, a suitable choice of the phase α allows us to cast the CP transformations of the relevant fields in the simple form:

$$
\Phi^{(I)} \xrightarrow{g\mathcal{CP}} \bar{\Phi}^{(I)}, \qquad u_i \xrightarrow{g\mathcal{CP}} \bar{u}_i\,.
\tag{3.17}
$$

Working in the vicinity of the fixed point, we only need a few terms of the expansion of such matrices in powers of the symmetry-breaking order parameter $u(\tau)$. In general, the kinetic terms originating from the most general Kähler potential allowed by CP and Siegel modular invariance are not canonical. After moving to the basis where such terms are canonical, it is not difficult to prove that, when the theory is invariant under G_0 and CP, charged lepton and neutrino mass matrices, $m_{\bar{e}e}(u,\bar{u})\equiv m_e(u,\bar{u})^\dagger m_e(u,\bar{u})$ and $m_\nu(u,\bar{u})$, should transform as shown in table 4.

[5] We used the identities $\det(C_i\tau+D_i)=\det(C_i\tau_0+D_i)\frac{\det(1-e^{i\alpha}h_i u)}{\det(1-e^{i\alpha}u)}$ and $A_i-\tau_0 C_i=(\tau_0 C_i^t+D_i^t)^{-1}$.

	G_0	CP
$m_\nu(u,\bar{u})$	$\boldsymbol{r}_s\Omega^*\ m_\nu(u,\bar{u})\ \Omega^\dagger$	$m_\nu(u,\bar{u})^*$
$m_\nu(u,\bar{u})^{-1}$	$\boldsymbol{r}_s^{-1}\Omega\ m_\nu(u,\bar{u})^{-1}\ \Omega^T$	$m_\nu(u,\bar{u})^{-1*}$
$m_e(u,\bar{u})$	$\boldsymbol{r}_s\Omega_c^*\ m_e(u,\bar{u})\ \Omega^\dagger$	$m_e(u,\bar{u})^*$
$m_{\bar{e}e}(u,\bar{u})$	$\Omega\ m_{\bar{e}e}(u,\bar{u})\ \Omega^\dagger$	$[m_{\bar{e}e}(u,\bar{u})]^*$

Table 4. Transformation properties of the lepton mass matrices under the group G_0 and CP. We have defined $m_{\bar{e}e}(u,\bar{u}) \equiv m_e(u,\bar{u})^\dagger m_e(u,\bar{u})$. In the column "$G_0$", we show the various mass matrices evaluated at $(h_i u, h_i \bar{u})$, while in the column "CP" they are evaluated at $(\bar{u}_i, u_i)$. We allow for the possibility that the superpotential w transforms as a generic singlet $\boldsymbol{r}_s$ of $N_n(H)$.

The unitary matrices in table 4 read:

$$\Omega \equiv \Omega_{H_u}\Omega_L\,, \qquad \Omega_c \equiv \Omega^*_{H_u}\Omega_{H_d}\Omega_{E^c}\,. \tag{3.18}$$

If the neutrino mass matrix arises from the seesaw mechanism, it may occur that $m_\nu(0,0)$ is singular.[6] In such a case it is convenient to enforce the transformations on the inverse $[m_\nu(u,\bar{u})]^{-1}$, also reported in table 4. Once Ω and Ω_c are specified, table 4 can be used to get the most general parametrization of $m_\nu(u,\bar{u})$ and $m_{\bar{e}e}(u,\bar{u})$ in the vicinity of τ_0.

4 Siegel modular invariant models near the fixed points

In this section we analyze the fixed points 2, 3, 4 and 5, embedded in the region Σ_2, whose normalizer $N(H)$ is generated by the elements G_i $(i = 1, \ldots, 4)$ of eq. (2.18). We show that CP is conserved at each of these fixed points. We find out the stability group G_0 and its generators. Moving to the basis where G_0 acts linearly, we determine the transformation properties of moduli and fields under G_0 and CP. Finally, we find the decomposition of any irreducible triplet of the finite Siegel modular group $N_n(H)$ under the subgroup G_0.

4.1 Fixed point 2

We start by analyzing the fixed point

$$\tau_0 = \begin{pmatrix} \eta & \frac{1}{2}(\eta-1) \\ \frac{1}{2}(\eta-1) & \eta \end{pmatrix}\,, \qquad \eta = \frac{1}{3}(1+i2\sqrt{2})\,. \tag{4.1}$$

CP remains unbroken at τ_0, since[7]

$$-\bar{\tau}_0 = \gamma\tau_0\,, \qquad \gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}\,. \tag{4.2}$$

It is convenient to define the CP action on τ as in eq. (3.12), with the matrix γ of eq. (4.2), such that $\tau_{\rm CP}$ belongs to the region conjugate to Σ_2 and $\tau_{0CP} = \tau_0$.

[6] That is the limit of $m_\nu(u,\bar{u})$ when u goes to zero does not exist or is infinite.
[7] The matrix γ is not unique: both γ and $\gamma' = \gamma h$, with $h\tau_0 = \tau_0$, satisfy eq. (4.2).

4.1.1 Stability group

The stability group G_0 of τ_0, subgroup of the flavour group $N(H)$ defined in eq. (2.18), is isomorphic to D_4, generated by the elements a and b, satisfying $a^4 = b^2 = (ab)^2 = 1$. In terms of the generators G_i $(i = 1, 2, 3, 4)$ of $N(H)$, a and b read:

$$a = G_4 G_2 G_3 G_2 \,, \qquad b = (G_3 G_2)^3 \,. \tag{4.3}$$

The irreducible representations of D_4 are four singlets $\mathbf{1}_+$, $\mathbf{1}_-$, $\mathbf{1}'_+$, $\mathbf{1}'_-$ and one doublet $\mathbf{2}$. Further details about D_4 are given in appendix A.

4.1.2 New basis

To realize a linear and unitary action of the stability group D_4 on τ and an antiunitary action of CP, we perform the field redefinition of eq. (3.13), choosing

$$e^{2i\alpha} = -\frac{1}{3} - i\frac{2\sqrt{2}}{3} \,. \tag{4.4}$$

Moreover, we use the parametrization:

$$u = \begin{pmatrix} \frac{1}{\sqrt{3}}u_1 + i\sqrt{\frac{2}{3}}u_3 & i\sqrt{\frac{2}{3}}u_1 + \frac{1}{\sqrt{3}}u_3 \\ i\sqrt{\frac{2}{3}}u_1 + \frac{1}{\sqrt{3}}u_3 & \frac{1}{\sqrt{3}}u_1 + i\sqrt{\frac{2}{3}}u_3 \end{pmatrix} . \tag{4.5}$$

The phase α reveals useful in simplifying the action of CP on $u_{1,3}$. Under the stability group D_4, the new fields u split into the sum of an invariant singlet, u_1, and a component u_3 transforming as $\mathbf{1}'_-$, see appendix A:

$$\begin{aligned} &u_1 \xrightarrow{a} +u_1, \qquad u_1 \xrightarrow{b} +u_1, \qquad u_1 \sim \mathbf{1}_+ \\ &u_3 \xrightarrow{a} -u_3, \qquad u_3 \xrightarrow{b} +u_3, \qquad u_3 \sim \mathbf{1}'_- \,. \end{aligned} \tag{4.6}$$

Under CP, we get

$$u_1 \xrightarrow{g\mathcal{CP}} \bar{u}_1 \,, \qquad u_3 \xrightarrow{g\mathcal{CP}} \bar{u}_3 \,. \tag{4.7}$$

The action of D_4 on the new field $\Phi^{(I)}$ of eq. (3.13) reads

$$\Phi^{(I)} \xrightarrow{\gamma} \Omega_I(\gamma)\Phi^{(I)} \,, \qquad \Omega_I(\gamma) = j(\gamma, \tau_0)^{-k_I} \rho_I(\gamma) \,. \tag{4.8}$$

where $j(\gamma, \tau) = \det(C\tau + D)$. In particular, we have:

$$j(a, \tau_0)^{-1} = +1 \,, \qquad j(b, \tau_0)^{-1} = -1, \tag{4.9}$$

which reproduces the representation $\mathbf{1}_-$ of the stability group. We see that $\Omega_I(\gamma) = j(\gamma, \tau_0)^{-k_I} \rho_I(\gamma)$ is also a representation of the stability group, the direct product of $(\mathbf{1}_-)^{k_I} = (\mathbf{1}_-, \mathbf{1}_+)$ and $\rho_I(\gamma)$. In a bottom-up approach, by varying all possible representations of the matter multiplets, we can absorb the factor $(\mathbf{1}_-)^{k_I}$ into the choice of $\rho_I(\gamma)$. From eq. (2.8) and the expression of generators in eq. (4.3), we know the CP transformation X'_I satisfies the following consistency conditions on the stability group D_4:

$$X'_I \rho^*_I(a) X'^{-1}_I = \rho_I(a), \quad X'_I \rho^*_I(b) X'^{-1}_I = \rho_I(b^{-1}) \,, \tag{4.10}$$

which lead to

$$X'_I = \mathbb{1} \tag{4.11}$$

in the basis of appendix A, no matter whether $\Phi^{(I)}$ transforms as singlets or doublet of D_4. Hence the CP transformation simply maps $\Phi^{(I)}$ to its conjugate.

4.1.3 Triplet decomposition

From the 168 triplet representations of the finite Siegel modular groups $N_n(H)$, the three-dimensional representation matrices of the D_4 generators a and b can be obtained via the relations in eq. (4.3). This allows the decomposition of the three-dimensional irreducible representations of $N_n(H)$ under the stability group D_4. We find that each irreducible triplet ρ_L decomposes into one of the following sum of three singlets of D_4

$$\rho_L \sim \begin{cases} \mathbf{1}_- \oplus \mathbf{1}'_+ \oplus \mathbf{1}'_+ \\ \mathbf{1}_+ \oplus \mathbf{1}'_- \oplus \mathbf{1}'_- \\ \mathbf{1}'_+ \oplus \mathbf{1}_- \oplus \mathbf{1}_- \\ \mathbf{1}'_- \oplus \mathbf{1}_+ \oplus \mathbf{1}_+ \end{cases} . \tag{4.12}$$

We end up with

$$\Omega_L(\gamma) = j(\gamma, \tau_0)^{-k_I} \rho_L(\gamma)\,, \tag{4.13}$$

where $j(\gamma, \tau_0)^{-1} \sim \mathbf{1}_-$. Finally, we get $\Omega \equiv \Omega_{H_u}\Omega_L$ by multiplying Ω_L and the generic singlet associated with Ω_{H_u}.

4.2 Fixed point 3

The fixed point

$$\tau_0 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \tag{4.14}$$

belongs to both Σ_1 and Σ_2. It preserves CP using the standard definition of eq. (3.12), where $\gamma = \mathbb{1}$. We assume that in the theory under examination, the matrix τ is restricted to Σ_2 and the flavor group is the corresponding normalizer $N(H)$ defined in eq. (2.18).

4.2.1 Stability group

The stability group G_0 of τ_0, subgroup of the flavour group $N(H)$ defined in eq. (2.18), is isomorphic to the Pauli group, namely the central product of D_4 and Z_4: $D_4 \circ Z_4$. G_0 is generated by the elements a and b and c, satisfying $a^4 = c^4 = b^2 = (ab)^2 = 1$, $a^2 = c^2$, $ac = ca$, $bc = cb$. The elements (a, b) generate D_4 and c generates Z_4. The elements of D_4 commute with those of Z_4. In terms of the generators G_i $(i = 1, 2, 3, 4)$ of $N(H)$, a, b and c read:

$$a = G_3 G_4\,, \qquad b = (G_3 G_2)^3\,, \qquad c = G_3\,. \tag{4.15}$$

The irreducible representations of $D_4 \circ Z_4$ are eight singlets and two doublets. Further details about $D_4 \circ Z_4$ are given in appendix B.

4.2.2 New basis

We move to the new basis of eq. (3.13), choosing $\alpha = 0$. Using the parametrization:

$$u = \begin{pmatrix} u_1 & u_3 \\ u_3 & u_1 \end{pmatrix}, \tag{4.16}$$

we find that, under the stability group $D_4 \circ Z_4$, the new multiplet u splits into the sum of two singlets: $\mathbf{1}_{-+-} \oplus \mathbf{1}_{++-}$, see appendix B:

$$\begin{aligned} &u_1 \xrightarrow{a} -u_1\,, \qquad u_1 \xrightarrow{b} +u_1\,, \qquad u_1 \xrightarrow{c} -u_1\,, \qquad u_1 \sim \mathbf{1}_{-+-} \\ &u_3 \xrightarrow{a} +u_3\,, \qquad u_3 \xrightarrow{b} +u_3\,, \qquad u_3 \xrightarrow{c} -u_3\,, \qquad u_3 \sim \mathbf{1}_{++-}\ . \end{aligned} \tag{4.17}$$

Under CP, with $\gamma = \mathbb{1}$ in eq. (3.12), we get

$$u_1 \xrightarrow{\mathcal{CP}} \bar{u}_1\,, \qquad u_3 \xrightarrow{\mathcal{CP}} \bar{u}_3\,. \tag{4.18}$$

The action of $D_4 \circ Z_4$ on the new matter fields $\Phi^{(I)}$ of eq. (3.13) is

$$\Phi^{(I)} \xrightarrow{\gamma} \Omega_I(\gamma)\Phi^{(I)}\,, \qquad \Omega_I(\gamma) = j(\gamma,\tau_0)^{-k_I}\rho_I(\gamma)\,, \tag{4.19}$$

where $j(\gamma,\tau) = \det(C\tau + D)$. In particular, we have:

$$j(a,\tau_0)^{-1} = +1\,, \qquad j(b,\tau_0)^{-1} = -1\,, \qquad j(c,\tau_0)^{-1} = -1\,, \tag{4.20}$$

which reproduces the representation $\mathbf{1}_{+--}$ of the stability group. We see that $\Omega_I(\gamma) = j(\gamma,\tau_0)^{-k_I}\rho_I(\gamma)$ is also a representation of the stability group, the direct product of $(\mathbf{1}_{+--})^{k_I}$ and $\rho_I(\gamma)$. In a bottom-up approach, by varying all possible representations of the matter multiplets, we can absorb the factor $(\mathbf{1}_{+--})^{k_I}$ into the choice of $\rho_I(\gamma)$. The consistency conditions of CP transformation X'_I on the stability group $D_4 \circ Z_4$ are given by

$$X'_I\,\rho_I^*(a)X'^{-1}_I = \rho_I(a^{-1}), \quad X'_I\,\rho_I^*(b)X'^{-1}_I = \rho_I(b^{-1}), \quad X'_I\,\rho_I^*(c)X'^{-1}_I = \rho_I(c^{-1})\,. \tag{4.21}$$

In the basis of appendix B, we have $X'_I = \mathbb{1}$ regardless of the transformation of $\Phi^{(I)}$ under $D_4 \circ Z_4$. As a result, the CP transformation of the matter field is $\Phi^{(I)} \xrightarrow{\mathcal{CP}} \bar{\Phi}^{(I)}$.

4.2.3 Triplet decomposition

We find that each irreducible triplet of $N_n(H)$ decomposes under $D_4 \circ Z_4$ into a direct sum of three singlets, wherein two of them are identical. There are only eight distinct cases as follows,

$$\rho_L \sim \begin{cases} \mathbf{1}_{+++} \oplus \mathbf{1}_{-+-} \oplus \mathbf{1}_{-+-} \\ \mathbf{1}_{-+-} \oplus \mathbf{1}_{+++} \oplus \mathbf{1}_{+++} \\ \mathbf{1}_{+--} \oplus \mathbf{1}_{--+} \oplus \mathbf{1}_{--+} \\ \mathbf{1}_{--+} \oplus \mathbf{1}_{+--} \oplus \mathbf{1}_{+--} \\ \mathbf{1}_{+-+} \oplus \mathbf{1}_{---} \oplus \mathbf{1}_{---} \\ \mathbf{1}_{---} \oplus \mathbf{1}_{+-+} \oplus \mathbf{1}_{+-+} \\ \mathbf{1}_{-++} \oplus \mathbf{1}_{++-} \oplus \mathbf{1}_{++-} \\ \mathbf{1}_{++-} \oplus \mathbf{1}_{-++} \oplus \mathbf{1}_{-++} \end{cases}. \tag{4.22}$$

JHEP05(2024)052

We end up with

$$\Omega_L(\gamma) = j(\gamma,\tau_0)^{-k_I}\rho_L(\gamma)\,, \tag{4.23}$$

where $j(\gamma,\tau_0)^{-1} \sim \mathbf{1}_{+--}$. Finally, we get $\Omega \equiv \Omega_{H_u}\Omega_L$ by multiplying Ω_L and the generic singlet associated with Ω_{H_u}.

4.3 Fixed point 4

The fixed point

$$\tau_0 = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \tag{4.24}$$

belongs to both Σ_1 and Σ_2. It preserves CP using the definition of eq. (3.12), with

$$\gamma = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.25}$$

We assume that in the theory under examination the matrix τ is restricted to Σ_2 and the flavor group is the corresponding normalizer $N(H)$ defined in eq. (2.18).

4.3.1 Stability group

The stability group G_0 of τ_0, subgroup of the flavour group $N(H)$ defined in eq. (2.18), is isomorphic to $D_4 \times Z_3$ generated by the elements a and b and c, satisfying $a^4 = b^2 = (ab)^2 = c^3 = 1$, $ca = ac$, $cb = bc$. The elements (a, b) generate D_4 and c generates Z_3. In terms of the generators G_i, $(i = 1, 2, 3, 4)$ of $N(H)$, eq. (2.18), a, b and c read:

$$a = (G_2G_3)^3G_4\,, \qquad b = G_3^2G_4\,, \qquad c = (G_3G_1)^2. \tag{4.26}$$

The irreducible representations of $D_4 \times Z_3$ are 12 singlets and 3 doublets. Further details about $D_4 \times Z_3$ are given in appendix C.

4.3.2 New basis

We move to the new basis of eq. (3.13), choosing $\alpha = 0$. Using the parametrization:

$$u = \begin{pmatrix} u_1 & u_3 \\ u_3 & u_1 \end{pmatrix}, \tag{4.27}$$

we find that, under the stability group $G_0 = D_4 \times Z_3$, the new multiplet u transforms as the sum of two singlets: $u_1 \sim \mathbf{1}_{++1}$ and $u_3 \sim \mathbf{1}_{--1}$, see appendix C.

$$\begin{aligned} &u_1 \xrightarrow{a} +u_1\,, \qquad u_1 \xrightarrow{b} +u_1\,, \qquad u_1 \xrightarrow{c} \omega u_1 \\ &u_3 \xrightarrow{a} -u_3\,, \qquad u_3 \xrightarrow{b} -u_3\,, \qquad u_3 \xrightarrow{c} \omega u_3\,, \end{aligned} \tag{4.28}$$

JHEP05(2024)052

where $\omega = -1/2 + i\sqrt{3}/2$. Under CP, with γ in eq. (4.25), we get

$$u_1 \xrightarrow{g\mathcal{CP}} \bar{u}_1 \,, \qquad u_2 \xrightarrow{g\mathcal{CP}} \bar{u}_2 \,. \tag{4.29}$$

The action of $D_4 \times Z_3$ on the new matter fields $\Phi^{(I)}$ of eq. (3.13) is

$$\Phi^{(I)} \xrightarrow{\gamma} \Omega_I(\gamma)\Phi^{(I)} \,, \qquad \Omega_I(\gamma) = j(\gamma,\tau_0)^{-k_I}\rho_I(\gamma) \,. \tag{4.30}$$

where $j(\gamma,\tau) = \det(C\tau + D)$. In particular, we have:

$$j(a,\tau_0)^{-1} = 1 \,, \qquad j(b,\tau_0)^{-1} = -1 \,, \qquad j(c,\tau_0)^{-1} = \omega \,, \tag{4.31}$$

which reproduces the representation $\mathbf{1}_{+-1}$ of the stability group. We see that $\Omega_I(\gamma) = j(\gamma,\tau_0)^{-k_I}\rho_I(\gamma)$ is also a representation of the stability group, the direct product of $(\mathbf{1}_{+-1})^{k_I}$ and $\rho_I(\gamma)$. In a bottom-up approach, by varying all possible representations of the matter multiplets, we can absorb the factor $(\mathbf{1}_{+-1})^{k_I}$ into the choice of $\rho_I(\gamma)$. The CP transformation X'_I fulfills the following consistency conditions on the stability group $D_4 \times Z_3$:

$$X'_I\,\rho^*_I(a)X'^{-1}_I = \rho_I(a) \,, \quad X'_I\,\rho^*_I(b)X'^{-1}_I = \rho_I(b^{-1}) \,, \quad X'_I\,\rho^*_I(c)X'^{-1}_I = \rho_I(c^{-1}) \,. \tag{4.32}$$

A basis for the $D_4 \times Z_3$ generators exists, where the consistency conditions of CP transformation restricted to the stability group $D_4 \times Z_3$ are solved by $X'_I = \mathbb{1}$, see comment at the end of appendix C.

4.3.3 Triplet decomposition

We find that each irreducible triplet of $N_n(H)$ decomposes under $D_4 \times Z_3$ into a direct sum of three singlets which transform in the same way under the $D_4 \times Z_3$ subgroup. There are only four distinct cases as follows,

$$\rho_L \sim \begin{cases} \mathbf{1}_{++0} \oplus \mathbf{1}_{++1} \oplus \mathbf{1}_{++2} \\ \mathbf{1}_{+-0} \oplus \mathbf{1}_{+-1} \oplus \mathbf{1}_{+-2} \\ \mathbf{1}_{-+0} \oplus \mathbf{1}_{-+1} \oplus \mathbf{1}_{-+2} \\ \mathbf{1}_{--0} \oplus \mathbf{1}_{--1} \oplus \mathbf{1}_{--2} \end{cases} . \tag{4.33}$$

We end up with

$$\Omega_L(\gamma) = j(\gamma,\tau_0)^{-k_I}\rho_L(\gamma) \,, \tag{4.34}$$

where $j(\gamma,\tau_0)^{-1} \sim \mathbf{1}_{+-1}$. Finally, we get $\Omega \equiv \Omega_{H_u}\Omega_L$ by multiplying Ω_L and the generic singlet associated with Ω_{H_u}. Since each triplet of $N_n(H)$ is decomposed into three singlets of $D_4 \times Z_3$, the CP symmetry transforms the lepton fields into their conjugate.

4.4 Fixed point 5

The last case to be examined is the fixed point

$$\tau_0 = \frac{i}{\sqrt{3}}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} . \tag{4.35}$$

It preserves CP using the standard definition of eq. (3.12), where $\gamma = \mathbb{1}$.

4.4.1 Stability group

The stability group G_0 of τ_0, subgroup of the flavour group $N(H)$ defined in eq. (2.18), is isomorphic to D_4, generated by the elements a and b, satisfying $a^4 = b^2 = (ab)^2 = 1$. In terms of the generators G_i $(i = 1, 2, 3, 4)$ of $N(H)$, a and b read:

$$a = G_3 G_4 \,, \qquad b = (G_3 G_2)^3 \,. \tag{4.36}$$

The irreducible representations of D_4 are four singlets $\mathbf{1}_+$, $\mathbf{1}_-$, $\mathbf{1}'_+$, $\mathbf{1}'_-$ and one doublet $\mathbf{2}$. Further details about D_4 are given in appendix A.

4.4.2 New basis

We move to the new basis of eq. (3.13), choosing $\alpha = 0$. We define:

$$u = \begin{pmatrix} u_1 & u_3 \\ u_3 & u_1 \end{pmatrix} \,, \tag{4.37}$$

and we find that, under the stability group D_4, the new fields u split into the sum of an invariant singlet, u_3, and a component u_1 transforming as $\mathbf{1}'_-$, see appendix A:

$$\begin{aligned} &u_1 \xrightarrow{a} -u_1 \,, \qquad u_1 \xrightarrow{b} +u_1 \,, \qquad u_1 \sim \mathbf{1}'_- \\ &u_3 \xrightarrow{a} +u_3 \,, \qquad u_3 \xrightarrow{b} +u_3 \,, \qquad u_3 \sim \mathbf{1}_+ \,. \end{aligned} \tag{4.38}$$

Under CP, we get

$$u_1 \xrightarrow{\mathcal{CP}} \bar{u}_1 \,, \qquad u_3 \xrightarrow{\mathcal{CP}} \bar{u}_3 \,. \tag{4.39}$$

The action of D_4 on the new matter fields $\Phi^{(I)}$ of eq. (3.13) is

$$\Phi^{(I)} \xrightarrow{\gamma} \Omega_I(\gamma)\Phi^{(I)} \,, \qquad \Omega_I(\gamma) = j(\gamma, \tau_0)^{-k_I} \rho_I(\gamma). \tag{4.40}$$

where $j(\gamma, \tau) = \det(C\tau + D)$. In particular, we have:

$$j(a, \tau_0)^{-1} = +1 \,, \qquad j(b, \tau_0)^{-1} = -1 \,, \tag{4.41}$$

which reproduces the representation $\mathbf{1}_-$ of the stability group. We see that $\Omega_I(\gamma) = j(\gamma, \tau_0)^{-k_I} \rho_I(\gamma)$ is also a representation of the stability group, the direct product of $(\mathbf{1}_-)^{k_I} = (\mathbf{1}_-, \mathbf{1}_+)$ and $\rho_I(\gamma)$. In a bottom-up approach, by varying all possible representations of the matter multiplets, we can absorb the factor $(\mathbf{1}_-)^{k_I}$ into the choice of $\rho_I(\gamma)$. The consistency conditions of CP transformation X'_I on the stability group D_4 are given by

$$X'_I\, \rho^*_I(a) X'^{-1}_I = \rho_I(a^{-1}), \qquad X'_I\, \rho^*_I(b) X'^{-1}_I = \rho_I(b^{-1}) \,. \tag{4.42}$$

A basis for the D_4 generators exists, where the consistency conditions of CP transformation restricted to the stability group D_4 are solved by $X'_I = \mathbb{1}$, see comment at the end of appendix A. Notice that this basis does not coincide with the one allowing to choose $X'_I = \mathbb{1}$ at the fixed point 2, despite the stability group for the fixed points 2 and 5 is the same.

JHEP05(2024)052

4.4.3 Triplet decomposition

We need the decomposition of the irreducible triplets ρ_L of $N_n(H)$, to which lepton doublets L are assigned, within the stability group D_4. With the help of appendix A, by considering the representation matrices of the two generators, a and b, we find that each irreducible triplet of $N_n(H)$ decomposes under D_4 into a direct sum of three singlets, wherein two of them are identical. There are only four distinct cases, illustrated in eq. (4.43).

$$\rho_L \sim \begin{cases} \mathbf{1}_- \oplus \mathbf{1}'_+ \oplus \mathbf{1}'_+ \\ \mathbf{1}_+ \oplus \mathbf{1}'_- \oplus \mathbf{1}'_- \\ \mathbf{1}'_+ \oplus \mathbf{1}_- \oplus \mathbf{1}_- \\ \mathbf{1}'_- \oplus \mathbf{1}_+ \oplus \mathbf{1}_+ \end{cases} . \tag{4.43}$$

We end up with

$$\Omega_L(\gamma) = j(\gamma, \tau_0)^{-k_I} \rho_L(\gamma) \,, \tag{4.44}$$

where $j(\gamma, \tau_0)^{-1} \sim \mathbf{1}_-$. Finally, we get $\Omega \equiv \Omega_{H_u}\Omega_L$ by multiplying Ω_L and the generic singlet associated with Ω_{H_u}.

5 Neutrino mass matrix

In this section, we discuss the patterns of the lepton mass matrices of a generic theory whose moduli, belonging to the region Σ_2 parameterized by

$$\tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_1 \end{pmatrix} \tag{5.1}$$

are close to one of the fixed points examined previously. We assume the most general CP-invariant $\mathcal{N} = 1$ local supersymmetric action and enforce Siegel modular invariance by asking that the superpotential w transforms as in eq. (3.8) with $\boldsymbol{r}_s$ either trivial or non-trivial singlet of the finite modular group $N_n(H)$, and thus also a singlet of the stability group G_0. In such a theory the level n, the weights k_I and the representations ρ_I of the matter multiplets, and even the specific form of the Kähler potential are the most general ones consistent with the requirement of CP and modular invariance. In this completely general framework, we make a single assumption: the lepton doublets L are assigned to an irreducible triplet ρ_L of the finite modular group $N_n(H)$. In a bottom-up approach this assumption usually minimizes the number of free parameters of the theory.

Working with the local coordinates $(u, \Phi^{(I)})$ of eq. (3.13), on which the stability group G_0 has a linear action, the lepton mass matrices $m_{\bar{e}e}$ and m_ν must transform as required by table 4. Making use of the decomposition of each irreducible triplet of $N_n(H)$ and the transformation laws of u under G_0, we can easily fulfill this requisite, which can be implemented order by order in the expansion in powers of $|u|$, a small quantity when τ is close to the fixed point. We provide the general expression of $m_{\bar{e}e}$ and m_ν at the first nontrivial order in $|u|$.

As a general result, we find that in each fixed point, the mass matrices $m_{\bar{e}e}$ and m_ν are sensitive only to $\boldsymbol{r}_s$ and not to the decomposition of ρ_L under G_0. Moreover, the mass

JHEP05(2024)052

matrices $m_{\bar{e}e}$ are the same for all $\boldsymbol{r}_s$, while the neutrino mass matrices m_ν(or m_ν^{-1}) depend on $\boldsymbol{r}_s$. The examined fixed points fall into two classes. The fixed points 2, 3 and 5 give rise to similar patterns, which are different from those arising around the fixed point 4.

5.1 Fixed points 2, 3, and 5

From table 4 and the analysis of the previous section, we find the following pattern for $m_{\bar{e}e}(u,\bar{u})$:

$$m_{\bar{e}e}(u,\bar{u}) = m_{0e}^2 \begin{pmatrix} y_{11}^0 & y_{12}\,x & y_{13}\,x \\ y_{12}^*\,x & y_{22}^0 & y_{23}^0 \\ y_{13}^*\,x & y_{23}^0 & y_{33}^0 \end{pmatrix} + \ldots\,, \tag{5.2}$$

where $x = |u_3|$ for the fixed point 2 and $x = |u_1|$ for the fixed points 3 and 5. The overall real coefficient m_{0e}^2 has the dimension of (mass)^2. Dots stand for higher orders in the x expansion. The coefficients $y_{ij}^0(y_{ij})$ are real (complex) numbers, independent of the moduli. They are not constrained in the present analysis, though they are expected to be of the same order. To discuss the lepton mixing matrix, we diagonalize $m_{\bar{e}e}(u,\bar{u})$:

$$U_e^\dagger m_{\bar{e}e}(u,\bar{u}) U_e = \mathtt{diag}[m_{\bar{e}e}(u,\bar{u})]\,. \tag{5.3}$$

Up to a permutation matrix P related to the ordering of the charged lepton masses, U_e has the pattern:

$$U_e = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(x) & \mathcal{O}(x) \\ \mathcal{O}(x) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(x) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}\,. \tag{5.4}$$

Next, we examine the neutrino mass matrices. We find that they fall into three classes, depending on the singlet representation $\boldsymbol{r}_s$ under which w transforms, see table 5.

- **A** The following pattern is obtained when $\boldsymbol{r}_s$ embeds one of the G_0 singlets listed in table 5, first line. The overall factor $m_{0\nu}$ is a real mass parameter.

$$m_\nu(u,\bar{u}) = m_{0\nu} \begin{pmatrix} x_{11}^0 & x_{12}\,x & x_{13}\,x \\ \cdot & x_{22}^0 & x_{23}^0 \\ \cdot & \cdot & x_{33}^0 \end{pmatrix} + \ldots\,. \tag{5.5}$$

  The coefficients $x_{ij}^0(x_{ij})$ are real (complex) numbers, independent of the moduli $u_{1,3}$, except for the choice $\mathbf{1}_{++-}$ at the fixed point 3, where both x_{ij}^0 and x_{ij} are proportional to a linear combination of u_3 and $\bar{u}_3$. In any case, $x_{ij}^0(x_{ij})$ are expected to be of the same order, and we do not need to treat separately this special case.

- **B** The following pattern is obtained when $\boldsymbol{r}_s$ embeds one of the G_0 singlets listed in table 5, second line. The overall factor $m_{0\nu}$ is a real mass parameter.

$$m_\nu(u,\bar{u}) = m_{0\nu} \begin{pmatrix} x_{11}\,x & x_{12}^0 & x_{13}^0 \\ \cdot & x_{22}\,x & x_{23}\,x \\ \cdot & \cdot & x_{33}\,x \end{pmatrix} + \ldots\,. \tag{5.6}$$

pattern	$\boldsymbol{r}_s$(FP2)	$\boldsymbol{r}_s$(FP3)	$\boldsymbol{r}_s$(FP5)
A - eq. (5.5)	$\mathbf{1}_+$	$\mathbf{1}_{+++}$, $\mathbf{1}_{++-}$	$\mathbf{1}_+$
B - eq. (5.6)	$\mathbf{1}'_-$	$\mathbf{1}_{-+-}$, $\mathbf{1}_{-++}$	$\mathbf{1}'_-$
C	$\mathbf{1}_-$, $\mathbf{1}'_+$	$\mathbf{1}_{+--}$, $\mathbf{1}_{+-+}$, $\mathbf{1}_{---}$, $\mathbf{1}_{--+}$	$\mathbf{1}_-$, $\mathbf{1}'_+$

Table 5. Patterns of neutrino mass matrices $m_\nu(u,\bar{u})$, or their inverse $m_\nu(u,\bar{u})^{-1}$, and their dependence on the singlet representation of the superpotential w, for the fixed points $i = 2, 3, 5$. We have displayed the singlet representations $\boldsymbol{r}_s$(`FPi`) of the stability group G_0. Each representation $\boldsymbol{r}_s$(`FPi`) can be embedded in one (or more) representations $\boldsymbol{r}_s$ of $N_n(H)$.

The coefficients $x^0_{ij}(x_{ij})$ are real (complex) numbers, independent of the moduli $u_{1,3}$, except for the choice $\mathbf{1}_{-++}$ at the fixed point 3, where both x^0_{ij} and x_{ij} are proportional to a linear combination of u_3 and $\bar{u}_3$. In any case, $x^0_{ij}(x_{ij})$ are expected to be of the same order, and we do not need to discuss separately this particular case.

Since $m_\nu(u,\bar{u})$ has rank two at $x = 0$, we also consider the expansion of $[m_\nu(u,\bar{u})]^{-1}$, which is identical to the one in eq. (5.6), with the replacement $m_\nu(u,\bar{u}) \to [m_\nu(u,\bar{u})]^{-1}$ and $m_{0\nu} \to m_{0\nu}^{-1}$. The two possibilities arising from the analysis of $m_\nu(u,\bar{u})$ and $[m_\nu(u,\bar{u})]^{-1}$ are physically distinct. At $x = 0$, the pattern of $[m_\nu(u,\bar{u})]$ predicts a vanishing mass, while that of $[m_\nu(u,\bar{u})]^{-1}$ predicts a divergent neutrino mass. We understand such an infinite mass in terms of an extra degree of freedom of the full theory becoming massless at the fixed point. It is natural to interpret such a degree of freedom as a right-handed neutrino. Thus, the pattern of $[m_\nu(u,\bar{u})]^{-1}$ is expected to arise in the context of the seesaw mechanism when a right-handed neutrino, whose mass depends on the moduli, becomes massless at the fixed point.

- **C** In this case, the only solution to the constraint specified in table 4, is a vanishing neutrino mass matrix $m_\nu(u,\bar{u}) = 0$, to any order in the expansion in powers of u_1 and u_3. We dismiss this unphysical possibility.

From eq. (5.4) we see that moving to the basis where $m_{\bar{e}e}(u,\bar{u})$ is diagonal, up to a common permutation matrix of rows and columns and up to higher-order terms in the expansion, the neutrino mass matrix maintains the same pattern shown in eqs. (5.5) and (5.6). To first order in x, the effect of the basis change can be absorbed in the coefficients x^0_{ij}, x_{ij}. The same conclusion holds for the inverse $m_\nu(u,\bar{u})^{-1}$ and, without losing generality, we can discuss the neutrino mass spectrum, mixing angles and phases by directly analyzing the matrices (5.5) and (5.6). These two patterns coincide with those occurring in single-modulus SL(2, $\mathbb{Z}$)-invariant theories close to the fixed point $\tau_0 = i$ [43, 44]. We briefly recapitulate the results here and we refer the reader to the literature for more details.[8]

In table 6 we summarize the predictions of Siegel modular invariant models for lepton masses in the vicinity of one of the fixed points, up to possible permutations affecting the mixing matrix. When the pattern in eq. (5.5) is realized, both normal (NO) and inverted

[8] In particular, the predictions for the lepton masses and lepton mixing matrix as well as mixing parameters have been presented in appendices D.1.1 and D.1.2 and section 6 of ref. [44].

JHEP05(2024)052

		mass ordering	$\frac{\Delta m^2_{sol}}{\Delta m^2_{atm}}$	$\sin^2\theta_{12}$	$\sin^2\theta_{13}$	$\sin^2\theta_{23}$
A - eq. (5.5)	$m_\nu(0,0)$ regular	NO/IO	$\mathcal{O}(1)$	$\mathcal{O}(x^2)$	$\mathcal{O}(x^2)$	$\mathcal{O}(1)$
B - eq. (5.6)	$m_\nu(0,0)$ regular	IO	$\mathcal{O}(x)$	$\frac{1}{2}(1+\mathcal{O}(x))$	$\mathcal{O}(x^2)$	$\mathcal{O}(1)$
B - eq. (5.6)	$m_\nu(0,0)$ singular	NO	$\mathcal{O}(x^3)$	$\frac{1}{2}(1+\mathcal{O}(x))$	$\mathcal{O}(x^2)$	$\mathcal{O}(1)$
D - eq. (5.10)	$m_\nu(0,0)$ regular	NO/IO	$\mathcal{O}(x)$	$\frac{1}{2}(1+\mathcal{O}(x))$	$\mathcal{O}(x^2)$	$\mathcal{O}(x^2)$
E - eq. (5.11)	$m_\nu(0,0)$ regular	NO/IO	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$

Table 6. Synopsis of predictions in modular invariant flavor models of leptons, when the modulus τ falls in the vicinity of the fixed points and ρ_L is an irreducible triplet. For the fixed point 4, $x = |u_1|$.

ordering (IO) of the neutrino mass spectrum can be realized. However, $\Delta m^2_{sol}/\Delta m^2_{atm}$ is generically expected to be of order one. Moreover $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$ are expected to be of the same order, contrary to observation. Therefore this pattern can only be reconciled with the data at the price of tuning the coefficients x^0_{ij} and x_{ij}.

When the pattern in eq. (5.6) is realized, and $m_\nu(0,0)$ has a vanishing eigenvalue, an inverted ordering of neutrino masses is predicted. To reproduce the observed values of $\sin^2\theta_{13}$ and $\sin^2\theta_{12}$, x should be close to 0.15. This is in tension with the value of x required by $r = \Delta m^2_{sol}/\Delta m^2_{atm} = \mathcal{O}(x)$, experimentally close to 0.03. This pattern can match the data too, at the price of tuning the coefficients x^0_{ij} and x_{ij}.

When the outcome is the pattern in eq. (5.6), a particularly appealing scenario occurs when $[m_\nu(0,0)]^{-1}$ has a vanishing eigenvalue, which can occur within the seesaw mechanism. In this case, all the data can be reproduced by parameters x^0_{ij} and x_{ij} of the same order of magnitude, by adjusting the overall scale $m_{0\nu}$ and choosing x close to 0.1. No tuning of the unknown order-one parameters is needed in this case. At the fixed point, $\Delta m^2_{sol}/\Delta m^2_{atm} = \sin^2\theta_{13} = \sin^2\theta_{12} - 1/2 = 0$ and CP is conserved. Nonvanishing values of these three quantities and CP-violating effects all originate from a small departure of τ from the fixed point. This is the most successful pattern among all those discussed in this paper.

5.2 Fixed point 4

From table 4 and the analysis of the previous section, we find the following pattern for $m_{\bar{e}e}(u,\bar{u})$:

$$m_{\bar{e}e} = m^2_{0e} \begin{pmatrix} y^0_{11} & y^{01}_{12}\,\bar{u}_1 & y^{10}_{13}\,u_1 \\ y^{01}_{12}\,u_1 & y^0_{22} & y^{01}_{23}\bar{u}_1 \\ y^{10}_{13}\,\bar{u}_1 & y^{01}_{23}\,u_1 & y^0_{33} \end{pmatrix} + \dots . \tag{5.7}$$

The overall real coefficient m^2_{0e} has the dimension of (mass)^2. Dots stand for higher orders in the moduli expansion. The parameters $y^{(0,01,10)}{}_{ij}$ are real, independent of the moduli and expected to be of the same order. To discuss the lepton mixing matrix, we diagonalize $m_{\bar{e}e}(u,\bar{u})$:

$$U_e^\dagger m_{\bar{e}e}(u,\bar{u}) U_e = \mathtt{diag}[m_{\bar{e}e}(u,\bar{u})]\,, \tag{5.8}$$

where U_e is a nearly diagonal matrix:

$$U_e = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1)\ \bar{u}_1 & \mathcal{O}(1)\ u_1 \\ \mathcal{O}(1)\ u_1 & \mathcal{O}(1) & \mathcal{O}(1)\ \bar{u}_1 \\ \mathcal{O}(1)\ \bar{u}_1 & \mathcal{O}(1)\ u_1 & \mathcal{O}(1) \end{pmatrix}. \tag{5.9}$$

Next, we examine the neutrino mass matrices. We find that they fall into three classes, depending on the singlet representation $\boldsymbol{r}_s$ under which w transforms.

- **D** Up to cyclic permutations of rows and columns, when $\boldsymbol{r}_s$ embeds one of the G_0 singlets $\mathbf{1}_{++n}$ $(n=0,1,2)$, we get the following pattern, $m_{0\nu}$ denoting a real mass parameter:

$$m_\nu(u,\bar{u}) = m_{0\nu} \begin{pmatrix} x_{11}^{0} & x_{12}^{01}\ \bar{u}_1 & x_{13}^{10}\ u_1 \\ \cdot & x_{22}^{10}\ u_1 & x_{23}^{0} \\ \cdot & \cdot & x_{33}^{01}\ \bar{u}_1 \end{pmatrix} + \ldots. \tag{5.10}$$

  The coefficients ${x^{(0,01,10)}}_{ij}$ are real and independent of the moduli $u_{1,3}$.

- **E** Up to cyclic permutations of rows and columns, when $\boldsymbol{r}_s$ embeds one of the G_0 singlets $\mathbf{1}_{--n}$ $(n=0,1,2)$, we get the following pattern, $m_{0\nu}$ denoting a real mass parameter:

$$m_\nu(u,\bar{u}) = m_{0\nu} \begin{pmatrix} x_{11}^{01}\ \bar{u}_3 & x_{12}^{10}\ u_3 & x_{13}^{11}\ u_1\bar{u}_3 + x_{13}^{\prime 11}\ \bar{u}_1 u_3 \\ \cdot & x_{22}^{11}\ u_1\bar{u}_3 + x_{22}^{\prime 11}\ \bar{u}_1 u_3 & x_{23}^{01}\ \bar{u}_3 \\ \cdot & \cdot & x_{33}^{10}\ u_3 \end{pmatrix} + \ldots. \tag{5.11}$$

  The coefficients ${x^{(01,10,11)}}_{ij}$ and $x_{ij}^{\prime 11}$ are real and independent of the moduli $u_{1,3}$.

- **F** When $\boldsymbol{r}_s$ embeds one of the G_0 singlets $\mathbf{1}_{+-n}$ or $\mathbf{1}_{-+n}$ $(n=0,1,2)$, to all orders in the moduli expansion the neutrino mass matrix vanishes, an option we discard as unphysical.

  Given the form of U_e in eq. (5.9), in the basis where $m_{\bar{e}e}(u,\bar{u})$ is diagonal, up to a common permutation matrix of rows and columns and up to higher-order terms in the expansion, the neutrino mass matrix has always the same pattern shown in eqs. (5.10) and (5.11). To first order in u_1 and $\bar{u}_1$, the effect of the basis change can be absorbed in the coefficients ${x^{(01,10,11)}}_{ij}$ and $x_{ij}^{\prime 11}$. Thus, without losing generality, we can discuss the neutrino mass spectrum, mixing angles and phases by directly analyzing the matrices (5.10) and (5.11).

The pattern in eq. (5.10) coincides with the one occurring in single-modulus $\mathrm{SL}(2,\mathbb{Z})$-invariant theories close to the fixed point $\tau_0 = \omega$ [44]. We refer the reader to the literature for more details.[9] In table 6 we summarize the predictions of Siegel modular invariant models for lepton masses in the vicinity of the fixed point 4, up to possible permutations affecting the mixing matrix. Within the pattern in eq. (5.10), both normal and inverted ordering of the

[9] In particular, the predictions for the lepton masses and lepton mixing matrix as well as mixing parameters have been presented in appendix D.2 and section 6 of ref. [44].

neutrino mass spectrum can be realized. However, $\Delta m^2_{sol}/\Delta m^2_{atm} = \mathcal{O}(x)$ would demand $x \approx 0.03$, which is inadequate to describe $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$, both expected of $\mathcal{O}(x^2)$. A considerable tuning of the order-one coefficients is required to reconcile this pattern with the data. Finally, the pattern in eq. (5.11) predicts neutrino masses of the same order of magnitude and mixing angle of approximately the same size, as a numerical simulation shows. While data can be reproduced by fitting the unknown order-one coefficients, this pattern does not suggest any explanation for the smallness of $\Delta m^2_{sol}/\Delta m^2_{atm}$ and/or $\sin^2\theta_{13}$.

5.3 Summary

Excluding the unphysical case of a vanishing neutrino mass matrix, the four patterns of eqs. ((5.5), (5.6), (5.10), (5.11)) exhaust all possible cases that can arise from a CP and Siegel modular invariant locally supersymmetric theory when the moduli are close to a fixed point. The predictions are summarized in table 6, where the scaling properties of the leptonic mixing angles and the ratio between solar and atmospheric squared mass differences are shown. The distance from the fixed point is parametrized by a small variable x. The only pattern that accommodates all the data without tuning of the unknown parameters is **B** of eq. (5.6), when $m_\nu(0,0)$ is singular. It can be realized by the seesaw mechanism when one of the right-handed neutrinos happens to be massless at the fixed point. All the other patterns require an adjustment of the parameters to overcome the wrong scaling of one or several observable quantities. An explicit model belonging to the class analyzed in this section has been discussed in ref. [74] and, for completeness, is illustrated in the appendix D, where we show that it matches pattern **A** of eq. (5.5).

Many properties of the mass spectrum and the mixing matrix follow mainly from the decomposition of the representation Ω into irreducible components. This idea was developed in ref. [90], where the decompositions of Ω, and its charged lepton counterpart Ω_c, compatible with a realistic leading-order pattern of lepton masses and mixing angles have been classified. While our results apply to a specific context and are not aimed to cover the case of the most general flavour group acting linearly on matter fields, they have been obtained under less restrictive assumptions. First, we go beyond the zeroth order approximation by including the correction linear in the moduli. Second, we also consider the case where the neutrino mass matrix develops diverging eigenvalues at the fixed point, as happens in the seesaw mechanism when a right-handed neutrino mass vanishes at the symmetric point. Third, consistently with the freedom permitted by supergravity, we allow an overall phase factor in the transformation of neutrino mass matrices.

6 Conclusion and outlook

To explore theories depending on more than one modulus, we have analyzed a class of locally supersymmetric models of lepton masses invariant under CP and under a subgroup $N(H)$ of the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$. We have concentrated on the subgroup $N(H)$ whose finite copies $N_n(H)$ contain three-dimensional irreducible representations, to which we assign lepton electroweak doublets. For consistency, the moduli space is restricted to a subset Σ_2 of the Siegel upper half plane, spanned by two complex moduli and invariant under $N(H)$. We have identified 168 irreducible triplets of $N_n(H)$, which we conjecture to exhaust the number of inequivalent three-dimensional irreducible representations. There are four inequivalent fixed points in Σ_2, each left invariant by a specific finite group G_0. By exploiting the decomposition of each irreducible triplet of $N_n(H)$ under G_0, and a convenient basis for moduli and matter fields, we have built all possible patterns of neutrino mass matrices, consisting of a series expansion around each fixed point, of which we keep the first nontrivial term. The leading-order contribution is invariant under both G_0 and CP, which are spontaneously broken by the vacuum expectation value of an order parameter, measuring the distance from the fixed point. After moving to the basis where kinetic terms are canonical and the charged lepton mass matrix is diagonal, we can read neutrino masses and lepton mixing angles.

Apart from the unrealistic case of a vanishing neutrino mass matrix, only five patterns are found. Four of them coincide with those arising in $\mathrm{SL}(2,\mathbb{Z})$-invariant single-modulus theories in the vicinity of the fixed points $\tau_0 = i$ and $\tau_0 = -1/2 + i\sqrt{3}/2$. In each pattern, all physical quantities scale with the distance of the moduli from the fixed point in a way that is largely independent of the details of the theory. All the patterns but a single one require tuning the free parameters to match such a scaling with the smallness of $\Delta m^2_{sol}/\Delta m^2_{atm}$ and $\sin^2\theta_{13}$ and the largeness of $\sin^2\theta_{12}$ and $\sin^2\theta_{23}$. The matrix that best describes the data and needs no tuning is pattern **B** of eq. (5.6), when $m_\nu(0,0)$ is singular. This is the same pattern preferred by the majority of single-modulus models when τ is close to the self-dual point $\tau_0 = i$.

We stress the generality of this result. Except for the assumption that the lepton doublets transform under $N_n(H)$ as any of the 168 irreducible triplets, our finding is independent of the level, of the weights of matter multiplets, and the form of the Kähler potential. In particular, we are not forced to assume a minimal or flavor universal Kähler potential: our conclusion holds for the most general Kähler potential compatible with Siegel modular invariance. It would be very difficult and time-consuming to reproduce our results by inspecting one-by-one all models compatible with our assumptions. Today very few models of this type exist in the literature [73, 74, 91, 92]. When moduli fall in the vicinity of a fixed point, we find agreement with the present analysis.

On the other hand, our analysis has some limitations. First, the dependence on the chosen level, weights, and Kähler potential affect the unknown coefficients of the patterns we found. We have assumed that such dependence does not conspire to produce hierarchies among these coefficients, but we cannot provide a mathematical proof of this statement, which seems to work well in the case of single-modulus theories. Second, we have no dynamical justification for working close to a fixed point. Here again, we rely on the statistics accumulated in the single modulus theories and on the few semianalytical results derived from the minimization of modular invariant energy densities. Finally, new patterns may arise if we relax the assumption that the lepton doublets transform as irreducible triplets of the finite modular group. This is a realistic possibility, especially in the context of string theory compactifications where typically the modular group is part of a bigger eclectic group [32–38]. Allowing for reducible representations would presumably open many new unexplored possibilities. Both the two-dimensional region Σ_1 or the entire Siegel upper half plane might be relevant in a more general analysis. A complete classification of all representations of dimensionality one and two would be required to undertake such a demanding task, which we leave for future work.

	$1C_1$	$1C_2$	$2C_2$	$2C'_2$	$2C_4$
	$\{b^2\}$	$\{a^2\}$	$\{b, a^2b\}$	$\{ab, a^3b\}$	$\{a, a^3\}$
$\mathbf{1}_+$	$+1$	$+1$	$+1$	$+1$	$+1$
$\mathbf{1}_-$	$+1$	$+1$	-1	-1	$+1$
$\mathbf{1}'_-$	$+1$	$+1$	$+1$	-1	-1
$\mathbf{1}'_+$	$+1$	$+1$	-1	$+1$	-1
$\mathbf{2}$	$+2$	-2	0	0	0

Table 7. Character table of D_4, where mC_n denotes a conjugacy class with m elements of order n.

Acknowledgments

We thank warmly Gianguido Dall'Agata and Davide Cassani, for a stimulating correspondence. This work is supported by the INFN. GJD is supported by the National Natural Science Foundation of China under Grant Nos. 12375104 and 11975224. XGL is supported by the National Science Foundation, under Grant No. PHY-1915005.

A Group theory of D_4

The dihedral group D_4 is the symmetry group of a square, generated by the two elements a and b, satisfying $a^4 = b^2 = (ab)^2 = 1$. The element a represents a rotation of $\pi/2$ of the square around its center, while the element b is a reflection around a symmetry axis. The group has 5 conjugacy classes and 5 irreducible representations: four singlets $\mathbf{1}_+$, $\mathbf{1}_-$, $\mathbf{1}'_+$, $\mathbf{1}'_-$ and one doublet $\mathbf{2}$. The representation matrices of the generators a and b are

$$\begin{aligned}
\mathbf{1}_+ &: \quad a = +1, \qquad b = +1\,, \\
\mathbf{1}_- &: \quad a = +1, \qquad b = -1\,, \\
\mathbf{1}'_- &: \quad a = -1, \qquad b = +1\,, \\
\mathbf{1}'_+ &: \quad a = -1, \qquad b = -1\,, \\
\mathbf{2} &: \quad a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned} \tag{A.1}$$

The character table is shown in table 7. The product of representations decomposes as follows:

$$\begin{aligned}
&\mathbf{1}_m \otimes \mathbf{1}_n = \mathbf{1}_{mn}, \qquad \mathbf{1}_m \otimes \mathbf{1}'_n = \mathbf{1}'_{mn}, \qquad \mathbf{1}'_m \otimes \mathbf{1}'_n = \mathbf{1}_{mn} \\
&\mathbf{1}_m \otimes \mathbf{2} = \mathbf{2}, \qquad \mathbf{1}'_m \otimes \mathbf{2} = \mathbf{2}\,, \qquad \mathbf{2} \otimes \mathbf{2} = \mathbf{1}_+ \oplus \mathbf{1}_- \oplus \mathbf{1}'_+ \oplus \mathbf{1}'_-\,.
\end{aligned} \tag{A.2}$$

We define:

$$\mathbf{1}_\pm = \beta_\pm \qquad \mathbf{1}'_\pm = \beta'_\pm \qquad \mathbf{2} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

and we get:

$$\mathbf{1}_+ \otimes \mathbf{2} = \mathbf{2} = \begin{pmatrix} \beta_+\alpha_1 \\ \beta_+\alpha_2 \end{pmatrix}, \qquad \mathbf{1}_- \otimes \mathbf{2} = \mathbf{2} = \begin{pmatrix} \beta_-\alpha_2 \\ -\beta_-\alpha_1 \end{pmatrix}. \tag{A.3}$$

$$\mathbf{1}'_+ \otimes \mathbf{2} = \mathbf{2} = \begin{pmatrix} \beta'_+\alpha_2 \\ \beta'_+\alpha_1 \end{pmatrix}, \qquad \mathbf{1}'_- \otimes \mathbf{2} = \mathbf{2} = \begin{pmatrix} \beta'_-\alpha_1 \\ -\beta'_-\alpha_2 \end{pmatrix}. \tag{A.4}$$

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1}_+ \oplus \mathbf{1}_- \oplus \mathbf{1}'_+ \oplus \mathbf{1}'_-, \qquad \begin{cases} \mathbf{1}_+ = \alpha_1\beta_1 + \alpha_2\beta_2 \\ \mathbf{1}_- = \alpha_1\beta_2 - \alpha_2\beta_1 \\ \mathbf{1}'_- = \alpha_1\beta_1 - \alpha_2\beta_2 \\ \mathbf{1}'_+ = \alpha_1\beta_2 + \alpha_2\beta_1 \end{cases}. \tag{A.5}$$

An equivalent basis (a', b') for the D_4 generators is obtained by the similarity transformation acting on doublets:

$$a' = U^{-1} a\, U = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad b' = U^{-1} b\, U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{A.6}$$

where U is the matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}. \tag{A.7}$$

Singlet representations are unchanged in this new basis, where the consistency conditions of CP transformation in eq. (4.42) are solved by $X'_I = \mathbb{1}$, both for singlets and doublets when working at the fixed point 5. Of course, this new basis is inconvenient when dealing with the fixed point 2, since a nontrivial matrix X'_I would be required in the CP transformation law of doublets.

B Group theory of $D_4 \circ Z_4$

The Pauli group is the central product of D_4 and Z_4: $D_4 \circ Z_4$, which has the GAP id [16,13] and is isomorphic to $(Z_2 \times Z_4) \rtimes Z_2$. It is generated by the elements a, b and c, satisfying $a^4 = c^4 = b^2 = (ab)^2 = 1$, $a^2 = c^2$, $ac = ca$, $bc = cb$. The elements (a, b) generate D_4 and c generates Z_4. The elements of D_4 commute with those of Z_4. They have in common the element $a^2 = c^2$ and for this reason the central product does not coincide with the direct product. The group has ten irreducible representations: eight singlets and two doublets. The generators a, b and c in each irreducible representation are represented by

$$\begin{aligned}
\mathbf{1}_{+++}:&\quad a = +1, && b = +1, && c = +1\,,\\
\mathbf{1}_{+--}:&\quad a = +1, && b = -1, && c = -1\,,\\
\mathbf{1}_{+-+}:&\quad a = +1, && b = -1, && c = +1\,,\\
\mathbf{1}_{---}:&\quad a = -1, && b = -1, && c = -1\,,\\
\mathbf{1}_{--+}:&\quad a = -1, && b = -1, && c = +1\,,\\
\mathbf{1}_{-+-}:&\quad a = -1, && b = +1, && c = -1\,,
\end{aligned}$$

	$1C_1$	$1C_2$	$1C_4$	$1C'_4$	$2C_2$	$2C'_2$	$2C''_2$	$2C_4$	$2C'_4$	$2C''_4$
	$\{b^2\}$	$\{c^2\}$	$\{c\}$	$\{a^2c\}$	$\{b, a^2b\}$	$\{ab, abc^2\}$	$\{ac, ac^3\}$	$\{a, ac^2\}$	$\{bc, a^2bc\}$	$\{abc, abc^3\}$
$\mathbf{1}_{+++}$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
$\mathbf{1}_{+--}$	$+1$	$+1$	-1	-1	-1	-1	-1	$+1$	$+1$	$+1$
$\mathbf{1}_{+-+}$	$+1$	$+1$	$+1$	$+1$	-1	-1	$+1$	$+1$	-1	-1
$\mathbf{1}_{---}$	$+1$	$+1$	-1	-1	-1	$+1$	$+1$	-1	$+1$	-1
$\mathbf{1}_{--+}$	$+1$	$+1$	$+1$	$+1$	-1	$+1$	-1	-1	-1	$+1$
$\mathbf{1}_{-+-}$	$+1$	$+1$	-1	-1	$+1$	-1	$+1$	-1	-1	$+1$
$\mathbf{1}_{-++}$	$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1	-1	$+1$	-1
$\mathbf{1}_{++-}$	$+1$	$+1$	-1	-1	$+1$	$+1$	-1	$+1$	-1	-1
$\mathbf{2}$	$+2$	-2	$-2i$	$+2i$	0	0	0	0	0	0
$\mathbf{2}'$	$+2$	-2	$+2i$	$-2i$	0	0	0	0	0	0

Table 8. Character table of $D_4 \circ Z_4$.

$$\begin{aligned}
\mathbf{1}_{-++}: &\quad a=-1, \qquad b=+1, \qquad c=+1\,,\\
\mathbf{1}_{++-}: &\quad a=+1, \qquad b=+1, \qquad c=-1\,,\\
\mathbf{2}: &\quad a=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}, \quad b=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \quad c=\begin{pmatrix} -i & 0\\ 0 & -i\end{pmatrix}\,,\\
\mathbf{2}': &\quad a=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}, \quad b=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \quad c=\begin{pmatrix} i & 0\\ 0 & i\end{pmatrix}\,.
\end{aligned} \tag{B.1}$$

The character table is shown in table 8. The product of representations decomposes as follows:

$$\begin{aligned}
\mathbf{1}_{mnp}\otimes\mathbf{1}_{m'n'p'} &= \mathbf{1}_{(mm')(nn')(pp')}\,,\\
\mathbf{2}\otimes\mathbf{2}=\mathbf{2}'\otimes\mathbf{2}' &= \mathbf{1}_{---}\oplus\mathbf{1}_{+--}\oplus\mathbf{1}_{-+-}\oplus\mathbf{1}_{++-}\,,\\
\mathbf{2}\otimes\mathbf{2}' &= \mathbf{1}_{+++}\oplus\mathbf{1}_{-++}\oplus\mathbf{1}_{+-+}\oplus\mathbf{1}_{--+}\,,\\
\begin{pmatrix}\mathbf{2}\\ \mathbf{2}'\end{pmatrix}\otimes(\mathbf{1}_{+++},\mathbf{1}_{-++},\mathbf{1}_{+-+},\mathbf{1}_{--+}) &= \begin{pmatrix}\mathbf{2}\\ \mathbf{2}'\end{pmatrix}\,,\\
\begin{pmatrix}\mathbf{2}\\ \mathbf{2}'\end{pmatrix}\otimes(\mathbf{1}_{---},\mathbf{1}_{+--},\mathbf{1}_{-+-},\mathbf{1}_{++-}) &= \begin{pmatrix}\mathbf{2}'\\ \mathbf{2}\end{pmatrix}\,.
\end{aligned} \tag{B.2}$$

The tensor products between singlets and doublets are given by

$$\begin{aligned}
\mathbf{1}_{+++}\otimes\mathbf{2}=\mathbf{2}=\alpha\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}, &\qquad \mathbf{1}_{+++}\otimes\mathbf{2}'=\mathbf{2}'=\alpha\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix},\\
\mathbf{1}_{++-}\otimes\mathbf{2}=\mathbf{2}'=\alpha\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}, &\qquad \mathbf{1}_{++-}\otimes\mathbf{2}'=\mathbf{2}=\alpha\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix},\\
\mathbf{1}_{+--}\otimes\mathbf{2}=\mathbf{2}'=\alpha\begin{pmatrix}\beta_1\\ -\beta_2\end{pmatrix}, &\qquad \mathbf{1}_{+--}\otimes\mathbf{2}'=\mathbf{2}=\alpha\begin{pmatrix}\beta_1\\ -\beta_2\end{pmatrix},\\
\mathbf{1}_{+-+}\otimes\mathbf{2}=\mathbf{2}=\alpha\begin{pmatrix}\beta_1\\ -\beta_2\end{pmatrix}, &\qquad \mathbf{1}_{+-+}\otimes\mathbf{2}'=\mathbf{2}'=\alpha\begin{pmatrix}\beta_1\\ -\beta_2\end{pmatrix},\\
\mathbf{1}_{---}\otimes\mathbf{2}=\mathbf{2}'=\alpha\begin{pmatrix}\beta_2\\ -\beta_1\end{pmatrix}, &\qquad \mathbf{1}_{---}\otimes\mathbf{2}'=\mathbf{2}=\alpha\begin{pmatrix}\beta_2\\ -\beta_1\end{pmatrix},
\end{aligned}$$

JHEP05(2024)052

$$\mathbf{1}_{--+} \otimes \mathbf{2} = \mathbf{2} = \alpha \begin{pmatrix} \beta_2 \\ -\beta_1 \end{pmatrix}, \qquad \mathbf{1}_{--+} \otimes \mathbf{2}' = \mathbf{2}' = \alpha \begin{pmatrix} \beta_2 \\ -\beta_1 \end{pmatrix},$$
$$\mathbf{1}_{-+-} \otimes \mathbf{2} = \mathbf{2}' = \alpha \begin{pmatrix} \beta_2 \\ \beta_1 \end{pmatrix}, \qquad \mathbf{1}_{-+-} \otimes \mathbf{2}' = \mathbf{2} = \alpha \begin{pmatrix} \beta_2 \\ \beta_1 \end{pmatrix},$$
$$\mathbf{1}_{-++} \otimes \mathbf{2} = \mathbf{2} = \alpha \begin{pmatrix} \beta_2 \\ \beta_1 \end{pmatrix}, \qquad \mathbf{1}_{-++} \otimes \mathbf{2}' = \mathbf{2}' = \alpha \begin{pmatrix} \beta_2 \\ \beta_1 \end{pmatrix}. \tag{B.3}$$

The tensor products between two doublets are

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1}_{+--} \oplus \mathbf{1}_{---} \oplus \mathbf{1}_{-+-} \oplus \mathbf{1}_{++-}, \qquad \begin{cases} \mathbf{1}_{+--} = \alpha_1\beta_2 - \alpha_2\beta_1 \\ \mathbf{1}_{---} = \alpha_1\beta_1 - \alpha_2\beta_2 \\ \mathbf{1}_{-+-} = \alpha_1\beta_1 + \alpha_2\beta_2 \\ \mathbf{1}_{++-} = \alpha_1\beta_2 + \alpha_2\beta_1 \end{cases},$$
$$\mathbf{2} \otimes \mathbf{2}' = \mathbf{1}_{+++} \oplus \mathbf{1}_{+-+} \oplus \mathbf{1}_{--+} \oplus \mathbf{1}_{-++}, \qquad \begin{cases} \mathbf{1}_{+++} = \alpha_1\beta_2 + \alpha_2\beta_1 \\ \mathbf{1}_{+-+} = \alpha_1\beta_2 - \alpha_2\beta_1 \\ \mathbf{1}_{--+} = \alpha_1\beta_1 - \alpha_2\beta_2 \\ \mathbf{1}_{-++} = \alpha_1\beta_1 + \alpha_2\beta_2 \end{cases},$$
$$\mathbf{2}' \otimes \mathbf{2}' = \mathbf{1}_{+--} \oplus \mathbf{1}_{---} \oplus \mathbf{1}_{-+-} \oplus \mathbf{1}_{++-}, \qquad \begin{cases} \mathbf{1}_{+--} = \alpha_1\beta_2 - \alpha_2\beta_1 \\ \mathbf{1}_{---} = \alpha_1\beta_1 - \alpha_2\beta_2 \\ \mathbf{1}_{-+-} = \alpha_1\beta_1 + \alpha_2\beta_2 \\ \mathbf{1}_{++-} = \alpha_1\beta_2 + \alpha_2\beta_1 \end{cases}, \tag{B.4}$$

C Group theory of $D_4 \times Z_3$

The group $D_4 \times Z_3$ is generated by the elements a and b and c, satisfying $a^4 = b^2 = (ab)^2 = c^3 = 1$, $ca = ac$, $cb = bc$. The elements (a, b) generate D_4 and c generates Z_3. This group has 15 conjugacy classes as follows,

$$\begin{aligned} 1C_1 &= \{1\}\,, \\ 1C_2 &= \left\{a^2\right\}\,, \\ 1C_3 &= \{c\}\,, \\ 1C_3' &= \left\{c^2\right\}\,, \\ 1C_6 &= \left\{a^2c\right\}\,, \\ 1C_6' &= \left\{a^2c^2\right\}\,, \\ 2C_2 &= \left\{b, a^2b\right\}\,, \\ 2C_2' &= \left\{ab, a^3b\right\}\,, \\ 2C_6 &= \left\{bc, a^2bc\right\}\,, \\ 2C_6' &= \left\{bc^2, a^2bc^2\right\}\,, \\ 2C_6'' &= \left\{abc, a^3bc\right\}\,, \end{aligned}$$

	$1C_1$	$1C_2$	$1C_3$	$1C'_3$	$1C_6$	$1C'_6$	$2C_2$	$2C'_2$	$2C_6$	$2C'_6$	$2C''_6$	$2C^{(3)}_6$	$2C_4$	$2C_{12}$	$2C'_{12}$
$\mathbf{1}_{++0}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_{++1}$	1	1	ω	ω^2	ω	ω^2	1	1	ω	ω^2	ω	ω^2	1	ω	ω^2
$\mathbf{1}_{++2}$	1	1	ω^2	ω	ω^2	ω	1	1	ω^2	ω	ω^2	ω	1	ω^2	ω
$\mathbf{1}_{+-0}$	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1
$\mathbf{1}_{+-1}$	1	1	ω	ω^2	ω	ω^2	-1	-1	$-\omega$	$-\omega^2$	$-\omega$	$-\omega^2$	1	ω	ω^2
$\mathbf{1}_{+-2}$	1	1	ω^2	ω	ω^2	ω	-1	-1	$-\omega^2$	$-\omega$	$-\omega^2$	$-\omega$	1	ω^2	ω
$\mathbf{1}_{-+0}$	1	1	1	1	1	1	1	-1	1	1	-1	-1	-1	-1	-1
$\mathbf{1}_{-+1}$	1	1	ω	ω^2	ω	ω^2	1	-1	ω	ω^2	$-\omega$	$-\omega^2$	-1	$-\omega$	$-\omega^2$
$\mathbf{1}_{-+2}$	1	1	ω^2	ω	ω^2	ω	1	-1	ω^2	ω	$-\omega^2$	$-\omega$	-1	$-\omega^2$	$-\omega$
$\mathbf{1}_{--0}$	1	1	1	1	1	1	-1	1	-1	-1	1	1	-1	-1	-1
$\mathbf{1}_{--1}$	1	1	ω	ω^2	ω	ω^2	-1	1	$-\omega$	$-\omega^2$	ω	ω^2	-1	$-\omega$	$-\omega^2$
$\mathbf{1}_{--2}$	1	1	ω^2	ω	ω^2	ω	-1	1	$-\omega^2$	$-\omega$	ω^2	ω	-1	$-\omega^2$	$-\omega$
$\mathbf{2}_0$	2	-2	2	2	-2	-2	0	0	0	0	0	0	0	0	0
$\mathbf{2}_1$	2	-2	2ω	$2\omega^2$	-2ω	$-2\omega^2$	0	0	0	0	0	0	0	0	0
$\mathbf{2}_2$	2	-2	$2\omega^2$	2ω	$-2\omega^2$	-2ω	0	0	0	0	0	0	0	0	0

Table 9. Character table of $D_4 \times Z_3$.

$$\begin{aligned} 2C^{(3)}_6 &= \left\{abc^2, a^3bc^2\right\}\,, \\ 2C_4 &= \left\{a, a^3\right\}\,, \\ 2C_{12} &= \left\{ac, a^3c\right\}\,, \\ 2C'_{12} &= \left\{ac^2, a^3c^2\right\}\,. \end{aligned} \tag{C.1}$$

The irreducible representations of $D_4 \times Z_3$ are 12 singlets and 3 doublets, the representation matrices of the generators are

$$\mathbf{1}_{++n} : a = +1, \qquad b = +1, \qquad c = \omega^n\,, \tag{C.2}$$

$$\mathbf{1}_{+-n} : a = +1, \qquad b = -1, \qquad c = \omega^n\,, \tag{C.3}$$

$$\mathbf{1}_{-+n} : a = -1, \qquad b = +1, \qquad c = \omega^n\,, \tag{C.4}$$

$$\mathbf{1}_{--n} : a = -1, \qquad b = -1, \qquad c = \omega^n\,, \tag{C.5}$$

$$\mathbf{2}_n : a = \begin{pmatrix} +i & 0 \\ 0 & -i \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} \omega^n & 0 \\ 0 & \omega^n \end{pmatrix}, \tag{C.6}$$

with $n = 0, 1, 2$ and $\omega = e^{2\pi i/3}$. The character table is listed in table 9.

The Kronecker products of different representations are as follows:

$$\begin{aligned} &\mathbf{1}_{mnp} \otimes \mathbf{1}_{m'n'p'} = \mathbf{1}_{(mm')(nn')[p+p']}\,, \quad \mathbf{1}_{mnp} \otimes \mathbf{2}_{p'} = \mathbf{2}_{[p+p']}\,, \\ &\mathbf{2}_p \otimes \mathbf{2}_{p'} = \mathbf{1}_{++[p+p']} \oplus \mathbf{1}_{+-[p+p']} \oplus \mathbf{1}_{-+[p+p']} \oplus \mathbf{1}_{--[p+p']}\,, \end{aligned} \tag{C.7}$$

where the integer $[n] \equiv n \pmod 3$. The tensor products between singlets and doublets are given by

$$\begin{aligned} &\mathbf{1}_{++p} \otimes \mathbf{2}_{p'} = \mathbf{2}_{[p+p']} = \alpha \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \qquad \mathbf{1}_{+-p} \otimes \mathbf{2}_{p'} = \mathbf{2}_{[p+p']} = \alpha \begin{pmatrix} \beta_1 \\ -\beta_2 \end{pmatrix}, \\ &\mathbf{1}_{-+p} \otimes \mathbf{2}_{p'} = \mathbf{2}_{[p+p']} = \alpha \begin{pmatrix} \beta_2 \\ \beta_1 \end{pmatrix}, \qquad \mathbf{1}_{--p} \otimes \mathbf{2}_{p'} = \mathbf{2}_{[p+p']} = \alpha \begin{pmatrix} \beta_2 \\ -\beta_1 \end{pmatrix}. \end{aligned} \tag{C.8}$$

JHEP05(2024)052

The tensor products between the doublets are given by

$$\mathbf{2}_p \otimes \mathbf{2}_{p'} = \mathbf{1}_{++[p+p']} \oplus \mathbf{1}_{+-[p+p']} \oplus \mathbf{1}_{-+[p+p']} \oplus \mathbf{1}_{--[p+p']}, \qquad \begin{cases} \mathbf{1}_{++[p+p']} = \alpha_1\beta_2 + \alpha_2\beta_1 \\ \mathbf{1}_{+-[p+p']} = \alpha_1\beta_2 - \alpha_2\beta_1 \\ \mathbf{1}_{-+[p+p']} = \alpha_1\beta_1 + \alpha_2\beta_2 \\ \mathbf{1}_{--[p+p']} = \alpha_1\beta_1 - \alpha_2\beta_2 \end{cases} . \tag{C.9}$$

An equivalent basis (a', b', c') for the $D_4 \times Z_3$ generators is obtained by the similarity transformation acting on doublets:

$$a' = U^{-1} a\, U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad b' = U^{-1} b\, U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad c' = U^{-1} c\, U = c, \tag{C.10}$$

where U is the matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}. \tag{C.11}$$

Singlet representations are unchanged in this new basis, where the consistency conditions of CP transformation in eq. (4.32) are solved by $X_I' = \mathbb{1}$, both for singlets and doublets.

D Lepton model of ref. [74]

To verify the above general analysis, we take the lepton model in ref. [74] as an example,

$$\begin{aligned} &\rho_{E^c} = \mathbf{2} \oplus \mathbf{1}, \quad \rho_L = \mathbf{3}', \quad \rho_{H_u} = \rho_{H_d} = \mathbf{1}\,, \\ &k_{H_u} = k_{H_d} = 0, \quad k_{E^c_D} = -3, \quad k_{E^c_3} = k_L = -1\,. \end{aligned} \tag{D.1}$$

In this model, the superpotential w is required to be modular invariant, i.e. $\boldsymbol{r}_s = \mathbf{1}$. The weighted representation matrices can be obtained from the appendix of ref. [74], which happen to have the following diagonal form:

$$\Omega_L(a) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Omega_L(b) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{D.2}$$

$$\Omega_{E^c}(a) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Omega_{E^c}(b) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{D.3}$$

Namely, they are decomposed under D_4 into

$$\Omega_L \sim \mathbf{1}'_+ \oplus \mathbf{1}_- \oplus \mathbf{1}_-\,, \qquad \Omega_{E^c} \sim \mathbf{1}'_+ \oplus \mathbf{1}_- \oplus \mathbf{1}_-\,, \tag{D.4}$$

they indeed belong to one of the four cases in eq. (4.43). The linearized moduli $u_{1,3}$ are transformed as $\mathbf{1}'_- \oplus \mathbf{1}_+$. As we analyzed in section 5, the charged lepton mass matrix m_e

JHEP05(2024)052

and neutrino mass matrix m_ν have the pattern **A** of eq. (5.5):

$$m_{\bar{e}e}(u_1,u_3)\simeq\begin{pmatrix}1+u_3+u_3^2+u_1^2 & u_1+u_1u_3 & u_1+u_1u_3\\ u_1+u_1u_3 & 1+u_3+u_3^2+u_1^2 & 1+u_3+u_3^2+u_1^2\\ u_1+u_1u_3 & 1+u_3+u_3^2+u_1^2 & 1+u_3+u_3^2+u_1^2\end{pmatrix}+\mathcal{O}(u_i^3)$$
$$\simeq\begin{pmatrix}1 & u_1 & u_1\\ u_1 & 1 & 1\\ u_1 & 1 & 1\end{pmatrix}+\dots. \tag{D.5}$$

$$m_\nu(u_1,u_3)\simeq\begin{pmatrix}1+u_3+u_3^2+u_1^2 & u_1+u_1u_3 & u_1+u_1u_3\\ u_1+u_1u_3 & 1+u_3+u_3^2+u_1^2 & 1+u_3+u_3^2+u_1^2\\ u_1+u_1u_3 & 1+u_3+u_3^2+u_1^2 & 1+u_3+u_3^2+u_1^2\end{pmatrix}+\mathcal{O}(u_i^3)$$
$$\simeq\begin{pmatrix}1 & u_1 & u_1\\ u_1 & 1 & 1\\ u_1 & 1 & 1\end{pmatrix}+\dots. \tag{D.6}$$

Here we omit the unknown constant coefficients. Through the u-expansion of the Siegel modular forms:

$$\begin{aligned}
&((1-u_1)^2-u_3^2)^{-2}Y^{(2)}_{\mathbf{3},1}(u)=15.68u_1+\mathcal{O}(u_i^3)\,,\\
&((1-u_1)^2-u_3^2)^{-2}Y^{(2)}_{\mathbf{3},2}(u)=1.21+25.49u_1^2-6.37u_3^2+\mathcal{O}(u_i^3)\,,\\
&((1-u_1)^2-u_3^2)^{-2}Y^{(2)}_{\mathbf{3},3}(u)=-2.56+5.54u_3-27.04u_1^2+\mathcal{O}(u_i^3)\,,\\
&((1-u_1)^2-u_3^2)^{-2}Y^{(2)}_{\mathbf{1}}(u)=1.81+3.92u_3+19.12u_3^2+19.12u_1^2+\mathcal{O}(u_i^3)\,.
\end{aligned}\tag{D.7}$$

We can obtain the u-expansion form of mass matrices in ref. [74],

$$\begin{aligned}
m_e(u_1,u_3)&\simeq\begin{pmatrix}2.57 & 71.84u_1 & -101.60u_1\\ -71.84u_1 & 23.12 & 18.17\\ 0.18u_1 & 0.014 & -0.03\end{pmatrix}v_d+\dots.\\
m_\nu(u_1,u_3)&\simeq\begin{pmatrix}2.01 & 22.17u_1 & 15.68u_1\\ 22.17u_1 & -1.40 & 1.21\\ 15.68u_1 & 1.21 & 7.97\end{pmatrix}\frac{v_u^2}{\Lambda}+\dots,
\end{aligned}\tag{D.8}$$

where the free Lagrangian parameters are fixed at their best-fit values.[10] This result is consistent with our above general analysis. Since this pattern is not optimal for reproducing the data, the coefficients multiplying u_1 are not of the same order, but appear to be tuned to overcome the bad scaling shown in table 6.

[10] The best-fit values for Lagrangian parameters are $\alpha=1,\ \beta=-0.83991,\ \gamma=0.01176,\ g_1=1,\ g_2=1.58030$. And the best-fit values of the moduli vacua are $\tau_1=\tau_2=-0.03376+1.11329i, \tau_3=-0.02376+0.50670i$, i.e. $u_1=u_2=-0.00418+0.01298i, u_3=-0.02895+0.00474i$.

References

[1] F. Feruglio, *Are neutrino masses modular forms?*, in *From My Vast Repertoire …: Guido Altarelli's Legacy*, A. Levy et al. eds., World Scientific (2019), p. 227–266 [DOI:10.1142/9789813238053_0012] [arXiv:1706.08749] [INSPIRE].

[2] S. Ferrara, D. Lust, A.D. Shapere and S. Theisen, *Modular Invariance in Supersymmetric Field Theories*, *Phys. Lett. B* **225** (1989) 363 [INSPIRE].

[3] S. Ferrara, D. Lust and S. Theisen, *Target Space Modular Invariance and Low-Energy Couplings in Orbifold Compactifications*, *Phys. Lett. B* **233** (1989) 147 [INSPIRE].

[4] J.C. Criado and F. Feruglio, *Modular Invariance Faces Precision Neutrino Data*, *SciPost Phys.* **5** (2018) 042 [arXiv:1807.01125] [INSPIRE].

[5] F. Feruglio, A. Strumia and A. Titov, *Modular invariance and the QCD angle*, *JHEP* **07** (2023) 027 [arXiv:2305.08908] [INSPIRE].

[6] S. Hamidi and C. Vafa, *Interactions on Orbifolds*, *Nucl. Phys. B* **279** (1987) 465 [INSPIRE].

[7] L.J. Dixon, D. Friedan, E.J. Martinec and S.H. Shenker, *The Conformal Field Theory of Orbifolds*, *Nucl. Phys. B* **282** (1987) 13 [INSPIRE].

[8] J. Lauer, J. Mas and H.P. Nilles, *Duality and the Role of Nonperturbative Effects on the World Sheet*, *Phys. Lett. B* **226** (1989) 251 [INSPIRE].

[9] J. Lauer, J. Mas and H.P. Nilles, *Twisted sector representations of discrete background symmetries for two-dimensional orbifolds*, *Nucl. Phys. B* **351** (1991) 353 [INSPIRE].

[10] J. Erler, D. Jungnickel and J. Lauer, *Dependence of Yukawa couplings on the axionic background moduli of Z(N) orbifolds*, *Phys. Rev. D* **45** (1992) 3651 [INSPIRE].

[11] S. Stieberger, *Moduli and twisted sector dependence on $Z_N \times Z_M$ orbifold couplings*, *Phys. Lett. B* **300** (1993) 347 [hep-th/9211027] [INSPIRE].

[12] J. Erler, D. Jungnickel, M. Spalinski and S. Stieberger, *Higher twisted sector couplings of Z_N orbifolds*, *Nucl. Phys. B* **397** (1993) 379 [hep-th/9207049] [INSPIRE].

[13] S. Stieberger, D. Jungnickel, J. Lauer and M. Spalinski, *Yukawa couplings for bosonic Z(N) orbifolds: Their moduli and twisted sector dependence*, *Mod. Phys. Lett. A* **7** (1992) 3059 [hep-th/9204037] [INSPIRE].

[14] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys. Rept.* **244** (1994) 77 [hep-th/9401139] [INSPIRE].

[15] D. Cremades, L.E. Ibanez and F. Marchesano, *Yukawa couplings in intersecting D-brane models*, *JHEP* **07** (2003) 038 [hep-th/0302105] [INSPIRE].

[16] R. Blumenhagen, M. Cvetic, P. Langacker and G. Shiu, *Toward realistic intersecting D-brane models*, *Ann. Rev. Nucl. Part. Sci.* **55** (2005) 71 [hep-th/0502005] [INSPIRE].

[17] S.A. Abel and M.D. Goodsell, *Realistic Yukawa Couplings through Instantons in Intersecting Brane Worlds*, *JHEP* **10** (2007) 034 [hep-th/0612110] [INSPIRE].

[18] R. Blumenhagen, B. Kors, D. Lust and S. Stieberger, *Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes*, *Phys. Rept.* **445** (2007) 1 [hep-th/0610327] [INSPIRE].

[19] F. Marchesano, *Progress in D-brane model building*, *Fortsch. Phys.* **55** (2007) 491 [hep-th/0702094] [INSPIRE].

[20] I. Antoniadis, A. Kumar and B. Panda, *Fermion Wavefunctions in Magnetized branes: Theta identities and Yukawa couplings*, *Nucl. Phys. B* **823** (2009) 116 [arXiv:0904.0910] [INSPIRE].

[21] T. Kobayashi, S. Nagamoto and S. Uemura, *Modular symmetry in magnetized/intersecting D-brane models*, *PTEP* **2017** (2017) 023B02 [arXiv:1608.06129] [INSPIRE].

[22] T. Kobayashi and H. Otsuka, *Classification of discrete modular symmetries in Type IIB flux vacua*, *Phys. Rev. D* **101** (2020) 106017 [arXiv:2001.07972] [INSPIRE].

[23] D. Cremades, L.E. Ibanez and F. Marchesano, *Computing Yukawa couplings from magnetized extra dimensions*, *JHEP* **05** (2004) 079 [hep-th/0404229] [INSPIRE].

[24] H. Abe, K.-S. Choi, T. Kobayashi and H. Ohki, *Non-Abelian Discrete Flavor Symmetries from Magnetized/Intersecting Brane Models*, *Nucl. Phys. B* **820** (2009) 317 [arXiv:0904.2631] [INSPIRE].

[25] S. Kikuchi et al., *Revisiting modular symmetry in magnetized torus and orbifold compactifications*, *Phys. Rev. D* **102** (2020) 105010 [arXiv:2005.12642] [INSPIRE].

[26] S. Kikuchi et al., *Modular symmetry by orbifolding magnetized $T^2 \times T^2$: realization of double cover of Γ_N*, *JHEP* **11** (2020) 101 [arXiv:2007.06188] [INSPIRE].

[27] S. Ramos-Sánchez and M. Ratz, *Heterotic Orbifold Models*, arXiv:2401.03125 [INSPIRE].

[28] H.P. Nilles and S. Ramos-Sánchez, *The Flavor Puzzle: Textures and Symmetries*, arXiv:2308.14810 [INSPIRE].

[29] T. Kobayashi and M. Tanimoto, *Modular flavor symmetric models*, arXiv:2307.03384 [INSPIRE].

[30] G.-J. Ding and S.F. King, *Neutrino Mass and Mixing with Modular Symmetry*, arXiv:2311.09282 [INSPIRE].

[31] M.-C. Chen, S. Ramos-Sánchez and M. Ratz, *A note on the predictions of models with modular flavor symmetries*, *Phys. Lett. B* **801** (2020) 135153 [arXiv:1909.06910] [INSPIRE].

[32] A. Baur, H.P. Nilles, A. Trautner and P.K.S. Vaudrevange, *Unification of Flavor, CP, and Modular Symmetries*, *Phys. Lett. B* **795** (2019) 7 [arXiv:1901.03251] [INSPIRE].

[33] H.P. Nilles, S. Ramos-Sánchez and P.K.S. Vaudrevange, *Eclectic Flavor Groups*, *JHEP* **02** (2020) 045 [arXiv:2001.01736] [INSPIRE].

[34] H.P. Nilles, S. Ramos-Sánchez and P.K.S. Vaudrevange, *Lessons from eclectic flavor symmetries*, *Nucl. Phys. B* **957** (2020) 115098 [arXiv:2004.05200] [INSPIRE].

[35] H.P. Nilles, S. Ramos-Sánchez and P.K.S. Vaudrevange, *Eclectic flavor scheme from ten-dimensional string theory — I. Basic results*, *Phys. Lett. B* **808** (2020) 135615 [arXiv:2006.03059] [INSPIRE].

[36] A. Baur et al., *The eclectic flavor symmetry of the $\mathbb{Z}_2$ orbifold*, *JHEP* **02** (2021) 018 [arXiv:2008.07534] [INSPIRE].

[37] H.P. Nilles, S. Ramos-Sánchez and P.K.S. Vaudrevange, *Eclectic flavor scheme from ten-dimensional string theory — II detailed technical analysis*, *Nucl. Phys. B* **966** (2021) 115367 [arXiv:2010.13798] [INSPIRE].

[38] A. Baur et al., *Completing the eclectic flavor scheme of the $\mathbb{Z}_2$ orbifold*, *JHEP* **06** (2021) 110 [arXiv:2104.03981] [INSPIRE].

[39] M.-C. Chen et al., *Quasi-eclectic modular flavor symmetries*, *Phys. Lett. B* **824** (2022) 136843 [arXiv:2108.02240] [INSPIRE].

JHEP05(2024)052

[40] A. Baur et al., *The first string-derived eclectic flavor model with realistic phenomenology*, JHEP **09** (2022) 224 [arXiv:2207.10677] [INSPIRE].

[41] G.-J. Ding et al., *Neutrino mass and mixing models with eclectic flavor symmetry* $\Delta(27) \rtimes T'$, JHEP **05** (2023) 144 [arXiv:2303.02071] [INSPIRE].

[42] C.-C. Li and G.-J. Ding, *Eclectic flavor group* $\Delta(27) \rtimes S_3$ *and lepton model building*, JHEP **03** (2024) 054 [arXiv:2308.16901] [INSPIRE].

[43] F. Feruglio, *Universal Predictions of Modular Invariant Flavor Models near the Self-Dual Point*, Phys. Rev. Lett. **130** (2023) 101801 [arXiv:2211.00659] [INSPIRE].

[44] F. Feruglio, *Fermion masses, critical behavior and universality*, JHEP **03** (2023) 236 [arXiv:2302.11580] [INSPIRE].

[45] A. Font, L.E. Ibanez, D. Lust and F. Quevedo, *Supersymmetry Breaking From Duality Invariant Gaugino Condensation*, Phys. Lett. B **245** (1990) 401 [INSPIRE].

[46] M. Cvetic et al., *Target space duality, supersymmetry breaking and the stability of classical string vacua*, Nucl. Phys. B **361** (1991) 194 [INSPIRE].

[47] P.P. Novichkov, J.T. Penedo and S.T. Petcov, *Modular flavour symmetries and modulus stabilisation*, JHEP **03** (2022) 149 [arXiv:2201.02020] [INSPIRE].

[48] J.M. Leedom, N. Righi and A. Westphal, *Heterotic de Sitter beyond modular symmetry*, JHEP **02** (2023) 209 [arXiv:2212.03876] [INSPIRE].

[49] K. Ishiguro, T. Kobayashi and H. Otsuka, *Landscape of Modular Symmetric Flavor Models*, JHEP **03** (2021) 161 [arXiv:2011.09154] [INSPIRE].

[50] K. Ishiguro, H. Okada and H. Otsuka, *Residual flavor symmetry breaking in the landscape of modular flavor models*, JHEP **09** (2022) 072 [arXiv:2206.04313] [INSPIRE].

[51] V. Knapp-Pérez et al., *Matter matters in moduli fixing and modular flavor symmetries*, Phys. Lett. B **844** (2023) 138106 [arXiv:2304.14437] [INSPIRE].

[52] K. Ishiguro, T. Kai, T. Kobayashi and H. Otsuka, *Flux Landscape with enhanced symmetry not on* $SL(2,\mathbb{Z})$ *elliptic points*, JHEP **02** (2024) 099 [arXiv:2311.12425] [INSPIRE].

[53] L. Kofman et al., *Beauty is attractive: Moduli trapping at enhanced symmetry points*, JHEP **05** (2004) 030 [hep-th/0403001] [INSPIRE].

[54] S. Enomoto, S. Iida, N. Maekawa and T. Matsuda, *Beauty is more attractive: particle production and moduli trapping with higher dimensional interaction*, JHEP **01** (2014) 141 [arXiv:1310.4751] [INSPIRE].

[55] S. Kikuchi, T. Kobayashi, K. Nasu and Y. Yamada, *Moduli trapping mechanism in modular flavor symmetric models*, JHEP **08** (2023) 081 [arXiv:2307.13230] [INSPIRE].

[56] F. Feruglio, V. Gherardi, A. Romanino and A. Titov, *Modular invariant dynamics and fermion mass hierarchies around* $\tau = i$, JHEP **05** (2021) 242 [arXiv:2101.08718] [INSPIRE].

[57] P.P. Novichkov, J.T. Penedo and S.T. Petcov, *Fermion mass hierarchies, large lepton mixing and residual modular symmetries*, JHEP **04** (2021) 206 [arXiv:2102.07488] [INSPIRE].

[58] S.T. Petcov and M. Tanimoto, A_4 *modular flavour model of quark mass hierarchies close to the fixed point* $\tau = \omega$, Eur. Phys. J. C **83** (2023) 579 [arXiv:2212.13336] [INSPIRE].

[59] S. Kikuchi et al., *Quark hierarchical structures in modular symmetric flavor models at level 6*, Phys. Rev. D **107** (2023) 055014 [arXiv:2301.03737] [INSPIRE].

[60] Y. Abe, T. Higaki, J. Kawamura and T. Kobayashi, *Quark masses and CKM hierarchies from S_4' modular flavor symmetry*, *Eur. Phys. J. C* **83** (2023) 1140 [arXiv:2301.07439] [INSPIRE].

[61] S. Kikuchi et al., *Quark mass hierarchies and CP violation in $A_4 \times A_4 \times A_4$ modular symmetric flavor models*, *JHEP* **07** (2023) 134 [arXiv:2302.03326] [INSPIRE].

[62] Y. Abe, T. Higaki, J. Kawamura and T. Kobayashi, *Quark and lepton hierarchies from S_4' modular flavor symmetry*, *Phys. Lett. B* **842** (2023) 137977 [arXiv:2302.11183] [INSPIRE].

[63] S.T. Petcov and M. Tanimoto, *A_4 modular flavour model of quark mass hierarchies close to the fixed point $\tau = i\infty$*, *JHEP* **08** (2023) 086 [arXiv:2306.05730] [INSPIRE].

[64] Y. Abe, T. Higaki, J. Kawamura and T. Kobayashi, *Fermion hierarchies in* SU(5) *grand unification from Γ_6' modular flavor symmetry*, *JHEP* **08** (2023) 097 [arXiv:2307.01419] [INSPIRE].

[65] I. de Medeiros Varzielas, M. Levy, J.T. Penedo and S.T. Petcov, *Quarks at the modular S_4 cusp*, *JHEP* **09** (2023) 196 [arXiv:2307.14410] [INSPIRE].

[66] S.T. Petcov, *On the normalisation of the modular forms in modular invariant theories of flavour*, *Phys. Lett. B* **850** (2024) 138540 [arXiv:2311.04185] [INSPIRE].

[67] K. Ishiguro, T. Kobayashi, S. Nishimura and H. Otsuka, *Modular forms and hierarchical Yukawa couplings in heterotic Calabi-Yau compactifications*, arXiv:2402.13563 [INSPIRE].

[68] P. Mayr and S. Stieberger, *Moduli dependence of one loop gauge couplings in* $(0,2)$ *compactifications*, *Phys. Lett. B* **355** (1995) 107 [hep-th/9504129] [INSPIRE].

[69] S. Stieberger, $(0,2)$ *heterotic gauge couplings and their M theory origin*, *Nucl. Phys. B* **541** (1999) 109 [hep-th/9807124] [INSPIRE].

[70] K. Ishiguro, T. Kobayashi and H. Otsuka, *Spontaneous CP violation and symplectic modular symmetry in Calabi-Yau compactifications*, *Nucl. Phys. B* **973** (2021) 115598 [arXiv:2010.10782] [INSPIRE].

[71] A. Baur et al., *Siegel modular flavor group and CP from string theory*, *Phys. Lett. B* **816** (2021) 136176 [arXiv:2012.09586] [INSPIRE].

[72] K. Ishiguro, T. Kobayashi and H. Otsuka, *Symplectic modular symmetry in heterotic string vacua: flavor, CP, and R-symmetries*, *JHEP* **01** (2022) 020 [arXiv:2107.00487] [INSPIRE].

[73] G.-J. Ding, F. Feruglio and X.-G. Liu, *Automorphic Forms and Fermion Masses*, *JHEP* **01** (2021) 037 [arXiv:2010.07952] [INSPIRE].

[74] G.-J. Ding, F. Feruglio and X.-G. Liu, *CP symmetry and symplectic modular invariance*, *SciPost Phys.* **10** (2021) 133 [arXiv:2102.06716] [INSPIRE].

[75] E. Gottschling, *Über die Fixpunkte der Siegelschen Modulgruppe*, *Math. Ann.* **143** (1961) 111.

[76] E. Gottschling, *Über die Fixpunktuntergruppen der Siegelschen Modulgruppe*, *Math. Ann.* **143** (1961) 399.

[77] E. Gottschling, *Die Uniformisierbarkeit der Fixpunkte eigentlich diskontinuierlicher Gruppen von biholomorphen Abbildungen*, *Math. Ann.* **169** (1967) 26.

[78] M. Dine, R.G. Leigh and D.A. MacIntire, *Of CP and other gauge symmetries in string theory*, *Phys. Rev. Lett.* **69** (1992) 2030 [hep-th/9205011] [INSPIRE].

[79] K.-W. Choi, D.B. Kaplan and A.E. Nelson, *Is CP a gauge symmetry?*, *Nucl. Phys. B* **391** (1993) 515 [hep-ph/9205202] [INSPIRE].

JHEP05(2024)052

[80] R.G. Leigh, *The Strong CP problem, string theory and the Nelson-Barr mechanism*, in the proceedings of the *International Workshop on Recent Advances in the Superworld*, Woodlands, U.S.A., April 13–16 (1993) [hep-ph/9307214] [INSPIRE].

[81] M. Koecher, *Zur Theorie der Modulformenn-ten Grades. I*, *Math. Z.* **59** (1953) 399.

[82] A. Fiorentino, *On modular and cusp forms with respect to the congruence subgroup, over which the map given by the gradients of odd Theta functions in genus* 2 *factors, and related topics*, Ph.D. thesis, Università di Roma "La Sapienza", Roma, Italy (2013).

[83] N.M. Katz and P.H. Tiep, *Rigid local systems and finite symplectic groups*, *Finite Fields and Their Applications* **59** (2019) 134 [arXiv:2002.05863].

[84] P.P. Novichkov, J.T. Penedo, S.T. Petcov and A.V. Titov, *Generalised CP Symmetry in Modular-Invariant Models of Flavour*, *JHEP* **07** (2019) 165 [arXiv:1905.11970] [INSPIRE].

[85] A. Baur, H.P. Nilles, A. Trautner and P.K.S. Vaudrevange, *A String Theory of Flavor and $\mathcal{CP}$*, *Nucl. Phys. B* **947** (2019) 114737 [arXiv:1908.00805] [INSPIRE].

[86] X.-G. Liu, C.-Y. Yao, B.-Y. Qu and G.-J. Ding, *Half-integral weight modular forms and application to neutrino mass models*, *Phys. Rev. D* **102** (2020) 115035 [arXiv:2007.13706] [INSPIRE].

[87] C.-Y. Yao, X.-G. Liu and G.-J. Ding, *Fermion masses and mixing from the double cover and metaplectic cover of the A_5 modular group*, *Phys. Rev. D* **103** (2021) 095013 [arXiv:2011.03501] [INSPIRE].

[88] Y. Almumin et al., *Metaplectic Flavor Symmetries from Magnetized Tori*, *JHEP* **05** (2021) 078 [arXiv:2102.11286] [INSPIRE].

[89] K. Ishiguro, T. Kai, H. Okada and H. Otsuka, *Flavor, CP and metaplectic modular symmetries in Type IIB chiral flux vacua*, *JHEP* **12** (2023) 136 [arXiv:2305.19155] [INSPIRE].

[90] Y. Reyimuaji and A. Romanino, *Can an unbroken flavour symmetry provide an approximate description of lepton masses and mixing?*, *JHEP* **03** (2018) 067 [arXiv:1801.10530] [INSPIRE].

[91] S. Kikuchi et al., $\mathrm{Sp}(6, Z)$ *modular symmetry in flavor structures: quark flavor models and Siegel modular forms for* $\widetilde{\Delta}(96)$, arXiv:2310.17978 [INSPIRE].

[92] M. Ricky Devi, *Neutrino Masses and Higher Degree Siegel Modular Forms*, arXiv:2401.16257 [INSPIRE].

JHEP05(2024)052