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On scalar charges and black hole thermodynamics

Romina Ballesteros,[a,b] **Carmen Gómez-Fayrén,**[a] **Tomás Ortín**[a] **and Matteo Zatti**[a]

[a] *Instituto de Física Teórica UAM/CSIC, C/ Nicolás Cabrera, 13–15, C.U. Cantoblanco, E-28049 Madrid, Spain*

[b] *Instituto de Física, Pontificia Universidad Católica de Valparaíso, Av. Brasil 2950, Valparaíso, Chile*

E-mail: romina.ballesteros@estudiante.uam.es, carmen.gomez-fayren@estudiante.uam.es, tomas.ortin@csic.es, matteo.zatti@estudiante.uam.es

Abstract: We revisit the first law of black hole thermodynamics in 4-dimensional theories containing scalar and Abelian vector fields coupled to gravity using Wald's formalism and a new definition of scalar charge as an integral over a 2-surface which satisfies a Gauss law in the background of stationary black-hole spacetimes. We focus on ungauged supergravity-inspired theories with symmetric sigma models whose symmetries generate electric-magnetic dualities leaving invariant their equations of motion. Our manifestly duality-invariant form of the first law is compatible with the one obtained by of Gibbons, Kallosh and Kol. We also obtain the general expression for the scalar charges of a stationary black hole in terms of the other physical parameters of the solution and the position of the horizon, generalizing the expression obtained by Pacilio for dilaton black holes.

Keywords: Black Holes, Global Symmetries, Space-Time Symmetries, Black Holes in String Theory

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1 Introduction

It is widely believed that one of the defining characteristics of classical black holes is that they have no "hair". The concept of black-hole hair is a very broad one but, for the stationary black holes we will be concerned with in this paper it can be defined as any parameter that enters the metric and which cannot be eliminated through a coordinate transformation which is not a function of the charges of the theory which are conserved by virtue of a local symmetry (mass, angular momenta, electric charges) or a topological property (magnetic charges) or the asymptotic values of the scalars (*moduli*).

Scalar charges, typically defined through the asymptotic behavior at spatial infinity of the scalars in the black-hole spacetime, are not protected by any conservation law. In ungauged theories the only local symmetries scalar fields transform under are diffeomorphisms but the conserved charges associated to them are the gravitational ones: mass and linear and angular momenta. Scalar fields only transform under global symmetries of the action or of the equations of motion only to which we will refer to as dualities. However, the charges associated to those symmetries in stationary black-hole spacetimes vanish identically. They seem to have nothing to do with the conventionally-defined black-hole scalar charges. Gauging the global symmetries does not help because the gauge symmetry would be associated to some 1-form gauge fields and the conserved charges would have the interpretation of electric and magnetic charges.

Therefore, according to our definition of hair, scalar charges are understood as hair and, according to the *no-hair conjecture*, no black-hole solutions with regular horizons (henceforth to be referred to as "regular black holes") carrying scalar charges should be expected. Any scalar charges possessed by gravitationally collapsing matter should be radiated away in the black-hole formation. However, there are many regular black hole solutions carrying non-vanishing scalar charges such as dilaton black holes and their generalizations.[1]

The solution to this apparent counterexample of the no-hair conjecture lies in the distinction between primary and secondary hair [2]: in all the regular black-hole solutions with non-vanishing scalar charges, those charges are not independent parameters but very specific functions of the independent conserved charges which are allowed by the no-hair conjecture and they are (by definition) secondary hair. In the solutions in which the scalar charges are truly independent parameters, such as the Janis-Newman-Winicour solution [3] or the Agnese-La Camera solutions [4] and their generalizations [1], there are no regular horizons but naked singularities unless the scalar charge takes the value of the specific function of the conserved charges we mentioned above (simply zero in the JNW solution). This kind of scalar hair is, by definition, primary hair and it is the one which would actually be forbidden by the conjecture.

The scalar charges which are allowed by the no-hair conjecture remain, nevertheless, quite mysterious: what are the values of the scalar charges allowed in a given theory? Why are those values allowed and no others? And, even more basic: is there a coordinate-independent definition of scalar charge?

This mystery only deepened when Gibbons, Kallosh and Kol (GKK) showed in ref. [5] (see also ref. [6]) that the allowed scalar charges occur in the first law of black hole mechanics [7] as thermodynamical potentials conjugate to the variations of the moduli. While it is not clear which kind of physical process may result in a change of the moduli,[2] it is a fact that varying the black-hole entropy formulae of known solutions with respect to the moduli one finds the scalar charges as coefficients of those variations.

Wald's formalism [8–10] opened a new venue for the study of black-hole thermodynamics that can be used to explore the role of scalar charges into it. The main observation, realized in the context of purely gravitational (matter-free) theories invariant under diffeomorphisms is that the properties of the Noether $(d-2)$-form charge associated to the invariance under diffeomorphisms (*Noether-Wald charge*) can be used to prove the first law of black-hole thermodynamics.

In theories with matter, this law includes work terms proportional to the variations of conserved charges and the GKK scalar term proportional to the variations of the moduli. In the last few years we have extended the formalism to handle theories in which there are matter fields with gauge symmetries coupled to gravity showing how the electric work terms appear [11–13],[3] showing how extended black-hole thermodynamics arises in this context [24, 25], how to include magnetic charges in the first law [26] and how to construct Komar integrals from which Smarr formulae can be derived [24, 27]. In all those cases each new work term in the first laws is associated to a gauge symmetry or an equivalent topological property. Since, as we have seen, scalar charges are not associated to neither,

[1]For a review with many references, see ref. [1].

[2]The same could be said about magnetic charges.

[3]A slightly different approach to the one taken in those papers, which is the one used here as well, is the point of view of "invariance up to gauge transformations", taken in refs. [16, 19–23].

it is unclear how the GKK work term can be recovered in Wald's formalism.[4] The absence of a good coordinate-independent definition for the scalar charge complicates this problem.

In this paper we are going to show how this problem can be solved taking into account hitherto ignored contributions to the integrals at spatial infinity and using a definition of scalar charge as the integral of a $(d-2)$-form which is manifestly coordinate and gauge independent and which satisfies a Gauss law in stationary black-hole spacetimes. This definition relies in the existence of conserved charges associated to global symmetries and in the existence of a timelike Killing vector whose Killing horizon coincides with the black-hole's event horizon and whose action leaves invariant all the physical fields. Therefore, there is a scalar charge associated to each global symmetry, and, therefore, the number of charges may or may not coincide with the number of scalar fields.

In this paper we have studied 4-dimensional theories[5] whose scalar kinetic terms are described by symmetric sigma models in which the scalar fields map spacetime into a target space which is a symmetric Riemannian homogeneous space G/H. These kinetic terms are very common in supergravity theories. Furthermore, our theories include Abelian 1-forms and we are going to assume that the couplings of the scalars to those 1-forms are such that the equations of motion, enhanced with the Bianchi identities satisfied by the 2-form field strengths are invariant under the duality group G.[6] Again, this is a fairly common situation in supergravity and include simple theories such as the Einstein-Maxwell-Dilaton ones. In these theories we can associate a conserved scalar charge to each of the generators of G, even if some of the transformations (the electric-magnetic duality rotations in particular) do not leave the action invariant. As a result, according to our definition, there are always more scalar charges than scalars. Nevertheless, we are going to show that the conventional scalar charges can be recovered as combinations of the ones we have defined and we are going to check these relations in particular black-hole solutions.

In this framework we are going to proof the first law of black-hole thermodynamics recovering the GKK results and, as a byproduct, we are going to find a general expression for the scalar charges in terms of the conserved charges and the position of the horizon, thus answering one of the long-standing questions posed above.[7] Observe that, since our definition of scalar charge satisfies a Gauss Law, the value obtained for those charges is the same whether we calculate the integrals over the horizon or at infinity.

This paper is organized as follows: in section 2 we review the kind of theories that we are considering, their duality symmetries, the Gaillard-Zumino theorem [36] and the construction of the *Noether-Gaillard-Zumino* (NGZ) currents which will be used in section 3

[4] In extended thermodynamics there are work terms associated to the variation of dimensionful constants which, apparently, unrelated to gauge symmetries. However, those constants can be dualized into $(d-1)$-form potentials with a gauge freedom (for the cosmological constant, see refs. [30, 31]) and this description leads to the work terms [24, 25, 28, 32, 33].

[5] The extension to higher dimensions and higher-rank forms is straightforward using the results of ref. [34] for the Noether-Gaillard-Zumino currents.

[6] The general form of the theories that we consider is identically to that of the theories considered by GKK in ref. [5] but, in our approach it is crucial to know the global symmetries of the theory.

[7] After completion of this work, we found that a similar definition of scalar charge and similar result had been found in ref. [35] in the context of the dilaton black holes of the EMD theories.

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to define the scalar charges. In section 4 we derive the first law recovering the GKK results and the general expression of scalar charges in terms of conserved charges and the position of the horizon. In sections 5 and 6 we test our results on dilaton and axion-dilaton black holes respectively. Section 7 contains a discussion of our results.

2 The theory

In this section we are going to review the theories we are going to consider and their duality symmetries. Most of this material can be found elsewhere, but here we adapt it to our needs and conventions.

Throughout this paper we are going to consider 4-dimensional ungauged supergravity-inspired theories containing n_S scalar fields ϕ^x that parametrize a symmetric coset space G/H and n_V 1-form fields $A^\Lambda = A^\Lambda{}_\mu dx^\mu$ with 2-form field strengths

$$F^\Lambda = dA^\Lambda\,, \tag{2.1}$$

coupled to gravity which we will describe through the Vierbein $e^a = e^a{}_\mu dx^\mu$. Up to two derivatives, they can be described by the generic action

$$\begin{aligned} S &= \frac{1}{16\pi G_N^{(4)}}\int\left[-\star(e^a\wedge e^b)\wedge R_{ab} + \frac{1}{2}g_{xy}d\phi^x\wedge\star d\phi^y - \frac{1}{2}I_{\Lambda\Sigma}F^\Lambda\wedge\star F^\Sigma - \frac{1}{2}R_{\Lambda\Sigma}F^\Lambda\wedge F^\Sigma\right] \\ &\equiv \int \mathbf{L}\,, \end{aligned} \tag{2.2}$$

where the kinetic matrix $I = (I_{\Lambda\Sigma})$ is negative-definite and we are going to assume that the positive-definite σ-model metric $g_{xy}(\phi)$ is invariant under the action of G (the duality group) which also leaves invariant the set of all equations of motion plus the Bianchi identities of the theory. This assumption will be translated into conditions for the scalar-dependent matrices $I = (I_{\Lambda\Sigma})$ and $R = (R_{\Lambda\Sigma})$ shortly.

We will set $G_N^{(4)} = 1$ and we will ignore the normalization factor $(16\pi)^{-1}$ for the time being.

The equations of motion are defined by (here φ stands for all the fields of the theory)

$$\delta S = \int\left\{\mathbf{E}_a\wedge\delta e^a + \mathbf{E}_x\delta\phi^x + \mathbf{E}_\Lambda\delta A^\Lambda + d\mathbf{\Theta}(\varphi,\delta\varphi)\right\}\,, \tag{2.3}$$

and given by

$$\begin{aligned} \mathbf{E}_a &= \imath_a\star(e^b\wedge e^c)\wedge R_{bc} + \frac{1}{2}g_{xy}\left(\imath_a d\phi^x\star d\phi^y + d\phi^x\wedge\imath_a\star d\phi^y\right) \\ &\quad - \frac{1}{2}I_{\Lambda\Sigma}\left(\imath_a F^\Lambda\wedge\star F^\Sigma - F^\Lambda\wedge\imath_a\star F^\Sigma\right)\,, \end{aligned} \tag{2.4a}$$

$$\mathbf{E}_x = -g_{xy}\left\{d\star d\phi^y + \Gamma_{zw}{}^y d\phi^z\wedge\star d\phi^w\right\} - \frac{1}{2}\partial_x I_{\Lambda\Sigma}F^\Lambda\wedge\star F^\Sigma - \frac{1}{2}\partial_x R_{\Lambda\Sigma}F^\Lambda\wedge F^\Sigma\,, \tag{2.4b}$$

$$\mathbf{E}_\Lambda = dF_\Lambda\,, \tag{2.4c}$$

where we have defined the dual 2-form field strength

$$F_\Lambda \equiv I_{\Lambda\Sigma}\star F^\Sigma + R_{\Lambda\Sigma}F^\Sigma\,. \tag{2.5}$$

Furthermore,

$$\mathbf{\Theta}(\varphi,\delta\varphi) = -\star(e^a\wedge e^b)\wedge\delta\omega_{ab} + g_{xy}\star d\phi^x\delta\phi^y - F_\Lambda\wedge\delta A^\Lambda\,. \tag{2.6}$$

The original and dual 2-forms can be combined into a symplectic vector of 2-forms[8]

$$\left(F^M\right)\equiv\begin{pmatrix}F^\Lambda\\ F_\Lambda\end{pmatrix}, \tag{2.7}$$

and the Bianchi identities of the original 2-form field strength F^Λ

$$dF^\Lambda = 0\,, \tag{2.8}$$

and the Maxwell equations $\mathbf{E}_\Lambda = 0$ can be written as

$$dF^M = 0\,. \tag{2.9}$$

These equations can be interpreted as Bianchi identities implying the local existence of 1-form potentials

$$\left(A^M\right)\equiv\begin{pmatrix}A^\Lambda\\ A_\Lambda\end{pmatrix}, \tag{2.10}$$

such that

$$F^M = dA^M\,. \tag{2.11}$$

The set of equations (2.9) is invariant under arbitrary $\mathrm{GL}(2n_V,\mathbb{R})$ transformations

$$F^{M\,\prime} = S^M{}_N F^N\,, \tag{2.12}$$

but we have to take into account the rest of the equations and an important constraint: the components of F^M are not independent and, therefore, F^M satisfies the following *twisted self-duality constraint*

$$\star F^M = -\Omega^{MN}\mathcal{M}_{NP}F^P\,, \tag{2.13}$$

where $\mathcal{M}_{MN}$ is the $2n_V\times 2n_V$ symmetric symplectic matrix

$$\begin{aligned}\mathcal{M} = (\mathcal{M}_{MN}) &= \begin{pmatrix}I+RI^{-1}R & -RI^{-1}\\ & \\ -I^{-1}R & I^{-1}\end{pmatrix},\\ \mathcal{M}^{-1} = \left(\mathcal{M}^{MN}\right) &= \Omega^{-1\,T}\mathcal{M}\Omega = \begin{pmatrix}I^{-1} & I^{-1}R\\ & \\ RI^{-1} & I+RI^{-1}R\end{pmatrix},\end{aligned} \tag{2.14}$$

and

$$\Omega = (\Omega_{MN}) = \begin{pmatrix}0 & \mathbb{1}_{n_V\times n_V}\\ & \\ -\mathbb{1}_{n_V\times n_V} & 0\end{pmatrix}, \qquad \Omega^{-1} = \left(\Omega^{PN}\right). \tag{2.15}$$

[8]The symplectic nature of this vector will be proven shortly.

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As a consequence, the set of Maxwell equations and Bianchi identities will only be invariant under the subset of $\mathrm{GL}(2n_V,\mathbb{R})$ transformations that preserve this constraint, which is possible provided that $\mathcal{M}$ transforms as

$$\mathcal{M}' = \left(\Omega^{-1}S\Omega\right)\mathcal{M}S^{-1}\,, \qquad S = \left(S^M{}_N\right)\,. \tag{2.16}$$

It is convenient to analyze the invariance of the Einstein equations first.

Using the identity

$$\mathcal{M}_{MN}\imath_a F^M \wedge \star F^N = I_{\Lambda\Sigma}\left(\imath_a F^\Lambda \wedge \star F^\Sigma - F^\Lambda \wedge \imath_a \star F^\Sigma\right)\,, \tag{2.17}$$

and the twisted self-duality constraint eq. (2.13), the energy-momentum tensor of the 1-forms can be written in the form

$$-\,\Omega_{MN}\imath_a \star F^M \wedge \star F^N\,, \tag{2.18}$$

which is left invariant by the transformations that leave invariant Ω

$$S^T\Omega S = \Omega\,, \tag{2.19}$$

that is, by transformations that belong to $\mathrm{Sp}(2n_V,\mathbb{R})$ [36]. Defining the $n_V \times n_V$ blocks of the symplectic matrix S

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\,, \tag{2.20}$$

the symplectic nature of S implies the following conditions for them:

$$A^TC - C^TA = 0\,, \tag{2.21a}$$

$$B^TD - D^TB = 0\,, \tag{2.21b}$$

$$D^TA - B^TC = \mathbb{1}_{n_V\times n_V}\,. \tag{2.21c}$$

It is not difficult to see that, if S symplectic, so is S^T. The symplectic nature of S^T implies

$$BA^T - AB^T = 0\,, \tag{2.22a}$$

$$DC^T - CD^T = 0\,, \tag{2.22b}$$

$$DA^T - CB^T = \mathbb{1}_{n_V\times n_V}\,. \tag{2.22c}$$

On the other hand, eq. (2.19) implies

$$\Omega^{-1}S\Omega = S^{-1\,T}\,, \tag{2.23}$$

and, going back to eq. (2.16), we find that

$$\mathcal{M}^{-1\,\prime} = S\mathcal{M}^{-1}S^T\,. \tag{2.24}$$

Defining the $n_V \times n_V$, symmetric, *period matrix*

$$\mathcal{N} = R + iI\,, \tag{2.25}$$

it can be seen that the transformation of $\mathcal{M}$ eq. (2.24) is equivalent to the following generalized fractional-linear transformations of $\mathcal{N}$:

$$\mathcal{N}' = (C + D\mathcal{N})\,(A + B\mathcal{N})^{-1}\,. \tag{2.26}$$

It is clear that these transformations of the period matrix are associated to transformations of the scalars which we are going to study in their infinitesimal form. The transformations of the scalars that leave the equations of motion invariant must necessarily be generated by the Killing vectors of the σ-model metric g_{xy}, which we are going to denote by $\{k_A{}^x(\phi)\}$.[9] In some cases it is convenient to include in this set some vectors which are identically zero so that the index A can be used to label also transformations of the 1-form fields that do not involve the scalars, if necessary. Of course, additional conditions involving the kinetic matrices (hence, the period matrix) need to be satisfied.

The infinitesimal transformations of the 1-form fields are

$$\begin{aligned} S &\sim \mathbb{1}_{2n_V\times 2n_V} + \alpha^A T_A\,,\\ T_A &= \left(T_A{}^M{}_N\right) = \begin{pmatrix} T_A{}^\Lambda{}_\Sigma & T_A{}^{\Lambda\Sigma} \\ T_{A\,\Lambda\Sigma} & T_{A\,\Lambda}{}^\Sigma \end{pmatrix}. \end{aligned} \tag{2.28}$$

S is symplectic if

$$T_A{}^T\Omega + \Omega T_A = 0\,, \quad \Rightarrow (\Omega T_A)^T = \Omega T_A\,, \tag{2.29}$$

(so $\Omega_{MP}T_A{}^P{}_N$ is symmetric in MN) which implies, for the block matrices

$$\begin{aligned} T_{A\,\Lambda\Sigma} &= T_{A\,\Sigma\Lambda}\,,\\ T_A{}^\Lambda{}_\Sigma &= -T_{A\Sigma}{}^\Lambda\,,\\ T_A{}^{\Lambda\Sigma} &= T_A{}^{\Sigma\Lambda}\,. \end{aligned} \tag{2.30}$$

Then, the infinitesimal form of eq. (2.26) is

$$\delta_A\mathcal{N}_{\Lambda\Sigma} = T_{A\,\Lambda\Sigma} + T_{A\Lambda}{}^\Omega\mathcal{N}_{\Omega\Sigma} - \mathcal{N}_{\Lambda\Omega}T_A{}^\Omega{}_\Sigma - \mathcal{N}_{\Lambda\Gamma}T_A{}^{\Gamma\Omega}\mathcal{N}_{\Omega\Sigma}\,, \tag{2.31}$$

and, for the kinetic matrices,

$$\delta_A R_{\Lambda\Sigma} = T_{A\,\Lambda\Sigma} + T_{A\Lambda}{}^\Omega R_{\Omega\Sigma} - R_{\Lambda\Omega}T_A{}^\Omega{}_\Sigma - R_{\Lambda\Gamma}T_A{}^{\Gamma\Omega}R_{\Omega\Sigma} + I_{\Lambda\Gamma}T_A{}^{\Gamma\Omega}I_{\Omega\Sigma}\,, \tag{2.32a}$$

$$\delta_A I_{\Lambda\Sigma} = T_{A\Lambda}{}^\Omega I_{\Omega\Sigma} - I_{\Lambda\Omega}T_A{}^\Omega{}_\Sigma - 2R_{(\Lambda|\Gamma}T_A{}^{\Gamma\Omega}I_{\Omega|\Sigma)}\,. \tag{2.32b}$$

[9]These transformations leave exactly invariant the energy-momentum tensor of the scalars, which is the only piece of the Einstein equations that we had not studied and transform covariantly the first two terms of the scalar equations of motion:

$$\delta_A\left\{d\star d\phi^x + \Gamma_{yz}{}^x d\phi^y\wedge\star d\phi^z\right\} = \partial_w k_A^x\left\{d\star d\phi^w + \Gamma_{yz}{}^w d\phi^y\wedge\star d\phi^z\right\}\,. \tag{2.27}$$

Then, it can be easily seen that the whole scalar equations of motion transform as

$$\delta_A \mathbf{E}_x = -\partial_x k_A{}^y \mathbf{E}_y \,, \tag{2.33}$$

under the transformations

$$\delta_A \phi^x = k_A{}^x \,, \qquad \delta_A F^M = T_A{}^M{}_N F^N \,, \tag{2.34}$$

provided that

$$k_A{}^x \partial_x \mathcal{N} = \delta_A \mathcal{N} \,, \tag{2.35}$$

where $\delta_A \mathcal{N}$ is the infinitesimal generalized fractional-linear transformation in eq. (2.31) (or equivalently, in eqs. (2.32a) and (2.32b) for the kinetic matrices). This equivariance condition of the kinetic matrices is the condition we announced when we defined the theory.

3 A definition of scalar charge

Not all the symmetries of the equations of motion that we have studied are symmetries of the action: those generated by $T_A{}^{\Lambda\Sigma}$ do not leave the action invariant. Those generated by $T_{A\,\Lambda\Sigma}$ leave it invariant up to a total derivative. However, as shown in ref. [36], there is an on-shell conserved current for each of them, the so-called *Noether-Gaillard-Zumino (NGZ) current*. The simplest way to construct them is by contracting the scalar equations of motion with the Killing vectors that generate them. Using the Killing vector equation and the equivariance conditions eqs. (2.32a) and (2.32b) we get [34]

$$\begin{aligned} k_A{}^x \mathbf{E}_x &= -d \star \hat{k}_A - \frac{1}{2}\Omega_{MP} T_A{}^P{}_N F^M \wedge F^N \\ &= -d \left[\star \hat{k}_A + \frac{1}{2}\Omega_{MP} T_A{}^P{}_N A^M \wedge F^N \right] + \frac{1}{2}\Omega_{MP} T_A{}^P{}_N A^M \wedge \mathbf{E}^N \,, \end{aligned} \tag{3.1}$$

where we have collected in a symplectic vector of 3-forms the Maxwell equations and Bianchi identities:

$$\left(\mathbf{E}^M \right) \equiv \begin{pmatrix} \mathbf{E}^\Lambda \\ \mathbf{E}_\Lambda \end{pmatrix} \,, \tag{3.2}$$

and where we have denoted by $\hat{k}_A$ the pullback of the 1-form dual to the target space Killing vector k_A

$$\hat{k}_A \equiv k_A{}^x g_{xy} d\phi^y \,. \tag{3.3}$$

Therefore, we find that the NGZ currents

$$\star j_A \equiv - \star \hat{k}_A - \frac{1}{2}\Omega_{MP} T_A{}^P{}_N A^M \wedge F^N \,, \tag{3.4}$$

are conserved on-shell

$$d \star j_A = k_A{}^x \mathbf{E}_x - \frac{1}{2}\Omega_{MP} T_A{}^P{}_N A^M \wedge \mathbf{E}^N \doteq 0 \,. \tag{3.5}$$

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The conservation of these currents follows from a global symmetry and the associated charges are expressed as integrals over spacelike hypersurfaces (volumes)

$$q_A \sim \int_{\Sigma^3} \star j_A \,. \tag{3.6}$$

However, it is not difficult to see that in static black hole solutions with non-trivial scalar fields ϕ^x whose charges Σ^x are conventionally defined[10] through the asymptotic behavior of the field at spatial infinity

$$\phi^x \overset{r\to\infty}{\sim} \phi^x_\infty + \frac{\Sigma^x}{r}\,, \tag{3.7}$$

the NGZ charges not only do not reproduce the charges Σ^x (or combinations of them) but vanish identically.

In stationary black hole spacetimes, though, there is another definition of scalar charge that satisfies a Gauss law. Let us assume that all the fields are invariant under the isometry generated by the spacetime vector k, $\delta_k\varphi = 0$. This implies, in particular, that k is a Killing vector and, for us, it will be the Killing vector associated to the Black hole's Killing horizon. For the scalar fields it means that their Lie derivatives with respect to that vector vanishes

$$\delta_k\phi^x = -\pounds_k\phi^x = -\imath_k d\phi^x = 0\,. \tag{3.8}$$

As shown in [11–13],[11] for the 1-form fields, it means that their Lie derivatives with respect to k plus a gauge transformation with parameter

$$\chi_k = \imath_k A - P_k\,, \tag{3.9}$$

where the *Maxwell momentum map* P_k satisfies the *Maxwell momentum map equation*[12]

$$\imath_k F + dP_k = 0\,, \tag{3.11}$$

vanish identically:

$$\begin{aligned}\delta_k A^M &= -\pounds_k A^M + d\chi^M_k = -\left(\imath_k d + d\imath_k\right) A^M + d\left(\imath_k A^M - P_k{}^M\right)\\ &= -\left(\imath_k F^M + dP_k{}^M\right) = 0\,,\end{aligned} \tag{3.12}$$

by virtue of the Maxwell momentum map equation (3.11).

[10] See, for instance, ref. [5].

[11] The work terms for the electric charges associated to p-forms were found using the covariant phase space formalsm in ref. [14]. See also [15] for a different, equivalent, approach based on the mathematics of principal bundles. The importance of the gauge- and diffeomorphism invariance of the charges and potentials that occur in the laws of black-hole thermodynamics has been stressed in [16, 18] and in the 5th chapter of [17].

[12] The local existence of a P_k satisfying this equation follows from the assumption:

$$\delta_k F = -\pounds_k F = -d\imath_k F = 0\,. \tag{3.10}$$

If all the fields are invariant under δ_k, so must the NGZ currents be. Furthermore, since the NGZ currents are not gauge invariant, we must use this definition for δ_k:

$$\begin{aligned}
\delta_k \star j_A &= -\pounds_k \star j_A + \delta_{\chi_k} \star j_A \\
&= -\left(\imath_k d + d\imath_k\right) \star j_A - \frac{1}{2}\Omega_{MP} T_A{}^P{}_N \delta_{\chi_k} A^M \wedge F^N \\
&\doteq -d\imath_k \star j_A - \frac{1}{2}\Omega_{MP} T_A{}^P{}_N d\chi_k{}^M \wedge F^N \\
&\doteq d\left\{-\imath_k \star j_A - \frac{1}{2}\Omega_{MP} T_A{}^P{}_N \chi_k{}^M F^N\right\} \\
&= 0\,,
\end{aligned} \tag{3.13}$$

by assumption.

The expression in brackets is a 2-form that satisfies a Gauss law. Massaging it a bit, we find the following manifestly gauge-invariant expression for it:

$$\mathbf{Q}_A[k] = \imath_k \star \hat{k}_A + \Omega_{MP} T_A{}^P{}_N P_k{}^M F^N\,. \tag{3.14}$$

Now, integrating over 2-dimensional, spacelike, closed surfaces (and restoring the normalization) we get the charges associated to the NGZ currents:

$$Q_{A,k} = \frac{1}{16\pi G_N^{(4)}} \int_{\Sigma^2} \left\{\imath_k \star \hat{k}_A + \Omega_{MP} T_A{}^P{}_N P_k{}^M F^N\right\}\,. \tag{3.15}$$

This is our proposal for scalar charges. Observe that under a duality transformation generated by k_A, T_A with Lie brackets and commutation relations

$$[k_A, k_B] = -f_{AB}{}^C k_C\,, \qquad [T_A, T_B] = +f_{AB}{}^C T_C\,, \tag{3.16}$$

these charges transform in the adjoint representation of the duality group:

$$\delta_A Q_{B,k} = -f_{AB}{}^C Q_{C,k}\,. \tag{3.17}$$

In what follows we are going to show in several examples corresponding to static dilaton and axidilaton black holes that their values are non-vanishing and reproduce the values of the conventionally-defined scalar charges eq. (3.7) but, before we set to do that, let us observe that this definition depends on the value of the momentum map over the integration surface. The Maxwell momentum map is defined only up to an additive constant. This constant can be chosen so that $P_k{}^M\big|_\infty = 0$. That is the choice that allows us to recover the values of the conventionally-defined scalar charges eq. (3.7). However, other choices are possible. The form of the first law that we are going to find includes an additional term that takes into account that possibility so that the first law is invariant under a change of asymptotic value of the Maxwell momentum maps.

It is also worth stressing that in the case in which we are considering (a symmetric σ-model) there are always more symmetries than scalar fields. Therefore, there are more 2-forms $\mathbf{Q}_A[k]$ satisfying a Gauss law than scalars. Obviously, not all of them will be independent. In any case, the conservation laws of those currents can be used to reconstruct the equations of motion of the scalars using the identity

$$\delta_x{}^y = g^{AB} k_{A\,x} k_B{}^y\,, \tag{3.18}$$

in which g^{AB} is the Killing metric of the duality group G.

It also follows that there are more scalar charges than scalars, but we are going to see that the conventionally-defined scalar charges Σ^x can be expressed in terms of the charges $Q_{A,k}$ that we have just defined.

It is worth mentioning that there is a slightly different procedure that allows us to obtain the same expression eq. (3.14) and that was used in the case of dilaton black holes in ref. [35]. In that case there is only one scalar and one target space Killing vector $k = 1$ that generates the constant shifts of the scalar which are compensated by rescalings of the vector field (see section 5). In our case, we have to project the scalar equations with the different Killing vectors $k_A{}^x$ first, as in the first line of eq. (3.1). Then, we take the inner product of the resulting equation with $\imath_k$

$$\imath_k k_A{}^x \mathbf{E}_x = -\imath_k d\star \hat{k}_A - \Omega_{MP} T_A{}^P{}_N \imath_k F^M \wedge F^N\,. \tag{3.19}$$

If all the fields are invariant under the diffeomorphism generated by k

$$\begin{aligned} -\imath_k d\star \hat{k}_A &= d\imath_k \star \hat{k}_A\,, \\ \imath_k F^M &= -dP_k{}^M\,, \end{aligned} \tag{3.20}$$

and, integrating by parts we arrive to $d\mathbf{Q}_A[k] = 0$.

4 First law and scalar charges

Taking into account the results obtained in refs. [11–13, 26] for the inclusion of matter fields, in Wald's formalism [8–10], the first law of black hole thermodynamics for a non-extremal black hole whose bifurcate horizon coincides with the Killing horizon of the Killing vector field $k = \partial_t + \Omega\partial_\varphi$, can be derived by integrating the on-shell identity

$$d\mathbf{W}[k] \doteq 0\,, \tag{4.1}$$

where

$$\mathbf{W}[k] \equiv \delta\mathbf{Q}[k] + \imath_k \mathbf{\Theta}(\varphi, \delta\varphi) - \varpi_k\,, \tag{4.2}$$

over a spacelike hypersurface with boundaries at spatial infinity (S^2_∞) and at the bifurcation sphere $\mathcal{BH}$ and applying the Stokes theorem.

In the above identity $\mathbf{Q}[k]$ is the Noether-Wald charge for the Killing vector k, $\mathbf{\Theta}(\varphi, \delta\varphi)$ is the *presymplectic* $(d-1)$*-form* defined in ref. [8] and $\varpi_k$is defined by[13]

$$\delta_{\Lambda_k} \mathbf{\Theta}(\varphi, \delta\varphi) \equiv d\varpi_k\,. \tag{4.3}$$

[13]This term arises when the effect of the induced gauge transformations are correctly taken into account as in ref. [26]. In eq. (4.3) δ_{Λ_k} stands for all the gauge transformations induced by the isometry generated by k.

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Furthermore, it is assumed that the variations of the fields $\delta\varphi$ satisfy the linearized equations of motion in the black-hole's background.

The first law, thus, follows from the identity

$$\int_{S^2{}_\infty} \mathbf{W}[k] = \int_{\mathcal{BH}} \mathbf{W}[k]\,. \tag{4.4}$$

In previous works, following ref. [10], we assumed that, almost by definition, the first integral simply gives the variation of the conserved charges associated to the Killing vector k, that is,

$$\delta M - \Omega\delta J\,. \tag{4.5}$$

A closer look reveals that, in presence of matter fields, it contains additional terms that contribute to the first law [26]. In particular, as we are going to see, it contains terms related to the scalar charges that we have just defined.

A standard calculation along the lines of refs. [11–13, 26] gives

$$\mathbf{Q}[k] = \star(e^a \wedge e^b)P_{k\,ab} - P_k{}^\Lambda F_\Lambda\,, \tag{4.6}$$

where $P_{k\,ab}$ is the *Lorentz momentum map* defined in ref. [11] and coincides with the *Killing bivector*

$$P_{k\,ab} = \nabla_a k_b\,, \tag{4.7}$$

and $P_k{}^\Lambda$ is the Maxwell momentum map defined in eq. (3.11). A quick calculation gives

$$\delta\mathbf{Q}[k] = P_{k\,ab}\delta\star(e^a\wedge e^b) + \star(e^a\wedge e^b)\delta P_{k\,ab} - F_\Lambda\delta P_k{}^\Lambda - P_k{}^\Lambda\delta F_\Lambda\,. \tag{4.8}$$

The presymplectic 3-form is given in eq. (2.6) and another short calculation gives

$$\begin{aligned}\imath_k\mathbf{\Theta} = &-\imath_k\star(e^a\wedge e^b)\wedge\delta\omega_{ab} - \star(e^a\wedge e^b)\wedge\delta\imath_k\omega_{ab} + g_{xy}\imath_k\star d\phi^x\delta\phi^y\\ &-\frac{1}{2}\imath_k F_\Lambda\wedge\delta A^\Lambda - \frac{1}{2}F_\Lambda\wedge\delta\imath_k A^\Lambda\,.\end{aligned} \tag{4.9}$$

Since, on-shell, the dual 1-forms obey the same equations as the original ones, we can define the *dual (magnetic) momentum maps* $P_{k\,\Lambda}$ through the equation

$$\imath_k F_\Lambda + dP_{k\,\Lambda} = 0\,, \tag{4.10}$$

and, substituting this definition in the above expression and integrating by parts, we get

$$\begin{aligned}\imath_k\mathbf{\Theta} = &-\imath_k\star(e^a\wedge e^b)\wedge\delta\omega_{ab} - \star(e^a\wedge e^b)\wedge\delta\imath_k\omega_{ab} + g_{xy}\imath_k\star d\phi^x\delta\phi^y\\ &+P_{k\,\Lambda}\wedge\delta F^\Lambda - F_\Lambda\wedge\delta\imath_k A^\Lambda\,,\end{aligned} \tag{4.11}$$

up to an irrelevant total derivative.

Another simple calculation gives [26]

$$\begin{aligned}\delta_{\Lambda_k}\mathbf{\Theta} &= (\delta_{\sigma_k} + \delta_{\chi_k})\mathbf{\Theta}\\ &= -\delta_{\sigma_k}\left[\star(e^a\wedge e^b)\wedge\delta\omega_{ab}\right] - F_\Lambda\wedge\delta_{\chi_k}\delta A^\Lambda\\ &= -\star(e^a\wedge e^b)\wedge\mathcal{D}\delta\sigma_{k\,ab} - F_\Lambda\wedge d\delta\chi_k{}^\Lambda\\ &= d\left\{-\star(e^a\wedge e^b)\wedge\delta\sigma_{k\,ab} - F_\Lambda\delta\chi_k{}^\Lambda\right\}\,,\end{aligned} \tag{4.12}$$

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where the parameters of the induced Lorentz and Maxwell gauge transformations are, respectively

$$\sigma_k{}^{ab} = \imath_k \omega^{ab} - P_k{}^{ab}\,, \tag{4.13a}$$

$$\chi_k{}^{\Lambda} = \imath_k A^{\Lambda} - P_k{}^{\Lambda}\,. \tag{4.13b}$$

Therefore,

$$-\,\varpi_k = \star(e^a \wedge e^b) \wedge \delta\sigma_{k\,ab} + F_{\Lambda}\delta\chi_k{}^{\Lambda}\,. \tag{4.14}$$

Combining all these partial results, we arrive at

$$\begin{aligned} \mathbf{W}[k] = {} & P_{k\,ab}\delta \star (e^a \wedge e^b) - \imath_k \star (e^a \wedge e^b) \wedge \delta\omega_{ab} \\ & - P_k{}^{\Lambda}\delta F_{\Lambda} + P_{k\,\Lambda}\delta F^{\Lambda} + g_{xy}\imath_k \star d\phi^x \delta\phi^y\,. \end{aligned} \tag{4.15}$$

Let us consider the integral of $\mathbf{W}[k]$ at spatial infinity first, restoring the global factor $1/(16\pi G_N^{(4)})$. The first two terms give the gravitational contribution

$$\frac{1}{16\pi G_N^{(4)}} \int_{S^2_\infty} \left\{ P_{k\,ab}\delta \star (e^a \wedge e^b) - \imath_k \star (e^a \wedge e^b) \wedge \delta\omega_{ab} \right\} = \delta M - \Omega\delta J\,, \tag{4.16}$$

while the third and fourth give[14]

$$\frac{1}{16\pi G_N^{(4)}} \int_{S^2_\infty} \left\{ -P_k{}^{\Lambda}\delta F_{\Lambda} + P_{k\,\Lambda}\delta F^{\Lambda} \right\} = -\Phi^{\Lambda}_{\infty}\delta q_{\Lambda} + \Phi_{\Lambda\,\infty}\delta p^{\Lambda} = -\Omega_{MN}\Phi^M_\infty \delta q^N\,, \tag{4.17}$$

where Φ^{Λ}_{∞} and $\Phi_{\Lambda\,\infty}$ are the values of the electrostatic and magnetostatic potentials at spatial infinity.

Let us consider the last term. In the previous cases only conserved charges are involved and it is natural to use the definition of scalar charges we have proposed here to rewrite that term. Using the identity $g_{xy} = g^{AB}k_{A\,x}k_{B\,y}$

$$g_{xy}\imath_k \star d\phi^x \delta\phi^y = g^{AB}\imath_k \star \hat{k}_A k_{B\,y}\delta\phi^y = \left(\mathbf{Q}_A[k] - \Omega_{MP} T_A{}^P{}_N P_k{}^M F^N \right)\delta^A\,. \tag{4.18}$$

where we have defined

$$\delta^A \equiv g^{AB}k_{B\,y}\delta\phi^y\,, \tag{4.19}$$

Restoring the global factor $1/(16\pi G_N^{(4)})$, we find

$$\int_{S^2_\infty} \left(\mathbf{Q}_A[k] - \frac{1}{16\pi G_N^{(4)}} \Omega_{MP} T_A{}^P{}_N P_k{}^M F^N \right)\delta^A = \left(\mathcal{Q}_{A\,k} - \Omega_{MP} T_A{}^P{}_N \Phi^M_\infty q^N \right)\delta^A_\infty\,. \tag{4.20}$$

Then,

$$\int_{S^2_\infty} \mathbf{W}[k] = \delta M - \Omega\delta J - \Omega_{MN}\Phi^M_\infty \delta q^N + \left(\mathcal{Q}_A - \Omega_{MP} T_A{}^P{}_N \Phi^M_\infty q^N \right)\delta^A_\infty\,. \tag{4.21}$$

[14]The electric and magnetic Maxwell momentum maps can be identified with the electrostatic and magnetostatic potentials Φ^{Λ} and Φ_{Λ}, respectively.

The bifurcation surface is defined by the property $k = 0$ and, on it,

$$P_{k\,ab} \stackrel{\mathcal{BH}}{=} \kappa n_{ab}\,, \tag{4.22}$$

where n^{ab} is the binormal to the horizon wit the normalization $n^{ab}n_{ab} = -2$ and κ is the surface gravity. Therefore,

$$\begin{aligned}\int_{\mathcal{BH}} \mathbf{W}[k] &= \frac{1}{16\pi G_N^{(4)}} \int_{\mathcal{BH}} \left\{ P_{k\,ab}\delta\star(e^a\wedge e^b) - P_k{}^{\Lambda}\delta F_{\Lambda} + P_{k\,\Lambda}\delta F^{\Lambda}\right\} \\ &= \frac{\kappa\delta A_{\mathcal{H}}}{2\pi G_N^{(4)}} - \Phi^{\Lambda}_{\mathcal{H}}\delta q_{\Lambda} + \Phi_{\Lambda\,\mathcal{H}}\delta p^{\Lambda}\,,\end{aligned} \tag{4.23}$$

where $A_{\mathcal{H}}$ is the area of the horizon and $\Phi^{\Lambda}_{\mathcal{H}}$ and $\Phi_{\Lambda\,\mathcal{H}}$ are the values of the electrostatic and magnetostatic potentials over the horizon (constant according to the generalized zeroth law).

We arrive at our main result:[15]

$$\delta M = \frac{\kappa\delta A_{\mathcal{H}}}{8\pi G_N^{(4)}} + \Omega\delta J - \Omega_{MN}\left(\Phi^M_{\mathcal{H}} - \Phi^M_{\infty}\right)\delta q^N - \left(\mathcal{Q}_{A\,k} - \Omega_{MP}T_A{}^P{}_N\Phi^M_{\infty}q^N\right)\delta^A_{\infty}\,. \tag{4.24}$$

In this expression the object δ^A_{∞} is unusual, but it just reflects the different forms in which the dualities of the theory can modify the values of the moduli at infinity, which are also naturally associated to the charges that we have defined.

The last term involving Φ^M_{∞} is also unusual, but it has to be there if we are going to allow for potentials which do not vanish at infinity. In the examples that we are going to study explicitly, $\Phi^M_{\infty} = 0$ and the scalar charges take the expected value. Furthermore, in that case, the scalar term can be brought to the form found in ref. [5] (up to the normalization of the charges):

$$-\mathcal{Q}_{A\,k}\delta^A_{\infty} = -\mathcal{Q}_{A\,k}g^{AB}k_{B}{}^x_{\infty}g_{xy\,\infty}\delta\phi^y_{\infty} = -\frac{1}{4}\Sigma^x g_{xy\,\infty}\delta\phi^y_{\infty}\,, \tag{4.25}$$

where the scalar charges defined through the asymptotic expansions, Σ^x are related to the ones associated to the duality symmetries Q_A by

$$\Sigma^x = 4\mathcal{Q}_A g^{AB}k_{B}{}^x_{\infty}\,. \tag{4.26}$$

Finally, observe that, on the bifurcation surface

$$\mathbf{Q}_A[k] \stackrel{\mathcal{BH}}{=} \Omega_{MP}T_A{}^P{}_N P_{k\,\mathcal{H}}^M F^N\,, \tag{4.27}$$

and, therefore

$$\mathcal{Q}_{A\,k} = -\Omega_{MP}T_A{}^P{}_N\Phi^M_{\mathcal{H}}q^N\,. \tag{4.28}$$

This formula, which is our second main result, gives a universal relation between the scalar charges of a black hole and the electric and magnetic charges and potentials evaluated

[15] The overall sign of the electric and magnetic terms is unconventional. It is due to the definition of F_{Λ} with a negative-definite kinetic matrix $I_{\Lambda\Sigma}$. It can be easily be changed, but the relative sign between the electric and magnetic terms can only be changed at the expense of losing explicit symplectic invariance.

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on the horizon generalizing the result found in ref. [35] in a gauge-invariant way. Observe that The existence of a bifurcate Killing horizon is crucial: in other spacetime backgrounds the scalar charges may take arbitrary values in agreement with the no-hair "theorem" and the interpretation of the non-trivial scalar fields of these black-hole solutions as secondary scalar hair [2].[16]

If we plug that formula back into the first law we arrive at

$$\delta M = \frac{\kappa \delta A_{\mathcal{H}}}{8\pi G_N^{(4)}} + \Omega \delta J - \Omega_{MN} \left(\Phi^M_{\mathcal{H}} - \Phi^M_\infty\right) \delta q^N - \Omega_{MP} T_A{}^P{}_N \left(\Phi^M_{\mathcal{H}} - \Phi^M_\infty\right) q^N \delta^A_\infty \,, \tag{4.29}$$

which is manifestly independent of the choice of asymptotic value of the potentials.

Notice that the right-hand side of this expression only contains the variations of quantities which are independent physical parameters of the black-hole solutions. The variations of the scalar charges cannot and do not appear. The scalar charges actually pleay the roles of thermodynamical potentials.

In the next two sections we are going to compare the scalar charges we have defined with those obtained through the asymptotic expansion and the first law that we have obtained with the first law obtained through the variation of the entropy with respect to the physical parameters in two sets of solutions: static, electrically-charged black holes and static axion-dilaton black holes.

5 Static dilaton black hole solutions

Dilaton black holes are solutions of the family of models defined by the action[17]

$$S[e, A, \phi] = \frac{1}{16\pi} \int \left\{ - \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \frac{1}{2} e^{-a\phi} F \wedge \star F \right\} \,, \tag{5.1}$$

which depends on the real parameter a and determines the strength of the coupling of the dilaton and the Maxwell field. The static black-hole solutions[18] of this model were found in refs. [37–39] and can be written in the form

$$\begin{aligned} ds^2 &= H^{-\frac{2}{1+a^2}} W dt^2 - H^{\frac{2}{1+a^2}} \left[W^{-1} dr^2 + r^2 d\Omega^2_{(2)} \right] \,, \\ A_t &= \alpha e^{a\phi_\infty/2} (H^{-1} - 1) \,, \\ e^{-\phi} &= e^{-\phi_\infty} H^{\frac{2a}{1+a^2}} \,, \end{aligned} \tag{5.2}$$

where the functions H and W take the form

$$H = 1 + \frac{h}{r} \,, \qquad W = 1 + \frac{\omega}{r} \,, \tag{5.3}$$

[16] Static, spherically-symmetric solutions of pure gravity and dilaton gravity with primary scalar hair (i.e. scalar fields with charges which are independent parameters of the solutions) can be found in refs. [3, 4] (see also the higher-dimensional generalizations in chapter 16 of ref. [1]) and are singular.

[17] Sometimes they are called Einstein-Maxwell-Dilaton (EMD) actions. We set $G_N^{(4)} = 1$ throughout all this section.

[18] Related solutions with primary scalar hair were found in ref. [4].

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and the integration constants h, ω, α satisfy the following relation

$$\omega = h\Big[1 - (1+a^2)(\alpha/2)^2\Big] \,. \tag{5.4}$$

In terms of the physical parameters M, q, ϕ_∞ (ADM mass, electric charge and modulus) and the coupling constant a, the integration constants h, ω and α are given by

$$\begin{aligned}
h &= -\frac{a^2+1}{a^2-1}\left\{M - \sqrt{M^2 + 4(a^2-1)e^{a\phi_\infty}q^2}\right\} \,, \\
\omega &= -\frac{2}{a^2-1}\left\{a^2 M - \sqrt{M^2 + 4(a^2-1)e^{a\phi_\infty}q^2}\right\} \,, \\
\alpha &= -4qe^{a\phi_\infty/2}/h \,,
\end{aligned} \tag{5.5}$$

for $a \neq 1$ and

$$\begin{aligned}
h &= \frac{4e^{\phi_\infty}q^2}{M} \,, \\
\omega &= -2\frac{M^2 - 2e^{\phi_\infty}q^2}{M} \,, \\
\alpha &= -e^{-\phi_\infty/2}M/q \,.
\end{aligned} \tag{5.6}$$

The scalar charge Σ, computed using the conventional asymptotic definition

$$\phi \sim \phi_\infty + \frac{\Sigma}{r} \,, \tag{5.7}$$

takes the value

$$\Sigma = -\frac{2ah}{a^2+1} \,. \tag{5.8}$$

We have chosen the sign of the square roots in h and ω so as to always have $h > 0$ and ω negative if certain non-extremality conditions are met: for all values of a

$$M^2 > \frac{4}{a^2+1}e^{a\phi_\infty}q^2 \,. \tag{5.9}$$

In that case, there is an event horizon at

$$r = -\omega \equiv r_0 \,, \tag{5.10}$$

with Bekenstein-Hawking entropy

$$S = \pi r_0^{\frac{2a^2}{a^2+1}}(r_0+h)^{\frac{2}{a^2+1}} \,, \tag{5.11}$$

and Hawking temperature

$$T = \frac{r_0}{4S} \,. \tag{5.12}$$

We can derive the first law for these families of black holes by varying the entropy with respect to all the independent physical parameters, including the modulus ϕ_∞:

$$\delta S = \frac{1}{T}\left[\delta M + \frac{4}{(a^2+1)\alpha}e^{a\phi_\infty/2}\delta q + \frac{1}{4}\Sigma\delta\phi_\infty\right] \,, \tag{5.13}$$

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for $a^2 \neq 1$ and

$$\delta S = \frac{1}{T}\left[\delta M + \frac{2}{\alpha} e^{\phi_\infty/2}\delta q + \frac{1}{4}\Sigma\delta\phi_\infty\right], \tag{5.14}$$

for $a^2 = 1$.

In the above expressions Σ is the scalar charge defined through the asymptotic expansion eq. (5.7).

These theories are invariant under the global transformations generated by

$$\delta\phi = -1\,, \qquad \delta A = -\frac{a}{2}A\,, \tag{5.15}$$

and eq. (3.14) takes the form

$$\mathbf{Q}[k] = -\frac{1}{16\pi}\left\{\imath_k \star d\phi + \frac{a}{2}P_k e^{-a\phi} \star F\right\} = -\frac{ah}{8\pi(a^2+1)}\omega_{(2)}\,, \tag{5.16}$$

where $\omega_{(2)}$ is the volume form of the round 2-sphere of unit radius. It is evident that these 2-forms satisfy a Gauss law and they give the same value when they are integrated over 2-spheres of any radius:

$$\mathcal{Q}_k = -\frac{ah}{2(a^2+1)} = -\frac{1}{4}\Sigma\,, \tag{5.17}$$

as expected according to our general arguments. This is, essentially, the result obtained by Pacilio in ref. [35].

6 Static axion-dilaton black hole solutions

The so-called axion-dilaton model is just a generalization to an arbitrary number of vector fields n_V of pure, ungauged, $\mathcal{N}=4, d=4$ supergravity [40], although this model can also be embedded in $\mathcal{N}=2, d=4$ supergravity for $n_V = 2$.

We can introduce it as a model with two real scalars $\phi^1 = a$ (the axion) and $\phi^2 = \phi$ (the dilaton) which are naturally combined into the complex scalar (*axidilaton*)

$$\lambda = a + ie^{-2\phi}\,, \tag{6.1}$$

and where the σ-model metric and the period matrix are given by

$$(g_{xy}) = \begin{pmatrix} e^{4\phi} & 0 \\ 0 & 4 \end{pmatrix}, \qquad \mathcal{N}_{\Lambda\Sigma} = -\lambda\delta_{\Lambda\Sigma}\,. \tag{6.2}$$

The most general non-extremal, static, black-hole solution of the axion-dilaton model was presented in ref. [41] and it is a generalization of the solutions presented in refs. [37, 42–46].[19] A very useful feature of this solution is that it is written in terms of its physical parameters only: the ADM mass M, the asymptotic value of the axidilaton $\lambda_\infty = a_\infty + ie^{-2\phi_\infty}$, the complex electromagnetic charges Γ^Λ (a combination of the real

[19] The most general stationary, non-extremal black-hole solution of this theory was presented in ref. [47].

electric charges q_Λ, the real magnetic charges p_Λ and the moduli λ_∞) and the complex axi-dilaton charge $\Upsilon = \Sigma + i\Delta$. All these parameters are defined by the asymptotic expansions ($G_N^{(4)} = 1$)

$$g_{tt} \sim 1 - \frac{2M}{r}\,, \tag{6.3a}$$

$$\lambda \sim \lambda_\infty - ie^{-2\phi_\infty}\frac{2\Upsilon}{r}\,, \tag{6.3b}$$

$$\frac{1}{2}\left[F^\Lambda{}_{tr} + i\star F^\Lambda{}_{tr}\right] \sim \frac{e^{+\phi_\infty}\Gamma^\Lambda}{r^2} = \frac{e^{+2\phi_\infty}(q_\Lambda - \lambda^*_\infty p^\Lambda)}{r^2}\,. \tag{6.3c}$$

The asymptotic behavior of λ implies for those of a and ϕ

$$a \sim a_\infty + \frac{2e^{-2\phi_\infty}\Im\mathfrak{m}\Upsilon}{r}\,, \tag{6.4a}$$

$$\phi \sim \phi_\infty + \frac{\Re\mathfrak{e}\Upsilon}{r}\,, \tag{6.4b}$$

so

$$\Sigma^1 = 2e^{-2\phi_\infty}\Im\mathrm{m}(\Upsilon)\,, \qquad \Sigma^2 = \Re\mathrm{e}(\Upsilon)\,. \tag{6.5}$$

The axidilaton charge is a function of the rest of the physical parameters:

$$\Upsilon = -\frac{2}{M}\Gamma^{\Lambda\,*}\Gamma^{\Lambda\,*}\,. \tag{6.6}$$

The ADM mass can be defined more rigorously as a conserved quantity through the ADM [48], the Abbott-Deser [49] or many other formalisms. The electric and magnetic charges can also be defined as conserved charges by standard methods refs. [50, 51] as

$$p^\Lambda \equiv \frac{1}{16\pi G_N^{(4)}}\int F^\Lambda\,, \tag{6.7a}$$

$$q_\Lambda \equiv \frac{1}{16\pi G_N^{(4)}}\int F_\Lambda\,. \tag{6.7b}$$

In contrast, as we have stressed, the scalar charges are conventionally defined through the above asymptotic expansion which is not based on any conservation (*Gauss*) law. Our goal in this section will be to show that the definition of scalar charges that we have proposed in section 3 gives exactly the same result for the static solutions of the axion-dilaton model.

The most economical way of presenting this kind of solutions is through the time components of the original and dual 1-form fields $A^\Lambda{}_t$ and $A_{\Lambda\,t}$, respectively. They contain enough information to recover the rest of the components of each of them.[20] The solution is, then, [41]

$$\begin{aligned}
ds^2 &= e^{2U}dt^2 - e^{-2U}dr^2 - R^2 d\Omega^2_{(2)}\,,\\
\lambda &= \frac{\lambda_\infty r + \lambda^*_\infty\Upsilon}{r+\Upsilon}\,,\\
A^\Lambda{}_t &= 2e^{\phi_\infty}R^{-2}[\Gamma^\Lambda(r+\Upsilon) + \text{c.c.}]\,,\\
A_{\Lambda\,t} &= -2e^{\phi_\infty}R^{-2}[\Gamma^\Lambda(\lambda_\infty r + \lambda^*_\infty\Upsilon) + \text{c.c.}]\,.
\end{aligned} \tag{6.8}$$

[20] They are computed explicitly in ref. [27].

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The functions e^{2U} and R are given by

$$
\begin{aligned}
e^{2U} &= R^{-2}(r - r_+)(r - r_-)\,, \\
R^2 &= r^2 - |\Upsilon|^2\,,
\end{aligned}
\tag{6.9}
$$

and the parameters $r_\pm$ that appear in e^{2U} (actually, the positions of the outer and inner horizons when they take real values, i.e. when $r_0^2 > 0$) are given by

$$
r_\pm = M \pm r_0\,, \quad \text{with} \quad r_0^2 = M^2 + |\Upsilon|^2 - 4\Gamma^\Lambda\Gamma^{\Lambda\,*}\,. \tag{6.10}
$$

Since we are just interested in the thermodynamics of these black holes, we only need their Hawking temperature and Bekenstein-Hawking entropy, which are given by

$$
T = \frac{r_0}{2S}\,, \tag{6.11a}
$$

$$
S = 2\pi \left\{ M^2 + M r_0 - 2\Gamma^{\Lambda\,*}\Gamma^{\Lambda\,*} \right\}\,. \tag{6.11b}
$$

Varying S with respect to the physical charges M, q_Λ, p^Λ and the moduli λ_∞ we get the first law:

$$
\delta M = T\delta S + \Phi^\Lambda \delta q_\Lambda - \Phi_\Lambda \delta p^\Lambda - \frac{1}{2}\Im\mathrm{m}(\Upsilon) e^{2\phi_\infty}\delta a_\infty - \Re\mathrm{e}(\Upsilon)\delta\phi_\infty\,. \tag{6.12}
$$

The last two terms can be rewritten in two different fashions:

$$
\begin{aligned}
-\frac{1}{2}\Im\mathrm{m}(\Upsilon) e^{2\phi_\infty}\delta a_\infty - \Re\mathrm{e}(\Upsilon)\delta\phi_\infty &= -\frac{1}{2}\Im\mathrm{m}\left(\Upsilon^* \frac{\delta\lambda_\infty}{e^{-2\phi_\infty}}\right) \\
&= -\frac{1}{4} g_{xy}(\phi_\infty)\Sigma^x \delta\phi^y_\infty\,,
\end{aligned}
\tag{6.13}
$$

where Σ^1, Σ^2 are the asymptotic scalar charges defined in eqs. (6.5). Both expressions are manifestly duality-invariant.[21]

We are now going to see how the scalar charges Σ^x are related to those defined in section 3 and how the scalar term in the first law agrees with the one in eq. (4.24).

The Killing vectors of the target-space metric are

$$
k_1 = a\partial_a - \frac{1}{2}\partial_\phi\,, \quad k_2 = \frac{1}{2}(1 - a^2 + e^{-4\phi})\partial_a + \frac{1}{2}a\partial_\phi\,, \quad k_3 = \frac{1}{2}(1 + a^2 - e^{-4\phi})\partial_a - \frac{1}{2}a\partial_\phi\,, \tag{6.14}
$$

and their Lie brackets satisfy the sl$(2,\mathbb{R})$ $\sim$so$(2,1)$ algebra

$$
[k_A, k_B] = \varepsilon_{ABD}\eta^{DC} k_C\,, \tag{6.15}
$$

where $(\eta_{AB}) = (\eta^{AB})$ =diag$(++-)$ is the SO$(2,1)$ invariant metric.

The SL$(2,\mathbb{R})$ matrices which act on the 1-form fields are tensor products

$$
S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \mathbb{1}_{n_V \times n_V}\,, \quad \text{with} \quad AD - BC = 1\,. \tag{6.16}
$$

[21] $e^{2\phi}\delta\lambda$ and Υ are multiplied by the same phase under SL$(2,\mathbb{R})$ transformations.

The generators (always $\otimes \mathbb{1}_{n_V \times n_V}$) are

$$T_1 = -\frac{1}{2}\sigma^3\,, \qquad T_2 = -\frac{1}{2}\sigma^1\,, \qquad T_3 = \frac{i}{2}\sigma^2\,, \tag{6.17}$$

and their commutation relations are

$$[T_A, T_B] = -\varepsilon_{ABD}\eta^{DC}k_C\,. \tag{6.18}$$

It is somewhat simpler to work with the bases $k_1, k_\pm = k_2 \pm k_3$ and $T_1, T_\pm = T_2 \pm T_3$. We compute separately the $\imath_k \star \hat{k}_A$ and $\Omega_{MP}T_A{}^P{}_N P_k^M F^N$ contributions, which in this case correspond to

$$\begin{aligned}
\Omega_{MP}T_1{}^P{}_N P_k^M F^N &= \frac{1}{2}\left(P_k^\Lambda F_\Lambda + P_{k\,\Lambda}F^\Lambda\right)\,,\\
\Omega_{MP}T_+{}^P{}_N P_k^M F^N &= -P_k^\Lambda F^\Lambda\,,\\
\Omega_{MP}T_-{}^P{}_N P_k^M F^N &= P_{k\,\Lambda}F_\Lambda\,.
\end{aligned} \tag{6.19}$$

In this case we can use as potentials the time components of the original and dual vector fields given in eqs. (6.8)

$$P_k^M = A^M{}_t\,, \tag{6.20}$$

which vanish identically at infinity.

We are only interested in the pullback of these 2-forms over 2-spheres.[22] The results, after a long calculation are ($G_N^{(4)} = 1$)

$$\begin{aligned}
\mathbf{Q}_{1\,k} &= -\frac{1}{8\pi}e^{2\phi_\infty}\Im\mathfrak{m}(\lambda_\infty^*\Upsilon)\omega_{(2)}\,,\\
\mathbf{Q}_{+\,k} &= -\frac{1}{8\pi}e^{2\phi_\infty}\Im\mathfrak{m}(\Upsilon)\omega_{(2)}\,,\\
\mathbf{Q}_{-\,k} &= \frac{1}{8\pi}e^{2\phi_\infty}\Im\mathfrak{m}(\lambda_\infty^{*\,2}\Upsilon)\omega_{(2)}\,,
\end{aligned} \tag{6.21}$$

Again, it is evident that these 2-forms satisfy a Gauss law and they give the same value when they are integrated over 2-spheres of any radius, namely

$$\begin{aligned}
\mathcal{Q}_{1\,k} &= -\frac{1}{2}e^{2\phi_\infty}\Im\mathfrak{m}(\lambda_\infty^*\Upsilon)\,,\\
\mathcal{Q}_{+\,k} &= -\frac{1}{2}e^{2\phi_\infty}\Im\mathfrak{m}(\Upsilon)\,,\\
\mathcal{Q}_{-\,k} &= \frac{1}{2}e^{2\phi_\infty}\Im\mathfrak{m}(\lambda_\infty^{*\,2}\Upsilon)\,.
\end{aligned} \tag{6.22}$$

It is now trivial to see that the asymptotic charges eqs. (6.5) are recovered using eq. (4.26) with the Killing vectors given above and

$$\left(g^{AB}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 1/2 & 0 & 0 \end{pmatrix}\,, \tag{6.23}$$

for the $1, +, -$ basis.

The first law eq. (4.24) is recovered with

$$\delta_{1\,\infty} = \frac{1}{2}e^{4\phi_\infty}\delta|\lambda|^2_\infty\,, \qquad \delta_{+\,\infty} = e^{4\phi_\infty}\delta a_\infty\,, \qquad \delta_{-\,\infty} = -\frac{1}{2}e^{4\phi_\infty}(\lambda_\infty^{*\,2}\delta\lambda_\infty + \text{c.c.})\,. \tag{6.24}$$

[22] There are additional tr components that we ignore since they do not contribute to the integrals.

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7 Discussion

Some final comments on our results are in order.

First of all, it is unclear how to give coordinate-independent definitions of scalar charges satisfying a Gauss law in absence of global symmetries. This limitation led us to focus on theories with enough global symmetries to account for all the possible scalar charges. On the other hand, there are not many examples of black-hole solutions in theories with no or very few global isometries. Most of the general recipes elaborated to construct black holes in $\mathcal{N} = 2, d = 4$ theories, for instance, [52] are only valid for extremal black holes, which lie outside the scope of our methods. Working with non-extremal black holes is much more difficult [53] although some general methods have been developed [54] and they should be revisited to study this problem.

In general, a Gauss law is not equivalent to a full conservation law. In our case, the restriction to backgrounds with timelike Killing vectors makes it trivially equivalent to a conservation law in those particular backgrounds, but not in general. We expect, however, that the existence of a rigorous definition can be used to study the evolution of scalar charge or at least its behavior under perturbations.

It is worth stressing the relation between the value of the scalar charge and the existence of a regular bifurcate Killing horizon. In absence of such a horizon there does not seem to be a restriction on that value. It is because of this relation that it can be understood as secondary black-hole hair.

The general procedure that has allowed us to define a $(d-2)$-form satisfying a Gauss law starting from the $(d-1)$-form (Noether current) associated to a global symmetry can probably used in more general settings (fermionic matter, for instance).

As we mentioned before, it should be stressed that these results can be generalized to higher-rank fields and higher dimensions. The NGZ currents have been determined in ref. [34] and one simply has to follow the same steps. It also seems that it should also be possible to find $(d-2)$-forms satisfying Gauss laws starting from any standard Noether current $(d-1)$-forms associated to a global symmetry.

Concerning the first law, in order to recover the GKK scalar term it has been essential to realize that the integral of $\mathbf{W}[k]$ at spatial infinity gives more than just the variations of the gravitational charges at infinity. Often, these contributions have been ignored or set to zero via convenient boundary conditions at spatial infinity. Often, the integral on the bifurcation surface has been also identified with the $T\delta S$ term of the firm law ignoring other contributions (work terms). We think it is now clear that there are different contributions to the first law coming from that integral as well and that the only one which is associate to the entropy is the one that takes the form of a conserved Lorentz charge, as we have pointed out in refs. [12, 13, 26]. Actually, the title of ref. [9] should be replaced by "Black hole entropy is the (Lorentz) Noether charge."

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