Published for SISSA by Springer

Received: *March 17, 2023*
Accepted: *May 3, 2023*
Published: *May 15, 2023*

Integrated negative geometries in ABJM

Johannes M. Henn,[a] **Martín Lagares**[a,b] **and Shun-Qing Zhang**[a]

[a] *Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, 80805 München, Germany*
[b] *Instituto de Física La Plata, La Plata, Argentina*

E-mail: henn@mpp.mpg.de, martinlagares95@gmail.com, sqzhang@mpp.mpg.de

Abstract: We study, in the context of the three-dimensional $\mathcal{N} = 6$ Chern-Simons-matter (ABJM) theory, the infrared-finite functions that result from performing $L - 1$ loop integrations over the L-loop integrand of the logarithm of the four-particle scattering amplitude. Our starting point are the integrands obtained from the recently proposed all-loop projected amplituhedron for the ABJM theory. Organizing them in terms of negative geometries ensures that no divergences occur upon integration if at least one loop variable is left unintegrated. We explicitly perform the integrations up to $L = 3$, finding both parity-even and -odd terms. Moreover, we discuss a prescription to compute the cusp anomalous dimension Γ_{cusp} of ABJM in terms of the integrated negative geometries, and we use it to reproduce the first non-trivial order of Γ_{cusp}. Finally, we show that the leading singularities that characterize the integrated results are conformally invariant.

Keywords: Scattering Amplitudes, Supersymmetric Gauge Theory

ArXiv ePrint: 2303.02996

https://doi.org/10.1007/JHEP05(2023)112

Contents

1 Introduction

Over the last decades collider physics has pushed the threshold of precision in experimental data for particle physics to unprecedented values. Naturally, these results have stimulated the study of scattering amplitudes, leading to substantial developments in the field over the last years. In particular, lots of efforts have been put in trying to explain the surprising simplicity that those observables often show, with the Parke-Taylor formula [1] being the first of many examples. Computations with hundreds of Feynman diagrams often lead to results that can be written within a line, suggesting an underlying simplicity that should be understood. A case in point is the discovery of positive geometries [2–8], that provide a geometrical interpretation for scattering amplitudes in numerous quantum field theories.

In this new geometric picture, the S-matrix of the four-dimensional $\mathcal{N} = 4$ super Yang-Mills (sYM) theory is viewed, in the planar limit, as the *volume* of a mathematical object whose boundaries encode the physical singularities of the amplitudes. More precisely, it has been shown that the canonical form of a certain positive geometry, the so called *Amplituhedron*, gives the tree-level amplitudes and all-loop integrands for an arbitrary n-particle scattering process in the planar $\mathcal{N} = 4$ sYM theory [2]. Interestingly, in this new framework concepts such as locality and unitarity are no longer fundamental principles to be assumed, but rather they are derived properties. This geometric formulation of scattering amplitudes has also been applied in other contexts, such as the bi-adjoint ϕ^3 theory [9], cosmology [10] or, as it will be of interest for us, the three-dimensional $\mathcal{N} = 6$ Chern-Simons-matter theory known as ABJM [11–13]. For recent reviews about the study of amplitudes in terms of positive geometries see [14, 15].

Recently, the authors of [16] proposed a novel way to express the $\mathcal{N} = 4$ sYM amplituhedron as a sum over *negative* geometries. The latter, characterized by a change of sign in the defining inequalities, naturally give rise to the logarithm of the amplitude: its integrand at each loop order is simply given by summing over a certain subset of negative geometries, represented by *connected graphs* in the pictorial representation of [16].

As is well known, infrared divergences exponentiate in planar Yang-Mills theories. In particular, in conformal field theories the logarithm of an amplitude has only double poles $1/\epsilon^2$ in the dimensional regulator $D = 4 - 2\epsilon$ (with the coefficient being the cusp anomalous dimension), while the L-loop amplitude has poles of order $1/\epsilon^{2L}$. In the $\mathcal{N} = 4$ sYM theory there is a naturally related quantity that is completely free of divergences. It arises when one considers the integration of the L-loop negative geometry over $L - 1$ of the loop variables, i.e. leaving one of the loop variables unintegrated [16–25]. Remarkably, this object is infrared (IR) finite, and all the divergences concentrate on the L-th loop integral. This can be seen as a consequence of organizing the results as a sum of negative geometries [16]. Moreover, one can show that the result of integrating $L - 1$ of the loop variables can be expressed in terms of a function of $3n - 11$ conformal cross-ratios. This is the same number of kinematic variables as for QCD n-point amplitudes. This similarity, together with a conjectured duality with pure Yang-Mills all-plus helicity-amplitudes [24], motivates further studies of these finite observables. We will focus on the four-particle case, for which one gets a function $\mathcal{F}(z)$ of a single cross-ratio.

JHEP05(2023)112

An exciting outcome of the study of $\mathcal{F}(z)$ comes when taking into account the duality between scattering amplitudes and Wilson loops in $\mathcal{N} = 4$ sYM [26–29]. Interestingly, this duality allows to recover the L-loop contribution to the cusp anomalous dimension Γ_{cusp} from the $(L-1)$-loop term in the perturbative expansion of $\mathcal{F}(z)$ [16, 21, 23]. This prescription has been used to compute the full four-loop contribution to Γ_{cusp} both in $\mathcal{N} = 4$ sYM and in QCD, including the first non-planar corrections [23].

Besides of the fact that it is IR-finite, many other interesting properties and results have been found for $\mathcal{F}(z)$. As shown by [16], in $\mathcal{N} = 4$ sYM one can perform a non-perturbative sum over a particular subset of negative geometries (more precisely, ladder- and tree-type diagrams), opening the door for a full all-loop computation of $\mathcal{F}(z)$. Such results would allow a comparison with the non-perturbative derivation of Γ_{cusp} coming from integrability [30, 31]. Also surprisingly, the leading singularities of these integrated negative geometries enjoy a (hidden) conformal symmetry [24, 25]. Furthermore, identities relating $\mathcal{F}(z)$ to all-plus amplitudes in pure Yang-Mills theory have been found [24, 25]. Finally, one can also note that the perturbative expansion of $\mathcal{F}(z)$ respects a uniform transcendentality principle [23].

Taking into account the previous considerations, it seems natural to pose the question of how the above results generalize to other theories, ultimately aiming for a generalization to QCD. In this regard, the three-dimensional ABJM theory [32] emerges as a reasonable candidate, given its well-known similarities with $\mathcal{N} = 4$ sYM. Much progress has been made in understanding the properties of scattering amplitudes in this three-dimensional case. The four-particle scattering amplitude is known up to three-loops [33–37], and there is a BDS-like conjecture for the all-loop result [37, 38]. Moreover, even non-planar corrections have been computed [39, 40]. For $n = 6$ and $n = 8$ particles the current frontier is two-loops [41, 42], and there are one-loop results for scattering processes with arbitrary number of particles [43]. Furthermore, the Wilson loops/scattering amplitudes duality is believed to hold for the four-particle case, but has been shown to fail when the number of particles increases [34, 36, 44, 45]. There is also evidence of dual-superconformal [34, 46–48] and Yangian symmetry [33, 49].

The geometric formulation of amplitudes in terms of positive geometries was first extended at tree level to the ABJM theory in [11, 12]. Recently, the authors of [13] proposed an all-loop *projected amplituhedron* for ABJM by imposing a symplectic condition on the amplituhedron of $\mathcal{N} = 4$ sYM. This conjecture has been checked up to $L = 5$ loops for the four-particle case. Along the lines of [16], the projected amplituhedron allows for a decomposition in terms of negative geometries. More importantly, comparing to the four-dimensional case, in three dimensions a smaller number of negative geometries contribute to the integrand of the logarithm of the amplitude. More precisely, only those geometries associated to *bipartite* graphs contribute, allowing for a significant simplification in the perturbative expansion of the integrand. We should note that the study of integrated negative geometries in ABJM is interesting towards an all-loop computation of the ABJM cusp anomalous dimension [40, 50–52]. Non-perturbative results would clear the way for the all-loop computation of the interpolating function $h(\lambda)$ of ABJM [53–61], whose knowledge is crucial to exploit the results coming from integrability. An all-loop expression for $h(\lambda)$ was proposed in [60, 61].

JHEP05(2023)112

In this paper we focus on the ABJM theory, and we explicitly perform the $(L-1)$-loop integrations of the four-particle negative geometries for the $L \leq 3$ cases, showing that the integrated results are given by finite and uniform-transcendental polylogarithmic functions. In an analogous way to the five-particle case of $\mathcal{N}=4$ sYM [25], we find it convenient to organize the integrated results in parity-even and -odd terms, which are described by two functions $\mathcal{F}(z)$ and $\mathcal{G}(z)$, respectively. As we will see, it is straightforward to show that only the former contributes to the cusp anomalous dimension Γ_{cusp} after the last loop integration. Furthermore, we use our results to compute the first non-trivial contribution to Γ_{cusp}, finding perfect agreement with the literature [51]. Finally, we discover that the leading singularities of the integrated results also possess a hidden conformal symmetry, in a similar manner to what was found in the four-dimensional case [24, 25].

The paper is organized as follows. In section 2 we review the role that negative geometries play in the construction of integrands in $\mathcal{N}=4$ sYM and in ABJM, and we finish with a discussion of how dual conformal invariance constrains the expressions that come from integrating these geometries. Then, in section 3 we perform, up to two loops, the explicit integration of the negative geometries of ABJM. In section 4 we discuss how one can compute the cusp anomalous dimension of ABJM as a consequence of applying a functional on the integrated negative geometries. In section 5 we turn to the analysis of the transcendental weight properties of our results. Section 6 is devoted to the symmetry analysis of the leading singularities that characterize the integrated results. We give our conclusions in section 7. Finally, there are three appendices that complement the results discussed in the main body of the paper.

2 Integrands from negative geometries

In this paper we will analyze, within the context of the ABJM theory, the behaviour of L-loop integrands for the logarithm of the amplitude after performing $L-1$ of the corresponding loop integrations. Therefore, it is instructive to review how one can express the aforementioned integrands in terms of canonical forms of negative geometries.

Let us consider a D-dimensional gauge theory, with focus on a four-particle scattering process of particles with momenta p_i, $1 \leq i \leq 4$. To describe the external kinematics, we will either use dual-space coordinates (i.e. $p_i = x_{i+1} - x_i$ with $1 \leq i \leq 4$) or momentum-twistor notation [62]. We define

$$\mathcal{M} := \frac{\mathcal{A}}{\mathcal{A}_{\text{tree}}}\,, \tag{2.1}$$

where $\mathcal{A}$ is the color-ordered maximally helicity-violating (MHV) scattering amplitude and $\mathcal{A}_{\text{tree}}$ is its corresponding tree-level value. We will refer to $\mathcal{M}$ as the scattering amplitude, for simplicity. We define the L-loop integrand $\mathcal{I}_L$ for $\mathcal{M}$ as[1]

$$\mathcal{M}\Big|_{\text{L loops}} := \left(\prod_{j=5}^{4+L} \int \frac{d^D x_j}{i\pi^{D/2}}\right) \mathcal{I}_L\,, \tag{2.2}$$

[1]The Amplituhedron is defined for integer dimensions, i.e. $D=4$ and $D=3$ in the $\mathcal{N}=4$ SYM and ABJM cases, respectively. When considering the amplitude, an infrared regulator needs to be specified, for example dimensional regularization. Since we consider finite quantities, considering the integer-dimensional Amplituhedron integrands is sufficient for our purposes.

JHEP05(2023)112

where $x_5, x_6, \ldots, x_{4+L}$ describe the loop variables. Similarly, we take the L-loop integrand $\mathcal{L}_L$ for the logarithm of the scattering amplitude to be defined as

$$\log \mathcal{M}\Big|_{\text{L loops}} := \left(\prod_{j=5}^{4+L} \int \frac{d^D x_j}{i\pi^{D/2}}\right) \mathcal{L}_L \,. \tag{2.3}$$

We now turn to the computation of the integrands $\mathcal{I}_L$ and $\mathcal{L}_L$. In order to introduce the main ideas, let us focus first on the $\mathcal{N}=4$ sYM theory. There are many ways of obtaining four-point integrands at high loop orders, including generalized unitarity [63], on-shell recursion relations [64], soft-collinear consistency conditions [65, 66], and a connection to correlation functions [67], for example. A conceptual breakthrough was achieved in [2], where it was proposed that in the $\mathcal{N}=4$ sYM theory the integrands $\mathcal{I}_L$ are proportional to the canonical form of a positive geometry known as the *Amplituhedron*. To be more precise, let us take Z_a^I, $a=1,\ldots,4$ to be the four-dimensional momentum-twistors that describe the external kinematic data of the scattering process, and let us consider the region in momentum-twistor space described by the constraint

$$\langle 1234\rangle > 0\,, \tag{2.4}$$

with $\langle 1234\rangle = \epsilon_{IJKL} Z_1^I Z_2^J Z_3^K Z_4^L$. Moreover, we shall take L lines $l_5 := AB$, $l_6 := CD$, $l_7 := EF$, $\ldots$, in momentum-twistor space such that for each one of them we impose

$$\langle l_i 12\rangle > 0, \quad \langle l_i 23\rangle > 0, \quad \langle l_i 34\rangle > 0, \quad \langle l_i 14\rangle > 0\,, \tag{2.5}$$

$$\langle l_i 13\rangle < 0, \quad \langle l_i 24\rangle < 0\,. \tag{2.6}$$

Finally, let us demand that each pair of different lines satisfies the mutual positivity constraint

$$\langle l_i l_j\rangle > 0\,. \tag{2.7}$$

Then, the four-particle L-loop MHV Amplituhedron is defined as the set of points in momentum-twistor space that are subjected to the constraints given in (2.4)–(2.7). One can associate to the Amplituhedron a unique canonical differential form Ω with logarithmic singularities on the boundaries of the space. Let us introduce the notation

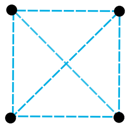
$$\Omega = \sum_{L=1}^{\infty} \lambda^L\, \Omega_L\,, \tag{2.8}$$

where λ is the 't Hooft coupling (we are working on the planar limit). Then, the L-loop integrand $\mathcal{I}_L$ for the scattering amplitude is simply given as

$$\mathcal{I}_L = n_L\, \Omega_L\,, \tag{2.9}$$

where n_L is a normalization factor which we discuss in the appendix B.

In the following it will prove useful to take into account the pictorial representation introduced in [16] to describe positive geometries. We will use a node to indicate a one-loop amplituhedron associated to a certain loop variable, i.e. a geometry satisfying the constraints (2.4)–(2.6), and a dashed light-blue line to describe a mutual positivity condition between a pair of loop variables. As an example, the four-loop amplituhedron will be drawn as

 (2.10)

As pointed out in [16], it turns out to be very convenient to consider also mutual negativity conditions between loop variables. That is, constraints given by

$$\langle l_i l_j \rangle < 0\,, \tag{2.11}$$

for which we will use thick red lines, e.g.

 (2.12)

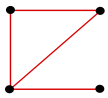
In order to understand the advantages of using negative geometries for the computation of integrands, let us introduce the notation

$$\tilde{\Omega} := \log \Omega\,, \tag{2.13}$$

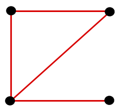
with

$$\tilde{\Omega} = \sum_{L=1}^{\infty} \lambda^L \, \tilde{\Omega}_L\,. \tag{2.14}$$

Then, as described in [16], one can expand $\tilde{\Omega}$ in terms of *connected* negative geometries. More precisely,

$$\tilde{\Omega} = \sum_{\substack{\text{all connected} \\ \text{graphs } G}} (-1)^{E(G)} \lambda^L$$  (2.15)

where $E(G)$ is the number of edges of a graph G and L is the corresponding number of vertices. Therefore,

$$\tilde{\Omega}_L = \sum_{\substack{\text{all connected graphs G} \\ \text{with } L \text{ vertices}}} (-1)^{E(G)}$$  (2.16)

The integrand $\mathcal{L}_L$ is obtained from $\tilde{\Omega}_L$ as

$$\mathcal{L}_L = \tilde{n}_L \, \tilde{\Omega}_L\,. \tag{2.17}$$

We refer again to the appendix B for the discussion on the computation of the relative normalizations $\tilde{n}_L$.

JHEP05(2023)112

2.1 Projected amplituhedron for the ABJM theory

Following the ideas of [13], we will now discuss how the previous concepts generalize to the three-dimensional ABJM theory. The four-particle amplituhedron of the ABJM theory can be obtained from projecting the amplituhedron of the $\mathcal{N}=4$ sYM theory to three dimensions by means of a symplectic constraint. More specifically, the corresponding positive geometry is defined by considering, in addition to the conditions given in (2.4)–(2.7), the constraints

$$\Sigma_{IJ} Z_i^I Z_{i+1}^J = 0\,, \tag{2.18}$$

for the external kinematic data and

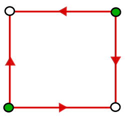
$$\Sigma_{IJ} A^I B^J = 0\,, \tag{2.19}$$

for the loop variables, with Σ being a symplectic matrix given as

$$\Sigma = \begin{pmatrix} 0 & \epsilon_{2\times 2} \\ \epsilon_{2\times 2} & 0 \end{pmatrix}, \tag{2.20}$$

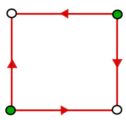
where $\epsilon_{2\times 2}$ is a totally anti-symmetric tensor.

One major simplification occurs in ABJM when considering the expansion of $\tilde{\Omega}_L$ into negative geometries, namely that only *bipartite* (connected) graphs are required [13]. The latter are defined as those graphs where, after assigning an orientation to each edge, each node is either a sink or a source. Examples of bipartite graphs are



where green nodes represent sources and white nodes correspond to sinks. Taking into account the above simplification, the expansion (2.16) now becomes

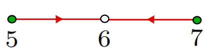
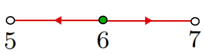
$$\tilde{\Omega}_L = \sum_{\substack{\text{all connected and bipartite} \\ \text{graphs G with } L \text{ vertices}}} (-1)^{E(G)} \tag{2.21}$$



Using (2.21), the canonical forms $\tilde{\Omega}_L$ were computed in [13] up to $L=5$. In particular, for the first three loop orders one gets

$$\tilde{\Omega}_1 = \frac{c\,\epsilon_5}{s_5 t_5}$$

$\tilde{\Omega}_2 = -$  $-$  $= -\dfrac{2c^2}{D_{56}}\left(\dfrac{1}{s_5 t_6} + \dfrac{1}{t_5 s_6}\right)$

$\tilde{\Omega}_3 =$  $+$  $+\,2\,\text{perms.} \;=\; \dfrac{4c^2\epsilon_6}{s_5 t_6 s_7 D_{56} D_{67}} + (s \leftrightarrow t) + 2\,\text{perms.}$ (2.22)

with

$$s_i := \langle l_i 12\rangle\langle l_i 34\rangle\,, \qquad t_i := \langle l_i 23\rangle\langle l_i 14\rangle\,, \qquad D_{ij} := -\langle l_i l_j\rangle\,, \tag{2.23}$$

$$c := \langle 1234\rangle\,, \qquad \epsilon_i := \sqrt{\langle l_i 13\rangle\langle l_i 24\rangle\langle 1234\rangle}\,, \tag{2.24}$$

and where we are again using the notation $l_5 := AB$, $l_6 := CD$, $l_7 := EF$, $\ldots$, for the loop lines, with the permutations being over all nonequivalent configurations of these variables. Let us note that for simplicity of notation we are omitting the $d^3 l_i$ factors in the differential forms.

Finally, as shown in appendix B, the relative normalizations $\tilde{n}_L$ defined in (2.17) are given by

$$\tilde{n}_1 = \frac{i}{2\sqrt{\pi}}\,, \qquad \tilde{n}_2 = \frac{\tilde{n}_1^2}{2!}\,, \qquad \tilde{n}_3 = \frac{\tilde{n}_1^3}{3!}\,. \tag{2.25}$$

Let us note that in order to get (2.25) we are assuming the standard convention

$$\lambda := \frac{N}{k} \tag{2.26}$$

for the 't Hooft coupling of ABJM, with N being the number of colors and k the Chern-Simons level. In the following we are going to use the differential forms (2.22) along with the normalizations (2.25) as the starting point for performing the loop integrations.

2.2 Constraints from dual conformal invariance

Before discussing the explicit integration of the negative geometries, let us analyze the constraints that dual conformal invariance imposes on the integrated expressions. This symmetry can be understood as a consequence of the duality between scattering amplitudes and Wilson loops. The latter is conjectured to hold in $\mathcal{N} = 4$ sYM and to partially extend to the ABJM case [26–29, 34, 36, 44, 45]. Indeed, the dual conformal invariance of scattering amplitudes is simply the conformal invariance of the Wilson loops in the dual picture. We will return to the Wilson loops/scattering amplitudes duality in section 4.

We will begin again by reviewing the $\mathcal{N} = 4$ sYM case. In four dimensions, the dual conformal invariance of the logarithm of the amplitude implies that there exists a function $\mathcal{F}_{L-1}$ such that [17]

$$\left(\prod_{j=6}^{4+L} \int \frac{d^4 x_j}{i\pi^2}\right) \mathcal{L}_L = \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \frac{\mathcal{F}_{L-1}(z)}{\pi^2}\,, \tag{2.27}$$

where the cross-ratio z, defined as

$$z = \frac{x_{25}^2 x_{45}^2 x_{13}^2}{x_{15}^2 x_{35}^2 x_{24}^2}\,, \tag{2.28}$$

is the only dual conformally invariant cross-ratio that can be built using the external kinematic data and the unintegrated loop variable x_5. The function $\mathcal{F}_{L-1}(z)$ has been computed in the literature up to $L = 4$ [20, 21, 23]. Moreover, the above analysis has also been extended to higher-point scattering processes, in which the integrated results depend on functions of more than one cross-ratio. In particular, the five-particle case was studied up to $L = 3$ [25], while for an arbitrary number of particles the current threshold is $L = 2$ [24].

Let us consider now the above ideas within the context of the ABJM theory. Interestingly, in the three-dimensional case the expression (2.27) is incomplete. To see this, it is convenient to use five-dimensional notation to describe the coordinates of the dual space (see for example [34, 41] and appendix A). One can then see that the most general expression that one can construct in order to generalize (2.27) to the three-dimensional case is

$$\left(\prod_{j=6}^{4+L}\int\frac{d^3X_j}{i\pi^{3/2}}\right)\mathcal{L}_L=\left(\frac{X_{13}^2X_{24}^2}{X_{15}^2X_{25}^2X_{35}^2X_{45}^2}\right)^{\frac{3}{4}}\frac{\mathcal{F}_{L-1}(z)}{\sqrt{\pi}}+\frac{\epsilon(1,2,3,4,5)}{X_{15}^2X_{25}^2X_{35}^2X_{45}^2}\frac{\mathcal{G}_{L-1}(z)}{\sqrt{\pi}}\,, \tag{2.29}$$

where capital letters refer to five-dimensional coordinates and

$$\epsilon(1,2,3,4,5):=\epsilon_{\mu\nu\rho\sigma\eta}X_1^\mu X_2^\nu X_3^\rho X_4^\sigma X_5^\eta\,. \tag{2.30}$$

Therefore, when going to three dimensions we have to include in (2.27) an additional parity-odd term given by a function $\mathcal{G}_{L-1}(z)$. This is analogous to what was found in $\mathcal{N}=4$ sYM for the five-particle case [25]. We will discuss the computation of $\mathcal{F}(z)$ and $\mathcal{G}(z)$ in the next section.

3 Perturbative analysis

We will turn now to the explicit $L-1$ loop integrations (up to $L=3$) of the negative geometries obtained from the projected amplituhedron for ABJM [13], seeking for the perturbative expansion of the $\mathcal{F}$ and $\mathcal{G}$ functions defined in (2.29).

3.1 Tree level

Let us begin by computing the tree-level values of $\mathcal{F}(z)$ and $\mathcal{G}(z)$. From (2.22) we have

$$\tilde{\Omega}_1=\frac{\langle 1234\rangle^{3/2}\sqrt{\langle l_5 13\rangle\langle l_5 24\rangle}}{\langle l_5 12\rangle\langle l_5 23\rangle\langle l_5 34\rangle\langle l_5 14\rangle}. \tag{3.1}$$

Using the Schouten identity

$$\langle l_5 13\rangle\langle l_5 24\rangle=\langle l_5 12\rangle\langle l_5 34\rangle+\langle l_5 23\rangle\langle l_5 14\rangle\,, \tag{3.2}$$

we can rewrite (3.1) in terms of five-dimensional dual coordinates (for a nice discussion on that see for example [68]) as follows,

$$\tilde{\Omega}_1=\frac{\sqrt{X_{13}^2X_{24}^2\,(X_{24}^2X_{15}^2X_{35}^2+X_{13}^2X_{25}^2X_{45}^2)}}{X_{15}^2X_{25}^2X_{35}^2X_{45}^2}\,. \tag{3.3}$$

In order to compute $\mathcal{F}_0(z)$ and $\mathcal{G}_0(z)$ it is important to note that the integration is over the three-dimensional Minkowski-space. However, (3.3) was derived within the Amplituhedron region defined in (2.4)–(2.7), and therefore we need to extend its definition. Indeed, one can see that naively integrating (3.3) over the whole kinematic space gives a non-zero result, in contradiction with what is expected for the one-loop four-particle amplitude of ABJM [48, 69]. This issue can be resolved by taking into account the identity[2]

$$\epsilon(1,2,3,4,5)=\frac{1}{4}\sqrt{-X_{13}^2X_{24}^2\,(X_{24}^2X_{15}^2X_{35}^2+X_{13}^2X_{25}^2X_{45}^2)}\,, \tag{3.4}$$

[2]Any possible overall sign ambiguity that could arise when using (3.4) should be absorbed in the sign of the overall normalization of the amplitude.

which can be probed to be valid for real values of the dual-coordinates.[3] Then, taking into account the normalization presented in (2.25) and using (3.4) to rewrite (3.3) in terms of the five-dimensional Levi-Civita tensor we arrive at

$$\mathcal{L}_1 = -\frac{2}{\sqrt{\pi}} \frac{\epsilon(1,2,3,4,5)}{X_{15}^2 X_{25}^2 X_{35}^2 X_{45}^2} \,. \tag{3.5}$$

Therefore, comparing to the definition in (2.29), we conclude

$$\mathcal{F}_0(z) = 0 \,, \quad \text{and} \quad \mathcal{G}_0(z) = -2 \,. \tag{3.6}$$

3.2 One loop

As discussed in the previous sections, the integrand $\mathcal{L}_2$ that is obtained from the canonical form $\tilde{\Omega}_2$ given in (2.22) reads

$$\begin{aligned} \mathcal{L}_2 &= \frac{c^2}{4\pi \, D_{56}} \left(\frac{1}{s_5 t_6} + \frac{1}{t_5 s_6} \right) = \\ &= \frac{X_{13}^2 X_{24}^2}{4\pi \, X_{56}^2} \left(\frac{1}{X_{15}^2 X_{26}^2 X_{35}^2 X_{46}^2} + \frac{1}{X_{16}^2 X_{25}^2 X_{36}^2 X_{45}^2} \right) . \end{aligned} \tag{3.7}$$

To obtain $\mathcal{F}_1(z)$ and $\mathcal{G}_1(z)$ we should now perform one of the loop integrations, which we choose to be the one over X_6 (i.e. we take X_5 to be the frozen loop variable). This integral turns out to be a triangle integral with three massive legs. Consequently, using the results of appendix C, we arrive at

$$\begin{aligned} \int \frac{d^3 X_6}{i\pi^{3/2}} \mathcal{L}_2 &= \frac{X_{13}^2 X_{24}^2}{4\pi} \int \frac{d^3 X_6}{i\pi^{3/2}} \frac{1}{X_{56}^2} \left(\frac{1}{X_{15}^2 X_{26}^2 X_{35}^2 X_{46}^2} + \frac{1}{X_{16}^2 X_{25}^2 X_{36}^2 X_{45}^2} \right) \\ &= -\frac{\sqrt{\pi}}{4} \left(\frac{X_{13}^2 X_{24}^2}{X_{15}^2 X_{25}^2 X_{35}^2 X_{45}^2} \right)^{3/4} \left(z^{1/4} + \frac{1}{z^{1/4}} \right) . \end{aligned} \tag{3.8}$$

Therefore, from (2.29) we deduce

$$\mathcal{F}_1(z) = -\frac{\pi}{4} \left(z^{1/4} + \frac{1}{z^{1/4}} \right) , \qquad \text{and} \quad \mathcal{G}_1(z) = 0 \,. \tag{3.9}$$

3.3 Two loops

The integrand $\mathcal{L}_3$, which can be obtained from (2.22) and (2.25), is

$$\mathcal{L}_3 = -\frac{i}{12\pi^{3/2}} \frac{c^2 \epsilon_6}{s_5 t_6 s_7 D_{56} D_{67}} + (s \leftrightarrow t) + 2 \, \text{perms.} \tag{3.10}$$

In order to compute $\mathcal{F}_2$ and $\mathcal{G}_2$ we will perform two of the loop integrations. We will choose to integrate over X_6 and X_7, and again we will keep X_5 frozen. Let us begin by considering the first term of the r.h.s. in (3.10), which is explicitly given by

$$\mathcal{L}_3^{(1)} = \frac{1}{3\pi^{3/2}} \frac{X_{13}^2 \, \epsilon(1,2,3,4,6)}{X_{15}^2 X_{35}^2 X_{26}^2 X_{46}^2 X_{17}^2 X_{37}^2 X_{56}^2 X_{67}^2} \,. \tag{3.11}$$

[3] Let us note that both sides of eq. (3.4) are complex-valued in the Amplituhedron region. Consequently, to get the one-loop amplitude one should perform an analytic continuation to the region in which all dual-coordinates are real-valued.

JHEP05(2023)112

First, the integral over X_7 is again a triangle integral; therefore, we get

$$\int \frac{d^3X_7}{i\pi^{3/2}}\,\mathcal{L}_3^{(1)} = \frac{1}{3}\,\frac{X_{13}\,\epsilon(1,2,3,4,6)}{X_{15}^2X_{35}^2X_{26}^2X_{46}^2X_{56}^2X_{16}X_{36}}\,. \tag{3.12}$$

We will turn now to the integration over X_6. To compute this integral we will make use of the results derived in appendix C. In particular, we have

$$\int \frac{d^3X_6}{i\pi^{3/2}}\,\frac{\epsilon(1,2,3,4,6)}{X_{26}^2X_{46}^2X_{56}^2X_{16}X_{36}} = \frac{\epsilon(1,2,3,4,5)}{(X_{15}^2X_{25}^2X_{35}^2X_{45}^2X_{24}^2)^{1/2}}\,\frac{\mathcal{H}(z)}{\sqrt{\pi z}}\,, \tag{3.13}$$

where the weight-two function $\mathcal{H}(z)$ takes the form

$$\begin{aligned}\mathcal{H}(z) = \sqrt{\frac{z}{1+z}}\Big(&\pi^2 + 2\,\log\!\left(\sqrt{z}+\sqrt{1+z}\right)\log(4z)\\ &+ \mathrm{Li}_2\!\left[-2\!\left(z+\sqrt{z(1+z)}\right)\right] - \mathrm{Li}_2\!\left[-2\!\left(z-\sqrt{z(1+z)}\right)\right]\Big)\,.\end{aligned} \tag{3.14}$$

Consequently,

$$\int \frac{d^3X_6}{i\pi^{3/2}}\int \frac{d^3X_7}{i\pi^{3/2}}\,\mathcal{L}_3^{(1)} = \frac{1}{3\sqrt{\pi}}\,\frac{\epsilon(1,2,3,4,5)}{X_{15}^2X_{25}^2X_{35}^2X_{45}^2}\,\mathcal{H}(z)\,. \tag{3.15}$$

For the $X_6 \leftrightarrow X_7$ permutation, i.e. for

$$\mathcal{L}_3^{(2)} := \frac{1}{3\pi^{3/2}}\,\frac{X_{13}^2\,\epsilon(1,2,3,4,7)}{X_{15}^2X_{35}^2X_{27}^2X_{47}^2X_{16}^2X_{36}^2X_{57}^2X_{67}^2}\,, \tag{3.16}$$

we similarly get

$$\int \frac{d^3X_6}{i\pi^{3/2}}\int \frac{d^3X_7}{i\pi^{3/2}}\,\mathcal{L}_3^{(2)} = \frac{1}{3\sqrt{\pi}}\,\frac{\epsilon(1,2,3,4,5)}{X_{15}^2X_{25}^2X_{35}^2X_{45}^2}\,\mathcal{H}(z)\,. \tag{3.17}$$

Finally, let us consider the $X_5 \leftrightarrow X_6$ permutation. That is, let us take

$$\mathcal{L}_3^{(3)} := \frac{1}{3\pi^{3/2}}\,\frac{X_{13}^2\,\epsilon(1,2,3,4,5)}{X_{16}^2X_{36}^2X_{25}^2X_{45}^2X_{17}^2X_{37}^2X_{56}^2X_{67}^2}\,. \tag{3.18}$$

We can see that the integrals of $\mathcal{L}_3^{(3)}$ over X_6 and X_7 are simply two triangle integrals, and therefore

$$\int \frac{d^3X_6}{i\pi^{3/2}}\int \frac{d^3X_7}{i\pi^{3/2}}\,\mathcal{L}_3^{(3)} = \frac{\pi^{3/2}}{3}\,\frac{\epsilon(1,2,3,4,5)}{X_{15}^2X_{26}^2X_{35}^2X_{45}^2}\,. \tag{3.19}$$

Adding the corresponding $(s \leftrightarrow t)$ terms, we finally get

$$\begin{aligned}\mathcal{F}_2(z) &= 0\,,\\ \mathcal{G}_2(z) &= \frac{2}{3}\left[\mathcal{H}(z) + \mathcal{H}\!\left(\frac{1}{z}\right) + \pi^2\right].\end{aligned} \tag{3.20}$$

4 Cusp anomalous dimension

We will discuss now how to obtain the cusp anomalous dimension Γ_{cusp} of ABJM from $\mathcal{F}(z)$ and $\mathcal{G}(z)$. Let us being then by recalling that Γ_{cusp} is defined as [70]

$$\Gamma_{\text{cusp}} = \mu \frac{d \log Z_{\text{cusp}}}{d\mu} \,, \tag{4.1}$$

where μ is the renormalization scale of the theory and Z_{cusp} is the renormalization factor introduced to renormalize the vacuum expectation value of Wilson loops with light-like cusps.

In $\mathcal{N} = 4$ sYM, an all-loop prediction for Γ_{cusp} was obtained following integrability ideas [30]. This result was expressed as a function of an interpolating function $h(\lambda)$, which governs the dispersion relation of magnons in the integrability picture [53–61]. At weak coupling, it was shown that

$$\Gamma_{\text{cusp}}(h) = 4h^2 - \frac{4}{3}\pi^2 h^4 + \frac{44}{45}\pi^4 h^6 + \dots \,. \tag{4.2}$$

Crucially, in $\mathcal{N} = 4$ sYM the interpolating function was proven to simply be

$$h(\lambda) = \frac{\sqrt{\lambda}}{4\pi} \,, \tag{4.3}$$

at all loops [71].

In terms of the $\mathcal{N} = 4$ sYM result, the cusp anomalous dimension of ABJM was proposed to be [50]

$$\Gamma^{\text{ABJM}}_{\text{cusp}} = \frac{1}{4} \Gamma^{\mathcal{N}=4}_{\text{cusp}} \Big|_{h^{\mathcal{N}=4} \to h^{\text{ABJM}}} \,. \tag{4.4}$$

However, the interpolating function $h(\lambda)$ of ABJM has proven to be much less trivial than its $\mathcal{N} = 4$ sYM counterpart. An all-loop proposal was made in [60, 61], giving

$$\lambda = \frac{\sinh(2\pi h)}{2\pi} \, {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\sinh^2(2\pi h)\right) . \tag{4.5}$$

Therefore, in the weak-coupling limit we get

$$\Gamma_{\text{cusp}}(\lambda) = \lambda^2 - \pi^2 \lambda^4 + \frac{49\pi^4}{30} \lambda^6 + \dots \,. \tag{4.6}$$

The above proposal is consistent with the leading-order perturbative result computed in [51].

Wilson loops seem to be intimately related to scattering amplitudes within the context of the AdS/CFT correspondence. First observed in $\mathcal{N} = 4$ sYM and then partially extended to ABJM, there is a duality that relates scattering amplitudes and polygonal light-like Wilson loops [26–29, 34, 36, 44, 45]. To be more specific, let us focus again on a four-particle MHV scattering process characterized by four points x_i in the dual coordinate space. Moreover, let us consider a tetragonal light-like Wilson loop W_4 whose vertices locate at the x_i points. Then, the duality identifies

$$\log \mathcal{M} \sim \log \langle W_4 \rangle \,, \tag{4.7}$$

at the level of the integrands.[4] We should note that, while in $\mathcal{N}=4$ sYM the duality is believed to hold for arbitrary number of particles, in ABJM the duality has only been observed for the four-particle case and has been proven to fail for higher numbers of particles [34, 36, 44, 45].

In order to exploit the Wilson loops/scattering amplitudes duality to relate $\Gamma_{\rm cusp}$ to negative geometries we should recall that the renormalization theory of light-like Wilson loops [70] implies

$$\log\langle W_4\rangle = -2\sum_{L=1}^{\infty}\frac{\lambda^L\,\Gamma_{\rm cusp}^{(L)}}{(L\epsilon)^2}+\mathcal{O}(1/\epsilon)\,, \tag{4.8}$$

where $\Gamma_{\rm cusp}^{(L)}$ is the L-loop coefficient of the cusp anomalous dimension. Therefore, we get

$$\int\frac{d^DX_5}{i\pi^{D/2}}\left[\left(\frac{X_{13}^2X_{24}^2}{X_{15}^2X_{25}^2X_{35}^2X_{45}^2}\right)^{\frac{3}{4}}\frac{\mathcal{F}(z)}{\sqrt{\pi}}+\frac{\epsilon(1,2,3,4,5)}{X_{15}^2X_{25}^2X_{35}^2X_{45}^2}\frac{\mathcal{G}(z)}{\sqrt{\pi}}\right]=-2\sum_{L=1}^{\infty}\frac{\lambda^L\,\Gamma_{\rm cusp}^{(L)}}{\epsilon^2}+\mathcal{O}(1/\epsilon)\,, \tag{4.9}$$

with $D=3-2\epsilon$ and where, as in the $\mathcal{N}=4$ sYM case [21], we have included for dimensional reasons an L^2 factor in the L-loop contribution. Then, after defining

$$I_{\mathcal{F}}[\mathcal{F}_{L-1}] := \left[-\frac{1}{2\sqrt{\pi}}\int\frac{d^DX_5}{i\pi^{D/2}}\left(\frac{X_{13}^2X_{24}^2}{X_{15}^2X_{25}^2X_{35}^2X_{45}^2}\right)^{\frac{3}{4}}\mathcal{F}_{L-1}(z)\right]_{1/\epsilon^2\text{ term}}\,, \tag{4.10}$$

$$I_{\mathcal{G}}[\mathcal{G}_{L-1}] := \left[-\frac{1}{2\sqrt{\pi}}\int\frac{d^DX_5}{i\pi^{D/2}}\frac{\epsilon(1,2,3,4,5)}{X_{15}^2X_{25}^2X_{35}^2X_{45}^2}\,\mathcal{G}_{L-1}(z)\right]_{1/\epsilon^2\text{ term}}\,, \tag{4.11}$$

we have

$$\Gamma_{\rm cusp}^{(L)} = I_{\mathcal{F}}[\mathcal{F}_{L-1}]+I_{\mathcal{G}}[\mathcal{G}_{L-1}]\,. \tag{4.12}$$

As shown in appendix C, using Feynman parametrization we get

$$I_{\mathcal{F}}[z^p] = -\frac{2}{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}+p\right)\Gamma\left(\frac{3}{4}-p\right)}\,, \tag{4.13}$$

$$I_{\mathcal{G}}[z^p] = 0\,. \tag{4.14}$$

Therefore, we see that (4.12) together with (4.13) and (4.14) gives us a prescription to compute $\Gamma_{\rm cusp}$ from the knowledge of $\mathcal{F}(z)$. In particular, the results of section 3 allow us to recover the leading-order contribution to the cusp anomalous dimension of ABJM, i.e.

$$\Gamma_{\rm cusp}(\lambda) = \lambda^2+\mathcal{O}(\lambda^4)\,, \tag{4.15}$$

in accordance with eq. (4.6).

5 Transcendental weight properties of the results

In this section we discuss the transcendental weight properties of our results, and more generally of loop corrections in the three-dimensional ABJM theory.

[4] To be precise one should specify a prescription to compute the integrands. At the amplitude's side of the duality the integrand is fixed by using dual-space coordinates and requiring the correct pole structure, while at the Wilson loop's side one should use the method of the lagrangian insertions to build the integrand [72].

JHEP05(2023)112

5.1 Preliminaries

The appearance of series expansions in the coupling constant λ or in the dimensional regularization parameter ϵ is ubiquitous in the context of quantum field theories. In this framework, the analysis of the transcendental degree properties of the different terms in a given expansion has proven to be a powerful tool for the study of scattering amplitudes and Feynman integrals. As an example, the method of canonical differential equations [73] relies on insight about which loop integrands integrate to uniform transcendental weight functions. In this section we will turn to the study of the transcendental degree properties of the results we have presented so far.

Let us begin by recalling that the *transcendental weight* (also called *degree of transcendentality*) T of a function f is defined as the number of iterated integrals that are needed to compute f [73], e.g. $T(\mathrm{Li}_n) = n$. Moreover, one can extend the definition of T to transcendental numbers, i.e. numbers that can not be obtained as the solution of a polynomial equation with rational coefficients. For example, $T(\pi) = 1$ and $T(\zeta_n) = n$. A series expansion is said to have uniform degree of transcendentality (often abbreviated as UT) when all its terms have the same degree T. Moreover, when discussing Laurent expansions in the dimensional regularization parameter ϵ, it is natural to assign weight -1 to ϵ (see [74] for a review).

5.2 Transcendental weight properties of three-dimensional scattering amplitudes and Feynman integrals in the literature

Scattering amplitudes in $\mathcal{N} = 4$ sYM are conjectured to have uniform transcendental weight [30, 75–78] (and, as a consequence, the same holds for the cusp anomalous dimension Γ_{cusp}). Furthermore, it has been observed that the leading transcendental weight terms of Γ_{cusp} agree between $\mathcal{N} = 4$ sYM and QCD [79, 80]. The specific weight at a given loop order L depends on the choice of the effective coupling constant. In $\mathcal{N} = 4$ sYM, a natural normalization choice for the effective coupling is $g^2 := g^2_{\mathrm{YM}}/(16\pi^2)$. With that normalization choice, the L-loop coefficients of scattering amplitudes are observed to have weight $2L$. See [74] for more details.

Similar uniform weight properties of scattering amplitudes and Γ_{cusp} have been observed in ABJM [34, 41–43, 50, 81]. Since the specific weight at each loop order depends on the choice of effective coupling (and whether its weight is counted or not), let us recall that the standard choice used in the literature is

$$\lambda = \frac{N}{k} \,. \tag{5.1}$$

With the choice (5.1) of the effective coupling, we find that the results presented in the literature for scattering amplitudes and Wilson loops are consistent with L-loop coefficients having weight L. This is also the case for the conjectured all-loop formula for the cusp anomalous dimension (see eq. (4.6)) when multiplied by $1/\epsilon^2$ (recalling that it appears in scattering amplitudes in this combination).

JHEP05(2023)112

Let us note that we could alternatively use the following effective coupling,

$$\tilde{\lambda} = \frac{\lambda}{\sqrt{\pi}} \,. \tag{5.2}$$

This would lead to a transcendental weight of $3L/2$ at L loops. This may be a natural choice, as in this case one could say that the weight is $D_0 L/2$ at L loops, with $D = D_0 - 2\epsilon$, which then applies both to $\mathcal{N} = 4$ sYM and ABJM.

Ultimately, the transcendental weight properties of amplitudes can be traced back to properties of Feynman loop integrals. Here we wish to remind readers of what is known, and to establish what our conventions for loop integrals are. When computing Feynman integrals, the following measure is commonly used,

$$\frac{d^D k}{i\pi^{D/2}} \,, \tag{5.3}$$

for each loop integration. This convention has the effect that when switching to Feynman parametrization there are no explicit factors of π. Of course, when computing QFT observables the choice of measure cannot be seen in an isolated way, but is related to the choice of effective coupling, such as eq. (5.1) or eq. (5.2).

It has been observed that with the convention (5.3) the maximal weight of L-loop Feynman integrals is $D_0 L/2$ if D_0 is even (again we are taking $D = D_0 - 2\epsilon$). This is in agreement with [82]. For example, the well-known four-point amplitude in $\mathcal{N} = 4$ SYM is given by g^2 times the following box integral (for $D = 4 - 2\epsilon$),

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{st}{(k-p_1)^2 k^2 (k+p_2)^2 (k+p_2+p_3)^2} \,, \tag{5.4}$$

which has uniform weight 2, in agreement the statements made above.

Much less is known about integrals in odd space-time dimensions D_0. It is interesting to inspect the integrals computed in this paper, see section 3 and appendix C. Using the integration measure (5.3) and the alternative convention (5.2) for the coupling constant[5] we find that the one-loop integral (3.8) (see also the triangle integral in (C.1) and the epsilon integral in (C.3)) and the two-loop integral (3.15) have weight $D_0 L/2$. This appears to lie within the bound proposed by [82].

5.3 Transcendental weight properties of $\mathcal{F}$ and $\mathcal{G}$

Let us now turn to our tree-level, one-loop and two-loop results for the functions $\mathcal{F}$ and $\mathcal{G}$, given in eqs. (3.6), (3.9) and (3.20), respectively. Putting them together, we have

$$\mathcal{F}(z) = \sum_{L=1}^{\infty} \mathcal{F}_{L-1}(z) \lambda^L \,, \tag{5.5}$$

$$\mathcal{G}(z) = \sum_{L=1}^{\infty} \mathcal{G}_{L-1}(z) \lambda^L \,, \tag{5.6}$$

[5] With the alternative convention $\tilde{\lambda} = \frac{\lambda}{\sqrt{\pi}}$ one gets that the normalizations $\tilde{n}_L$ become

$$n_1 = i \,, \qquad n_2 = \tilde{n}_2 = \frac{\tilde{n}_1^2}{2!} \,, \qquad n_3 = \tilde{n}_3 = \frac{\tilde{n}_1^3}{3!} \,.$$

This change in the normalization has to be taken into account when revisiting the integrals (3.8) and (3.15).

with

$$\frac{\mathcal{F}(z)}{\lambda} = -\frac{\pi}{4}\left(z^{1/4} + \frac{1}{z^{1/4}}\right)\lambda + \mathcal{O}\left(\lambda^3\right), \tag{5.7}$$

$$\frac{\mathcal{G}(z)}{\lambda} = -2 + \frac{2}{3}\left[\mathcal{H}(z) + \mathcal{H}\left(\frac{1}{z}\right) + \pi^2\right]\lambda^2 + \mathcal{O}\left(\lambda^3\right). \tag{5.8}$$

We see that the coefficients of the λ^L powers have weight L, in agreement with the discussion of the previous subsection. Equivalently, when using the effective coupling $\tilde{\lambda}$ from eq. (5.2) one would find weight $3L/2$ at L loops.

Finally, let us comment on the function space we found. This is best analyzed by the *symbol* [83], which is an important concept related to a transcendental function. Let us consider a transcendental function f of weight n whose total derivative can be writen as

$$df = \sum_i g_i \, d\log\omega_i \,, \tag{5.9}$$

where the g_i are functions of weight $n-1$ and the ω_i are rational functions called *letters*. The set of all letters of a transcendental function is known as its *alphabet*. Then, the symbol of f is defined recursively as

$$\mathcal{S}(f) = \sum_i \mathcal{S}(g_i) \otimes \omega_i \,. \tag{5.10}$$

The knowledge of the symbol of a transcendental function, combined with other information, is often enough to bootstrap the result of the corresponding iterated integral. As an example, this program has been used to compute several scattering amplitudes in $\mathcal{N}=4$ sYM [76, 84–89]. With this into consideration, obtaining the alphabet of the results presented in section 3 could open the door for a bootstrap computation of higher-loop terms.

It is therefore interesting to determine the alphabet of letters of our two-loop functions, given in eq. (3.14). It can be readily read off using the definitions (5.9) and (5.10). In terms of the variables z one finds that the letters have a square root dependence. The latter can be removed by changing variables as follows,

$$z = \frac{4q^2}{(1-q^2)^2}\,, \tag{5.11}$$

with $0 < q < 1$. Using this variable, we find the symbol

$$\mathcal{S}\left(\sqrt{\frac{z}{1+z}}H(z)\right) = \frac{q^2}{(1-q^2)^2} \otimes \frac{1+q}{1-q}\,. \tag{5.12}$$

In other words, the letters that compose the alphabet at two loops are

$$\vec{\omega} = \{q, 1-q, 1+q\}\,. \tag{5.13}$$

6 Conformal invariance of leading singularities

In this section we study the symmetry properties of the leading singularities that characterize the integrated negative geometries of ABJM, as previously explored in [24, 25] for the

$\mathcal{N}=4$ sYM case. We begin our analysis by a short review of the four-dimensional results, and then we turn to the discussion of the conformal invariance of the three-dimensional leading singularities. To that end, we discuss separately the parity-even and -odd terms that appear in (2.29) after the integration of the negative geometries.

6.1 Review of four-dimensional results

Let us start by reviewing the hidden conformal symmetry that was observed in $\mathcal{N}=4$ sYM at tree level. In this case one has

$$\mathcal{L}_1 = -\frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}\,. \tag{6.1}$$

More generally, in the generic L-loop case one can write

$$\left(\prod_{j=6}^{4+L}\int\frac{d^4x_j}{i\pi^2}\right)\mathcal{L}_L = \sum_{i=1}^{k} R_{L-1,i}\,T_{L-1,i}\,, \tag{6.2}$$

where k is some integer, $T_{L-1,i}$ are transcendental functions, and $R_{L-1,i}$ are rational functions known as *leading singularities*. As an example, when applying the definition (6.2) to (6.1) we get

$$R_0 = \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}\,, \quad \text{and} \quad T_0 = -1\,. \tag{6.3}$$

At this point it is useful to take advantage of the conformal covariance of the l.h.s. of (6.2), which allows us to go to the frame at which $x_5 \to \infty$. The convenience of this frame relies on the fact that now we can write all functions using four-particle kinematic notation. To be more precise, let us define the leading singularities $r_{L,i}$ in the $x_5 \to \infty$ frame as

$$r_{L,i} := \lim_{x_5\to\infty}(x_5^2)^4\,R_{L,i}\,. \tag{6.4}$$

Then, at tree level we get

$$r_0 = x_{13}^2 x_{24}^2 = s\,t\,, \tag{6.5}$$

where $s := (p_1+p_2)^2 = x_{13}^2$ and $t := (p_1+p_4)^2 = x_{24}^2$ are the well-known Mandelstam variables. Moreover, in terms of four-dimensional spinor-helicity variables we have

$$r_0 = \langle 12\rangle\langle 14\rangle[12][14]\,, \tag{6.6}$$

where $\langle ij\rangle = \epsilon_{ab}\,\lambda_i^a\lambda_j^b$ and $[ij] = \epsilon_{\dot a\dot b}\,\tilde\lambda_i^{\dot a}\tilde\lambda_j^{\dot b}$, and with the spinor-helicity variables λ^a and $\tilde\lambda^{\dot a}$ being defined as

$$p^{a\dot a} = p_\mu(\sigma^\mu)^{a\dot a} = \lambda^a\tilde\lambda^{\dot a}\,. \tag{6.7}$$

Now, in order to discuss the conformal invariance of the leading singularities defined above let us recall that in four-particle kinematics the generator of special conformal transformations is written as

$$K_{a\dot a} = \sum_{i=1}^{4}\frac{\partial^2}{\partial\lambda^a\partial\tilde\lambda^{\dot a}}\,. \tag{6.8}$$

JHEP05(2023)112

Therefore, we can see that the leading singularity (6.6) is not invariant under special conformal transformations. Instead, as observed in [24], in order to get a conformally invariant quantity one should multiply the leading singularity by the Parke-Taylor factor PT, defined as

$$\mathrm{PT} = \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \,. \tag{6.9}$$

That is, when normalizing the leading singularity r_0 as

$$\hat{r}_0 := \mathrm{PT}\, r_0 = \frac{[12][41]}{\langle 23\rangle\langle 34\rangle} \,, \tag{6.10}$$

one gets a conformally invariant function. Finally, we should note that, as shown in [24], these results generalize to higher-point tree-level leading singularities.

6.2 Leading singularities of parity-even terms

In order to generalize the above results to the ABJM case, let us first review how the conformal generators look when written in terms of three-dimensional kinematic variables. First, let us introduce three-dimensional spinor-helicity variables as

$$p^{ab} = \lambda^a \lambda^b \,, \tag{6.11}$$

with

$$p^{ab} = (\sigma^\mu)^{ab} p_\mu = \begin{pmatrix} p^0 - p^1 & p^2 \\ p^2 & p^0 + p^1 \end{pmatrix} . \tag{6.12}$$

Moreover, let us define the Mandelstam variables s_{ij} as

$$s_{ij} := (p_i + p_j)^2 = \langle ij\rangle^2 \,. \tag{6.13}$$

Then, one can write the conformal generators of the one-particle representation of the $\mathfrak{osp}(6|4)$ superalgebra of ABJM [33] as

$$\begin{aligned} P^{ab} &= \lambda^a \lambda^b \,, & L^a_b &= \lambda^a \partial_b - \frac{1}{2}\delta^a_b \lambda^c \partial_c \,, \\ K_{ab} &= \partial_a \partial_b \,, & D &= \frac{1}{2}\lambda^a \partial_a + \frac{1}{2} \,, \end{aligned}$$

were L^a_b are the generators of rotations, D is the dilatation operator, and K_{ab} is the generator of special conformal transformations. As for multi-particle representations, one can construct the generators by adding up the corresponding single-particle operators. In particular, for the four-particle case the three-dimensional generalization of (6.8) reads

$$K_{ab} = \sum_{i=1}^{4} \partial^i_a \partial^i_b \,. \tag{6.14}$$

We can turn now to the symmetry analysis of the three-dimensional leading singularities. We should recall that in (2.29) we found that in ABJM the result of performing $L-1$ loop integrations over the L-loop integrand $\mathcal{L}_L$ can be separated into parity-even and -odd

JHEP05(2023)112

terms. In order to discuss the conformal properties of the integrated geometries, we will find instructive to study those terms separately.

Let us begin by considering the parity-even terms $\mathcal{P}_e$. In section 3 we found that these terms were given by

$$\frac{\mathcal{P}_e}{\lambda} = -\frac{\sqrt{\pi}}{4}\left(\frac{X_{13}^2 X_{24}^2}{X_{15}^2 X_{25}^2 X_{35}^2 X_{45}^2}\right)^{3/4}\left(z^{1/4} + \frac{1}{z^{1/4}}\right)\lambda + \mathcal{O}(\lambda^3)\,. \tag{6.15}$$

Therefore, at one-loop order the leading singularities are

$$R_{e,1} = \left(\frac{X_{13}^2 X_{24}^2}{X_{15}^2 X_{25}^2 X_{35}^2 X_{45}^2}\right)^{3/4} z^{1/4}\,, \qquad \text{and} \qquad R_{e,2} = \left(\frac{X_{13}^2 X_{24}^2}{X_{15}^2 X_{25}^2 X_{35}^2 X_{45}^2}\right)^{3/4} \frac{1}{z^{1/4}}\,.$$

As in the $\mathcal{N}=4$ sYM case, we will take the $x_5 \to \infty$ limit and we will define

$$r_{e,i} := \lim_{x_5\to\infty} (x_5^2)^3\, R_{e,i}\,, \tag{6.16}$$

such that

$$r_{e,1} = s\sqrt{t}\,, \qquad \text{and} \qquad r_{e,2} \;= t\sqrt{s}\,.$$

When going to spinor-helicity notation we have to be careful with the sign that comes from taking the square root in (6.13). However, given that a constant overall sign in the leading singularities is not important when discussing their symmetry properties, from now on we will ignore it, simply assuming a plus sign. We remind the reader that the sign could be different depending on the kinematic region, and therefore our conclusions regarding conformal invariance will only be valid locally. Then, we can write

$$r_{e,1} = \langle 12\rangle^2\langle 14\rangle\,, \qquad \text{and} \qquad r_{e,2} = \langle 12\rangle\langle 14\rangle^2\,. \tag{6.17}$$

As a first test, let us consider what happens when we multiply (6.17) by the Parke-Taylor factor given in (6.9). We get

$$\mathrm{PT}\, r_{e,1} = \frac{\langle 12\rangle}{\langle 23\rangle\langle 34\rangle\langle 14\rangle}\,, \qquad \text{and} \qquad \mathrm{PT}\, r_{e,2} = \frac{\langle 14\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle}\,, \tag{6.18}$$

which are not invariant under the action of the special conformal generators given in (6.14). In order to understand why (6.18) fails to be conformally invariant we should recall in $\mathcal{N}=4$ sYM the Parke-Taylor factor appears within the tree-level amplitudes as

$$\mathcal{A}_4^{\mathcal{N}=4\,\mathrm{sYM}} = \mathrm{PT}\,\delta^{(4)}(P)\,\delta^{(8)}(Q)\,. \tag{6.19}$$

Comparing with tree-level amplitudes in ABJM, which are given by [33]

$$\mathcal{A}_4^{\mathrm{ABJM}} = -\frac{\delta^{(3)}(P)\,\delta^{(6)}(Q)}{\langle 12\rangle\langle 14\rangle}\,, \tag{6.20}$$

we find natural to define a three-dimensional Parke-Taylor factor as

$$\mathrm{PT}^{(3)} := \frac{1}{\langle 12\rangle\langle 14\rangle}\,, \tag{6.21}$$

and to normalize the three-dimensional leading singularities as

$$\hat{r} = \mathrm{PT}^{(3)}\, r\,. \tag{6.22}$$

Indeed, with this normalization we get

$$\hat{r}_{e,1} = \sqrt{s} = \langle 12\rangle\,, \qquad \text{and} \qquad \hat{r}_{e,2} = \sqrt{t} = \langle 14\rangle\,, \tag{6.23}$$

which are invariant under the generators (6.14) of special conformal transformations. Therefore, we conclude that the one-loop leading singularities of the parity-even terms of (2.29) become conformally invariant when normalized by the three-dimensional Parke-Taylor factor (6.21) and when evaluated in the $x_5 \to \infty$ frame.

6.3 Leading singularities of parity-odd terms

Let us turn now to the study of the parity-odd terms $\mathcal{P}_o$ of (2.29). In order to analyze the conformal invariance of their leading singularities, we must identify first how to take the $x_5 \to \infty$ limit in expressions that include contractions $\epsilon(1,2,3,4,5)$ with the five-dimensional Levi-Civita tensor. To that end, it is instructive to recall the identity

$$\epsilon(1,2,3,4,5)^2 = -\frac{x_{13}^2 x_{24}^2}{16}\left(x_{24}^2 x_{15}^2 x_{35}^2 + x_{13}^2 x_{25}^2 x_{45}^2\right), \tag{6.24}$$

which can be found in the discussion of appendix A. As in the previous section, we will ignore the sign ambiguity that comes from taking square roots. Our equalities should be understood up to a possible overall sign, and our conclusions about symmetry invariance will only be valid locally. Then, we get[6]

$$\lim_{x_5\to\infty}\left(x_5^2\right)^{-1}\epsilon(1,2,3,4,5) = \frac{1}{4}\sqrt{-st(s+t)}\,. \tag{6.25}$$

Having discussed how to correctly take the $x_5 \to \infty$ limit, we can now safely turn to the analysis of the symmetry properties of the parity-odd terms $\mathcal{P}_o$. From (5.8) we have

$$\frac{\mathcal{P}_o}{\lambda} = \frac{\epsilon(1,2,3,4,5)}{\sqrt{\pi}\, X_{15}^2 X_{25}^2 X_{35}^2 X_{45}^2}\left[-2 + \frac{2}{3}\left(\mathcal{H}(z) + \mathcal{H}\left(\frac{1}{z}\right) + \pi^2\right)\lambda^2\right] + \mathcal{O}(\lambda^3)\,. \tag{6.26}$$

Then, up to the loop order we have studied we get that the leading singularities are

$$\begin{aligned} R_{o,1} &= \frac{4\,\epsilon(1,2,3,4,5)}{X_{15}^2 X_{25}^2 X_{35}^2 X_{45}^2}\,,\\ R_{o,2} &= R_{o,1}\sqrt{-\frac{z}{1+z}}\,,\\ R_{o,3} &= R_{o,1}\sqrt{-\frac{1}{1+z}}\,. \end{aligned}$$

[6] It is interesting to note that (6.25) can also be obtain from the contraction

$$\epsilon(1,2,3,4,I) = \frac{1}{2}\sqrt{-st(s+t)}\,,$$

where $I = (1,0,\vec{0})$ corresponds to a point in infinity [41]. That is, we can alternatively write

$$\lim_{x_5\to\infty}\left(x_5^2\right)^{-1}\epsilon(1,2,3,4,5) = \frac{1}{2}\,\epsilon(1,2,3,4,I)\,.$$

Therefore, in the $x_5 \to \infty$ we get

$$\begin{aligned} r_{o,1} &= \sqrt{-st(s+t)}\,, \\ r_{o,2} &= s\sqrt{t}\,, \\ r_{o,3} &= t\sqrt{s}\,, \end{aligned} \tag{6.27}$$

and, after normalizing with the three-dimensional Parke-Taylor factor, we have

$$\begin{aligned} \hat{r}_{o,1} &= \sqrt{-s-t} = \langle 13\rangle\,, \\ \hat{r}_{o,2} &= \sqrt{s} = \langle 12\rangle\,, \\ \hat{r}_{o,3} &= \sqrt{t} = \langle 14\rangle\,, \end{aligned} \tag{6.28}$$

We see that the expressions in (6.28) are conformally invariant, in a similar way to what was observed for the parity-even terms.

7 Conclusions

In this paper we have studied, for the three-dimensional $\mathcal{N} = 6$ Chern-Simons-matter (ABJM) theory, the result of performing $L-1$ loop integrations over the L-loop integrand of the logarithm of the four-particle scattering amplitude. We have used the negative geometries that come from the projected amplituhedron for the ABJM theory [13] as the starting point for constructing the integrands. We have found that the dual conformal symmetry of the amplitudes allows for the presence of both parity-even and -odd terms in the integrated results, in a similar way to what was described for the five-particle case in the $\mathcal{N} = 4$ super Yang-Mills (sYM) theory [25]. We have performed the explicit integrations up to $L = 3$, and we have found that the results are given by infrared-finite quantities with uniform degree of transcendentality, as it was also observed for the analogous quantities in $\mathcal{N} = 4$ sYM. Moreover, we have constructed functionals that allow one to compute the ABJM cusp anomalous dimension Γ_{cusp} using the integrated negative geometries as the input, and by doing so we have recovered the known first non-trivial order of Γ_{cusp} [50, 51]. Finally, we have discussed the symmetry properties of the leading singularities associated to the integrated results. We have found that the leading singularities have a hidden conformal symmetry (in the frame in which the unintegrated loop variable goes to infinity, and after normalization with a three-dimensional generalization of the Parke-Taylor factor), extending the four-dimensional analysis of [24, 25].

There are a number of exciting open questions. In the $\mathcal{N} = 4$ super Yang-Mills theory, a useful dual perspective is provided by the duality between scattering amplitudes and Wilson loops. This allows one to think of loop integrands as derivatives of Wilson loop correlators w.r.t. the coupling. More precisely, the derivatives produce Lagrangian insertions, and it is natural to consider

$$\frac{\langle W_4\, L(x_5)\rangle}{\langle W_4\rangle}\,, \tag{7.1}$$

where W_4 is the dual polygonal Wilson loop, L is the Lagrangian of the theory, and x_5 is the unintegrated loop variable. It would be desirable to extend this to the ABJM case. However, an immediate difficulty is that the Lagrangian in ABJM (and in Chern-Simons theories in general) is not gauge invariant, as the variation of the action includes a non-trivial topological term.

Another interesting direction that arises from our results is the question about their generalization to scattering processes with higher numbers of particles. To that end, one could expect to apply the idea of the projected amplituhedron proposed in [13] to compute the corresponding negative geometries for higher multiplicities, to then study the properties of the integrated results as we have performed at four points. Furthermore, in view of the Grassmanian formulas proposed for the ABJM theory [49, 90, 91], it would be interesting to analyze the symmetry properties of the leading singularities in terms of a Grassmanian formulation, as it was done in [24] for the $\mathcal{N} = 4$ sYM case.

Considering the relation between integrated negative geometries in $\mathcal{N} = 4$ sYM and all-plus amplitudes in the pure Yang Mills theory [24, 25], one intriguing question to address is a possible generalization of this result to the ABJM case. In this regard, one should take into account that an analogous relation between ABJM and the pure Chern-Simons theory does not seem possible, as the latter is a topological theory and therefore has a vanishing S-matrix. However, it would be interesting to investigate a possible relation between ABJM and less supersymmetric Chern-Simons-matter theories.

Furthermore, it would be interesting to carry on the integrations to higher loop orders. For $L \geq 4$ it seems to be far less trivial how to perform the integrations by first principles. However, many useful methods have been developed over the last years to overcome the difficulties that arise when computing Feynman integrals. In particular, the method of differential equations [73] appears as a promising tool to solve the $L = 4$ case, which in turn would allow to reproduce the next-to-leading non-trivial order of the ABJM cusp anomalous dimension.

Finally, an interesting problem to investigate is whether one can sum infinite series of negative-geometry diagrams. This question was addressed in $D = 4$ in [16], where the all-loop sum of *ladder* and *tree* diagrams was performed. The crucial observation used in [16] is that in $D = 4$ the Laplace operator $\square = \partial^\mu \partial_\mu$ acts on the propagator as

$$\square \frac{1}{x^2} = -4i\pi^2\, \delta^{(4)}(x)\,. \tag{7.2}$$

This allows one to recursively relate diagrams that differ only on one loop integration, and ends up giving second-order differential equations for $\mathcal{F}_{\text{ladder}}(z)$ and $\mathcal{F}_{\text{tree}}(z)$. It is useful to note that eq. (7.2) naturally arises in Fourier space, as in this context the four-dimensional propagator is simply $1/k^2$. Unfortunately, in $D = 3$ there is a mismatch of dimensions in momentum space, which prevents us from using the Laplace equation trick. Nevertheless, we find it likely that the finite integrals have other special properties that may lead to simplifications. It would be interesting to work further in this direction. An all-loop sum of negative-geometry diagrams could be a first step towards obtaining non-perturbative results for the cusp anomalous dimension Γ_{cusp}. Moreover, it could set the stage for a non-perturbative computation of the interpolating function $h(\lambda)$, for which all-loop proposals exist [60, 61].

JHEP05(2023)112

Acknowledgments

We would like to thank Paolo Benincasa, Jungwon Lim, Antonela Matijašić, Julian Miczajka and Emery Sokatchev for useful discussions. ML would like to especially thank Diego Correa for insightful comments and supervision. ML is supported by fellowships from CONICET (Argentina) and DAAD (Germany). This research was supported by the Munich Institute for Astro-, Particle and BioPhysics (MIAPbP) which is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy — EXC-2094 — 390783311. This research received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 725110), *Novel structures in scattering amplitudes.*

Note added. After this work was completed we learned that Song He, Chia-Kai Kuo, Zhenjie Li and Yao-Qi Zhang independently obtained similar results [92]. We would like to thank them for correspondence and for confirming agreement with our two-loop result, see (3.14) and (3.20).

A Five-dimensional notation

When working with three-dimensional dual-coordinates it turns out useful to consider the embedding of the three-dimensional Minkowski space into the five-dimensional projective light-cone. One of the main advantages of this parametrization lies in the fact it allows to write three-dimensional dual-conformal invariants simply as five-dimensional expressions that respect Lorentz and scale invariance.

To be more precise, let us consider a five-dimensional Minkowski space with $(-,-,+,+,+)$ signature and with coordinates (X^1,X^2,X^3,X^4,X^5). Then the light-cone is defined by the constraint

$$-(X^1)^2-(X^2)^2+(X^3)^2+(X^4)^2+(X^5)^2=0\,. \tag{A.1}$$

Let us note that the constraint (A.1) is invariant under a rescaling of the coordinates, and therefore defines a projective space with 3 degrees of freedom, as expected. It is useful to switch to light-cone coordinates (X^+,X^-,X^2,X^4,X^5), with X^+ and X^- given by

$$X^+=\frac{X^1+X^3}{\sqrt{2}}\,,\qquad \text{and}\qquad X^-=\frac{X^1-X^3}{\sqrt{2}}\,, \tag{A.2}$$

so that (A.1) becomes

$$-\,2\,X^+X^- -(X^2)^2+(X^4)^2+(X^5)^2=0\,. \tag{A.3}$$

The embedding of the three-dimensional Minkowski space[7] with coordinates (x^0,x^1,x^2) into the five-dimensional space can be defined as

$$(X^+,X^-,X^2,X^4,X^5)=\left(\frac{x^\mu x_\mu}{2},1,x^0,x^1,x^2\right). \tag{A.4}$$

[7]We are using the $(-,+,+)$ signature for the three-dimensional Minkowski space.

It is straightforward then to check that (A.4) satisfies (A.3). Moreover, under this parametrization we have

$$(X_i - X_j)^2 = -2X_i.X_j = (x_i - x_j)^2 \,, \tag{A.5}$$

where x_i and x_j are points in the three-dimensional space and X_i and X_j are their corresponding images under the mapping (A.4).

In order to simplify notation, we will write the contraction of the dual coordinates with the five-dimensional Levi-Civita tensor as

$$\epsilon(1,2,3,4,5) := \epsilon_{\mu\nu\rho\sigma\eta} X_1^\mu X_2^\nu X_3^\rho X_4^\sigma X_5^\eta \,. \tag{A.6}$$

Let us recall some properties of (A.6). In the first place, one can rewrite (A.6) in terms of three-dimensional dual-coordinates as [34]

$$\epsilon(1,2,3,4,5) = \frac{1}{2}\Big(x_{51}^2 \epsilon_{\mu\nu\rho} x_{21}^\mu x_{31}^\nu x_{41}^\rho + x_{31}^2 \epsilon_{\mu\nu\rho} x_{51}^\mu x_{21}^\nu x_{41}^\rho\Big) \,. \tag{A.7}$$

Also, the product of two contractions is given by

$$\epsilon(1,2,3,4,5)\,\epsilon(1,2,3,4,6) = -\frac{X_{13}^4 X_{24}^4}{32}\left(\frac{X_{15}^2 X_{36}^2 + X_{16}^2 X_{35}^2}{X_{13}^2} + \frac{X_{25}^2 X_{46}^2 + X_{26}^2 X_{45}^2}{X_{24}^2} - X_{56}^2\right) \,. \tag{A.8}$$

In particular, we have

$$\epsilon(1,2,3,4,5)^2 = -\frac{X_{13}^2 X_{24}^2}{16}(X_{24}^2 X_{15}^2 X_{35}^2 + X_{13}^2 X_{25}^2 X_{45}^2) \,. \tag{A.9}$$

Finally, following [41] we can define a measure on the five-dimensional light-cone as

$$d^3 X := \int \frac{d^5 X}{\mathrm{Vol[GL(1)]}}\, \delta(X^2) \,, \tag{A.10}$$

where the factor $\delta(X^2)$ is included to satisfy the constraint given in (A.1), while the denominator Vol[GL(1)] eliminates the redundancy coming from the projective invariance of the light-cone. Therefore, we get

$$\int d^3 X \equiv \int d^3 x \,. \tag{A.11}$$

B Normalization of negative geometries

As shown in (2.9) and (2.17), there are relative normalizations n_L and $\tilde{n}_L$ between the integrands $\mathcal{I}_L$ and $\mathcal{L}_L$ and the canonical forms Ω_L and $\tilde{\Omega}_L$. We will discuss their computation in this section.

As a first step, we should note that the definitions (2.2) and (2.3) imply

$$\mathcal{I}_1 = \mathcal{L}_1 \,, \tag{B.1}$$

$$\mathcal{I}_2 = \mathcal{L}_2 + \frac{1}{2}\mathcal{L}_1^2 \,, \tag{B.2}$$

$$\mathcal{I}_3 = \mathcal{L}_3 + \mathcal{L}_2\,\mathcal{L}_1 + \frac{1}{6}\mathcal{L}_1^3 \,. \tag{B.3}$$

JHEP05(2023)112

On the other hand, from (2.16) we get

$$\Omega_1 = \tilde{\Omega}_1 \,, \tag{B.4}$$

$$\Omega_2 = \tilde{\Omega}_2 + \tilde{\Omega}_1^2 \,, \tag{B.5}$$

$$\Omega_3 = \tilde{\Omega}_3 + 3\,\tilde{\Omega}_2\tilde{\Omega}_1 + \tilde{\Omega}_1^3 \,. \tag{B.6}$$

Therefore, using the definitions (2.9) and (2.17) and the expansions (B.1)–(B.6) we have

$$\mathcal{L}_1 = n_1\,\tilde{\Omega}_1 \,, \tag{B.7}$$

$$\mathcal{L}_2 = n_2\tilde{\Omega}_2 + \left(n_2 - \frac{n_1^2}{2}\right)\tilde{\Omega}_1^2 \,, \tag{B.8}$$

$$\mathcal{L}_3 = n_3\tilde{\Omega}_3 + \tilde{\Omega}_1\,\tilde{\Omega}_2(3n_3 - n_2 n_1) + \tilde{\Omega}_1^3\left(n_3 - n_2 n_1 + \frac{n_1^3}{3}\right). \tag{B.9}$$

To fix the values of the n_L coefficients we will follow the ideas of [65]. These authors used the fact that the integrands $\mathcal{L}_L$ in planar $\mathcal{N}=4$ sYM should behave as $\mathcal{O}(1/\delta)$ in the limit

$$\langle l_5 12\rangle \sim \delta \,, \qquad \text{and} \qquad \langle l_5 23\rangle \sim \delta \,, \tag{B.10}$$

while all other brackets remain non-vanishing. This property makes sure that infrared divergences exponentiate (after integration). A similar analysis can be done in the ABJM case. Therefore, noticing that (2.22) implies

$$\tilde{\Omega}_L \sim \mathcal{O}(1/\delta) \qquad \text{for} \qquad 1 \leq L \leq 3 \,, \tag{B.11}$$

in the limit (B.10), and demanding the same behaviour for the l.h.s. of (B.7)–(B.9), we get

$$n_1 = \tilde{n}_1 \,, \qquad n_2 = \tilde{n}_2 = \frac{\tilde{n}_1^2}{2!} \,, \qquad n_3 = \tilde{n}_3 = \frac{\tilde{n}_1^3}{3!} \,. \tag{B.12}$$

Finally, comparing the explicit formulas for $\mathcal{I}_1$ and $\mathcal{I}_2$ given in [34, 45] to the expressions for Ω_1 and Ω_2 obtained from the results of [13] we get

$$\tilde{n}_1 = \frac{i}{2\sqrt{\pi}} \,. \tag{B.13}$$

C Useful integrals

We present here several useful integrals for computing the perturbative results of section 3, as well as the integrals that give us the functionals $I_{\mathcal{F}}$ and $I_{\mathcal{G}}$ in section 4.

C.1 Triangle integral

Let us begin with a triangle integral in three dimensions and with three massive legs. This integral first appears in the one-loop analysis in (3.8), and it is explicitly given by

$$\mathcal{T} := \int \frac{d^3 X_6}{i\pi^{3/2}} \frac{1}{X_{26}^2 X_{46}^2 X_{56}^2} \,. \tag{C.1}$$

Using the standard Feynman parametrization one gets

$$\mathcal{T} = \frac{\pi^{3/2}}{\sqrt{X_{25}^2 X_{45}^2 X_{24}^2}} \,. \tag{C.2}$$

It is interesting to note that the functional form of the result (C.2) can also be obtained from noticing that the integral (C.1) has dual conformal invariance.

C.2 Five-leg integral with an epsilon numerator

Let us consider now the integral

$$\mathcal{E} := \int \frac{d^3 X_6}{i\pi^{3/2}} \frac{\epsilon(1,2,3,4,6)}{X_{26}^2 X_{46}^2 X_{56}^2 X_{16} X_{36}} \,, \tag{C.3}$$

which shows up at the two-loop computation in (3.13). Introducing Feynman parameters, we have

$$\mathcal{E} = \frac{\epsilon_{\mu\nu\rho\sigma\eta} X_1^\mu X_2^\nu X_3^\rho X_4^\sigma}{\pi \, \mathrm{Vol[GL(1)]}} \left(\prod_{i=1}^{5} \int_0^\infty d\alpha_i \right) (\alpha_1 \alpha_3)^{-1/2} \, \partial_Y^\eta \left[\int \frac{d^3 X_6}{i\pi^{3/2}} \frac{1}{(-2Y.X_6)^3} \right] , \tag{C.4}$$

where we have defined

$$Y := \sum_{i=1}^{5} \alpha_i \, X_i \,, \tag{C.5}$$

and we have used (A.5). Then, performing the space-time integral we have

$$\mathcal{E} = \frac{3}{4\sqrt{\pi} \, \mathrm{Vol[GL(1)]}} \left(\prod_{i=1}^{5} \int_0^\infty d\alpha_i \right) \frac{(\alpha_1 \alpha_3)^{-1/2} \, \epsilon(1,2,3,4,Y)}{(-Y^2)^{\frac{5}{2}}} \,. \tag{C.6}$$

At this point is useful to define

$$\beta_i := \alpha_i X_{i5}^2 \quad \text{for} \quad i = 1, \ldots, 4 \,, \tag{C.7}$$

and to mod out the GL(1) invariance by setting

$$\sum_{i=1}^{4} \beta_i = 1 \,. \tag{C.8}$$

Then, performing the integral over α_5 we get

$$\mathcal{E} = \frac{\epsilon(1,2,3,4,5)}{\sqrt{\pi} \, (X_{15}^2 X_{35}^2)^{1/2} X_{25}^2 X_{45}^2} \left(\prod_{i=1}^{4} \int_0^\infty d\beta_i \right) \delta\left(\sum_{i=1}^{4} \beta_i - 1 \right) \frac{(\beta_1 \beta_3)^{-1/2}}{\left(\beta_1 \beta_3 \frac{X_{13}^2}{X_{15}^2 X_{35}^2} + \beta_2 \beta_4 \frac{X_{24}^2}{X_{25}^2 X_{45}^2} \right)^{\frac{1}{2}}} \,. \tag{C.9}$$

The number of remaining integrals can be further simplified by defining

$$\begin{aligned} \beta_1 &:= \gamma_1 \gamma_2 \,, & \beta_2 &:= \gamma_1 (1 - \gamma_2) \,, \\ \beta_3 &:= (1 - \gamma_1) \gamma_3 \,, & \beta_4 &:= (1 - \gamma_1)(1 - \gamma_3) \,. \end{aligned} \tag{C.10}$$

JHEP05(2023)112

Let us note that the constraint (C.8) is trivially satisfied by the γ's. In terms of these variables we get

$$\mathcal{E} = \frac{\epsilon(1,2,3,4,5)}{(X_{15}^2 X_{25}^2 X_{35}^2 X_{45}^2 X_{24}^2)^{1/2}} \frac{\mathcal{H}(z)}{\sqrt{\pi}\, z}\,, \tag{C.11}$$

with

$$\mathcal{H}(z) := \sqrt{z} \int_0^1 d\gamma_2 \int_0^1 d\gamma_3 \, \frac{(\gamma_2\gamma_3)^{-1/2}}{[\gamma_2\gamma_3 z + (1-\gamma_2)(1-\gamma_3)]^{\frac{1}{2}}}\,. \tag{C.12}$$

Let us focus on the integral (C.12), which as we shall see can be solved by iterating Feynman parametrizations. Making the change of variables

$$\gamma_2 \to \frac{1}{1+\gamma_2}\,, \qquad \gamma_3 \to \frac{\gamma_3}{1+\gamma_3}\,, \tag{C.13}$$

and introducing Feynman parameters one gets

$$\mathcal{H}(z) = \frac{\sqrt{z}}{\pi} \left(\prod_{i=1}^{3} \int_0^\infty d\eta_i \right) \frac{1}{\sqrt{\eta_3}\,(\eta_1+\eta_2)(\eta_1+1)(\eta_2+\eta_3+z)}\,. \tag{C.14}$$

Moreover, taking

$$\eta_3 \to \eta_3^2\,, \tag{C.15}$$

and making a further Feynman parametrization we have

$$\mathcal{H}(z) = \int_0^\infty d\nu_1 \int_0^\infty d\nu_2 \, \frac{\sqrt{z}}{(\nu_1+\nu_2)(\nu_1+1)\sqrt{\nu_2+z}}\,. \tag{C.16}$$

Finally, defining

$$\theta := \sqrt{\nu_2 + z}\,, \tag{C.17}$$

and integrating over θ we arrive at

$$\begin{aligned}\mathcal{H}(z) = \sqrt{\frac{z}{1+z}} \Big(& \pi^2 + 2\log\!\left(\sqrt{z} + \sqrt{1+z}\right)\log(4z) \\ & + \mathrm{Li}_2\!\left[-2\left(z + \sqrt{z(1+z)}\right)\right] - \mathrm{Li}_2\!\left[-2\left(z - \sqrt{z(1+z)}\right)\right]\Big)\,.\end{aligned} \tag{C.18}$$

C.3 $I_{\mathcal{F}}$ functional

Let us consider the integral that appears in the definition (4.10) of the $I_{\mathcal{F}}$ functional, i.e.

$$I_{\mathcal{F}}[z^p] \sim -\frac{1}{2\sqrt{\pi}} \int \frac{d^D X_5}{i\pi^{D/2}} \left(\frac{X_{13}^2 X_{24}^2}{X_{15}^2 X_{25}^2 X_{35}^2 X_{45}^2} \right)^{\frac{3}{4}} z^p\,, \tag{C.19}$$

with $p \in \mathbb{Z}$, $D = 3 - 2\epsilon$, and where to simplify notation we have chosen to use the symbol $\sim$ to indicate that we are only retaining the leading $1/\epsilon^2$ divergence. Using Feynman parametrization we get

$$\begin{aligned} I_{\mathcal{F}}[z^p] \sim & -\frac{X_{13}^{3/2+2p} X_{24}^{3/2-2p}}{\sqrt{\pi}\,\Gamma\!\left(\frac{3}{4}+p\right)^2 \Gamma\!\left(\frac{3}{4}-p\right)^2} \frac{1}{\mathrm{Vol}[\mathrm{GL}(1)]} \left(\prod_{i=1}^{4} \int_0^\infty d\alpha_i \right) \\ & \times \int \frac{d^D X_5}{i\pi^{D/2}} \frac{(\alpha_1\alpha_3)^{-\frac{1}{4}+p}(\alpha_2\alpha_4)^{-\frac{1}{4}-p}}{(-2X_5.W)^3}\,, \end{aligned}$$

with

$$W := \sum_{i=1}^{4} \alpha_i X_i \,. \tag{C.20}$$

Working as with the $\mathcal{E}$ integral discussed in the previous section we arrive at

$$\begin{aligned} I_{\mathcal{F}}[z^p] \sim & -\frac{\Gamma\left(\frac{3}{2}+\epsilon\right)\Gamma(-\epsilon)^2}{2\sqrt{\pi}\,\Gamma\left(\frac{3}{4}+p\right)^2\Gamma\left(\frac{3}{4}-p\right)^2\Gamma(-2\epsilon)} \\ & \times \int_0^1 d\gamma_2 \int_0^1 d\gamma_3 \, \frac{(\gamma_2\gamma_3)^{-\frac{1}{4}+p}[(1-\gamma_2)(1-\gamma_3)]^{-\frac{1}{4}-p}}{[\gamma_2\gamma_3+(1-\gamma_2)(1-\gamma_3)]^{3/2+\epsilon}} \,. \end{aligned}$$

At this point is useful to use the Mellin-Barnes formula, which allows us to write

$$\begin{aligned} I_{\mathcal{F}}[z^p] \sim & -\frac{1}{2\sqrt{\pi}\,\Gamma\left(\frac{3}{4}+p\right)^2\Gamma\left(\frac{3}{4}-p\right)^2\Gamma(-2\epsilon)} \\ & \times \int_{\zeta-i\infty}^{\zeta+i\infty} \frac{dz}{2\pi i}\,\Gamma\left(z+\frac{3}{2}+\epsilon\right)\Gamma(-z)\Gamma^2\left(\frac{3}{4}+p+z\right)\Gamma^2\left(-\frac{3}{4}-p-z-\epsilon\right), \end{aligned} \tag{C.21}$$

with

$$-\frac{3}{4}-p<\zeta<-\frac{3}{4}-p-\epsilon \,. \tag{C.22}$$

To compute the leading divergence of (C.21) we have chosen to follow the method described in [93], which was later automatized in a *Mathematica* package by [94]. Then, we finally get

$$I_{\mathcal{F}}[z^p] = -\frac{2}{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}+p\right)\Gamma\left(\frac{3}{4}-p\right)} \,. \tag{C.23}$$

C.4 $I_{\mathcal{G}}$ functional

Finally, let us focus now on the integral that defines the $I_{\mathcal{G}}$ functional in (4.11). That is, we will consider

$$I_{\mathcal{G}}[z^p] \sim -\frac{1}{2\sqrt{\pi}} \int \frac{d^D X_5}{i\pi^{D/2}} \frac{\epsilon(1,2,3,4,5)}{X_{15}^2 X_{25}^2 X_{35}^2 X_{45}^2}\, z^p \,, \tag{C.24}$$

where again we are using the symbol $\sim$ to indicate that we are only keeping the $1/\epsilon^2$ contribution. Introducing Feynman parameters we get

$$\begin{aligned} I_{\mathcal{G}}[z^p] \sim & -\frac{X_{13}^{2p} X_{24}^{-2p}\,\epsilon_{\mu\nu\rho\sigma\eta} X_1^\mu X_2^\nu X_3^\rho X_4^\sigma}{2\sqrt{\pi}\,\Gamma^2(1+p)\,\Gamma^2(1-p)\,\mathrm{Vol[GL(1)]}} \\ & \times \left(\prod_{i=1}^{4}\int_0^\infty d\alpha_i\right)\left(\frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}\right)^p \partial_W^\eta \left[\int \frac{d^D X_5}{i\pi^{D/2}} \frac{1}{(-2X_5.W)^3}\right], \end{aligned}$$

where W was defined in (C.20). The integral

$$\epsilon_{\mu\nu\rho\sigma\eta} X_1^\mu X_2^\nu X_3^\rho X_4^\sigma\, \partial_W^\eta \left[\int \frac{d^D X_5}{i\pi^{D/2}} \frac{1}{(-2X_5.W)^3}\right], \tag{C.25}$$

was solved in [34, 41] using a regularization scheme that allows one to dimensionally regularize the integral without losing the projective invariance that comes from the constraint (A.1), getting as a result that (C.25) is $\mathcal{O}(\epsilon)$. Therefore,

$$I_{\mathcal{G}}[z^p] = 0 \,. \tag{C.26}$$

JHEP05(2023)112

References

[1] S.J. Parke and T.R. Taylor, *An Amplitude for n Gluon Scattering*, Phys. Rev. Lett. **56** (1986) 2459 [INSPIRE].

[2] N. Arkani-Hamed and J. Trnka, *The Amplituhedron*, JHEP **10** (2014) 030 [arXiv:1312.2007] [INSPIRE].

[3] N. Arkani-Hamed and J. Trnka, *Into the Amplituhedron*, JHEP **12** (2014) 182 [arXiv:1312.7878] [INSPIRE].

[4] S. Franco, D. Galloni, A. Mariotti and J. Trnka, *Anatomy of the Amplituhedron*, JHEP **03** (2015) 128 [arXiv:1408.3410] [INSPIRE].

[5] N. Arkani-Hamed, H. Thomas and J. Trnka, *Unwinding the Amplituhedron in Binary*, JHEP **01** (2018) 016 [arXiv:1704.05069] [INSPIRE].

[6] N. Arkani-Hamed, C. Langer, A. Yelleshpur Srikant and J. Trnka, *Deep Into the Amplituhedron: Amplitude Singularities at All Loops and Legs*, Phys. Rev. Lett. **122** (2019) 051601 [arXiv:1810.08208] [INSPIRE].

[7] D. Damgaard, L. Ferro, T. Lukowski and M. Parisi, *The Momentum Amplituhedron*, JHEP **08** (2019) 042 [arXiv:1905.04216] [INSPIRE].

[8] L. Ferro and T. Lukowski, *The Loop Momentum Amplituhedron*, arXiv:2210.01127 [INSPIRE].

[9] N. Arkani-Hamed, Y. Bai, S. He and G. Yan, *Scattering Forms and the Positive Geometry of Kinematics, Color and the Worldsheet*, JHEP **05** (2018) 096 [arXiv:1711.09102] [INSPIRE].

[10] N. Arkani-Hamed, P. Benincasa and A. Postnikov, *Cosmological Polytopes and the Wavefunction of the Universe*, arXiv:1709.02813 [INSPIRE].

[11] Y.-T. Huang, R. Kojima, C. Wen and S.-Q. Zhang, *The orthogonal momentum amplituhedron and ABJM amplitudes*, JHEP **01** (2022) 141 [arXiv:2111.03037] [INSPIRE].

[12] S. He, C.-K. Kuo and Y.-Q. Zhang, *The momentum amplituhedron of SYM and ABJM from twistor-string maps*, JHEP **02** (2022) 148 [arXiv:2111.02576] [INSPIRE].

[13] S. He, C.-K. Kuo, Z. Li and Y.-Q. Zhang, *All-Loop Four-Point Aharony-Bergman-Jafferis-Maldacena Amplitudes from Dimensional Reduction of the Amplituhedron*, Phys. Rev. Lett. **129** (2022) 221604 [arXiv:2204.08297] [INSPIRE].

[14] L. Ferro and T. Lukowski, *Amplituhedra, and beyond*, J. Phys. A **54** (2021) 033001 [arXiv:2007.04342] [INSPIRE].

[15] E. Herrmann and J. Trnka, *The SAGEX review on scattering amplitudes Chapter 7: Positive geometry of scattering amplitudes*, J. Phys. A **55** (2022) 443008 [arXiv:2203.13018] [INSPIRE].

[16] N. Arkani-Hamed, J. Henn and J. Trnka, *Nonperturbative negative geometries: amplitudes at strong coupling and the amplituhedron*, JHEP **03** (2022) 108 [arXiv:2112.06956] [INSPIRE].

[17] L.F. Alday, E.I. Buchbinder and A.A. Tseytlin, *Correlation function of null polygonal Wilson loops with local operators*, JHEP **09** (2011) 034 [arXiv:1107.5702] [INSPIRE].

[18] T. Adamo, *Correlation functions, null polygonal Wilson loops, and local operators*, *JHEP* **12** (2011) 006 [arXiv:1110.3925] [INSPIRE].

[19] O.T. Engelund and R. Roiban, *On correlation functions of Wilson loops, local and non-local operators*, *JHEP* **05** (2012) 158 [arXiv:1110.0758] [INSPIRE].

[20] L.F. Alday, P. Heslop and J. Sikorowski, *Perturbative correlation functions of null Wilson loops and local operators*, *JHEP* **03** (2013) 074 [arXiv:1207.4316] [INSPIRE].

[21] L.F. Alday, J.M. Henn and J. Sikorowski, *Higher loop mixed correlators in $N = 4$ SYM*, *JHEP* **03** (2013) 058 [arXiv:1301.0149] [INSPIRE].

[22] R. Hernandez and J.M. Nieto, *Holographic correlation functions of hexagon Wilson loops with one local operator*, *Phys. Lett. B* **726** (2013) 417 [arXiv:1301.7220] [INSPIRE].

[23] J.M. Henn, G.P. Korchemsky and B. Mistlberger, *The full four-loop cusp anomalous dimension in $\mathcal{N} = 4$ super Yang-Mills and QCD*, *JHEP* **04** (2020) 018 [arXiv:1911.10174] [INSPIRE].

[24] D. Chicherin and J.M. Henn, *Symmetry properties of Wilson loops with a Lagrangian insertion*, *JHEP* **07** (2022) 057 [arXiv:2202.05596] [INSPIRE].

[25] D. Chicherin and J. Henn, *Pentagon Wilson loop with Lagrangian insertion at two loops in $\mathcal{N} = 4$ super Yang-Mills theory*, *JHEP* **07** (2022) 038 [arXiv:2204.00329] [INSPIRE].

[26] L.F. Alday and J.M. Maldacena, *Gluon scattering amplitudes at strong coupling*, *JHEP* **06** (2007) 064 [arXiv:0705.0303] [INSPIRE].

[27] J.M. Drummond, G.P. Korchemsky and E. Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops*, *Nucl. Phys. B* **795** (2008) 385 [arXiv:0707.0243] [INSPIRE].

[28] A. Brandhuber, P. Heslop and G. Travaglini, *MHV amplitudes in $N = 4$ super Yang-Mills and Wilson loops*, *Nucl. Phys. B* **794** (2008) 231 [arXiv:0707.1153] [INSPIRE].

[29] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *On planar gluon amplitudes/Wilson loops duality*, *Nucl. Phys. B* **795** (2008) 52 [arXiv:0709.2368] [INSPIRE].

[30] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and Crossing*, *J. Stat. Mech.* **0701** (2007) P01021 [hep-th/0610251] [INSPIRE].

[31] D. Correa, J. Maldacena and A. Sever, *The quark anti-quark potential and the cusp anomalous dimension from a TBA equation*, *JHEP* **08** (2012) 134 [arXiv:1203.1913] [INSPIRE].

[32] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *$N = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [arXiv:0806.1218] [INSPIRE].

[33] T. Bargheer, F. Loebbert and C. Meneghelli, *Symmetries of Tree-level Scattering Amplitudes in N=6 Superconformal Chern-Simons Theory*, *Phys. Rev. D* **82** (2010) 045016 [arXiv:1003.6120] [INSPIRE].

[34] W.-M. Chen and Y.-T. Huang, *Dualities for Loop Amplitudes of N=6 Chern-Simons Matter Theory*, *JHEP* **11** (2011) 057 [arXiv:1107.2710] [INSPIRE].

[35] T. Bargheer, N. Beisert, F. Loebbert and T. McLoughlin, *Conformal Anomaly for Amplitudes in $\mathcal{N} = 6$ Superconformal Chern-Simons Theory*, *J. Phys. A* **45** (2012) 475402 [arXiv:1204.4406] [INSPIRE].

[36] M. Leoni, A. Mauri and A. Santambrogio, *On the amplitude/Wilson loop duality in $N = 2$ SCQCD*, *Phys. Lett. B* **747** (2015) 325 [arXiv:1502.07614] [INSPIRE].

[37] M.S. Bianchi and M. Leoni, *On the ABJM four-point amplitude at three loops and BDS exponentiation*, *JHEP* **11** (2014) 077 [arXiv:1403.3398] [INSPIRE].

[38] M.S. Bianchi, M. Leoni and S. Penati, *An All Order Identity between ABJM and $N = 4$ SYM Four-Point Amplitudes*, *JHEP* **04** (2012) 045 [arXiv:1112.3649] [INSPIRE].

[39] M.S. Bianchi et al., *ABJM amplitudes and WL at finite N*, *JHEP* **09** (2013) 114 [arXiv:1306.3243] [INSPIRE].

[40] L. Bianchi and M.S. Bianchi, *Nonplanarity through unitarity in the ABJM theory*, *Phys. Rev. D* **89** (2014) 125002 [arXiv:1311.6464] [INSPIRE].

[41] S. Caron-Huot and Y.-T. Huang, *The two-loop six-point amplitude in ABJM theory*, *JHEP* **03** (2013) 075 [arXiv:1210.4226] [INSPIRE].

[42] S. He, Y.-T. Huang, C.-K. Kuo and Z. Li, *The two-loop eight-point amplitude in ABJM theory*, *JHEP* **02** (2023) 065 [arXiv:2211.01792] [INSPIRE].

[43] M.S. Bianchi et al., *One Loop Amplitudes In ABJM*, *JHEP* **07** (2012) 029 [arXiv:1204.4407] [INSPIRE].

[44] J.M. Henn, J. Plefka and K. Wiegandt, *Light-like polygonal Wilson loops in 3d Chern-Simons and ABJM theory*, *JHEP* **08** (2010) 032 [*Erratum ibid.* **11** (2011) 053] [arXiv:1004.0226] [INSPIRE].

[45] M.S. Bianchi et al., *Scattering Amplitudes/Wilson Loop Duality In ABJM Theory*, *JHEP* **01** (2012) 056 [arXiv:1107.3139] [INSPIRE].

[46] Y.-T. Huang and A.E. Lipstein, *Dual Superconformal Symmetry of $N = 6$ Chern-Simons Theory*, *JHEP* **11** (2010) 076 [arXiv:1008.0041] [INSPIRE].

[47] D. Gang et al., *Tree-level Recursion Relation and Dual Superconformal Symmetry of the ABJM Theory*, *JHEP* **03** (2011) 116 [arXiv:1012.5032] [INSPIRE].

[48] M.S. Bianchi et al., *Scattering in ABJ theories*, *JHEP* **12** (2011) 073 [arXiv:1110.0738] [INSPIRE].

[49] S. Lee, *Yangian Invariant Scattering Amplitudes in Supersymmetric Chern-Simons Theory*, *Phys. Rev. Lett.* **105** (2010) 151603 [arXiv:1007.4772] [INSPIRE].

[50] N. Gromov and P. Vieira, *The all loop AdS_4/CFT_3 Bethe ansatz*, *JHEP* **01** (2009) 016 [arXiv:0807.0777] [INSPIRE].

[51] L. Griguolo, D. Marmiroli, G. Martelloni and D. Seminara, *The generalized cusp in ABJ(M) $N = 6$ Super Chern-Simons theories*, *JHEP* **05** (2013) 113 [arXiv:1208.5766] [INSPIRE].

[52] L. Bianchi et al., *Two-loop cusp anomaly in ABJM at strong coupling*, *JHEP* **10** (2014) 013 [arXiv:1407.4788] [INSPIRE].

[53] N. Beisert, *The Analytic Bethe Ansatz for a Chain with Centrally Extended $\mathfrak{su}(2|2)$ Symmetry*, *J. Stat. Mech.* **0701** (2007) P01017 [nlin/0610017] [INSPIRE].

[54] J.A. Minahan and K. Zarembo, *The Bethe ansatz for superconformal Chern-Simons*, *JHEP* **09** (2008) 040 [arXiv:0806.3951] [INSPIRE].

[55] J.A. Minahan, W. Schulgin and K. Zarembo, *Two loop integrability for Chern-Simons theories with N=6 supersymmetry*, *JHEP* **03** (2009) 057 [arXiv:0901.1142] [INSPIRE].

[56] D. Bak, D. Gang and S.-J. Rey, *Integrable Spin Chain of Superconformal $U(M) \times \overline{U(N)}$ Chern-Simons Theory*, *JHEP* **10** (2008) 038 [arXiv:0808.0170] [INSPIRE].

[57] J.A. Minahan, O. Ohlsson Sax and C. Sieg, *Magnon dispersion to four loops in the ABJM and ABJ models*, *J. Phys. A* **43** (2010) 275402 [arXiv:0908.2463] [INSPIRE].

JHEP05(2023)112

[58] J.A. Minahan, O. Ohlsson Sax and C. Sieg, *Anomalous dimensions at four loops in N=6 superconformal Chern-Simons theories*, Nucl. Phys. B **846** (2011) 542 [arXiv:0912.3460] [INSPIRE].

[59] M. Leoni et al., *Superspace calculation of the four-loop spectrum in $N = 6$ supersymmetric Chern-Simons theories*, JHEP **12** (2010) 074 [arXiv:1010.1756] [INSPIRE].

[60] N. Gromov and G. Sizov, *Exact Slope and Interpolating Functions in $N = 6$ Supersymmetric Chern-Simons Theory*, Phys. Rev. Lett. **113** (2014) 121601 [arXiv:1403.1894] [INSPIRE].

[61] A. Cavaglià, N. Gromov and F. Levkovich-Maslyuk, *On the Exact Interpolating Function in ABJ Theory*, JHEP **12** (2016) 086 [arXiv:1605.04888] [INSPIRE].

[62] A. Hodges, *Eliminating spurious poles from gauge-theoretic amplitudes*, JHEP **05** (2013) 135 [arXiv:0905.1473] [INSPIRE].

[63] Z. Bern, J.J.M. Carrasco, H. Johansson and R. Roiban, *The Five-Loop Four-Point Amplitude of $N = 4$ super-Yang-Mills Theory*, Phys. Rev. Lett. **109** (2012) 241602 [arXiv:1207.6666] [INSPIRE].

[64] N. Arkani-Hamed et al., *The All-Loop Integrand For Scattering Amplitudes in Planar $N = 4$ SYM*, JHEP **01** (2011) 041 [arXiv:1008.2958] [INSPIRE].

[65] J.L. Bourjaily et al., *The Soft-Collinear Bootstrap: $N = 4$ Yang-Mills Amplitudes at Six and Seven Loops*, JHEP **03** (2012) 032 [arXiv:1112.6432] [INSPIRE].

[66] J.L. Bourjaily, P. Heslop and V.-V. Tran, *Amplitudes and Correlators to Ten Loops Using Simple, Graphical Bootstraps*, JHEP **11** (2016) 125 [arXiv:1609.00007] [INSPIRE].

[67] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *Constructing the correlation function of four stress-tensor multiplets and the four-particle amplitude in $N = 4$ SYM*, Nucl. Phys. B **862** (2012) 450 [arXiv:1201.5329] [INSPIRE].

[68] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo and J. Trnka, *Local Integrals for Planar Scattering Amplitudes*, JHEP **06** (2012) 125 [arXiv:1012.6032] [INSPIRE].

[69] A. Agarwal, N. Beisert and T. McLoughlin, *Scattering in Mass-Deformed $N \geq 4$ Chern-Simons Models*, JHEP **06** (2009) 045 [arXiv:0812.3367] [INSPIRE].

[70] I.A. Korchemskaya and G.P. Korchemsky, *On lightlike Wilson loops*, Phys. Lett. B **287** (1992) 169 [INSPIRE].

[71] N. Gromov and A. Sever, *Analytic Solution of Bremsstrahlung TBA*, JHEP **11** (2012) 075 [arXiv:1207.5489] [INSPIRE].

[72] B. Eden, G.P. Korchemsky and E. Sokatchev, *From correlation functions to scattering amplitudes*, JHEP **12** (2011) 002 [arXiv:1007.3246] [INSPIRE].

[73] J.M. Henn, *Multiloop integrals in dimensional regularization made simple*, Phys. Rev. Lett. **110** (2013) 251601 [arXiv:1304.1806] [INSPIRE].

[74] J.M. Henn, *What Can We Learn About QCD and Collider Physics from $N = 4$ Super Yang-Mills?*, Ann. Rev. Nucl. Part. Sci. **71** (2021) 87 [arXiv:2006.00361] [INSPIRE].

[75] Z. Bern, L.J. Dixon and V.A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, Phys. Rev. D **72** (2005) 085001 [hep-th/0505205] [INSPIRE].

[76] L.J. Dixon, J.M. Drummond and J.M. Henn, *Bootstrapping the three-loop hexagon*, JHEP **11** (2011) 023 [arXiv:1108.4461] [INSPIRE].

JHEP05(2023)112

[77] N. Arkani-Hamed et al., *Grassmannian Geometry of Scattering Amplitudes*, Cambridge University Press (2016) [DOI:10.1017/CBO9781316091548].

[78] A.V. Kotikov and L.N. Lipatov, *On the highest transcendentality in $N = 4$ SUSY*, *Nucl. Phys. B* **769** (2007) 217 [hep-th/0611204] [INSPIRE].

[79] A.V. Kotikov and L.N. Lipatov, *DGLAP and BFKL equations in the $N = 4$ supersymmetric gauge theory*, *Nucl. Phys. B* **661** (2003) 19 [hep-ph/0208220] [INSPIRE].

[80] A.V. Kotikov, L.N. Lipatov, A.I. Onishchenko and V.N. Velizhanin, *Three loop universal anomalous dimension of the Wilson operators in $N = 4$ SUSY Yang-Mills model*, *Phys. Lett. B* **595** (2004) 521 [hep-th/0404092] [INSPIRE].

[81] M.S. Bianchi, G. Giribet, M. Leoni and S. Penati, *Light-like Wilson loops in ABJM and maximal transcendentality*, *JHEP* **08** (2013) 111 [arXiv:1304.6085] [INSPIRE].

[82] H.S. Hannesdottir, A.J. McLeod, M.D. Schwartz and C. Vergu, *Implications of the Landau equations for iterated integrals*, *Phys. Rev. D* **105** (2022) L061701 [arXiv:2109.09744] [INSPIRE].

[83] A.B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, *Classical Polylogarithms for Amplitudes and Wilson Loops*, *Phys. Rev. Lett.* **105** (2010) 151605 [arXiv:1006.5703] [INSPIRE].

[84] L.J. Dixon, J.M. Drummond and J.M. Henn, *Analytic result for the two-loop six-point NMHV amplitude in $N = 4$ super Yang-Mills theory*, *JHEP* **01** (2012) 024 [arXiv:1111.1704] [INSPIRE].

[85] L.J. Dixon, J.M. Drummond, C. Duhr and J. Pennington, *The four-loop remainder function and multi-Regge behavior at NNLLA in planar $N = 4$ super-Yang-Mills theory*, *JHEP* **06** (2014) 116 [arXiv:1402.3300] [INSPIRE].

[86] L.J. Dixon, M. von Hippel and A.J. McLeod, *The four-loop six-gluon NMHV ratio function*, *JHEP* **01** (2016) 053 [arXiv:1509.08127] [INSPIRE].

[87] S. Caron-Huot, L.J. Dixon, A. McLeod and M. von Hippel, *Bootstrapping a Five-Loop Amplitude Using Steinmann Relations*, *Phys. Rev. Lett.* **117** (2016) 241601 [arXiv:1609.00669] [INSPIRE].

[88] J.M. Drummond, G. Papathanasiou and M. Spradlin, *A Symbol of Uniqueness: The Cluster Bootstrap for the 3-Loop MHV Heptagon*, *JHEP* **03** (2015) 072 [arXiv:1412.3763] [INSPIRE].

[89] L.J. Dixon et al., *Heptagons from the Steinmann Cluster Bootstrap*, *JHEP* **02** (2017) 137 [arXiv:1612.08976] [INSPIRE].

[90] Y.-T. Huang and C.K. Wen, *ABJM amplitudes and the positive orthogonal grassmannian*, *JHEP* **02** (2014) 104 [arXiv:1309.3252] [INSPIRE].

[91] Y.-T. Huang, C. Wen and D. Xie, *The Positive orthogonal Grassmannian and loop amplitudes of ABJM*, *J. Phys. A* **47** (2014) 474008 [arXiv:1402.1479] [INSPIRE].

[92] S. He, C.-K. Kuo, Z. Li and Y.-Q. Zhang, *Emergent unitarity, all-loop cuts and integrations from the ABJM amplituhedron*, arXiv:2303.03035 [INSPIRE].

[93] J.B. Tausk, *Nonplanar massless two loop Feynman diagrams with four on-shell legs*, *Phys. Lett. B* **469** (1999) 225 [hep-ph/9909506] [INSPIRE].

[94] M. Czakon, *Automatized analytic continuation of Mellin-Barnes integrals*, *Comput. Phys. Commun.* **175** (2006) 559 [hep-ph/0511200] [INSPIRE].

JHEP05(2023)112