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Integrability vs. RG flow in G × G and G × G/H sigma models

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Abstract: We consider a class of 2d σ-models on products of group spaces that provide new examples of a close connection between integrability and stability under the RG flow. We first study the integrable $G \times G$ model derived from the affine Gaudin construction (for which the 1-loop β-functions were found in arXiv:2010.07879) and show that its condition of integrability is preserved also by the 2-loop RG flow. We then investigate the RG flow in the gauged $G \times G/H$ model, in particular the integrable $T^{1,1}$ model found in arXiv:2010.05573. We also construct a new class of integrable $G \times G/H$ models in the case when the subgroup H is abelian. In the simplest case of $G = SU_2$, $H = U_1$ this leads to an integrable σ-model on the $T^{1,q}$ space (with a particular B-field). This model is also shown to be stable under the 2-loop RG flow, and we relate this property to its invariance under T-duality in an isometric U_1 direction. This $T^{1,q}$ model may be interpreted as an integrable deformation of the GMM model (of two coupled WZW theories with generic levels) away from the conformal point.

Keywords: Integrable Field Theories, Renormalization Group, Sigma Models

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JHEP05(2021)076

Contents

1 Introduction

It is expected that classically integrable 2d σ-models should be stable under the renormalization group flow, the intuition being that hidden symmetries will constrain the RG evolution. Constraints on coupling constants required for integrability should thus be RG-invariant. At the leading 1-loop order, this has been observed for some time (see, e.g., [1–3]). It was recently found on various examples [4, 5] that the RG stability for integrable theories extends also to higher-loop orders (provided the classical actions are supplemented by particular finite counterterms or if RG evolution is considered on a larger configuration space).

The aim of this paper is to explore the connection between integrability and the RG flow on some new examples — integrable $G \times G$ and $G \times G/H$ models that were derived from the affine Gaudin construction [6–8]. These models may be viewed as generalizations of the PCM_k,[1]

$$\mathcal{L}_{\mathrm{PCM}_k} = \mathcal{L}_{\mathrm{PCM}} + k\,\mathcal{L}_{\mathrm{WZ}}(g)\,, \qquad \mathcal{L}_{\mathrm{PCM}} = -\frac{1}{2}\,\mathrm{h}\,\mathrm{Tr}[J_+ J_-]\,, \qquad J \equiv g^{-1}dg\,, \quad g \in G\,, \tag{1.1}$$

[1] In our conventions the action is $S = \frac{1}{4\pi\alpha'}\int d^2\xi\,\mathcal{L}$ with the "string" notation for the loop counting parameter $\alpha' \equiv \hbar$ that may be set to 1 in some of the equations below. We also use $\partial_\pm = \partial_0 \pm \partial_1$.

JHEP05(2021)076

i.e. the principal chiral model (with inverse coupling h) with the WZ term (with "level" k). The conformal WZW model is obtained at the special points $\mathrm{h} = \pm k$.

The PCM_k admits various integrable deformations (see, e.g., [9–13]), which have been interpreted [6, 7] as particular cases of integrable affine Gaudin models. The affine Gaudin construction also produces natural generalizations of the PCM_k to integrable models on products of group spaces $G^N = G \times \ldots \times G$ [6, 7].

Here we shall consider a subclass of such models defined by

$$\mathcal{L} = -\frac{1}{2}\,\rho_{ij}\,\mathrm{Tr}[J_+^{(i)} J_-^{(j)}] + k_i\,\mathcal{L}_{\mathrm{WZ}}(g^{(i)})\,, \tag{1.2}$$
$$J^{(i)} \equiv {g^{(i)}}^{-1} dg^{(i)}\,, \qquad g^{(i)} \in G^N\,, \qquad i = 1, \ldots, N\,.$$

We denote by $J^{(i)}$ the Maurer-Cartan 1-form corresponding to i-th copy of G and ρ_{ij} is a *constant* coupling matrix (summation over repeated i, j is assumed).

The PCM_k (1.1) corresponds to the special case $N = 1$ (with $\rho_{11} = \mathrm{h}$ and $k_1 = k$), and is integrable for any values of its couplings. However, for $N > 1$, the model (1.2) is classically integrable only for *special* couplings (ρ_{ij}, k_i) that correspond to the affine Gaudin models [6, 7]. These are selected as the solutions of certain polynomial equations. We will focus on the first non-trivial case of $N = 2$, i.e. on $G \times G$ models.

As we shall find in section 2, the classical integrability condition for $G \times G$ theories (1.2) is automatically stable under the *2-loop* RG flow in a particular subtraction scheme (extending the 1-loop results of [14]). Here the 2-loop stability is obtained without the need for any finite counterterms.

The model (1.2) is a special case of the 2d σ-model

$$S = \frac{1}{4\pi\alpha'}\int d^2\xi\,\mathcal{L} = \frac{1}{4\pi\alpha'}\int d^2\xi\,[G_{mn}(x) + B_{mn}(x)]\,\partial_+ x^m \partial_- x^n\,. \tag{1.3}$$

This is a "two-coupling" theory, so the 2-loop β-functions for (G, B) generally depend on a choice of a renormalization scheme [15, 16]. There exists a special 2-loop scheme [15–17] that effectively treats G_{mn} and B_{mn} as symmetrically as possible (with the respective β-functions being the symmetric and antisymmetric parts of a single tensor expression). We shall refer to this G–B symmetric scheme as the "GB scheme". Explicitly, in this scheme one finds for the 2-loop β-functions [15, 16] (see also [18–20])[2]

$$\begin{aligned}\frac{d}{d\tau}(G_{mn} + B_{mn}) &= \alpha'\,\beta^{(1)}_{mn} + \alpha'^2\,\beta^{(2)}_{mn} + \ldots \\ &= \alpha'\,\widehat{R}_{mn} + \alpha'^2\,\frac{1}{2}\Big[\widehat{R}^{klp}{}_{n}\widehat{R}_{mklp} - \frac{1}{2}\widehat{R}^{lpk}{}_{m}\widehat{R}_{nklp} + \frac{1}{2}\widehat{R}_{kmnl}H^{kpq}H^{l}{}_{pq}\Big] + \ldots\,.\end{aligned} \tag{1.4}$$

[2]Here τ is the RG parameter. In general, the β-functions may contain also diffeomorphism and B-gauge transformation terms corresponding to freedom of field renormalizations and shifts of the Lagrangian by total derivatives depending on RG scale. We omit these terms since they automatically vanish in the examples considered below due to manifest global $G_L \times G_L$ symmetry.

JHEP05(2021)076

Here $H_{mnk} = 3\partial_{[m}B_{nk]}$ and $\widehat{R}$ is the curvature of the generalized connection $\widehat{\Gamma}^k{}_{mn} = \Gamma^k{}_{mn}(G) - \frac{1}{2}H^k{}_{mn}$. Applied to the case of the PCM$_k$ in (1.1), the expression in (1.4) gives (here we set $\alpha' = 1$)

$$\frac{d}{d\tau}\mathrm{h} = \mathrm{c}_{_G}\left(1 - \frac{k^2}{\mathrm{h}^2}\right)\left[1 + \frac{\mathrm{c}_{_G}}{2\mathrm{h}}\left(1 - \frac{3\,k^2}{\mathrm{h}^2}\right)\right], \qquad \frac{d}{d\tau}k = 0\,, \tag{1.5}$$

so that the position of the WZW fixed point $\mathrm{h} = \pm k$ remains unchanged at the 2-loop order. The 2-loop PCM$_k$ β-function (1.5) was found in [17] using a scheme equivalent (at the 2-loop level) to the one of [15, 16] that leads to (1.4).[3]

The GB scheme is naturally "adapted" to the vicinity of the WZW conformal point: the derivative $\partial_{\mathrm{h}}\beta_{\mathrm{h}}\big|_{\mathrm{h}=k}$ of the β-function for h at the fixed point correctly reproduces [17] the anomalous dimension of the $\mathrm{Tr}(J_+J_-)$ operator (PCM Lagrangian) as computed [22] using the underlying infinite dimensional Kac-Moody symmetry of the WZW model. Thus this scheme is apparently consistent with the preservation of the KM symmetry in the vicinity of the conformal point.

It is then natural to expect that this scheme should also play a special role in a more general class of integrable models (1.2) containing WZW models as special limits,[4] and should facilitate preservation of the hidden integrable structure of these models at the quantum level. We will indeed see evidence for this below: the classical integrability conditions for the $G \times G$ model (1.2) will be automatically preserved by the 2-loop RG evolution provided one uses the β-functions in the GB scheme (1.4).

We shall also study, in section 3, a gauged analog of the models (1.2) defined on a coset space $G \times G/H$. This theory, which was recently derived from affine Gaudin models in [8], may be viewed as a generalization of the standard G/H symmetric space σ-model, also including WZ terms. For these $G \times G/H$ theories to be gauge invariant, the corresponding couplings must satisfy certain linear relations. In addition, for a gauge invariant model to be classically integrable, the couplings should further satisfy certain polynomial relations [8].

We will compute the RG flow for these integrable $G \times G/H$ theories, finding that they are stable under the 1-loop RG flow. However, at the 2-loop level, RG stability does not automatically arise and, in general, requires certain finite redefinitions of the couplings. These are equivalent to adding specific finite counterterms, which may be interpreted as required for preservation of integrability at the quantum level (this is analogous to what was observed on other examples in [4, 5]). There are still a few special cases, in particular the integrable $T^{1,1}$ model of [8], that are automatically stable under the 2-loop RG flow (see section 3.2).

[3] To recall, part of the scheme freedom comes from the prescription of how one treats the antisymmetric 2d tensor ε^{ab} appearing in the B-term in (1.3) in dimensional regularization. Ref. [17] used 't Hooft-Veltman prescription of treating ε^{ab} as effectively 2-dimensional. In [15, 16] it was assumed that, in $d = 2 + \epsilon$ dimensions, $\varepsilon^{ab}\varepsilon_{cd} = f(\epsilon)(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c)$ where $f = 1 + f_1\epsilon + \dots$, and then the GB scheme corresponds to the choice $f_1 = -1$. As noted in [21], the scheme used in [17] is equivalent (at least at the 2-loop level) to $f(d) = \frac{1}{d-1} = 1 - \epsilon + \dots$, i.e. to the choice $f_1 = -1$ [15, 16] of the GB scheme (1.4).

[4] Similar logic was recently used in [23] in the discussion of the 2-loop RG evolution of a "squashed" SU_2 variant of PCM$_k$.

JHEP05(2021)076

In section 4 we shall present a new integrable σ-model with target space metric $T^{1,q} = SU_2 \times SU_2/U_1$ [24–27] and a particular B-field. The model admits as a special limit the conformal GMM model with unequal levels [28–30]. Our central observation is that, in the case of the subgroup H in $G \times G/H$ being abelian, the gauge invariance conditions of [8] are too restrictive and there is also a second "branch" of gauge invariant theories. This allows a natural generalization of the integrable $T^{1,1}$ model of [8] to $T^{1,q}$ with a general parameter q. We demonstrate that the resulting $T^{1,q}$ model is classically integrable, admitting a Lax representation. We observe that the $T^{1,q}$ model is self-dual under T-duality in one isometry direction, and argue that this property forces it to be stable under the RG flow. We verify this fact explicitly by computing the corresponding 2-loop RG flow of the two coupling constants.

A few concluding remarks will be made in section 5. In appendix A we shall discuss the integrability conditions for the G^N model (1.2). In appendix B we shall provide the explicit formulae for the 2-loop β-functions of the $G \times G$ and $G \times G/H$ models and explain how they were derived.

2 $G \times G$ models

As was mentioned in the Introduction, the G^N model (1.2) is classically integrable for *special* values of its couplings (ρ_{ij}, k_i) satisfying certain polynomial relations, which originate from the affine Gaudin construction [6, 7]. For such values of the couplings the model admits a Lax connection of the form

$$L_+ = \alpha_i \, J_+^{(i)} \,, \qquad\qquad L_- = \beta_i \, J_-^{(i)} \,, \tag{2.1}$$

whose flatness condition, $F_{+-}(L) \equiv \partial_+ L_- - \partial_- L_+ + [L_+, L_-] = 0$, is equivalent to the equations of motion following from (1.2). Moreover, the affine Gaudin construction guarantees that the Poisson brackets of the Lax matrix $L_\sigma = \frac{1}{2}(L_+ - L_-)$ can be written in a 'twist' form, i.e. a special form of the standard non-ultralocal r/s Poisson bracket [31, 32]. This implies the existence of a tower of conserved commuting higher-spin charges [33].

Below we shall consider the simplest $N = 2$ case of the G^N model (1.2) for a simple Lie group G. We shall parametrize the 2×2 matrix ρ_{ij} in (1.2) in terms of the 4 components s, t, u, b as follows

$$\mathcal{L} = -\frac{1}{2} \begin{pmatrix} s & t+b \\ t-b & u \end{pmatrix}_{ij} \mathrm{Tr}[J_+^{(i)} J_-^{(j)}] + k_i \, \mathcal{L}_{\mathrm{WZ}}(g^{(i)}) \,. \tag{2.2}$$

Then the affine Gaudin condition for integrability is the vanishing of a cubic polynomial [6, 7, 14],

$$f(s,t,u,b,k_1,k_2) \equiv -t\,(s+t)\,(t+u) + b^2\,(s+t+u) + t\,k_1\,k_2 + b\,(u\,k_1 - s\,k_2) = 0 \,. \tag{2.3}$$

Let us note that the affine Gaudin conditions for integrability (e.g. (2.3) in the $N = 2$ case) are certainly sufficient for integrability. However, it is not a priori clear if they are necessary, since there could also be integrable theories of the form (1.2) that are unrelated

JHEP05(2021)076

to the affine Gaudin construction of [6, 7]. In appendix A we presented a check that the condition (2.3) is also *necessary* for the integrability of the $G \times G$ model (2.2), assuming the natural ansatz (2.1) for the corresponding Lax connection.

2.1 RG flow in $G \times G$ models

The general G^N model (1.2) has global $(G_L)^N \times G_R$ symmetry acting as

$$g^{(i)} \to u_L^{(i)} g^{(i)} u_R \,, \qquad\qquad (u_L^{(i)}, u_R) \in (G_L)^N \times G_R \,. \tag{2.4}$$

In fact, (1.2) is the most general 2-derivative local Lagrangian having this symmetry. This implies that only ρ_{ij} can run under the RG flow (with the WZ parameters k_i not renormalized as usual).[5]

Starting with the σ-model couplings (G_{mn}, B_{mn}) corresponding to the $G \times G$ model (2.2) and computing the corresponding 2-loop β-functions in the GB scheme (1.4), we find

$$\begin{aligned} &\frac{d}{d\tau}\rho_{ij} = \alpha' \, \beta_{ij}^{(1)} + \alpha'^2 \, \beta_{ij}^{(2)} + \dots \,, \\ &\beta_{ij}^{(1)} = \mathrm{c}_{_G} \, F_{ij}^{(4)}(s,t,u,b,k_1,k_2) \,, \qquad \beta_{ij}^{(2)} = \mathrm{c}_{_G}^2 \, (su - t^2)^{-5} \, F_{ij}^{(9)}(s,t,u,b,k_1,k_2) \,, \end{aligned} \tag{2.5}$$

where the matrices $F^{(4)}, F^{(9)}$ are homogeneous polynomials of degrees 4 and 9 in their arguments and $\mathrm{c}_{_G}$ is the dual Coxeter number of the group G, as in the PCM_k case in (1.5). The explicit expressions for $F^{(4)}, F^{(9)}$ are given in appendix B.1 and also in some special cases below.

Remarkably, despite the complicated expressions for the β-functions, one is able to verify that the integrability condition (2.3) is, in fact, preserved by the 2-loop RG flow:

$$\frac{df}{d\tau}\Big|_{f=0} = \Big(\alpha' \, \beta_{ij}^{(1)} + \alpha'^2 \, \beta_{ij}^{(2)} + \dots\Big) \, \frac{\partial f}{\partial \rho_{ij}}\Big|_{f=0} = \alpha' \times 0 + \alpha'^2 \times 0 + \dots \tag{2.6}$$

The vanishing of the 1-loop $\mathcal{O}(\alpha')$ term in (2.6) was already established in [14], and the vanishing of the 2-loop term is a new non-trivial result. Let us stress that this property of the integrability condition (2.3) not being deformed at the 2-loop level is specific to the GB scheme (1.4).

2.2 Some special cases

Let us consider some particular examples of the integrable $G \times G$ models (2.2), (2.3).

2.2.1 $\rho_{21} = 0$ and the $G \times G$ model related to λ-model

The most general integrable model with $\rho_{21} = 0$ corresponds to the following choice of the parameters in (2.2) (this case was also considered in appendix C of [14])

$$b = t \,, \qquad u = -k_2 \,. \tag{2.7}$$

[5] As in the PCM_k case, the RG invariance of k_i follows from the fact that the corresponding field strength $H = dB$ is covariantly constant.

JHEP05(2021)076

After the redefinition $(g^{(1)}, g^{(2)}) \equiv (g, \widetilde{g}^{-1})$, the corresponding Lagrangian (2.2) depending on s, t, k_1, k_2 may be written as (cf. (1.1))

$$\mathcal{L} = [s\mathcal{L}_{\rm PCM}(g) + k_1\mathcal{L}_{\rm WZ}(g)] - k_2\,[\mathcal{L}_{\rm PCM}(\widetilde{g}) + \mathcal{L}_{\rm WZ}(\widetilde{g})] + t\,{\rm Tr}\,[J_+(g)K_-(\widetilde{g})]\,, \tag{2.8}$$

where $J = g^{-1}dg$, $K(\widetilde{g}) = d\widetilde{g}\,\widetilde{g}^{-1}$. This is just a PCM_{k_1} and WZW model (with level $-k_2$) coupled via the $J_+(g)K_-(\widetilde{g})$ term. In this case the global G_R symmetry in (2.4) is enlarged to a chiral symmetry $G_R(\xi^+)$,[6]

$$(g,\,\widetilde{g}) \to (u\,g\,v,\;\; v^{-1}\widetilde{g}\,w(\xi^+))\,, \qquad\qquad (u, v, w(\xi^+)) \in G_L \times G_L \times G_R(\xi^+)\,. \tag{2.9}$$

These symmetries protect the structure of (2.8) under renormalization so that only the parameters s and t are expected to run with the RG scale. Indeed, in this case the RG equations (2.5) take the following explicit form[7]

$$\begin{aligned}
\frac{d}{d\tau}s =& \frac{{\rm c}_{_G}\,(s-k_1)}{(k_2 s + t^2)^2}\left[k_2^2(k_1+s) + 4k_2t^2 - 2t^3\right] \\
&+ \frac{{\rm c}_{_G}^2\,(s-k_1)}{2\,(k_2 s + t^2)^5}\left[2k_2t^5\left(38t^2 - 11k_1t + 2s^2 + 41st\right)\right. \\
&+ 2k_2^3t^3\left(-8k_1s - 42k_1t + 9k_1^2 - 5s^2 + 18st\right) - 2k_2^2t^4\left(-7k_1s - 46k_1t + s^2 + 48st + 28t^2\right) \\
&\left.+k_2^5(k_1+s)\left(s^2 - 3k_1^2\right) + 2k_2^4t^2(3k_1+2s)(3s-5k_1) - 4t^7(5s+6t)\right]\,,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\frac{d}{d\tau}t =& \frac{{\rm c}_{_G}\,t(t-k_2)}{(k_2 s + t^2)^2}\left[k_2(k_1-s) + 2t(s+t)\right] \\
&+ \frac{{\rm c}_{_G}^2\,t(t-k_2)}{2\,(k_2 s + t^2)^5}\left[4t^5\left(t(4t-k_1) + 5s^2 + 10st\right) - k_2^4(s-k_1)\left(s^2 - 3k_1^2\right)\right. \\
&+ 2k_2^2t^2\left(s^2(28t-3k_1) + 2st(13t-16k_1) + 6k_1t(k_1-4t) + 3s^3\right) \\
&\left.-2k_2t^3\left(-k_1t(13s+19t) + 2s^3 + 31s^2t + 45st^2 + 10t^3\right) - 2k_2^3t(s-k_1)(5st - 3k_1(s+4t))\right]\,.
\end{aligned} \tag{2.11}$$

At the obvious fixed point $s = k_1$, $t = k_2$, the model (2.8) becomes [14] the sum of two decoupled WZW models, $\mathcal{L} = (k_1+k_2)\mathcal{L}_{\rm WZW}(g) - k_2\mathcal{L}_{\rm WZW}(g\widetilde{g})$. As discussed in [14], the fixed points are all decoupled WZW models of this type. The RG trajectories either interpolate between such WZW-type fixed points or flow to them in the IR from the asymptotically free UV fixed point $s, t \to \infty$.

An interesting special case of (2.8) is $s = k_1 = -k_2 \equiv -k'$, when it becomes

$$\mathcal{L} = -k'\,(\mathcal{L}_{\rm WZW}(g) + \mathcal{L}_{\rm WZW}(\widetilde{g}) - \lambda'\,{\rm Tr}\,[J_+(g)K_-(\widetilde{g})])\,, \qquad\qquad \lambda' \equiv k'^{-1}t\,. \tag{2.12}$$

This particular $G \times G$ model appears from the "tripled" version [5] of the λ-model [34, 35] after removing the decoupled WZW part. It is also a special case of the "doubly λ-deformed" model of [36–39]. Here the β-functions (2.10), (2.11) reduce to just λ' running as

$$\frac{d}{d\tau}\lambda' = \frac{2{\rm c}_{_G}\,\lambda'^2}{k'(1+\lambda')^2} + \frac{4{\rm c}_{_G}^2\,\lambda'^4(1-2\lambda')}{k'^2(1-\lambda')(1+\lambda')^5}\,. \tag{2.13}$$

[6] We denote by $G(\xi^+)$ right G multiplications depending on light-cone coordinate $\xi^+ = \frac{1}{2}(\xi^0 + \xi^1)$.

[7] As in (1.5), here we set the loop counting parameter α' to be 1. The 1-loop terms in (2.10), (2.11) match those in [14] (after reversing the sign of the WZ terms $k_i \to -k_i$ to match the conventions).

JHEP05(2021)076

This is the 2-loop β-function [5] for the λ-model based on the group G with parameters (k, λ) related to (k', λ') as $k' = k + 2\mathrm{c}_G$, $\lambda' = \frac{k}{k+2\mathrm{c}_G}\lambda^{-1}$.[8]

2.2.2 $k_1 = k_2 = 0$

Setting the WZ levels to zero, $k_1 = k_2 = 0$, the integrability condition (2.3) implies that

$$b = b(s,t,u) \equiv \left[\frac{t(t+s)(t+u)}{t+s+u}\right]^{1/2} . \tag{2.14}$$

We thus obtain from (2.2) an integrable $G \times G$ model with 3 independent couplings s, t, u,

$$\mathcal{L} = -\frac{1}{2}\begin{pmatrix} s & t + b(s,t,u) \\ t - b(s,t,u) & u \end{pmatrix}_{ij} \mathrm{Tr}[J_+^{(i)} J_-^{(j)}] . \tag{2.15}$$

Since k_i do not run, this special case of the model (2.2), (2.3) should also be stable under the RG flow, i.e. (2.15) should be renormalizable with only s, t, u running. Indeed, using for convenience the redefined couplings $(s,t,u) \to (x,y,z)$ with

$$x = s + t + u\,, \qquad y = \frac{s}{t}\,, \qquad z = \frac{u}{t}\,, \tag{2.16}$$

the 2-loop β-functions (2.5) become

$$\begin{aligned} \frac{d}{d\tau}x &= 2\mathrm{c}_G - \frac{\mathrm{c}_G^2}{2x(yz-1)^3}\Big[16 + 32(y+z) + 16(y^2+z^2) + 88yz + 68yz(y+z) \\ &\quad + 12yz(y^2+z^2+5yz) + 8y^2z^2(y+z) - y^2z^2(y+z)^2\Big]\,, \end{aligned} \tag{2.17}$$

$$\frac{d}{d\tau}y = F(x;y,z)\,, \qquad \frac{d}{d\tau}z = F(x;z,y)\,, \tag{2.18}$$

$$\begin{aligned} F(x;y,z) &\equiv \frac{y(y+1)(y+2)}{(zy-1)^2}\left(\frac{\mathrm{c}_G}{x}\Big[1 - zy - 3(z+1)^2\Big]\right. \\ &- \frac{\mathrm{c}_G^2}{2x^2}\frac{1}{(zy-1)^3}\Big[-z^6y^2 - y^6(3zy - 38z - 44) + 2z(y+1)(26zy + 101y + 58) + 20(y+1)^2 \\ &- z^4(y(3y((y-14)y - 100) - 296) - 38) - z^3(y(y(y((y-4)y - 178) - 728) - 708) - 152) \\ &\left. + 2z^2(y+1)(y(y(5y+89)+262)+105)\Big]\right)\,. \end{aligned} \tag{2.19}$$

The obvious symmetry between s and u in (2.15) is translated into the symmetry of the RG equations under $y \leftrightarrow z$.

The fact that these 2-loop β-functions are much simpler than the general (not necessarily integrable) case of (2.5) (see also appendix B.1) suggests that a substantial simplification happens upon specifying the couplings to be at the integrable locus $f = 0$ in (2.3) (this was already observed at the 1-loop order in [14]).

[8] See also [40]. For the 1-loop beta functions of the λ-models based on G and G/H, see [41] and [42] respectively.

JHEP05(2021)076

3 $G \times G/H$ models

Let H be a subgroup of G such that G/H is a symmetric space (we assume that both G and H are simple real Lie groups). Then the Lagrangian for the gauged $G \times G/H$ model of [8] takes the form[9]

$$\mathcal{L} = -\frac{1}{2}\,\rho_{ij}\,\mathrm{Tr}[P_+^{(i)} P_-^{(j)}] - \frac{1}{2}\,r_{ij}\,\mathrm{Tr}[I_+^{(i)} I_-^{(j)}] + k_i\,\mathcal{L}_{\mathrm{WZ}}(g^{(i)})\,, \tag{3.1}$$

$$P^{(i)} = P_{G/H} J^{(i)}\,, \qquad I^{(i)} = P_H J^{(i)}\,, \qquad J^{(i)} = {g^{(i)}}^{-1} dg^{(i)}\,. \tag{3.2}$$

Here $P_{G/H}$ and P_H are projectors to the corresponding parts of the algebra of G, and ρ_{ij}, r_{ij} are constant 2×2 matrices. The global symmetry consists of left multiplication $G_L \times G_L$, as well as the discrete $\mathbb{Z}_2$ corresponding to the symmetric space structure of G/H. The action for (3.1) is required to be gauge invariant under the local right action by an element of H (acting the same on both $g^{(i)}$)

$$g^{(i)} \to g^{(i)}\mathrm{w}\,, \qquad \mathrm{w}(\xi^+, \xi^-) \in H\,. \tag{3.3}$$

For general choices of G and H, gauge invariance imposes the linear constraints [8][10]

$$k_1 = -k_2 \equiv k\,, \qquad r_{ij} = \begin{pmatrix} r & -r-k \\ -r+k & r \end{pmatrix}. \tag{3.4}$$

The remaining free parameters of the gauge invariant model are then r, k and the 2×2 matrix ρ_{ij}.

Requiring integrability imposes further constraints which, as for the G^N models (1.2), can be obtained from the affine Gaudin construction. The following parametrization of the 6 constants r, k, ρ_{ij} in terms of 4 parameters K, x, ζ_+, ζ_- was shown in [8] to be sufficient for integrability

$$r = r_{11} = r_{22} = K\frac{\zeta_-^2 - \zeta_+^2}{(1-x^2)^2}\,, \quad r_{12} = 2K\frac{(1-\zeta_+^2)(x^2-\zeta_-^2)}{(1-x^2)^3}\,, \quad r_{21} = -2K\frac{(1-\zeta_-^2)(x^2-\zeta_+^2)}{(1-x^2)^3}\,, \tag{3.5}$$

$$\rho_{11} = K\frac{1-2\zeta_+^2+\zeta_-^2\zeta_+^2}{(1-x^2)^2}\,, \qquad \rho_{12} = x\,r_{12} = 2K\frac{x(1-\zeta_+^2)(x^2-\zeta_-^2)}{(1-x^2)^3}\,, \tag{3.6}$$

$$\rho_{21} = x^{-1} r_{21} = -2K\frac{(1-\zeta_-^2)(x^2-\zeta_+^2)}{x(1-x^2)^3}\,, \qquad \rho_{22} = K\frac{x^4-2\zeta_+^2x^2+\zeta_-^2\zeta_+^2}{x^2(1-x^2)^2}\,, \tag{3.7}$$

$$k = k_1 = -k_2 = -K\frac{2x^2+2\zeta_-^2\zeta_+^2-(1+x^2)(\zeta_-^2+\zeta_+^2)}{(1-x^2)^3}\,. \tag{3.8}$$

[9] Our conventions in (3.1) are related to the ones of [8] by $r_{ij} \to 2\rho_{ij}^{(0)}$, $\rho_{ij} \to 2\rho_{ij}^{(1)}$ and the opposite sign for the WZ terms, i.e. $k_i \to -k_i$.

[10] The special case of abelian H will be discussed below in section 4.

JHEP05(2021)076

This parametrization is simply equivalent to the gauge invariance conditions (3.4) combined with the two extra polynomial integrability conditions

$$\begin{aligned} f_1 &\equiv r^2 - k^2 - \rho_{12}\rho_{21} = 0\,, \\ f_2 &\equiv (r-k)^4\rho_{12} + (r-k)^2(r-k-2\rho_{11})(r-k+2\rho_{22})\rho_{21} \\ &\quad - 2(r-k)(\rho_{11}+\rho_{22})\rho_{12}\rho_{21}^2 + (\rho_{12}+\rho_{21})\rho_{12}\rho_{21}^3 = 0\,. \end{aligned} \tag{3.9}$$

Two simple solutions of these conditions are found by setting $r = k$ (i.e. $r_{21} = 0$ in (3.4)) and either $\rho_{21} = 0$ or $\rho_{12} = 0$.

3.1 RG flow in $G \times G/H$ models

The structure of the gauge invariant $G \times G/H$ action (3.1), (3.4) is protected by the right H gauge symmetry (3.3) and the global $G_L \times G_L$ and $\mathbb{Z}_2$ symmetry. This rules out all counterterms except those corresponding to renormalizations of the 6 couplings r, k, ρ_{ij} (of which k is not renormalized as usual). Let us parametrize ρ_{ij} as in (2.2),

$$\rho_{ij} = \begin{pmatrix} s & t+b \\ t-b & u \end{pmatrix}. \tag{3.10}$$

Computing the β-functions (1.4) corresponding to the σ-model couplings (G_{mn}, B_{mn}) for the model (3.1), (3.4), (3.10), we find for the 1-loop β-functions of the 5 running couplings

$$\frac{d}{d\tau}h_p = \alpha'\beta^{(1)}_{h_p}\,, \qquad h_p \equiv (r,s,t,b,u)\,, \tag{3.11}$$

$$\begin{aligned} \beta^{(1)}_r &= \frac{\mathrm{c}_G - \mathrm{c}_H}{(t^2-su)^2}\Big(r^2s^2 - 2b^2t^2 - 2r^2t^2 + 2t^4 - 2b^2su - 2st^2u + r^2u^2 \\ &\qquad + 4bstk + 4btuk - s^2k^2 - 2t^2k^2 - u^2k^2\Big) + \mathrm{c}_H\left(1 - \frac{k^2}{r^2}\right), \end{aligned} \tag{3.12}$$

$$\beta^{(1)}_s = \frac{\mathrm{c}_G}{r(t^2-su)}\Big[b^2s - st^2 + r^2u + 2r(t^2-su) - 2btk + uk^2\Big]\,, \tag{3.13}$$

$$\beta^{(1)}_t = \frac{\mathrm{c}_G}{r(t^2-su)}\Big[-b^2t + b(s+u)k + t(r^2 - su - k^2)\Big]\,, \tag{3.14}$$

$$\beta^{(1)}_b = \frac{\mathrm{c}_G}{r(t^2-su)}\Big[-b(t^2+su) + t(s+u)k\Big]\,, \tag{3.15}$$

$$\beta^{(1)}_u = \frac{\mathrm{c}_G}{r(t^2-su)}\Big[r^2s + b^2u - t^2u + 2r(t^2-su) - 2btk + sk^2\Big]\,. \tag{3.16}$$

We observe that the integrability conditions (3.9) are stable under the 1-loop RG flow (3.12)–(3.16),

$$\frac{\partial f_a}{\partial\tau}\Big|_{f_1=f_2=0} = \alpha'\beta^{(1)}_{h_p}\frac{\partial f_a}{\partial h_p}\Big|_{f_1=f_2=0} + \mathcal{O}(\alpha'^2) = 0 + \mathcal{O}(\alpha'^2)\,, \qquad a = 1,2\,. \tag{3.17}$$

However, it turns out that (as for some examples discussed in [4, 5]) this property of RG stability does not, in general, extend to the 2-loop order. Computing the 2-loop β-functions

JHEP05(2021)076

for the model (3.1), (3.4) in the GB scheme (1.4) (given explicitly in appendix B.2), we find that the subleading correction to (3.17) is non-zero at general values of the couplings,

$$\beta^{(2)}_{h_p}\frac{\partial f_a}{\partial h_p}\Big|_{f_1=f_2=0}\neq 0\,,\qquad\qquad a=1,2\,. \tag{3.18}$$

Moreover, we checked that (3.18) is also non-vanishing in arbitrary covariant 2-loop subtraction schemes.[11]

As in other examples [4, 5], one may expect to restore the property of RG stability at the 2-loop order by adding certain finite quantum α'-corrections to the target space geometry. Because of the global and local symmetries, the only possible corrections would correspond to redefinitions $h_p\to\bar h_p$ of the couplings $h_p=(r,s,t,b,u)$,

$$h_p=\bar h_p+\alpha' Q_p(\bar h)+\dots\,. \tag{3.19}$$

Such redefinitions may be interpreted as quantum corrections to the integrability conditions (3.9): if the original couplings h_p satisfied $f_a(h)=0$, then the corrected ones $\bar h_p$ would satisfy a *corrected* version of the integrability conditions,

$$\bar f_a(\bar h)=0\,,\qquad \bar f_a\equiv f_a+\alpha'\,Q_p\,\partial_{h_p}f_a+\dots\,. \tag{3.20}$$

3.2 Some special RG-stable cases

There are still special exceptional cases of the integrable $G\times G/H$ model (3.1), (3.4), (3.9) that are automatically stable under the 2-loop RG flow in the GB scheme. Two of them are discussed below.

3.2.1 $G\times G/H$ model related to G/H λ-model

One solution of the integrability conditions (3.9) is

$$r=k\,,\quad \rho_{21}=0\,,\quad \rho_{11}=\rho_{22}=k\,,\qquad \text{i.e.}\qquad r=s=u=k\,,\quad t=b\,, \tag{3.21}$$

on which (3.1), (3.4) become (redefining $(g,\widetilde g)\equiv(g^{(1)},(g^{(2)})^{-1})$)

$$\begin{aligned}&\mathcal{L}=-k'\Big(\mathcal{L}_{\rm WZW}(g)+\mathcal{L}_{\rm WZW}(\widetilde g)-{\rm Tr}\Big[J_+(g)\Big(P_H+\lambda'\,P_{G/H}\Big)K_-(\widetilde g)\Big]\Big)\,,\\ &J_+(g)\equiv g^{-1}\partial_+ g\,,\qquad K_-(\widetilde g)\equiv\partial_-\widetilde g\,\widetilde g^{-1}\,,\qquad k'\equiv -k\,,\qquad \lambda'\equiv k'^{-1}t\,.\end{aligned} \tag{3.22}$$

This model is a "gauged" version of (2.12), similarly being constructed from a combination of two WZW Lagrangians coupled by a current-current term. This particular $G\times G/H$ model appears from the "tripled" formulation [5] of the G/H λ-model [34, 35] (after removing a decoupled third WZW part). Compared to generic $G\times G/H$ models, the $G\times G$ global symmetry is enhanced to a chiral gauge symmetry $G(\xi^-)\times G(\xi^+)$ acting as (see footnote 6)

$$(g,\,\widetilde g)\to\Big(u(\xi^-)\,g,\;\widetilde g\,v(\xi^+)\Big)\,,\qquad\qquad \Big(u(\xi^-),\,v(\xi^+)\Big)\in G(\xi^-)\times G(\xi^+)\,. \tag{3.23}$$

[11] More precisely, we considered arbitrary subtraction schemes related to the GB scheme (1.4) by covariant redefinitions of G_{mn} and B_{mn}.

JHEP05(2021)076

This symmetry protects the structure of (3.22), allowing only the coupling λ' to run. The 1- and 2-loop β-functions in (3.12)–(3.16) and appendix B.2 lead to the following RG equation for λ'

$$\frac{d}{d\tau}\lambda' = \frac{\mathrm{c}_{_G}\lambda'}{k'} + \frac{\mathrm{c}_{_G}\lambda'\left[c_{_H} - (2\mathrm{c}_{_G} - c_{_H})\lambda'^2\right]}{k'^2(1-\lambda'^2)}\,. \tag{3.24}$$

This is the 2-loop β-function [5] for the λ-model based on the symmetric space G/H with parameters (k,λ) related to (k',λ') by $k' = k + 2\mathrm{c}_{_G}$, $\lambda' = \frac{k}{k+2\mathrm{c}_{_G}}(\lambda^{-1} + 2\mathrm{c}_{_G})$.

3.2.2 Integrable deformation of GMM model on $G \times G/H$ and $T^{1,1}$ model

Let us consider a particular solution of the integrability conditions (3.9) that was studied in [8],

$$r = k\,, \qquad \rho_{12} = \rho_{21} = 0\,, \quad \text{i.e.} \quad t = b = 0\,. \tag{3.25}$$

The Lagrangian of the corresponding theory (3.1), (3.4) is given by

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\left[\mathrm{h}\,P_+P_- + \widetilde{\mathrm{h}}\,\widetilde{P}_+\widetilde{P}_-\right] - \frac{1}{2}k\,\mathrm{Tr}\left[I_+I_- + \widetilde{I}_+\widetilde{I}_- - 2I_+\widetilde{I}_-\right] + k\left[\mathcal{L}_{\mathrm{WZ}}(g) - \mathcal{L}_{\mathrm{WZ}}(\widetilde{g})\right], \tag{3.26}$$

where we have set[12]

$$(g^{(1)}, g^{(2)}) \equiv (g, \widetilde{g})\,, \quad \widetilde{P}_\pm = P_\pm(\widetilde{g})\,, \quad \widetilde{I}_\pm = I_\pm(\widetilde{g})\,, \qquad \mathrm{h} \equiv \rho_{11} = s\,, \quad \widetilde{\mathrm{h}} \equiv \rho_{22} = u\,. \tag{3.27}$$

This is an integrable deformation of the special point $\mathrm{h} = \widetilde{\mathrm{h}} = k$ that corresponds to the conformal GMM model [28, 29] on the homogeneous space $G \times G/H$ with equal levels.

Specializing the 1-loop and 2-loop β-functions in (3.12)–(3.16) and appendix B.2 to this case, we find that the model (3.26) is automatically stable under 2-loop renormalization with only h and $\widetilde{\mathrm{h}}$ running,

$$\begin{aligned}
\frac{d}{d\tau}\mathrm{h} &= 2\mathrm{c}_{_G}\left(1 - \frac{k}{\mathrm{h}}\right)\left(1 + \frac{1}{\mathrm{h}}\left[2(\mathrm{c}_{_G} - \mathrm{c}_{_H}) - (3\mathrm{c}_{_G} - 2\mathrm{c}_{_H})\frac{k}{\mathrm{h}}\right]\right),\\
\frac{d}{d\tau}\widetilde{\mathrm{h}} &= 2\mathrm{c}_{_G}\left(1 - \frac{k}{\widetilde{\mathrm{h}}}\right)\left(1 + \frac{1}{\widetilde{\mathrm{h}}}\left[2(\mathrm{c}_{_G} - \mathrm{c}_{_H}) - (3\mathrm{c}_{_G} - 2\mathrm{c}_{_H})\frac{k}{\widetilde{\mathrm{h}}}\right]\right), \qquad \frac{d}{d\tau}k = 0\,.
\end{aligned} \tag{3.28}$$

Remarkably, the RG evolution of h and $\widetilde{\mathrm{h}}$ is decoupled. Note that the structure of their β-functions is similar to the one in the PCM_k case (1.5). As expected, the GMM model $\mathrm{h} = \widetilde{\mathrm{h}} = k$ is a fixed point.

Let us consider the simplest example of this theory (3.26) with $G = SU_2$ and $H = U_1$ and choose the parametrization

$$g = e^{\phi_1 T_1} e^{\theta_1 T_2} e^{\psi T_3}\,, \qquad \widetilde{g} = e^{-\phi_2 T_1} e^{-\theta_2 T_2} e^{-\widetilde{\psi} T_3}\,, \tag{3.29}$$

[12] Ref. [8] used the notation $(k,\,\mathrm{h},\,\widetilde{\mathrm{h}}) \equiv (\lambda^2,\,\lambda_2^2,\,\lambda_1^2)$.

JHEP05(2021)076

where the SU_2 generators are $T_A = \frac{i}{2}\sigma_A$ and the generator of $H = U_1$ is T_3. We shall fix the H gauge freedom by setting $\widetilde{\psi} = 0$. As a result, we get an integrable 5-dimensional σ-model (cf. (1.3))

$$\begin{aligned}\mathcal{L} = (G_{mn} + B_{mn})\partial_+ x^m \partial_- x^n = \frac{1}{4}k\Big[&\partial_+\psi\partial_-\psi + \cos^2\theta_1\,\partial_+\phi_1\partial_-\phi_1 + \cos^2\theta_2\,\partial_+\phi_2\partial_-\phi_2 \\ &+ 2\cos\theta_1\,\partial_+\phi_1\partial_-\psi + 2\cos\theta_2\,\partial_+\psi\partial_-\phi_2 + 2\cos\theta_1\cos\theta_2\,\partial_+\phi_1\partial_-\phi_2\Big] \\ &+ \frac{1}{4}\mathrm{h}\Big[\partial_+\theta_1\partial_-\theta_1 + \sin^2\theta_1\,\partial_+\phi_1\partial_-\phi_1\Big] + \frac{1}{4}\widetilde{\mathrm{h}}\Big[\partial_+\theta_2\partial_-\theta_2 + \sin^2\theta_2\,\partial_+\phi_2\partial_-\phi_2\Big]\,.\end{aligned} \tag{3.30}$$

The resulting target space geometry corresponds to the $T^{1,1}$ metric and a particular B-field [8][13]

$$\begin{aligned}ds^2_{T^{1,1}} = G_{mn}dx^m dx^n = &\frac{1}{4}k\,(d\psi + \cos\theta_1\,d\phi_1 + \cos\theta_2\,d\phi_2)^2 \\ &+ \frac{1}{4}\mathrm{h}\,(d\theta_1^2 + \sin^2\theta_1\,d\phi_1^2) + \frac{1}{4}\widetilde{\mathrm{h}}\,(d\theta_2^2 + \sin^2\theta_2\,d\phi_2^2)\,,\end{aligned} \tag{3.31}$$

$$B = \frac{1}{2}B_{mn}dx^m \wedge dx^n = \frac{1}{4}k\,(d\psi + \cos\theta_1\,d\phi_1)\wedge(d\psi + \cos\theta_2\,d\phi_2)\,. \tag{3.32}$$

The 3 parameters $\mathrm{h}, \widetilde{\mathrm{h}}, k$ of (3.26) are thus mapped to the 3 parameters of the $T^{1,1}$ metric in [24–27].[14] The 2-loop RG equations (3.28) become in this case ($\mathrm{c}_{_G} = 2,\ \mathrm{c}_{_H} = 0$)

$$\frac{d}{d\tau}\mathrm{h} = 4\Big(1 - \frac{k}{\mathrm{h}}\Big)\Big[1 + \frac{4}{\mathrm{h}}\Big(1 - \frac{3\,k}{2\,\mathrm{h}}\Big)\Big]\,,\quad \frac{d}{d\tau}\widetilde{h} = 4\Big(1 - \frac{k}{\widetilde{\mathrm{h}}}\Big)\Big[1 + \frac{4}{\widetilde{\mathrm{h}}}\Big(1 - \frac{3\,k}{2\,\widetilde{\mathrm{h}}}\Big)\Big]\,. \tag{3.33}$$

As we shall discuss in section 4, the 2-loop RG stability of this $T^{1,1}$ model may be understood as consequence of the fact that the σ-model (3.30) is self-dual under T-duality in the ψ-direction.

4 Integrable $T^{1,q}$ model

Let us now introduce a new integrable σ-model with target space metric $T^{1,q}$ and a particular B-field, which is a one-parameter generalization of the $T^{1,1}$ model (3.30) of [8]. Its special conformal case will be the $SU_2 \times SU_2/U_1$ GMM model, now with *unequal* WZ levels [28–30] (with their ratio related to the parameter q).

Our central observation is that, starting with the $G\times G/H$ model (3.1) and considering the case when the subgroup H is *abelian*, the gauge invariance condition (3.4) of [8] is too

[13] Due to differing conventions, the B-field here is opposite in sign to that in [8]. This difference is not significant, and can be removed by a parity transformation.

[14] To recall, the $T^{1,1}$ metric (3.31) is an Einstein space if $\mathrm{h} = \widetilde{\mathrm{h}} = \frac{3}{2}k$. It then serves as a base of a Ricci flat 6d conifold with metric $dr^2 + r^2 ds^2_{T^{1,1}}$ if we formally set $k = \frac{1}{9}$ so that $R_{ij} = 4g_{ij}$. In general, the non-zero components of the Ricci tensor of the cone geometry $ds^2 = G_{mn}(X)dX^m dX^n = dr^2 + r^2 g_{ij}(x)dx^i dx^j$ (with $i = 1, \dots, d$) are $R_{ij}(G) = R_{ij}(g) - (d-1)g_{ij}$. Thus it vanishes if g_{ij} is an Einstein metric with a particular value of the scalar curvature $R(g) = d(d-1)$ (this condition is satisfied, e.g., for a unit-radius sphere S^d when G_{mn} is flat).

JHEP05(2021)076

restrictive. At the particular point $\rho_{12}=\rho_{21}=0$, there is also a second "branch" of gauge invariant models,[15]

$$\begin{aligned}&\mathcal{L}=-\frac{1}{2}\,\rho_{ij}\,\mathrm{Tr}[P_+^{(i)}P_-^{(j)}]-\frac{1}{2}\,r_{ij}\,\mathrm{Tr}[I_+^{(i)}I_-^{(j)}]+k_i\,\mathcal{L}_{\rm WZ}(g^{(i)})\,,\\&\rho_{12}=\rho_{21}=0\,,\qquad r_{ij}=\begin{pmatrix} r & q(-r-k_1)\\ q(-r+k_1) & q^2 r\end{pmatrix}\,,\qquad q^2\equiv -k_2/k_1\,,\end{aligned}\tag{4.1}$$

where k_1, k_2 are assumed to be of opposite sign. The action for (4.1) is invariant under the modified gauge transformation[16]

$$(g^{(1)},\,g^{(2)})\to(g^{(1)}\mathrm{w}^q,\,g^{(2)}\mathrm{w})\,,\qquad\qquad \mathrm{w}=\mathrm{w}(\xi^+,\xi^-)\in H\,.\tag{4.2}$$

Here w^q is the q-th power of the abelian group element w. In the case when the abelian H is compact then, to make w^q single-valued, one should assume that $q=\sqrt{\left|\frac{k_2}{k_1}\right|}$ is an integer.[17] At the value $q=1$ (i.e. $k_1=-k_2$), this model intersects with the gauge invariant $G\times G/H$ model (3.1), (3.4) considered above.

We claim that model (4.1) is integrable (admitting a Lax representation) if[18]

$$r=k\,.\tag{4.3}$$

In this case it becomes a generalization of (3.26) to the case of unequal levels $k,\widetilde{k}$,

$$\mathcal{L}=-\frac{1}{2}\,\mathrm{Tr}\left[\mathrm{h}\,P_+P_-+\widetilde{\mathrm{h}}\,\widetilde{P}_+\widetilde{P}_-\right]-\frac{1}{2}\,\mathrm{Tr}\left[k\,I_+I_-+\widetilde{k}\,\widetilde{I}_+\widetilde{I}_--2\sqrt{k\widetilde{k}}\,I_+\widetilde{I}_-\right]+k\mathcal{L}_{\rm WZ}(g)-\widetilde{k}\mathcal{L}_{\rm WZ}(\widetilde{g})\,,\tag{4.4}$$

where we have set (cf. (3.27))

$$\begin{aligned}&(g^{(1)},g^{(2)})\equiv(g,\widetilde{g})\,,\quad \widetilde{P}_\pm=P_\pm(\widetilde{g}),\quad \widetilde{I}_\pm=I_\pm(\widetilde{g}),\\&\mathrm{h}\equiv\rho_{11},\qquad \widetilde{\mathrm{h}}\equiv\rho_{22}\,,\qquad k\equiv k_1,\qquad \widetilde{k}\equiv -k_2\,,\quad q=\sqrt{\frac{\widetilde{k}}{k}}\,.\end{aligned}\tag{4.5}$$

The fact that the $\widetilde{k}=k$ limit (3.26) is an integrable theory provides a first check of the integrability of (4.4). Indeed, starting from the Lax connections [8] for (3.26) (with z

[15] Note that, in generic cases, WZ terms present a topological obstruction to gauging [43]. There are, however, special "anomaly-free" subgroups of the WZ term's global symmetry $G_L\times G_R$ that can be gauged [44], satisfying $\mathrm{Tr}_L[T_AT_B]-\mathrm{Tr}_R[T_AT_B]=0$. This condition is satisfied here by the gauge transformations (3.3) and (4.2) on both "branches" of theories, due to cancellation between the two copies of G in $G\times G/H$.

[16] The reason for the restriction of H to be abelian if $q\neq 1$ is that the variation of the Lagrangian (3.1) under (4.2) with $\mathrm{w}\in H$ will be proportional to $(q-1)\mathcal{L}_{\rm WZ}(\mathrm{w})$, which vanishes for abelian H for any q. We also need to assume $\rho_{12}=\rho_{21}$ to prevent mixing between $P^{(1)}$ and $P^{(2)}$ terms, which transform differently under w and w^q respectively.

[17] More generally, one could consider a "twisted" action of the abelian subgroup, $(g^{(1)},\,g^{(2)})\to(g^{(1)}\mathrm{w}^q,\,g^{(2)}\mathrm{w}^p)$ characterized by integers p,q satisfying $q^2/p^2=-k_2/k_1$. In the $SU_2\times SU_2/U_1$ example discussed below, that would lead to the $T^{p,q}$ model.

[18] The case $r=-k$ is also integrable since it is related to (4.3) by parity.

JHEP05(2021)076

as spectral parameter),[19]

$$L_+(z) = I_+ + z^{-1}P_+\,, \quad L_-(z) = \frac{1}{kz^2 - \mathrm{h}}\Big[(k-\mathrm{h})(I_- + zP_-) + k(z^2-1)\widetilde{I}_-\Big]\,, \tag{4.6}$$

$$\widetilde{L}_-(z) = \widetilde{I}_- + z\widetilde{P}_-\,, \qquad \widetilde{L}_+(z) = \frac{1}{kz^{-2} - \widetilde{\mathrm{h}}}\Big[(k-\widetilde{\mathrm{h}})(\widetilde{I}_+ + z^{-1}\widetilde{P}_+) + k(z^{-2}-1)I_+\Big]\,, \tag{4.7}$$

we have found the following Lax connections for (4.4) by replacing some factors of k by $\widetilde{k}$,

$$L_+(z) = I_+ + z^{-1}P_+\,, \quad L_-(z) = \frac{1}{kz^2 - \mathrm{h}}\left[(k-\mathrm{h})(I_- + zP_-) + \sqrt{k\widetilde{k}}(z^2-1)\widetilde{I}_-\right], \tag{4.8}$$

$$\widetilde{L}_-(z) = \widetilde{I}_- + z\widetilde{P}_-\,, \qquad \widetilde{L}_+(z) = \frac{1}{\widetilde{k}z^{-2} - \widetilde{\mathrm{h}}}\left[(\widetilde{k}-\widetilde{\mathrm{h}})(\widetilde{I}_+ + z^{-1}\widetilde{P}_+) + \sqrt{k\widetilde{k}}(z^{-2}-1)I_+\right]. \tag{4.9}$$

Assuming the simplest case $G = SU_2$, $H = U_1$ (see footnote 29), using the same coordinate parametrization of this $SU_2 \times SU_2/U_1$ model as in (3.29), and fixing again the $H = U_1$ gauge as $\widetilde{\psi} = 0$, we find the following generalization of (3.30)

$$\begin{aligned}\mathcal{L} = (G_{mn} + B_{mn})\partial_+ x^m \partial_- x^n = \frac{1}{4}k\Big[&\partial_+\psi\partial_-\psi + \cos^2\theta_1\,\partial_+\phi_1\partial_-\phi_1 + q^2\cos^2\theta_2\,\partial_+\phi_2\partial_-\phi_2 \\ &+ 2\cos\theta_1\,\partial_+\phi_1\partial_-\psi + 2q\cos\theta_2\,\partial_+\psi\partial_-\phi_2 + 2q\cos\theta_1\cos\theta_2\,\partial_+\phi_1\partial_-\phi_2\Big] \\ &+ \frac{1}{4}\mathrm{h}\Big[\partial_+\theta_1\partial_-\theta_1 + \sin^2\theta_1\,\partial_+\phi_1\partial_-\phi_1\Big] + \frac{1}{4}\widetilde{\mathrm{h}}\Big[\partial_+\theta_2\partial_-\theta_2 + \sin^2\theta_2\,\partial_+\phi_2\partial_-\phi_2\Big]\,,\quad q = \sqrt{\frac{\widetilde{k}}{k}}\,.\end{aligned} \tag{4.10}$$

The resulting target space metric is that of the $T^{1,q}$ space [24–27] and the B-field is a natural generalization of the one in (3.32),

$$\begin{aligned} ds^2_{T^{1,q}} = G_{mn}dx^m dx^n &= \frac{1}{4}k\,(d\psi + \cos\theta_1\,d\phi_1 + q\cos\theta_2\,d\phi_2)^2 \\ &\quad + \frac{1}{4}\mathrm{h}\,(d\theta_1^2 + \sin^2\theta_1\,d\phi_1^2) + \frac{1}{4}\widetilde{\mathrm{h}}\,(d\theta_2^2 + \sin^2\theta_2\,d\phi_2^2)\,, \end{aligned} \tag{4.11}$$

$$B = \frac{1}{2}B_{mn}dx^m \wedge dx^n = \frac{1}{4}k\,(d\psi + \cos\theta_1\,d\phi_1) \wedge (d\psi + q\cos\theta_2\,d\phi_2)\,. \tag{4.12}$$

Like the $q = 1$ case [8] in (3.30), the presence of the B-field is crucial here for integrability (the $T^{1,q}$ σ-model without B-field is not integrable [47, 48]). The coordinate form of the

[19] These Lax connections were obtained in [8] from the affine Gaudin Lax connection of the general integrable $G \times G/H$ model (3.1), (3.4), (3.9) by taking the limit $r = k$, $\rho_{12} = \rho_{21} = 0$. It was found that certain components of the Lax connection degenerate to zero and thus the flatness condition of the resulting connection $L_\pm$ does not imply some of the equations of motion. However, one can consider a generalized limiting procedure by infinitely rescaling the spectral parameter while taking this limit, thus obtaining a second connection $\widetilde{L}_\pm$ that "misses" a different subset of equations of motion. The flatness conditions of the two Lax connections together encode the full set of the equations of motion. The fact of having two separate Lax connections may seem unusual but should be sufficient for the integrability applications: for example, each Lax connection will lead to its own family of conserved charges. Note also that for $k = 0$ the two Lax connections (4.6), (4.7) become the familiar ones of the two decoupled G/H σ-models so the fact of having two connections may not be totally surprising (we thank B. Hoare for this comment).

JHEP05(2021)076

Lax connections (4.8), (4.9) is (T_A are the SU_2 generators in (3.29))

$$
\begin{aligned}
L'_+(z) &= \cos\theta_1\,\partial_+\phi_1\,T_3 + z^{-1}\,(\partial_+\theta_1\,T_2 + \sin\theta_1\,\partial_+\phi_1\,T_1)\ , \\
L'_-(z) &= \frac{1}{kz^2-\mathrm{h}}\Big[(k-\mathrm{h})\,(\cos\theta_1\,\partial_-\phi_1\,T_3 + z(\partial_-\theta_1\,T_2 + \sin\theta_1\,\partial_-\phi_1\,T_1)) \\
&\qquad -(z^2-1)\,\Big(\sqrt{k\widetilde{k}}\,\cos\theta_2\,\partial_-\phi_2 + k\,\partial_-\psi\Big)\Big]\,, \qquad (4.13) \\
\widetilde{L}_+(z) &= \frac{1}{\widetilde{k}z^{-2}-\widetilde{h}}\Big[(\widetilde{k}-\widetilde{\mathrm{h}})\,\Big(-\cos\theta_2\,\partial_-\phi_2\,T_3 + z^{-1}(-\partial_+\theta_2\,T_2 + \sin\theta_3\,\partial_+\phi_3\,T_1)\Big) \\
&\qquad +\sqrt{k\widetilde{k}}(z^{-2}-1)\,(\cos\theta_1\,\partial_+\phi_1 + \partial_+\psi)\Big]\,, \\
\widetilde{L}_-(z) &= -\cos\theta_2\,\partial_-\phi_2\,T_3 + z\,(-\partial_-\theta_2\,T_2 + \sin\theta_2\,\partial_-\phi_2\,T_1)\,. \qquad (4.14)
\end{aligned}
$$

To simplify the expressions we followed [8] here in replacing $L_\pm$ by its gauge transformed version $L'_\pm = \mathrm{w}^{-1}L_\pm\mathrm{w} + \mathrm{w}^{-1}\partial_\pm\mathrm{w}$, with $\mathrm{w} = \exp(-\psi T_3)$.

At the special point $\mathrm{h} = k,\ \widetilde{\mathrm{h}} = \widetilde{k} = q^2k$, the model (4.10) becomes the $SU_2\times SU_2/U_1$ case of the conformal GMM model with levels $k_1 = k,\ k_2 = -\widetilde{k}$. It was pointed out in [30] that the $SU_2\times SU_2/U_1$ GMM model corresponds to the $T^{1,q}$ metric and a particular B-field, and its 2-loop conformality was explicitly checked (see also [45]). The general GMM model has a current algebra symmetry [28, 29] and is also integrable in the Lax connection sense [46]. What we have shown above is that it admits an integrable extension (4.10) away from the conformal point $\mathrm{h} = k,\ \widetilde{\mathrm{h}} = \widetilde{k}$.

4.1 Stability under the 2-loop RG flow

Let us now show that the integrable $T^{1,q}$ model (4.10) is stable under the 2-loop RG flow.

The general gauge invariant model (4.1) (with the $r = k$ condition (4.3) relaxed) must be stable under the RG with only $(r, \mathrm{h}, \widetilde{\mathrm{h}})$ as running couplings.[20] This is due to its H gauge invariance and global $G_L\times G_L$ symmetry prohibiting any new counterterm structures. We shall see that the $T^{1,q}$ model, obtained by fixing $r = k$, is a "fixed line" of its RG flow.

[20] At the point $q = 1$ or $k_1 = -k_2$ where the two "branches" of gauge invariant theories (4.1), (3.4) intersect, one may worry that the couplings ρ_{12}, ρ_{21} may also run, since this is no longer prevented by the gauge invariance. However, in the abelian H case this is forbidden by an extra global "center" symmetry [8, 24–27], $(g, \widetilde{g}) \to (g\,z, \widetilde{g})\,,\ \ z\in \mathrm{Z}(H)\subset H$ preserving the non-mixing of the coset parts of the current P and $\widetilde{P}$ in (3.26). Note that this symmetry alone would not be sufficient to explain the stability of the $T^{1,q}$ model since it does not prevent r from running.

JHEP05(2021)076

Relaxing $r = k$ has the effect of replacing $k \to r$ in the metric, with k still appearing in the B-field (cf. (4.11), (4.12))[21]

$$ds^2 = \frac{1}{4} r \, (d\psi + \cos\theta_1 \, d\phi_1 + q \cos\theta_2 \, d\phi_2)^2 + \frac{1}{4} \mathrm{h} \, (d\theta_1^2 + \sin^2\theta_1 \, d\phi_1^2) + \frac{1}{4} \widetilde{\mathrm{h}} \, (d\theta_2^2 + \sin^2\theta_2 \, d\phi_2^2) \ ,$$

$$B = \frac{1}{4} k \, (d\psi + \cos\theta_1 \, d\phi_1) \wedge (d\psi + q \cos\theta_2 \, d\phi_2) \, , \qquad q = \sqrt{\frac{\widetilde{k}}{k}} \, . \tag{4.15}$$

The corresponding 2-loop β-functions in the GB scheme following from (1.4) are ($k, \widetilde{k}$ do not run)[22]

$$\frac{d}{d\tau} r = 2(r^2 - k^2) \left[\frac{1}{\mathrm{h}^2} + \frac{\widetilde{k}}{k} \frac{1}{\widetilde{\mathrm{h}}^2} + \frac{r^2 - 3k^2}{r} \left(\frac{1}{\mathrm{h}^4} + \frac{\widetilde{k}^2}{k^2} \frac{1}{\widetilde{\mathrm{h}}^4} \right) \right] , \tag{4.16}$$

$$\frac{d}{d\tau} \mathrm{h} = 4 \left[1 - \frac{r}{2\mathrm{h}} \left(1 + \frac{k^2}{r^2} \right) \right] + \frac{2}{\mathrm{h}^3 r^2} \left[8\mathrm{h}^2 r^2 - 8\mathrm{h} k^2 r - 12 \mathrm{h} r^3 + 4 r^2 k^2 + 3k^4 + 5 r^4 \right.$$
$$\left. + \frac{\widetilde{k}}{k} \frac{\mathrm{h}^2}{\widetilde{\mathrm{h}}^2} (r^2 - k^2)(3r^2 - k^2) \right] , \tag{4.17}$$

$$\frac{d}{d\tau} \widetilde{\mathrm{h}} = 4 \left[1 - \frac{\widetilde{k}}{k} \frac{r}{2\widetilde{\mathrm{h}}} \left(1 + \frac{k^2}{r^2} \right) \right] + \frac{2}{\widetilde{\mathrm{h}}^3 r^2} \left[8\widetilde{\mathrm{h}}^2 r^2 - 8 \widetilde{\mathrm{h}} \widetilde{k} k r - 12 \frac{\widetilde{k}}{k} \widetilde{\mathrm{h}} r^3 + 4 r^2 \widetilde{k}^2 + 3 \widetilde{k}^2 k^2 + 5 \frac{\widetilde{k}^2}{k^2} r^4 \right.$$
$$\left. + \frac{\widetilde{k}}{k} \frac{\widetilde{\mathrm{h}}^2}{\mathrm{h}^2} (r^2 - k^2)(3r^2 - k^2) \right] . \tag{4.18}$$

Thus $r = \pm k$ are fixed lines of (4.16), both at 1-loop and 2-loop order. The couplings $(r, \mathrm{h}, \widetilde{\mathrm{h}})$ grow linearly with $\tau \to \infty$ in the UV (reflecting asymptotic freedom), while they decrease to the GMM fixed point $(r, \mathrm{h}, \widetilde{\mathrm{h}}) = (k, k, \widetilde{k})$ in the IR.[23]

Specialising to the fixed line $r = k$ of (4.16), the expressions (4.17), (4.18) simplify, giving the 2-loop β-functions of the integrable $T^{1,q}$ model (4.4), (4.10),

$$\frac{d}{d\tau} \mathrm{h} = 4 \left(1 - \frac{k}{\mathrm{h}} \right) \left[1 + \frac{4}{\mathrm{h}} \left(1 - \frac{3\,k}{2\,\mathrm{h}} \right) \right] , \quad \frac{d}{d\tau} \widetilde{h} = 4 \left(1 - \frac{\widetilde{k}}{\widetilde{\mathrm{h}}} \right) \left[1 + \frac{4}{\widetilde{\mathrm{h}}} \left(1 - \frac{3\,\widetilde{k}}{2\,\widetilde{\mathrm{h}}} \right) \right] . \tag{4.19}$$

[21] Rescaling $r \to r'k$ and $\psi \to \frac{1}{\sqrt{k}} \psi'$ this background can be put into the form symmetric under $k \leftrightarrow \widetilde{k}$, $\mathrm{h} \leftrightarrow \widetilde{\mathrm{h}}$:

$$ds^2 = \frac{1}{4} r' \left(d\psi' + \sqrt{k} \cos\theta_1 \, d\phi_1 + \sqrt{\widetilde{k}} \cos\theta_2 \, d\phi_2 \right)^2 + \frac{1}{4} \mathrm{h} \, (d\theta_1^2 + \sin^2\theta_1 \, d\phi_1^2) + \frac{1}{4} \widetilde{\mathrm{h}} \, (d\theta_2^2 + \sin^2\theta_2 \, d\phi_2^2) \, ,$$
$$B = \frac{1}{4} \left(d\psi' + \sqrt{k} \cos\theta_1 \, d\phi_1 \right) \wedge \left(d\psi' + \sqrt{\widetilde{k}} \cos\theta_2 \, d\phi_2 \right) .$$

[22] Note that for $k = \widetilde{k}$ (i.e. $q = 1$) this system of RG equations is obviously symmetric under interchanging h and $\widetilde{\mathrm{h}}$. Note also that setting $k = \widetilde{k} \to 0$ with $\frac{k}{\widetilde{k}} = 1$ and $h = \mathrm{h}$ the 1-loop β-functions become $\frac{d}{d\tau} r = 4 \frac{r^2}{h^2}$, $\frac{d}{d\tau} h = 4 - 2 \frac{r}{h}$ so that $\frac{d}{d\tau} \frac{r}{h} = 6 \frac{r}{h^2} \left(\frac{r}{h} - \frac{2}{3} \right)$. The point $\frac{r}{h} = \frac{2}{3}$ corresponds to the case when the $T^{1,1}$ metric is an Einstein space (cf. footnote 14), i.e. $R_{mn} = \Lambda G_{mn}$ with $\Lambda = \frac{16}{9r}$.

[23] As was argued in [28, 29], the GMM model on $G \times G'/H$ is an exact CFT, assuming at least one of the cosets G/H or G'/H is a symmetric space (as is indeed the case for the $SU_2 \times SU_2/U_1$ model).

JHEP05(2021)076

These are a natural generalization of the β-functions for the $T^{1,1}$ model in (3.33) to the case of $\widetilde{k} \neq k$. Like in (3.33), the RG evolution of h and $\widetilde{\mathrm{h}}$ happens to be decoupled (while this is not the case for $r \neq k$ in (4.17), (4.18)).

Let us note that, in addition to the $r = k$ case of the $T^{1,q}$ model, the σ-model corresponding to (4.15) admits *another* integrable limit, $\widetilde{k} = 0$. In this case it factorizes into a squashed S^3 with WZ term and a round S^2.[24] Then the β-functions (4.16) and (4.17) both become the same as those of this squashed S^3 model in [23] (for 1-loop β-functions see [49–51]).[25] Taking further limits, the β-functions (4.16), (4.17), (4.18) agree with other previously known expressions:

(i) Setting $\widetilde{k} = 0$ and $r = \mathrm{h}$, we get from (4.15) the direct sum of the PCM_k (round S^3 with a WZ term) and the S^2 σ-model. In this case (4.16), (4.17) are indeed equivalent to the β-function of PCM_k, i.e. (1.5) with $\mathrm{c}_{_G} = 2$.

(ii) Setting $\widetilde{k} = 0$ (i.e. $q = 0$) and then $k = 0$, we instead get the direct sum of a squashed S^3 (with no WZ term) and a round S^2. The β-functions for r and h agree with those of the "squashed" PCM in [5] (with $G = SU_2$ and the "squashing" parameter $\varepsilon = \frac{r}{\mathrm{h}}$):

$$\frac{d}{d\tau} r = 2r^2 \mathrm{h}^{-2} + r^3 \mathrm{h}^{-4} , \qquad \frac{d}{d\tau} \mathrm{h} = 4 \left(1 - \frac{1}{2} r \mathrm{h}^{-1}\right) + 2\mathrm{h}^{-3} \left(8\mathrm{h}^2 - 12\mathrm{h}r + 5r^2\right) .$$

4.2 Covariance under T-duality

One can argue that the RG stability of the integrable $T^{1,q}$ model (4.10), i.e. the presence of the fixed line $r = k$ of (4.16), is related to its property of being self-dual under T-duality in the isometric ψ-direction. To see this, let us write the Lagrangian (4.10) in the following form[26]

$$\mathcal{L} = \frac{1}{4} k \left[(\partial_+ \psi + U_+)(\partial_- \psi + V_-) - \frac{1}{2} U_+ V_- \right] + \widehat{\mathcal{L}} , \tag{4.20}$$

$$U_\pm \equiv 2 \cos\theta_1 \, \partial_\pm \phi_1 , \qquad V_\pm \equiv 2q \cos\theta_2 \, \partial_\pm \phi_2 , \tag{4.21}$$

$$\begin{aligned} \widehat{\mathcal{L}} \equiv \frac{1}{4} \mathrm{h} \Big[\partial_+\theta_1 \partial_- \theta_1 + \Big(\sin^2\theta_1 + \frac{k}{\mathrm{h}} \cos^2\theta_1 \Big) \partial_+\phi_1 \partial_- \phi_1 \Big] \\ + \frac{1}{4} \widetilde{\mathrm{h}} \Big[\partial_+\theta_2 \partial_- \theta_2 + \Big(\sin^2\theta_2 + \frac{kq^2}{\widetilde{\mathrm{h}}} \cos^2\theta_2 \Big) \partial_+\phi_2 \partial_- \phi_2 \Big] . \end{aligned} \tag{4.22}$$

Starting from the interpolating Lagrangian (obtained by $\partial_\pm \psi \to A_\pm$ and adding the condition $\partial_+ A_- - \partial_- A_+ = 0$ with a Lagrange multiplier $\bar{\psi}$)

$$\mathcal{L}_{\text{int}} = \frac{1}{4} k \left[(A_+ + U_+)(A_- + V_-) - \frac{1}{2} U_+ V_- - \bar{\psi}(\partial_+ A_- - \partial_- A_+) \right] + \widehat{\mathcal{L}} , \tag{4.23}$$

[24] The β-function (4.18) for the coefficient $\widetilde{\mathrm{h}}$ then matches that of the S^2 σ-model, i.e. a special case of the G/H symmetric space β-function [5] (here $G/H = \mathrm{SO}(3)/\mathrm{SO}(2)$, i.e. $\mathrm{c}_{_G} = 2$, $\mathrm{c}_{_H} = 0$):

$$\frac{d}{d\tau} \widetilde{h} = 2\mathrm{c}_{_G} + 4\mathrm{c}_{_G}(\mathrm{c}_{_G} - \mathrm{c}_{_H}) \widetilde{h}^{-1} = 4 + 16 \widetilde{h}^{-1} .$$

[25] The relation to the notation used in [23] is $\eta = k\mathrm{h}^{-1}$, $\lambda^2 = 2\pi \mathrm{h}^{-1}$, $\kappa = 1 - r\mathrm{h}^{-1}$.

[26] Note that $\widehat{\mathcal{L}}$ becomes simply quadratic in the fields at the GMM point $\mathrm{h} = k$, $\widetilde{\mathrm{h}} = \widetilde{k} = q^2 k$.

JHEP05(2021)076

and integrating out $A_\pm$, we obtain the following T-dual Lagrangian

$$\bar{\mathcal{L}} = \frac{1}{4}k\left[(\partial_+\bar{\psi}+U_+)(\partial_-\bar{\psi}-V_-)+\frac{1}{2}U_+V_-\right]+\widehat{\mathcal{L}}\,. \tag{4.24}$$

This is the same as the original theory (4.20), with $\psi \to \bar{\psi}$ and a coordinate redefinition $\phi_2 \to -\phi_2$ (under which $V_- \to -V_-$).

To appreciate the special structure of (4.20), let us relax the condition $r = k$ and go back to the general model (4.1) corresponding to the background (4.15). Using again the notation (4.21), we find the following generalization of (4.20)

$$\mathcal{L} = \frac{r+k}{8}\left[(\partial_+\psi+U_+)(\partial_-\psi+V_-)-\frac{1}{2}U_+V_-\right]+\frac{r-k}{8}\left[(\partial_-\psi+U_-)(\partial_+\psi+V_+)-\frac{1}{2}U_-V_+\right]+\widehat{\mathcal{L}}\,. \tag{4.25}$$

Applying the T-duality $\psi \to \bar{\psi}$ to (4.25) we get, instead of (4.24),

$$\bar{\mathcal{L}} = \frac{1}{4}r\left[\left(\partial_+\bar{\psi}+\frac{r+k}{2r}U_++\frac{r-k}{2r}V_+\right)\left(\partial_-\bar{\psi}-\frac{r-k}{2r}U_--\frac{r+k}{2r}V_-\right)+\frac{r+k}{4r}U_+V_-+\frac{r-k}{4r}U_-V_+\right]+\widehat{\mathcal{L}}\,. \tag{4.26}$$

For general values of r and k, (4.26) is different from (4.25); the only self-dual theory where (4.25) and (4.26) coincide is the $T^{1,q}$ model (4.20) corresponding to $r = k$ (or its parity-conjugate $r = -k$).

By the standard path integral argument, the T-dual models (4.25), (4.26) should be quantum-equivalent.[27] Since the model (4.25) is stable under the RG due to its symmetries, with the 3 running couplings $r, \mathrm{h}, \widetilde{\mathrm{h}}$, its T-dual (4.26) must also be stable. Given that the self-dual points $r = \pm k$ are part of both RG-stable families (4.25) and (4.26), then they must also remain in both families after the renormalization. Hence $r = \pm k$ must be fixed lines of the RG flow. This was indeed confirmed above by the explicit computation of the β-functions leading to (4.16).

5 Concluding remarks

In this paper we discussed some new instances of a close connection between the conditions of integrability and a consistent restriction of the RG flow to a subspace of couplings.

We have found the 2-loop β-functions of the 6-parameter $G \times G$ model (2.2) and have shown that its integrability condition (2.3) is automatically preserved by the RG flow. In [14], the 1-loop β-functions for this integrable model were written in a universal form in terms of the twist function, revealing a hidden simplicity. It would be interesting to see if

[27] In general, the T-duality transformation rules may be subject to quantum α' corrections [52–54] that may be attributed to extra finite counterterms resulting from integration over the auxiliary gauge field $A_\pm$ (see, e.g., [4]). If the kinetic term of the isometric coordinate is non-trivial, i.e. the term quadratic in $A_\pm$ is $A_+M(x)A_-$, then the leading quantum correction to the effective Lagrangian is represented by the term $\Delta\mathcal{L} \sim \alpha'\,\partial_+ \log M\,\partial_- \log M$ (as well as a shift of the dilaton [55, 56]). In the case of (4.25) we have $M = 1$ and thus this correction is absent.

JHEP05(2021)076

the complicated expressions we have found for the 2-loop β-functions (see appendix B.1) simplify on the "integrable surface" once expressed in terms of the twist function.[28]

We also studied the 6-parameter gauged $G\times G/H$ model (3.1), (3.4), which is integrable under the conditions (3.9). The latter were found to be stable under the 1-loop RG flow but, in general, require a certain deformation (i.e. the addition of finite counterterms) at the 2-loop level to preserve integrability. It is possible that there exists an extended target space formulation of the $G \times G/H$ model in which no additional 2-loop counterterms are needed (as was demonstrated for the λ-model examples in [5]).

We have found that there are still some special cases in which integrable $G \times G/H$ models are automatically stable under the 2-loop RG flow. One simple example is the $T^{1,1}$ model of [8]. We also constructed a new class of integrable $G \times G/H$ models (4.4) in the case when the subgroup H is abelian (see (4.1), (4.3), (4.4)). For $G = SU_2$ and $H = U_1$, this led to an integrable $T^{1,q}$ model generalizing the $T^{1,1}$ model, which we also found to be stable under the 2-loop RG flow for any value of the parameter q. This model may be interpreted as an integrable deformation of the conformal GMM model with unequal levels [28, 29]. Since the GMM model admits a $G \times G'/H$ generalization (with $G \neq G'$), this raises the question of whether there is a larger class of integrable $G \times G'/H$ models that flow to such conformal theories.[29] Another open question is whether the integrable $T^{1,q}$ model admits a description in terms of affine Gaudin models (like the $T^{1,1}$ case) or if it is outside of that formalism.

Given a σ-model with running couplings, it can be promoted to a conformal theory (and thus embedded into string theory) by adding two light-cone directions u and v, replacing the RG "time" in the coupling constants by u and adding a dilaton linear in v [59, 60]. Fixing the light-cone gauge on u, one then gets back the original σ-model with "local" couplings depending on 2d time according to the RG equations. It would be interesting to study whether the connection between the classical Lax integrability of such local-coupling models and the RG evolution of couplings observed in [61] applies also to the models discussed in this paper.

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[28] One may try to follow the method of [14] at the 2-loop order, computing the Riemann tensor and then the 2-loop β-function in terms of the twist function. It would also be interesting to investigate the connection to the "doubled" approach of [57, 58] which studied the model (2.2), (2.3) with additional integrable η- or λ-deformation parameters turned on.

[29] One obvious possibility is to consider some analytic continuations, e.g., take G' to be a different real form of the complexification of G (assuming the resulting σ-model couplings G, B remain real). For example, the counterpart of the $SU_2 \times SU_2/U_1$ model would be $SL_2(\mathbb{R}) \times SU_2/U_1$.

JHEP05(2021)076

A Deriving the integrability conditions for the G^N model

It was shown in [6, 7] that the coupled model (1.2) is integrable for particular choices of the couplings (ρ_{ij}, k_i) corresponding to realisations of the affine Gaudin models. Here we shall try to demonstrate the converse statement: these affine Gaudin models are the *only* integrable cases of the coupled models (1.2).

We will assume a natural ansatz (2.1) for the Lax connection, valued in Lie(G) (here we explicitly indicate the summation over $i = 1, \ldots, N$)[30]

$$L_+ = \sum_i \alpha_i(z)\, J_+^{(i)}\,, \qquad\qquad L_- = \sum_i \beta_i(z)\, J_-^{(i)}\,, \tag{A.1}$$

where z is the spectral parameter. The curvature of this Lax connection takes the form

$$F_{+-}(L) = \sum_i \Big(\beta_i(1-\alpha_i)\ \partial_+ J_-^{(i)} - \alpha_i(1-\beta_i)\ \partial_- J_+^{(i)}\Big) + \sum_{i\neq j} \alpha_i\beta_j\ [J_+^{(i)}, J_-^{(j)}]\,. \tag{A.2}$$

The equations of motion of the model (1.2) are (for G^N with arbitrary N)

$$E_i \equiv \sum_j \Big((\rho_{ij} - \delta_{ij}k_j)\ \partial_+ J_-^{(j)} + (\rho_{ij} + \delta_{ij}k_j)\ \partial_- J_+^{(j)} + \rho_{ij}\ [J_+^{(i)}, J_-^{(j)}] + \rho_{ji}\ [J_-^{(i)}, J_+^{(j)}]\Big) = 0\,. \tag{A.3}$$

We note that (A.2) and (A.3) are the *unique* ways to write these expressions without any terms of the form $[J_+^{(i)}, J_-^{(i)}]$, which have been eliminated using the identity $F_{+-}(J^{(i)}) = 0$.

If the model (1.2) is integrable then,[31] for some $v^i(z)$, we have

$$F_{+-}(L) = \sum_i v^i(z)\, E_i\,, \tag{A.4}$$

which implies that

$$\begin{aligned} \beta_j(1-\alpha_j) &= \sum_i v^i(\rho_{ij} - \delta_{ij}k_j)\,, \qquad & \alpha_j(1-\beta_j) &= \sum_i v^i(-\rho_{ji} - \delta_{ij}k_j)\,, \\ \alpha_i\beta_j &= (v^i - v^j)\rho_{ij}\,, \quad i\neq j \ \text{ (no summation)}\,. \end{aligned} \tag{A.5}$$

This is a system of $N+N+(N^2-N) = N^2+N$ equations. Fixing the freedom to redefine the spectral parameter by setting $v^1 = z$, there are $N+N+(N-1) = 3N-1$ "artificial" variables, $\alpha_i, \beta_i, v^{i\neq 1}$. After solving for these, there are $(N^2+N)-(3N-1) = N^2-2N+1$ remaining equations to be solved for the N^2+N variables ρ_{ij}, k_i. After solving all the

[30] While (A.1) is the natural ansatz for the Lax connection arising from affine Gaudin models, it does degenerate at certain points in coupling space. For example, taking ρ_{ij} to be diagonal (i.e. decoupled PCM$_k$ models), one instead requires a Lax connection valued in Lie$(G)^N$. Thus it would also be interesting to consider other ansatze for the Lax connection.

[31] Here we are assuming integrability and deriving necessary conditions on the couplings. Thus we do not need to worry about whether the $v^i(z)$ in (A.4) are independent functions (which would be relevant for the converse question).

JHEP05(2021)076

equations, this leaves $(N^2+N)-(N^2-2N+1)=3N-1$ free parameters for the integrable theory (including the WZ levels, which may be continuous for non-compact groups).[32]

Thus the space of integrable models is $(3N-1)$-dimensional, which coincides with the number of free parameters following from the affine Gaudin construction (see [14] and refs. therein).

Specializing to the $N=2$ case of $G\times G$, this counting suggests a 5-dimensional space of integrable models. Then the 6 free parameters (s,t,u,b,k_1,k_2) in (2.2) should be subject to only one relation to ensure integrability. Solving the equations (A.5) in this case, one indeed obtains the condition (2.3) originally found from the affine Gaudin construction.

To summarize, for general N, the space of integrable models has the same dimension as the space of affine Gaudin models. It remains to understand if there may still be extra branches of integrable theories not corresponding to the affine Gaudin models (cf. the $G\times G/H$ models, where this seems to be the case for abelian H, see section 4). For the $N=2$ case of $G\times G$ models, we found exact matching between the space of integrable models (A.5) and the space of affine Gaudin models satisfying the condition (2.3).

B Explicit form of the 2-loop β-functions

Here we shall provide the explicit formulae for the 2-loop β-functions of the general $G\times G$ and $G\times G/H$ models that were used in the main text.[33] We will also briefly explain how they were derived.

B.1 $G\times G$ model

For the $G\times G$ model (2.2), let us use the notation

$$\rho_{ij}=\mathrm{h}_{(ij)}+b_{[ij]}=\begin{pmatrix}s & t\\ t & u\end{pmatrix}+\begin{pmatrix}0 & b\\ -b & 0\end{pmatrix}. \tag{B.1}$$

Let the 2×2 matrix n_{ij} be the "square root" of $\mathrm{h}_{ij}=\mathrm{h}_{(ij)}$, and let m_{ij} be its inverse,

$$n_{ik}\,n_{jk}=\mathrm{h}_{ij}\,,\qquad m_{ik}\,n_{kj}=\delta_{ij}\,,\qquad n_{ij}=n_{(ij)},\quad m_{ij}=m_{(ij)}\,. \tag{B.2}$$

The target space metric of the σ-model (2.2) is "diagonalised" by the vielbein 1-form[34]

$$\begin{aligned}&E^{Ai}=n_{ik}\,J^{(k)A},\qquad A=1,\dots,\dim G\,,\quad i=1,2\,,\\ &J^{(k)}\equiv T_A J^{(k)A}=(g^{(k)})^{-1}dg^{(k)}\,.\end{aligned} \tag{B.3}$$

[32] One might worry that the integrability constraints on the couplings ρ_{ij},k_i resulting from (A.4) might depend on the spectral parameter $v^1=z$. However, this will not happen because there is a rescaling ambiguity $E_i\to c_iE_i$ in the definition of the equations of motion (A.3). One may thus rescale E_1 to effectively set $v^1=z=1$ in (A.4). Since the constraints on the couplings from (A.4) must be invariant under such rescalings, then they must not depend on z.

[33] The formulae derived in this appendix are also available in the Mathematica file attached to this paper as supplementary material.

[34] We use the generators T_A satisfying $[T_A,T_B]=if^C{}_{AB}T_C$ and we define $f_{ABC}=\mathrm{Tr}[T_CT_D]f^D{}_{AB}$. For simple groups G, the structure constants satisfy $f^A{}_{BC}f^B{}_{AD}=-2\mathrm{c}_G\mathrm{Tr}[T_CT_D]$ and $f^D{}_{EA}f^E{}_{GB}f^G{}_{DC}=\mathrm{c}_Gf_{ABC}$, where c_G is the dual Coxeter number of G. Note that $if^A{}_{BC}$ in our present conventions is equivalent to $f^A{}_{BC}$ in the conventions of [5].

JHEP05(2021)076

Then the coefficients of the metric $ds^2 = G_{Ai,Bj}E^{Ai}E^{Bj}$ and the 3-form $H = dB = \frac{1}{6}H_{Ai,Bj,Ck}E^{Ai}\wedge E^{Bj}\wedge E^{Ck}$ are given by[35]

$$G_{Ai,Bj} = -\frac{1}{2}\,\mathrm{Tr}[T_A T_B]\,\delta_{ij}\,, \tag{B.4}$$

$$H_{Ai,Bj,Ck} = \frac{i}{2}f_{ABC}\,[k_l\,m_{il}\,m_{jl}\,m_{kl} + b_{lp}(m_{ip}\,m_{jl}\,m_{kl} + m_{il}\,m_{jp}\,m_{kl} + m_{il}\,m_{jl}\,m_{kp})]\,. \tag{B.5}$$

From Cartan's structure equation $dE^{Ai}+\widehat{\omega}^{Ai}{}_{Bj}\wedge E^{Bj}=T^{Ai}$ with torsion $T^{Ai}=\frac{1}{2}H^{Ai}{}_{Bj,Ck}E^{Bj}\wedge E^{Ck}$, we obtain the torsionful spin connection

$$\widehat{\omega}^{Ai}{}_{Bj} = \frac{i}{2}f^{A}{}_{BC}M_{ijk}\,, \tag{B.6}$$

$$\begin{aligned} M_{ijk} \equiv\; & m_{il}m_{jl}n_{kl} - m_{il}n_{jl}m_{kl} - n_{il}m_{jl}m_{kl} \\ & + k_l\,m_{il}\,m_{jl}\,m_{kl} + b_{lp}(m_{ip}\,m_{jl}\,m_{kl} + m_{il}\,m_{jp}\,m_{kl} + m_{il}\,m_{jl}\,m_{kp})\,. \end{aligned} \tag{B.7}$$

The torsionful Riemann curvature $\widehat{R}^{Ai}{}_{Bj} \equiv \frac{1}{2}\widehat{R}^{Ai}{}_{Bj,Ck,Dl}E^{Ck}\wedge E^{Dl} = d\widehat{\omega}^{Ai}{}_{Bj} + \widehat{\omega}^{Ai}{}_{Ck}\wedge \widehat{\omega}^{Ck}{}_{Bj}$ is then found in terms of M_{ijk} to be

$$\begin{aligned} \widehat{R}^{Ai}{}_{Bj,Ck,Dl} = \frac{1}{4}\Big[& 2f^{A}{}_{BE}f^{E}{}_{CD}M_{ijp}n_{pq}m_{kq}m_{lq} \\ & + f^{A}{}_{CE}f^{E}{}_{BD}M_{ipk}M_{pjl} - f^{A}{}_{DE}f^{E}{}_{BC}M_{ipl}M_{pjk}\Big]\,. \end{aligned} \tag{B.8}$$

It is then straightforward to substitute (B.5), (B.7), (B.8) into the 2-loop β-functions in the GB scheme (1.4), obtaining explicit formulae for the RG equations $\frac{d}{d\tau}\rho_{ij} = \beta_{ij}(n_{11}, n_{12}, n_{22}, b, k_1, k_2)$ depending on the components of n_{ij}. Using a computer symbolic algebra package (e.g. Mathematica) it is easy to rewrite these expressions in terms of the components s, t, u of the "square" coupling $\mathrm{h}_{ij} = n_{ik}n_{jk}$ in (B.1), with all the square roots cancelling out as the Riemann tensor and the H-field must clearly be rational functions of h_{ij}. We thus obtain the β-functions in the form given in (2.5),

$$\begin{aligned} &\frac{d}{d\tau}\rho_{ij} = \alpha'\,\beta^{(1)}_{ij} + \alpha'^2\,\beta^{(2)}_{ij} + \dots\,, \\ &\beta^{(1)}_{ij} = \mathrm{c}_G(su-t^2)^{-2}\,F^{(4)}_{ij}(s,t,u,b,k_1,k_2)\,, \qquad \beta^{(2)}_{ij} = \mathrm{c}_G^2(su-t^2)^{-5}\,F^{(9)}_{ij}(s,t,u,b,k_1,k_2)\,, \end{aligned} \tag{B.9}$$

where the explicit form of the homogeneous polynomials $F^{(4)}_{ij}$ and $F^{(9)}_{ij}$ is:

$$F^{(4)}{}_{11} = -b^2\,s^2 - 4\,b^2\,s\,t - 2\,b^2\,t^2 + s^2\,t^2 + 2\,s\,t^3 + 2\,t^4 - 2\,b^2\,s\,u - 2\,s\,t^2\,u + s^2\,u^2 + 2\,b\,t^2\,k_1 + 4\,b\,t\,u\,k_1 - u^2\,k_1^2$$

$$F^{(4)}{}_{12} = 2\,b^2\,s\,t + 3\,b^2\,t^2 + 2\,b\,s\,t^2 + 2\,b\,t^3 - t^4 + 2\,b^2\,s\,u + b\,s^2\,u + 2\,b^2\,t\,u + 4\,b\,s\,t\,u + s^2\,t\,u + 2\,b\,t^2\,u + 3\,s\,t^2\,u + b\,s\,u^2 + s\,t\,u^2 - 2\,b\,t\,u\,k_1 - t^2\,u\,k_1 - b\,u^2\,k_1 - t\,u^2\,k_1 + b\,s^2\,k_2 + 2\,b\,s\,t\,k_2 + s^2\,t\,k_2 + s\,t^2\,k_2 - t^2\,k_1\,k_2$$

$$F^{(4)}{}_{21} = 2\,b^2\,s\,t + 3\,b^2\,t^2 - 2\,b\,s\,t^2 - 2\,b\,t^3 - t^4 + 2\,b^2\,s\,u - b\,s^2\,u + 2\,b^2\,t\,u - 4\,b\,s\,t\,u + s^2\,t\,u - 2\,b\,t^2\,u + 3\,s\,t^2\,u - b\,s\,u^2 + s\,t\,u^2 - 2\,b\,t\,u\,k_1 + t^2\,u\,k_1 - b\,u^2\,k_1 + t\,u^2\,k_1 + b\,s^2\,k_2 + 2\,b\,s\,t\,k_2 - s^2\,t\,k_2 - s\,t^2\,k_2 - t^2\,k_1\,k_2$$

$$F^{(4)}{}_{22} = -2\,b^2\,t^2 + 2\,t^4 - 2\,b^2\,s\,u - 4\,b^2\,t\,u - 2\,s\,t^2\,u + 2\,t^3\,u - b^2\,u^2 + s^2\,u^2 + t^2\,u^2 - 4\,b\,s\,t\,k_2 - 2\,b\,t^2\,k_2 - s^2\,k_2^2$$

[35] The overall factor of i in (B.5) simply reflects the fact that the vielbein (B.3) is imaginary. This makes no difference and could be eliminated by just multiplying $E^{Ai} \to iE^{Ai}$.

JHEP05(2021)076

$$F^{(9)}{}_{11} =$$

$$\frac{1}{2}\Big(22\,b^4 s^3 t^2 + 98\,b^4 s^2 t^3 + 115\,b^4 s\,t^4 - 20\,b^2 s^3 t^4 + 30\,b^4 t^5 - 78\,b^2 s^2 t^5 - 98\,b^2 s\,t^6 - 2\,s^3 t^6 - 36\,b^2 t^7 + 7\,s\,t^8 + 6\,t^9 + 5\,b^4 s^4 u + 64\,b^4 s^3 t\,u +$$
$$214\,b^4 s^2 t^2 u - 6\,b^2 s^4 t^2 u + 200\,b^4 s\,t^3 u - 24\,b^2 s^3 t^3 u + 30\,b^4 t^4 u - 64\,b^2 s^2 t^4 u + s^4 t^4 u - 100\,b^2 s\,t^5 u - 12\,s^3 t^5 u - 36\,b^2 t^6 u - 26\,s^2 t^6 u -$$
$$12\,s\,t^7 u + 6\,t^8 u + 22\,b^4 s^3 u^2 - b^2 s^5 u^2 + 94\,b^4 s^2 t\,u^2 - 6\,b^2 s^4 t\,u^2 + 70\,b^4 s\,t^2 u^2 + 4\,b^2 s^3 t^2 u^2 + s^5 t^2 u^2 + 44\,b^2 s^2 t^3 u^2 + 12\,s^4 t^3 u^2 + 4\,b^2 s\,t^4 u^2 +$$
$$13\,s^3 t^4 u^2 - 6\,s^2 t^5 u^2 - 18\,s\,t^6 u^2 + 8\,b^4 s^2 u^3 - 4\,b^2 s^4 u^3 - 16\,b^2 s^3 t\,u^3 + 12\,b^2 s^2 t^2 u^3 + 6\,s^4 t^2 u^3 + 12\,s^3 t^3 u^3 + 16\,s^2 t^4 u^3 - 7\,b^2 s^3 u^4 -$$
$$5\,s^3 t^2 u^4 + s^4 u^5 - 30\,b^3 s\,t^4 k_1 - 42\,b^3 t^5 k_1 + 10\,b\,s\,t^6 k_1 + 18\,b\,t^7 k_1 - 40\,b^3 s^2 t^2 u\,k_1 - 196\,b^3 s\,t^3 u\,k_1 - 156\,b^3 t^4 u\,k_1 + 12\,b\,s^2 t^4 u\,k_1 +$$
$$44\,b\,s\,t^5 u\,k_1 + 68\,b\,t^6 u\,k_1 - 2\,b^3 s^3 u^2 k_1 - 68\,b^3 s^2 t\,u^2 k_1 - 238\,b^3 s\,t^2 u^2 k_1 - 4\,b\,s^3 t^2 u^2 k_1 - 108\,b^3 t^3 u^2 k_1 - 4\,b\,s^2 t^3 u^2 k_1 - 10\,b\,s\,t^4 u^2 k_1 +$$
$$52\,b\,t^5 u^2 k_1 - 20\,b^3 s^2 u^3 k_1 - 72\,b^3 s\,t\,u^3 k_1 - 4\,b\,s^3 t\,u^3 k_1 - 4\,b\,s^2 t^2 u^3 k_1 - 48\,b\,s\,t^3 u^3 k_1 + 14\,b\,s^2 t\,u^4 k_1 + 30\,b^2 t^4 u\,k_1^2 - 2\,t^6 u\,k_1^2 +$$
$$34\,b^2 s\,t^2 u^2 k_1^2 + 126\,b^2 t^3 u^2 k_1^2 - s\,t^4 u^2 k_1^2 - 10\,t^5 u^2 k_1^2 + 2\,b^2 s^2 u^3 k_1^2 + 48\,b^2 s\,t\,u^3 k_1^2 + 96\,b^2 t^2 u^3 k_1^2 + 4\,s\,t^3 u^3 k_1^2 - 12\,t^4 u^3 k_1^2 + 15\,b^2 s\,u^4 k_1^2 +$$
$$13\,s\,t^2 u^4 k_1^2 - 4\,s^2 u^5 k_1^2 - 24\,b\,t^2 u^3 k_1^3 - 30\,b\,t\,u^4 k_1^3 + 3\,u^5 k_1^4 + 18\,b^3 s^4\,t\,k_2 + 82\,b^3 s^3 t^2 k_2 + 100\,b^3 s^2 t^3 k_2 - 18\,b\,s^4 t^3 k_2 + 30\,b^3 s\,t^4 k_2 -$$
$$58\,b\,s^3 t^4 k_2 - 58\,b\,s^2 t^5 k_2 - 18\,b\,s\,t^6 k_2 + 8\,b^3 s^4 u\,k_2 + 44\,b^3 s^3 t\,u\,k_2 + 40\,b^3 s^2 t^2 u\,k_2 + 4\,b\,s^4 t^2 u\,k_2 - 4\,b\,s^2 t^4 u\,k_2 + 2\,b^3 s^3 u^2 k_2 + 4\,b\,s^4 t\,u^2 k_2 +$$
$$4\,b\,s^3 t^2 u^2 k_2 - 34\,b^2 s^2 t^3 k_1 k_2 - 64\,b^2 s\,t^4 k_1 k_2 - 18\,b^2 t^5 k_1 k_2 + 10\,s^2 t^5 k_1 k_2 + 16\,s\,t^6 k_1 k_2 + 6\,t^7 k_1 k_2 - 8\,b^2 s^3 t\,u\,k_1 k_2 - 64\,b^2 s^2 t^2 u\,k_1 k_2 -$$
$$64\,b^2 s\,t^3 u\,k_1 k_2 - 4\,s^3 t^3 u\,k_1 k_2 + 4\,s\,t^5 u\,k_1 k_2 - 4\,b^2 s^3 u^2 k_1 k_2 - 14\,b^2 s^2 t\,u^2 k_1 k_2 - 4\,s^3 t^2 u^2 k_1 k_2 - 4\,s^2 t^3 u^2 k_1 k_2 + 6\,b\,t^5 k_1^2 k_2 + 20\,b\,s\,t^3 u\,k_1^2 k_2 +$$
$$24\,b\,t^4 u\,k_1^2 k_2 + 4\,b\,s^2 t\,u^2 k_1^2 k_2 + 18\,b\,s\,t^2 u^2 k_1^2 k_2 - 6\,t^3 u^2 k_1^3 k_2 + 3\,b^2 s^5 k_2^2 + 12\,b^2 s^4 t\,k_2^2 + 10\,b^2 s^3 t^2 k_2^2 - 3\,s^5 t^2 k_2^2 - 6\,s^4 t^3 k_2^2 - 3\,s^3 t^4 k_2^2 +$$
$$2\,b^2 s^4 u\,k_2^2 - 6\,b\,s^3 t^2 k_1 k_2^2 - 8\,b\,s^2 t^3 k_1 k_2^2 - 4\,b\,s^3 t\,u\,k_1 k_2^2 + s\,t^4 k_1^2 k_2^2 + 2\,s^2 t^2 u\,k_1^2 k_2^2\Big)$$

$$F^{(9)}{}_{12} =$$

$$\frac{1}{2}\Big(-32\,b^4 s^2 t^3 - 79\,b^4 s\,t^4 - 22\,b^3 s^2 t^4 - 38\,b^4 t^5 - 44\,b^3 s\,t^5 + 18\,b^2 s^2 t^5 - 16\,b^3 t^6 + 54\,b^2 s\,t^6 + 2\,b\,s^2 t^6 + 32\,b^2 t^7 + 4\,b\,s\,t^7 - 6\,s^2 t^7 - 15\,s\,t^8 -$$
$$10\,t^9 - 22\,b^4 s^3 t\,u - 149\,b^4 s^2 t^2 u - 52\,b^3 s^3 t^2 u - 236\,b^4 s\,t^3 u - 200\,b^3 s^2 t^3 u - 40\,b^2 s^3 t^3 u - 79\,b^4 t^4 u - 210\,b^3 s\,t^4 u - 77\,b^2 s^2 t^4 u - 44\,b^3 t^5 u +$$
$$34\,b^2 s\,t^5 u + 6\,b\,s^2 t^5 u + 10\,s^3 t^5 u + 54\,b^2 t^6 u + 18\,b\,s\,t^6 u + 18\,s^2 t^6 u + 4\,b\,t^7 u - 2\,s\,t^7 u - 15\,t^8 u - 15\,b^4 s^3 u^2 - 7\,b^3 s^4 u^2 - 104\,b^4 s^2 t\,u^2 -$$
$$80\,b^3 s^3 t\,u^2 - 5\,b^2 s^4 t\,u^2 - 149\,b^4 s\,t^2 u^2 - 240\,b^3 s^2 t^2 u^2 - 86\,b^2 s^3 t^2 u^2 - 3\,b\,s^4 t^2 u^2 - 32\,b^4 t^3 u^2 - 200\,b^3 s\,t^3 u^2 - 204\,b^2 s^2 t^3 u^2 - 12\,b\,s^3 t^3 u^2 -$$
$$5\,s^4 t^3 u^2 - 22\,b^3 t^4 u^2 - 77\,b^2 s\,t^4 u^2 - 18\,b\,s^2 t^4 u^2 - 3\,s^3 t^4 u^2 + 18\,b^2 t^5 u^2 + 6\,b\,s\,t^5 u^2 + 16\,s^2 t^5 u^2 + 2\,b\,t^6 u^2 + 18\,s\,t^6 u^2 - 6\,t^7 u^2 - 15\,b^4 s^2 u^3 -$$
$$20\,b^3 s^3 u^3 + b^2 s^4 u^3 + b\,s^5 u^3 - 22\,b^4 s\,t\,u^3 - 80\,b^3 s^2 t\,u^3 - 24\,b^2 s^3 t\,u^3 + 2\,b\,s^4 t\,u^3 + s^5 t\,u^3 - 52\,b^3 s\,t^2 u^3 - 86\,b^2 s^2 t^2 u^3 - 40\,b^2 s\,t^3 u^3 -$$
$$12\,b\,s^2 t^3 u^3 - 4\,s^3 t^3 u^3 - 3\,s^2 t^4 u^3 + 10\,s\,t^5 u^3 - 7\,b^3 s^2 u^4 + b^2 s^3 u^4 - 5\,b^2 s^2 t\,u^4 + 2\,b\,s^3 t\,u^4 - 3\,b\,s^2 t^2 u^4 - 5\,s^2 t^3 u^4 + b\,s^3 u^5 + s^3 t\,u^5 +$$
$$12\,b^3 t^5 k_1 + 6\,b^2 t^6 k_1 - 8\,b\,t^7 k_1 - 2\,t^8 k_1 + 56\,b^3 s\,t^3 u\,k_1 + 101\,b^3 t^4 u\,k_1 + 49\,b^2 s\,t^4 u\,k_1 + 67\,b^2 t^5 u\,k_1 + 10\,b\,s\,t^5 u\,k_1 - 27\,b\,t^6 u\,k_1 + 3\,s\,t^6 u\,k_1 -$$
$$5\,t^7 u\,k_1 + 22\,b^3 s^2 t\,u^2 k_1 + 148\,b^3 s\,t^2 u^2 k_1 + 27\,b^2 s^2 t^2 u^2 k_1 + 152\,b^3 t^3 u^2 k_1 + 152\,b^2 s\,t^3 u^2 k_1 + 16\,b\,s^2 t^3 u^2 k_1 + 117\,b^2 t^4 u^2 k_1 + 55\,b\,s\,t^4 u^2 k_1 -$$
$$s^2 t^4 u^2 k_1 - 26\,b\,t^5 u^2 k_1 + 7\,s\,t^5 u^2 k_1 - 5\,t^6 u^2 k_1 + 12\,b^3 s^2 u^3 k_1 - b^2 s^3 u^3 k_1 + 82\,b^3 s\,t\,u^3 k_1 + 24\,b^2 s^2 t\,u^3 k_1 + 54\,b^3 t^2 u^3 k_1 + 121\,b^2 s\,t^2 u^3 k_1 +$$
$$26\,b\,s^2 t^2 u^3 k_1 + 50\,b^2 t^3 u^3 k_1 + 72\,b\,s\,t^3 u^3 k_1 - 2\,s^2 t^3 u^3 k_1 - 6\,b\,t^4 u^3 k_1 + 7\,s\,t^4 u^3 k_1 - 2\,t^5 u^3 k_1 + 9\,b^3 s\,u^4 k_1 + 5\,b^2 s^2 u^4 k_1 + 31\,b^2 s\,t\,u^4 k_1 +$$
$$8\,b\,s^2 t\,u^4 k_1 + 25\,b\,s\,t^2 u^4 k_1 - 2\,s^2 t^2 u^4 k_1 + 3\,s\,t^3 u^4 k_1 - b\,s^2 u^5 k_1 - s^2 t\,u^5 k_1 - 32\,b^2 t^3 u^2 k_1^2 - 19\,b\,t^4 u^2 k_1^2 + 3\,t^5 u^2 k_1^2 - 10\,b^2 s\,t\,u^3 k_1^2 -$$
$$63\,b^2 t^2 u^3 k_1^2 - 8\,b\,s\,t^2 u^3 k_1^2 - 46\,b\,t^3 u^3 k_1^2 - 6\,s\,t^3 u^3 k_1^2 + 3\,t^4 u^3 k_1^2 - 6\,b^2 s\,u^4 k_1^2 - 24\,b^2 t\,u^4 k_1^2 - 8\,b\,s\,t\,u^4 k_1^2 - 24\,b\,t^2 u^4 k_1^2 - 9\,s\,t^2 u^4 k_1^2 -$$
$$3\,b\,s\,u^5 k_1^2 - 3\,s\,t\,u^5 k_1^2 + 6\,b\,t\,u^4 k_1^3 + 3\,t^2 u^4 k_1^3 + 3\,b\,u^5 k_1^3 + 3\,t\,u^5 k_1^3 - 54\,b^3 s^3 t^2 k_2 - 152\,b^3 s^2 t^3 k_2 - 50\,b^2 s^3 t^3 k_2 - 101\,b^3 s\,t^4 k_2 - 117\,b^2 s^2 t^4 k_2 +$$
$$6\,b\,s^3 t^4 k_2 - 12\,b^3 t^5 k_2 - 67\,b^2 s\,t^5 k_2 + 26\,b\,s^2 t^5 k_2 + 2\,s^3 t^5 k_2 - 6\,b^2 t^6 k_2 + 27\,b\,s\,t^6 k_2 + 5\,s^2 t^6 k_2 + 8\,b\,t^7 k_2 + 5\,s\,t^7 k_2 + 2\,t^8 k_2 - 9\,b^3 s^4 u\,k_2 -$$
$$82\,b^3 s^3 t\,u\,k_2 - 31\,b^2 s^4 t\,u\,k_2 - 148\,b^3 s^2 t^2 u\,k_2 - 121\,b^2 s^3 t^2 u\,k_2 - 25\,b\,s^4 t^2 u\,k_2 - 56\,b^3 s\,t^3 u\,k_2 - 152\,b^2 s^2 t^3 u\,k_2 - 72\,b\,s^3 t^3 u\,k_2 - 3\,s^4 t^3 u\,k_2 -$$
$$49\,b^2 s\,t^4 u\,k_2 - 55\,b\,s^2 t^4 u\,k_2 - 7\,s^3 t^4 u\,k_2 - 10\,b\,s\,t^5 u\,k_2 - 7\,s^2 t^5 u\,k_2 - 3\,s\,t^6 u\,k_2 - 12\,b^3 s^3 u^2 k_2 - 5\,b^2 s^4 u^2 k_2 + b\,s^5 u^2 k_2 - 22\,b^3 s^2 t\,u^2 k_2 -$$
$$24\,b^2 s^3 t\,u^2 k_2 - 8\,b\,s^4 t\,u^2 k_2 + s^5 t\,u^2 k_2 - 27\,b^2 s^2 t^2 u^2 k_2 - 26\,b\,s^3 t^2 u^2 k_2 + 2\,s^4 t^2 u^2 k_2 - 16\,b\,s^2 t^3 u^2 k_2 + 2\,s^3 t^3 u^2 k_2 + s^2 t^4 u^2 k_2 +$$
$$b^2 s^3 u^3 k_2 + 41\,b^2 s\,t^4 k_1 k_2 + 32\,b^2 t^5 k_1 k_2 + 26\,b\,s\,t^5 k_1 k_2 + 20\,b\,t^6 k_1 k_2 - s\,t^6 k_1 k_2 + 49\,b^2 s^2 t^2 u\,k_1 k_2 + 128\,b^2 s\,t^3 u\,k_1 k_2 + 30\,b\,s^2 t^3 u\,k_1 k_2 +$$
$$41\,b^2 t^4 u\,k_1 k_2 + 76\,b\,s\,t^4 u\,k_1 k_2 + 7\,s^2 t^4 u\,k_1 k_2 + 26\,b\,t^5 u\,k_1 k_2 + 4\,s\,t^5 u\,k_1 k_2 - t^6 u\,k_1 k_2 + 3\,b^2 s^3 u^2 k_1 k_2 + 32\,b^2 s^2 t\,u^2 k_1 k_2 - 2\,b\,s^3 t\,u^2 k_1 k_2 +$$
$$49\,b^2 s\,t^2 u^2 k_1 k_2 + 12\,b\,s^2 t^2 u^2 k_1 k_2 + 30\,b\,s\,t^3 u^2 k_1 k_2 + 8\,s^2 t^3 u^2 k_1 k_2 + 7\,s\,t^4 u^2 k_1 k_2 + 3\,b^2 s^2 u^3 k_1 k_2 - 2\,b\,s^2 t\,u^3 k_1 k_2 - 19\,b\,t^4 u^2 k_1^2 k_2 -$$
$$7\,t^5 u^2 k_1^2 k_2 - 14\,b\,s\,t^2 u^2 k_1^2 k_2 - 24\,b\,t^3 u^2 k_1^2 k_2 - 2\,s\,t^3 u^2 k_1^2 k_2 - 10\,t^4 u^2 k_1^2 k_2 - 6\,b\,s\,t\,u^3 k_1^2 k_2 + s\,t^2 u^3 k_1^2 k_2 + 3\,t^2 u^3 k_1^3 k_2 - 24\,b^2 s^4 t\,k_2^2 -$$
$$63\,b^2 s^3 t^2 k_2^2 - 24\,b\,s^4 t^2 k_2^2 - 32\,b^2 s^2 t^3 k_2^2 - 46\,b\,s^3 t^3 k_2^2 - 19\,b\,s^2 t^4 k_2^2 + 3\,s^3 t^4 k_2^2 + 3\,s^2 t^5 k_2^2 - 6\,b^2 s^4 u\,k_2^2 - 3\,b\,s^5 u\,k_2^2 - 10\,b^2 s^3 t\,u\,k_2^2 -$$
$$8\,b\,s^4 t\,u\,k_2^2 - 3\,s^5 t\,u\,k_2^2 - 8\,b\,s^3 t^2 u\,k_2^2 - 9\,s^4 t^2 u\,k_2^2 - 6\,s^3 t^3 u\,k_2^2 + 24\,b\,s^2 t^3 k_1 k_2^2 + 19\,b\,s\,t^4 k_1 k_2^2 + 10\,s^2 t^4 k_1 k_2^2 + 7\,s\,t^5 k_1 k_2^2 + 6\,b\,s^3 t\,u\,k_1 k_2^2 +$$
$$14\,b\,s^2 t^2 u\,k_1 k_2^2 - s^3 t^2 u\,k_1 k_2^2 + 2\,s^2 t^3 u\,k_1 k_2^2 - 2\,t^5 k_1^2 k_2^2 - 4\,s\,t^3 u\,k_1^2 k_2^2 - 3\,b\,s^5 k_2^3 - 6\,b\,s^4 t\,k_2^3 - 3\,s^5 t\,k_2^3 - 3\,s^4 t^2 k_2^3 + 3\,s^3 t^2 k_1 k_2^3\Big)$$

$$F^{(9)}{}_{21} =$$

$$\frac{1}{2}\Big(-32\,b^4 s^2 t^3 - 79\,b^4 s\,t^4 + 22\,b^3 s^2 t^4 - 38\,b^4 t^5 + 44\,b^3 s\,t^5 + 18\,b^2 s^2 t^5 + 16\,b^3 t^6 + 54\,b^2 s\,t^6 - 2\,b\,s^2 t^6 + 32\,b^2 t^7 - 4\,b\,s\,t^7 - 6\,s^2 t^7 - 15\,s\,t^8 -$$
$$10\,t^9 - 22\,b^4 s^3 t\,u - 149\,b^4 s^2 t^2 u + 52\,b^3 s^3 t^2 u - 236\,b^4 s\,t^3 u + 200\,b^3 s^2 t^3 u - 40\,b^2 s^3 t^3 u - 79\,b^4 t^4 u + 210\,b^3 s\,t^4 u - 77\,b^2 s^2 t^4 u + 44\,b^3 t^5 u +$$
$$34\,b^2 s\,t^5 u - 6\,b\,s^2 t^5 u + 10\,s^3 t^5 u + 54\,b^2 t^6 u - 18\,b\,s\,t^6 u + 18\,s^2 t^6 u - 4\,b\,t^7 u - 2\,s\,t^7 u - 15\,t^8 u - 15\,b^4 s^3 u^2 + 7\,b^3 s^4 u^2 - 104\,b^4 s^2 t\,u^2 +$$
$$80\,b^3 s^3 t\,u^2 - 5\,b^2 s^4 t\,u^2 - 149\,b^4 s\,t^2 u^2 + 240\,b^3 s^2 t^2 u^2 - 86\,b^2 s^3 t^2 u^2 + 3\,b\,s^4 t^2 u^2 - 32\,b^4 t^3 u^2 + 200\,b^3 s\,t^3 u^2 - 204\,b^2 s^2 t^3 u^2 + 12\,b\,s^3 t^3 u^2 -$$
$$5\,s^4 t^3 u^2 + 22\,b^3 t^4 u^2 - 77\,b^2 s\,t^4 u^2 + 18\,b\,s^2 t^4 u^2 - 3\,s^3 t^4 u^2 + 18\,b^2 t^5 u^2 - 6\,b\,s\,t^5 u^2 + 16\,s^2 t^5 u^2 - 2\,b\,t^6 u^2 + 18\,s\,t^6 u^2 - 6\,t^7 u^2 - 15\,b^4 s^2 u^3 +$$
$$20\,b^3 s^3 u^3 + b^2 s^4 u^3 - b\,s^5 u^3 - 22\,b^4 s\,t\,u^3 + 80\,b^3 s^2 t\,u^3 - 24\,b^2 s^3 t\,u^3 - 2\,b\,s^4 t\,u^3 + s^5 t\,u^3 + 52\,b^3 s\,t^2 u^3 - 86\,b^2 s^2 t^2 u^3 - 40\,b^2 s\,t^3 u^3 +$$
$$12\,b\,s^2 t^3 u^3 - 4\,s^3 t^3 u^3 - 3\,s^2 t^4 u^3 + 10\,s\,t^5 u^3 + 7\,b^3 s^2 u^4 + b^2 s^3 u^4 - 5\,b^2 s^2 t\,u^4 - 2\,b\,s^3 t\,u^4 + 3\,b\,s^2 t^2 u^4 - 5\,s^2 t^3 u^4 - b\,s^3 u^5 + s^3 t\,u^5 +$$
$$12\,b^3 t^5 k_1 - 6\,b^2 t^6 k_1 - 8\,b\,t^7 k_1 + 2\,t^8 k_1 + 56\,b^3 s\,t^3 u\,k_1 + 101\,b^3 t^4 u\,k_1 - 49\,b^2 s\,t^4 u\,k_1 - 67\,b^2 t^5 u\,k_1 + 10\,b\,s\,t^5 u\,k_1 - 27\,b\,t^6 u\,k_1 - 3\,s\,t^6 u\,k_1 +$$
$$5\,t^7 u\,k_1 + 22\,b^3 s^2 t\,u^2 k_1 + 148\,b^3 s\,t^2 u^2 k_1 - 27\,b^2 s^2 t^2 u^2 k_1 + 152\,b^3 t^3 u^2 k_1 - 152\,b^2 s\,t^3 u^2 k_1 + 16\,b\,s^2 t^3 u^2 k_1 - 117\,b^2 t^4 u^2 k_1 + 55\,b\,s\,t^4 u^2 k_1 +$$
$$s^2 t^4 u^2 k_1 - 26\,b\,t^5 u^2 k_1 - 7\,s\,t^5 u^2 k_1 + 5\,t^6 u^2 k_1 + 12\,b^3 s^2 u^3 k_1 + b^2 s^3 u^3 k_1 + 82\,b^3 s\,t\,u^3 k_1 - 24\,b^2 s^2 t\,u^3 k_1 + 54\,b^3 t^2 u^3 k_1 - 121\,b^2 s\,t^2 u^3 k_1 +$$
$$26\,b\,s^2 t^2 u^3 k_1 - 50\,b^2 t^3 u^3 k_1 + 72\,b\,s\,t^3 u^3 k_1 + 2\,s^2 t^3 u^3 k_1 - 6\,b\,t^4 u^3 k_1 - 7\,s\,t^4 u^3 k_1 + 2\,t^5 u^3 k_1 + 9\,b^3 s\,u^4 k_1 - 5\,b^2 s^2 u^4 k_1 - 31\,b^2 s\,t\,u^4 k_1 +$$
$$8\,b\,s^2 t\,u^4 k_1 + 25\,b\,s\,t^2 u^4 k_1 + 2\,s^2 t^2 u^4 k_1 - 3\,s\,t^3 u^4 k_1 - b\,s^2 u^5 k_1 + s^2 t\,u^5 k_1 - 32\,b^2 t^3 u^2 k_1^2 + 19\,b\,t^4 u^2 k_1^2 + 3\,t^5 u^2 k_1^2 - 10\,b^2 s\,t\,u^3 k_1^2 -$$
$$63\,b^2 t^2 u^3 k_1^2 + 8\,b\,s\,t^2 u^3 k_1^2 + 46\,b\,t^3 u^3 k_1^2 - 6\,s\,t^3 u^3 k_1^2 + 3\,t^4 u^3 k_1^2 - 6\,b^2 s\,u^4 k_1^2 - 24\,b^2 t\,u^4 k_1^2 + 8\,b\,s\,t\,u^4 k_1^2 + 24\,b\,t^2 u^4 k_1^2 - 9\,s\,t^2 u^4 k_1^2 +$$
$$3\,b\,s\,u^5 k_1^2 - 3\,s\,t\,u^5 k_1^2 + 6\,b\,t\,u^4 k_1^3 - 3\,t^2 u^4 k_1^3 + 3\,b\,u^5 k_1^3 - 3\,t\,u^5 k_1^3 - 54\,b^3 s^3 t^2 k_2 - 152\,b^3 s^2 t^3 k_2 + 50\,b^2 s^3 t^3 k_2 - 101\,b^3 s\,t^4 k_2 + 117\,b^2 s^2 t^4 k_2 +$$
$$6\,b\,s^3 t^4 k_2 - 12\,b^3 t^5 k_2 + 67\,b^2 s\,t^5 k_2 + 26\,b\,s^2 t^5 k_2 - 2\,s^3 t^5 k_2 + 6\,b^2 t^6 k_2 + 27\,b\,s\,t^6 k_2 - 5\,s^2 t^6 k_2 + 8\,b\,t^7 k_2 - 5\,s\,t^7 k_2 - 2\,t^8 k_2 - 9\,b^3 s^4 u\,k_2 -$$
$$82\,b^3 s^3 t\,u\,k_2 + 31\,b^2 s^4 t\,u\,k_2 - 148\,b^3 s^2 t^2 u\,k_2 + 121\,b^2 s^3 t^2 u\,k_2 - 25\,b\,s^4 t^2 u\,k_2 - 56\,b^3 s\,t^3 u\,k_2 + 152\,b^2 s^2 t^3 u\,k_2 - 72\,b\,s^3 t^3 u\,k_2 + 3\,s^4 t^3 u\,k_2 +$$
$$49\,b^2 s\,t^4 u\,k_2 - 55\,b\,s^2 t^4 u\,k_2 + 7\,s^3 t^4 u\,k_2 - 10\,b\,s\,t^5 u\,k_2 + 7\,s^2 t^5 u\,k_2 + 3\,s\,t^6 u\,k_2 + 12\,b^3 s^3 u^2 k_2 + 5\,b^2 s^4 u^2 k_2 + b\,s^5 u^2 k_2 - 22\,b^3 s^2 t\,u^2 k_2 +$$
$$24\,b^2 s^3 t\,u^2 k_2 - 8\,b\,s^4 t\,u^2 k_2 - s^5 t\,u^2 k_2 + 27\,b^2 s^2 t^2 u^2 k_2 - 26\,b\,s^3 t^2 u^2 k_2 - 2\,s^4 t^2 u^2 k_2 - 16\,b\,s^2 t^3 u^2 k_2 - 2\,s^3 t^3 u^2 k_2 - s^2 t^4 u^2 k_2 -$$
$$b^2 s^3 u^3 k_2 + 41\,b^2 s\,t^4 k_1 k_2 + 32\,b^2 t^5 k_1 k_2 - 26\,b\,s\,t^5 k_1 k_2 - 20\,b\,t^6 k_1 k_2 - s\,t^6 k_1 k_2 + 49\,b^2 s^2 t^2 u\,k_1 k_2 + 128\,b^2 s\,t^3 u\,k_1 k_2 - 30\,b\,s^2 t^3 u\,k_1 k_2 +$$
$$41\,b^2 t^4 u\,k_1 k_2 - 76\,b\,s\,t^4 u\,k_1 k_2 + 7\,s^2 t^4 u\,k_1 k_2 - 26\,b\,t^5 u\,k_1 k_2 + 4\,s\,t^5 u\,k_1 k_2 - t^6 u\,k_1 k_2 + 3\,b^2 s^3 u^2 k_1 k_2 + 32\,b^2 s^2 t\,u^2 k_1 k_2 + 2\,b\,s^3 t\,u^2 k_1 k_2 +$$
$$49\,b^2 s\,t^2 u^2 k_1 k_2 - 12\,b\,s^2 t^2 u^2 k_1 k_2 - 30\,b\,s\,t^3 u^2 k_1 k_2 + 8\,s^2 t^3 u^2 k_1 k_2 + 7\,s\,t^4 u^2 k_1 k_2 + 3\,b^2 s^2 u^3 k_1 k_2 + 2\,b\,s^2 t\,u^3 k_1 k_2 - 19\,b\,t^4 u^2 k_1^2 k_2 +$$
$$7\,t^5 u^2 k_1^2 k_2 - 14\,b\,s\,t^2 u^2 k_1^2 k_2 - 24\,b\,t^3 u^2 k_1^2 k_2 + 2\,s\,t^3 u^2 k_1^2 k_2 + 10\,t^4 u^2 k_1^2 k_2 - 6\,b\,s\,t\,u^3 k_1^2 k_2 - s\,t^2 u^3 k_1^2 k_2 + 3\,t^2 u^3 k_1^3 k_2 - 24\,b^2 s^4 t\,k_2^2 -$$
$$63\,b^2 s^3 t^2 k_2^2 + 24\,b\,s^4 t^2 k_2^2 - 32\,b^2 s^2 t^3 k_2^2 + 46\,b\,s^3 t^3 k_2^2 + 3\,s^3 t^4 k_2^2 + 3\,s^2 t^5 k_2^2 - 6\,b^2 s^4 u\,k_2^2 + 3\,b\,s^5 u\,k_2^2 - 10\,b^2 s^3 t\,u\,k_2^2 +$$
$$8\,b\,s^4 t\,u\,k_2^2 - 3\,s^5 t\,u\,k_2^2 + 8\,b\,s^3 t^2 u\,k_2^2 - 9\,s^4 t^2 u\,k_2^2 + 6\,s^3 t^3 u\,k_2^2 - 24\,b\,s^2 t^3 k_1 k_2^2 + 19\,b\,s\,t^4 k_1 k_2^2 + 10\,s^2 t^4 k_1 k_2^2 + 7\,s\,t^5 k_1 k_2^2 + 6\,b\,s^3 t\,u\,k_1 k_2^2 +$$
$$14\,b\,s^2 t^2 u\,k_1 k_2^2 + s^3 t^2 u\,k_1 k_2^2 - 2\,s^2 t^3 u\,k_1 k_2^2 - 2\,t^5 k_1^2 k_2^2 - 4\,s\,t^3 u\,k_1^2 k_2^2 - 3\,b\,s^5 k_2^3 - 6\,b\,s^4 t\,k_2^3 - 3\,s^5 t\,k_2^3 - 3\,s^4 t^2 k_2^3 + 3\,s^3 t^2 k_1 k_2^3\Big)$$

$$
\begin{aligned}
&F^{(9)}{}_{22} = \\
&\frac{1}{2}\,(30\,b^4\,s\,t^4 + 30\,b^4\,t^5 - 36\,b^2\,s\,t^6 - 36\,b^2\,t^7 + 6\,s\,t^8 + 6\,t^9 + 70\,b^4\,s^2\,t^2\,u + 200\,b^4\,s\,t^3\,u + 115\,b^4\,t^4\,u + 4\,b^2\,s^2\,t^4\,u - 100\,b^2\,s\,t^5\,u - 98\,b^2\,t^6\,u - \\
&\quad 18\,s^2\,t^6\,u - 12\,s\,t^7\,u + 7\,t^8\,u + 8\,b^4\,s^3\,u^2 + 94\,b^4\,s^2\,t\,u^2 + 214\,b^4\,s\,t^2\,u^2 + 12\,b^2\,s^3\,t^2\,u^2 + 98\,b^4\,t^3\,u^2 + 44\,b^2\,s^2\,t^3\,u^2 - 64\,b^2\,s\,t^4\,u^2 + 16\,s^3\,t^4\,u^2 - \\
&\quad 78\,b^2\,t^5\,u^2 - 6\,s^2\,t^5\,u^2 - 26\,s\,t^6\,u^2 + 22\,b^4\,s^2\,u^3 - 7\,b^2\,s^4\,u^3 + 64\,b^4\,s\,t\,u^3 - 16\,b^2\,s^3\,t\,u^3 + 22\,b^4\,t^2\,u^3 + 4\,b^2\,s^2\,t^2\,u^3 - 5\,s^4\,t^2\,u^3 - 24\,b^2\,s\,t^3\,u^3 + \\
&\quad 12\,s^3\,t^3\,u^3 - 20\,b^2\,t^4\,u^3 + 13\,s^2\,t^4\,u^3 - 12\,s\,t^5\,u^3 - 2\,t^6\,u^3 + 5\,b^4\,s\,u^4 - 4\,b^2\,s^3\,u^4 + s^5\,u^4 - 6\,b^2\,s^2\,t\,u^4 - 6\,b^2\,s\,t^2\,u^4 + 6\,s^3\,t^2\,u^4 + 12\,s^2\,t^3\,u^4 + \\
&\quad s\,t^4\,u^4 - b^2\,s^2\,u^5 + s^2\,t^2\,u^5 - 30\,b^3\,t^4\,u\,k_1 + 18\,b\,t^6\,u\,k_1 - 40\,b^3\,s\,t^2\,u^2\,k_1 - 100\,b^3\,t^3\,u^2\,k_1 + 4\,b\,s\,t^4\,u^2\,k_1 + 58\,b\,t^5\,u^2\,k_1 - 2\,b^3\,s^2\,u^3\,k_1 - 44\,b^3\,s\,t\,u^3\,k_1 - \\
&\quad 82\,b^3\,t^2\,u^3\,k_1 - 4\,b\,s^2\,t^2\,u^3\,k_1 + 58\,b\,t^4\,u^3\,k_1 - 8\,b^3\,s\,u^4\,k_1 - 18\,b^3\,t\,u^4\,k_1 - 4\,b\,s^2\,t\,u^4\,k_1 - 4\,b\,s\,t^2\,u^4\,k_1 + 18\,b\,t^3\,u^4\,k_1 + 10\,b^2\,t^2\,u^3\,k_1^2 - 3\,t^4\,u^3\,k_1^2 + \\
&\quad 2\,b^2\,s\,u^4\,k_1^2 + 12\,b^2\,t\,u^4\,k_1^2 - 6\,t^3\,u^4\,k_1^2 + 3\,b^2\,u^5\,k_1^2 - 3\,t^2\,u^5\,k_1^2 + 108\,b^3\,s^2\,t^3\,k_2 + 156\,b^3\,s\,t^4\,k_2 + 42\,b^3\,t^5\,k_2 - 52\,b\,s^2\,t^5\,k_2 - 68\,b\,s\,t^6\,k_2 - 18\,b\,t^7\,k_2 + \\
&\quad 72\,b^3\,s^3\,t\,u\,k_2 + 238\,b^3\,s^2\,t^2\,u\,k_2 + 196\,b^3\,s\,t^3\,u\,k_2 + 48\,b\,s^3\,t^3\,u\,k_2 + 30\,b^3\,t^4\,u\,k_2 + 10\,b\,s^2\,t^4\,u\,k_2 - 44\,b\,s\,t^5\,u\,k_2 - 10\,b\,t^6\,u\,k_2 + 20\,b^3\,s^3\,u^2\,k_2 + \\
&\quad 68\,b^3\,s^2\,t\,u^2\,k_2 - 14\,b\,s^4\,t\,u^2\,k_2 + 40\,b^3\,s\,t^2\,u^2\,k_2 + 4\,b\,s^3\,t^2\,u^2\,k_2 + 4\,b\,s^2\,t^3\,u^2\,k_2 - 12\,b\,s\,t^4\,u^2\,k_2 + 2\,b^3\,s^2\,u^3\,k_2 + 4\,b\,s^3\,t\,u^3\,k_2 + 4\,b\,s^2\,t^2\,u^3\,k_2 - \\
&\quad 18\,b^2\,t^5\,k_1\,k_2 + 6\,t^7\,k_1\,k_2 - 64\,b^2\,s\,t^3\,u\,k_1\,k_2 - 64\,b^2\,t^4\,u\,k_1\,k_2 + 4\,s\,t^5\,u\,k_1\,k_2 + 16\,t^6\,u\,k_1\,k_2 - 14\,b^2\,s^2\,t\,u^2\,k_1\,k_2 - 64\,b^2\,s\,t^2\,u^2\,k_1\,k_2 - 34\,b^2\,t^3\,u^2\,k_1\,k_2 - \\
&\quad 4\,s^2\,t^3\,u^2\,k_1\,k_2 + 10\,t^5\,u^2\,k_1\,k_2 - 4\,b^2\,s^2\,u^3\,k_1\,k_2 - 8\,b^2\,s\,t\,u^3\,k_1\,k_2 - 4\,s^2\,t^2\,u^3\,k_1\,k_2 - 4\,s\,t^3\,u^3\,k_1\,k_2 + 8\,b\,t^3\,u^2\,k_1^2\,k_2 + 4\,b\,s\,t\,u^3\,k_1^2\,k_2 + 6\,b\,t^2\,u^3\,k_1^2\,k_2 + \\
&\quad 96\,b^2\,s^3\,t^2\,k_2^2 + 126\,b^2\,s^2\,t^3\,k_2^2 + 30\,b^2\,s\,t^4\,k_2^2 - 12\,s^3\,t^4\,k_2^2 - 10\,s^2\,t^5\,k_2^2 - 2\,s\,t^6\,k_2^2 + 15\,b^2\,s^4\,u\,k_2^2 + 48\,b^2\,s^3\,t\,u\,k_2^2 + 34\,b^2\,s^2\,t^2\,u\,k_2^2 + 13\,s^4\,t^2\,u\,k_2^2 + \\
&\quad 4\,s^3\,t^3\,u\,k_2^2 - s^2\,t^4\,u\,k_2^2 + 2\,b^2\,s^3\,u^2\,k_2^2 - 4\,s^5\,u^2\,k_2^2 - 24\,b\,s^4\,k_1\,k_2^2 - 6\,b\,t^5\,k_1\,k_2^2 - 18\,b\,s^2\,t^2\,u\,k_1\,k_2^2 - 20\,b\,s\,t^3\,u\,k_1\,k_2^2 - 4\,b\,s^2\,t\,u^2\,k_1\,k_2^2 + t^4\,u\,k_1^2\,k_2^2 + \\
&\quad 2\,s\,t^2\,u^2\,k_1^2\,k_2^2 + 30\,b\,s^4\,t\,k_2^3 + 24\,b\,s^3\,t^2\,k_2^3 - 6\,s^2\,t^3\,k_1\,k_2^3 + 3\,s^5\,k_2^4)
\end{aligned}
$$

B.2 $G \times G/H$ model

The computation of the β-functions for the gauge invariant $G \times G/H$ model (3.1), (3.4) is similar to the $G \times G$ case above, except that one has to correctly handle the gauge invariance.

We shall again use the notation (B.1) and (B.2), with the symmetric "square root" of $\mathrm{h}_{ij} = \rho_{(ij)}$ being n_{ij}, and its inverse being m_{ij}. In the computation below, we shall denote certain combinations of n_{ij} and m_{ij} by

$$
\begin{aligned}
p_{ij} &= m_{1i}m_{1j} + m_{2i}m_{2j}\,, & \nu_{ij} &= n_{1i}m_{1j} - n_{2i}m_{2j}\,,\\
\chi_{ij} &= \frac{k}{2}p_{ij} + \frac{b}{2}(m_{1i}m_{2j} + m_{2i}m_{1j})\,, & \lambda_{ij} &= m_{1i}m_{1j} - m_{2i}m_{2j}\,.
\end{aligned}
\tag{B.10}
$$

We shall split up the generators T_A of G into $T_\alpha \in \mathrm{Lie}(H)$ and $T_a \in \mathrm{Lie}(G)/\mathrm{Lie}(H)$ (which are orthogonal with respect to the Killing form).

Assuming from the beginning that the matrix r_{ij} satisfies the gauge invariance condition (3.4),[36] the target space metric of the σ-model (3.1) is diagonalized by the vielbein[37]

$$
\begin{aligned}
&E^M = (e^\alpha, e^{\overline{\alpha}}, e^{ai}) = \Big(\sqrt{r}(I^{(1)\alpha} - I^{(2)\alpha})\,,\ I^{(1)\alpha} + I^{(2)\alpha}\,,\ n_{ik}P^{(k)a}\Big),\\
&I^{(k)} \equiv T_\alpha I^{(k)\alpha} = P_H\big[(g^{(k)})^{-1}dg^{(k)}\big]\,, \qquad P^{(k)} \equiv T_a P^{(k)a} = P_{G/H}\big[(g^{(k)})^{-1}dg^{(k)}\big].
\end{aligned}
\tag{B.11}
$$

In this frame, the metric $ds^2 = G_{MN}E^M E^N$ and the 3-form $H = \frac{1}{6}H_{MNP}E^M \wedge E^N \wedge E^P$ have the following non-zero components

$$
G_{\alpha\beta} = -\frac{1}{2}\,\mathrm{Tr}[T_\alpha T_\beta]\,, \qquad G_{ai,bj} = -\frac{1}{2}\,\mathrm{Tr}[T_a T_b]\,, \tag{B.12}
$$

$$
H_{\alpha\beta\gamma} = \frac{i}{2}kr^{-3/2}f_{\alpha\beta\gamma}\,, \qquad H_{\alpha,bi,cj} = ir^{-1/2}\chi_{ij}f_{\alpha bc}\,. \tag{B.13}
$$

[36] Alternatively, one could obtain the same results by starting with r_{ij} unconstrained, i.e. without gauge invariance imposed. One could first compute the torsionful Riemann tensor for the target space geometry (3.1) with general r_{ij}, ρ_{ij}. The gauge invariance condition (3.4) would then be imposed and the resulting Riemann tensor projected onto the non-degenerate directions of the metric G_{MN}.

[37] The index M denotes all tangent space directions. In the $G \times G$ case in (B.3) we had $M = (Ai)$, while here $M = (\alpha, \bar{\alpha}, ai)$. Both G and H are assumed to be simple.

JHEP05(2021)076

The H gauge invariance is reflected in the vanishing of all $\overline{\alpha}$ components of G_{MN} and H_{MNP}, and, in particular, the fact that G_{MN} is degenerate as a result. One could explicitly fix a gauge, eliminating some target space directions and removing this degeneracy. Instead, we find it more convenient to lift the degeneracy with a small parameter ϵ acting as a regulator,[38]

$$G_{\overline{\alpha}\overline{\beta}} = -\frac{1}{2}\epsilon \,\mathrm{Tr}[T_\alpha T_\beta]\,. \tag{B.14}$$

Computing the torsionful Riemann tensor as in subsection B.1, one finds that it has a finite $\epsilon \to 0$ limit. This means that the resulting Riemann tensor for $\epsilon = 0$ is unambiguous (since there are no divergences that could create finite-term ambiguities). Finally, we project out the $\overline{\alpha}$ directions to obtain the non-zero components

$$\begin{aligned}
\widehat{R}^{\alpha}{}_{\beta\delta\varepsilon} &= -\frac{1}{4r} f^{\alpha}{}_{\beta\gamma} f^{\gamma}{}_{\delta\varepsilon} + \frac{k^2}{4r^3}(f^{\alpha}{}_{\delta\gamma} f^{\gamma}{}_{\beta\varepsilon} - f^{\alpha}{}_{\varepsilon\gamma} f^{\gamma}{}_{\beta\delta})\,,\\
\widehat{R}^{\alpha}{}_{\beta,dk,el} &= \left(\frac{k}{2r}\lambda_{kl} - \frac{1}{2}p_{kl}\right) f^{\alpha}{}_{\beta\gamma} f^{\gamma}{}_{de} + \frac{1}{r} A_{kj} A_{lj} (f^{\alpha}{}_{dc} f^{c}{}_{\beta e} - f^{\alpha}{}_{ec} f^{c}{}_{\beta d})\,,\\
\widehat{R}^{\alpha}{}_{bi,\delta,el} &= -\frac{1}{2r} A_{ji} g_{jl} f^{\alpha}{}_{bc} f^{c}{}_{\delta e} - \frac{k}{2r^2} A_{li} f^{\alpha}{}_{\delta\gamma} f^{\gamma}{}_{be} + \frac{1}{r} A_{lj} C_{ji} f^{\alpha}{}_{ec} f^{c}{}_{b\delta}\,,\\
\widehat{R}^{ai}{}_{bj,dk,el} &= \left(C_{ij}\lambda_{kl} - \frac{1}{2}\delta_{ij} p_{kl}\right) f^{a}{}_{b\gamma} f^{\gamma}{}_{de} + \frac{1}{r} A_{ki} A_{lj} f^{a}{}_{d\gamma} f^{\gamma}{}_{be} - \frac{1}{r} A_{li} A_{kj} f^{a}{}_{e\gamma} f^{\gamma}{}_{bd}\,,\\
\widehat{R}^{ai}{}_{bk,\delta,\varepsilon} &= -\frac{1}{4r}\delta_{ik} f^{a}{}_{b\gamma} f^{\gamma}{}_{\delta\varepsilon} + \frac{1}{r} C_{ij} C_{jk} (f^{a}{}_{\delta c} f^{c}{}_{b\varepsilon} - f^{a}{}_{\varepsilon c} f^{c}{}_{b\delta})\,,\\
A_{ij} &\equiv \frac{1}{4}(\nu_{ji} - \nu_{ij}) - \chi_{ij} + \frac{r}{2}\lambda_{ij}\,, \qquad C_{ij} \equiv -\frac{1}{4}(\nu_{ji} + \nu_{ij}) + \chi_{ij} + \frac{r}{2}\lambda_{ij}\,.
\end{aligned} \tag{B.15}$$

All that remains is to substitute (B.13), (B.15) into the 2-loop β-functions (1.4) in the GB scheme. The resulting expression is hard to evaluate as it contains hundreds of terms, each proportional to a contraction of the form $(ffff)_{\alpha\beta}$ or $(ffff)_{ab}$ where each f denotes a component $f_{\gamma\delta\varepsilon}$ or $f_{\gamma de}$ of the structure constants, and indices are contracted using the Killing form $\mathrm{Tr}[T_a T_b]$. One can show, however, that there are only 11 independent such contractions after accounting for the antisymmetry of the structure constants. This allows for the efficient evaluation of the resulting β-functions for r and ρ_{ij}. These are first obtained depending on n_{11}, n_{12}, n_{22} but, as discussed in subsection B.1, they may be rewritten in terms of s, t, u with all square roots cancelling. As a result, we obtain the 2-loop generalization of the 1-loop β-functions in (3.12)–(3.16)

$$\frac{d}{d\tau} h_p = \alpha' \beta^{(1)}_{h_p} + \alpha'^2 \beta^{(2)}_{h_p}\,, \qquad\qquad h_p \equiv (r, s, t, b, u)\,, \tag{B.16}$$

[38] The use of the "regulator" ϵ is a short-cut for the following gauge-fixing procedure. Fixing an "axial" gauge $iX^u(I^{(1)}_u + I^{(2)}_u) = y(\xi) \in \mathrm{Lie}\, H$ ($u = 1, 2$ is the 2d index), the path integral should be independent of the choice of the constant 2d vector X^u and the algebra-valued function of 2d coordinates $y(\xi)$. Inserting the δ-function of the gauge fixing condition into the path integral and then integrating over X^u and y with a Gaussian measure, i.e. $e^{-\frac{1}{2}X^u X_u - \frac{1}{2}\epsilon \int d^2\xi\, \mathrm{Tr}[yy]}$, the result should be independent of ϵ (here we assume Euclidean 2d signature but the same is true also in Minkowski signature after an analytic continuation). Integrating first over y we get $\int d^2X \exp\left[-\frac{1}{2}X^u X^v \left(\delta_{uv} - \epsilon\,\mathrm{Tr}[(I^{(1)}_u + I^{(2)}_u)(I^{(1)}_v + I^{(2)}_v)]\right)\right]$. Integrating over X_u restores the 2d Euclidean invariance and the result to leading order in the $\varepsilon \to 0$ limit is equivalent to simply adding the regulator term $\Delta\mathcal{L} = -\frac{1}{2}\epsilon\,\mathrm{Tr}[(I^{(1)}_u + I^{(2)}_u)^2]$ corresponding to (B.14).

JHEP05(2021)076

where $\beta^{(2)}_{h_p}$ are given by the following expressions:

$$
\begin{aligned}
&\beta^{(2)}{}_r = \\
&\frac{c_H^2\,(3k^4-4k^2r^2+r^4)}{2r^5} \\
&+\frac{1}{r^3\,(t^2-su)^4}\,(c_G-c_H)\,c_H \\
&(-2bk^3t\,(2(s+u)\,(t^2-su)^2+4r^2(s+u)\,(s^2+2t^2+u^2)+3r\,(s^3u+su^3+s^2(-t^2+u^2)-t^2(t^2+u^2))) - \\
&\quad 2bkrt\,(-4r^3(s^3-st^2-t^2u+u^3)+2r(s+u)\,(-5t^4+6st^2u-s^2u^2+4b^2(t^2+su))-(t^2-su)\,(-3t^4+3st^2u+b^2(t^2+3su))+ \\
&\qquad r^2(5t^4+s^3u-t^2u^2+su^3-s^2(t^2+5u^2)))+ \\
&\quad k^4((t^2-su)^2(s^2+2t^2+u^2)+r^2(s^4+4s^2t^2+2t^4+4st^2u+4t^2u^2+u^4)+r(-s^3t^2+s^4u+3s^2t^2u-t^2u(3t^2+u^2)+s(-3t^4+3t^2u^2+u^4)))+ \\
&\quad r^2(r^4(s^4-4s^2t^2+2t^4+4st^2u-4t^2u^2+u^4)+r(-5t^4+5st^2u+b^2(3t^2+su))(-s^2u+t^2u+s(t^2-u^2))+ \\
&\qquad 2((t^2+su)\,(t^3-stu)^2+b^4(t^4+6st^2u+s^2u^2)-b^2(2t^6+5st^4u-8s^2t^2u^2+s^3u^3))- \\
&\qquad 3r^3(-s^3t^2+s^4u-s^2t^2u+t^2u(t^2-u^2)+s(t^4-t^2u^2+u^4))+ \\
&\qquad r^2(-4s^3t^2u+3s^4u^2-4st^2u(t^2+u^2)+t^4(4t^2+u^2)+s^2(t^4+3u^4)-2b^2(3s^2t^2-2t^4+s^3u+3t^2u^2+s(-6t^2u+u^3))))+ \\
&\quad k^2(-2r^4(s^4-2t^4+u^4)+2(t^2-su)^2(t^4-st^2u+b^2(t^2+su))+3r(-t^4+st^2u+b^2(3t^2+su))(s^2u-t^2u+s(-t^2+u^2))+ \\
&\qquad 2r^3(-s^3t^2+s^4u-3s^2t^2u+3t^4u-t^2u^3+s(3t^4-3t^2u^2+u^4))+ \\
&\qquad 2r^2(-5t^6+6s^3t^2u-2s^4u^2-4t^4u^2+6st^2u(t^2+u^2)-s^2(4t^4+t^2u^2+2u^4)+2b^2(3t^4+s^3u+5t^2u^2+s^2(5t^2+u^2)+s(8t^2u+u^3)))))) \\
&+\frac{1}{2r\,(t^2-su)^4}\,(c_G-c_H)^2 \\
&(2(t^2+4su)\,(t^3-stu)^2-24bk^3t(s+u)\,(s^2+2t^2+u^2)+6b^4(t^4+6st^2u+s^2u^2)-8b^2(t^6+4st^4u-5s^2t^2u^2)+ \\
&\quad r^4(s^4-4s^2t^2+2t^4+4st^2u-4t^2u^2+u^4)+3k^4(s^4+4s^2t^2+2t^4+4st^2u+4t^2u^2+u^4)-8rt^2(-2b^2+t^2-su)\,(-s^2u+t^2u+s(t^2-u^2))- \\
&\quad 8bkt(-2r^2(s^3-st^2-t^2u+u^3)+2(s+u)\,(-2t^4+st^2u+s^2u^2+3b^2(t^2+su))+r(-s^3u+s^2(t^2-2u^2)+t^2(4t^2+u^2)-s(2t^2u+u^3)))- \\
&\quad 4r^2(b^2(3s^2t^2-2t^4+s^3u+3t^2u^2+s(-6t^2u+u^3))-t^2(-s^3u+t^2(t^2+u^2)+s^2(t^2+3u^2)-s(4t^2u+u^3)))- \\
&\quad 4k^2(r^2(s^4-2t^4+u^4)+4rt^2(s^2u-t^2u+s(-t^2+u^2))+t^2(2t^4-3s^3u+3t^2u^2+s^2(3t^2-4u^2)+s(2t^2u-3u^3))- \\
&\qquad 3b^2(3t^4+s^3u+5t^2u^2+s^2(5t^2+u^2)+s(8t^2u+u^3)))) \\
&\\
&\beta^{(2)}{}_s = \\
&\frac{1}{2r^4\,(t^2-su)^3}\,c_H\,c_G \\
&(b^4r^2s(5t^2+3su)-2b^3krt(s(t^2-su)+4r(s^2+t^2+2su))+k^4(-u(t^2-su)^2+r(t^2-su)\,(t^2+u^2)+r^2(st^2+s^2u+4t^2u+2u^3))+ \\
&\quad r^2(5r^2u(t^2-su)^2+s^2t^2u(-t^2+su)+r^4(st^2+3s^2u-8t^2u+4u^3)+r^3(-2s^2t^2+t^4+7t^2u^2+3su(t^2-3u^2))- \\
&\qquad rt^2(t^4+s^3u+st^2u-s^2(t^2+2u^2)))+ \\
&\quad k^2(s(t^3-stu)^2-2r^4(st^2+2s^2u+2t^2u-u^3)-2r^2(s^3t^2+2st^4-2t^4u+5st^2u^2-2s^2u^3)+rt^2(-3t^4+s^3u+5st^2u-s^2(t^2+2u^2))+ \\
&\qquad 2r^3(s^2t^2+3t^4+4t^2u^2-su(t^2+3u^2)))- \\
&\quad 2bkt(k^2(-(t^2-su)^2+2r^2(s^2+2t^2+2su+3u^2)+r(st^2-s^2u+2t^2u-2su^2))+ \\
&\qquad r(st^2(-t^2+su)-2r^3(s^2+4su-u^2)+r^2(s^2u+10t^2u+3s(t^2-2u^2))+r(t^4-6st^2u+s^2(-4t^2+u^2))))+ \\
&\quad b^2(-r^2(2r^2(3st^2+s^2u-2t^2u+2su^2)+s(5t^4+2st^2u+s^2u^2)+r(-9t^4-s^3u+3st^2u+s^2(t^2-2u^2)))+ \\
&\qquad k^2(-s(t^2-su)^2-r(-3t^4+s^3u+st^2u-s^2(t^2-2u^2))+2r^2(s^3+s^2u+10t^2u+3s(3t^2+u^2))))) \\
&-\frac{1}{2r^2\,(t^2-su)^3}\,c_G^2 \\
&(2s^2t^2u(-t^2+su)+4b^4s(2t^2+su)-12b^3kt(s^2+t^2+2su)+k^4(2st^2+s^2u+6t^2u+3u^3)+r^2(s^3t^2-2st^4+4t^4u-11st^2u^2+8s^2u^3)+ \\
&\quad 2r^3(-2s^2t^2+t^4+4t^2u^2+3su(t^2-2u^2))+r^4(2st^2+3s^2u+5u(-2t^2+u^2))-2rt^2(t^4+s^3u-2st^2u+s^2(-t^2+u^2))- \\
&\quad 2bkt(-2t^2(3s^2+t^2+2su)+3k^2(s^2+2t^2+2su+3u^2)+r^2(-3s^2+4t^2-4su+7u^2)-2r(-st^2+s^2u-6t^2u+6su^2))- \\
&\quad k^2(3s^3t^2+2t^4u+s^2(4r^2u-2t^2u)-4r^2u(t^2+u^2)-2r(5t^4+4t^2u^2)+s(8t^4+8ru^3+t^2u(10r+u)))- \\
&\quad b^2(-k^2(3s^3+26st^2+4s^2u+30t^2u+9su^2)+2s(4t^4+st^2u+s^2u^2)+2r(-5t^4-s^3u+4st^2u+s^2(t^2+u^2))+ \\
&\qquad r^2(s^3+2s^2u-12t^2u+s(4t^2+u^2))))
\end{aligned}
$$

$$
\begin{aligned}
&\beta^{(2)}{}_t = \\
&-\frac{1}{2\,r^4\,(t^2-s\,u)^3}\,c_H\,c_G \\
&\quad (b^4 r^2 t\,(3t^2+5su) - b^3 k r\,(t^4 - s^2u^2 + 4r\,(3st^2+s^2u+3t^2u+su^2)) + \\
&\quad\; b^2 t\,(r^2\,(-2t^4+4st^2u+6s^2u^2+2r^2\,(2s^2+t^2-su+2u^2)) + r\,(3st^2-7s^2u+3t^2u-7su^2)) + \\
&\quad\;\; k^2\,(-(t^2-su)^2 - 3r\,(s^2u-t^2u+s\,(-t^2+u^2)) + 2r^2\,(5s^2+9su+5\,(t^2+u^2)))) + \\
&\quad\; b\,(k^3\,((s+u)\,(t^2-su)^2 - 2r^2\,(s+u)\,(s^2+6t^2+u^2) + r\,(s^3u+s^2\,(-t^2+u^2) - t^2\,(3t^2+u^2) + s\,(2t^2u+u^3))) + \\
&\quad\;\; k r\,(t^6 - 4st^4u + 3s^2t^2u^2 - 2r^3\,(s+u)\,(s^2+4t^2-2su+u^2) + r^2\,(-9t^4+3s^3u+t^2u^2+s^2\,(t^2-u^2)+3s\,(6t^2u+u^3)) - \\
&\quad\;\;\; r\,(-3t^4u+s^3u^2+s\,(-3t^4+10t^2u^2)+s^2\,(10t^2u+u^3)))) + \\
&\quad\; t\,(k^4\,(-(t^2-su)^2 + r^2\,(2s^2+3t^2+su+2u^2)) + r\,(st^2-s^2u+t^2u-su^2)) + \\
&\quad\;\; r^2\,(t^6 - st^4u + r^4\,(-4s^2+7t^2+su-4u^2) + r^3\,(s+u)\,(s^2-5t^2+3su+u^2) + r\,t^2\,(st^2-s^2u+t^2u-su^2) + \\
&\quad\;\;\; r^2\,(s^2t^2-2t^4-s^3u+2st^2u+t^2u^2-su^3)) + \\
&\quad\;\; k^2\,(-r^3\,(s+u)^3 + (t^2-2su)\,(t^2-su)^2 + 2r^4\,(s^2+3t^2-su+u^2) + r\,t^2\,(-s^2u+t^2u+s\,(t^2-u^2)) + \\
&\quad\;\;\; r^2\,(3s^3u+s^2\,(-t^2+u^2) - t^2\,(t^2+u^2) + s\,(4t^2u+3u^3)))))) \\
&+\frac{1}{2\,r^2\,(t^2-s\,u)^3}\,c_G^2 \\
&\quad (4b^4\,(t^3+2stu) - 6b^3 k\,(s^2u+3t^2u+s\,(3t^2+u^2)) + \\
&\quad\; b^2 t\,(-2t^4+6st^2u+8s^2u^2+r^2\,(3s^2-10su+3u^2) + k^2\,(15s^2+16t^2+26su+15u^2) - 7r\,(-st^2+s^2u-t^2u+su^2)) - \\
&\quad\; b k\,(3k^2\,(s+u)\,(s^2+6t^2+u^2) - 2t^2\,(st^2-7s^2u+t^2u-7su^2) + r^2\,(s^3-5s^2u-5su^2+u^3) - 2r\,(-s^2t^2-8t^4+s^3u+8st^2u-t^2u^2+su^3)) + \\
&\quad\; t\,(2t^4\,(t^2-su) + r^4\,(-5s^2+8t^2+2su-5u^2) + r^3\,(s+u)\,(2s^2-7t^2+3su+2u^2) + k^4\,(3s^2+4t^2+2su+3u^2) + \\
&\quad\;\; r\,t^2\,(-st^2+s^2u-t^2u+su^2) + r^2\,(2s^2t^2-3s^3u+2st^2u+2t^2u^2-3su^3) + \\
&\quad\;\; k^2\,(3su\,(s^2+2t^2+u^2) - 2r^2\,(s^2-2t^2+2su+u^2) - 3r\,(-st^2+s^2u-t^2u+su^2))))
\end{aligned}
$$

$$
\begin{aligned}
&\beta^{(2)}{}_b = \\
&\frac{1}{2\,r^4\,(t^2-s\,u)^3}\,c_H\,c_G \\
&\quad (b^3 r^2\,(-t^4 - 10st^2u - 5s^2u^2 + 2r\,(s^2u+t^2u+s\,(t^2+u^2))) + \\
&\quad\; b^2 k r t\,(t^4+2st^2u-3s^2u^2-4r^2\,(s^2+t^2+3su+u^2) + 8r\,(2s^2u+t^2u+s\,(t^2+2u^2))) + \\
&\quad\; b\,(k^2\,(2su\,(t^2-su)^2 + r^3\,(s+u)\,(s^2+10t^2+u^2) + 4rt^2\,(-st^2+s^2u-t^2u+su^2) - r^2\,(9s^2t^2+3t^4+3s^3u+20st^2u+s^2u^2+9t^2u^2+3su^3)) - \\
&\quad\;\; r^2\,((t^3-stu)^2 + r^3\,(s+u)\,(s^2-2t^2+u^2) + 6rt^2\,(-s^2u+t^2u+s\,(t^2-u^2)) - r^2\,(3t^4+s^3u-5t^2u^2+s^2\,(-5t^2+u^2)+s\,(4t^2u+u^3)))) + \\
&\quad\; k t\,(k^2\,(-(s+u)\,(t^2-su)^2 - 2r^3\,(s^2+2t^2+u^2) + 2r^2\,(s+u)\,(s^2+2t^2+u^2) + r\,(-s^3u+s^2\,(t^2+u^2) + t^2\,(3t^2+u^2) - s\,(4t^2u+u^3))) + \\
&\quad\;\; r\,(2r^3\,(s-u)^2\,(s+u) + r\,(s+u)\,(t^2-su)^2 - (t^3-stu)^2 + 2r^4\,(s^2-2t^2+u^2) - r^2\,(-5t^4+3s^3u-3t^2u^2+s^2\,(-3t^2+u^2)+s\,(4t^2u+3u^3)))))) \\
&+\frac{1}{2\,r^2\,(t^2-s\,u)^3}\,c_G^2 \\
&\quad (-2b^2 k t\,(rt^2+8st^2-rsu+10s^2u+8t^2u+10su^2) + \\
&\quad\; k t\,(r^3\,(-4s^2+6t^2+2su-4u^2) - 2r\,(t^2-su)\,(-k^2+s^2+3t^2+u^2) - r^2\,(s+u)\,(s^2+2t^2-4su+u^2) - k^2\,(s+u)\,(3s^2+8t^2-2su+3u^2)) + \\
&\quad\; b^3\,(2\,(t^4+8st^2u+3s^2u^2) + r\,(-s^2u+t^2u+s\,(t^2-u^2))) + \\
&\quad\; b\,(2\,(t^3-stu)^2 + r^3\,(s+u)\,(2s^2-3t^2-su+2u^2) - r^2\,(-5s^2t^2+2t^4+3s^3u+2st^2u-5t^2u^2+3su^3) + \\
&\quad\;\; k^2\,(15s^2t^2+10t^4+3s^3u+26st^2u+15t^2u^2+3su^3) + r\,(k^2-5t^2)\,(s^2u-t^2u+s\,(-t^2+u^2))))
\end{aligned}
$$

$$
\begin{aligned}
&\beta^{(2)}{}_u = \\
&-\frac{1}{2\,r^4\,(t^2-s\,u)^3}\,c_H\,c_G \\
&\quad (2b k^3 t\,(-(t^2-su)^2 + 2r^2\,(3s^2+2t^2+2su+u^2) + r\,(2st^2-2s^2u+t^2u-su^2)) - \\
&\quad\; k^4\,(-s\,(t^2-su)^2 - r\,(s^2+t^2)\,(-t^2+su) + r^2\,(2s^3+t^2u+s\,(4t^2+u^2))) + \\
&\quad\; 2bkrt\,((b^2-t^2)\,u\,(t^2-su) + 2r^3\,(s^2-4su-u^2) + r^2\,(-6s^2u+3t^2u+s\,(10t^2+u^2)) + r\,(t^4+s^2u^2-2t^2u\,(3s+2u)+4b^2\,(t^2+u\,(2s+u)))) + \\
&\quad\; k^2\,((b^2-t^2)\,u\,(t^2-su)^2 + r^4\,(-2s^3+2t^2u+4s\,(t^2+u^2)) + 2r^3\,(-4s^2t^2+3s^3u+st^2u-t^2\,(3t^2+u^2)) - \\
&\quad\;\; 2r^2\,(2st^4-5s^2t^2u-2t^4u+2s^3u^2-t^2u^3+b^2\,(10st^2+3s^2u+9t^2u+su^2+u^3)) + \\
&\quad\;\; r\,(t^2-su)\,(3t^4+t^2u\,(-2s+u) - b^2\,(3t^2+u\,(2s+u)))) + \\
&\quad\; r^2\,(-r^4\,(4s^3-8st^2+t^2u+3su^2) + r^3\,(-7s^2t^2-t^4+9s^3u-3st^2u+2t^2u^2) - (b^2-t^2)\,u\,(su\,(t^2-su)+b^2\,(5t^2+3su)) + \\
&\quad\;\; r^2\,(-5s\,(t^2-su)^2 + 2b^2\,(-2st^2+2s^2u+3t^2u+su^2)) + r\,(t^6+t^4\,(s-u)\,u+st^2u^2\,(-2s+u)+b^2\,(-9t^4-su^2\,(2s+u)+t^2u\,(3s+u)))))) \\
&-\frac{1}{2\,r^2\,(t^2-s\,u)^3}\,c_G^2 \\
&\quad (-6b k^3 t\,(3s^2+2t^2+2su+u^2) + r^4\,(5s^3-10st^2+2t^2u+3su^2) + 2r^3\,(4s^2t^2+t^4-6s^3u+3st^2u-2t^2u^2) + \\
&\quad\; 2\,(b^2-t^2)\,u\,(su\,(t^2-su)+2b^2\,(2t^2+su)) + 2r\,(t^2-su)\,(-t^4+t^2u\,(s+u)+b^2\,(5t^2+(s-u)\,u)) + k^4\,(3s^3+2t^2u+s\,(6t^2+u^2)) + \\
&\quad\; k^2\,(-2r\,(4s^2+5t^2)\,(-t^2+su) + 4r^2 s\,(s^2+t^2-u^2) - t^2\,(2st^2+s^2u+8t^2u-2su^2+3u^3) + b^2\,(30st^2+9s^2u+26t^2u+4su^2+3u^3)) - \\
&\quad\; 2bkt\,(r^2\,(7s^2+4t^2-4su-3u^2) + 6b^2\,(t^2+2su+u^2) - 2t^2\,(t^2+2su+3u^2) - 2r\,(6s^2u-t^2u+s\,(-6t^2+u^2))) - \\
&\quad\; r^2\,(-4st^4+11s^2t^2u+2t^4u-8s^3u^2-t^2u^3+b^2\,(s^2u+4t^2u+u^3+2s\,(-6t^2+u^2))))
\end{aligned}
$$

JHEP05(2021)076

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JHEP05(2021)076