Subleading-power corrections to the radiative leptonic $B \to \gamma \ell \nu$ decay in QCD

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ABSTRACT: Applying the method of light-cone sum rules with photon distribution amplitudes, we compute the subleading-power correction to the radiative leptonic $B \to \gamma \ell \nu$ decay from the twist-two hadronic photon contribution at next-to-leading order in QCD; and further evaluate the higher-twist "resolved photon" corrections at leading order in α_s, up to twist-four accuracy. QCD factorization for the vacuum-to-photon correlation function with an interpolating current for the B-meson is established explicitly at leading power in Λ/m_b employing the evanescent operator approach. Resummation of the parametrically large logarithms of m_b^2/Λ^2 entering the hard function of the leading-twist factorization formula is achieved by solving the QCD evolution equation for the light-ray tensor operator at two loops. The leading-twist hadronic photon effect turns out to preserve the symmetry relation between the two $B \to \gamma$ form factors due to the helicity conservation, however, the higher-twist hadronic photon corrections can yield symmetry-breaking effect already at tree level in QCD. Using the conformal expansion of photon distribution amplitudes with the non-perturbative parameters estimated from QCD sum rules, the twist-two hadronic photon contribution can give rise to approximately 30% correction to the leading-power "direct photon" effect computed from the perturbative QCD factorization approach. In contrast, the subleading-power corrections from the higher-twist two-particle and three-particle photon distribution amplitudes are estimated to be of $\mathcal{O}(3 \sim 5\%)$ with the light-cone sum rule approach. We further predict the partial branching fractions of $B \to \gamma \ell \nu$ with a photon-energy cut $E_\gamma \geq E_{\rm cut}$, which are of interest for determining the inverse moment of the leading-twist B-meson distribution amplitude thanks to the forthcoming high-luminosity Belle II experiment at KEK.

Contents

1 Introduction

Exploring the subleading-power contributions to exclusive B-meson decays in effective field theories are of essential importance to understand general properties of the heavy quark expansion and its higher-order behaviours in QCD and to achieve precision determinations of CKM matrix elements with a wealth of data accumulated at the B factories and at the LHC phenomenologically. In these respects, the radiative leptonic decay $B \to \gamma \ell \nu$ with an energetic photon in the final state is widely believed to provide a clean probe of the strong interaction dynamics of a heavy quark system and to put stringent constraints on the inverse moment of the leading-twist B-meson distribution amplitude (DA). Factorization properties of $B \to \gamma \ell \nu$ have been investigated extensively at leading power in Λ/m_b with distinct QCD techniques [1, 2] and with the soft-collinear effective theory (SCET) [3–5] which established the corresponding QCD factorization formula to all orders in perturbation theory.

Subleading-power corrections to the $B \to \gamma \ell \nu$ transition form factors were discussed in QCD factorization at tree level [6], where the symmetry-preserving form factor $\xi(E_\gamma)$ was introduced to parameterize the non-local SCET matrix element without integrating

out the hard-collinear scale. Systematic studies on the higher-power terms of the radiative leptonic B-meson decay amplitude in the heavy quark expansion are, however, still absent in the framework of SCET beyond the leading-order in α_s. Applying the dispersion relations and the parton-hadron duality, an alternative approach without identifying manifest structures of the subleading-power effective operators was proposed [7] to estimate the power suppressed soft contributions at tree level and was further extended [8] to compute the soft-overlap contribution at next-to-leading-order (NLO) in QCD. Consequently, there will be a price to pay for the dispersion approach when taking into account the hadronic photon corrections and the end-point contributions (the so-called Feynman mechanism) by implementing the non-perturbative modifications of the QCD spectral densities, as two additional non-perturbative parameters (vector meson mass m_ρ and effective threshold parameter s_0) are introduced when compared to the direct QCD calculation. It is then evident that evaluating the higher-power terms in the expansion of Λ/m_b individually with direct QCD approaches is of particular interest to deepen our understanding of perturbative QCD factorization for hard exclusive reactions.

The major objective of this paper is to perform QCD calculations of the subleading-power corrections induced by the hadronic component of the energetic photon at NLO in the strong coupling constant. QCD factorization formula for the two-particle hadronic photon correction to the $B \to \gamma \ell \nu$ amplitude was demonstrated to be invalidated by the rapidity divergence in the convolution integral of the hard scattering kernel with the light-cone DAs of the B-meson and of the photon [5]. Employing the technique of light-cone sum rules (LCSR) with the two-particle photon DAs, the power suppressed "resolved photon" contribution was computed at twist-four accuracy and at leading-order (LO) in α_s [9–11], and was further updated [12] by including the NLO correction to the leading-twist hadronic photon DA contribution and by calculating the higher-twist correction from the three-particle photon DAs at tree level. However, QCD factorization for the vacuum-to-photon correlation function with an interpolating current for the B-meson is not explicitly demonstrated with the operator-product-expansion (OPE) technique at one loop in [12], where the renormalization scheme dependence of γ_5 for the QCD amplitude in dimensional regularization was not addressed in any detail. It is therefore necessary to perform an independent calculation of the twist-two hadronic photon correction to the $B \to \gamma \ell \nu$ form factors at NLO in α_s by compensating the above-mentioned gaps. To this end, we will apply the standard perturbative matching procedure including the evanescent SCET operators to establish QCD factorization formulae for the vacuum-to-B-meson correlation function with the Dirac matrix γ_5 defined in naive dimensional regularization (NDR) (see [13, 14] for an overview, and [15] for a discussion in the context of the pion-photon transition form factor).

The presentation is organized as follows. We first summarize the theoretical status on QCD calculations of the $B \to \gamma \ell \nu$ form factors with different techniques based upon the heavy quark expansion and discuss the origin of subleading-power corrections in section 2. To construct the sum rules for the leading-twist hadronic photon correction, we then establish QCD factorization for the correlation function defined with an interpolating current for the B-meson and with the weak transition current $[\bar{u} \gamma_\mu (1 - \gamma_5) b]$ in section 3, where the master formula of the hard matching coefficient entering the factorization formula at one

loop will be derived with the implementation of the infrared (IR) subtraction including the evanescent SCET operator. With the aid of the evolution equation of the twist-two photon DA at two loops, summation of the parametrically large logarithms of m_b^2/Λ^2 in the hard function will be further preformed at next-to-leading-logarithmic (NLL) accuracy applying the momentum-space renormalization group (RG) approach. The NLL resummation improved LCSR for the twist-two hadronic correction to the $B \to \gamma$ form factors will be also presented here, taking advantage of the dispersion relation technique and the parton-hadron duality ansatz. The subleading-power corrections to the $B \to \gamma \ell \nu$ decay amplitude from both the two-particle and three-particle higher-twist photon DAs displayed in [16] will be computed with the LCSR approach at tree level in section 4, where a comparison of our results with that obtained in [11, 12] will be also presented. Phenomenological impacts of the various subleading-power corrections with the non-perturbative parameters of the photon DAs determined from QCD sum rules [17] will be explored in section 5, including the dependence of the partial branching fractions of $B \to \gamma \ell \nu$, with the phase-space cut of the photon energy, on the inverse moment λ_B. A summary of our main observations and future perspectives will be presented in section 6. We further collect spectral representations of the convolution integrals entering the leading-twist factorization formulae for the vacuum-to-photon correlation function at one-loop accuracy and the operator-level definitions of the higher-twist photon DAs up to the twist-four in appendices A and B, respectively.

2 Theoretical overview of $B \to \gamma \ell \nu$ decay

The radiative leptonic $B \to \gamma \ell \nu$ decay amplitude is defined by the following matrix element

$$\mathcal{A}(B^- \to \gamma \, \ell \, \nu) = \frac{G_F \, V_{ub}}{\sqrt{2}} \, \langle \gamma(p) \, \ell(p_\ell) \, \nu(p_\nu) \, | \, [\bar{\ell} \, \gamma_\mu \, (1 - \gamma_5) \, \nu] \, \, [\bar{u} \, \gamma^\mu \, (1 - \gamma_5) \, b] \, | \, B^-(p_B) \rangle \, . \tag{2.1}$$

Following [8] we will work in the rest frame of the B-meson with momentum $p_B = m_B \, v$ and introduce two light-cone vectors n_μ and $\bar{n}_\mu$ with the definitions

$$p_\mu = \frac{n \cdot p}{2} \, \bar{n}_\mu \equiv E_\gamma \, \bar{n}_\mu \, , \qquad v_\mu = \frac{n_\mu + \bar{n}_\mu}{2} \, . \tag{2.2}$$

Expanding $\mathcal{A}(B^- \to \gamma \, \ell \, \nu)$ to the leading order in electromagnetic interaction and employing the Ward identity due to the conservation of vector current leads to [6, 8]

$$\mathcal{A}(B^- \to \gamma \, \ell \, \nu) \to \frac{G_F \, V_{ub}}{\sqrt{2}} \, (i \, g_{em} \, \epsilon_\nu^*) \, v \cdot p \left\{ -i \, \epsilon_{\mu\nu\rho\sigma} \, n^\rho \, v^\sigma \, F_V(n \cdot p) + g_{\mu\nu} \, F_A(n \cdot p) \right\} , \tag{2.3}$$

where the contribution due to photon radiation off the final-state lepton has been taken into account by the redefinition of the axial form factor $F_A(n \cdot p)$.

At leading power in Λ/m_b the QCD factorization formula for the $B \to \gamma$ form factors can be readily derived with the SCET technique [3, 4]

$$F_{V,\mathrm{LP}}(n \cdot p) = F_{A,\mathrm{LP}}(n \cdot p) = \frac{Q_u \, m_B}{n \cdot p} \, \tilde{f}_B(\mu) \, C_\perp(n \cdot p, \mu) \int_0^\infty d\omega \, \frac{\phi_B^+(\omega, \mu)}{\omega} \, J_\perp(n \cdot p, \omega, \mu) \, . \tag{2.4}$$

The hard function $C_\perp$ arises from matching the QCD weak current $\bar{u}\,\gamma_{\mu\perp}\,(1-\gamma_5)\,b$ onto the corresponding SCET current and the one-loop expression is given by [18, 19]

$$C_\perp = 1 - \frac{\alpha_s\,C_F}{4\,\pi}\left[2\,\ln^2\frac{\mu}{n\cdot p} + 5\,\ln\frac{\mu}{m_b} - 2\,\mathrm{Li}_2\left(1-\frac{1}{r}\right) - \ln^2 r\right.$$
$$\left. + \frac{3r-2}{1-r}\,\ln r + \frac{\pi^2}{12} + 6\right], \tag{2.5}$$

with $r = n\cdot p/m_b$. The hard-collinear function $J_\perp$ entering the SCET factorization formula (2.4) reads [3, 4, 8]

$$J_\perp = 1 + \frac{\alpha_s\,C_F}{4\,\pi}\left[\ln^2\frac{\mu^2}{n\cdot p\,(\omega - \bar{n}\cdot p)} - \frac{\pi^2}{6} - 1\right] + \mathcal{O}(\alpha_s^2). \tag{2.6}$$

Setting μ as a hard-collinear scale of order $\sqrt{\Lambda\,m_b}$ and performing the NLL resummation of the parametrically large logarithms in the hard function yields

$$F_{V,\,\mathrm{LP}}(n\cdot p) = F_{A,\,\mathrm{LP}}(n\cdot p)$$
$$= \frac{Q_u\,m_B}{n\cdot p\,\lambda_B(\mu)}\left[U_2(n\cdot p, \mu_{h2}, \mu)\,\tilde{f}_B(\mu_{h2})\right]\left[U_1(n\cdot p, \mu_{h1}, \mu)\,C_\perp(n\cdot p, \mu_{h1})\right]$$
$$\times\left\{1 + \frac{\alpha_s(\mu)\,C_F}{4\,\pi}\left[\sigma_2(\mu) + 2\,\ln\frac{\mu^2}{n\cdot p\,\mu_0}\,\sigma_1(\mu) + \ln^2\frac{\mu^2}{n\cdot p\,\mu_0} - \frac{\pi^2}{6} - 1\right]\right\}, \tag{2.7}$$

where the convolution integral of ω has been expressed as moments of the B-meson DA defined in [6] and the manifest expressions of the evolution functions U_1 and U_2 can be found in [8].

The subleading-power corrections from photon radiation off the heavy quark and from higher-twist B-meson DAs were addressed [6] by computing the two diagrams for the tree $b\,\bar{u} \to \gamma\,W^*$ amplitude in QCD. Since the factorization property of the non-local subleading-power correction from photon radiation off the light quark has not been explored yet, we will only focus on the local subleading-power contribution to the $B \to \gamma\ell\nu$ amplitude at tree level

$$F_{V,\,\mathrm{NLP}}^{\mathrm{LC}}(n\cdot p) = -F_{A,\,\mathrm{NLP}}^{\mathrm{LC}}(n\cdot p) = \frac{Q_u\,f_B\,m_B}{(n\cdot p)^2} + \frac{Q_b\,f_B\,m_B}{n\cdot p\,m_b}. \tag{2.8}$$

As discussed in [5] the subleading-power contribution can be further generated by the effective matrix element $\langle\gamma(p)|\mathcal{O}|B^-(p_B)\rangle$ with the SCET operator $\mathcal{O} \supset [\bar{q}_s\,h_v]_s\,[\bar{\xi}\,\xi]_c$ containing no photon field, due to the unsuppressed interactions of photons with any numbers of collinear quark and gluon fields. The collinear matrix element $\langle\gamma(p)|[\bar{\xi}\,\xi]_c|0\rangle$ defines the photon DAs on the light cone, making the photon behave in analogy to an energetic vector meson. Consequently, these terms are also referred to as the "hadronic (resolved) photon" contributions in different contexts. QCD calculations of such power suppressed corrections to the $B \to \gamma\ell\nu$ decay form factors will be carried out, to the twist-four photon DAs accuracy, with the LCSR approach in the following.

3 Leading-twist hadronic photon correction in QCD

To obtain the sum rules for the form factors $F_V(n \cdot p)$ and $F_A(n \cdot p)$, we construct the vacuum-to-photon correlation function with an interpolating current for the B-meson

$$\Pi_\mu(p, q) = \int d^4x \, e^{i \, q \cdot x} \, \langle \gamma(p) | T\{\bar{u}(x) \, \gamma_{\mu\perp} \, (1 - \gamma_5) \, b(x), \, \bar{b}(0) \, \gamma_5 \, u(0)\} | 0 \rangle \,, \qquad (3.1)$$

where $q = p_\ell + p_\nu$ refers to the four-momentum of the lepton-neutrino pair. QCD factorization for the correlation function (3.1) can be demonstrated with the technique of the light-cone OPE at $(p + q)^2 \ll m_b^2$ and $q^2 \ll m_b^2$. For definiteness, we will employ the following power counting scheme

$$n \cdot p \sim \mathcal{O}(m_b), \qquad |n \cdot (p + q) - m_b| \sim \mathcal{O}(\Lambda) \,. \qquad (3.2)$$

The primary task of this section is to compute the perturbative matching coefficient entering the leading-twist factorization formula for (3.1) at NLO, with the evanescent operator approach.

3.1 The twist-two hadronic photon correction at tree level

QCD factorization for the twist-two contribution to the correlation function (3.1) can be justified by investigating the QCD amplitude

$$F_\mu(p, q) = \int d^4x \, e^{i \, q \cdot x} \, \langle q(z \, p) \, \bar{q}(\bar{z} \, p) | T\{\bar{u}(x) \, \gamma_{\mu\perp} \, (1 - \gamma_5) \, b(x), \, \bar{b}(0) \, \gamma_5 \, u(0)\} | 0 \rangle \,, \qquad (3.3)$$

where z indicates the momentum fraction carried by the collinear quark and $\bar{z} \equiv 1 - z$. Evaluating the tree diagram displayed in figure 1 leads to

$$\begin{aligned}
F_\mu^{(0)}(p, q) &= \frac{i}{2} \frac{\bar{n} \cdot q}{z(p + q)^2 + \bar{z} \, q^2 - m_b^2 + i0} \, \bar{u}(z \, p) \, \gamma_{\mu\perp} \, \slashed{n} \, (1 + \gamma_5) \, v(\bar{z} \, p) \\
&= \frac{i}{2} \frac{\bar{n} \cdot q}{z'(p + q)^2 + \bar{z}' \, q^2 - m_b^2 + i0} * \langle O_{A, \mu}(z, z') \rangle^{(0)} \,,
\end{aligned} \qquad (3.4)$$

where the convolution integral of z' is represented by an asterisk. $\langle O_{A, \mu}(z, z') \rangle^{(0)}$ indicates the partonic matrix element of the SCET operator $O_{A, \mu}$ at tree level
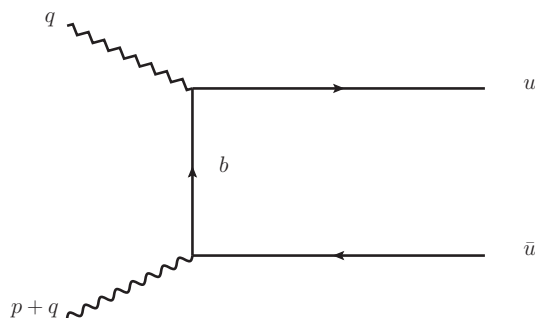
$$\langle O_{A, \mu}(z, z') \rangle = \langle q(z \, p) \, \bar{q}(\bar{z} \, p) | O_{A, \mu}(z') | 0 \rangle = \bar{\xi}(z \, p) \, \gamma_{\mu\perp} \, \slashed{n} \, (1 + \gamma_5) \, \xi(z \, p) \, \delta(z - z') + \mathcal{O}(\alpha_s), \qquad (3.5)$$

where the general definition of the collinear operator in moment space reads

$$\begin{aligned}
O_{j, \mu}(z') &= \frac{n \cdot p}{2 \, \pi} \int d\tau \, e^{-i \, z' \, \tau \, n \cdot p} \, \bar{\xi}(\tau \, n) \, W_c(\tau \, n, 0) \, \Gamma_j \, \xi(0) \,, \\
\Gamma_j &= \gamma_{\mu\perp} \, \slashed{n} \, (1 + \gamma_5) \,.
\end{aligned} \qquad (3.6)$$

The collinear Wilson line with the convention of the covariant derivative $D_\mu \equiv \partial_\mu - i \, g_s \, A_\mu$ is defined as

$$W_c(\tau n, 0) = \mathrm{P} \left\{ \mathrm{Exp} \left[i \, g_s \int_0^\tau d\lambda \, n \cdot A_c(\lambda \, n) \right] \right\} \,. \qquad (3.7)$$

Figure 1. Diagrammatical representation of the leading-order (LO) contribution to the QCD amplitude $F_\mu(p, q)$ defined in (3.3).

To establish the hard-collinear factorization for the QCD amplitude (3.3), we further decompose the SCET operator $O_{A,\mu}$ into the light-ray operators defining the photon DAs displayed in [16]

$$O_{A,\mu} = O_{1,\mu} + O_{2,\mu} + O_{E,\mu}, \tag{3.8}$$

with

$$\Gamma_1 = \gamma_{\mu\perp}\, \not{n}\,, \qquad \Gamma_2 = \frac{n^\nu}{2}\, \epsilon_{\mu\nu\alpha\beta}\, \sigma^{\alpha\beta}\,, \qquad \Gamma_E = \gamma_{\mu\perp}\, \not{n}\, \gamma_5 - \frac{n^\nu}{2}\, \epsilon_{\mu\nu\alpha\beta}\, \sigma^{\alpha\beta}\,. \tag{3.9}$$

It is evident that $O_{E,\mu}$ is an evanescent operator vanishing in four-dimensional space. Expanding the operator matching equation including the evanescent operator

$$F_\mu(p, q) = \sum_i C_i(z', (p+q)^2, q^2) * \langle O_{1,\mu}(z, z')\rangle\,, \tag{3.10}$$

at the LO in the strong coupling constant, gives rise to

$$C_1^{(0)} = C_2^{(0)} = C_E^{(0)} = \frac{i}{2}\, \frac{\bar{n} \cdot q}{z'(p+q)^2 + \bar{z}'\, q^2 - m_b^2 + i0}\,. \tag{3.11}$$

Taking advantage of the definition of the leading-twist photon DA [16]

$$\langle\gamma(p)|\bar{\xi}(x)\, W_c(x, 0)\, \sigma_{\alpha\beta}\, \xi(0)|0\rangle$$
$$= i\, g_{\rm em}\, Q_q\, \chi(\mu)\, \langle\bar{q}q\rangle(\mu)\, (p_\beta\, \epsilon_\alpha^* - p_\alpha\, \epsilon_\beta^*) \int_0^1 dz\, e^{i\, z\, p \cdot x}\, \phi_\gamma(z, \mu) + \mathcal{O}(x^2)\,. \tag{3.12}$$

we can readily derive the tree-level factorization formula for the correlation function (3.1)

$$\Pi_\mu(p, q) = \frac{i}{2}\, g_{\rm em}\, Q_u\, \chi(\mu)\, \langle\bar{q}q\rangle(\mu)\, \epsilon^{*\,\alpha}(p) \left[g_{\mu\alpha}^\perp - i\, \epsilon_{\mu\alpha\nu\beta}\, n^\nu\, v^\beta\right]$$
$$\times \int_0^1 dz\, \phi_\gamma(z, \mu)\, \frac{n \cdot p\, \bar{n} \cdot q}{z(p+q)^2 + \bar{z}\, q^2 - m_b^2 + i0} + \mathcal{O}(\alpha_s)\,. \tag{3.13}$$

Employing the definition of the B-meson decay constant in QCD

$$\langle B^-(p_B)|\bar{b}\, \gamma_5\, u|0\rangle = -i\, \frac{f_B\, m_B^2}{m_b + m_u}\,, \tag{3.14}$$

we can derive the hadronic dispersion relation of (3.1) as follows

$$\Pi_\mu(p,q) = \frac{i}{2}\, g_{\rm em}\, \frac{f_B\, m_B^2}{m_b + m_u}\, \epsilon^{*\,\alpha}(p)\, \left[g_{\mu\alpha}^\perp\, F_{A,\,\rm photon}^{\rm 2PLT}(n\cdot p) - i\,\epsilon_{\mu\alpha\nu\beta}\, n^\nu\, v^\beta\; F_{V,\,\rm photon}^{\rm 2PLT}(n\cdot p) \right]$$

$$\times\, \frac{n\cdot p}{(p+q)^2 - m_B^2 + i0} + \int_{s_0}^\infty ds\, \frac{\rho_\mu^h(s, q^2)}{s - (p+q)^2 - i0}\,, \tag{3.15}$$

where s_0 is the effective threshold of the B-meson channel. The tree-level LCSR for the $B \to \gamma\ell\nu$ form factors can be obtained by matching the factorization formula (3.13) and (3.15) with the aid of the parton-hadron duality approximation and the Borel transformation

$$\frac{f_B\, m_B}{m_b + m_u}\, F_{V,\,\rm photon}^{\rm 2PLT}(n\cdot p) = \frac{f_B\, m_B}{m_b + m_u}\, F_{A,\,\rm photon}^{\rm 2PLT}(n\cdot p)$$

$$= Q_u\, \chi(\mu)\, \langle \bar{q}q \rangle(\mu) \int_{z_0}^1 \frac{dz}{z}\, \exp\left[-\frac{m_b^2 - \bar{z}\, q^2}{z\, M^2} + \frac{m_B^2}{M^2} \right]\, \phi_\gamma(z, \mu)$$

$$+\, \mathcal{O}(\alpha_s)\,, \tag{3.16}$$

with $z_0 = (m_b^2 - q^2)/(s - q^2)$. With the power counting scheme for the threshold parameter and the Borel mass entering the sum rules (3.16)

$$\left(s_0 - m_b^2 \right) \sim M^2 \sim \mathcal{O}(m_b\,\Lambda)\,, \qquad \bar{z}_0 \sim \Lambda/m_b\,, \tag{3.17}$$

the heavy-quark scaling of the hadronic photon correction at leading twist can be established

$$F_{V,\,\rm photon}^{\rm 2PLT} \sim F_{A,\,\rm photon}^{\rm 2PLT} \sim \mathcal{O}\left(\frac{\Lambda}{m_b} \right)^{3/2}\,, \tag{3.18}$$

which is indeed suppressed by a factor of Λ/m_b compared with the direct photon contribution (see [5] for more details)

$$\left\langle \gamma(p) \left| \bar{q}_s\, \slashed{A}_{\perp(\gamma)}\, \frac{1}{i\,\bar{n}\cdot\overleftarrow{D}_s}\, \frac{\slashed{\bar{n}}}{2}\, \gamma_\mu\, (1 - \gamma_5)\, h_v \right| B^-(p_B) \right\rangle \sim \mathcal{O}\left(\frac{\Lambda}{m_b} \right)^{1/2}\,. \tag{3.19}$$
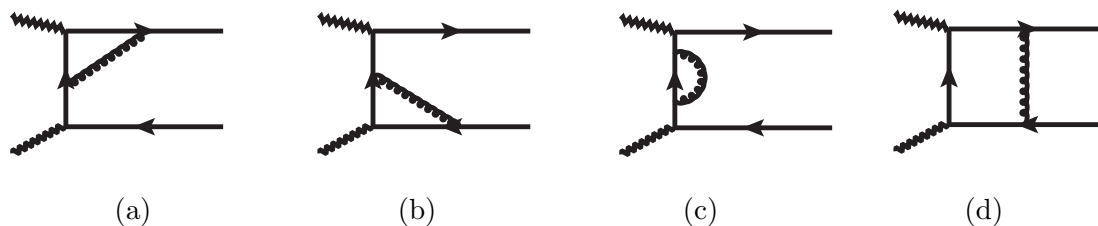
3.2 The twist-two hadronic photon correction at one loop

In this subsection we will proceed to derive the NLO sum rules for the twist-two hadronic photon correction to the $B \to \gamma$ form factors and to perform resummation of the large logarithms of m_b^2/μ^2 in the hard function at NLL accuracy. To this end, we will need to demonstrate QCD factorization for the vacuum-to-photon correlation function (3.1) at one loop, applying the technique of the light-cone OPE. For the sake of determining the NLO matching coefficients entering the factorization formulae of $\Pi_\mu(p, q)$, we will first evaluate the one-loop diagrams for the QCD matrix element $F_\mu(p, q)$ displayed in figure 2.

The one-loop QCD correction to the weak vertex diagram shown in figure 2(a) can be readily computed as

$$F_{\mu,\,\rm weak}^{(1)} = \frac{g_s^2 C_F}{z(p+q)^2 + \bar{z}q^2 - m_b^2 + i0} \int \frac{d^D l}{(2\pi)^D}\, \frac{1}{[(zp+l)^2 + i0][(zp+q+l)^2 - m_b^2 + i0][l^2 + i0]}$$

$$\bar{u}(zp)\gamma_\nu(z\,\slashed{p} + \slashed{l})\gamma_{\mu\perp}(1 - \gamma_5)(z\,\slashed{p} + \slashed{q} + \slashed{l} + m_b)\gamma^\nu(z\,\slashed{p} + \slashed{l} + m_b)\gamma_5 v(\bar{z}p)\,, \tag{3.20}$$

Figure 2. Diagrammatical representation of the NLO contribution to the QCD amplitude $F_\mu(p,q)$ defined in (3.3).

where the external partons are already taken to be on the mass-shell due to the insensitivity of the hard matching coefficients on the IR physics. With the power counting scheme specified in (3.2), one can identify the leading-power contributions of the scalar integral

$$I_1 = \int \frac{d^D l}{(2\pi)^D} \frac{1}{[(z\,p+l)^2+i0][(z\,p+q+l)^2-m_b^2+i0][l^2+i0]}, \qquad (3.21)$$

from the hard and collinear regions as expected. Applying the method of regions [20], the collinear contribution of I_1 vanishes in dimensional regularization due to the resulting scaleless integral. The collinear contribution of I_1 may not vanish with a different regularization scheme, however, it will be always cancelled by the corresponding IR subtraction term. Reducing the Dirac algebra of $F^{(1)}_{\mu,\,\mathrm{weak}}$ with the NDR scheme of the Dirac matrix γ_5 and preforming the loop-momentum integration leads to

$$
\begin{aligned}
F^{(1),h}_{\mu,\,\mathrm{weak}} = \frac{\alpha_s C_F}{4\pi} & \left\{ \left[\frac{2(1-r_2)}{r_2-r_1} \ln \frac{1-r_1}{1-r_2} - 1 \right] \left[\frac{1}{\epsilon} + \ln \frac{\mu^2}{m_b^2} - \frac{\ln[(1-r_1)(1-r_2)]}{2} - \frac{r_1-3r_2}{4(1-r_2)} + 2 \right] \right. \\
& + \frac{1}{r_1-r_2} \left[2(1-r_2) \mathrm{Li}_2 \left(1 - \frac{1-r_1}{1-r_2} \right) - \frac{2[r_1(r_1-2)+r_2]}{r_1} \ln(1-r_1) \right. \\
& \left. \left. + \frac{2[r_1(r_2-2)+r_2]}{r_2} \ln(1-r_2) + \frac{r_2}{2} \right] - \frac{r_1-3r_2}{4(1-r_2)} - 3 \right\} F^{(0)}_\mu, \qquad (3.22)
\end{aligned}
$$

where $r_1 = (z\,p+q)^2/m_b^2$ and $r_2 = q^2/m_b^2$.

Along the same vein, the one-loop QCD correction to the B-meson vertex diagram displayed in figure 2(b) can be written as

$$
\begin{aligned}
F^{(1)}_{\mu,B} = & -\frac{g_s^2 C_F}{z(p+q)^2+\bar{z}\,q^2-m_b^2+i0} \int \frac{d^D l}{(2\pi)^D} \frac{1}{[(\bar{z}p-l)^2+i0][(zp+q+l)^2-m_b^2+i0][l^2+i0]} \\
& \bar{u}(z\,p)\,\gamma_{\mu\perp}\,(1-\gamma_5)\,(z\,\slashed{p}+\slashed{q}+m_b)\,\gamma_\nu\,(z\,\slashed{p}+\slashed{q}+\slashed{l}+m_b)\,\gamma_5\,(\bar{z}\,\slashed{p}-\slashed{l})\,\gamma^\nu\,v(\bar{z}p), \qquad (3.23)
\end{aligned}
$$

which again depends on the precise prescription of γ_5 in the complex D-dimensional space. It is straightforward to verify that the leading-power contributions to the B-meson vertex diagram also arise from the hard and collinear regions. Evaluating the hard contribution

to $F_{\mu,B}^{(1)}$ with the method of regions in the NDR scheme of γ_5 yields

$$F_{\mu,B}^{(1),h} = -\frac{\alpha_s C_F}{4\pi}\left\{ 2\left[\frac{1-r_3}{r_1-r_3}\ln\frac{1-r_1}{1-r_3}-1\right]\left[\frac{1}{\epsilon}+\ln\frac{\mu^2}{m_b^2}-\frac{\ln[(1-r_1)(1-r_3)]}{2}+\frac{3r_1-r_3}{2(1-r_3)}\right]\right.$$
$$-\frac{2}{r_1-r_3}\left[(1-r_3)\,\mathrm{Li}_2\left(1-\frac{1-r_1}{1-r_3}\right)+(3r_1-r_3-1)\left(\frac{\ln(1-r_1)}{r_1}-\frac{\ln(1-r_3)}{r_3}\right)\right]$$
$$\left.-\frac{1-3r_1}{1-r_3}+3\right\} F_\mu^{(0)}, \tag{3.24}$$

with $r_3 = (p+q)^2/m_b^2$.

The self-energy correction to the intermediate bottom-quark propagator displayed in figure 2(c) can be computed as

$$F_{\mu,wfc}^{(1)} = -\frac{\alpha_s C_F}{4\pi}\left\{\frac{7-r_1}{1-r_1}\left[\frac{1}{\epsilon}+\ln\frac{\mu^2}{m_b^2}-\ln(1-r_1)+\frac{1}{2}\right]\right\} F_\mu^{(0)}. \tag{3.25}$$

Furthermore, the wave function renormalization of the external quarks will be cancelled precisely by the corresponding collinear subtraction term and hence will not contribute to the perturbative matching coefficients.

Now we turn to compute the one-loop correction to the box diagram displayed in figure 2(d)

$$F_{\mu,box}^{(1)} = -g_s^2 C_F \int \frac{d^D l}{(2\pi)^D}\frac{1}{[(zp+l)^2+i0][(zp+q+l)^2-m_b^2+i0][(\bar u p-l)^2+i0][l^2+i0]}$$
$$\bar u(zp)\gamma_\nu\,(z\,\slashed p+\slashed l)\,\gamma_{\mu\perp}(1-\gamma_5)\,(z\,\slashed p+\slashed q+\slashed l+m_b)\,\gamma_5\,(\bar z\,\slashed p-\slashed l)\,\gamma^\nu\,v(\bar z p)$$
$$= -i\,g_s^2 C_F F_\mu^{(0)}\int\frac{d^D l}{(2\pi)^D}\frac{m_b^2\,(r_1-1)}{[(zp+l)^2+i0][(zp+q+l)^2-m_b^2+i0][(\bar u p-l)^2+i0][l^2+i0]}$$
$$\times (D-4)\left\{-\frac{D-4}{D-2}\,l_\perp^2+\frac{\bar n\cdot l}{\bar n\cdot q}\left[\bar n\cdot l\,n\cdot(u p+q)+l^2\right]\right\}, \tag{3.26}$$

where the reduction of the Dirac algebra is achieved with the NDR scheme of γ_5 in the second step and $l_\perp^2 \equiv g_{\mu\nu}^\perp l^\mu l^\nu$. Performing the loop-momentum integration we find that the one-loop box diagram only contributes at $\mathcal{O}(\epsilon)$, vanishing in four dimensional space. Such observation is in analogy to the hard-collinear factorization for the hadronic photon correction to the pion-photon form factor at leading-twist accuracy [15].

Adding up different pieces together, we obtain the one-loop QCD correction to the four-point QCD matrix element as follows

$$F_\mu^{(1)}(p,q) = T_{A,\,\mathrm{hard}}^{(1)}(z',(p+q)^2,q^2) * \langle O_{A,\mu}(z,z')\rangle^{(0)} + \dots$$
$$= \sum_{i=1,2,E} T_{i,\,\mathrm{hard}}^{(1)}(z',(p+q)^2,q^2) * \langle O_{i,\mu}(z,z')\rangle^{(0)} + \dots, \tag{3.27}$$

where the explicit expression of the NLO hard amplitude is given by

$$
\begin{aligned}
T_{i,\mathrm{hard}}^{(1)}\big|_{\mathrm{NDR}} = \frac{\alpha_s\, C_F}{4\pi} \Bigg\{ &(-2) \left[\frac{1-r_2}{r_1-r_2} \ln\frac{1-r_1}{1-r_2} + \frac{1-r_3}{r_1-r_3} \ln\frac{1-r_1}{1-r_3} + \frac{3}{1-r_1} \right] \left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{m_b^2} \right) \\
&+ \frac{2\,(1-r_2)}{r_1-r_2} \mathrm{Li}_2 \left(1 - \frac{1-r_1}{1-r_2} \right) + \frac{2\,(1-r_3)}{r_1-r_3} \mathrm{Li}_2 \left(1 - \frac{1-r_1}{1-r_3} \right) \\
&+ \left(\frac{1-r_2}{r_1-r_2} + \frac{1-r_3}{r_1-r_3} \right) \ln^2(1-r_1) - \frac{1-r_2}{r_1-r_2} \ln^2(1-r_2) - \frac{1-r_3}{r_1-r_3} \ln^2(1-r_3) \\
&+ \left[\frac{2}{r_1(r_3-r_1)} + \frac{2(r_3-2)}{r_3-r_1} + \frac{6}{1-r_1} - \frac{2-r_2}{r_1-r_2} + \frac{4}{r_1} - 4 \right] \ln(1-r_1) \\
&+ \left(\frac{2-r_2}{r_1-r_2} - \frac{4}{r_2} + 2 \right) \ln(1-r_2) + \left[\frac{2}{r_3(r_1-r_3)} + \frac{2\,(r_1-2)}{r_1-r_3} - \frac{6}{r_3} \right] \ln(1-r_3) \\
&+ \frac{r_2}{2\,(r_1-r_2)} - \frac{3}{1-r_1} - \frac{15}{2} \Bigg\} C_{i,\mathrm{hard}}^{(0)},
\end{aligned} \tag{3.28}
$$

where the parameter z in the definition of r_1 should be apparently understood as z'.

We are now in a position to derive the master formulae for the hard functions $C_{1,2}(z',(p+q)^2,q^2)$ by implementing the ultraviolet (UV) renormalization and the IR subtraction. Expanding the operator matching condition (3.10) at $\mathcal{O}(\alpha_s)$ gives rise to

$$
\sum_i T_i^{(1)}(z',(p+q)^2,q^2) * \langle O_{i,\mu}(z,z') \rangle^{(0)}
$$
$$
= \sum_i \left[C_i^{(1)}(z',(p+q)^2,q^2) * \langle O_{i,\mu}(z,z') \rangle^{(0)} + C_i^{(0)}(z',(p+q)^2,q^2) * \langle O_{i,\mu}(z,z') \rangle^{(1)} \right]. \tag{3.29}
$$

The UV renormalized one-loop SCET matrix elements $\langle O_{i,\mu} \rangle^{(1)}$ can be further written as [21]

$$
\langle O_{i,\mu} \rangle^{(1)} = \sum_j \left[M_{ij,\mathrm{bare}}^{(1),R} + Z_{ij}^{(1)} \right] \langle O_{j,\mu} \rangle^{(0)}, \tag{3.30}
$$

where $M_{ij,\mathrm{bare}}^{(1),R}$ are the bare matrix elements dependent on the IR regularization scheme and $Z_{ij}^{(1)}$ are the UV renormalization constants at one loop. When both UV and IR divergences are coped with dimensional regularization, the bare SCET matrix elements vanish due to the resulting scaleless integrals from the corresponding one-loop diagrams. Comparing the coefficients of $\langle O_{i,\mu} \rangle^{(0)}$ ($i=1,2$) on both sides of (3.29) with the aid of (3.30) yields

$$
C_i^{(1)} = T_i^{(1)} - \sum_{j=1,2,E} C_j^{(0)} * Z_{ji}^{(1)}. \tag{3.31}
$$

The SCET operators $O_{1,\mu}$ and $O_{2,\mu}$ do not mix into each other, which can be verified explicitly by computing the one-loop correction to the two SCET matrix elements

$$
\langle O_{i,\mu} \rangle^{(1)} = Z_{ii}^{(1)} \langle O_{i,\mu} \rangle^{(0)}, \text{ with } i=1,2. \tag{3.32}
$$

The collinear subtraction term $Z_{ii}^{(1)} \langle O_{i,\mu} \rangle^{(0)}$ and the UV renormalization of the QCD pseudoscalar current $\bar{b}\,\gamma_5\,u$ will remove the divergent terms of the NLO QCD amplitude $T_i^{(1)}$
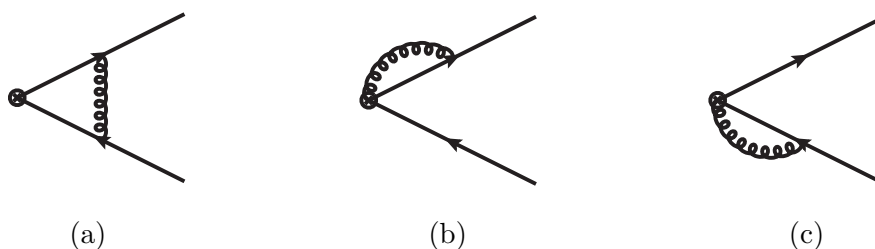
Figure 3. The one-loop diagrams for the SCET matrix element $\langle O_{E,\mu} \rangle$.

to guarantee that the perturbative matching coefficients entering the factorization formulae of the correlation function (3.1) are free of singularities and are entirely from the hard-scale dynamics of Π_μ. We further turn to determine the IR subtraction term $Z_{Ei}^{(1)}$ $(i = 1, 2)$ originated from the renormalization mixing of the evanesceneet operators $O_{E,\mu}$ into the physical SCET operators $O_{1,\mu}$ and $O_{2,\mu}$. As discussed in [21–23], the renormalization constants $Z_{Ei}^{(1)}$ $(i = 1, 2)$ will be determined by implementing the prescription that the IR finite matrix element of the evanescent operator $O_{E,\mu}$ vanishes, when applying dimensional regularization only to the UV divergences and regularizing the IR singularities with any other scheme different from the dimensions of spacetime. In accordance with (3.30) this amounts to

$$Z_{Ei}^{(1)} = -M_{Ei,\,\text{bare}}^{(1),\,\text{off}}. \tag{3.33}$$

Inserting (3.33) into (3.31) leads to the following master formula

$$C_i^{(1)} = T_i^{(1)} - C_i^{(0)} * Z_{ii}^{(1)} + C_E^{(0)} * M_{Ei,\,\text{bare}}^{(1),\,\text{off}} = T_{i,\,\text{hard}}^{(1),\,\text{reg}} + C_E^{(0)} * M_{Ei,\,\text{bare}}^{(1),\,\text{off}}, \tag{3.34}$$

where $T_{i,\,\text{hard}}^{(1),\,\text{reg}}$ is the regularized terms of the NLO hard contribution to the QCD matrix element F_μ as presented in (3.28) and $i = 1, 2$.

We proceed to compute the one-loop matrix element of the evanescent SCET operator $\langle O_{E,\mu} \rangle^{(1)}$ by evaluating the effective diagrams shown in figure 3. Employing the SCET Feynman rules, we find that only the diagram (a) with a collinear-gluon exchange between two collinear quarks could give rise to a non-trivial contribution to $M_{Ei,\,\text{bare}}^{(1),\,\text{off}}$. Evaluating this one-loop SCET diagram explicitly yields

$$\langle O_{E,\mu}(z, z') \rangle^{(1)} \supset -i\,g_s^2\,C_F \int \frac{d^D l}{(2\,\pi)^D} \frac{1}{[(z\,p + l)^2 + i0][(l - \bar{z}\,p)^2 + i0][l^2 + i0]}$$
$$\bar{u}(z\,p)\,\gamma_{\nu\perp}\,\slashed{l}_\perp\,\Gamma_3\,\slashed{l}_\perp\,\gamma_\perp^\nu\,v(z\,p)\,\delta\left(z' - z - \frac{n \cdot l}{n \cdot p}\right), \tag{3.35}$$

which only generates a non-vanishing contribution proportional to the SCET matrix element $\langle O_{E,\mu} \rangle^{(0)}$ at $\mathcal{O}(\epsilon)$ with the NDR scheme of γ_5. Explicitly, we obtain

$$C_E^{(0)} * M_{Ei,\,\text{bare}}^{(1),\,\text{off}} = 0\,, \quad \text{with } i = 1, 2\,, \tag{3.36}$$

from which the one-loop hard matching coefficients can be written as

$$C_i^{(1)} = T_{i,\,\text{hard}}^{(1),\,\text{reg}}\,. \tag{3.37}$$

Now we are ready to demonstrate the factorization-scale independence of the factorization formula for the vacuum-to-photon correlation function (3.1)

$$\Pi_\mu(p,q) = \frac{i}{2}\, g_{\rm em}\, Q_u\, \chi(\mu)\, \langle \bar{q}q \rangle(\mu)\, \epsilon^{*\,\alpha}(p) \left[g_{\mu\alpha}^\perp - i\,\epsilon_{\mu\alpha\nu\beta}\, n^\nu\, v^\beta \right] \int_0^1 dz\, \phi_\gamma(z,\mu)$$

$$\frac{n \cdot p\, \bar{n} \cdot q}{z(p+q)^2 + \bar{z}\, q^2 - m_b^2 + i0} \left[1 + \frac{C_i^{(1)}(z,(p+q)^2,q^2)}{C_i^{(0)}(z,(p+q)^2,q^2)} \right] + \mathcal{O}(\alpha_s^2). \qquad (3.38)$$

To this end, we need to make use of the evolution equation for the leading-twist photon DA

$$\mu^2 \frac{d}{d\mu^2} \left[\chi(\mu)\, \langle \bar{q}q \rangle(\mu)\, \phi_\gamma(z,\mu) \right] = \int_0^1 dz'\, \widetilde{V}(z,z') \left[\chi(\mu)\, \langle \bar{q}q \rangle(\mu)\, \phi_\gamma(z',\mu) \right], \qquad (3.39)$$

with the renormalization kernel $\widetilde{V}(z,z')$ expanded perturbatively in QCD

$$\widetilde{V}(z,z') = \sum_{n=0} \left(\frac{\alpha_s}{4\pi} \right)^{n+1} \widetilde{V}_n(z,z'), \qquad (3.40)$$

and the RG equation for the bottom-quark mass [24, 25]

$$\frac{d\, m_b(\mu)}{d\ln\mu} = -\sum_{n=0} \left(\frac{\alpha_s(\mu)}{4\pi} \right)^{n+1} \gamma_m^{(n)}, \qquad \gamma_m^{(0)} = 6\, C_F. \qquad (3.41)$$

The explicit expression of the one-loop evolution kernel $\widetilde{V}_0$ is given by [26, 27]

$$\widetilde{V}_0(z,z') = 2\, C_F \left[\frac{\bar{z}}{\bar{z}'} \frac{1}{z-z'}\, \theta(z-z') + \frac{z}{z'} \frac{1}{z'-z}\, \theta(z'-z) \right]_+ - C_F\, \delta(z-z'), \qquad (3.42)$$

where the plus function is defined as

$$\left[f(z,z') \right]_+ = f(z,z') - \delta(z-z') \int_0^1 dt\, f(t,z'). \qquad (3.43)$$

It is then straightforward to write down

$$\frac{d\Pi_\mu(p,q)}{d\ln\mu} = \frac{i}{2}\, g_{\rm em}\, Q_u\, \chi(\mu)\, \langle \bar{q}q \rangle(\mu)\, \epsilon^{*\,\alpha}(p) \left[g_{\mu\alpha}^\perp - i\,\epsilon_{\mu\alpha\nu\beta}\, n^\nu\, v^\beta \right] \int_0^1 dz\, \phi_\gamma(z,\mu)$$

$$\frac{n \cdot p\, \bar{n} \cdot q}{z(p+q)^2 + \bar{z}\, q^2 - m_b^2 + i0} \left\{ \frac{\alpha_s\, C_F}{4\pi}\, 6 + \mathcal{O}(\alpha_s^2) \right\}. \qquad (3.44)$$

The residual μ dependence at one loop arises from the UV renormalization of the pseudoscalar QCD current defining the correlation function (3.1). Distinguishing the renormalization scale μ, due to the non-conservation of the pseudoscalar current in QCD, from the factorization scale μ governing the RG evolution in SCET, we are led to conclude that the factorization formula (3.38) of $\Pi_\mu(p,q)$ is indeed independent of the scale μ at one-loop accuracy.

According to the QCD factorization formula (3.38) for the correlation function $\Pi_\mu(p,q)$, we cannot avoid the parametrically large logarithms of $\mathcal{O}(\ln(m_b^2/\Lambda^2))$ by adopting a universal scale μ in the hard matching coefficient and in the photon DA. We will perform

resummmation of the above-mentioned large logarithms at NLL accuracy by applying the two-loop RG equation of the twist-two photon DA and by setting the factorization scale as $\mu \sim m_b$. The NLO evolution kernel $\widetilde{V}_1$ in QCD can be decomposed as follows [28–30]

$$\widetilde{V}_1(z, z') = \frac{N_f}{2} C_F \widetilde{V}_N(z, z') + C_F C_A \widetilde{V}_G(z, z') + C_F^2 \widetilde{V}_F(z, z'), \tag{3.45}$$

where the explicit expressions of the evolution functions can be found in [29]. Symmetry properties of the RG evolution equation (3.39) imply the series expansion of the leading-twist photon DA in terms of the Gegenbauer polynomials

$$\phi_\gamma(z, \mu) = 6 z \bar{z} \sum_{n=0}^{\infty} a_n(\mu) C_n^{3/2}(2z - 1). \tag{3.46}$$

The two-loop evolution of the Gegenbauer moment $a_n(\mu)$ can then be obtained as follows

$$\chi(\mu) \langle \bar{q}q \rangle(\mu) a_n(\mu) = E_{T,n}^{\mathrm{NLO}}(\mu, \mu_0) \chi(\mu_0) \langle \bar{q}q \rangle(\mu_0) a_n(\mu_0)$$
$$+ \frac{\alpha_s(\mu)}{4\pi} \sum_{k=0}^{n-2} E_{T,n}^{\mathrm{LO}}(\mu, \mu_0) d_{T,n}^k(\mu, \mu_0) \chi(\mu_0) \langle \bar{q}q \rangle(\mu_0) a_n(\mu_0), \tag{3.47}$$

where $k, n = 0, 2, 4, \ldots$ and the explicit expressions of the RG functions $E_{T,n}^{\mathrm{(N)LO}}$ and the off-diagonal mixing coefficients can be found in appendix A of [15]. In contrast to the LO evolution in QCD, the Gegenbauer coefficients $a_n(\mu)$ do not renormalize multiplicatively at NLO accuracy. Inserting (3.46) and (3.47) into the NLO factorization formula (3.38) gives rise to the NLL resummation improved expression

$$\Pi_\mu(p, q) = g_{\mathrm{em}} Q_u n \cdot p \, \chi(\mu) \langle \bar{q}q \rangle(\mu) \epsilon^{*\alpha}(p) \left[g_{\mu\alpha}^\perp - i \epsilon_{\mu\alpha\nu\beta} n^\nu v^\beta \right]$$
$$\times \sum_{n=0}^{\infty} a_n(\mu) K_n((p+q)^2, q^2) + \mathcal{O}(\alpha_s^2), \tag{3.48}$$

where the perturbative function $K_n((p+q)^2, q^2)$ is determined by

$$K_n = \int_0^1 dz \left[C_i^{(0)}(z, (p+q)^2, q^2) + C_i^{(1)}(z, (p+q)^2, q^2) \right] \left[6 z \bar{z} C_n^{3/2}(2z - 1) \right]. \tag{3.49}$$

To construct the sum rules for the twist-two hadronic photon correction to the $B \to \gamma \ell \nu$ form factors, we need to derive the dispersion representation for the NLL factorization formula (3.48). Applying the spectral representations of the convolution integrals collected in appendix A, we can readily obtain

$$\Pi_\mu(p, q) = \frac{i}{2} g_{\mathrm{em}} Q_u n \cdot p \, \bar{n} \cdot q \, \chi(\mu) \langle \bar{q}q \rangle(\mu) \epsilon^{*\alpha}(p) \left[g_{\mu\alpha}^\perp - i \epsilon_{\mu\alpha\nu\beta} n^\nu v^\beta \right]$$
$$\times \int_0^\infty \frac{ds}{s - (p+q)^2 - i0} \left[\rho^{(0)}(s, q^2) + \frac{\alpha_s C_F}{4\pi} \rho^{(1)}(s, q^2) \right], \tag{3.50}$$

where the LO spectral function $\rho^{(0)}(s, q^2)$ is given by

$$\rho^{(0)}(s, q^2) = -\frac{1}{s - q^2} \phi_\gamma \left(\frac{m_b^2 - q^2}{s - q^2}, \mu \right) \theta(s - m_b^2). \tag{3.51}$$

The resulting NLO spectral function $\rho^{(1)}(s, q^2)$ is rather involved and can be written as

$$
\begin{aligned}
\rho^{(1)}(s,q^2) = (-2)\ln\left(\frac{\mu^2}{m_b^2}\right)\frac{1}{r_3-r_2}\Bigg\{ & \int_0^1 dz\Bigg[\ln\left(\frac{z\,r_3+\bar{z}\,r_2-1}{1-r_2}\right)\theta(z\,r_3+\bar{z}\,r_2-1) \\
& +\ln\left|\frac{z\,r_3+\bar{z}\,r_2-1}{r_3-1}\right|\,(\theta(z\,r_3+\bar{z}\,r_2-1)-\theta(r_3-1))\Bigg]\phi'_\gamma(z,\mu) \\
& +\int_0^1 dz\left[\frac{\theta(z\,r_3+\bar{z}\,r_2-1)}{z}-\frac{\theta(z\,r_3+\bar{z}\,r_2-1)-\theta(r_3-1)}{\bar{z}}\right]\phi_\gamma(z,\mu) \\
& +\frac{3}{r_3-r_2}\theta(r_3-1)\,\phi'_\gamma\left(\frac{1-r_2}{r_3-r_2},\mu\right)-\frac{1}{r_3-r_2}\frac{2\pi^2}{3}\phi_\gamma\left(\frac{1-r_2}{r_3-r_2},\mu\right)\theta(r_3-1)\Bigg\} \\
& +2\int_0^1 dz\,\phi_\gamma(z,\mu)\Bigg\{\left[\frac{1}{z\,r_3+\bar{z}\,r_2-1}-\frac{1}{z}\frac{1}{r_3-r_2}\right]\ln\left(1+\frac{z\,r_3+\bar{z}\,r_2-1}{1-r_2}\right) \\
& \times\theta(z\,r_3+\bar{z}\,r_2-1)-\left[\frac{1}{z\,r_3+\bar{z}\,r_2-1}+\frac{1}{\bar{z}}\frac{1}{r_3-r_2}\right]\ln\left(1+\frac{z\,r_3+\bar{z}\,r_2-1}{1-r_3}\right) \\
& \times\theta\left(\frac{z\,r_3+\bar{z}\,r_2-1}{1-r_3}\right)\Bigg\}+\int_0^1 dz\,\frac{2}{r_3-r_2}\left[\ln^2(z\,r_3+\bar{z}\,r_2-1)-\frac{\pi^2}{3}\right] \\
& \times\theta(z\,r_3+\bar{z}\,r_2-1)\,\phi'_\gamma(z,\mu)+\int_0^1 dz\,\phi_\gamma(z,\mu)\left(\frac{1}{z}-\frac{1}{\bar{z}}\right) \\
& \times\left[\frac{2}{r_3-r_2}\ln(z\,r_3+\bar{z}\,r_2-1)\theta(z\,r_3+\bar{z}\,r_2-1)+\delta(r_3-r_2)\ln^2(1-r_2)\right] \\
& +\frac{1}{r_3-r_2}\phi_\gamma\left(\frac{1-r_2}{r_3-r_2},\mu\right)\theta(r_3-1)\left[\ln^2(1-r_2)+\ln^2(r_3-1)-\pi^2\right] \\
& +2\int_0^1 dz\,\phi_\gamma(z,\mu)\left[\mathcal{P}\frac{1}{z\,r_3+\bar{z}\,r_2-1}-\frac{1}{\bar{z}}\frac{1}{r_2-r_3}\right]\ln(r_3-1)\theta(r_3-1) \\
& +\frac{\theta(r_3-1)}{r_3-r_2}\left[\frac{4}{r_3}\ln(r_3-1)+\frac{(2-r_2)(8-3r_2)}{r_2(4-r_2)}\ln(1-r_2)\right]\phi_\gamma\left(\frac{1-r_2}{r_3-r_2},\mu\right) \\
& +3\frac{\theta(r_3-1)}{(r_3-r_2)^2}\left[\phi'_\gamma\left(\frac{1-r_2}{r_3-r_2},\mu\right)+2\ln(r_3-1)\phi'_\gamma(z=1,\mu)\right] \\
& -\int_0^1 dz\,\ln(z\,r_3+\bar{z}\,r_2-1)\frac{\theta(z\,r_3+\bar{z}\,r_2-1)}{r_3-r_2}\left[\frac{6}{r_3-r_2}\frac{d^2}{dz^2}+\frac{r_2}{1-r_2}\frac{d}{dz}\right]\phi_\gamma(z,\mu) \\
& -\theta(r_3-1)\int_0^1 dz\,\phi_\gamma(z,\mu)\Bigg\{\theta\left(z-\frac{1-r_2}{r_3-r_2}\right)\left[\frac{r_2(r_2-2)(1+z)+2\,z}{r_2(1-r_2)z\bar{z}}\frac{1}{r_2-r_3}\right. \\
& +\frac{2(z-2\bar{z}r_2)}{\bar{z}r_2}\frac{1}{z\,r_3+\bar{z}\,r_2}\Bigg]+\frac{2(1-r_2)}{\bar{z}r_2}\frac{1}{r_3-r_2}-\frac{2(1-3\bar{z}r_2)}{\bar{z}r_2(1-\bar{z}r_2)}\frac{1}{r_3} \\
& -\frac{4z}{1-\bar{z}r_2}\mathcal{P}\frac{1}{z\,r_3+\bar{z}\,r_2-1}\Bigg\}+\frac{r_2}{1-r_2}\delta(r_2-r_3)\int_0^1 dz\,\frac{\phi_\gamma(z,\mu)}{z} \\
& -\frac{1}{r_3-r_2}\frac{16\,r_2-15}{2(1-r_2)}\phi_\gamma\left(\frac{1-r_2}{r_3-r_2},\mu\right)\theta(r_3-1),
\end{aligned}
\tag{3.52}
$$

where $\mathcal{P}$ indicates the principle-value prescription. Finally, the NLL sum rules for the hadronic photon correction to the $B \to \gamma$ form factors at leading twist can be written as

$$\frac{f_B \, m_B}{m_b + m_u} \, F_{V,\,\mathrm{photon}}^{2\mathrm{PLT}}(n \cdot p) = \frac{f_B \, m_B}{m_b + m_u} \, F_{A,\,\mathrm{photon}}^{2\mathrm{PLT}}(n \cdot p)$$

$$= -Q_u \, \chi(\mu) \, \langle \bar{q}q \rangle(\mu) \, \bar{n} \cdot q \int_0^{s_0} ds \exp\left[-\frac{s - m_B^2}{M^2}\right]$$

$$\times \left[\rho^{(0)}(s, q^2) + \frac{\alpha_s \, C_F}{4\,\pi} \, \rho^{(1)}(s, q^2)\right] + \mathcal{O}(\alpha_s^2). \qquad (3.53)$$

It is evident that the twist-two hadronic photon correction preserves the symmetry relation of the two form factors F_V and F_A at leading power in Λ/m_b.

4 Higher-twist hadronic photon corrections in QCD

In this section we will aim at computing the higher-twist hadronic photon corrections to the $B \to \gamma\ell\nu$ decay form factors at LO in α_s, up to the twist-four accuracy, from the LCSR approach. Following the discussion on a general classification of the photon DAs [16], we will need to take into account the subleading-power contributions arising from the light-cone matrix elements of both the two-body and three-body collinear operators. To achieve this goal, we first demonstrate QCD factorization for the two-particle and three-particle higher-twist contributions to the vacuum-to-photon correlation function (3.1) and then construct the tree-level sum rules for the form factors F_V and F_A following the standard strategy.

4.1 Higher-twist two-particle corrections

Employing the light-cone expansion of the bottom-quark propagator and keeping the subleading-power contributions to the correlation function (3.1) leads to

$$\Pi_\mu(p, q) \supset \int \frac{d^4 k}{(2\pi)^4} \int d^4 x \, e^{i\,(q-k)\cdot x} \, \frac{k^\nu}{k^2 - m_b^2} \, \langle \gamma(p)| \bar{u}(x) \, \sigma_{\mu\nu} \, (1 + \gamma_5) \, u(0)|0\rangle$$

$$- i \int \frac{d^4 k}{(2\pi)^4} \int d^4 x \, e^{i\,(q-k)\cdot x} \, \frac{m_b}{k^2 - m_b^2} \, \langle \gamma(p)| \bar{u}(x) \, \gamma_\mu \, (1 - \gamma_5) \, u(0)|0\rangle. \qquad (4.1)$$

Making use of the definitions of the higher-twist photon DAs displayed in appendix B, it is straightforward to write down

$$\Pi_\mu(p, q) \supset \frac{i}{4} \, g_{\mathrm{em}} \, Q_q \, (p \cdot q) \int_0^1 dz \left\{ \epsilon_\mu^* \left[\frac{\rho_{A,2}^{2\mathrm{PHT}}((p+q)^2, q^2, z)}{[(z\,p+q)^2 - m_b^2 + i0]^2} + \frac{\rho_{A,3}^{2\mathrm{PHT}}((p+q)^2, q^2, z)}{[(z\,p+q)^2 - m_b^2 + i0]^3} \right] \right.$$

$$\left. - i \varepsilon_{\mu\nu\alpha\beta} \, \epsilon^{*\,\nu} \, n^\alpha \, v^\beta \left[\frac{\rho_{V,2}^{2\mathrm{PHT}}((p+q)^2, q^2, z)}{[(z\,p+q)^2 - m_b^2 + i0]^2} + \frac{\rho_{V,3}^{2\mathrm{PHT}}((p+q)^2, q^2, z)}{[(z\,p+q)^2 - m_b^2 + i0]^3} \right] \right\}, \qquad (4.2)$$

where the explicit expressions of the invariant functions $\rho_{V(A),\,i}^{2\mathrm{PHT}}$ ($i = 2, 3$) are given by

$$\rho_{V,2}^{2\mathrm{PHT}}((p+q)^2, q^2, z) = 2\,m_b \, f_{3\gamma}(\mu) \, \psi^{(a)}(z, \mu) - \langle \bar{q}q \rangle(\mu) \, \mathbb{A}(z, \mu),$$

$$\rho_{V,3}^{2\mathrm{PHT}}((p+q)^2, q^2, z) = -2\,m_b^2 \, \langle \bar{q}q \rangle(\mu) \, \mathbb{A}(z, \mu),$$

$$\rho_{A,2}^{2\mathrm{PHT}}((p+q)^2, q^2, z) = 4\,m_b \, f_{3\gamma}(\mu) \, \bar{\psi}^{(v)}(z, \mu) + \left[\mathbb{A}(z, \mu) - 2\,\bar{h}_\gamma(z, \mu)\right] \langle \bar{q}q \rangle(\mu),$$

$$\rho_{A,3}^{2\mathrm{PHT}}((p+q)^2, q^2, z) = -2\,m_b^2 \, \langle \bar{q}q \rangle(\mu) \, \left[\mathbb{A}(z, \mu) - 2\,\bar{h}_\gamma(z, \mu)\right]. \qquad (4.3)$$

The two new functions $\bar{\psi}^{(v)}(z,\mu)$ and $\bar{h}_\gamma(z,\mu)$ introduced in (4.3) are defined by

$$\bar{\psi}^{(v)}(z,\mu) = 2 \int_0^z d\alpha\, \psi^{(v)}(\alpha,\mu)\,, \qquad \bar{h}_\gamma(z,\mu) = -4 \int_0^z d\alpha(z-\alpha)\, h_\gamma(\alpha,\mu)\,. \qquad (4.4)$$

The resulting LCSR for the two-particle higher-twist hadronic photon corrections to the $B \to \gamma\ell\nu$ form factors can be further derived as follows

$$-\frac{f_B\, m_B}{m_b + m_u} \exp\left(-\frac{m_B^2}{M^2}\right)\, F_{V(A),\,\mathrm{photon}}^{\mathrm{2PHT,\,LL}}(n \cdot p)$$

$$= \frac{Q_q}{4} \left\{ \frac{1}{m_b^2 - q^2} \exp\left(-\frac{s_0}{M^2}\right)\, \rho_{V(A),2}^{\mathrm{2PHT}}\left(s_0, q^2, z = f_1(s_0, q^2)\right)\right.$$

$$+ \int_{m_b^2}^{s_0} \frac{ds}{m_b^2 - q^2}\, \frac{1}{M^2}\, \exp\left(-\frac{s}{M^2}\right)\, \rho_{V(A),2}^{\mathrm{2PHT}}\left(s, q^2, z = f_1(s, q^2)\right)$$

$$+ \frac{1}{(s_0 - q^2)^2} \exp\left(-\frac{s_0}{M^2}\right) \frac{d}{dz}\left[\frac{1}{2\,z}\, \rho_{V(A),3}^{\mathrm{2PHT}}(s_0, q^2, z)\right]\bigg|_{z=f_1(s_0,q^2)}$$

$$+ \frac{s_0 - q^2}{2\,(m_b^2 - q^2)^2} \frac{d}{ds_0}\left[\exp\left(-\frac{s_0}{M^2}\right)\, \rho_{V(A),3}^{\mathrm{2PHT}}(s_0, q^2, z)\right]\bigg|_{z=f_1(s_0,q^2)}$$

$$\left.- \int_{m_b^2}^{s_0} ds\, \frac{s - q^2}{2\,(m_b^2 - q^2)^2} \frac{d^2}{ds^2}\left[\exp\left(-\frac{s}{M^2}\right)\, \rho_{V(A),3}^{\mathrm{2PHT}}(s, q^2, z)\right]\bigg|_{z=f_1(s,q^2)}\right\}\,, \qquad (4.5)$$

where we have defined $f_1(s, q^2) = (m_b^2 - q^2)/(s - q^2)$ to compactify the above expressions.

Several comments on the tree-level sum rules for the higher-twist corrections to the form factors F_V and F_A presented in (4.5) are in order.

- It is evident from (4.3) that the higher-twist two-particle hadronic photon corrections can lead to the symmetry-breaking contributions to the $B \to \gamma\ell\nu$ form factors already at tree level, in agreement with the observation made in [12]. However, it needs to be pointed out that the subleading-twist effects do not always violate the symmetry relation of the two $B \to \gamma$ form factors at leading power in Λ/m_b [8].

- Applying the power-counting scheme for the threshold parameter (3.17) and the end-point behaviours of the two-particle photon DAs $\bar{\psi}^{(v)}(z,\mu)$, $\psi^{(a)}(z,\mu)$, $\mathbb{A}(z,\mu)$ and $\bar{h}_\gamma(z,\mu)$, we can readily identify the heavy-quark scaling for the two-particle higher-twist corrections

$$F_{V,\,\mathrm{photon}}^{\mathrm{2PHT,\,LL}}(n \cdot p) \sim F_{A,\,\mathrm{photon}}^{\mathrm{2PHT,\,LL}}(n \cdot p) \sim \left(\frac{\Lambda}{m_b}\right)^{3/2}\,, \qquad (4.6)$$

which is of the same power as the twist-two hadronic photon contribution obtained in (3.18) and is suppressed by only one factor of Λ/m_b compared with the "direct" photon contribution (3.19). We are then led to conclude that there is generally no correspondence between the heavy-quark expansion and the twist expansion for the $B \to \gamma\ell\nu$ form factors in the LCSR approach (see [31] for a discussion in the context of the pion-photon form factor).
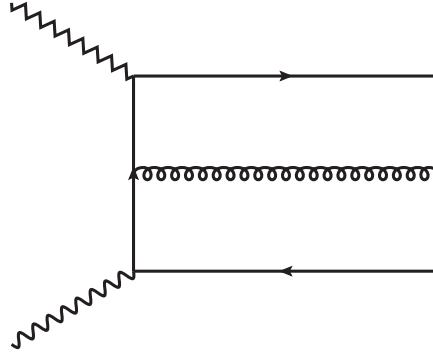
Figure 4. Diagrammatical representation of the three-particle contribution to the QCD amplitude $F_\mu(p,q)$ defined in (3.3) at tree level.

4.2 Higher-twist three-particle corrections

We will proceed to compute the higher-twist three-particle hadronic photon corrections to the $B \to \gamma \ell \nu$ form factors at tree level with the sum rule technique. Following the standard strategy, we first compute the three-particle contribution to the four-point QCD amplitude $F_\mu(p,q)$ (3.3) displayed in figure 4. Keeping the one-gluon part for the light-cone expansion of the bottom-quark propagator in the background gluon/photon field [32, 33]

$$\langle 0|T\{\bar{b}(x),b(0)\}|0\rangle$$
$$\supset ig_s \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \int_0^1 dv \left[\frac{vx_\mu}{k^2-m_b^2} G^{\mu\nu}(vx)\gamma_\nu - \frac{\not{k}+m_b}{2(k^2-m_b^2)^2} G^{\mu\nu}(vx)\sigma_{\mu\nu} \right]$$
$$+ig_{\mathrm{em}}Q_b \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \int_0^1 dv \left[\frac{vx_\mu}{k^2-m_b^2} F^{\mu\nu}(vx)\gamma_\nu - \frac{\not{k}+m_b}{2(k^2-m_b^2)^2} F^{\mu\nu}(vx)\sigma_{\mu\nu} \right], \quad (4.7)$$

and employing the definitions of the three-particle photon DAs in appendix B, we obtain

$$\Pi_\mu(p,q) \supset i\,g_{\mathrm{em}}\,Q_q\,(p{\cdot}q) \int_0^1 dv \int [\mathcal{D}\alpha_i] \left\{ \epsilon^*_\mu \left[\frac{\rho^{\mathrm{3P}}_{A,2}((p+q)^2,q^2,\alpha_i,v)}{[((\alpha_q+v\,\alpha_g)\,p+q)^2-m_b^2+i\,0]^2} \right. \right.$$
$$+ \frac{\rho^{\mathrm{3P}}_{A,3}((p+q)^2,q^2,\alpha_i,v)}{[((\alpha_q+v\,\alpha_g)\,p+q)^2-m_b^2+i\,0]^3} \right] - i\,\varepsilon_{\mu\nu\alpha\beta}\,\epsilon^{*\nu}\,n^\alpha\,v^\beta$$
$$\times \left. \left[\frac{\rho^{\mathrm{3P}}_{V,2}((p+q)^2,q^2,\alpha_i,v)}{[((\alpha_q+v\,\alpha_g)\,p+q)^2-m_b^2+i\,0]^2} + \frac{\rho^{\mathrm{3P}}_{V,3}((p+q)^2,q^2,\alpha_i,v)}{[((\alpha_q+v\,\alpha_g)\,p+q)^2-m_b^2+i\,0]^3} \right] \right\}, \quad (4.8)$$

where the integration measure is defined as

$$\int [\mathcal{D}\alpha_i] \equiv \int_0^1 d\alpha_q \int_0^1 d\alpha_{\bar{q}} \int_0^1 d\alpha_g\, \delta\left(1-\alpha_q-\alpha_{\bar{q}}-\alpha_g\right). \quad (4.9)$$

The resulting expressions for the invariant functions $\rho_{V(A),i}^{\rm 3P}$ $(i=2,\,3)$ are given by

$$\rho_{V,2}^{\rm 3P} = -\left[S(\alpha_i,\mu) + S_\gamma(\alpha_i,\mu)\right] + (1-2\,v)\,\widetilde{S}(\alpha_i,\mu) - 2\,v\,\left[T_1(\alpha_i,\mu) - T_2(\alpha_i,\mu)\right]\,,$$
$$\rho_{V,3}^{\rm 3P} = 0\,,$$
$$\rho_{A,2}^{\rm 3P} = -(1-2\,v)\,\left[S(\alpha_i,\mu) + S_\gamma(\alpha_i,\mu)\right] + \widetilde{S}(\alpha_i,\mu) + T_1(\alpha_i,\mu) + (1-2\,v)\,T_2(\alpha_i,\mu)\,,$$
$$\rho_{A,3}^{\rm 3P} = 2\left[(p+q)^2 - q^2\right]\left\{(2\,v-1)\,\bar{T}_3(\alpha_i,\mu) - \left[\bar{T}_4(\alpha_i,\mu) + \bar{T}_4^\gamma(\alpha_i,\mu)\right]\right\}\,, \qquad (4.10)$$

where we have introduced the following notations

$$\bar{T}_{3(4)}(\alpha_i,\mu) = \int_0^{\alpha_q} d\alpha_q'\, T_{3(4)}(\alpha_q',\alpha_{\bar{q}},\alpha_g,\mu)\,, \quad \bar{T}_4^\gamma(\alpha_i,\mu) = \int_0^{\alpha_q} d\alpha_q'\, T_4^\gamma(\alpha_q',\alpha_{\bar{q}},\alpha_g,\mu)\,. \tag{4.11}$$

Implementing the continuum subtraction with the aid of the parton-hadron duality relation and preforming the Borel transformation in the variable $(p+q)^2 \to s$ gives rise to the desired sum rules for the three-particle hadronic photon corrections at tree level

$$-\frac{f_B\, m_B}{m_b+m_u} \exp\left(-\frac{m_B^2}{M^2}\right) F_{V(A),\,\rm photon}^{\rm 3P,LL}(n\cdot p)$$

$$= Q_q\,\langle\bar{q}q\rangle(\mu)\left\{ \int_0^{f_1(s_0,q^2)} d\alpha_q \int_{f_2(\alpha_q,s_0,q^2)}^{1-\alpha_q} \frac{d\alpha_g}{\alpha_g} \frac{\theta(1-\alpha_q-f_2(\alpha_q,s_0,q^2))}{m_b^2-q^2} \exp\left(-\frac{s_0}{M^2}\right)\right.$$

$$\times \rho_{V(A),2}^{\rm 3P}\left(s_0,q^2,\alpha_q,\alpha_{\bar{q}},\alpha_g,v=\frac{f_2(\alpha_q,s_0,q^2)}{\alpha_g}\right) + \int_{m_b^2}^{s_0} ds \int_0^{f_1(s,q^2)} d\alpha_q \int_{f_2(\alpha_q,s,q^2)}^{1-\alpha_q} \frac{d\alpha_g}{\alpha_g}$$

$$\times \frac{\theta(1-\alpha_q-f_2(\alpha_q,s,q^2))}{m_b^2-q^2} \frac{1}{M^2} \exp\left(-\frac{s}{M^2}\right) \rho_{V(A),2}^{\rm 3P}\left(s,q^2,\alpha_q,\alpha_{\bar{q}},\alpha_g,v=\frac{f_2(\alpha_q,s,q^2)}{\alpha_g}\right)$$

$$-\int_0^{f_1(s_0,q^2)} \frac{d\alpha_g}{\alpha_g} \frac{\exp\left(-s_0/M^2\right)}{2\,(m_b^2-q^2)(s_0-q^2)} \rho_{V(A),3}^{\rm 3P}\left(s_0,q^2,\alpha_q=f_1(s_0,q^2)-\alpha_g,\alpha_{\bar{q}},\alpha_g,v=1\right)$$

$$+\int_0^{1-f_1(s_0,q^2)} \frac{d\alpha_g}{\alpha_g} \frac{\exp\left(-s_0/M^2\right)}{2\,(m_b^2-q^2)(s_0-q^2)} \rho_{V(A),3}^{\rm 3P}\left(s_0,q^2,\alpha_q=1-f_1(s_0,q^2),\alpha_{\bar{q}},\alpha_g,v=0\right)$$

$$+\int_0^{f_1(s_0,q^2)} d\alpha_q \int_{f_2(\alpha_q,s_0,q^2)}^{1-\alpha_q} \frac{d\alpha_g}{\alpha_g^2} \frac{\theta(1-\alpha_q-f_2(\alpha_q,s_0,q^2))}{(s_0-q^2)^2} \exp\left(-\frac{s_0}{M^2}\right)$$

$$\times \frac{d}{dv}\left[\frac{1}{2\,(\alpha_q+v\,\alpha_g)} \rho_{V(A),3}^{\rm 3P}\left(s_0,q^2,\alpha_q,\alpha_{\bar{q}},\alpha_g,v\right)\right]\Bigg|_{v=f_2(\alpha_q,s_0,q^2)/\alpha_g}$$

$$+\int_0^{f_1(s_0,q^2)} d\alpha_q \int_{f_2(\alpha_q,s_0,q^2)}^{1-\alpha_q} \frac{d\alpha_g}{\alpha_g} \frac{s_0-q^2}{2\,(m_b^2-q^2)} \theta(1-\alpha_q-f_2(\alpha_q,s_0,q^2))$$

$$\times \frac{d}{ds_0}\left[\exp\left(-\frac{s_0}{M^2}\right) \rho_{V(A),3}^{\rm 3P}\left(s_0,q^2,\alpha_q,\alpha_{\bar{q}},\alpha_g,v\right)\right]\Bigg|_{v=f_2(\alpha_q,s_0,q^2)/\alpha_g}$$

$$-\int_{m_b^2}^{s_0} ds \int_0^{f_1(s,q^2)} d\alpha_q \int_{f_2(\alpha_q,s,q^2)}^{1-\alpha_q} \frac{d\alpha_g}{\alpha_g} \frac{s-q^2}{2\,(m_b^2-q^2)} \theta(1-\alpha_q-f_2(\alpha_q,s,q^2))$$

$$\left.\times \frac{d^2}{ds^2}\left[\exp\left(-\frac{s}{M^2}\right) \rho_{V(A),3}^{\rm 3P}\left(s,q^2,\alpha_q,\alpha_{\bar{q}},\alpha_g,v\right)\right]\Bigg|_{v=f_2(\alpha_q,s,q^2)/\alpha_g}\right\}\,, \qquad (4.12)$$

where for brevity we have introduced the auxiliary function $f_2(\alpha_q, s, q^2)$ defined by

$$f_2(\alpha_q, s, q^2) = \frac{m_b^2 - q^2}{s - q^2} - \alpha_q \,. \qquad (4.13)$$

In accordance with the power counting scheme for the threshold parameter and the end-point behaviours of the three-particle photon DAs entering the sum rules (4.12), we can deduce the heavy-quark scaling of the three-particle hadronic photon corrections

$$F_{V,\,\text{photon}}^{\text{3P, LL}}(n \cdot p) \sim F_{A,\,\text{photon}}^{\text{3P, LL}}(n \cdot p) \sim \left(\frac{\Lambda}{m_b} \right)^{5/2} \,, \qquad (4.14)$$

which is suppressed by one factor of Λ/m_b compared with the higher-twist two-particle contributions to the $B \to \gamma \ell \nu$ form factors at tree level as presented in (4.5). It remains interesting to verify whether the NLO QCD corrections to the three-particle hadronic photon contributions can give rise to a dynamically enhancement to remove the power-suppression mechanism of the LO contributions (see [34, 35] for a discussion in the context of the NLO sum rules for the $B \to \pi$ form factors) and we will leave explicit QCD calculations of the yet higher-order corrections for future work.

Collecting the different pieces together, the resulting expressions for the $B \to \gamma \ell \nu$ form factors including the subleading-power contributions from the tree-level $b \, \bar{u} \to \gamma \, W^*$ amplitude in QCD and from the hadronic photon corrections can be written as

$$
\begin{aligned}
F_V(n \cdot p) &= F_{V,\,\text{LP}}(n \cdot p) + F_{V,\,\text{NLP}}^{\text{LC}}(n \cdot p) + F_{V,\,\text{photon}}^{\text{2PLT}}(n \cdot p) + F_{V,\,\text{photon}}^{\text{2PHT, LL}}(n \cdot p) + F_{V,\,\text{photon}}^{\text{3P, LL}}(n \cdot p)\,, \\
F_A(n \cdot p) &= F_{A,\,\text{LP}}(n \cdot p) + F_{A,\,\text{NLP}}^{\text{LC}}(n \cdot p) + F_{A,\,\text{photon}}^{\text{2PLT}}(n \cdot p) + F_{A,\,\text{photon}}^{\text{2PHT, LL}}(n \cdot p) + F_{A,\,\text{photon}}^{\text{3P, LL}}(n \cdot p) \\
&\quad + \frac{Q_\ell \, f_B}{v \cdot p}\,,
\end{aligned} \qquad (4.15)
$$

where the last term proportional to the electric charge of the lepton comes from the redefinition of the axial-vector form factor as discussed in section 2. The detailed expressions of the individual terms displayed on the right-hand side of (4.15) are given by (2.7), (2.8), (3.53), (4.5) and (4.12), respectively. We mention in passing that the LCSR calculations of the hadronic photon corrections to the $B \to \gamma \ell \nu$ decay form factors presented here suffer from the systematic uncertainty due to the parton-hadron duality ansatz in the B-meson channel. Future development of the subleading-power contributions to the radiative leptonic B-meson decays in the framework of SCET including a proper treatment of the rapidity divergences will be in demand for a model-independent QCD analysis.

Several comments on the general structure of the $B \to \gamma \ell \nu$ form factors (4.15) are in order.

- Both the point-like (short-distance) and the hadronic (long-distance) photon contributions to the $B \to \gamma \ell \nu$ amplitude were computed with same correlation function (3.1) in [12] (see also [9, 10]), where the leading-power point-like photon contribution was represented by the triangle quark diagrams. By contrast, we apply the QCD factorization approach for the evaluation of the short-distance photon effect and employ the LCSR method with the photon DAs for the computation of the

subleading power hadronic photon corrections. Since both the short-distance and long-distance photon contributions can be defined by hadronic matrix elements of the corresponding effective operators in SCET [5], we are allowed to compute the different hadronic matrix elements contributing to the $B \to \gamma \ell \nu$ amplitude with the aid of distinct QCD techniques, and such "hybrid" computation scheme as implemented in this work is free of the double-counting issue. However, it needs to be pointed out that an additional source of the systematic uncertainty could be introduced in the "hybrid" approach, due to the scheme dependence of separating the leading and sub-leading power contributions.

- The subleading power contributions to the $B \to \gamma \ell \nu$ transition form factors were also computed from the dispersion approach [7, 8, 36] by investigating the $B \to \gamma^* \ell \nu$ amplitude with the parton-hadron duality ansatz. In particular, the higher-twist corrections to the form factors due to the higher Fock states of the B-meson and to the transverse momentum of the light-quark in the valence state were computed at tree level [36], where the soft contributions from the twist-five and -six B-meson DAs were also estimated in the factorization approximation.

5 Numerical analysis

We are now ready to explore the phenomenological implications of the hadronic photon corrections to the $B \to \gamma \ell \nu$ amplitude computed from the LCSR approach. To this end, we will proceed by specifying the nonperturbative models of the two-particle and three-particle photon DAs, the first inverse moment $\lambda_B(\mu)$ and the logarithmic moments $\sigma_1(\mu)$ and $\sigma_2(\mu)$ of the leading-twist B-meson DA, and by determining the Borel mass and the hadronic threshold parameter entering the sum rules for the subleading-power resolved photon contributions. Having at our disposal the theory predictions for the form factors F_V and F_A, we will further explore the opportunity of constraining the inverse moment $\lambda_B(\mu)$ taking advantage of the improved measurements at the Belle II experiment in the near future.

5.1 Theory inputs

In analogy to the leading-twist photon DA, we employ the conformal expansion for the twist-three DAs defined by the chiral-even light-cone matrix elements

$$
\begin{aligned}
\psi^{(v)}(z, \mu) &= 5 \left(3 \xi^2 - 1\right) + \frac{3}{64} \left(15 \omega_\gamma^V(\mu) - 5 \omega_\gamma^A(\mu)\right) \left(3 - 30 \xi^2 + 35 \xi^4\right), \\
\psi^{(a)}(z, \mu) &= \frac{5}{2} \left(1 - \xi^2\right) \left(5 \xi^2 - 1\right) \left(1 + \frac{9}{16} \omega_\gamma^V(\mu) - \frac{3}{16} \omega_\gamma^A(\mu)\right), \\
V(\alpha_i, \mu) &= 540 \, \omega_\gamma^V(\mu) \left(\alpha_q - \alpha_{\bar{q}}\right) \alpha_q \, \alpha_{\bar{q}} \, \alpha_g^2, \\
A(\alpha_i, \mu) &= 360 \, \alpha_q \, \alpha_{\bar{q}} \, \alpha_g^2 \left[1 + \frac{\omega_\gamma^A(\mu)}{2} \left(7 \alpha_g - 3\right)\right],
\end{aligned}
\tag{5.1}
$$

with $\xi = 2z - 1$, and for the chiral-odd twist-four DAs

$$
\begin{aligned}
\mathbb{A}(z,\mu) =\ & 40\, z^2\, \bar{z}^2 \left[3\,\kappa(\mu) - \kappa^+(\mu) + 1 \right] + 8 \left[\zeta_2^+(\mu) - 3\,\zeta_2(\mu) \right] \left[z\,\bar{z}\,(2 + 13\, z\,\bar{z}) \right. \\
& \left. + 2\, z^3\,(10 - 15\, z + 6\, z^2)\,\ln z + 2\,\bar{z}^3\,(10 - 15\,\bar{z} + 6\,\bar{z}^2)\,\ln\bar{z} \right], \\
h_\gamma(z,\mu) =\ & -10 \left(1 + 2\,\kappa^+(\mu) \right) C_2^{1/2}(2\,z - 1), \\
S(\alpha_i,\mu) =\ & 30\,\alpha_g^2 \bigg\{ \left(\kappa(\mu) + \kappa^+(\mu) \right) (1 - \alpha_g) + (\zeta_1 + \zeta_1^+)(1 - \alpha_g)(1 - 2\,\alpha_g) \\
& + \zeta_2(\mu) \left[3\,(\alpha_{\bar{q}} - \alpha_q)^2 - \alpha_g\,(1 - \alpha_g) \right] \bigg\}, \\
\widetilde{S}(\alpha_i,\mu) =\ & -30\,\alpha_g^2 \bigg\{ \left(\kappa(\mu) - \kappa^+(\mu) \right) (1 - \alpha_g) + (\zeta_1 - \zeta_1^+)(1 - \alpha_g)(1 - 2\,\alpha_g) \\
& + \zeta_2(\mu) \left[3\,(\alpha_{\bar{q}} - \alpha_q)^2 - \alpha_g\,(1 - \alpha_g) \right] \bigg\}, \\
S_\gamma(\alpha_i,\mu) =\ & 60\,\alpha_g^2\,(\alpha_q + \alpha_{\bar{q}}) \left[4 - 7\,(\alpha_{\bar{q}} + \alpha_q) \right], \\
T_1(\alpha_i,\mu) =\ & -120 \left(3\,\zeta_2(\mu) + \zeta_2^+(\mu) \right) (\alpha_{\bar{q}} - \alpha_q)\,\alpha_{\bar{q}}\,\alpha_q\,\alpha_g, \\
T_2(\alpha_i,\mu) =\ & 30\,\alpha_g^2\,(\alpha_{\bar{q}} - \alpha_q) \left[\left(\kappa(\mu) - \kappa^+(\mu) \right) + \left(\zeta_1(\mu) - \zeta_1^+(\mu) \right) (1 - 2\,\alpha_g) + \zeta_2(\mu)\,(3 - 4\,\alpha_g) \right], \\
T_3(\alpha_i,\mu) =\ & -120 \left(3\,\zeta_2(\mu) - \zeta_2^+(\mu) \right) (\alpha_{\bar{q}} - \alpha_q)\,\alpha_{\bar{q}}\,\alpha_q\,\alpha_g, \\
T_4(\alpha_i,\mu) =\ & 30\,\alpha_g^2\,(\alpha_{\bar{q}} - \alpha_q) \left[\left(\kappa(\mu) + \kappa^+(\mu) \right) + \left(\zeta_1(\mu) + \zeta_1^+(\mu) \right) (1 - 2\,\alpha_g) + \zeta_2(\mu)\,(3 - 4\,\alpha_g) \right], \\
T_4^\gamma(\alpha_i,\mu) =\ & 60\,\alpha_g^2\,(\alpha_q - \alpha_{\bar{q}}) \left[4 - 7\,(\alpha_{\bar{q}} + \alpha_q) \right],
\end{aligned}
\tag{5.2}
$$

defined by the chiral-odd light-ray matrix elements. Here, we have truncated the conformal expansion of the photon light-cone DAs up to the next-to-leading conformal spin (i.e., "P"-wave). The renormalization-scale dependence of the twist-three parameters can be written as

$$
f_{3\gamma}(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right)^{\gamma_f/\beta_0} f_{3\gamma}(\mu_0), \qquad \gamma_f = -\frac{C_F}{3} + 3\,C_A, \qquad \beta_0 = 11 - \frac{2\,n_f}{3},
$$

$$
\begin{pmatrix} \omega_\gamma^V(\mu) - \omega_\gamma^A(\mu) \\ \omega_\gamma^V(\mu) + \omega_\gamma^A(\mu) \end{pmatrix} = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right)^{\Gamma_\omega/\beta_0} \begin{pmatrix} \omega_\gamma^V(\mu_0) - \omega_\gamma^A(\mu_0) \\ \omega_\gamma^V(\mu_0) + \omega_\gamma^A(\mu_0) \end{pmatrix},
\tag{5.3}
$$

where the anomalous dimension matrix Γ_ω is given by [16, 37]

$$
\Gamma_\omega = \begin{pmatrix} 3\,C_F - \frac{2}{3}\,C_A & \frac{2}{3}\,C_F - \frac{2}{3}\,C_A \\ \frac{5}{3}\,C_F - \frac{4}{3}\,C_A & \frac{1}{2}\,C_F + C_A \end{pmatrix}.
\tag{5.4}
$$

Due to the Ferrara-Grillo-Parisi-Gatto theorem [38], the twist-four parameters corresponding to the "P"-wave conformal spin satisfy the following relations

$$
\zeta_1(\mu) + 11\,\zeta_2(\mu) - 2\,\zeta_2^+(\mu) = \frac{7}{2}.
\tag{5.5}
$$

$f_{3\gamma}(\mu_0)\,(\mathrm{GeV}^2)$	$\omega_\gamma^V(\mu_0)$	$\omega_\gamma^A(\mu_0)$	$\kappa(\mu_0)$	$\kappa^+(\mu_0)$	$\zeta_1(\mu_0)$	$\zeta_1^+(\mu_0)$	$\zeta_2^+(\mu_0)$
$-(4\pm2)\times10^{-3}$	3.8 ± 1.8	-2.1 ± 1.0	0.2 ± 0.2	0	0.4 ± 0.4	0	0

Table 1. Numerical values of the nonperturbative parameters entering the photon DAs at the renormalization scale $\mu_0 = 1.0\,\mathrm{GeV}$.

The scale evolution of the nonperturbative parameters at twist-four accuracy is given by

$$\kappa^+(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{(\gamma^+ - \gamma_{q\bar{q}})/\beta_0} \kappa^+(\mu_0)\,, \qquad \kappa(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{(\gamma^- - \gamma_{q\bar{q}})/\beta_0} \kappa(\mu_0)\,,$$

$$\zeta_1(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{(\gamma_{Q^{(1)}} - \gamma_{q\bar{q}})/\beta_0} \zeta_1(\mu_0)\,, \qquad \zeta_1^+(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{(\gamma_{Q^{(5)}} - \gamma_{q\bar{q}})/\beta_0} \zeta_1^+(\mu_0)\,,$$

$$\zeta_2^+(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{(\gamma_{Q^{(3)}} - \gamma_{q\bar{q}})/\beta_0} \zeta_2^+(\mu_0)\,, \tag{5.6}$$

where the anomalous dimensions of these twist-four parameters at one loop are given by [16]

$$\gamma^+ = 3\,C_A - \frac{5}{3}\,C_F\,, \qquad \gamma^- = 4\,C_A - 3\,C_F\,,$$

$$r_{q\bar{q}} = -3\,C_F\,, \qquad \gamma_{Q^{(1)}} = \frac{11}{2}\,C_A - 3\,C_F\,,$$

$$\gamma_{Q^{(3)}} = \frac{13}{3}\,C_F\,, \qquad \gamma_{Q^{(5)}} = 5\,C_A - \frac{8}{3}\,C_F\,. \tag{5.7}$$

Numerical values of the input parameters entering the photon DAs up to twist-four are collected in table 1, where we have assigned 100 % uncertainties for the estimates of the twist-four parameters from QCD sum rules [17]. The second Gegenbauer moment of the leading-twist photon DA will be further taken as $a_2(\mu_0) = 0.07 \pm 0.07$ as obtained in [16]. The magnetic susceptibility of the quark condensate $\chi(1\,\mathrm{GeV}) = (3.15 \pm 0.3)\,\mathrm{GeV}^{-2}$ computed from the QCD sum rule approach including the $\mathcal{O}(\alpha_s)$ corrections [16] and the quark condensate density $\langle\bar{q}q\rangle(1\,\mathrm{GeV}) = -(246^{+28}_{-19}\,\mathrm{MeV})^3$ determined by the GMOR relation [39] will be also employed for the numerical estimates in the following.

The key quantity entering the leading-power factorization formula of the $B \to \gamma\ell\nu$ form factors is the first inverse moment of the B-meson DA $\lambda_B(\mu)$, whose determination has been discussed extensively in the context of exclusive B-meson decays with distinct QCD approaches (see [34, 40] for more discussions). To illustrate the phenomenological consequences of the subleading-power corrections from the hadronic photon contributions we will take the interval $\lambda_B(1\,\mathrm{GeV}) = 354^{+38}_{-30}\,\mathrm{MeV}$ implied by the LCSR calculations of the semileptonic $B \to \pi$ form factors with B-meson DAs on the light-cone [34]. The renormalization-scale dependence of $\lambda_B(\mu)$ at one loop can be determined from the evolution equation of $\phi_B^+(\omega,\mu)$ [41]

$$\frac{\lambda_B(\mu)}{\lambda_B(\mu_0)} = 1 + \frac{\alpha_s(\mu_0)\,C_F}{4\,\pi}\,\ln\frac{\mu}{\mu_0}\left[2 - 2\ln\frac{\mu}{\mu_0} - 4\,\sigma_1(\mu_0)\right] + \mathcal{O}(\alpha_s^2)\,, \tag{5.8}$$

where the inverse-logarithmic moments $\sigma_n(\mu_0)$ are defined as [6]

$$\sigma_n(\mu_0) = \lambda_B(\mu) \int_0^\infty \frac{d\omega}{\omega} \, \ln^n\left(\frac{\mu_0}{\omega}\right) \phi_B^+(\omega, \mu_0)\,. \qquad (5.9)$$

Numerically we will employ $\sigma_1(1\,\text{GeV}) = 1.5 \pm 1$ consistent with the NLO QCD sum rule calculation [42] and $\sigma_2(1\,\text{GeV}) = 3 \pm 2$ from [6]. Furthermore, the static B-meson decay constant $\tilde{f}_B(\mu)$ will be expressed in terms of the QCD decay constant f_B

$$\tilde{f}_B(\mu) = f_B \left\{ 1 + \frac{\alpha_s(\mu)\, C_F}{4\,\pi} \left[3 \ln \frac{m_B}{\mu} - 2 \right] \right\}^{-1}, \qquad (5.10)$$

and the determination $f_B = (192.0 \pm 4.3)\,\text{MeV}$ from the FLAG Working Group [43] will be taken in the numerical analysis.

Following the discussions presented in [6, 34], the hard scales μ_{h1} and μ_{h2} entering the leading-power factorization formula will be chosen as $\mu_{h1} = \mu_{h2} \in [m_b/2,\ 2\,m_b]$ around the default value m_b and the factorization scale in (2.7) will be varied in the interval $1\,\text{GeV} \leq \mu \leq 2\,\text{GeV}$ with the central value $\mu = 1.5\,\text{GeV}$. In contrast, the factorization scale entering the LCSR for the hadronic photon corrections will be taken as $\mu \in [m_b/2,\ 2\,m_b]$ around the default choice m_b. In addition, we adopt the numerical values of the bottom quark mass $\overline{m_b}(\overline{m_b}) = 4.193^{+0.022}_{-0.033}\,\text{GeV}$ [44] in the $\overline{\text{MS}}$ scheme from non-relativistic sum rules. Finally, we turn to determine the Borel mass M^2 and the threshold parameter s_0 in the LCSR for the hadronic photon contributions. Applying the standard strategies presented in [34] (see also [45] for a review) gives rise to following intervals

$$s_0 = (37.5 \pm 2.5)\,\text{GeV}^2\,, \qquad M^2 = (18.0 \pm 3.0)\,\text{GeV}^2\,, \qquad (5.11)$$

which is consistent with the determinations from the LCSR of the $B \to \pi$ form factors [46].

5.2 Predictions for the $B \to \gamma \ell \nu$ form factors

We are now in a position to explore the phenomenological significance of the hadronic photon corrections to the $B \to \gamma \ell \nu$ form factors. To develop a better understanding of the heavy quark expansion for the bottom sector, we plot the photon-energy dependence of the leading-power contribution, the subleading-power local correction and the subleading-power two-particle and three-particle hadronic photon effects in figure 5. It is apparent that the twist-two hadronic photon contribution at NLL can generate sizeable destructive interference with the leading-power "direct photon" contribution: approximately $\mathcal{O}(30\%)$ for $n \cdot p \in [3\,\text{GeV}, m_B]$ with $\lambda_B(\mu_0) = 354\,\text{MeV}$. However, both the two-particle higher-twist and the three-particle hadronic photon contributions turn out to be numerically insignificant at tree level and will only shift the leading-power prediction by an amount of $\mathcal{O}(3 \sim 5)\%$ for $n \cdot p \in [3\,\text{GeV}, m_B]$. Furthermore, the subleading-power local contribution $F_{V,\text{NLP}}^{\text{LC}}$ at tree level displayed in (2.8) will give rise to $\mathcal{O}(3\%)$ correction at $n \cdot p = m_B$ and $\mathcal{O}(10\%)$ correction at $n \cdot p = 3\,\text{GeV}$. On account of the observed pattern for the separate terms contributing to the $B \to \gamma$ form factors numerically, we are led to conclude that the power suppressed contributions to the radiative leptonic B-meson decay are dominated by the leading-twist hadronic photon correction with the default theory inputs.
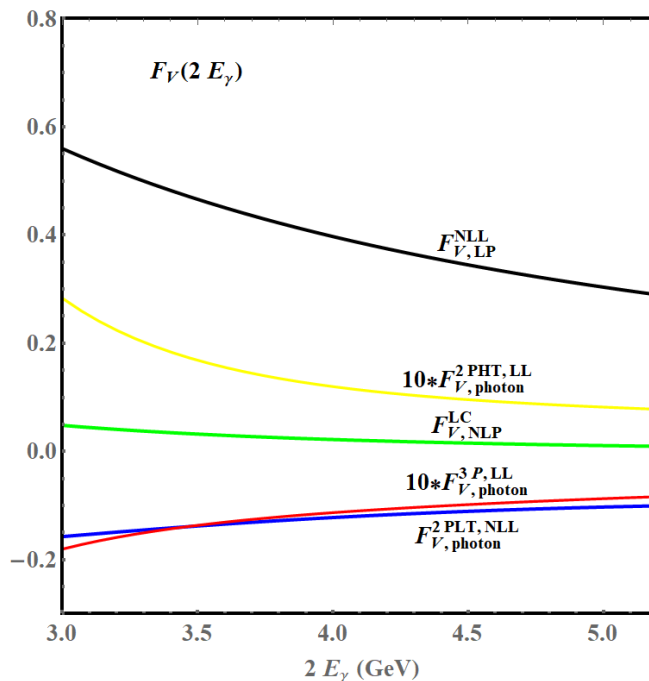
Figure 5. The photon-energy dependence of different terms contributing to the vector $B \to \gamma$ form factor $F_V(2\,E_\gamma)$ as displayed in (4.15) with the central values of theory inputs. The individual contributions correspond to the leading-power contribution at NLL computed from the QCD factorization approach ($F_{V,\,\mathrm{LP}}^{\mathrm{NLL}}$, black), the subleading-power local contribution at LO ($F_{V,\,\mathrm{NLP}}^{\mathrm{LC}}$, green), the two-particle leading-twist hadronic photon correction at NLL ($F_{V,\,\mathrm{photon}}^{2\mathrm{PLT,\,NLL}}$, blue), the two-particle higher-twist hadronic photon correction at leading-logarithmic (LL) accuracy ($F_{V,\,\mathrm{photon}}^{2\mathrm{PHT,\,LL}}$, yellow), the three-particle leading-twist hadronic photon correction at LL ($F_{V,\,\mathrm{photon}}^{3\mathrm{P,\,LL}}$, red).

We further turn to investigate the numerical impact of the perturbative correction at NLO and the QCD resummation of the parametrically large logarithms of m_b^2/Λ^2 for the leading-twist hadronic photon contribution computed from the LCSR technique. It is evident from figure 6 that the NLO QCD correction can decrease the tree-level prediction of the twist-two hadronic photon contribution by an amount of $\mathcal{O}\,(20 \sim 40)\%$ for the factorization scale varied in the interval $[3.0,\,5.0]\,\mathrm{GeV}$ and the NLL resummation effect can yield $\mathcal{O}\,(10\,\%)$ enhancement to the NLO QCD results within the same range of μ. Hence, the dominant radiative correction to the leading-twist hadronic photon contribution originates from the NLO QCD correction to the hard matching coefficient entering the factorization formula (3.38) rather than from resummation of the large logarithms m_b^2/Λ^2. However, the renormalization scale dependence of the resummation improved theory predictions in the allowed region indeed becomes weaker compared with the NLO calculation. We further plot the photon-energy dependence of the ratio $R_{FV,\,\mathrm{photon}}^{2\mathrm{PLT}}(n \cdot p) \equiv F_{V,\,\mathrm{photon}}^{2\mathrm{PLT,\,NLL}}(n \cdot p)/F_{V,\,\mathrm{photon}}^{2\mathrm{PLT,\,LL}}(n \cdot p)$ characterizing the perturbative QCD corrections at NLL in figure 6, where the theory uncertainties due to the variations of the renormalization scale μ are also displayed.

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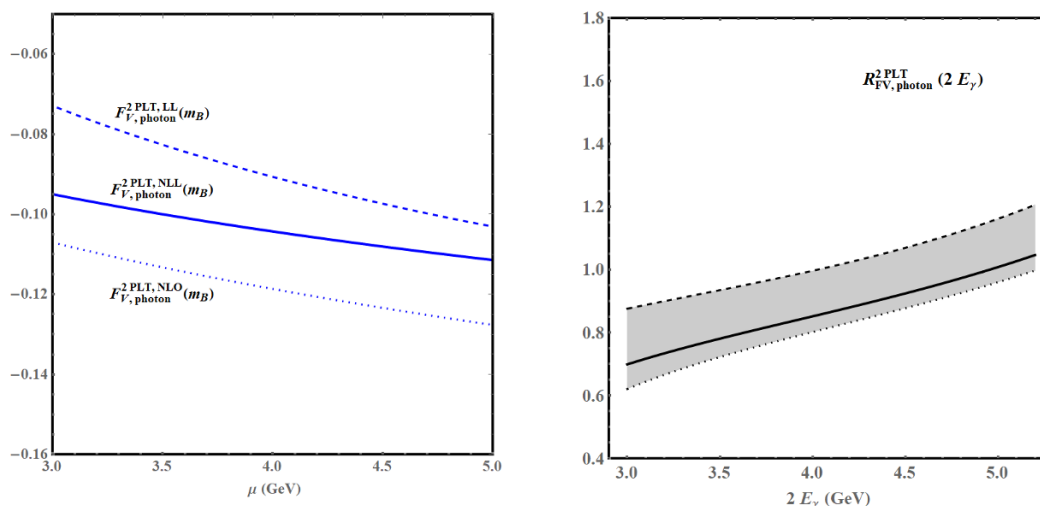
Figure 6. *Left*: the renormalization scale dependence of the twist-two hadronic photon correction to the vector $B \to \gamma$ form factor $F_V(m_B)$ at LL (dashed), NLO (dotted), and NLL (solid) accuracy, respectively. *Right*: the photon-energy dependence of the ratio $R_{FV,\,\text{photon}}^{2\text{PLT}}(n \cdot p) \equiv F_{V,\,\text{photon}}^{2\text{PLT, NLL}}(n \cdot p)/F_{V,\,\text{photon}}^{2\text{PLT, LL}}(n \cdot p)$ with the uncertainties from the variations of the renormalization scale μ.

Taking into account the fact that the QCD sum rule calculation of the second Gegenbauer moment of the twist-two photon DA $a_2(\mu_0)$ suffers from the large theory uncertainties due to the strong sensitivity to the input parameters [16], we plot the leading-twist hadronic photon correction to the vector form factor $F_V(n \cdot p)$ in a wide range of $a_2(\mu_0)$ in figure 7. One can readily observe that the variation of the Gegenbauer moment $a_2(\mu_0) \in [-0.2, 0.2]$ can only give rise to a minor impact on the theory prediction of the $B \to \gamma$ form factor $F_V(m_B)$ at maximal recoil numerically. However, the "P-wave" conformal spin contribution from the leading-twist photon DA will become significant for the evaluation of the form factor $F_V(n \cdot p)$ with the decrease of the photon energy: approximately $\mathcal{O}(35\%)$ at $n \cdot p = 3\,\text{GeV}$. To further understand the systematic uncertainty due to the truncation of the conformal expansion at "P-wave", we also display the theory predictions for the $B \to \gamma\ell\nu$ form factors including the "D-wave" effect from the fourth Gegenbauer moment $a_4(\mu_0)$ in figure 7. It is apparent that the sensitivity of the leading-twist hadronic photon contribution on $a_4(\mu_0)$ is rather weak numerically for $n\cdot p \in [3\,\text{GeV}, m_B]$ in the "reasonable" interval $-0.2 \leq a_4(\mu_0) \leq 0.2$. In the light of such observation, the yet higher Gegenbauer moments of the twist-two photon DA are not expected to bring about notable impact on the prediction of the subleading-power contribution to the $B \to \gamma\ell\nu$ form factors induced by the photon light-cone DAs.

We present our final predictions for the $B \to \gamma\ell\nu$ form factors including the newly computed two-particle and three-particle hadronic photon corrections with theory uncertainties in figure 8. The dominant theory uncertainties originate from the first inverse moment $\lambda_B(\mu_0)$, the factorization scale μ entering the leading-power "direct photon" contribution, and the second Gegenbauer moment $a_2(\mu_0)$ of the twist-two photon DA. However, the symmetry breaking effect between the two $B \to \gamma$ form factors due to the subleading-power
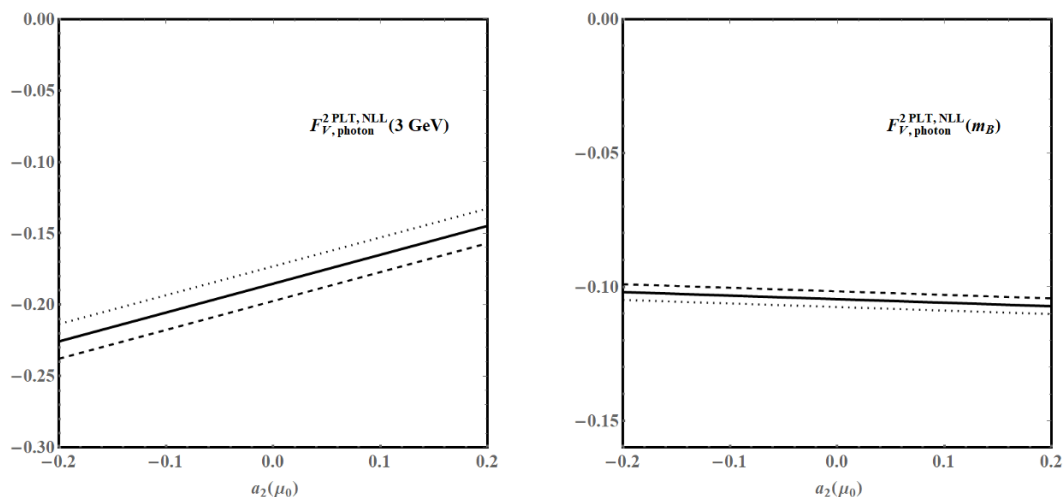
Figure 7. Dependence of the leading-twist hadronic photon correction at $n \cdot p = 3 \, \text{GeV}$ (left panel) and $n \cdot p = m_B$ (right panel) on the second Gegenbauer moment of the photon light-cone DA $a_2(\mu_0)$ with different values of the fourth Gegenbauer moment: $a_4(\mu_0) = 0.2$ (dashed), $a_4(\mu_0) = 0$ (solid) and $a_4(\mu_0) = -0.2$ (dotted).
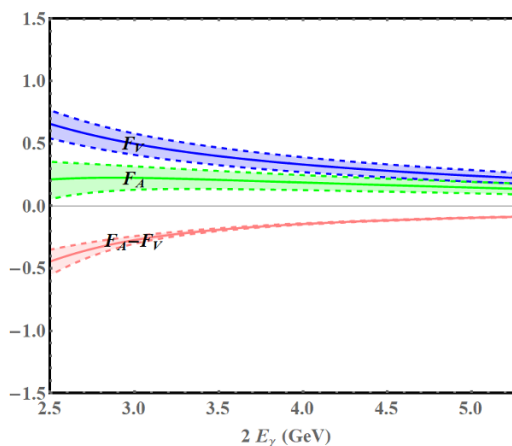
Figure 8. The photon-energy dependence of the $B \to \gamma \ell \nu$ form factors as well as their difference computed from (4.15) with the theory uncertainties from variations of different input parameters added in quadrature.

local contribution and the higher-twist hadronic photon corrections suffers from much less uncertainty than the individual form factors at $3 \, \text{GeV} \leq n \cdot p \leq m_B$. Having in our hands the theoretical predictions for the $B \to \gamma \ell \nu$ form factors, we proceed to discuss the theory constraints on the inverse moment $\lambda_B(\mu_0)$ taking advantage of the future measurements on the (partially) integrated branching fractions with a photon-energy cut to get rid of the soft photon radiation. It is straightforward to derive the differential decay width for
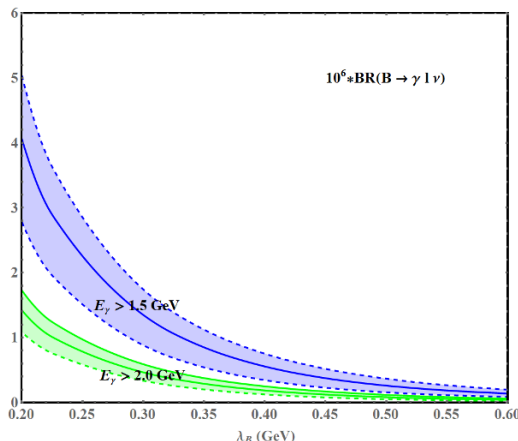
Figure 9. Dependence of the partial branching fractions $\mathcal{BR}(B \to \gamma \ell \nu, E_\gamma \geq E_{\text{cut}})$ on the first inverse moment $\lambda_B(\mu_0)$ for $E_{\text{cut}} = 1.5 \, \text{GeV}$ (blue band) and $E_{\text{cut}} = 2.0 \, \text{GeV}$ (green band).

$B \to \gamma \ell \nu$ in the rest frame of the B-meson (see also [6, 8])

$$\frac{d\,\Gamma(B \to \gamma \ell \nu)}{d\,E_\gamma} = \frac{\alpha_{em}\, G_F^2\, |V_{ub}|^2}{6\,\pi^2}\, m_B\, E_\gamma^3 \left(1 - \frac{2\,E_\gamma}{m_B}\right) \left[F_V^2(n \cdot p) + F_A^2(n \cdot p)\right], \quad (5.12)$$

and the integrated branching fractions with the phase-space cut on the photon energy read

$$\mathcal{BR}(B \to \gamma \ell \nu, E_\gamma \geq E_{\text{cut}}) = \tau_B \int_{E_{\text{cut}}}^{m_B/2} dE_\gamma \frac{d\,\Gamma(B \to \gamma \ell \nu)}{d\,E_\gamma}, \quad (5.13)$$

where τ_B indicates the lifetime of the B-meson. Our predictions for the partial branching fractions of the radiative leptonic decay $B \to \gamma \ell \nu$ including the hadronic photon corrections to the form factors are displayed in figure 9 with the variation of the inverse moment $\lambda_B(\mu_0)$ in the interval $[0.2, 0.6] \, \text{GeV}$. It can be observed that the integrated branching fractions $\mathcal{BR}(B \to \gamma \ell \nu, E_\gamma \geq E_{\text{cut}})$ grow rapidly with the decrease of the inverse moment due to the dependence of the two form factors on $1/\lambda_B(\mu_0)$ at leading-power in Λ/m_b. Since the photon-energy cut $E_\gamma \geq 1 \, \text{GeV}$ implemented in the Belle measurements [47] is not sufficiently large to perform perturbative QCD calculations of the $B \to \gamma$ form factors, we will not employ the experimental bound $\mathcal{BR}(B \to \gamma \ell \nu, E_\gamma \geq E_{\text{cut}}) < 3.5 \times 10^{-6}$ with the full Belle data sample reported in [47] for the determination of $\lambda_B(\mu_0)$ at the moment. Instead, we prefer to explore the solid theory constraints on the first inverse moment by comparing our predictions of the (partially) integrated branching fractions with the improved measurements at the Belle II experiment, with the tighter phase-space cut on the photon energy, thanks to the much higher designed luminosity of the SuperKEKB accelerator.

6 Conclusion

We computed perturbative QCD corrections to the leading-twist hadronic photon contribution to the $B \to \gamma \ell \nu$ form factors employing the LCSR method. QCD factorization for the vacuum-to-photon correlation function (3.1) has been demonstrated explicitly at one loop with the OPE technique and the NDR scheme of the Dirac matrix γ_5 including the evanescent SCET operator. The perturbative matching coefficient entering the NLO factorization formula (3.38) was obtained by applying the method of regions and the factorization-scale independence of the correlation function (3.1) was further verified at $\mathcal{O}(\alpha_s)$ with the evolution equations of the twist-two photon DA and of the bottom-quark mass. Resummation of the parametrically large logarithms of $\mathcal{O}(\ln(m_b^2/\Lambda^2))$ was achieved at NLL accuracy with the two-loop RG equation of the light-ray tensor operator. Implementing the continuum subtraction with the aid of the parton-hadron duality and the Borel transformation, the NLL resummation improved LCSR for the twist-two hadronic photon correction to the $B \to \gamma$ form factors was subsequently constructed with the spectral representations of the factorization formula (3.38). The subleading-power correction to the $B \to \gamma \ell \nu$ amplitude from the leading-twist photon DA was shown to preserve the symmetry relation between the two form factors due to the helicity conservation, in agreement with the observation made in [12].

Along the same vein, we proceed to compute the two-particle and three-particle higher-twist hadronic photon corrections to the $B \to \gamma \ell \nu$ form factors at tree level, up to the twist-four accuracy. The symmetry relation between the two form factors $F_V(n \cdot p)$ and $F_A(n \cdot p)$ was found to be violated by both the two-particle and three-particle higher-twist effects of the photon light-cone DAs. In addition, our calculations explicitly indicate that the correspondence between the heavy-quark expansion and the twist expansion is generally invalid for the soft contributions to the exclusive B-meson decays, in analogy to the similar pattern observed in the context of the pion-photon form factor [31].[1]

Adding up different pieces contributing to the $B \to \gamma \ell \nu$ amplitude, we further investigated the phenomenological impacts of the subleading-power hadronic photon contributions, employing the conformal expansion of the photon DAs at the "P-wave" accuracy. Numerically, the NLL twist-two hadronic photon correction was estimated to give rise to an approximately $\mathcal{O}(30\%)$ reduction of the leading-power contribution, computed from QCD factorization, with the default values of theory inputs. By contrast, the higher-twist hadronic photon contributions at LO in $\mathcal{O}(\alpha_s)$ was found to be of minor importance at $3 \, \text{GeV} \leq n \cdot p \leq m_B$, albeit with the rather conservative uncertainty ranges for the nonperturbative parameters collected in table 1. Moreover, we observed that the dominant radiative effect of the leading-twist hadronic photon contribution comes from the NLO QCD correction instead of the QCD resummation of the parametrically large logarithms m_b^2/Λ^2. To understand the systematic uncertainty from the truncation of the Gegenbauer expansion at the second order, we explored the numerical impact of the fourth moment of the leading-twist photon DA in a wide interval $\alpha_4(\mu_0) \in [-0.2, 0.2]$ and observed that

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[1]The large scale Q^2 in the pion-photon transition form factor $F_{\gamma^* \gamma \to \pi}(Q^2)$ plays a similar role of the heavy-quark mass in B-meson decays.

the dependence of the twist-two hadronic photon correction to the $B \to \gamma \ell \nu$ form factors on $\alpha_4(\mu_0)$ was rather moderate at $n \cdot p \geq 3\,\mathrm{GeV}$, at least in the framework of the LCSR method. Our main theory predictions for the $B \to \gamma \ell \nu$ form factors with the uncertainties from variations of different input parameters added in quadrature were displayed in figure 8 and the poor constraint on the first inverse moment of the B-meson DA $\lambda_B(\mu_0)$ brought about one of the major uncertainties for the theory calculations. In this respect, the improved measurements of the partial branching fractions $\mathcal{BR}(B \to \gamma \ell \nu, E_\gamma \geq E_{\mathrm{cut}})$ with the tighter phase-space cut on the photon energy to validate the perturbative QCD calculations from the Belle II experiment will be of value to provide solid constraints on the inverse moment $\lambda_B(\mu_0)$, when combined with the theory predictions including the power suppressed contributions of different origins.

Further improvements of the theory descriptions of the $B \to \gamma \ell \nu$ form factors in QCD can be pursued in distinct directions. First, it would be of interest to perform the NLO QCD corrections to the twist-three hadronic photon corrections with the LCSR approach for a systematic understanding of the higher-twist contributions. The technical challenge of accomplishing this task lies in the demonstration of QCD factorization for the vacuum-to-B-meson correlation function (3.1) in the presence of the non-trivial mixing of the two-particle and three-particle light-ray operators under the QCD renormalization. Second, exploring the subleading-power contributions to the radiative leptonic B-meson decay in the framework of SCET directly will be indispensable for deepening our understanding of factorization properties for more complicated exclusive B-meson decays, where the rapidity divergences of the convolution integrals entering the corresponding factorization formulae already emerge at leading power in the heavy quark expansion. Earlier attempts to address this ambitious question have been undertaken in different contexts (for an incomplete list, see for instance [5, 48–50]). Third, computing the subleading-power corrections to the $B \to \gamma$ form factors from the higher-twist B-meson DAs will be of both conceptual and phenomenological value to investigate general properties of the twist expansion in heavy-quark effective theory (see [8] for a preliminary discussion with an incomplete decomposition of the three-particle vacuum-to-B-meson matrix element on the light-cone). To this end, we will need to employ the RG equations for these higher-twist B-meson DAs at one loop following the discussions presented in [51], where the evolution equations of the twist-four B-meson DAs at one loop were demonstrated to be completely integrable and therefore can be solved exactly. We are therefore anticipating dramatic progress toward better understanding of the strong interaction dynamics of the radiative leptonic decay $B \to \gamma \ell \nu$ in QCD.

Acknowledgments

A Spectral representations

Here we collect the dispersion representations of the various convolution integrals entering the NLL factorization formula of the vacuum-to-photon correlation function (3.50) for the sake of constructing the LCSR for the $B \to \gamma\ell\nu$ form factors.

$$\frac{1}{\pi}\operatorname{Im}_s \int_0^1 dz\,\phi_\gamma(z,\mu)\,\frac{1}{z\,r_3+\bar{z}\,r_2-1+i0}\,\frac{1-r_2}{r_1-r_2}\ln\left(\frac{1-r_2}{r_1-r_2}\right)$$
$$= \int_0^1 dz\,\mathcal{P}\frac{1}{r_3-r_2}\ln\left(\frac{z\,r_3+\bar{z}\,r_2-1}{1-r_2}\right)\theta(z\,r_3+\bar{z}\,r_2-1)\,\phi'_\gamma(z,\mu)$$
$$+ \int_0^1 dz\,\mathcal{P}\frac{1}{r_3-r_2}\,\theta(z\,r_3+\bar{z}\,r_2-1)\,\frac{\phi_\gamma(z,\mu)}{z}\,. \tag{A.1}$$

$$\frac{1}{\pi}\operatorname{Im}_s \int_0^1 dz\,\phi_\gamma(z,\mu)\,\frac{1}{z\,r_3+\bar{z}\,r_2-1+i0}\,\frac{1-r_3}{r_1-r_3}\ln\left(\frac{1-r_1}{1-r_3}\right)$$
$$= \int_0^1 dz\,\mathcal{P}\frac{1}{r_3-r_2}\ln\left|\frac{1-z\,r_3-\bar{z}\,r_2}{1-r_3}\right|\,[\theta(z\,r_3+\bar{z}\,r_2-1)-\theta(r_3-1)]\,\phi'_\gamma(z,\mu)$$
$$- \int_0^1 dz\,\mathcal{P}\frac{1}{r_3-r_2}\,[\theta(z\,r_3+\bar{z}\,r_2-1)-\theta(r_3-1)]\,\frac{\phi_\gamma(z,\mu)}{\bar{z}}\,. \tag{A.2}$$

$$\frac{1}{\pi}\operatorname{Im}_s \int_0^1 dz\,\phi_\gamma(z,\mu)\,\frac{1}{z\,r_3+\bar{z}\,r_2-1+i0}\,\frac{1}{1-r_1} = \frac{\theta(r_3-1)}{(r_3-r_2)^2}\,\phi'_\gamma\left(\frac{1-r_2}{r_3-r_2},\mu\right)\,. \tag{A.3}$$

$$\frac{1}{\pi}\operatorname{Im}_s \int_0^1 dz\,\phi_\gamma(z,\mu)\,\frac{1}{z\,r_3+\bar{z}\,r_2-1+i0}\,\frac{1-r_2}{r_1-r_2}\operatorname{Li}_2\left(1-\frac{1-r_1}{1-r_2}\right)$$
$$= \frac{\theta(r_3-1)}{r_2-r_3}\frac{\pi^2}{6}\,\phi_\gamma\left(\frac{1-r_2}{r_3-r_2},\mu\right)+\int_0^1 dz\,\phi_\gamma(z,\mu)\left[\mathcal{P}\frac{1}{z\,r_3+\bar{z}\,r_2-1}-\mathcal{P}\frac{1}{z\,(r_3-r_2)}\right]$$
$$\times \ln\left(1-\frac{1-z\,r_3-\bar{z}\,r_2}{1-r_2}\right)\theta(z\,r_3+\bar{z}\,r_2-1)\,. \tag{A.4}$$

$$\frac{1}{\pi}\operatorname{Im}_s \int_0^1 dz\,\phi_\gamma(z,\mu)\,\frac{1}{z\,r_3+\bar{z}\,r_2-1+i0}\,\frac{1-r_3}{r_1-r_3}\operatorname{Li}_2\left(1-\frac{1-r_1}{1-r_3}\right)$$
$$= \frac{\theta(r_3-1)}{r_2-r_3}\frac{\pi^2}{6}\,\phi_\gamma\left(\frac{1-r_2}{r_3-r_2},\mu\right)+\int_0^1 dz\,\phi_\gamma(z,\mu)\left[\mathcal{P}\frac{1}{z\,r_3+\bar{z}\,r_2-1}-\mathcal{P}\frac{1}{\bar{z}\,(r_2-r_3)}\right]$$
$$\times \operatorname{sgn}(r_2-r_3)\ln\left(1-\frac{1-z\,r_3-\bar{z}\,r_2}{1-r_3}\right)\theta\left(\frac{z\,r_3+\bar{z}\,r_2-1}{1-r_3}\right)\,. \tag{A.5}$$

$$\frac{1}{\pi}\operatorname{Im}_s \int_0^1 dz\,\phi_\gamma(z,\mu)\,\frac{1}{z\,r_3+\bar{z}\,r_2-1+i0}\,\frac{1-r_2}{r_1-r_2}\ln^2(1-r_1)$$
$$= \int_0^1 dz\,\mathcal{P}\frac{1}{r_3-r_2}\left[\ln^2|1-z\,r_3-\bar{z}\,r_2|-\frac{\pi^2}{3}\right]\theta(z\,r_3+\bar{z}\,r_2-1)\,\phi'_\gamma(z,\mu)$$
$$+ \int_0^1 dz\,\frac{\phi_\gamma(z,\mu)}{z}\left[2\mathcal{P}\frac{1}{r_3-r_2}\ln|1-z\,r_3-\bar{z}\,r_2|\theta(z\,r_3+\bar{z}\,r_2-1)\right.$$
$$\left.+ \delta(r_3-r_2)\ln^2(1-r_2)\right]\,. \tag{A.6}$$

$$\frac{1}{\pi}\mathrm{Im}_s\int_0^1 dz\,\phi_\gamma(z,\mu)\,\frac{1}{z\,r_3+\bar z\,r_2-1+i0}\,\frac{1-r_3}{r_1-r_3}\,\ln^2(1-r_1)$$
$$=\int_0^1 dz\,\mathcal{P}\frac{1}{r_3-r_2}\left[\ln^2|1-z\,r_3-\bar z\,r_2|-\frac{\pi^2}{3}\right]\theta(z\,r_3+\bar z\,r_2-1)\,\phi_\gamma'(z,\mu)$$
$$-\int_0^1 dz\,\frac{\phi_\gamma(z,\mu)}{\bar z}\left[2\mathcal{P}\frac{1}{r_3-r_2}\ln|1-z\,r_3-\bar z\,r_2|\,\theta(z\,r_3+\bar z\,r_2-1)\right.$$
$$\left.+\,\delta(r_3-r_2)\ln^2(1-r_2)\right]. \tag{A.7}$$

$$\frac{1}{\pi}\mathrm{Im}_s\int_0^1 dz\,\phi_\gamma(z,\mu)\,\frac{1}{z\,r_3+\bar z\,r_2-1+i0}\,\frac{1-r_2}{r_1-r_2}\,\ln^2(1-r_2)$$
$$=\left[\frac{\theta(r_3-1)}{r_2-r_3}\,\phi_\gamma\left(\frac{1-r_2}{r_3-r_2},\mu\right)+\delta(r_3-r_2)\int_0^1 dz\,\frac{\phi_\gamma(z,\mu)}{z}\right]\ln^2(1-r_2). \tag{A.8}$$

$$\frac{1}{\pi}\mathrm{Im}_s\int_0^1 dz\,\phi_\gamma(z,\mu)\,\frac{1}{z\,r_3+\bar z\,r_2-1+i0}\,\frac{1-r_3}{r_1-r_3}\,\ln^2(1-r_3)$$
$$=\int_0^1 dz\,\phi_\gamma(z,\mu)\frac{r_2(r_2-2)(1+z)+2z}{r_2(1-r_2)z\bar z}\left[\delta(r_2-r_3)\ln(1-r_2)-\frac{\theta(r_3-1)}{r_2-r_3}\theta\left(z-\frac{1-r_2}{r_3-r_2}\right)\right]$$
$$+\frac{6}{r_3-r_2}\left\{\frac{\theta(r_3-1)}{r_3-r_2}\,\phi_\gamma'\left(\frac{1-r_2}{r_3-r_2},\mu\right)+\frac{\theta(r_3-1)}{r_3-r_2}\ln(r_3-1)\,\phi_\gamma'(z=1,\mu)\right.$$
$$\left.-\frac{1}{r_3-r_2}\int_0^1 dz\,\ln(z\,r_3+\bar z\,r_2-1)\,\theta(z\,r_3+\bar z\,r_2-1)\,\phi_\gamma''(z,\mu)\right\}$$
$$+\frac{r_2}{(1-r_2)(r_2-r_3)}\int_0^1 dz\,\ln(z\,r_3+\bar z\,r_2-1)\,\theta(z\,r_3+\bar z\,r_2-1)\,\phi_\gamma'(z,\mu)$$
$$-\int_0^1 dz\,\phi_\gamma(z,\mu)\,\frac{2\,(z-2\,\bar z\,r_2)}{\bar z\,r_2\,(z\,r_3+\bar z\,r_2)}\,\theta(z\,r_3+\bar z\,r_2-1). \tag{A.9}$$

$$\frac{1}{\pi}\mathrm{Im}_s\int_0^1 dz\,\phi_\gamma(z,\mu)\,\frac{1}{z\,r_3+\bar z\,r_2-1+i0}\left(\frac{2-r_2}{r_1-r_2}-\frac{4}{r_2}+2\right)$$
$$=\frac{\theta(r_3-1)}{r_2-r_3}\,\phi_\gamma\left(\frac{1-r_2}{r_3-r_2},\mu\right)\left[\frac{2-r_2}{4-r_2}-\frac{4}{r_2}+2\right]-\delta(r_3-r_2)\int_0^1 dz\,\frac{\phi_\gamma(z,\mu)}{z}. \tag{A.10}$$

$$\frac{1}{\pi}\mathrm{Im}_s\int_0^1 dz\,\phi_\gamma(z,\mu)\,\frac{1}{z\,r_3+\bar z\,r_2-1+i0}\left[\frac{2}{r_3(r_1-r_3)}+\frac{2\,(r_1-2)}{r_1-r_3}-\frac{6}{r_3}\right]\ln(1-r_3)$$
$$=\int_0^1 dz\,\phi_\gamma(z,\mu)\left\{\left[\frac{2(r_2-1)}{\bar z\,r_2}\delta(r_3-r_2)+\frac{4z\,\theta(r_3-1)}{(1-\bar z\,r_2)(r_3-r_2)}\delta\left(z-\frac{1-r_2}{r_3-r_2}\right)\right]\ln|1-r_3|\right.$$
$$\left.-\theta(r_3-1)\left[\frac{2\,(1-r_2)}{\bar z\,r_2}\frac{1}{r_3-r_2}-\frac{2\,(1-3\,\bar z\,r_2)}{r_2\,r_3\,\bar z\,(1-\bar z\,r_2)}-\frac{4z}{1-\bar z\,r_2}\mathcal{P}\frac{1}{z\,r_3+\bar z\,r_2-1}\right]\right\}. \tag{A.11}$$

$$\frac{1}{\pi}\mathrm{Im}_s\int_0^1 dz\,\phi_\gamma(z,\mu)\,\frac{1}{z\,r_3+\bar z\,r_2-1+i0}\left[\frac{r_2}{2\,(r_1-r_2)}-\frac{3}{1-r_1}-\frac{15}{2}\right]$$
$$=\frac{r_2}{2\,(1-r_2)}\delta(r_2-r_3)\int_0^1 dz\,\frac{\phi_\gamma(z,\mu)}{z}-\frac{3\theta(r_3-1)}{(r_3-r_2)^2}\,\phi_\gamma'\left(\frac{1-r_2}{r_3-r_2},\mu\right)$$
$$-\frac{16\,r_2-15}{2\,(1-r_2)}\frac{\theta(r_3-1)}{r_3-r_2}\,\phi_\gamma\left(\frac{1-r_2}{r_3-r_2},\mu\right). \tag{A.12}$$

In this appendix we will collect the operator-level definitions of the two-particle and three-particle photon DAs on the light-cone up to the twist-four accuracy as presented in [16].

$$\langle\gamma(p)|\bar{q}(x)\,W_c(x,0)\,\sigma_{\alpha\beta}\,q(0)|0\rangle$$

$$= i\,g_{\rm em}\,Q_q\,\langle\bar{q}q\rangle(\mu)\,(p_\beta\,\epsilon_\alpha^*-p_\alpha\,\epsilon_\beta^*)\int_0^1 dz\,e^{i\,z\,p\cdot x}\left[\chi(\mu)\,\phi_\gamma(z,\mu)+\frac{x^2}{16}\,\mathbb{A}(z,\mu)\right]$$

$$+\frac{i}{2}\,g_{\rm em}\,Q_q\,\frac{\langle\bar{q}q\rangle(\mu)}{q\cdot x}\,(x_\beta\,\epsilon_\alpha^*-x_\alpha\,\epsilon_\beta^*)\int_0^1 dz\,e^{i\,z\,p\cdot x}\,h_\gamma(z,\mu)\,. \tag{B.1}$$

$$\langle\gamma(p)|\bar{q}(x)\,W_c(x,0)\,\gamma_\alpha\,q(0)|0\rangle = -g_{\rm em}\,Q_q\,f_{3\gamma}(\mu)\,\epsilon_\alpha^*\int_0^1 dz\,e^{i\,z\,p\cdot x}\,\psi^{(v)}(z,\mu)\,. \tag{B.2}$$

$$\langle\gamma(p)|\bar{q}(x)\,W_c(x,0)\,\gamma_\alpha\,\gamma_5\,q(0)|0\rangle$$

$$= \frac{g_{\rm em}}{4}\,Q_q\,f_{3\gamma}(\mu)\,\varepsilon_{\alpha\beta\rho\tau}\,p^\rho\,x^\tau\,\epsilon^{*\beta}\int_0^1 dz\,e^{i\,z\,p\cdot x}\,\psi^{(a)}(z,\mu)\,. \tag{B.3}$$

$$\langle\gamma(p)|\bar{q}(x)\,W_c(x,0)\,g_s\,G_{\alpha\beta}(v\,x)\,q(0)|0\rangle$$

$$= i\,g_{\rm em}\,Q_q\,\langle\bar{q}q\rangle(\mu)\,(p_\beta\,\epsilon_\alpha^*-p_\alpha\,\epsilon_\beta^*)\int[\mathcal{D}\alpha_i]\,e^{i(\alpha_q+v\alpha_g)p\cdot x}\,S(\alpha_i,\mu)\,. \tag{B.4}$$

$$\langle\gamma(p)|\bar{q}(x)\,W_c(x,0)\,g_s\,\widetilde{G}_{\alpha\beta}(v\,x)\,i\,\gamma_5\,q(0)|0\rangle$$

$$= i\,g_{\rm em}\,Q_q\,\langle\bar{q}q\rangle(\mu)\,(p_\beta\,\epsilon_\alpha^*-p_\alpha\,\epsilon_\beta^*)\int[\mathcal{D}\alpha_i]\,e^{i(\alpha_q+v\alpha_g)p\cdot x}\,\widetilde{S}(\alpha_i,\mu)\,. \tag{B.5}$$

$$\langle\gamma(p)|\bar{q}(x)\,W_c(x,0)\,g_s\,\widetilde{G}_{\alpha\beta}(v\,x)\,\gamma_\rho\,\gamma_5\,q(0)|0\rangle$$

$$= -g_{\rm em}\,Q_q\,f_{3\gamma}(\mu)\,p_\rho\,(p_\beta\,\epsilon_\alpha^*-p_\alpha\,\epsilon_\beta^*)\int[\mathcal{D}\alpha_i]\,e^{i(\alpha_q+v\alpha_g)p\cdot x}\,A(\alpha_i,\mu)\,. \tag{B.6}$$

$$\langle\gamma(p)|\bar{q}(x)\,W_c(x,0)\,g_s\,G_{\alpha\beta}(v\,x)\,i\,\gamma_\rho\,q(0)|0\rangle$$

$$= g_{\rm em}\,Q_q\,f_{3\gamma}(\mu)\,p_\rho\,(p_\beta\,\epsilon_\alpha^*-p_\alpha\,\epsilon_\beta^*)\int[\mathcal{D}\alpha_i]\,e^{i(\alpha_q+v\alpha_g)p\cdot x}\,V(\alpha_i,\mu)\,. \tag{B.7}$$

$$\langle\gamma(p)|\bar{q}(x)\,W_c(x,0)\,g_{\rm em}\,Q_q\,F_{\alpha\beta}(v\,x)\,q(0)|0\rangle$$

$$= i\,g_{\rm em}\,Q_q\,\langle\bar{q}q\rangle(\mu)\,(p_\beta\,\epsilon_\alpha^*-p_\alpha\,\epsilon_\beta^*)\int[\mathcal{D}\alpha_i]\,e^{i(\alpha_q+v\alpha_g)p\cdot x}\,S_\gamma(\alpha_i,\mu)\,. \tag{B.8}$$

$$\langle\gamma(p)|\bar{q}(x)\,W_c(x,0)\,\sigma_{\rho\tau}\,g_s\,G_{\alpha\beta}(v\,x)\,q(0)|0\rangle$$

$$= -g_{\rm em}\,Q_q\,\langle\bar{q}q\rangle(\mu)\left[p_\rho\,\epsilon_\alpha^*\,g_{\tau\beta}^\perp-p_\tau\,\epsilon_\alpha^*\,g_{\rho\beta}^\perp-(\alpha\leftrightarrow\beta)\right]\int[\mathcal{D}\alpha_i]\,e^{i(\alpha_q+v\alpha_g)p\cdot x}\,T_1(\alpha_i,\mu)$$

$$- g_{\rm em}\,Q_q\,\langle\bar{q}q\rangle(\mu)\left[p_\alpha\,\epsilon_\rho^*\,g_{\tau\beta}^\perp-p_\beta\,\epsilon_\rho^*\,g_{\tau\alpha}^\perp-(\rho\leftrightarrow\tau)\right]\int[\mathcal{D}\alpha_i]\,e^{i(\alpha_q+v\alpha_g)p\cdot x}\,T_2(\alpha_i,\mu)$$

$$- g_{\rm em}\,Q_q\,\langle\bar{q}q\rangle(\mu)\,\frac{(p_\alpha\,x_\beta-p_\beta\,x_\alpha)(p_\rho\,\epsilon_\tau^*-p_\tau\,\epsilon_\rho^*)}{p\cdot x}\int[\mathcal{D}\alpha_i]\,e^{i(\alpha_q+v\alpha_g)p\cdot x}\,T_3(\alpha_i,\mu)$$

$$- g_{\rm em}\,Q_q\,\langle\bar{q}q\rangle(\mu)\,\frac{(p_\rho\,x_\tau-p_\tau\,x_\rho)(p_\alpha\,\epsilon_\beta^*-p_\beta\,\epsilon_\alpha^*)}{p\cdot x}\int[\mathcal{D}\alpha_i]\,e^{i(\alpha_q+v\alpha_g)p\cdot x}\,T_4(\alpha_i,\mu)\,. \tag{B.9}$$

$$\langle \gamma(p)|\bar{q}(x) W_c(x,0)\,\sigma_{\rho\tau}\,g_{\rm em}\,Q_q\,F_{\alpha\beta}(v\,x)\,q(0)|0\rangle$$

$$= -g_{\rm em}\,Q_q\,\langle\bar{q}q\rangle(\mu)\,\frac{(p_\rho\,x_\tau - p_\tau\,x_\rho)(p_\alpha\,\epsilon_\beta^* - p_\beta\,\epsilon_\alpha^*)}{p\cdot x}\int[\mathcal{D}\alpha_i]\,e^{i\,(\alpha_q + v\,\alpha_g)\,p\cdot x}\,T_4^\gamma(\alpha_i,\mu) + \dots$$

<div align="right">(B.10)</div>

Here, we have employed the following notations for the dual field strength tensor and the integration measure

$$\widetilde{G}_{\alpha\beta} = \frac{1}{2}\,\varepsilon_{\alpha\beta\rho\tau}\,G^{\rho\tau}\,,\quad \int[\mathcal{D}\alpha_i]\equiv\int_0^1 d\alpha_q\int_0^1 d\alpha_{\bar{q}}\int_0^1 d\alpha_g\,\delta\left(1 - \alpha_q - \alpha_{\bar{q}} - \alpha_g\right).\quad\text{(B.11)}$$

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