

Chiral trace relations in Ω -deformed $\mathcal{N} = 2$ theories

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ABSTRACT: We consider $\mathcal{N} = 2$ $SU(2)$ gauge theories in four dimensions (pure or mass deformed) and discuss the properties of the simplest chiral observables in the presence of a generic Ω -deformation. We compute them by equivariant localization and analyze the structure of the exact instanton corrections to the classical chiral ring relations. We predict exact relations valid at all instanton number among the traces $\langle \text{Tr} \varphi^n \rangle$, where φ is the scalar field in the gauge multiplet. In the Nekrasov-Shatashvili limit, such relations may be explained in terms of the available quantized Seiberg-Witten curves. Instead, the full two-parameter deformation enjoys novel features and the ring relations require non trivial additional derivative terms with respect to the modular parameter. Higher rank groups are briefly discussed emphasizing non-factorization of correlators due to the Ω -deformation. Finally, the structure of the deformed ring relations in the $\mathcal{N} = 2^*$ theory is analyzed from the point of view of the Alday-Gaiotto-Tachikawa correspondence proving consistency as well as some interesting universality properties.

KEYWORDS: Extended Supersymmetry, Nonperturbative Effects, Supersymmetric Effective Theories

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1 Introduction and results

Four dimensional $\mathcal{N} = 2$ super Yang-Mills (SYM) theories are a unique theoretical laboratory where non-perturbative effects are fully under control. This is achieved by combining the Seiberg-Witten (SW) description of the low-energy effective theory [1, 2] with the localization computation of instanton corrections [3–8]. Additional structure is available in superconformal theories with possible mass deformations [9]. In this case, a large variety of new tools have been developed as, in particular, the relation to integrable models [10] and the Alday-Gaiotto-Tachikawa (AGT) 2d/4d correspondence [11, 12].

The duality properties of these models determine important constraints [13]. Remarkable results may be obtained in $\mathcal{N} = 2^*$ theories where an adjoint hypermultiplet of mass m is present beside the gauge vector multiplet. The mass deformation interpolates the $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ transition. The S -duality of the $\mathcal{N} = 4$ theory is inherited for generic mass and the prepotential is constrained by a modular anomaly equation [14].¹ As a consequence of this structure, it is possible to encode the mass expansion of the prepotential in terms of quasi-modular functions of the gauge coupling and the vacuum expectation $\langle \varphi \rangle$ of the scalar in the gauge multiplet. The construction is general and holds for arbitrary gauge groups [37–39].

Recently, the authors of [40] have presented a detailed study of the modular properties of specific observables different from the prepotential, i.e. the chiral traces $\langle \text{Tr } \varphi^n \rangle$ in $\mathcal{N} = 2^*$ $U(N)$ gauge theories, where φ is the scalar field in the gauge multiplet. Supersymmetry implies that correlators of chiral operators factorize and can be expressed in terms of such traces that parametrize the (quantum) chiral ring. An important motivation for the analysis of these chiral observables is that they play a role in the physics of surface operators [41–43]. The associated infrared effects are indeed captured by a twisted two dimensional effective superpotential that can be computed from the expectation values of (higher order) chiral ring elements in the bulk four dimensional theory [44].

In our analysis, we present a discussion of special relations between higher order traces that are exact at all orders in the instanton expansion. In the simplest case of the $SU(2)$ gauge group, these are expressions for $\langle \text{Tr } \varphi^n \rangle$ in terms of $\langle \text{Tr } \varphi^2 \rangle$ for generic parameters of the gauge theory, i.e. the hypermultiplet mass m and the components of $\langle \varphi \rangle$. At the classical

¹The modular anomaly constraint appears in several contexts, e.g. in an Ω background [15–28], from the point of view of the AGT correspondence [29–32], in the large- N limit [24], and in SQCD models with fundamental matter [21, 22, 33, 34]. Recently, it has been shown to be associated with the all-order WKB expansion of quantum mechanical models [35, 36].

level, higher order traces are not independent quantities due to elementary properties of $SU(2)$ matrices. Thus, for even n (odd traces are zero)

$$\langle \text{Tr } \varphi^{2n} \rangle = P_n(u), \quad u = \langle \text{Tr } \varphi^2 \rangle, \quad (1.1)$$

where P_n is a degree n polynomial in u with constant coefficients.² In the following, we shall call *trace relations* constraints like (1.1). Non perturbative instanton effects introduce non-trivial corrections. In the pure gauge theory, the quantum trace relations read

$$\langle \text{Tr } \varphi^{2n} \rangle = P_n(u, q), \quad (1.2)$$

where now P_n is a new polynomial in u with coefficients depending polynomially on the instanton counting parameter q and computable from the gauge theory resolvent [8, 45, 46]. Moving to the $\mathcal{N} = 2^*$ theory, the structure of instanton corrections is completely different. It is again possible to write

$$\langle \text{Tr } \varphi^{2n} \rangle = P_n^*(u, q), \quad (1.3)$$

but now P_n^* is a polynomial in u whose coefficients may be expressed as polynomials in certain Eisenstein series $E_n(q)$ [47]. Thus, there are still exact trace relations, but the precise instanton dependence is more involved as predicted by S -duality [40]. Extension to higher rank groups is essentially the same, except for a larger number of independent coordinates playing the role of u .

It is quite interesting and natural to investigate the properties of chiral traces in Ω -deformed $\mathcal{N} = 2$ theories. This 4d Poincaré breaking deformation depends on two parameters $\varepsilon_1, \varepsilon_2$ and is the main ingredient in the localization approach of [3, 5, 48] where it is needed to regularize the multi-instanton moduli space [4, 6, 8, 49–53].³ The Ω -deformed prepotential expanded around the undeformed limit is the generating function of higher genus amplitudes of the $\mathcal{N} = 2$ topological string [18, 58–63] and satisfies a holomorphic anomaly equation [17, 64–66]. At finite $\varepsilon_1, \varepsilon_2$, the (deformed) partition function is also a fundamental object within the AGT correspondence [11]. In this context, the deformed $\mathcal{N} = 2$ instanton partition function is mapped to conformal blocks of a suitable CFT and may be tested perturbatively in the instanton number [67] and proved in various cases like $\mathcal{N} = 2^*$ theory [68] or linear quivers on sphere [69].⁴

In the simplest case of the $SU(2)$ gauge group the analysis of the ε -deformed chiral ring aims at finding suitable generalizations of (1.2) and (1.3) taking into account the deformation parameters $\varepsilon_1, \varepsilon_2$. Notice that a priori it is not at all trivial that such generalization exists in some reasonable simple form. In principle, this issue may be addressed by (at least) three different approaches. The first is based on the use of a quantized SW curve

²Eq. (1.4) is clearly nothing but an elementary relation among symmetric polynomials.

³The string interpretation of the Ω -background and its BPS excitations has been fully clarified in [54–57] where a geometric interpretation of localization in terms of a dilaton potential is provided giving also a clear picture of all possible deformation parameters.

⁴In the deformed $\mathcal{N} = 2^*$ $SU(2)$ gauge theory the CFT dual quantity is the one-point conformal block on the torus. In the $SU(2)$ theory with four fundamental flavours, it is the conformal block of four Liouville operators on a sphere [30–32, 70–80]. The pure gauge case is somewhat special and has been discussed in [81].

taking into account the ε -deformation. Investigations in this direction have been discussed in [82, 83]. Another option is to exploit the topological string in the spirit of [66, 84, 85]. Finally, one can exploit AGT correspondence. Very recently, this approach has been applied in the $N_f = 4$ theory [86, 87]. Important differences may be expected in the $\mathcal{N} = 2^*$ theory due to its peculiar modular structure. Reversing the logic, an explicit microscopic computation of the relations between the traces $\langle \text{Tr } \varphi^n \rangle$ may be used to prove the correctness of any proposal for the deformed SW curve, or as a test of AGT correspondence.

We looked for the existence of such relations by inspecting the explicit expressions of $\langle \text{Tr } \varphi^n \rangle$ computed from localization at high instanton number in various deformed $\mathcal{N} = 2$ theories. The results of our analysis show that the undeformed trace relations undergo important modifications when the Ω -background is switched on. To illustrate the changes in (1.2) and (1.3) it is convenient to provide some simple illustrative example. In pure $\mathcal{N} = 2$ SU(2) gauge theory, a prototypical relation we obtain is

$$\langle \text{Tr } \varphi^4 \rangle = \frac{1}{2} u^2 + 4q - \varepsilon_1 \varepsilon_2 u', \tag{1.4}$$

where $u = \langle \text{Tr } \varphi^2 \rangle$ and $X' = q \partial_q X$ where q is the instanton counting parameter. The undeformed limit contains the well known one-instanton correction to the classical chiral ring. The novel term, last in (1.4), is present for $\varepsilon_1 \varepsilon_2 \neq 0$, so it vanishes in the Nekrasov-Shatashvili (NS) limit [10] where one of the two ε parameters vanishes. In general, for higher order traces, the gravitational corrections survive even in the NS limit. An example, still in pure gauge theory, is the 6-th order trace relation

$$\langle \text{Tr } \varphi^6 \rangle = \frac{1}{4} u^3 + 6q u - \frac{3}{2} \varepsilon_1 \varepsilon_2 u u' + \varepsilon_1^2 \varepsilon_2^2 u'' + 6q (3 \varepsilon_1^2 + 4 \varepsilon_1 \varepsilon_2 + 3 \varepsilon_2^2), \tag{1.5}$$

where the last term remains if only one of the two ε parameters vanishes. In general, the NS limit of results like (1.5) may be treated perturbatively in the instanton number by means of the available deformed SW curves. However, the modular structure is not captured in an automatic way, although modularity is still at work. To explain this point, we can consider the simplest NS trace relation in the $\mathcal{N} = 2^*$ theory. We shall show that for $(\varepsilon_1, \varepsilon_2) = (\hbar, 0)$

$$\langle \text{Tr } \varphi^4 \rangle = \frac{u^2}{2} + \frac{\mathcal{C}_\hbar}{12} u (E_2 - 1) - \frac{\mathcal{C}_\hbar}{1440} \left[\mathcal{C}_\hbar (5 E_2^2 - 5 E_2 - E_4 + 1) + 12 (E_4 - 1) \hbar^2 \right], \tag{1.6}$$

where $\mathcal{C}_\hbar = 4m^2 - \hbar^2$. The relation (1.6) is exact, i.e. valid at all instanton numbers. The instanton parameter q is the argument of the Eisenstein series $E_k \equiv E_k(q)$. It seems non trivial to extend the undeformed methods of [47] to get relations like (1.6) in a systematic way. Even worse, in the fully deformed case, with $\varepsilon_1 \varepsilon_2 \neq 0$, there is no obvious way to construct a doubly deformed curve providing the $q \partial_q$ derivative terms in (1.4) and (1.5).

Remarkably, we show that in the AGT perspective these extra terms are instead natural. Loosely speaking, they turn out to be associated with higher powers of the Virasoro operator L_0 in the Liouville integrals of motion. This means that the trace relations for the $\mathcal{N} = 2$ theories on a generic Ω background may be considered as a test of AGT, in the same

way as they were in the $N_f = 4$ theory considered in [86, 87]. The identification of the chiral observables $\langle \text{Tr } \varphi^n \rangle$ with the conserved hamiltonians of the Liouville integrable system is of course expected a priori [10], although its precise dictionary requires a choice of basis. In general terms, a simple nice consequence of the AGT interpretation is the prediction of definite universality properties of certain leading derivative terms in the trace relations.

The plan of the paper is the following. We begin by discussing undeformed trace relations and their proof in section 2. Gravitational corrections, i.e. the dependence on the ε deformation, are presented in section 3 in the pure $\mathcal{N} = 2$ gauge theory. The $\mathcal{N} = 2^*$ theory is considered in section 4 by looking first at the NS limit, and later in section 5 by switching on a general background. Section 6 is devoted to the generalization of the previous discussion to $U(N)$ gauge groups. There are no remarkable differences with the exception of interesting violation to supersymmetric factorization of correlators. This is expected in the deformed theory [47] and we give explicit examples. Finally, section 7 presents a discussion of the trace relations in the $\mathcal{N} = 2^*$ theory from the point of view of the AGT correspondence checking the predicted universality property of some derivative terms. Several appendices present technical details, tools, and side comments.

2 Trace relations in undeformed theories

In this section, we consider the undeformed pure gauge $SU(2)$ $\mathcal{N} = 2$ theory and its $\mathcal{N} = 2^*$ mass deformation. We begin by briefly explaining how trace relations may be proved in general terms from the known resolvents of these theories. Then, we discuss the derivation of trace relations from the explicit localization computation of the chiral observables $\langle \text{Tr } \varphi^n \rangle$.

2.1 Trace relations from resolvent expansion

In the pure gauge theory, trace relations may be systematically obtained from the resolvent for chiral 1-point functions [45]

$$\left\langle \text{Tr } \frac{1}{z - \varphi} \right\rangle = \frac{2z}{\sqrt{(z^2 - e^2)^2 - 4q}}, \tag{2.1}$$

where q is the conventional instanton counting parameter. The coordinate e may be conveniently traded by the moduli space coordinate $\mathbf{u} = \langle \text{Tr } \varphi^2 \rangle$.⁵ Expanding at large z and comparing the two sides of (2.1) we immediately recover the well-known relations

$$\begin{aligned} \langle \text{Tr } \varphi^2 \rangle &= \mathbf{u}, & \langle \text{Tr } \varphi^4 \rangle &= \frac{1}{2} \mathbf{u}^2 + 4q, \\ \langle \text{Tr } \varphi^6 \rangle &= \frac{1}{4} \mathbf{u}^3 + 6q\mathbf{u}, & \langle \text{Tr } \varphi^8 \rangle &= \frac{1}{8} \mathbf{u}^4 + 6q\mathbf{u}^2 + 12q^2, \quad \text{etc.} \end{aligned} \tag{2.2}$$

The same strategy may be applied to the $\mathcal{N} = 2^*$ theory as in the analysis of [47]. For completeness, we briefly review the construction. The main tool is the D'Hoker-Phong

⁵Here, we use a boldface symbol for the scalar $\langle \text{Tr } \varphi^2 \rangle$ just to emphasize it better in the following equations.

formulation of the spectral curve [88]. In the SU(2) theory, following [5, 47] to which we defer the reader for a thorough discussion, one can introduce the resolvent

$$G(z) = \left\langle \text{Tr} \frac{1}{z - \varphi - \frac{i}{2} m_c} \right\rangle - \left\langle \text{Tr} \frac{1}{z - \varphi + \frac{i}{2} m_c} \right\rangle, \quad (2.3)$$

and aims at a generalization of the nice formula (2.1). The resolvent $G(z)$ is expected to be analytic with branch cuts $[\alpha_n^- \pm \frac{i}{2} m_c, \alpha_n^+ \pm \frac{i}{2} m_c]$, $n = 1, 2$. The function $\omega(z) = \frac{1}{2\pi i} \int_\infty^z G(y) dy$ maps the cut complex plane to the curve $(\omega(z), z)$ which is a double cover of an elliptic curve obtained by the identification (π factor omitted) $(\omega + 1, z) \sim (\omega, z)$ and $(\omega + \tau, z) \sim (\omega, z + i m_c)$ [89]. This is the double periodicity of the function [88] $f(\omega, z) = z^2 - 2 i m_c z h_1(\omega) + (i m_c)^2 h_2(\omega) - z_1^2$ where

$$h_1(\omega) = \frac{\vartheta_1'(\omega|\tau)}{\vartheta_1(\omega|\tau)}, \quad h_2(\omega) = \frac{\vartheta_1''(\omega|\tau)}{\vartheta_1(\omega|\tau)} = h_1'(\omega) + h_1^2(\omega) \quad (2.4)$$

The resolvent is obtained as the large z expansion of $G(z) = 2\pi i \omega'(z)$ where $\omega(z)$ is implicitly defined by

$$f(\omega(z), -2\pi i z) = 0. \quad (2.5)$$

This can be solved perturbatively at small $\omega \sim 1/z$.⁶ Comparing $G(z) = 2\pi i \omega'(z)$ with the large z expansion of (2.3) we obtain

$$\begin{aligned} \langle \text{Tr} \varphi^2 \rangle &= 4 f_1 m_c^2 - \frac{z_1^2}{2\pi^2}, \\ \langle \text{Tr} \varphi^4 \rangle &= \frac{1}{2} (\langle \text{Tr} \varphi^2 \rangle)^2 + 8 m_c^2 f_1 \langle \text{Tr} \varphi^2 \rangle + m_c^4 \left(\frac{4}{3} f_1 - 32 f_1^2 + \frac{8}{3} f_3 \right), \end{aligned} \quad (2.6)$$

and so on. Using the first equation to replace z_1 by $\mathbf{u} = \langle \text{Tr} \varphi^2 \rangle$ and setting $i m_c = m$ gives all the desired trace relations. The first cases are

$$\begin{aligned} \langle \text{Tr} \varphi^4 \rangle &= \frac{\mathbf{u}^2}{2} + \frac{1}{3} (\mathbf{E}_2 - 1) m^2 \mathbf{u} + \frac{1}{90} m^4 (-5 \mathbf{E}_2^2 + 5 \mathbf{E}_2 + \mathbf{E}_4 - 1), \\ \langle \text{Tr} \varphi^6 \rangle &= \frac{\mathbf{u}^3}{4} + \frac{1}{2} (\mathbf{E}_2 - 1) m^2 \mathbf{u}^2 + \frac{1}{4} m^4 (1 - \mathbf{E}_2) \mathbf{u} \\ &\quad + \frac{m^6 (-140 \mathbf{E}_2^3 + 525 \mathbf{E}_2^2 + 84 \mathbf{E}_2 (\mathbf{E}_4 - 5) - 105 \mathbf{E}_4 - 24 \mathbf{E}_6 + 80)}{7560}, \\ \langle \text{Tr} \varphi^8 \rangle &= \frac{\mathbf{u}^4}{8} + \frac{1}{2} (\mathbf{E}_2 - 1) m^2 \mathbf{u}^3 \\ &\quad + \left(\frac{7}{36} \mathbf{E}_2^2 - \frac{11}{12} \mathbf{E}_2 - \frac{1}{36} \mathbf{E}_4 + \frac{3}{4} \right) m^4 \mathbf{u}^2 \end{aligned}$$

⁶This is achieved by using the representation

$$h_1(\omega) = \pi \cot(\pi \omega) + 4\pi \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} \sin(2\pi n \omega) = \frac{1}{\omega} + \mathcal{O}(\omega).$$

Higher orders in ω involve the Eisenstein functions, see appendix A, since, for instance, $\mathbf{E}_2 = 1 - 24 f_1$, $\mathbf{E}_4 = 1 + 240 f_3$, and $\mathbf{E}_6 = 1 - 504 f_5$ where $f_p = \sum_{n=1}^{\infty} \frac{n^p q^n}{1 - q^n}$.

$$\begin{aligned}
& + \left(-\frac{1}{18}E_2^3 + \frac{7}{108}E_2^2 + \frac{1}{3}E_2 + \frac{7}{270}E_4E_2 - \frac{1}{108}E_4 - \frac{1}{135}E_6 - \frac{19}{54} \right) m^6 \mathbf{u} \\
& + \left(-\frac{E_2^4}{648} + \frac{11E_2^3}{324} + \frac{E_4E_2^2}{324} - \frac{35E_2^2}{324} - \frac{7E_2E_4}{324} - \frac{E_6E_2}{567} + \frac{E_2}{12} + \frac{E_4^2}{4536} \right. \\
& \quad \left. + \frac{71E_4}{3240} + \frac{E_6}{162} - \frac{17}{1080} \right) m^8.
\end{aligned} \tag{2.7}$$

These are non-trivial exact all-instanton relations. It is remarkable that they take such a simple form even though the separate $\langle \text{Tr } \varphi^n \rangle$ are definitely non trivial and, for instance, require an infinite series of corrections in the small mass m expansion.

2.2 Trace relations from localization

There is a simple trick to generate relations like (2.2) [47]. To illustrate it, we start from the explicit expression of $\langle \text{Tr } \varphi^n \rangle$ computed by localization methods that we briefly review in appendix B. In the $\mathcal{N} = 2$ pure gauge theory the explicit values of $\langle \text{Tr } \varphi^n \rangle$ take the form of an equivalent expansion in instanton number or large scalar field vacuum expectation value a . They read

$$\begin{aligned}
\langle \text{Tr } \varphi^2 \rangle &= 2a^2 + \frac{q}{a^2} + \frac{5q^2}{16a^6} + \frac{9q^3}{32a^{10}} + \frac{1469q^4}{4096a^{14}} + \frac{4471q^5}{8192a^{18}} + \dots, \\
\langle \text{Tr } \varphi^4 \rangle &= 2a^4 + 6q + \frac{9q^2}{8a^4} + \frac{7q^3}{8a^8} + \frac{2145q^4}{2048a^{12}} + \frac{1575q^5}{1024a^{16}} + \dots, \\
\langle \text{Tr } \varphi^6 \rangle &= 2a^6 + 15a^2q + \frac{135q^2}{16a^2} + \frac{125q^3}{32a^6} + \frac{16335q^4}{4096a^{10}} + \frac{44343q^5}{8192a^{14}} + \dots, \\
\langle \text{Tr } \varphi^8 \rangle &= 2a^8 + 28a^4q + \frac{161q^2}{4} + \frac{35q^3}{2a^4} + \frac{15337q^4}{1024a^8} + \frac{19173q^5}{1024a^{12}} + \dots,
\end{aligned} \tag{2.8}$$

with vanishing odd traces. We can use the first of these series expansion to write $a = a(\mathbf{u})$ with $\mathbf{u} \equiv \langle \text{Tr } \varphi^2 \rangle$. Replacing in the other equations we get for the pure gauge theory the (exact) relations (2.2). A similar procedure may be applied to the $\mathcal{N} = 2^*$ theory. A closed form cannot be obtained, but the full q dependence can be resummed. This is achieved by means of an educated Ansatz taking into account the modular properties encoding S -duality [14, 21, 22, 40]. Organizing $\langle \text{Tr } \varphi^n \rangle$ as a mass expansion, the results are, see also [40]⁷

$$\begin{aligned}
\langle \text{Tr } \varphi^2 \rangle &= 2a^2 + \frac{1}{6}(E_2 - 1)m^2 + \frac{(E_4 - E_2^2)m^4}{288a^2} + \frac{(-5E_2^3 + 3E_4E_2 + 2E_6)m^6}{17280a^4} + \mathcal{O}(m^8), \\
\langle \text{Tr } \varphi^4 \rangle &= 2a^4 + a^2(E_2 - 1)m^2 + \frac{1}{720}(5E_2^2 - 60E_2 + 13E_4 + 42)m^4 \\
&\quad + \frac{(-20E_2^3 + 15E_2^2 + 18E_4E_2 - 15E_4 + 2E_6)m^6}{8640a^2} + \mathcal{O}(m^8). \\
\langle \text{Tr } \varphi^6 \rangle &= 2a^6 + \frac{5}{2}a^4(E_2 - 1)m^2 + \frac{1}{96}a^2(35E_2^2 - 120E_2 + E_4 + 84)m^4
\end{aligned}$$

⁷Notice that $\langle \text{Tr } \varphi^n \rangle$ with $n \geq 6$ differs from what is obtained using (5.11) of [34], the discrepancy starting at order $\sim m^6 q^2$. This is due to the fact that the compact formulas in [34] are explicitly designed to interpolate the $n \leq 5$ cases. We thank A. Lerda for clarifications on this issue.

$$\begin{aligned}
& + \frac{(-525E_2^3 - 350E_2^2 + 21(39E_4 + 140)E_2 - 2(455E_4 + 57E_6 + 930))}{40320} m^6 \\
& + \mathcal{O}(m^8), \\
\langle \text{Tr } \varphi^8 \rangle & = 2a^8 + \frac{14}{3}a^6(E_2 - 1)m^2 + \frac{a^4}{72}(294 - 420E_2 + 133E_2^2 - 7E_4)m^4 \\
& + \frac{a^2}{4320}(-5580 + 8820E_2 - 3675E_2^2 - 105E_4 + 350E_2^3 + 252E_2E_4 - 62E_6)m^6 \\
& + \mathcal{O}(m^8), \tag{2.9}
\end{aligned}$$

where $E_n(q)$ are the Eisenstein series defined in appendix A. Repeating the trick of writing $a = a(\mathbf{u})$ starting from these expressions, we arrive at the previous trace relations in (2.7).

3 Gravitational corrections to trace relations in pure SU(2) $\mathcal{N} = 2$ theory

We now move to the more interesting case of ε -deformed pure SU(2) $\mathcal{N} = 2$ theory where we look for a generalization of the relations (2.2). Since the ε -deformed resolvent is not available, we try to work out such relations from the explicit localization results. On a generic Ω -background, the values of $\langle \text{Tr } \varphi^n \rangle$ from localization may be organized again in a large a expansion, but the expressions for generic $a, \varepsilon_1, \varepsilon_2$ are quite complicated. Just to illustrate some of the results, we show the expansion around the undeformed limit $\varepsilon = 0$ at third order for $\langle \text{Tr } \varphi^2 \rangle$

$$\begin{aligned}
\langle \text{Tr } \varphi^2 \rangle & = 2a^2 + \frac{q}{a^2} + \frac{5q^2}{16a^6} + \frac{9q^3}{32a^{10}} + \frac{1469q^4}{4096a^{14}} + \dots \\
& + \varepsilon_1\varepsilon_2 \left(-\frac{q^2}{8a^8} - \frac{q^3}{2a^{12}} - \frac{1647q^4}{1024a^{16}} + \dots \right) \\
& + (\varepsilon_1 + \varepsilon_2)^2 \left(\frac{q}{4a^4} + \frac{21q^2}{32a^8} + \frac{55q^3}{32a^{12}} + \frac{18445q^4}{4096a^{16}} + \dots \right) \\
& + (\varepsilon_1\varepsilon_2)^2 \left(\frac{11q^2}{256a^{10}} + \frac{351q^3}{512a^{14}} + \frac{171201q^4}{32768a^{18}} + \dots \right) \\
& + \varepsilon_1\varepsilon_2(\varepsilon_1 + \varepsilon_2)^2 \left(-\frac{35q^2}{64a^{10}} - \frac{689q^3}{128a^{14}} - \frac{269693q^4}{8192a^{18}} + \dots \right) \\
& + (\varepsilon_1\varepsilon_2)^3 \left(-\frac{7q^2}{512a^{12}} - \frac{879q^3}{1024a^{16}} - \frac{985823q^4}{65536a^{20}} + \dots \right) \\
& + (\varepsilon_1\varepsilon_2)^2(\varepsilon_1 + \varepsilon_2)^2 \left(\frac{325q^2}{1024a^{12}} + \frac{23631q^3}{2048a^{16}} + \frac{20930787q^4}{131072a^{20}} + \dots \right) + \dots \tag{3.1}
\end{aligned}$$

Similar expansions for higher order traces are collected in appendix C, but we stress again that we shall always work with the exact localization expressions of $\langle \text{Tr } \varphi^n \rangle$, i.e. not using the small $\varepsilon_1, \varepsilon_2$ expansion in (3.1) that was just a device for illustration.

3.1 Surprises from empirical trace relations

To find trace relations in this case, we can try to repeat the practical procedure that worked in the undeformed theories. In other words, we invert the relation between a and

$\langle \text{Tr } \varphi^2 \rangle$ and replace in the higher traces $\langle \text{Tr } \varphi^n \rangle$. Doing so, we don't find a closed expression generalizing (2.2). Nevertheless, we have been able to propose the following relations that we checked at the level of 10 instantons and, of course, without expansion in ε . Let us denote

$$\mathbf{u} = \langle \text{Tr } \varphi^2 \rangle, \quad X' = q \partial_q X, \quad X'' = (q \partial_q)^2 X, \quad \text{etc.} \quad (3.2)$$

Then, we find the following results

$$\begin{aligned} \langle \text{Tr } \varphi^3 \rangle &= 0, \\ \langle \text{Tr } \varphi^4 \rangle &= \frac{1}{2} \mathbf{u}^2 + 4q - \varepsilon_1 \varepsilon_2 \mathbf{u}', \\ \langle \text{Tr } \varphi^5 \rangle &= 10(\varepsilon_1 + \varepsilon_2)q, \\ \langle \text{Tr } \varphi^6 \rangle &= \frac{1}{4} \mathbf{u}^3 + 6q\mathbf{u} - \frac{3}{2} \varepsilon_1 \varepsilon_2 \mathbf{u} \mathbf{u}' + \varepsilon_1^2 \varepsilon_2^2 \mathbf{u}'' + 6q(3\varepsilon_1^2 + 4\varepsilon_1 \varepsilon_2 + 3\varepsilon_2^2), \\ \langle \text{Tr } \varphi^7 \rangle &= (\varepsilon_1 + \varepsilon_2) \left[21q\mathbf{u} + 7q(4\varepsilon_1^2 + 3\varepsilon_1 \varepsilon_2 + 4\varepsilon_2^2) \right], \\ \langle \text{Tr } \varphi^8 \rangle &= \frac{1}{8} \mathbf{u}^4 + 6q\mathbf{u}^2 + 12q^2 - \varepsilon_1^3 \varepsilon_2^3 \mathbf{u}''' + 2\varepsilon_1^2 \varepsilon_2^2 \mathbf{u} \mathbf{u}'' + \frac{3}{2} \varepsilon_1^2 \varepsilon_2^2 \mathbf{u}'^2 - 12\varepsilon_1 \varepsilon_2 q \mathbf{u}' \\ &\quad - \frac{3}{2} \varepsilon_1 \varepsilon_2 \mathbf{u}^2 \mathbf{u}' + (52\varepsilon_1^2 + 72\varepsilon_1 \varepsilon_2 + 52\varepsilon_2^2) q \mathbf{u} \\ &\quad + 8q(5\varepsilon_1^4 + 11\varepsilon_1^3 \varepsilon_2 + 15\varepsilon_1^2 \varepsilon_2^2 + 11\varepsilon_1 \varepsilon_2^3 + 5\varepsilon_2^4). \end{aligned} \quad (3.3)$$

These expressions show clearly the reason why the naive procedure of replacing a as a function of $\langle \text{Tr } \varphi^2 \rangle$ did not work. In (3.3) there are non trivial derivatives of \mathbf{u} , the r.h.s's are not polynomials in \mathbf{u} when $\varepsilon_1 \varepsilon_2 \neq 0$. We also remark that the relations (3.3) simplify, but do not trivialize in the Nekrasov-Shatashvili (NS) limit

$$\varepsilon_1 = \hbar, \quad \varepsilon_2 = 0. \quad (3.4)$$

In fact, all derivative terms vanish in this case and the trace relations read⁸

$$\begin{aligned} \langle \text{Tr } \varphi^3 \rangle &= 0, & \langle \text{Tr } \varphi^4 \rangle &= \frac{1}{2} \mathbf{u}^2 + 4q, \\ \langle \text{Tr } \varphi^5 \rangle &= 10\hbar q, & \langle \text{Tr } \varphi^6 \rangle &= \frac{1}{4} \mathbf{u}^3 + 6q\mathbf{u} + 18\hbar^2 q, \\ \langle \text{Tr } \varphi^7 \rangle &= \hbar(21q\mathbf{u} + 28\hbar^2 q), & \langle \text{Tr } \varphi^8 \rangle &= \frac{1}{8} \mathbf{u}^4 + 6q\mathbf{u}^2 + 12q^2 + 52\hbar^2 q\mathbf{u} + 40\hbar^4 q. \end{aligned} \quad (3.5)$$

We can show how the peculiar \hbar dependent corrections in (3.5) and deforming the previous (2.2) can be predicted from the deformed SW curve of pure gauge SU(2) theory. This is discussed in appendix D.

4 Gravitational corrections to trace relations in $\mathcal{N} = 2^*$. The NS limit

We begin the analysis of the $\mathcal{N} = 2^*$ theory in the Nekrasov-Shatashvili limit. Indeed, given our experience in the pure gauge theory, we expect major simplifications to occur when one of the deformation parameters vanishes. Later, we shall study the case of a fully deformed background.

⁸Similar corrections have been investigated in [90] in the limit $\varepsilon_1 = -\varepsilon_2$ and in the U(1) gauge theory.

4.1 Localization results

We can compute the chiral traces at some high instanton order from localization and then take the NS limit. The explicit expressions of $\langle \text{Tr } \varphi^n \rangle$ are rather involved. To illustrate them, we trade the hypermultiplet mass by the following combination

$$\mathcal{C}_h = 4m^2 - \hbar^2, \quad \hbar \equiv \varepsilon_1. \quad (4.1)$$

Then, for $\langle \text{Tr } \varphi^2 \rangle$ we obtain

$$\begin{aligned} \langle \text{Tr } \varphi^2 \rangle = & 2a^2 + \mathcal{C}_h \left(\frac{\mathcal{C}_h}{16a^2 - 4\hbar^2} - 1 \right) q + \mathcal{C}_h \left[\frac{\mathcal{C}_h^3(20a^2 + 7\hbar^2)}{256(a^2 - \hbar^2)(4a^2 - \hbar^2)^3} \right. \\ & - \frac{3\mathcal{C}_h^2}{16(a^2 - \hbar^2)(4a^2 - \hbar^2)} + \frac{3\mathcal{C}_h(2a^2 - \hbar^2)}{4(a^2 - \hbar^2)(4a^2 - \hbar^2)} - 3 \left. \right] q^2 \\ & + \mathcal{C}_h \left[\frac{\mathcal{C}_h^5(144a^4 + 232a^2\hbar^2 + 29\hbar^4)}{512(4a^2 - 9\hbar^2)(a^2 - \hbar^2)(4a^2 - \hbar^2)^5} - \frac{\mathcal{C}_h^4(28a^2 + 17\hbar^2)}{32(4a^2 - 9\hbar^2)(a^2 - \hbar^2)(4a^2 - \hbar^2)^3} \right. \\ & + \frac{\mathcal{C}_h^3(120a^4 - 74a^2\hbar^2 - \hbar^4)}{8(4a^2 - 9\hbar^2)(a^2 - \hbar^2)(4a^2 - \hbar^2)^3} - \frac{6\mathcal{C}_h^2}{(4a^2 - 9\hbar^2)(4a^2 - \hbar^2)} \\ & \left. + \frac{3\mathcal{C}_h(4a^2 - 3\hbar^2)}{(4a^2 - 9\hbar^2)(4a^2 - \hbar^2)} - 4 \right] q^3 + \mathcal{O}(q^4). \end{aligned} \quad (4.2)$$

The same expansion for $\langle \text{Tr } \varphi^3 \rangle$ is much simpler

$$\langle \text{Tr } \varphi^3 \rangle = \mathcal{C}_h \hbar \left(-\frac{3}{2}q - \frac{15}{2}q^2 - 15q^3 + \mathcal{O}(q^4) \right), \quad (4.3)$$

and clearly vanish for $\hbar \rightarrow 0$. Besides, it is independent on a . This will be false for the higher odd traces. The next even trace is similar to (4.2) and reads

$$\begin{aligned} \langle \text{Tr } \varphi^4 \rangle = & 2a^4 + \mathcal{C}_h \left[\frac{1}{4}\mathcal{C}_h \left(1 - \frac{2a^2}{\hbar^2 - 4a^2} \right) - 2(3a^2 + \hbar^2) \right] q \\ & + \mathcal{C}_h \left[\frac{\mathcal{C}_h^3(36a^4 - 13a^2\hbar^2 + 4\hbar^4)}{128(a^2 - \hbar^2)(4a^2 - \hbar^2)^3} - \frac{3\mathcal{C}_h^2(3a^2 - 2\hbar^2)}{8((a^2 - \hbar^2)(4a^2 - \hbar^2))} \right. \\ & \left. + \frac{3\mathcal{C}_h(16a^4 - 17a^2\hbar^2 + 3\hbar^4)}{4(a^2 - \hbar^2)(4a^2 - \hbar^2)} - 18(a^2 + \hbar^2) \right] q^2 \\ & + \mathcal{C}_h \left[\frac{\mathcal{C}_h^5(896a^6 + 240a^4\hbar^2 + 16a^2\hbar^4 + 63\hbar^6)}{1024(4a^2 - 9\hbar^2)(a^2 - \hbar^2)(4a^2 - \hbar^2)^5} \right. \\ & - \frac{\mathcal{C}_h^4(880a^4 - 664a^2\hbar^2 - 81\hbar^4)}{256((4a^2 - 9\hbar^2)(a^2 - \hbar^2)(4a^2 - \hbar^2)^3)} \\ & + \frac{\mathcal{C}_h^3(576a^6 - 1084a^4\hbar^2 + 427a^2\hbar^4 - 54\hbar^6)}{8(4a^2 - 9\hbar^2)(a^2 - \hbar^2)(4a^2 - \hbar^2)^3} \\ & - \frac{3\mathcal{C}_h^2(52a^4 - 121a^2\hbar^2 + 54\hbar^4)}{4((4a^2 - 9\hbar^2)(a^2 - \hbar^2)(4a^2 - \hbar^2))} \\ & \left. + \frac{2\mathcal{C}_h(76a^4 - 169a^2\hbar^2 + 36\hbar^4)}{(4a^2 - 9\hbar^2)(4a^2 - \hbar^2)} - 8(3a^2 + 7\hbar^2) \right] q^3 + \mathcal{O}(q^4), \end{aligned} \quad (4.4)$$

and so on. We can invert the relation (4.2) to express

$$a = a(\mathbf{u}), \quad \mathbf{u} \equiv \langle \text{Tr } \varphi^2 \rangle, \tag{4.5}$$

order by order in the instanton number. Then, we replace $a(\mathbf{u})$ in (4.4) and obtain the following quite simple expansions where the dependence on \mathbf{u} turns out to be simply polynomial (we add a few more chiral traces)

$$\begin{aligned} \langle \text{Tr } \varphi^4 \rangle &= \frac{1}{2} \mathbf{u}^2 + (-2q - 6q^2 - 8q^3 + \mathcal{O}(q^4)) \mathcal{C}_h \mathbf{u} \\ &\quad + \frac{1}{4} \mathcal{C}_h q (\mathcal{C}_h - 8\hbar^2) - \frac{1}{4} q^2 \mathcal{C}_h (\mathcal{C}_h + 72\hbar^2) - 7q^3 \mathcal{C}_h (\mathcal{C}_h + 8\hbar^2) + \mathcal{O}(q^4), \\ \langle \text{Tr } \varphi^5 \rangle &= \left(-\frac{15q}{2} - \frac{75q^2}{2} - 75q^3 + \mathcal{O}(q^4) \right) \hbar \mathcal{C}_h \mathbf{u} \\ &\quad + \left(\frac{5}{8} q (\mathcal{C}_h - 4\hbar^2) + \frac{5}{8} q^2 (3\mathcal{C}_h - 68\hbar^2) - \frac{5}{4} q^3 (11\mathcal{C}_h + 164\hbar^2) + \mathcal{O}(q^4) \right) \mathcal{C}_h \hbar, \\ \langle \text{Tr } \varphi^6 \rangle &= \frac{1}{4} \mathbf{u}^3 + (-3q - 9q^2 - 12q^3 + \mathcal{O}(q^4)) \mathcal{C}_h \mathbf{u}^2 \\ &\quad + \left(q \left(\frac{3\mathcal{C}_h}{8} - 15\hbar^2 \right) + q^2 \left(\frac{9\mathcal{C}_h}{8} - 135\hbar^2 \right) + q^3 \left(\frac{3\mathcal{C}_h}{2} - 420\hbar^2 \right) \right) \mathcal{C}_h \mathbf{u} \\ &\quad + \mathcal{C}_h \left(\frac{3}{8} q \hbar^2 (3\mathcal{C}_h - 8\hbar^2) - \frac{3}{16} q^2 (\mathcal{C}_h^2 - 78\mathcal{C}_h \hbar^2 + 528\hbar^4) \right. \\ &\quad \left. - \frac{3}{2} q^3 (\mathcal{C}_h^2 - 18\mathcal{C}_h \hbar^2 + 488\hbar^4) + \mathcal{O}(q^4) \right). \end{aligned} \tag{4.6}$$

We stress that we have obtained the relations (4.6) from explicit localization computations, see the explicit results in (4.2)–(4.4).⁹

As a cross-check of our calculations it is interesting to consider the proposal in [82]. The authors of this paper analyze the NS limit of the Nekrasov integrals by saddle point methods. The advantage is that they are able to work directly at $\varepsilon_2 = 0$. We apply the results of [82] to the $SU(2)$ $\mathcal{N} = 2^*$ case in appendix E with full agreement.

4.2 Empirical all-instanton trace relations in the NS limit

With some educated guess, it is possible to identify the power series in q in terms of Eisenstein series and their odd generalizations, see appendix A. The results up to $\langle \text{Tr } \varphi^7 \rangle$ are summarized in the following expressions

$$\begin{aligned} \langle \text{Tr } \varphi^3 \rangle &= -\frac{3}{2} \mathcal{C}_h E_3 \hbar, \\ \langle \text{Tr } \varphi^4 \rangle &= \frac{\mathbf{u}^2}{2} + \frac{1}{12} \mathbf{u} \mathcal{C}_h (E_2 - 1) - \frac{\mathcal{C}_h (\mathcal{C}_h (5E_2^2 - 5E_2 - E_4 + 1) + 12(E_4 - 1)\hbar^2)}{1440}, \\ \langle \text{Tr } \varphi^5 \rangle &= -\frac{15}{2} \mathbf{u} \mathcal{C}_h E_3 \hbar + \frac{5}{8} \mathcal{C}_h \hbar (2\mathcal{C}_h E_3' - \mathcal{C}_h E_5 - 4E_5 \hbar^2), \\ \langle \text{Tr } \varphi^6 \rangle &= \frac{\mathbf{u}^3}{4} + \frac{1}{8} \mathbf{u}^2 \mathcal{C}_h (E_2 - 1) - \frac{1}{64} \mathbf{u} \mathcal{C}_h (-\mathcal{C}_h + \mathcal{C}_h E_2 + 4E_4 \hbar^2 - 4\hbar^2) \end{aligned}$$

⁹These must be worked out at generic $\varepsilon_1 \equiv \hbar$, and with non zero ε_2 . Only at the end we can take the $\varepsilon_2 \rightarrow 0$ limit.

$$\begin{aligned}
& - \frac{\mathcal{C}_h^3 (140 E_2^3 - 525 E_2^2 - 84 E_2 (E_4 - 5) + 105 E_4 + 24 E_6 - 80)}{483840} \\
& + \frac{\mathcal{C}_h^2 \hbar^2 (7 E_2 E_4 - 2 E_6 + 25200 E_3^2 - 5)}{2240} + \frac{1}{168} \mathcal{C}_h \hbar^4 (E_6 - 1), \\
\langle \text{Tr } \varphi^7 \rangle = & - \frac{105}{8} \mathbf{u}^2 \mathcal{C}_h E_3 \hbar \\
& + \frac{7}{16} \mathbf{u} \mathcal{C}_h \hbar (18 \mathcal{C}_h E_3' - 5 \mathcal{C}_h E_2 E_3 + 5 \mathcal{C}_h E_3 - 15 \mathcal{C}_h E_5 - 60 E_5 \hbar^2) \\
& + \frac{7}{384} \mathcal{C}_h^3 \hbar \left[(5 E_2^2 - 5 E_2 - E_4 + 1) E_3 - 12 E_3' - 24 E_3'' + 48 E_5' - 12 E_7 \right] \quad (4.7) \\
& + \frac{7}{32} \mathcal{C}_h^2 \hbar^3 \left[(E_4 - 1) E_3 + 16 E_5' - 8 E_7 \right] - \frac{7}{2} \mathcal{C}_h \hbar^5 E_7.
\end{aligned}$$

The relations (4.7) are quite interesting because they are valid at all instanton numbers. They are the $\mathcal{N} = 2^*$ version of the much simpler relations (3.5) valid in the pure gauge theory. To prove them in a systematic way, we would need a deformed version of D'Hoker-Phong curve with the full q dependence packaged in Eisenstein series or related objects. Unfortunately, this is not available.

5 Trace relations in $\mathcal{N} = 2^*$ on a generic Ω -background

One can look for trace relations in the generic background with non zero $\varepsilon_1, \varepsilon_2$ parameters. Given our experience in the pure gauge theory, we expect these relations to involve derivatives of the moduli space coordinate $\mathbf{u} = \langle \text{Tr } \varphi^2 \rangle$ making them highly non-trivial. With some insight, we have been able to find them, testing always at high (≥ 10) explicit localization results.

5.1 Empirical trace relations for generic Ω -background

We denote as always $\mathbf{u} = \langle \text{Tr } \varphi^2 \rangle$ and introduce the notations

$$X' \equiv q \frac{d}{dq} X, \quad \mathcal{C} = 4m^2 - (\varepsilon_1 - \varepsilon_2)^2, \quad \mathfrak{p} = \varepsilon_1 \varepsilon_2, \quad \mathfrak{s} = \varepsilon_1 + \varepsilon_2. \quad (5.1)$$

The explicit instanton expansion of $\langle \text{Tr } \varphi^n \rangle$ is highly non-trivial at generic finite a, m, ε_i . Nevertheless, we found the following relations.

$n = 3$. This odd trace is computed by

$$\langle \text{Tr } \varphi^3 \rangle = -\frac{3}{2} \mathcal{C} \mathfrak{s} E_3. \quad (5.2)$$

In general, all odd traces must vanish in the undeformed limit $\varepsilon_i \rightarrow 0$. In our calculation, this will be always due to an explicit $\mathfrak{s} = \varepsilon_1 + \varepsilon_2$ prefactor, see also $\langle \text{Tr } \varphi^5 \rangle$ below.

$n = 4$. The first non trivial even trace is given by the compact expression

$$\langle \text{Tr } \varphi^4 \rangle = \frac{1}{12} \mathcal{C} (E_2 - 1) \mathbf{u} - \mathfrak{p} \mathbf{u}' + \frac{\mathbf{u}^2}{2} + N_4, \quad (5.3)$$

where N_4 does not depend on a and is given by

$$N_4 = -\frac{\mathcal{C}^2}{1440} - \frac{1}{120}\mathcal{C}(p - s^2) - \frac{1}{288}\mathcal{C}(\mathcal{C} - 3p)E_2^2 + \frac{\mathcal{C}^2 E_2}{288} + \frac{\mathcal{C}(\mathcal{C} - 3(p + 4s^2))}{1440}E_4. \quad (5.4)$$

The undeformed and NS limits of this relation reproduce (2.7) and (4.7), respectively. We checked this for the higher traces too finding always agreement. As we expected, there is a contribution $\sim \mathbf{u}'$ whenever $\varepsilon_1 \varepsilon_2 \neq 0$.

$n = 5$. In this case, we found

$$\langle \text{Tr } \varphi^5 \rangle = -\frac{15}{2}\mathcal{C} s E_3 \mathbf{u} + \frac{5}{4}\mathcal{C} s (\mathcal{C} - 4p) E_3' - \frac{5}{8}\mathcal{C} s (\mathcal{C} - 8p + 4s^2) E_5. \quad (5.5)$$

$n = 6$.

$$\begin{aligned} \langle \text{Tr } \varphi^6 \rangle = & \frac{\mathbf{u}^3}{4} + \frac{1}{8}\mathcal{C}(E_2 - 1)\mathbf{u}^2 - \frac{3}{2}p\mathbf{u}\mathbf{u}' - \frac{1}{4}\mathcal{C}p(E_2 - 1)\mathbf{u}' + p^2\mathbf{u}'' \\ & - \frac{1}{192}\mathcal{C}(p(-7E_2^2 - 5E_4 + 12) + 3\mathcal{C}(E_2 - 1) + 12s^2(E_4 - 1))\mathbf{u} + N_6, \end{aligned} \quad (5.6)$$

where N_6 is the following combination independent on a

$$\begin{aligned} N_6 = & \frac{\mathcal{C}^3}{6048} + \frac{\mathcal{C}^2(2p - 3s^2)}{1344} - \frac{1}{168}\mathcal{C}(p^2 - 3ps^2 + s^4) - \frac{\mathcal{C}(\mathcal{C} - 3p)(\mathcal{C} + 2p)}{3456}E_2^3 \\ & + \frac{\mathcal{C}^2(5\mathcal{C} - 8p)}{4608}E_2^2 + \frac{\mathcal{C}E_2(\mathcal{C}^2(E_4 - 5) + \mathcal{C}E_4(18s^2 - 17p) + 6E_4p(7p - 12s^2))}{5760} \\ & + \frac{45}{4}\mathcal{C}^2s^2E_3^2 - \frac{\mathcal{C}^2(\mathcal{C} - 8p)}{4608}E_4 \\ & - \frac{\mathcal{C}(3\mathcal{C}^2 + 54s^2(\mathcal{C} + 6p) - 71\mathcal{C}p + 186p^2 - 360s^4)}{60480}E_6. \end{aligned} \quad (5.7)$$

$n = 7$.

$$\begin{aligned} \langle \text{Tr } \varphi^7 \rangle = & -\frac{105}{8}\mathcal{C} s E_3 \mathbf{u}^2 + \frac{105}{4}\mathcal{C} p s E_3 \mathbf{u}' \\ & + \left(\frac{63}{8}\mathcal{C} s (\mathcal{C} - 4p) E_3' - \frac{35}{16}\mathcal{C}^2 s E_2 E_3 \right. \\ & \left. + \frac{35}{16}\mathcal{C}^2 s E_3 - \frac{105}{16}\mathcal{C} s (\mathcal{C} - 8p + 4s^2) E_5 \right) \mathbf{u} + N_7 \end{aligned} \quad (5.8)$$

where N_7 is given by:

$$\begin{aligned} N_7 = & -\frac{7}{16}\mathcal{C} s (\mathcal{C} - 7p)(\mathcal{C} - 4p) E_3'' - \frac{7}{32}\mathcal{C}^2 s (\mathcal{C} - 4p) E_3' \\ & + \frac{7}{8}\mathcal{C} s (\mathcal{C} - 4p) (\mathcal{C} - 8p + 4s^2) E_5' + \frac{35}{384}\mathcal{C}^2 s (\mathcal{C} - 3p) E_2^2 E_3 \\ & - \frac{35}{384}\mathcal{C}^3 s E_2 E_3 + \frac{7}{384}\mathcal{C}^2 s (\mathcal{C} + 12(p - s^2)) E_3 \\ & - \frac{7}{384}\mathcal{C}^2 s (\mathcal{C} - 3(p + 4s^2)) E_3 E_4 \\ & - \frac{7}{32}\mathcal{C} s (\mathcal{C}^2 + 8s^2(\mathcal{C} - 8p) - 12\mathcal{C}p + 48p^2 + 16s^4) E_7. \end{aligned} \quad (5.9)$$

6 Generalization to $U(N)$ theories

The results we have presented so far have been computed for theories with $SU(2)$ gauge group. It is interesting to extend the analysis to $U(N)$ theories in order to see whether new features arise. For completeness, we briefly discuss the special $U(1)$ case separately in appendix F.

6.1 The pure gauge case

6.1.1 Undeformed ring: classical trace relations in $GL(N)$

As is well known, the classical trace relations for a $N \times N$ matrix φ are obtained from the trivial remark that the characteristic polynomial $\det(z - \varphi)$ has degree N . Thus, we expand the expression

$$\begin{aligned} \det(z - \varphi) &= e^{\log \det(z - \varphi)} = e^{\text{Tr} \log(z - \varphi)} = z^N e^{\text{Tr} \log(1 - \frac{\varphi}{z})} \\ &= z^N e^{\text{Tr}(-\frac{\varphi}{z} - \frac{\varphi^2}{2z^2} + \dots)} \\ &= z^N \left[1 - \frac{t_1}{z} + \frac{\frac{t_1^2}{2} - \frac{t_2}{2}}{z^2} + \frac{-\frac{t_1^3}{6} + \frac{t_2 t_1}{2} - \frac{t_3}{3}}{z^3} + \frac{\frac{t_1^4}{24} - \frac{1}{4} t_2 t_1^2 + \frac{t_3 t_1}{3} + \frac{t_2^2}{8} - \frac{t_4}{4}}{z^4} + \dots \right], \end{aligned} \tag{6.1}$$

where we have denoted $t_n = \text{Tr} \varphi^n$. The combinations multiplying negative powers of z must be identically zero. In the 2×2 case this means that the independent quantities are t_1 and t_2 and we must have

$$\begin{aligned} t_3 &= \frac{3 t_1 t_2}{2} - \frac{t_1^3}{2}, & t_4 &= -\frac{t_1^4}{2} + t_2 t_1^2 + \frac{t_2^2}{2}, \\ t_5 &= \frac{5}{4} t_1 t_2^2 - \frac{t_1^5}{4}, & t_6 &= -\frac{3}{4} t_2 t_1^4 + \frac{3}{2} t_2^2 t_1^2 + \frac{t_2^3}{4}, \\ t_7 &= \frac{t_1^7}{8} - \frac{7}{8} t_2 t_1^5 + \frac{7}{8} t_2^2 t_1^3 + \frac{7}{8} t_2^3 t_1, & t_8 &= \frac{t_1^8}{8} - \frac{1}{2} t_2 t_1^6 - \frac{1}{4} t_2^2 t_1^4 + \frac{3}{2} t_2^3 t_1^2 + \frac{t_2^4}{8}, \end{aligned} \tag{6.2}$$

and so on. In the case of $U(3)$ we have three independent quantities $t_1, t_2,$ and t_3 and, for instance,

$$\begin{aligned} t_4 &= \frac{t_1^4}{6} - t_2 t_1^2 + \frac{4 t_3 t_1}{3} + \frac{t_2^2}{2}, \\ t_5 &= \frac{t_1^5}{6} - \frac{5}{6} t_2 t_1^3 + \frac{5}{6} t_3 t_1^2 + \frac{5 t_2 t_3}{6}, \\ t_6 &= \frac{t_1^6}{12} - \frac{1}{4} t_2 t_1^4 + \frac{1}{3} t_3 t_1^3 - \frac{3}{4} t_2^2 t_1^2 + t_2 t_3 t_1 + \frac{t_2^3}{4} + \frac{t_3^2}{3}. \end{aligned} \tag{6.3}$$

6.1.2 One-instanton corrections in undeformed $SU(N)$ pure gauge theory

In $SU(N)$ theory we have $t_1 = 0$ both at the classical and at the quantum level. The previous relations at classical level take the following simple form in the $SU(2)$ and $SU(3)$ cases

$$SU(2) : \quad t_3 = 0, \quad t_4 = \frac{1}{2} t_2^2, \tag{6.4}$$

and

$$\text{SU}(3) : \quad t_4 = \frac{t_2^2}{2}, \quad t_5 = \frac{5 t_2 t_3}{6}, \quad t_6 = \frac{t_2^3}{4} + \frac{t_3^2}{3}. \quad (6.5)$$

The quantum corrections to these relations should be captured by the relation generalizing (2.1), see for instance [45],

$$\left\langle \text{Tr} \frac{1}{z - \varphi} \right\rangle = \frac{P'(z)}{\sqrt{P^2(z) - 4\Lambda^{2N}}}, \quad P(z) = z^N + \sum_{\ell=2}^N u_\ell z^{N-\ell}. \quad (6.6)$$

The first instanton correction appears in $\text{Tr} \varphi^{2N}$ and can be read from the large z expansions

$$\begin{aligned} \text{SU}(2) : \quad & \left\langle \text{Tr} \frac{1}{z - \varphi} \right\rangle = \frac{2}{z} - \frac{2u_2}{z^3} + \frac{4\Lambda^4 + 2u_2^2}{z^5} + \dots, \\ \text{SU}(3) : \quad & \left\langle \text{Tr} \frac{1}{z - \varphi} \right\rangle = \frac{3}{z} - \frac{2u_2}{z^3} - \frac{3u_3}{z^4} + \frac{2u_2^2}{z^5} + \frac{5u_2u_3}{z^6} + \frac{6\Lambda^6 - 2u_2^3 + 3u_3^2}{z^7} + \dots \end{aligned} \quad (6.7)$$

Of course, the undeformed results in (6.7) are in agreement with the general prediction in eq. (6.4) of [40].

6.1.3 All instanton relations in the U(2) gauge theory on generic background

The expressions in section 3.1 admit a straightforward generalization to U(2). The only difference is the presence of the trivial trace

$$t_1 = \langle \text{Tr} \varphi \rangle = a_1 + a_2. \quad (6.8)$$

This expression is exact and does not receive instanton corrections.¹⁰ Thus, in terms of t_1 and $t_2 = \langle \text{Tr} \varphi^2 \rangle$ we obtain after some trial and error the following list of exact relations that we have checked at 10 instantons

$$\begin{aligned} \langle \text{Tr} \varphi^3 \rangle &= \frac{3t_1 t_2}{2} - \frac{t_1^3}{2}, \\ \langle \text{Tr} \varphi^4 \rangle &= 4q - \frac{t_1^4}{2} + t_1^2 t_2 + \frac{t_2^2}{2} - \varepsilon_1 \varepsilon_2 t_2', \\ \langle \text{Tr} \varphi^5 \rangle &= 10q(t_1 + \varepsilon_1 + \varepsilon_2) - \frac{t_1^5}{4} + \frac{5t_1 t_2^2}{4} - \frac{5}{2} \varepsilon_1 \varepsilon_2 t_1 t_2', \\ \langle \text{Tr} \varphi^6 \rangle &= 12q t_1^2 + 30q t_1 (\varepsilon_1 + \varepsilon_2) + 6q t_2 + 6q (3\varepsilon_1^2 + 4\varepsilon_2 \varepsilon_1 + 3\varepsilon_2^2) - \frac{3t_1^4 t_2}{4} + \frac{3t_1^2 t_2^2}{2} \\ &\quad - 3t_1^2 \varepsilon_1 \varepsilon_2 t_2' + \frac{t_2^3}{4} + \varepsilon_1^2 \varepsilon_2^2 t_2'' - \frac{3}{2} \varepsilon_1 \varepsilon_2 t_2 t_2', \\ \langle \text{Tr} \varphi^7 \rangle &= 7q t_1^3 + 42q t_1^2 (\varepsilon_1 + \varepsilon_2) + 21q t_1 t_2 + 21q t_1 (3\varepsilon_1^2 + 4\varepsilon_2 \varepsilon_1 + 3\varepsilon_2^2) \\ &\quad + 21q t_2 (\varepsilon_1 + \varepsilon_2) + 7q (4\varepsilon_1^3 + 7\varepsilon_2 \varepsilon_1^2 + 7\varepsilon_2^2 \varepsilon_1 + 4\varepsilon_2^3) + \frac{t_1^7}{8} - \frac{7t_1^5 t_2}{8} + \frac{7t_1^3 t_2^2}{8} \\ &\quad - \frac{7}{4} \varepsilon_1 \varepsilon_2 t_1^3 t_2' + \frac{7t_1 t_2^3}{8} + \frac{7}{2} \varepsilon_1^2 \varepsilon_2^2 t_1 t_2'' - \frac{21}{4} \varepsilon_1 \varepsilon_2 t_1 t_2 t_2', \end{aligned}$$

¹⁰This is clear from (B.9) whose linear part in z is only $\sum_{u=1}^N a_u$.

$$\begin{aligned}
 \langle \text{Tr } \varphi^8 \rangle = & 12 q^2 - 2 q t_1^4 + 28 q t_1^3 (\varepsilon_1 + \varepsilon_2) + 36 q t_1^2 t_2 + 4 q t_1^2 (25 \varepsilon_1^2 + 33 \varepsilon_2 \varepsilon_1 + 25 \varepsilon_2^2) \\
 & + 84 q t_1 t_2 (\varepsilon_1 + \varepsilon_2) + 28 q t_1 (4 \varepsilon_1^3 + 7 \varepsilon_2 \varepsilon_1^2 + 7 \varepsilon_2^2 \varepsilon_1 + 4 \varepsilon_2^3) + 6 q t_2^2 - 12 q \varepsilon_1 \varepsilon_2 t_2' \\
 & + q t_2 (52 \varepsilon_1^2 + 72 \varepsilon_2 \varepsilon_1 + 52 \varepsilon_2^2) + 8 q (5 \varepsilon_1^4 + 11 \varepsilon_2 \varepsilon_1^3 + 15 \varepsilon_2^2 \varepsilon_1^2 + 11 \varepsilon_2^3 \varepsilon_1 + 5 \varepsilon_2^4) \\
 & + \frac{t_1^8}{8} - \frac{t_1^6 t_2}{2} - \frac{t_1^4 t_2^2}{4} + \frac{1}{2} \varepsilon_1 \varepsilon_2 t_1^4 t_2' + \frac{3 t_1^2 t_2^3}{2} + 6 \varepsilon_1^2 \varepsilon_2^2 t_1^2 t_2'' - 9 \varepsilon_1 \varepsilon_2 t_1^2 t_2 t_2' + \frac{t_2^4}{8} \\
 & - \varepsilon_1^3 \varepsilon_2^3 t_2''' + 2 \varepsilon_1^2 \varepsilon_2^2 t_2 t_2'' + \frac{3}{2} \varepsilon_1^2 \varepsilon_2^2 (t_2')^2 - \frac{3}{2} \varepsilon_1 \varepsilon_2 t_2^2 t_2', \tag{6.9}
 \end{aligned}$$

where, as usual, $X' \equiv q \partial_q X$.

6.1.4 U(3) on a generic background: some surprise

Repeating the same kind of analysis in the case of U(3) we find something new. Now, the independent coordinates are

$$t_1 = \langle \text{Tr } \varphi \rangle = a_1 + a_2 + a_3, \quad t_2 = \langle \text{Tr } \varphi^2 \rangle, \quad t_3 = \langle \text{Tr } \varphi^3 \rangle. \tag{6.10}$$

The relations we find for $\langle \text{Tr } \varphi^4 \rangle$ and $\langle \text{Tr } \varphi^5 \rangle$ are similar to the previous ones and read

$$\begin{aligned}
 \langle \text{Tr } \varphi^4 \rangle &= \frac{t_1^4}{6} - t_1^2 t_2 + \frac{4 t_1 t_3}{3} + \frac{t_2^2}{2} - \varepsilon_1 \varepsilon_2 t_2', \tag{6.11} \\
 \langle \text{Tr } \varphi^5 \rangle &= \frac{t_1^5}{6} - \frac{5 t_1^3 t_2}{6} + \frac{5 t_1^2 t_3}{6} + \frac{5 t_2 t_3}{6} - \frac{5}{3} \varepsilon_1 \varepsilon_2 t_3'.
 \end{aligned}$$

As a check of (6.11), we can consider the undeformed limit $\varepsilon = 0$ and restrict to SU(3) setting $t_1 = 0$. This gives the classical relations (6.5). This is correct because, according to (6.7), the first instanton correction is $\sim q = \Lambda^6$ and appears at dimension 6, i.e. in $\langle \text{Tr } \varphi^6 \rangle$. Looking at $\langle \text{Tr } \varphi^6 \rangle$ for generic ε one can try to mimic (6.11) by fitting a generic dimension 6 combination of monomials in t_1, t_2, t_3 , and their derivatives.¹¹ Quite surprisingly, we could not find a solution in this way. Inspecting the low instanton corrections in full details, we noticed that the required missing structures appear in the following double trace expectation values

$$t_{n,m} = \langle \text{Tr } \varphi^n \text{Tr } \varphi^m \rangle, \tag{6.12}$$

for the dimension 6 cases $(n, m) = (1, 5), (2, 4), (3, 3)$. Usually, double traces like those in (6.12) are not relevant because in the undeformed limit the supersymmetry algebra implies factorization of such correlators. However, with non zero ε_1 and ε_2 , this is not the case.¹² Including $\langle \text{Tr } \varphi \text{Tr } \varphi^5 \rangle$, $\langle \text{Tr } \varphi^2 \text{Tr } \varphi^4 \rangle$, and $\langle (\text{Tr } \varphi^3)^2 \rangle$ in the analysis, we find the following simple relation

$$\langle \text{Tr } \varphi^6 \rangle = -6 q + \frac{1}{12} t_1^6 - \frac{1}{4} t_1^4 t_2 + \frac{1}{3} t_1^3 t_3 - \frac{3}{4} t_1^2 t_2^2 + t_1 t_2 t_3 + \frac{1}{4} t_2^3 + \frac{1}{3} t_{3,3}, \tag{6.13}$$

¹¹The derivative $q \partial_q$ increases effectively the dimension by 2 because such terms are always accompanied by explicit $\varepsilon_1 \varepsilon_2$ factors. This reduces the set of monomials to be considered. However, to be sure, we relaxed this hypothesis and checked it just at the end.

¹²This peculiar violation of factorization appears already in the NS limit, see for instance the discussion in section 2.3 of [47], e.g. their eq. (2.25).

where we emphasized the double trace term. Besides, another outcome of the analysis is the vanishing of the following auxiliary combinations $K_1 = K_2 = 0$, where

$$\begin{aligned}
 K_1 &= -\frac{1}{5} t_1^6 + t_1^4 t_2 - t_1^3 t_3 - t_1 t_2 t_3 + \frac{6}{5} \mathbf{t}_{1,5}, \\
 K_2 &= -\frac{1}{30} t_1^6 - \frac{1}{6} t_1^3 t_3 + t_1^2 t_2^2 - 2 \varepsilon_1 \varepsilon_2 t_1^2 t_2' - \frac{3}{2} t_1 t_2 t_3 + 3 \varepsilon_1 \varepsilon_2 t_1 t_3' + \frac{1}{5} \mathbf{t}_{1,5} \\
 &\quad - \frac{1}{2} t_2^3 - 2 \varepsilon_1^2 \varepsilon_2^2 t_2'' + 3 \varepsilon_1 \varepsilon_2 t_2 t_2' + \mathbf{t}_{2,4}.
 \end{aligned} \tag{6.14}$$

Again, we can look at (6.13) in the undeformed SU(3) limit. Using in this case $t_1 = 0$ and $\langle (\text{Tr } \varphi^3)^2 \rangle = \langle \text{Tr } \varphi^3 \rangle^2$, we recover the one-instanton correction to the classical relation in (6.14). Besides, the auxiliary relations (6.14) are found to vanish using again $t_{1,5} = t_1 t_5 = 0$, $t_{2,4} = t_2 t_4$ and replacing t_4 by means of (6.11). One can also study the NS limit $\varepsilon_2 \rightarrow 0$ with the methods of appendix (D) and again, one has full agreement exploiting the observation that the three dimension 6 double traces factorize in this limit, as we explicitly checked.

6.2 Mass deformation: the SU(3) $\mathcal{N} = 2^*$ theory on a generic background

We can analyze in a similar way the $\mathcal{N} = 2^*$ theory. As an illustrative example we take the gauge group to be SU(3). Recalling the definitions in (5.1), we have for $\langle \text{Tr } \varphi^4 \rangle$

$$\begin{aligned}
 \langle \text{Tr } \varphi^4 \rangle &= \frac{1}{2} t_2^2 + \frac{1}{16} \mathcal{C} (E_2 - 1) t_2 - p t_2' + \frac{1}{256} \mathcal{C}^2 E_2 \\
 &\quad - \frac{\mathcal{C} E_2^2 (9\mathcal{C} - 32p)}{1536} - \frac{\mathcal{C} (\mathcal{C} + 32 (p - s^2))}{2560} + \frac{\mathcal{C} E_4 (9\mathcal{C} - 32p - 48s^2)}{3840}.
 \end{aligned} \tag{6.15}$$

The expression (6.15) is similar in structure to the analogous one for gauge group SU(2) in (5.3). For $\langle \text{Tr } \varphi^5 \rangle$, we find

$$\begin{aligned}
 \langle \text{Tr } \varphi^5 \rangle &= \frac{5}{6} t_2 t_3 - \frac{5}{3} p t_3' - \frac{5}{32} \mathcal{C} t_3 + \frac{5}{32} \mathcal{C} E_2 t_3 - \frac{45}{8} \mathcal{C} s E_3 t_2 - \frac{45}{128} \mathcal{C}^2 s E_3 \\
 &\quad + \frac{45}{128} \mathcal{C}^2 s E_2 E_3 - \frac{15}{16} \mathcal{C} s (\mathcal{C} - 8p + 4s^2) E_5 + \frac{15}{16} \mathcal{C} s (3\mathcal{C} - 16p) E_3',
 \end{aligned} \tag{6.16}$$

to be compared with (5.5). Notice that there is no prefactor s in this case. This is because the trace is not trivial in the undeformed theory. Finally, for $\langle \text{Tr } \varphi^6 \rangle$ we find

$$\begin{aligned}
 \langle \text{Tr } \varphi^6 \rangle &= \frac{1}{4} t_2^3 + \frac{1}{3} \mathbf{t}_{3,3} - \frac{3}{2} p t_2 t_2' + p^2 t_2''' + \frac{7}{64} \mathcal{C} (E_2 - 1) t_2^2 - \frac{7}{32} \mathcal{C} p (E_2 - 1) t_2' \\
 &\quad - \frac{3}{512} \mathcal{C}^2 E_2 t_2 - \frac{27}{2} \mathcal{C} s E_3 t_3 + \frac{1}{768} \mathcal{C} (3\mathcal{C} - 2(p + 24s^2)) E_4 t_2 \\
 &\quad + \frac{\mathcal{C} (200p - 27\mathcal{C})}{3072} E_2^2 t_2 + \frac{\mathcal{C} (11\mathcal{C} + 64(s^2 - p))}{1024} t_2 \\
 &\quad + \frac{\mathcal{C} (653\mathcal{C}^2 - 738\mathcal{C} (83p - 198s^2) + 19680 (10p^2 - 9ps^2 - 18s^4))}{1269760} E_2 \\
 &\quad + \frac{\mathcal{C} (-313\mathcal{C}^2 + 8\mathcal{C} (949p - 2079s^2) - 2240 (10p^2 - 9ps^2 - 18s^4))}{253952} E_4 \\
 &\quad + \frac{9\mathcal{C} (135\mathcal{C}^2 + \mathcal{C} (5184s^2 - 4980p) + 1600 (10p^2 - 9ps^2 - 18s^4))}{3968} E_3^2
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\mathcal{C}(-99\mathcal{C}^2 + \mathcal{C}(4303p - 10014s^2) + 16(-878p^2 + 939ps^2 + 1382s^4))}{126976} \\
& + \frac{\mathcal{C}(393\mathcal{C}^2 + \mathcal{C}(5940s^2 - 3110p) + 800(10p^2 - 9ps^2 - 18s^4))}{253952} E_2^2 \\
& + \frac{\mathcal{C}(531\mathcal{C}^2 - 8\mathcal{C}(821p - 3051s^2) + 320(52p^2 - 363ps^2 + 18s^4))}{3809280} E_2 E_4 \\
& + \frac{75\mathcal{C}(27\mathcal{C}^2 + \mathcal{C}(2376s^2 - 996p) + 320(10p^2 - 9ps^2 - 18s^4))}{15872} E_3 \\
& - \frac{3\mathcal{C}(27\mathcal{C}^2 + \mathcal{C}(2376s^2 - 996p) + 320(10p^2 - 9ps^2 - 18s^4))}{15872} E_2 E_3 \\
& + \frac{9\mathcal{C}(27\mathcal{C}^2 + \mathcal{C}(2376s^2 - 996p) + 320(10p^2 - 9ps^2 - 18s^4))}{1984} E_3' \\
& - \frac{\mathcal{C}(72\mathcal{C}^2 + \mathcal{C}(599p + 198s^2) - 80(38p^2 + 3ps^2 + 6s^4))}{380928} E_2^3, \tag{6.17}
\end{aligned}$$

to be compared with (5.6) in the SU(2) case. Again, we have emphasized the double trace $t_{3,3}$ appearing in the first line of (6.17).

7 Trace relations in Ω -deformed $\mathcal{N} = 2^*$ and AGT correspondence

In this section, we attempt to understand the origin of the trace relations in the SU(2) $\mathcal{N} = 2^*$ theory on a generic Ω -background exploiting AGT correspondence. To this aim, we begin with a brief review of the analysis of the $N_f = 4$ theory recently presented by Fucito, Morales and Poghossian (FMP) [86, 87]. Then, in a similar spirit, we discuss what happens in the case of the $\mathcal{N} = 2^*$ theory.

7.1 Review of Fucito-Morales-Poghossian results

In the recent papers [86, 87], Fucito, Morales, and Poghossian discuss the chiral correlators $\langle \text{Tr } \varphi^n \rangle$ in the SU(2) $\mathcal{N} = 2$ gauge theory with $N_f = 4$ fundamental hypermultiplets in terms of four-point correlators in the Liouville theory with the insertion of the Liouville theory integrals of motion introduced in [69]. Here, we summarize their results to set up our notation and as a preliminary step for next application to the $\mathcal{N} = 2^*$ theory.

7.1.1 CFT side

The symmetry algebra of the Liouville theory is $\text{Vir} \times \text{Heis}$ with mode operators

$$\begin{aligned}
[L_m, L_n] &= (m - n) L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0}, \\
[a_m, a_n] &= \frac{m}{2} \delta_{m+n,0}, \quad [L_m, a_n] = 0. \tag{7.1}
\end{aligned}$$

The standard AGT parametrization of the central charge is $c = 1 + 6Q^2$ with $Q = b + b^{-1}$. Primary fields are $V_\alpha = V_\alpha^{\text{Vir}} V_\alpha^{\text{Heis}}$ where V_α^{Vir} is a primary with conformal dimension $\Delta(\alpha) = \alpha(Q - \alpha)$ and

$$V_\alpha^{\text{Heis}}(z) = \exp\left(2i(\alpha - Q) \sum_{n < 0} \frac{a_n}{n} z^{-n}\right) \exp\left(2i\alpha \sum_{n > 0} \frac{a_n}{n} z^{-n}\right). \tag{7.2}$$

As usual, primary states are defined by $|\alpha\rangle = V_\alpha(0)|0\rangle$ where $|0\rangle$ is the standard vacuum state. We are interested in conformal blocks, i.e. four point functions with the exchange of an α -primary. It can be shown that

$$G(\alpha_i, \alpha|z) = \langle \alpha_1 | V_{\alpha_2}(1) V_{\alpha_3}(z) | \alpha_4 \rangle_\alpha = (1-z)^{2\alpha_2(Q-\alpha_3)} G^{\text{Vir}}(\alpha_i, \alpha|z), \quad (7.3)$$

where $G^{\text{Vir}}(\alpha_i, \alpha|z)$ is the standard conformal block. Following FMP, we can introduce Liouville integrals of motion according to, cf. [69],

$$\begin{aligned} I_2 &= L_0 - \frac{c}{24} + 2 \sum_{k=1}^{\infty} a_{-k} a_k, \\ I_3 &= \sum_{k \in \mathbb{Z} \setminus \{0\}} a_{-k} L_k + 2iQ \sum_{k=1}^{\infty} k a_{-k} a_k + \frac{1}{3} \sum_{i+j+k=0} a_i a_j a_k, \\ I_4 &= 2 \sum_{k=1}^{\infty} L_{-k} L_k + L_0^2 - \frac{c+2}{12} L_0 + \text{other terms involving Heis.} \end{aligned} \quad (7.4)$$

These may be inserted in the conformal block to build the new quantities

$$G_n(\alpha_i, \alpha|z) = \langle \alpha_1 | V_{\alpha_2}(1) I_n V_{\alpha_3}(z) | \alpha_4 \rangle_\alpha. \quad (7.5)$$

Exploiting the Vir \times Heis algebra it is possible to prove relations like

$$G_2 = \left(z \partial_z + \Delta_3 + \Delta_4 - \frac{c}{24} \right) G. \quad (7.6)$$

and similar (more involved) ones for higher G_n 's.

7.1.2 Chiral traces and AGT correspondence

The AGT correspondence relates the four-point conformal block of the Liouville theory to the partition function of the $\mathcal{N} = 2$ supersymmetric SU(2) gauge theory with four fundamentals. The parameter q is identified with the harmonic ratio z parametrizing the positions of vertex insertions. The four dimensions Δ_i are functions of the masses while the internal dimension Δ is function of the vacuum expectation value a . Finally, the deformation parameters appear in the central charge formula according to the proportionality $\varepsilon_1 : \varepsilon_2 = b : b^{-1}$ [11]. This dictionary allows to rewrite relations like (7.6) in terms of gauge theory parameters once a precise correspondence between $\langle \text{Tr } \varphi^n \rangle$ and G_n is established. The relations proposed by FMP are

$$\begin{aligned} \langle \text{Tr } \varphi^2 \rangle &\equiv \mathbf{u} = -2 \frac{G_2}{G} - \frac{1}{12}, & \langle \text{Tr } \varphi^3 \rangle &= 6i \frac{G_3}{G}, \\ \langle \text{Tr } \varphi^4 \rangle &= 2p^2 \frac{G_4}{G} - \frac{p}{4} \mathbf{u} + \frac{s^2(p+s^2)}{8}. \end{aligned} \quad (7.7)$$

With this identifications, it is possible to use relations like (7.6) and write $\langle \text{Tr } \varphi^n \rangle$ in terms of multiple $q\partial_q$ derivatives of $\log Z$ where Z is the partition function of the gauge theory. In particular, Matone's relation [91] reads in this context

$$\langle \text{Tr } \varphi^2 \rangle = -2pq \partial_q \log Z. \quad (7.8)$$

We remark that the precise form of I_n is not enough to predict $\langle \text{Tr } \varphi^n \rangle$ because we need the map from the quantities G_n to the generators of the chiral ring, as in (7.7). Nevertheless, it is important to emphasize that the presence of higher powers of L_0 immediately implies the occurrence of further $q\partial_q$ -derivatives in higher traces.

7.2 The $\mathcal{N} = 2^*$ theory

7.2.1 Hints from Matone's relation

Let us analyze the $\mathcal{N} = 2^*$ theory on a generic Ω -deformation starting from the generalized Matone's relation [8, 91]

$$\langle \text{Tr } \varphi^2 \rangle = 2a^2 - 2\varepsilon_1 \varepsilon_2 q \partial_q \log Z_{\text{inst}}, \tag{7.9}$$

where Z_{inst} is the instanton partition function of the gauge theory. The AGT dictionary reads

$$c = 1 + 6Q^2, \quad Q = b + b^{-1}, \quad b = \sqrt{\varepsilon_2/\varepsilon_1}, \tag{7.10}$$

$$\Delta = \frac{Q^2}{4} - \frac{a^2}{\varepsilon_1 \varepsilon_2}, \quad \Delta_m = \frac{Q^2}{4} - \frac{m^2}{\varepsilon_1 \varepsilon_2}. \tag{7.11}$$

The instanton partition function may be computed in terms of the Vir algebra with central charge c according to¹³

$$Z_{\text{inst}} = \left[q^{-\frac{1}{24}} \eta(\tau) \right]^{2\Delta_m - 1} \mathcal{F}_{\Delta_m}^\Delta(q). \tag{7.12}$$

Here, the so-called torus one-point functions $\mathcal{F}_{\Delta_m}^\Delta(q)$ is

$$\mathcal{F}_{\Delta_m}^\Delta(q) = \text{Tr}_\Delta \left(\mathcal{O}_{\Delta_m} q^{L_0 - \Delta} \right), \tag{7.13}$$

where the trace is over all descendants of the Virasoro primary \mathcal{O}_Δ .¹⁴ After these preliminary definitions we can plug (7.12) into (7.9) to find¹⁵

$$\begin{aligned} \langle \text{Tr } \varphi^2 \rangle &= 2a^2 - 2\varepsilon_1 \varepsilon_2 \left[(2\Delta_m - 1) \frac{E_2 - 1}{24} + q \partial_q \log \mathcal{F}_{\Delta_m}^\Delta(q) \right] \\ &= (4m^2 - \varepsilon_1^2 - \varepsilon_2^2) \frac{E_2 - 1}{24} - \frac{\varepsilon_1 \varepsilon_2}{12} - 2\varepsilon_1 \varepsilon_2 \frac{\text{Tr}_\Delta \left(\mathcal{O}_{\Delta_m} (L_0 - \frac{c}{24}) q^{L_0 - \frac{c}{24}} \right)}{\text{Tr}_\Delta \left(\mathcal{O}_{\Delta_m} q^{L_0 - \frac{c}{24}} \right)}. \end{aligned} \tag{7.14}$$

¹³The Dedekind η function is $\eta = q^{\frac{1}{24}} \prod_{k=1}^\infty (1 - q^k)$. Notice that $q \frac{d}{dq} \log \prod_{k=1}^\infty (1 - q^k) = \frac{1}{24} (E_2 - 1)$.

¹⁴To check normalization, it is useful to recall the following expansions at small q or large intermediate dimension Δ

$$\mathcal{F}_{\Delta_m}^\Delta(q) = 1 + \mathcal{O}(q), \quad \mathcal{F}_{\Delta_m}^\Delta(q) = \frac{q^{\frac{1}{24}}}{\eta(\tau)} \left[1 + \mathcal{O} \left(\frac{1}{\Delta} \right) \right].$$

¹⁵We can check (7.14) in the large Δ limit or — what is the same — at large a in the gauge theory. Using the second expansion in (7.14) we have

$$\langle \text{Tr } \varphi^2 \rangle = 2a^2 - 2\varepsilon_1 \varepsilon_2 \left[(2\Delta_m - 2) \frac{E_2 - 1}{24} + \mathcal{O}(a^{-1}) \right] = 2a^2 + (4m^2 - (\varepsilon_1 - \varepsilon_2)^2) \frac{E_2 - 1}{24} + \mathcal{O}(a^{-1}).$$

This is in agreement with the undeformed limit in (2.9) and also with explicit similar expressions with non zero ε .

eq. (7.14) may be considered as the $\mathcal{N} = 2^*$ version of the first relation in (7.7). Here the role of G and G_2 is played by the one-point torus function with possible insertion of the Virasoro part of I_2 in (7.4). In other words, (7.14) is totally similar to (7.7) when written in the form

$$\langle \text{Tr } \varphi^2 \rangle = -2 \varepsilon_1 \varepsilon_2 \frac{G_2^*}{G^*} + (4m^2 - \varepsilon_1^2 - \varepsilon_2^2) \frac{E_2 - 1}{24} - \frac{\varepsilon_1 \varepsilon_2}{12}, \quad (7.15)$$

where, see (7.3) and (7.5),

$$G^* = \text{Tr}_\Delta(\mathcal{O}_{\Delta_m} q^{L_0 - \frac{c}{24}}), \quad G_2^* = \text{Tr}_\Delta(\mathcal{O}_{\Delta_m} I_2^{\text{Vir}} q^{L_0 - \frac{c}{24}}) = q \partial_q G^*. \quad (7.16)$$

7.2.2 A consistency check: prediction of the leading terms in $\langle \text{Tr } \varphi^n \rangle$

The trace relations for $\langle \text{Tr } \varphi^n \rangle$ have been fully written out in section 5.1 up to $n = 8$. Special *leading* terms are dimension n monomials according to the weights $[\mathbf{u}] = 2$ and $[q \partial_q] = 2$. This definition isolates simple non trivial parts of $\langle \text{Tr } \varphi^n \rangle$ that do not involve Eisenstein sums and are non zero for even n . The first cases are (dots stand for lower dimension terms)

$$\begin{aligned} \langle \text{Tr } \varphi^2 \rangle &= \mathbf{u}, & (\text{by definition}), \\ \langle \text{Tr } \varphi^4 \rangle &= -p \mathbf{u}' + \frac{1}{2} \mathbf{u}^2 + \dots, \\ \langle \text{Tr } \varphi^6 \rangle &= p^2 \mathbf{u}'' - \frac{3}{2} p \mathbf{u} \mathbf{u}' + \frac{1}{4} \mathbf{u}^3 + \dots, \\ \langle \text{Tr } \varphi^8 \rangle &= -p^3 \mathbf{u}''' + 2p^2 \mathbf{u} \mathbf{u}'' + \frac{3}{2} p^2 (\mathbf{u}')^2 - \frac{3}{2} p \mathbf{u}^2 \mathbf{u}' + \frac{1}{8} \mathbf{u}^4 + \dots \end{aligned} \quad (7.17)$$

It is natural to expect that these terms come from the genuine n -th order part of the integral of motion I_n , i.e. the part that cannot be modified with a mixing from lower order integrals of motion. In $\langle \text{Tr } \varphi^{2n} \rangle$ this is nothing but the operator L_0^n . Thus, a natural conjecture is

$$\langle \text{Tr } \varphi^{2n} \rangle_{\text{leading}} = 2 \frac{(-p q \partial_q)^n G_{\text{leading}}^*}{G_{\text{leading}}^*}, \quad G_{\text{leading}}^* = \exp\left(-\frac{U}{2p}\right), \quad q \partial_q U = \mathbf{u}. \quad (7.18)$$

In fact, we can check that (7.18) works perfectly. For instance,

$$\begin{aligned} \langle \text{Tr } \varphi^4 \rangle &= 2p^2 e^{\frac{U}{2p}} \left[e^{-\frac{U}{2p}} \right]'' = -p U'' + \frac{1}{2} (U')^2 = -p \mathbf{u}' + \frac{1}{2} \mathbf{u}^2, \\ \langle \text{Tr } \varphi^6 \rangle &= -2p^3 e^{\frac{U}{2p}} \left[e^{-\frac{U}{2p}} \right]''' = p^2 U''' - \frac{3}{2} p U' U'' + \frac{1}{4} (U')^3 \\ &= p^2 \mathbf{u}'' - \frac{3}{2} p \mathbf{u} \mathbf{u}' + \frac{1}{4} \mathbf{u}^3, \end{aligned} \quad (7.19)$$

and so on. Notice that (7.18) is a non trivial constraint as further discussed in appendix G.

Structure of subleading terms

The subleading terms in the trace relations depend on the precise mapping between the integrals of motion and the generators of the chiral ring. Nevertheless, they are captured by

suitable insertions of powers of L_0 . We can consider for instance the list of 27 \mathbf{u} -dependent contributions to $\langle \text{Tr } \varphi^8 \rangle$ in (5.10). Replacing \mathbf{u} by $(\log Z_{\text{inst}})'$ using (7.9) we see that $\langle \text{Tr } \varphi^8 \rangle$ is a linear combination of terms $\sim Z_{\text{inst}}^{(k)}/Z_{\text{inst}}$ that encode all the nonlinearities in \mathbf{u} and its derivatives. Up to a dependent mixing terms we have for instance

$$\begin{aligned} \langle \text{Tr } \varphi^8 \rangle = & 2 p^4 \frac{Z_{\text{inst}}^{(4)}}{Z_{\text{inst}}} - \mathcal{C} p^3 (E_2 - 1) \frac{Z_{\text{inst}}^{(3)}}{Z_{\text{inst}}} + (c_1 + c_2 E_2 + c_3 E_2^2 + c_4 E_4) p^2 \frac{Z_{\text{inst}}^{(2)}}{Z_{\text{inst}}} \\ & + (c_5 + c_6 E_2 + c_7 E_2^2 + c_8 E_4 + c_9 E_2^3 + c_{10} E_2 E_4 + c_{11} E_3^2 + c_{12} E_6) p \frac{Z_{\text{inst}}^{(1)}}{Z_{\text{inst}}} \\ & + \dots, \end{aligned} \tag{7.20}$$

where c_i may be expressed as linear combinations of the k_i in (5.10).

Universality of the leading terms. A further consistency check of the AGT interpretation of trace relations follows from the following argument. Let us consider AGT correspondence for pure gauge SU(2) theory. The relevant CFT object is the irregular conformal block [81]. To define it, one starts with the Whittaker vector

$$|\Delta, \Lambda^2\rangle = v_0 + \Lambda^2 v_1 + \Lambda^4 v_2 + \dots, \tag{7.21}$$

where $v_0 \equiv |\Delta\rangle$ is a Virasoro highest weight state and the components v_n are determined by the conditions

$$L_1 v_n = v_{n-1}, \quad L_2 v_n = 0. \tag{7.22}$$

The instanton partition function of the pure gauge SU(2) theory is then simply

$$Z_{\text{inst}} = \langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \rangle = \sum_{n=0}^{\infty} \Lambda^{4n} \|v_n\|^2, \tag{7.23}$$

where Λ^4 is identified with the instanton counting parameter. Of course, the Virasoro data Δ and c are translated in gauge theory parameters $a, \varepsilon_1, \varepsilon_2$ with the usual AGT dictionary, see (7.10) and (7.11). If we now assume that the chiral observables are obtained by insertion of integrals of motion, we have the following schematic relation for the leading terms

$$\langle \text{Tr } \varphi^{2n} \rangle_{\text{leading}} \sim \frac{\langle \Delta, \Lambda^2 | L_0^n | \Delta, \Lambda^2 \rangle}{\langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \rangle} \sim \frac{(q \partial_q)^n Z_{\text{inst}}}{Z_{\text{inst}}}. \tag{7.24}$$

This is same as in $\mathcal{N} = 2^*$ and leads to the conclusion

$$\langle \text{Tr } \varphi^{2n} \rangle_{\text{leading}} \Big|_{\text{pure gauge}} = \langle \text{Tr } \varphi^{2n} \rangle_{\text{leading}} \Big|_{\mathcal{N}=2^*}. \tag{7.25}$$

From inspection of the explicit leading terms in (3.3) and comparing with (7.17), we confirm that this is indeed true in our localization computation.

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A Eisenstein series

The Eisenstein series E_2 , E_4 , and E_6 [92] admit the representation

$$\begin{aligned}
 E_2(q) &= 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n = 1 - 24q - 72q^2 - 96q^3 - 168q^4 + \mathcal{O}(q^5) \\
 E_4(q) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \mathcal{O}(q^5) \\
 E_6(q) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \mathcal{O}(q^5),
 \end{aligned}
 \tag{A.1}$$

where $\sigma_k(n)$ is the divisor function $\sigma_k(n) = \sum_{d|n} d^k$. The Eisenstein series have well defined properties under the modular group $SL(2, \mathbb{Z})$. In particular, E_4 and E_6 are modular forms of weight 4 and 6, while E_2 is a quasi modular form of degree 2. For the purposes of this paper, we shall also introduce the following non-standard objects.

$$E_{2k+1} \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \sigma_{2k}(n) q^n,
 \tag{A.2}$$

or explicitly

$$\begin{aligned}
 E_3(q) &= \sum_{n=1}^{\infty} \sigma_2(n) q^n = q + 5q^2 + 10q^3 + 21q^4 + \mathcal{O}(q^5) \\
 E_5(q) &= \sum_{n=1}^{\infty} \sigma_4(n) q^n = q + 17q^2 + 82q^3 + 273q^4 + \mathcal{O}(q^5).
 \end{aligned}
 \tag{A.3}$$

These series are not natural from the point of view of modular transformations, but are somewhat expected in the spirit of the derivation discussed in section 2, see footnote 6.

B Chiral observables from localization

The deformed partition function as well the chiral traces may be computed systematically by localization, see for instance [93] and references therein. Here, we briefly discuss the illustrative case of the $\mathcal{N} = 2^*$ gauge theory with gauge group $U(N)$. Other simpler cases may be treated in quite similar way. Focusing on the algorithmic implementation the k -instanton corrections to the partition function $Z = 1 + \sum_{k=1}^{\infty} Z_k q^k$ are obtained as

$$Z_k(\mathbf{a}, m, \varepsilon_1, \varepsilon_2) = \sum_{|Y_1| + \dots + |Y_N| = k} Z(Y_1, \dots, Y_N),
 \tag{B.1}$$

where $\mathbf{a} = \langle \varphi \rangle = (a_1, \dots, a_N)$ and we sum over all N -tuples (Y_1, Y_2, \dots, Y_N) of Young tableaux with a total of k blocks ($|Y|$ is the number of blocks in a tableau). For each N -tuple (Y_1, \dots, Y_N) , we build the symbol

$$V = T_{a_1} V(Y_1) + \dots + T_{a_N} V(Y_N),
 \tag{B.2}$$

where

$$V(Y) = \sum_{ij} T_{\varepsilon_1}^{i-1} T_{\varepsilon_2}^{j-1}. \tag{B.3}$$

In these expressions the symbols T_x must be thought as Abelian characters with

$$T_x T_y = T_{x+y}, \quad T_x^n = T_{nx}, \quad T_x^* = T_{-x}, \quad \text{and so on.} \tag{B.4}$$

The sum over i, j in (B.3) is over the blocks of Y , $i \geq 1$ is the row and $j \geq 1$ is the column. After computing V , we introduce the universal object $W = T_{a_1} + \dots + T_{a_N}$, and evaluate

$$\begin{aligned} T_{\text{gauge}} &= -VV^*(1 - T_{\varepsilon_1})(1 - T_{\varepsilon_2}) + V^*W + VW^* T_{\varepsilon_1} T_{\varepsilon_2}, \\ T_{\text{matter}} &= T_{m - \frac{\varepsilon_1 + \varepsilon_2}{2}} T_{\text{gauge}}. \end{aligned} \tag{B.5}$$

These expressions may be obtained from the exact sequence associated with an instanton [6, 8, 94]. The results may be written in the form

$$T_{\text{gauge}} = \sum_i n_i T_{x_i}, \quad T_{\text{matter}} = \sum_i m_i T_{y_i}, \quad n_i, m_i \in \mathbb{Z}, \tag{B.6}$$

From (B.6), we can read the partition function associated with N -tuple of Young tableaux as

$$Z(Y_1, \dots, Y_N) = \frac{\prod_i y_i^{m_i}}{\prod_i x_i^{n_i}}. \tag{B.7}$$

For an observable \mathcal{O} we shall introduce a specific function $\mathcal{O}(\mathbf{Y})$ and evaluate

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \sum_{k=0}^{\infty} q^k \sum_{|\mathbf{Y}|=k} \mathcal{Z}(\mathbf{Y}) \mathcal{O}(\mathbf{Y}). \tag{B.8}$$

In the case of $\text{Tr } e^{z\varphi}$ the recipe is to use the following $\mathcal{O}(\mathbf{Y})$, see for instance [7],

$$\mathcal{O}(\mathbf{Y}) = \sum_{u=1}^N \left[e^{z a_u} + (1 - e^{z \varepsilon_1})(1 - e^{z \varepsilon_2}) \sum_{i_R=1}^{\# \text{ rows of } Y_u} \sum_{i_C=1}^{\# \text{ cols of } Y_u} e^{z(a_u + (i_R-1)\varepsilon_1 + (i_C-1)\varepsilon_2)} \right]. \tag{B.9}$$

Expanding in powers of z we compute $\langle \text{Tr } \varphi^n \rangle$. Multiple traces are evaluated in the same way by considering, e.g., the map $\langle \text{Tr } \varphi^n \text{ Tr } \varphi^m \rangle \rightarrow \mathcal{O}_n(\mathbf{Y}) \mathcal{O}_m(\mathbf{Y})$, and so on.

C Higher order traces in deformed $\text{SU}(2)$ $\mathcal{N} = 2$ theory

We list here some long expansions for higher traces in the deformed $\text{SU}(2)$ $\mathcal{N} = 2$ theory. These are organized in a double expansion at large a around the undeformed point $\varepsilon_1 = \varepsilon_2 = 0$.

$$\begin{aligned} \langle \text{Tr } \varphi^4 \rangle &= 2a^4 + 6q + \frac{9q^2}{8a^4} + \frac{7q^3}{8a^8} + \frac{2145q^4}{2048a^{12}} + \dots \\ &+ \varepsilon_1 \varepsilon_2 \left(-\frac{q}{a^2} - \frac{7q^2}{8a^6} - \frac{63q^3}{32a^{10}} - \frac{5315q^4}{1024a^{14}} + \dots \right) \end{aligned}$$

$$\begin{aligned}
& + (\varepsilon_1 + \varepsilon_2)^2 \left(\frac{q}{2a^2} + \frac{25q^2}{16a^6} + \frac{267q^3}{64a^{10}} + \frac{22529q^4}{2048a^{14}} + \dots \right) \\
& + (\varepsilon_1\varepsilon_2)^2 \left(\frac{43q^2}{128a^8} + \frac{373q^3}{128a^{12}} + \frac{288189q^4}{16384a^{16}} + \dots \right) \\
& + \varepsilon_1\varepsilon_2(\varepsilon_1 + \varepsilon_2)^2 \left(-\frac{q}{4a^4} - \frac{77q^2}{32a^8} - \frac{33q^3}{2a^{12}} - \frac{367069q^4}{4096a^{16}} + \dots \right) \\
& + (\varepsilon_1\varepsilon_2)^3 \left(-\frac{29q^2}{256a^{10}} - \frac{1939q^3}{512a^{14}} - \frac{1699071q^4}{32768a^{18}} + \dots \right) \\
& + (\varepsilon_1\varepsilon_2)^2(\varepsilon_1 + \varepsilon_2)^2 \left(\frac{885q^2}{512a^{10}} + \frac{40503q^3}{1024a^{14}} + \frac{30341215q^4}{65536a^{18}} + \dots \right) + \dots \quad (C.1)
\end{aligned}$$

$$\begin{aligned}
\langle \text{Tr } \varphi^6 \rangle & = 2a^6 + 15a^2q + \frac{135q^2}{16a^2} + \frac{125q^3}{32a^6} + \frac{16335q^4}{4096a^{10}} + \dots \\
& + \varepsilon_1\varepsilon_2 \left(-15q - \frac{15q^2}{4a^4} - \frac{105q^3}{16a^8} - \frac{2025q^4}{128a^{12}} + \dots \right) \\
& + (\varepsilon_1 + \varepsilon_2)^2 \left(\frac{75q}{4} + \frac{135q^2}{32a^4} + \frac{735q^3}{64a^8} + \frac{124575q^4}{4096a^{12}} + \dots \right) \\
& + (\varepsilon_1\varepsilon_2)^2 \left(\frac{q}{a^2} + \frac{545q^2}{256a^6} + \frac{5139q^3}{512a^{10}} + \frac{1645387q^4}{32768a^{14}} + \dots \right) \\
& + \varepsilon_1\varepsilon_2(\varepsilon_1 + \varepsilon_2)^2 \left(-\frac{3q}{4a^2} - \frac{405q^2}{64a^6} - \frac{639q^3}{16a^{10}} - \frac{1759671q^4}{8192a^{14}} + \dots \right) \\
& + (\varepsilon_1\varepsilon_2)^3 \left(-\frac{409q^2}{512a^8} - \frac{13887q^3}{1024a^{12}} - \frac{9539613q^4}{65536a^{16}} + \dots \right) \\
& + (\varepsilon_1\varepsilon_2)^2(\varepsilon_1 + \varepsilon_2)^2 \left(\frac{q}{4a^4} + \frac{7023q^2}{1024a^8} + \frac{213033q^3}{2048a^{12}} + \frac{142336817q^4}{131072a^{16}} + \dots \right) \\
& + \dots \quad (C.2)
\end{aligned}$$

$$\begin{aligned}
\langle \text{Tr } \varphi^7 \rangle & = (\varepsilon_1 + \varepsilon_2) \left(42a^2q + \frac{21q^2}{a^2} + \frac{105q^3}{16a^6} + \frac{189q^4}{32a^{10}} + \dots \right) \\
& + \varepsilon_1\varepsilon_2(\varepsilon_1 + \varepsilon_2) \left(-35q - \frac{21q^3}{8a^8} - \frac{21q^4}{2a^{12}} + \dots \right) \\
& + (\varepsilon_1 + \varepsilon_2)^3 \left(28q + \frac{21q^2}{4a^4} + \frac{441q^3}{32a^8} + \frac{1155q^4}{32a^{12}} + \dots \right) \\
& + (\varepsilon_1\varepsilon_2)^2(\varepsilon_1 + \varepsilon_2) \left(\frac{231q^3}{256a^{10}} + \frac{7371q^4}{512a^{14}} + \dots \right) \\
& + \varepsilon_1\varepsilon_2(\varepsilon_1 + \varepsilon_2)^3 \left(-\frac{735q^3}{64a^{10}} - \frac{14469q^4}{128a^{14}} + \dots \right) + \dots \quad (C.3)
\end{aligned}$$

$$\begin{aligned}
\langle \text{Tr } \varphi^8 \rangle & = 2a^8 + 28a^4q + \frac{161q^2}{4} + \frac{35q^3}{2a^4} + \frac{15337q^4}{1024a^8} + \dots \\
& + \varepsilon_1\varepsilon_2 \left(-70a^2q - \frac{217q^2}{4a^2} - \frac{567q^3}{16a^6} - \frac{31269q^4}{512a^{10}} + \dots \right) \\
& + (\varepsilon_1 + \varepsilon_2)^2 \left(105a^2q + \frac{497q^2}{8a^2} + \frac{1505q^3}{32a^6} + \frac{99561q^4}{1024a^{10}} + \dots \right)
\end{aligned}$$

$$\begin{aligned}
& + (\varepsilon_1 \varepsilon_2)^2 \left(28q + \frac{651 q^2}{64 a^4} + \frac{1197 q^3}{32 a^8} + \frac{1318933 q^4}{8192 a^{12}} + \dots \right) \\
& + \varepsilon_1 \varepsilon_2 (\varepsilon_1 + \varepsilon_2)^2 \left(-\frac{147q}{2} - \frac{385 q^2}{16 a^4} - \frac{4053 q^3}{32 a^8} - \frac{1244957 q^4}{2048 a^{12}} + \dots \right) \\
& + (\varepsilon_1 \varepsilon_2)^3 \left(-\frac{q}{a^2} - \frac{649 q^2}{128 a^6} - \frac{12021 q^3}{256 a^{10}} - \frac{6830463 q^4}{16384 a^{14}} + \dots \right) \\
& + (\varepsilon_1 \varepsilon_2)^2 (\varepsilon_1 + \varepsilon_2)^2 \left(\frac{q}{a^2} + \frac{5141 q^2}{256 a^6} + \frac{137995 q^3}{512 a^{10}} + \frac{87543287 q^4}{32768 a^{14}} + \dots \right) + \dots
\end{aligned} \tag{C.4}$$

D Proof of the trace relations in the NS limit

We adopt the proposal discussed in [83] for the deformed SW curve in the NS limit.¹⁶ To this aim, we introduce the function $y(z)$ obeying the difference equation

$$y(z) y(z + \hbar) - P(z) y(z) + q = 0, \quad P(z) = z^2 - e^2. \tag{D.1}$$

We solve (D.1) perturbatively in \hbar setting

$$y(z) = \frac{1}{2} \left[Y(z) + z^2 - e^2 \right], \quad Y(z) = \sum_{n=0}^{\infty} Y_n(z) \hbar^n. \tag{D.2}$$

The 0th order is the undeformed standard curve (in quartic form with no linear term) $Y_0(z) = Q(z) \equiv \sqrt{P^2(z) - 4q}$. The deformed SW differential is then written in terms of $\Psi(z) = \log [P(z) + Y(z)]$, and the chiral traces can be extracted from the resolvent $\Psi'(z + \hbar)$ taking into account the \hbar shift proposed in [83]. Plugging into (D.1) the expression (D.2) with the Ansatz

$$Y(z) = z^2 + \xi_1 z + \xi_0 + \xi_{-1} \frac{1}{z} + \dots, \tag{D.3}$$

we determine very easily the coefficients $\{\xi_n\}$ and, using $\mathbf{u} = 2 e^2$, we find

$$\begin{aligned}
\Psi'(z + \hbar) &= \frac{2}{z} + \frac{\mathbf{u}}{z^3} + \frac{1}{z^5} \left(\frac{\mathbf{u}^2}{2} + 4q \right) + \frac{10 q \hbar}{z^6} + \frac{1}{z^7} \left(\frac{\mathbf{u}^3}{4} + 6 q \mathbf{u} + 18 q \hbar^2 \right) \\
&+ \frac{1}{z^8} \left(21 q \mathbf{u} \hbar + 28 q \hbar^3 \right) + \frac{1}{z^9} \left(\frac{\mathbf{u}^4}{8} + 6 q \mathbf{u} + 12 q^2 + 52 q \mathbf{u} \hbar^2 + 40 q \hbar^4 \right) \\
&+ \dots
\end{aligned} \tag{D.4}$$

in agreement with (3.5). Notice that the terms in (3.5) have the same weight under the natural assignment

$$[\varphi] = 1, \quad [\mathbf{u}] = 2, \quad [q] = 4, \quad [\hbar] = 1. \tag{D.5}$$

¹⁶We alert the reader that we slightly change the notation compared with [83].

Just to give an example, this procedure gives immediately the next two relations extending the list (3.5). They read

$$\begin{aligned} \langle \text{Tr } \varphi^9 \rangle &= 9q(10q + 3\mathbf{u}^2)\hbar + 108q\mathbf{u}\hbar^3 + 54q\hbar^5, \\ \langle \text{Tr } \varphi^{10} \rangle &= \frac{\mathbf{u}^5}{16} + 5q\mathbf{u}^3 + 30q^2\mathbf{u} + 5q(86q + 17\mathbf{u}^2)\hbar^2 + 200q\mathbf{u}\hbar^4 + 70q\hbar^6, \end{aligned} \quad (\text{D.6})$$

and turns out to be perfectly satisfied by the explicit localization result for $\langle \text{Tr } \varphi^9 \rangle$ and $\langle \text{Tr } \varphi^{10} \rangle$.

E A cross check by saddle point methods

Trace relations in the NS limit of the $\text{SU}(2)$ $\mathcal{N} = 2^*$ case may be treated by the quantized curve proposed in [82]. It allows to deal with the NS limit of the Nekrasov integrals by saddle point methods working directly at $\varepsilon_2 = 0$. The practical perturbative algorithm is straightforward. We fix an integer $L \geq 1$ and introduce the quantities (where $a_1 = a = -a_2$)

$$x_{u,\ell} = x_{u,\ell}^{(0)} + \sum_{k=\ell}^L \xi_{u,\ell}^{(k)} q^k, \quad x_{u,\ell}^{(0)} = a_u + (\ell - 1)\hbar. \quad (\text{E.1})$$

The set $\{\xi_{u,\ell}^{(k)}\}$ is determined by replacing (E.1) into the saddle point equation

$$1 - q \frac{w_L(x) w_L(x + \hbar)}{w_L(x + m + \hbar) w_L(x - m)} \Big|_{x=x_{u,\ell}} = 0. \quad (\text{E.2})$$

where

$$w_L(x) = \frac{1}{P(x - L\varepsilon)} \prod_{u=1}^2 \prod_{\ell=1}^L \frac{x - x_{u,\ell} - \hbar}{x - x_{u,\ell}}, \quad P(x) = x^2 - a^2. \quad (\text{E.3})$$

Eq. (E.2) must be used as follows: first we evaluate the combination of w_L functions with generic x . Several cancellations occur and after algebraic simplification it is possible to set x to each of the $x_{u,\ell}$. Given L , we work out (E.2) at order q^L included. Increasing L , the previous solutions involving $\xi^{(k)}$ with $k < L$ are unchanged. So the method is iterative, we can start from $L = 1$, solve the constants $\xi^{(1)}$, use them in the Ansatz with $L = 2$ and so on. After these steps, the chiral traces are simply given by the universal formula

$$\langle \text{Tr } \varphi^n \rangle = \sum_u a_u^n + \sum_{u,\ell} \left[x_{u,\ell}^n - (x_{u,\ell} + \hbar)^n - (x_{u,\ell}^{(0)})^n + (x_{u,\ell}^{(0)} + \hbar)^n \right]. \quad (\text{E.4})$$

after the shift $m \rightarrow m - \frac{\varepsilon}{2}$. We have checked that (E.4) agrees with the previous computations up to at least 3 instantons and for n up to 10. For illustration, let us explain the procedure at the simple 1-instanton level. The 1-instanton solution (E.1) is simply

$$x_{1,1} = a + \frac{\mathcal{C}_\hbar(8a(2a + \hbar) - \mathcal{C}_\hbar)}{32a\hbar(2a + \hbar)} q + \mathcal{O}(q^2), \quad x_{2,1} = -a - \frac{\mathcal{C}_\hbar(8a(\hbar - 2a) + \mathcal{C}_\hbar)}{32a\hbar(2a - \hbar)} q + \mathcal{O}(q^2). \quad (\text{E.5})$$

Writing a in terms of $\mathbf{u} \equiv \langle \text{Tr } \varphi^2 \rangle$ and applying (E.4) we recover the same results as from Nekrasov calculation, i.e. for even n , we find

$$\begin{aligned} \langle \text{Tr } e^{z\varphi} \rangle &= 2 \cosh \left(\sqrt{\frac{\mathbf{u}}{2}} z \right) \\ &\quad - q \frac{\mathcal{C}_h z}{16 \sqrt{\mathbf{u}} \hbar (2\mathbf{u} - \hbar^2)} \left[\sqrt{2} \hbar \sinh \left(\sqrt{\frac{\mathbf{u}}{2}} z \right) (\mathcal{C}_h \cosh(z\hbar) + \mathcal{C}_h + 8(\hbar^2 - 2\mathbf{u})) \right. \\ &\quad \left. - 2\sqrt{\mathbf{u}} (\mathcal{C}_h - 8\mathbf{u} + 4\hbar^2) \cosh \left(\sqrt{\frac{\mathbf{u}}{2}} z \right) \sinh(z\hbar) \right] + \mathcal{O}(q^2), \end{aligned} \tag{E.6}$$

with a similar expression for odd n . As a check, we can compute the $z \rightarrow 0$ expansion of (E.6) and we find (dots denote $\mathcal{O}(q^2)$ contributions)

$$\begin{aligned} \langle \text{Tr } e^{z\varphi} \rangle &= 2 + \frac{z^2}{2!} \mathbf{u} + \frac{z^4}{4!} \left[\frac{\mathbf{u}^2}{2} + \mathbf{u}(-2\mathcal{C}_h q + \dots) + \frac{1}{4} \mathcal{C}_h q (\mathcal{C}_h - 8\hbar^2) + \dots \right] \\ &\quad + \frac{z^6}{6!} \left[\frac{\mathbf{u}^3}{4} + \mathbf{u}^2(-3\mathcal{C}_h q + \dots) + \mathbf{u} \left(\frac{3}{8} \mathcal{C}_h (\mathcal{C}_h - 40\hbar^2) q + \dots \right) \right. \\ &\quad \left. + q \left(\frac{9\mathcal{C}_h^2 \hbar^2}{8} - 3\mathcal{C}_h \hbar^4 \right) + \dots \right] + \mathcal{O}(z^8), \end{aligned} \tag{E.7}$$

in agreement with the 1-instanton contribution in (4.6).

F The special case of U(1) theories

In this appendix we briefly discuss the special U(1) case. Formally, we shall be using the same localization expressions valid for U(N). This is a useful check given the explicit results of [90] to be discussed in a moment.

F.1 The pure gauge theory

Let us recall the definitions

$$\mathbf{p} = \varepsilon_1 \varepsilon_2, \quad \mathbf{s} = \varepsilon_1 + \varepsilon_2. \tag{F.1}$$

An explicit calculation gives the following exact expressions¹⁷

$$\begin{aligned} \langle \text{Tr } \varphi^2 \rangle &= -2q, \\ \langle \text{Tr } \varphi^3 \rangle &= -3qs, \\ \langle \text{Tr } \varphi^4 \rangle &= 2q(\mathbf{p} - 2s^2) + 6q^2, \\ \langle \text{Tr } \varphi^5 \rangle &= 5qs(\mathbf{p} - s^2) + 25q^2s, \\ \langle \text{Tr } \varphi^6 \rangle &= q(-2\mathbf{p}^2 + 9\mathbf{p}s^2 - 6s^4) + q^2(75s^2 - 30\mathbf{p}) - 20q^3, \\ \langle \text{Tr } \varphi^7 \rangle &= -7q\left(s(\mathbf{p} - s^2)^2\right) + q^2(196s^3 - 182\mathbf{p}s) - 154q^3s, \\ \langle \text{Tr } \varphi^8 \rangle &= 2q(\mathbf{p}^3 - 8\mathbf{p}^2s^2 + 10\mathbf{p}s^4 - 4s^6) \\ &\quad + 14q^2(9\mathbf{p}^2 - 52\mathbf{p}s^2 + 34s^4) + 56q^3(5\mathbf{p} - 14s^2) + 70q^4. \end{aligned} \tag{F.3}$$

¹⁷Notice that to compare with [90] we need to send $q \rightarrow -q$. This sign flip for U(N) theories with odd N will be further discussed later.

Due to the fact that these expressions are polynomials in q , it is possible to find simple relations fully discussed in [90]. The authors of [90] considered the further special limit $\varepsilon_1 = -\varepsilon_2 = \hbar$, i.e. $p = -\hbar^2$ and $s = 0$. Then, one has simple relations like

$$\langle \text{Tr } \varphi^4 \rangle - \hbar^2 \langle \text{Tr } \varphi^2 \rangle = 6 q^2, \quad \langle \text{Tr } \varphi^6 \rangle - 5 \hbar^2 \langle \text{Tr } \varphi^4 \rangle + 4 \hbar^4 \langle \text{Tr } \varphi^2 \rangle = -20 q^3, \quad (\text{F.4})$$

and so on.

F.2 The $\mathcal{N} = 2^*$ theory

The $U(1)$ $\mathcal{N} = 2^*$ has not been considered in [90], but it is a simple extension. Now, the chiral observables $\langle \text{Tr } \varphi^n \rangle$ are not polynomials in q . Nevertheless, they are polynomials in m and ε . In particular, we recall the definition

$$\mathcal{C} = 4 m^2 - (\varepsilon_1 - \varepsilon_2)^2, \quad (\text{F.5})$$

and find

$$\begin{aligned} \langle \text{Tr } \varphi^2 \rangle &= \frac{\mathcal{C}}{48} (E_2 - 1), & \langle \text{Tr } \varphi^3 \rangle &= -\frac{3}{4} \mathcal{C} s E_3, \\ \langle \text{Tr } \varphi^4 \rangle &= \frac{\mathcal{C} (7\mathcal{C} + 16 (s^2 - p))}{3840} - \frac{\mathcal{C}^2}{384} E_2 + \frac{\mathcal{C} (\mathcal{C} + 2p)}{1152} E_2^2 - \frac{\mathcal{C} (\mathcal{C} - 28p + 48s^2)}{11520} E_4. \end{aligned}$$

Due to the fact that $\langle \text{Tr } \varphi^2 \rangle$ is known in closed form, the $U(1)$ gauge theory is rather trivial from the perspective of our investigation.

G A technical remark

Let us discuss the non-triviality of (7.18) by discussing the difficulties that arise in a brute force attempt to identify the terms in (7.17) from an explicit low instanton calculation. Let us write the 1-instanton expression of $\langle \text{Tr } \varphi^2 \rangle$ and $\langle \text{Tr } \varphi^4 \rangle$ introducing as before the shortcut $\mathcal{C} = 4m^2 - (\varepsilon_1 - \varepsilon_2)^2$. We find

$$\begin{aligned} \langle \text{Tr } \varphi^2 \rangle &= 2 a^2 + \frac{\mathcal{C} (-16 a^2 + \mathcal{C} - 4 p + 4 s^2)}{4 (4 a^2 - s^2)} q + \mathcal{O}(q^2), \\ \langle \text{Tr } \varphi^4 \rangle &= 2 a^4 + \frac{\mathcal{C} (-96 a^4 + 6 a^2 \mathcal{C} - 8 a^2 p - 8 a^2 s^2 - \mathcal{C} p - \mathcal{C} s^2 + 4 p^2 + 8 s^4)}{4 (4 a^2 - s^2)} q + \mathcal{O}(q^2). \end{aligned} \quad (\text{G.1})$$

We can replace a by a series in q enforcing $\langle \text{Tr } \varphi^2 \rangle = \mathbf{u}$. This gives

$$a = \sqrt{\frac{\mathbf{u}}{2}} - \frac{\mathcal{C} (\mathcal{C} - 4 p + 4 s^2 - 8 \mathbf{u})}{8 \sqrt{2 \mathbf{u}} (2 \mathbf{u} - s^2)} q + \mathcal{O}(q^2). \quad (\text{G.2})$$

Replacing this in the expansion of $\langle \text{Tr } \varphi^4 \rangle$ we obtain

$$\langle \text{Tr } \varphi^4 \rangle = \frac{\mathbf{u}^2}{2} + \frac{\mathcal{C} (\mathcal{C} (p + s^2 - 2 \mathbf{u}) - 4 (p^2 + 2 s^4 - 2 s^2 \mathbf{u} - 4 \mathbf{u}^2))}{4 (s^2 - 2 \mathbf{u})} q + \mathcal{O}(q^2). \quad (\text{G.3})$$

This expression is not polynomial in \mathbf{u} . Nevertheless, with insight, we can look for constants k_i such that

$$\langle \text{Tr } \varphi^4 \rangle - \left(k_1 \mathbf{u}' + k_2 \mathbf{u}^2 \right) = \text{linear in } \mathbf{u}. \tag{G.4}$$

The idea is that the remaining linear r.h.s. will be taken into account by subleading contributions to the trace relation. The condition (G.4) gives immediately $k_2 = \frac{1}{2}$. Looking at the pole at $\mathbf{u} = \frac{s^2}{2}$ in the one-instanton contribution, we fix $k_1 = -p$, in agreement with (7.17). If we now move to $\langle \text{Tr } \varphi^6 \rangle$, the one-instanton expression is

$$\begin{aligned} \langle \text{Tr } \varphi^6 \rangle = & \frac{1}{4} \mathbf{u}^3 + \frac{\mathcal{C}}{8(s^2 - 2\mathbf{u})} \left[48\mathbf{u}^3 - 6\mathbf{u}^2 (\mathcal{C} + 16p - 36s^2) \right. \\ & + \mathbf{u} (15\mathcal{C}p - 15\mathcal{C}s^2 - 44p^2 + 48ps^2 - 72s^4) \\ & \left. + (-2\mathcal{C}p^2 - 6\mathcal{C}ps^2 + 9\mathcal{C}s^4 + 8p^3 + 16p^2s^2 - 24s^6) \right] q + \mathcal{O}(q^2) \end{aligned} \tag{G.5}$$

Again, we can look for special simplifications in

$$\langle \text{Tr } \varphi^6 \rangle - \left(k_1 \mathbf{u}'' + k_2 \mathbf{u} \mathbf{u}' + k_3 \mathbf{u}^3 \right) \tag{G.6}$$

The choice $k_3 = \frac{1}{4}$ cancels the cubic term $\sim \mathbf{u}^3$. Vanishing of the residue in the one-instanton term gives only the constraint

$$k_2 = \frac{-4k_1 + 4p^2 - 3ps^2}{2s^2}. \tag{G.7}$$

Thus, for $\langle \text{Tr } \varphi^6 \rangle$ we need the two instanton expression. After some work, we see that in order to cancel the most singular term in the two instanton contribution around the pole at $\mathbf{u} = \frac{s^2}{2}$ we need $k_1 = -p^2$ and therefore, from (G.7), we get $k_2 = -\frac{3}{2}p$. All this is in agreement with (7.17). However, the meaning of this exercise is to show that the leading terms captured by (7.18) cannot be obtained for generic n by means of a fixed instanton calculation. The order of the expansion must increase as n grows.

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