Published for SISSA by Springer

Received: *February 23, 2016*
Accepted: *May 15, 2016*
Published: *May 27, 2016*

JHEP05(2016)166

Heavy-tailed chiral random matrix theory

Takuya Kanazawa

iTHES Research Group and Quantum Hadron Physics Laboratory, RIKEN, Wako, Saitama, 351-0198 Japan

E-mail: takuya.kanazawa@riken.jp

Abstract: We study an unconventional chiral random matrix model with a heavy-tailed probabilistic weight. The model is shown to exhibit chiral symmetry breaking with no bilinear condensate, in analogy to the Stern phase of QCD. We solve the model analytically and obtain the microscopic spectral density and the smallest eigenvalue distribution for an arbitrary number of flavors and arbitrary quark masses. Exotic behaviors such as non-decoupling of heavy flavors and a power-law tail of the smallest eigenvalue distribution are illustrated.

Keywords: Chiral Lagrangians, Matrix Models, Spontaneous Symmetry Breaking

ArXiv ePrint: 1602.05631

doi:10.1007/JHEP05(2016)166

Contents

1 Introduction

Random matrix theory has flourished as a versatile tool in theoretical and mathematical sciences over decades [1–3]. In Quantum Chromodynamics (QCD), the development of chiral random matrix theory (chRMT) [4–6] (also called the Wishart-Laguerre ensemble) that extends the traditional Wigner-Dyson classes has helped us gain a profound understanding of the link between dynamical mass generation of fermions and spectral statistics of the Dirac operator [7]. The chRMT has also been used as a simplistic Ginzburg-Landau-type model of QCD at finite temperature and density [8, 9]. Exact spectral correlations of a non-Hermitian Dirac operator at nonzero chemical potential were also worked out [10–15]. On the practical side, chRMT has enabled accurate determinations of low-energy constants in lattice QCD near the chiral limit [16, 17]. We refer to [18–22] for reviews on chRMT.

In RMT there are various choices for the probabilistic weight of random matrix elements. While the independent Gaussian distribution is the simplest from a mathematical point of view, it often turns out that distributions that deviate from Gaussian lead to the same spectral correlations in the limit of large matrices. This robustness of RMT is known as *universality* [23]. However, when the deformation of the weight is strong enough, results begin to differ from those of the Gaussian ensemble. Such random matrix ensembles with heavy-tailed weights have found applications to disordered conductors and financial statistics, as reviewed in [24]. In general, when the rotational invariance of matrices is broken as in the Lévy matrix ensemble [25], the models tend to be analytically intractable. Alternatively one can also consider heavy-tailed matrix ensembles with rotational invariance, at the expense of losing statistical independence of matrix elements. Both directions have been actively pursued [26–37], revealing a plethora of exotic behaviors not seen in Gaussian RMTs.

JHEP05(2016)166

So far, applications of chRMT to QCD have been mostly limited to the hadronic phase with $\langle\overline{\psi}\psi\rangle \neq 0$. Extensions to the high-density regime where the diquark condensate $\langle\psi\psi\rangle \neq 0$ preponderates were explored in a chRMT framework in [13–15, 38, 39]. While spontaneous symmetry breaking in all these cases can be characterized by a nonvanishing fermion *bilinear* condensate, symmetry breaking in general can also be triggered by higher-order condensates. In QCD it was stressed by Stern that chiral symmetry breaking with $F_\pi \neq 0$ does not necessitate $\langle\overline{\psi}\psi\rangle \neq 0$ [40, 41]. Indeed one can imagine a situation where a chiral condensate is forbidden by an anomaly-free discrete subgroup of $\mathrm{U}(1)_\mathrm{A}$ and the spontaneous breaking $\mathrm{SU}(N_f)_R \times \mathrm{SU}(N_f)_L \to \mathrm{SU}(N_f)_V$ is driven by a *quartic* condensate. (This pattern of symmetry breaking was studied by Dashen long time ago [42].) While this exotic phase that we call the *Stern phase* is ruled out by rigorous QCD inequalities at vanishing baryon density [43], there are arguments in favor of the Stern phase at finite density. First, color superconducting phases in dense QCD are examples of the Stern phase due to the fact that the leading *gauge-invariant* order parameter that breaks chiral symmetry is provided by four-quark condensates [44, 45]. Secondly, in phases with spatially modulated chiral condensates, the phonon fluctuations associated with translational symmetry breaking wipe out the spatial order and lead to a phase with quartic condensates [46, 47]. Other related arguments can be found in [48–53].

In this paper, we propose a heavy-tailed chRMT that corresponds to the Stern phase. To be precise, we show that our chRMT with $N \times N$ random matrices reproduces, in the large-N limit, the finite-volume partition function of the Stern phase with $K > 4$ in the ε-regime. (Here we label the Stern phase with an index K that specifies the unbroken subgroup of $\mathrm{U}(1)_\mathrm{A}$ [53].) This implies that all infinitely many sum rules for the Dirac eigenvalues in the Stern phase [48, 53] are obeyed by microscopic eigenvalues of random matrices in this chRMT. In the chiral limit, the chRMT considered here coincides with the model previously considered by Akemann and Vivo [35]. Here we solve the model at large N with arbitrary quark masses and analytically obtain the microscopic spectral density and the smallest eigenvalue distribution in dependence of quark masses. In comparison to [35], this work is new in the following aspects: we derive the chiral Lagrangian at large N, relate it to the Stern phase in finite-density QCD, compute nontrivial mass dependence of spectral functions, and discuss the heavy-mass limit.

This paper is structured as follows. In section 2 we define the model and discuss its relevance to QCD. Then we solve the model analytically in the large-N limit and obtain the microscopic spectral density and the smallest eigenvalue distribution, for an arbitrary number of flavors and arbitrary quark masses. Section 3 is devoted to conclusions and outlook.

2 Random matrix theory for the Stern phase of QCD

2.1 Definition of the model and the large-N limit

The matrix model considered in this paper is defined by the partition function

$$Z^{\mathrm{S}}_{N_f}(\{m_f\}) \equiv \int_{\mathbb{C}^{N\times N}} \mathrm{d}X \frac{1}{(1+\operatorname{tr} X^\dagger X)^{N^2+NN_f+1}} \prod_{f=1}^{N_f} \det\begin{pmatrix} m_f^* \mathbb{1}_N & X \\ -X^\dagger & m_f \mathbb{1}_N \end{pmatrix}, \tag{2.1}$$

where X is a complex $N \times N$ random matrix and $\mathrm{d}X$ denotes the flat Cartesian measure. This integral converges for arbitrary $N \geq 1$ and $N_f \geq 0$. The weight (2.1) evidently has three interesting properties: (i) it is invariant under unitary rotations $X \to V_1 X V_2$ for $V_{1,2} \in \mathrm{U}(N)$, (ii) the matrix elements are statistically correlated, and (iii) the distribution is heavy tailed, i.e., it does not decay exponentially for large matrix elements. This random matrix ensemble can be seen as an unquenched generalization of previous RMTs [29, 30, 32, 34, 35, 37] that had a heavy-tailed weight similar to (2.1) but with no determinants. If the weight function $1/(1+\operatorname{tr} X^\dagger X)^{N^2+NN_f+1}$ in (2.1) is replaced with a Gaussian distribution, the model reverts to the standard $\beta = 2$ chRMT called the chiral Gaussian unitary ensemble (chGUE) [4, 5].

In the large-N limit, our chRMT enjoys a sigma-model representation. To see this, we rewrite (2.1) up to a trivial multiplicative constant as

$$
\begin{aligned}
Z^{\mathrm{S}}_{N_f}(\{m_f\}) &\sim \int\limits_{\mathbb{C}^{N\times N}} \mathrm{d}X \int_{\mathbb{C}} \mathrm{d}^2 z \; \mathrm{e}^{-N(1+\operatorname{tr} X^\dagger X)|z|^2} |z|^{2N^2+2NN_f} \prod_{f=1}^{N_f} \det \begin{pmatrix} m_f^* \mathbb{1}_N & X \\ -X^\dagger & m_f \mathbb{1}_N \end{pmatrix} \\
&= \int_{\mathbb{C}} \mathrm{d}^2 z \; \mathrm{e}^{-N|z|^2} \int\limits_{\mathbb{C}^{N\times N}} \mathrm{d}X \; |z|^{2N^2} \mathrm{e}^{-N|z|^2 \operatorname{tr} X^\dagger X} \prod_{f=1}^{N_f} \det \begin{pmatrix} z^* m_f^* \mathbb{1}_N & zX \\ -z^* X^\dagger & z m_f \mathbb{1}_N \end{pmatrix} \\
&= \int_{\mathbb{C}} \mathrm{d}^2 z \; \mathrm{e}^{-N|z|^2} \int\limits_{\mathbb{C}^{N\times N}} \mathrm{d}W \, \mathrm{e}^{-N \operatorname{tr} W^\dagger W} \prod_{f=1}^{N_f} \det \begin{pmatrix} z^* m_f^* \mathbb{1}_N & W \\ -W^\dagger & z m_f \mathbb{1}_N \end{pmatrix} ,
\end{aligned}
\tag{2.2}
$$

where in the last step we introduced $W \equiv zX$. The final expression (2.2) is akin to the standard chGUE except that the mass term is multiplied by another Gaussian random variable z of order $1/\sqrt{N}$. This implies that we need a large-N limit with $m_f = \mathcal{O}(1/\sqrt{N})$, which is different from the conventional large-N limit with $m_f = \mathcal{O}(1/N)$. Following the standard route of bosonization [4], one can easily obtain

$$
Z^{\mathrm{S}}_{N_f}(\{m_f\}) \sim \int_{\mathbb{C}} \mathrm{d}^2 z \; \mathrm{e}^{-N|z|^2} \int\limits_{\mathrm{U}(N_f)} \mathrm{d}U \;\; \exp\left[N \operatorname{tr}(zMU + z^* M^\dagger U^\dagger) \right] \tag{2.3}
$$

$$
= \int\limits_{\mathrm{SU}(N_f)} \mathrm{d}U \;\; \exp\left[N \operatorname{tr}(MU) \operatorname{tr}(M^\dagger U^\dagger) \right] , \tag{2.4}
$$

where $\mathrm{d}U$ denotes the Haar measure and $M \equiv \operatorname{diag}(m_1, \ldots, m_{N_f})$. Equation (2.4) exactly coincides with the ε-regime finite-volume partition function of QCD in the Stern phase with $K > 4$ [43, 48, 53]. Notably, the sigma model (2.4) has no term at $\mathcal{O}(M)$ in the exponent, in contradistinction to the standard chiral Lagrangian.[1] This reflects that the chiral condensate $\langle \overline{\psi}\psi \rangle \to 0$ in the chiral limit. The present chRMT may serve as a toy model for

[1] Attempts to recover higher-order terms of chiral perturbation theory from chRMT have been made in [54, 55] with the purpose of studying lattice fermions. The models considered in [54, 55] include not only $\mathcal{O}(M^2)$ terms as in (2.4) but also the leading term $\operatorname{tr}(MU + M^\dagger U^\dagger)$. By contrast, the latter term is missing in (2.4), which puts the present model (2.1) in a different symmetry class from those in [54, 55].

spontaneous symmetry breaking driven by higher-order condensates. It follows from the coincidence of mass dependence between chRMT and QCD that infinitely many spectral sum rules of the Dirac operator in the Stern phase [48, 53] can be reproduced exactly from chRMT. This suggests that the universal behavior of small Dirac eigenvalues originating from chiral symmetry breaking could be probed by using this chRMT, which shares the same pattern of symmetry breaking as the Stern phase of QCD but is much simpler and analytically tractable. We end this subsection with two supplementary remarks.

- The partition function (2.4) has no dependence on the gauge-field topology. In the Stern phase with $K > 4$, topological sectors with nonzero winding numbers are suppressed in the leading order of the ε expansion [53]. This deprives us of a physical motivation to study the model (2.1) with rectangular X. Nevertheless, it could be mathematically interesting to investigate such extensions in future work.[2]

- The sign of the leading term in the exponent of (2.4) was fixed unambiguously by chRMT, despite that both signs are allowed by symmetries. Actually, the sign of the leading term *can* be flipped if we modify the fermion determinant in (2.2) as

$$\det\begin{pmatrix} z^* m_f^* \mathbb{1}_N & W \\ -W^\dagger & z m_f \mathbb{1}_N \end{pmatrix} \to \det\begin{pmatrix} z^* m_f^* \mathbb{1}_N & W \\ W^\dagger & z m_f \mathbb{1}_N \end{pmatrix} = |z|^{2N} \det\begin{pmatrix} m_f^* \mathbb{1}_N & W/z \\ W^\dagger/z^* & m_f \mathbb{1}_N \end{pmatrix}. \tag{2.5}$$

  However the Dirac operator is now *Hermitian*! This means that the anti-Hermiticity of the Dirac operator imposes a nontrivial constraint on the sign of the low energy constant. A similar observation was made in chRMT for Wilson fermions [56].

2.2 Microscopic spectral density

When $\forall m_f \in \mathbb{R}$, the partition function (2.2) becomes

$$Z^{\mathrm{S}}_{N_f}(M) = \int_{\mathbb{C}} \mathrm{d}^2 z\ \mathrm{e}^{-N|z|^2} |z|^{2NN_f} \int_{\mathbb{C}^{N\times N}} \mathrm{d}W\ \mathrm{e}^{-N \operatorname{tr} W^\dagger W} \prod_{f=1}^{N_f} \det(m_f \mathbb{1}_{2N} + \mathcal{D}_{\mathrm{S}}), \tag{2.6}$$

with the Dirac operator

$$\mathcal{D}_{\mathrm{S}} \equiv \frac{1}{|z|}\begin{pmatrix} \mathbf{0} & W \\ -W^\dagger & \mathbf{0} \end{pmatrix}. \tag{2.7}$$

Our primary interest is in the spectral statistics of $\mathcal{D}_{\mathrm{S}}$ on the scale $1/\sqrt{N}$. In this "microscopic domain", the eigenvalue density and eigenvalue correlations are expected to be *universal* in the sense that it is solely determined by the pattern of global symmetry breaking, with no dependence on specific details of UV interactions in QCD that cause the symmetry breaking. In the following, we derive the microscopic spectral density in the large-N limit, first in the chiral limit (section 2.2.1) and then for arbitrary quark masses (section 2.2.2), by making use of a formal similarity of (2.6) to the chGUE. Our notation is fixed as follows.

[2] For $N_f = 0$, this task has already been undertaken in [35].

- $\{\pm i\lambda_n^{\mathrm{S}}\}_{n=1}^N$ (with $\forall\lambda_n^{\mathrm{S}}\geq 0$) $\cdots$ Eigenvalues of $\mathcal{D}_{\mathrm{S}}$
- $\{\pm i\lambda_n\}_{n=1}^N$ (with $\forall\lambda_n\geq 0$) $\cdots$ Eigenvalues of $\begin{pmatrix} \mathbf{0} & W \\ -W^\dagger & \mathbf{0}\end{pmatrix}$.

Obviously, $\lambda_n^{\mathrm{S}}=\lambda_n/|z|$ for every n.

2.2.1 Chiral limit

Let us begin with the chiral limit. The spectral density of $\mathcal{D}_{\mathrm{S}}$ at finite N is defined as

$$R^{\mathrm{S}}_{N,N_f}(\lambda)\equiv\left\langle\sum_{n=1}^N\delta(\lambda-\lambda_n^{\mathrm{S}})\right\rangle \tag{2.8}$$

$$=\frac{\displaystyle\int_{\mathbb{C}}\mathrm{d}^2z\ \mathrm{e}^{-N|z|^2}\,|z|^{2NN_f}\int\mathrm{d}W\,\mathrm{e}^{-N\,\mathrm{tr}\,W^\dagger W}\left[\sum_{n=1}^N\delta(\lambda-\lambda_n^{\mathrm{S}})\right]\prod_{f=1}^{N_f}\det\mathcal{D}_{\mathrm{S}}}{\displaystyle\int_{\mathbb{C}}\mathrm{d}^2z\ \mathrm{e}^{-N|z|^2}\,|z|^{2NN_f}\int\mathrm{d}W\,\mathrm{e}^{-N\,\mathrm{tr}\,W^\dagger W}\prod_{f=1}^{N_f}\det\mathcal{D}_{\mathrm{S}}} \tag{2.9}$$

$$=\frac{\displaystyle\int_{\mathbb{C}}\mathrm{d}^2z\ \mathrm{e}^{-N|z|^2}\int\mathrm{d}W\,\mathrm{e}^{-N\,\mathrm{tr}\,W^\dagger W}\left[\sum_{n=1}^N\delta\left(\lambda-\frac{\lambda_n}{|z|}\right)\right]\det{}^{N_f}W^\dagger W}{\displaystyle\int_{\mathbb{C}}\mathrm{d}^2z\ \mathrm{e}^{-N|z|^2}\int\mathrm{d}W\,\mathrm{e}^{-N\,\mathrm{tr}\,W^\dagger W}\det{}^{N_f}W^\dagger W} \tag{2.10}$$

$$=\frac{\displaystyle\int_{\mathbb{C}}\mathrm{d}^2z\ \mathrm{e}^{-N|z|^2}\,|z|\,R_{N,N_f}(|z|\lambda)}{\displaystyle\int_{\mathbb{C}}\mathrm{d}^2z\ \mathrm{e}^{-N|z|^2}}\,, \tag{2.11}$$

where we have introduced the spectral density in massless chGUE

$$R_{N,N_f}(\lambda)\equiv\frac{\displaystyle\int\mathrm{d}W\,\mathrm{e}^{-N\,\mathrm{tr}\,W^\dagger W}\left[\sum_{n=1}^N\delta(\lambda-\lambda_n)\right]\det{}^{N_f}W^\dagger W}{\displaystyle\int\mathrm{d}W\,\mathrm{e}^{-N\,\mathrm{tr}\,W^\dagger W}\det{}^{N_f}W^\dagger W}\,. \tag{2.12}$$

The microscopic limit of (2.12) was derived in [5]. Now we use this result to obtain the microscopic spectral density for our chRMT,

$$\rho^{\mathrm{S}}_{N_f}(\zeta)\equiv\lim_{N\to\infty}\frac{1}{\sqrt{N}}R^{\mathrm{S}}_{N,N_f}\left(\frac{\zeta}{\sqrt{N}}\right) \tag{2.13}$$

$$=\int_{\mathbb{C}}\frac{\mathrm{d}^2z}{\pi}\ \mathrm{e}^{-|z|^2}\,|z|\lim_{N\to\infty}\frac{1}{N}R_{N,N_f}\left(\frac{|z|\zeta}{N}\right) \tag{2.14}$$

$$=4\int_0^\infty\mathrm{d}x\ x^2\,\mathrm{e}^{-x^2}\,\rho_{N_f}(2x\zeta)\,, \tag{2.15}$$

where we have introduced the microscopic spectral density for massless chGUE [5]

$$\rho_{N_f}(\zeta) \equiv \frac{\zeta}{2}\Big(J^2_{N_f}(\zeta) - J_{N_f+1}(\zeta)J_{N_f-1}(\zeta)\Big). \tag{2.16}$$

We remark that the integral expression (2.15) is a special case of eq. (4.7) in [35]. The integral in (2.15) can be performed analytically, resulting in a compact expression

$$\rho^{\rm S}_{N_f}(\zeta) = 2\zeta\, \mathrm{e}^{-2\zeta^2}\, I_{N_f}(2\zeta^2)\,. \tag{2.17}$$

Here $I_n(z)$ is the modified Bessel function of the first kind. In figure 1 we show $\rho^{\rm S}_{N_f}(\zeta)$ for several values of N_f. It converges to $1/\sqrt{\pi} = 0.564\dots$ as $\zeta \to \infty$. As N_f increases, the density of eigenvalues near zero is depleted because of the determinant in the measure. Comparing $\rho^{\rm S}_{N_f}(\zeta)$ with $\rho_{N_f}(\zeta)$, we notice that $\rho^{\rm S}_{N_f}(\zeta)$ is flat and monotonic (except for $N_f = 0$), showing no oscillatory behavior typical of $\rho_{N_f}(\zeta)$.[3] Let us recall that, in the chGUE, the oscillation is produced by peaks in the density of individual small eigenvalues. By contrast, as we will see in section 2.3, the density of individual eigenvalues near zero in our chRMT is so broad that their superposition smears out each peak completely. Aside from this difference, $\rho_{N_f}(\zeta)$ and $\rho^{\rm S}_{N_f}(\zeta)$ look similar, but once again we emphasize that $\rho^{\rm S}_{N_f}(\zeta)$ is the density at the scale $\lambda^{\rm S}_n \sim 1/\sqrt{N}$, whereas $\rho_{N_f}(\zeta)$ is the density at the scale $\lambda_n \sim 1/N$ — these two regimes are totally different. It was originally pointed out by Stern [40, 41] that chiral symmetry could be spontaneously broken when near-zero Dirac eigenvalues scale as $1/\sqrt{V_4}$ instead of $1/V_4$ in the thermodynamic limit $V_4 \to \infty$. Our finding within chRMT is fully consistent with Stern's perspective. While the Stern phase is ruled out by rigorous QCD inequalities in QCD at zero density [43], its realization in zero-dimensional chRMT is not prohibited. As a side remark, we mention that the spectral universality of random matrices on the scale $1/\sqrt{N}$ in the large-N limit of strong non-Hermiticity has been known since [58]. Its physical applications to dense QCD-like theories were discussed in [13–15, 22, 39, 59].

Usually one associates the approach of $\rho_{N_f}(\zeta)$ to a constant value at $\zeta \to \infty$ with a nonzero chiral condensate through the Banks-Casher relation [7]. It must be noted, however, that the same behavior of $\rho^{\rm S}_{N_f}(\zeta)$ does *not* imply a nonvanishing chiral condensate. The reason is that, in this model, the height of the macroscopic spectral density $R^{\rm S}_{N,N_f}(\lambda)$ at the origin scales as $\sqrt{N}$ for $N \gg 1$, implying that the chiral condensate $\displaystyle\lim_{\lambda\to 0}\lim_{N\to\infty}\frac{1}{N}R^{\rm S}_{N,N_f}(\lambda)$ vanishes as $\displaystyle\propto \frac{1}{\sqrt{N}}$.

An important remark on a preceding work is in order. In [35], with a mathematical motivation, Akemann and Vivo studied a deformed Wishart-Laguerre ensemble, which is essentially equivalent to (2.1) with $N_f = 0$. They derived the microscopic spectral density in the large-N limit analytically, not only for square X but also for rectangular X of size $(N+\nu)\times N$. In addition, their analysis exhausted all the three symmetry classes with Dyson index $\beta = 1, 2$ and 4. One can show that their model with $\nu > 0$ and $\beta = 2$ exactly

[3] A similar behavior was found in a random matrix model for critical statistics interpolating between chGUE and a Poisson ensemble [57].

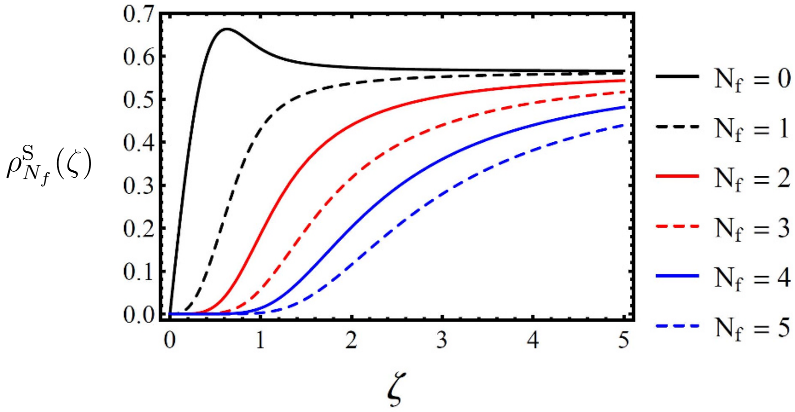

Figure 1. Microscopic spectral density (2.17) in the chiral limit for various N_f. See [35, figure 4] where the same spectra have been presented.

coincides with our unquenched model (2.1) with $N_f = \nu$ and $\forall m_f = 0$. This is known in chRMT as the duality between flavor and topology [18]. As a result, (2.15) above can be obtained from [35, eq. (4.7)] by letting $\nu = N_f$ and sending $\alpha \to 0$ there. We confirmed that our results agree with [35]. Nonetheless we have presented the full derivation above, firstly because the compact expression (2.17) is new, and secondly because the computation in the chiral limit is a useful preliminary step for the generalization to the case of arbitrary nonzero quark masses, which is a genuinely new result of this paper and will be worked out in the next subsection.

2.2.2 Nonzero masses

Reinstating quark masses in (2.10) and replacing z by $z/\sqrt{N}$, we obtain

$$R^{\mathrm{S}}_{N,N_f}(\lambda,\{m_f\}) = \frac{\int_{\mathbb{C}} \mathrm{d}^2 z\, \mathrm{e}^{-|z|^2} \int \mathrm{d}W \mathrm{e}^{-N\mathrm{tr}W^\dagger W} \left[\sum_{n=1}^{N} \delta\left(\lambda - \frac{\sqrt{N}\lambda_n}{|z|}\right)\right] \prod_{f=1}^{N_f} \det\left(\frac{|z|^2 m_f^2}{N}\mathbb{1}_N + W^\dagger W\right)}{\int_{\mathbb{C}} \mathrm{d}^2 z\, \mathrm{e}^{-|z|^2} \int \mathrm{d}W \mathrm{e}^{-N\mathrm{tr}W^\dagger W} \prod_{f=1}^{N_f} \det\left(\frac{|z|^2 m_f^2}{N}\mathbb{1}_N + W^\dagger W\right)} \tag{2.18}$$

$$= \frac{1}{\sqrt{N}} \frac{\int_0^\infty \mathrm{d}x\, x^2 \mathrm{e}^{-x^2} \int \mathrm{d}W \mathrm{e}^{-N\mathrm{tr}W^\dagger W} \left[\sum_{n=1}^{N} \delta\left(\frac{x\lambda}{\sqrt{N}} - \lambda_n\right)\right] \prod_{f=1}^{N_f} \det\left(\frac{x^2 m_f^2}{N}\mathbb{1}_N + W^\dagger W\right)}{\int_0^\infty \mathrm{d}x\, x \mathrm{e}^{-x^2} \int \mathrm{d}W \mathrm{e}^{-N\mathrm{tr}W^\dagger W} \prod_{f=1}^{N_f} \det\left(\frac{x^2 m_f^2}{N}\mathbb{1}_N + W^\dagger W\right)}. \tag{2.19}$$

The microscopic limit is achieved by sending N to infinity with $\sqrt{N}\lambda_n^{\mathrm{S}} \sim \sqrt{N} m_f \sim \mathcal{O}(1)$. With rescaled masses defined as $\mu_f \equiv \sqrt{N} m_f$, we now extend (2.13) to nonzero masses and obtain

$$\rho^{\rm S}_{N_f}(\zeta,\{\mu_f\})$$

$$\equiv \lim_{N\to\infty}\frac{1}{\sqrt{N}}R^{\rm S}_{N,N_f}\left(\frac{\zeta}{\sqrt{N}},\left\{\frac{\mu_f}{\sqrt{N}}\right\}\right) \tag{2.20}$$

$$= \lim_{N\to\infty}\frac{1}{N}\frac{\displaystyle\int_0^\infty {\rm d}x\, x^2{\rm e}^{-x^2}\int {\rm d}W{\rm e}^{-N{\rm tr}W^\dagger W}\left[\sum_{n=1}^N\delta\left(\frac{x\zeta}{N}-\lambda_n\right)\right]\prod_{f=1}^{N_f}\det\left[\left(\frac{x\mu_f}{N}\right)^2\mathbb{1}_N+W^\dagger W\right]}{\displaystyle\int_0^\infty {\rm d}x\, x{\rm e}^{-x^2}\int {\rm d}W{\rm e}^{-N{\rm tr}W^\dagger W}\prod_{f=1}^{N_f}\det\left[\left(\frac{x\mu_f}{N}\right)^2\mathbb{1}_N+W^\dagger W\right]} \tag{2.21}$$

$$= \lim_{N\to\infty}\frac{\displaystyle\int_0^\infty {\rm d}x\, x^2{\rm e}^{-x^2}S_{N,N_f}\left(\left\{\frac{x\mu_f}{N}\right\}\right)\frac{1}{N}R_{N,N_f}\left(\frac{x\zeta}{N},\left\{\frac{x\mu_f}{N}\right\}\right)}{\displaystyle\int_0^\infty {\rm d}x\, x{\rm e}^{-x^2}S_{N,N_f}\left(\left\{\frac{x\mu_f}{N}\right\}\right)}, \tag{2.22}$$

where we have introduced the partition function for chGUE (cf. [60–62])

$$S_{N,N_f}\left(\left\{\frac{x}{N}\mu_f\right\}\right) \equiv \int {\rm d}W\,{\rm e}^{-N\,{\rm tr}\,W^\dagger W}\prod_{f=1}^{N_f}\det\left[\left(\frac{x\mu_f}{N}\right)^2\mathbb{1}_N+W^\dagger W\right] \tag{2.23}$$

$$\sim \int_{{\rm U}(N_f)}{\rm d}U\,\exp\left[2x\;\;{\rm Re}\,{\rm tr}(\boldsymbol{\mu}U)\right] \qquad \text{for} \quad N\gg 1 \tag{2.24}$$

$$\propto \frac{1}{\Delta_{N_f}(\{-(2x\mu_f)^2\})}\det_{1\le i,j\le N_f}\left[(2x\mu_j)^{i-1}I_{i-1}(-2x\mu_j)\right] \tag{2.25}$$

with

$$\boldsymbol{\mu}\equiv{\rm diag}(\mu_1,\dots,\mu_{N_f}) \qquad \text{and} \qquad \Delta_{N_f}(\{a_i\})\equiv\prod_{i>j}(a_i-a_j)\,, \tag{2.26}$$

whereas the spectral density for massive chGUE is given by

$$R_{N,N_f}(\lambda,\{m_f\})\equiv\frac{\displaystyle\int {\rm d}W\,{\rm e}^{-N\,{\rm tr}\,W^\dagger W}\left[\sum_{n=1}^N\delta(\lambda-\lambda_n)\right]\prod_{f=1}^{N_f}\det(m_f^2\mathbb{1}_N+W^\dagger W)}{\displaystyle\int {\rm d}W\,{\rm e}^{-N\,{\rm tr}\,W^\dagger W}\prod_{f=1}^{N_f}\det(m_f^2\mathbb{1}_N+W^\dagger W)}\,. \tag{2.27}$$

The microscopic limit of (2.27) was computed in [63, 64] as

$$\lim_{N\to\infty}\frac{1}{N}R_{N,N_f}\left(\frac{x\zeta}{N};\left\{\frac{x\mu_f}{N}\right\}\right)=2\rho_{N_f}(2x\zeta,\{2x\mu_f\}) \tag{2.28}$$

with

$$\rho_{N_f}(z,\{m_f\}) \equiv -\frac{1}{2}\frac{\det\begin{bmatrix} J_{-1}(z) & zJ_0(z) & \cdots & z^{N_f+1}J_{N_f}(z) \\ J_0(z) & zJ_1(z) & \cdots & z^{N_f+1}J_{N_f+1}(z) \\ I_0(-m_1) & m_1I_1(-m_1) & \cdots & m_1^{N_f+1}I_{N_f+1}(-m_1) \\ \vdots & \vdots & \ddots & \vdots \\ I_0(-m_{N_f}) & m_{N_f}I_1(-m_{N_f}) & \cdots & m_{N_f}^{N_f+1}I_{N_f+1}(-m_{N_f}) \end{bmatrix}}{\prod_{f=1}^{N_f}(z^2+m_f^2)\det_{1\le i,j\le N_f}\left[m_j^{i-1}I_{i-1}(-m_j)\right]}. \tag{2.29}$$

In the chiral limit (2.29) reduces to (2.16). Now, substituting (2.25) and (2.28) into (2.22), we finally arrive at the microscopic spectral density of $\mathcal{D}_\mathrm{S}$ with N_f massive flavors,

$$\rho^\mathrm{S}_{N_f}(\zeta,\{\mu_f\}) = -\frac{2}{\prod_{f=1}^{N_f}(\zeta^2+\mu_f^2)}\frac{\int_0^\infty \mathrm{d}x\ \mathrm{e}^{-x^2}\,x^{(3-N_f)(2+N_f)/2}\det\left[\Xi_{N_f}(x,\zeta,\{\mu_f\})\right]}{\int_0^\infty \mathrm{d}x\ \mathrm{e}^{-x^2}\,x^{-N_f(N_f-1)/2+1}\det_{1\le i,j\le N_f}\left[\mu_j^{i-1}I_{i-1}(-2x\mu_j)\right]}, \tag{2.30}$$

with

$$\Xi_{N_f}(x,\zeta,\{\mu_f\}) \equiv \begin{bmatrix} J_{-1}(2x\zeta) & \zeta J_0(2x\zeta) & \cdots & \zeta^{N_f+1}J_{N_f}(2x\zeta) \\ J_0(2x\zeta) & \zeta J_1(2x\zeta) & \cdots & \zeta^{N_f+1}J_{N_f+1}(2x\zeta) \\ I_0(-2x\mu_1) & \mu_1I_1(-2x\mu_1) & \cdots & \mu_1^{N_f+1}I_{N_f+1}(-2x\mu_1) \\ \vdots & \vdots & \ddots & \vdots \\ I_0(-2x\mu_{N_f}) & \mu_{N_f}I_1(-2x\mu_{N_f}) & \cdots & \mu_{N_f}^{N_f+1}I_{N_f+1}(-2x\mu_{N_f}) \end{bmatrix}. \tag{2.31}$$

Let us examine the simplest case closely. For $N_f=1$, (2.30) can be simplified to

$$\rho^\mathrm{S}_{N_f=1}(\zeta,\mu) = -4\frac{\mathrm{e}^{-\mu^2}}{\zeta^2+\mu^2}\int_0^\infty \mathrm{d}x\ \mathrm{e}^{-x^2}\,x^3\det\begin{bmatrix} J_{-1}(2x\zeta) & \zeta J_0(2x\zeta) & \zeta^2J_1(2x\zeta) \\ J_0(2x\zeta) & \zeta J_1(2x\zeta) & \zeta^2J_2(2x\zeta) \\ I_0(-2x\mu) & \mu I_1(-2x\mu) & \mu^2I_2(-2x\mu) \end{bmatrix}. \tag{2.32}$$

In the limit $\mu\to 0$ (2.32) reproduces (2.15) for $N_f=1$, as it should. We plot $\rho^\mathrm{S}_{N_f=1}(\zeta,\mu)$ in figure 2 for several values of μ. Clearly $\rho^\mathrm{S}_{N_f=1}(\zeta,\mu)$ increases with μ. The asymptotic value at $\zeta\gg 1$ depends on μ, and appears to diverge as $\mu\to\infty$. This means that *a heavy flavor does not decouple* — $\rho^\mathrm{S}_{N_f=1}(\zeta,\mu)$ does not reduce to the quenched density at large μ.[4] This is quite unusual compared to what is known for standard chRMT [63, 64], where the microscopic spectral density approaches $1/\pi$ asymptotically for any number of flavors and any masses, and where the decoupling of heavy flavors holds in the sense that, when

[4]Non-decoupling of heavy flavors also occurs in non-Hermitian chRMT for dense QCD-like theories [15, 22]. In this case the origin of non-decoupling is physically understood: the Cooper pairing between quarks requires that we send masses of an *even number of flavors* to infinity simultaneously. Otherwise the Dirac spectrum becomes singular in the infinite-mass limit.

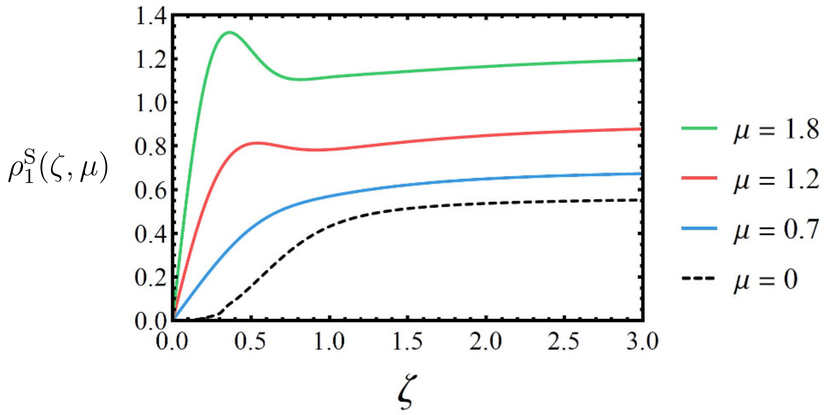

Figure 2. Microscopic spectral density (2.32) for $N_f = 1$.

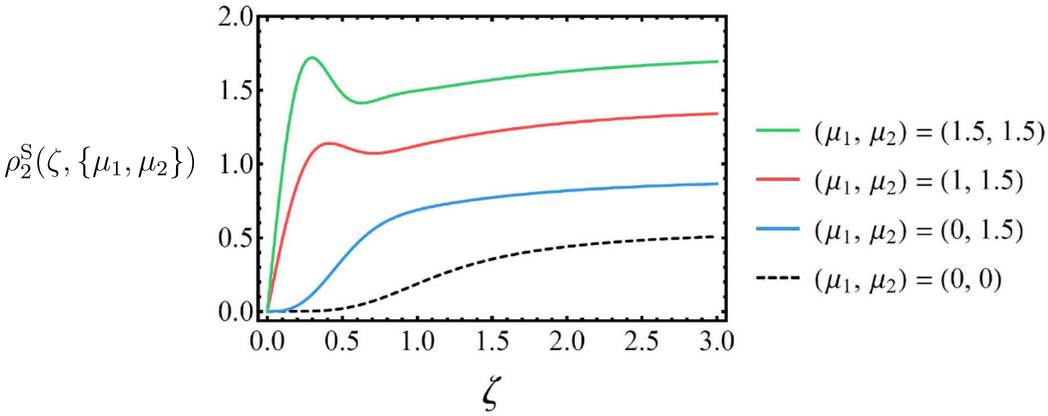

Figure 3. Microscopic spectral density for $N_f = 2$.

some of the masses are sent to infinity, the massive spectral density reduces to that for a reduced number of flavors. By contrast, figure 2 reveals that neither property persists in the Stern phase. For comparison, we also display the massive spectral density for $N_f = 2$ in figure 3, which exhibits a similar mass dependence to $N_f = 1$. We look into this curious behavior in more detail in the next subsection.

2.2.3 Large-mass limit

Why does a heavy flavor fail to decouple? Let us examine what happens to the spectral density when one of the masses is made large compared to the others. Our starting point is the ε-regime partition function (2.4) of the Stern phase for N_f light flavors,

$$Z^{\mathrm{S}}_{N_f}(\{\mu_f\}) \sim \int_{\mathrm{U}(N_f)} \mathrm{d}U \, \exp\left(|\operatorname{tr}(\boldsymbol{\mu} U)|^2\right) . \tag{2.33}$$

If μ_{N_f} is by far the largest among μ_f's, the fluctuation of U over $\mathrm{U}(N_f)$ would be effectively restricted to $\mathrm{U}(N_f-1)$, hence $U \simeq \begin{pmatrix} \tilde{U} & 0 \\ 0 & 1 \end{pmatrix}$ with $\tilde{U} \in \mathrm{U}(N_f-1)$. By plugging this into (2.33) and introducing a reduced mass matrix $\boldsymbol{\mu}_r \equiv \mathrm{diag}(\mu_1, \ldots, \mu_{N_f-1})$, we get

$$Z^{\mathrm{S}}_{N_f}(\{\mu_f\}) \sim \int\limits_{\mathrm{U}(N_f-1)} \mathrm{d}U \, \exp\left(|\mu_{N_f} + \mathrm{tr}(\boldsymbol{\mu}_r \tilde{U})|^2\right) \tag{2.34}$$

$$\sim \exp\left(\mu_{N_f}^2\right) \int\limits_{\mathrm{U}(N_f-1)} \mathrm{d}U \, \exp\left[2\mu_{N_f} \, \mathrm{Re}\,\mathrm{tr}(\boldsymbol{\mu}_r \tilde{U})\right] , \tag{2.35}$$

which is nothing but the partition function of chGUE with $N_f - 1$ flavors [4]. Hence one cannot recover $Z^{\mathrm{S}}_{N_f-1}$ from $Z^{\mathrm{S}}_{N_f}$ by sending one of the masses to infinity; this is how our naive expectation of decoupling fails. Instead, one ends up with the conventional chiral Lagrangian with an $\mathcal{O}(M)$ term whose coefficient is set by μ_{N_f}. This implies that a large explicit mass μ_{N_f} induces large chiral condensates $\langle \overline{\psi}_f \psi_f \rangle$ for the other $N_f - 1$ flavors. The generation of such *induced condensates* has been discussed in [49] for large-N_c QCD, and our analysis based on chRMT is totally consistent with [49].

As there is a generic correspondence between sigma models and spectral statistics, one can expect that the spectral density for the Stern phase at large μ_{N_f} would reduce to that of chGUE whose Gaussian distribution parameter is set by μ_{N_f}. In fact, when $\mu_{N_f} \gg 1$ and $\zeta \sim \mu_f \sim \mathcal{O}(1/\mu_{N_f}) \ll 1$ for $1 \leq f \leq N_f - 1$, there exists a relation

$$\rho^{\mathrm{S}}_{N_f}(\zeta, \{\mu_f\}) \simeq 2\mu_{N_f} \rho_{N_f-1}(2\mu_{N_f}\zeta, \{2\mu_{N_f}\mu_f\}) \, . \tag{2.36}$$

This can be shown from (2.30) by using the Laplace expansion of a determinant and approximating the modified Bessel function by its asymptotic form. The relation (2.36) explicitly provides a novel link between the spectral density in the Stern phase and that in chGUE. To assess the accuracy of (2.36), we display $\rho^{\mathrm{S}}_{N_f=1}(\zeta, \mu)$ for $\mu = 5$ and 10 in figure 4, together with the r.h.s. of (2.36). While the agreement is good for small ζ, deviations emerge for $\zeta \gtrsim 1/\mu$. An oscillatory behavior not present in the chiral limit gradually sets in as μ increases.

2.3 Smallest eigenvalue distribution

Next we turn to the smallest eigenvalue distribution in the large-N microscopic limit of the chRMT for the Stern phase. As in the previous sections, we work with the rescaled masses $\mu_f = \sqrt{N} m_f$. We first define the so-called *gap probability*

$$E_{N,N_f}(\zeta) \equiv \left\langle \prod_{n=1}^{N} \Theta\left(\sqrt{N}\lambda^{\mathrm{S}}_n - \zeta\right) \right\rangle , \tag{2.37}$$

which is the probability that *none* of $\{\sqrt{N}\lambda^{\mathrm{S}}_n\}_n$ falls into the interval $[0, \zeta]$. By definition, $\lim_{\zeta \to +0} E_{N,N_f}(\zeta) = 1$. The factor $\sqrt{N}$ in (2.37) indicates that we are probing the microscopic domain with $\lambda^{\mathrm{S}}_n \sim N^{-1/2}$. The importance of the gap probability stems from the relation

$$E_{N,N_f}(\zeta) = \int_{\zeta}^{\infty} \mathrm{d}\zeta_{\min} \; P_{N,N_f}(\zeta_{\min}; \{\mu_f\}) \tag{2.38}$$

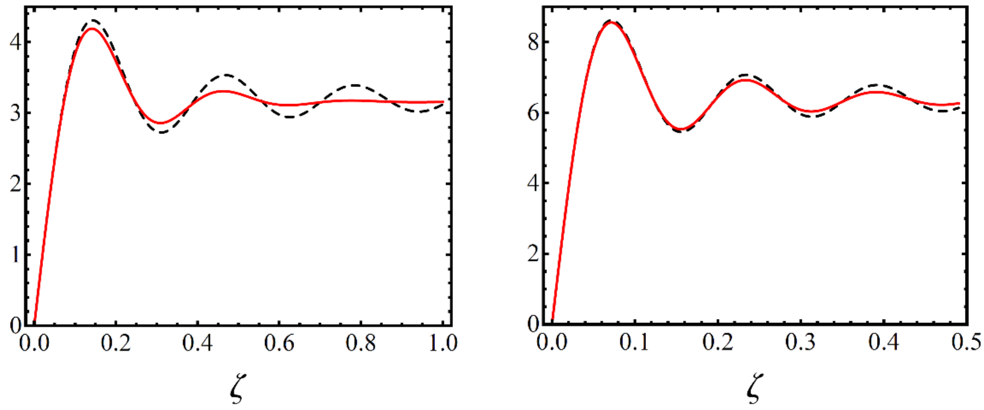

Figure 4. $\rho^{\rm S}_{N_f=1}(\zeta,\mu)$ [(2.32)] for $\mu = 5$ (left, red line) and $\mu = 10$ (right, red line) in comparison to the asymptotic form [RHS of (2.36)] (black dashed lines). Note the difference of scales in the two figures.

with P_{N,N_f} the smallest eigenvalue distribution. Now, by applying the method of [64], it is somewhat tedious but straightforward to show

$$P_{N,N_f}(\zeta;\{\mu_f\}) = -\frac{\mathrm{d}}{\mathrm{d}\zeta} E_{N,N_f}(\zeta) \tag{2.39}$$

$$= \lim_{\lambda\to+0} \frac{\zeta}{\lambda} \frac{\displaystyle\int_0^\infty \mathrm{d}x\, x^3 \mathrm{e}^{-(1+\zeta^2)x^2} S_{N,N_f}\left(\left\{\frac{x}{N}\sqrt{\mu_f^2+\zeta^2}\right\}\right) \frac{1}{N} R_{N,N_f}\left(\frac{\lambda}{N};\left\{\frac{x}{N}\sqrt{\mu_f^2+\zeta^2}\right\}\right)}{\displaystyle\int_0^\infty \mathrm{d}x\, x\mathrm{e}^{-x^2} S_{N,N_f}\left(\left\{\frac{x}{N}\mu_f\right\}\right)}, \tag{2.40}$$

where S_{N,N_f} and R_{N,N_f} are the partition function and the spectral density of chGUE, respectively, as defined in (2.23) and (2.27). Then it is easy to take the large-N microscopic limit by exploiting (2.25) and (2.28), with the result

$$P_{N_f}(\zeta;\{\mu_f\}) \equiv \lim_{N\to\infty} P_{N,N_f}(\zeta;\{\mu_f\}) \tag{2.41}$$

$$= 2\zeta \frac{\displaystyle\int_0^\infty \mathrm{d}x\, x^3\, \mathrm{e}^{-(1+\zeta^2)x^2} \frac{1}{\Delta_{N_f}(\{-(2x)^2(\mu_f^2+\zeta^2)\})} \Omega_{N_f}\left(\left\{2x\sqrt{\mu_f^2+\zeta^2}\right\}\right)}{\displaystyle\int_0^\infty \mathrm{d}x\, x\, \mathrm{e}^{-x^2} \frac{1}{\Delta_{N_f}(\{-(2x\mu_f)^2\})} \det_{1\le i,j\le N_f}\left[(2x\mu_j)^{i-1} I_{i-1}(-2x\mu_j)\right]} \tag{2.42}$$

$$= 2\zeta \frac{\displaystyle\int_0^\infty \mathrm{d}x\, x^{3-N_f(N_f-1)}\, \mathrm{e}^{-(1+\zeta^2)x^2}\, \Omega_{N_f}\left(\left\{2x\sqrt{\mu_f^2+\zeta^2}\right\}\right)}{\displaystyle\int_0^\infty \mathrm{d}x\, x^{1-N_f(N_f-1)}\, \mathrm{e}^{-x^2} \det_{1\le i,j\le N_f}\left[(2x\mu_j)^{i-1} I_{i-1}(-2x\mu_j)\right]}, \tag{2.43}$$

where

$$\Omega_{N_f}(\{m_f\}) \equiv -\lim_{\alpha\to 0}\frac{1}{\alpha}\frac{\det\begin{bmatrix} J_{-1}(\alpha) & \alpha J_0(\alpha) & \cdots & \alpha^{N_f+1}J_{N_f}(\alpha) \\ J_0(\alpha) & \alpha J_1(\alpha) & \cdots & \alpha^{N_f+1}J_{N_f+1}(\alpha) \\ I_0(-m_1) & m_1 I_1(-m_1) & \cdots & m_1^{N_f+1}I_{N_f+1}(-m_1) \\ \vdots & \vdots & \ddots & \vdots \\ I_0(-m_{N_f}) & m_{N_f}I_1(-m_{N_f}) & \cdots & m_{N_f}^{N_f+1}I_{N_f+1}(-m_{N_f}) \end{bmatrix}}{\prod_{f=1}^{N_f}(\alpha^2+m_f^2)} \tag{2.44}$$

$$= \det\begin{bmatrix} I_2(-m_1) & \cdots & m_1^{N_f-1}I_{N_f+1}(-m_1) \\ \vdots & \ddots & \vdots \\ I_2(-m_{N_f}) & \cdots & m_{N_f}^{N_f-1}I_{N_f+1}(-m_{N_f}) \end{bmatrix}. \tag{2.45}$$

This is a new result. For small N_f, integrals in (2.43) can be carried out analytically and yield simple expressions:

$$P_0(\zeta) = \frac{2\zeta}{(1+\zeta^2)^2}, \tag{2.46a}$$

$$P_1(\zeta;\mu) = \frac{2\zeta(\mu^2+\zeta^2)}{(1+\zeta^2)^3}\exp\left(\frac{(1-\mu^2)\zeta^2}{1+\zeta^2}\right), \tag{2.46b}$$

$$P_2(\zeta;\{\mu,\mu\}) = \frac{2\zeta}{(1+\zeta^2)^2}\exp\left(\frac{2(1-\mu^2)\zeta^2}{1+\zeta^2}\right)\frac{I_2\left(\frac{2(\mu^2+\zeta^2)}{1+\zeta^2}\right)}{I_0(2\mu^2)-I_1(2\mu^2)}. \tag{2.46c}$$

They are correctly normalized to 1 when integrated over $0 \le \zeta \le \infty$. The result for P_0 agrees with [35]. In P_2 the masses were set equal for simplicity.

A salient feature of (2.46) is that they decay only polynomially ($\propto \zeta^{-3}$) at large ζ, in contrast to a Gaussian decay in chGUE [64–66]. This long tail of $P_{N_f}(\zeta)$ could be a signal of weak eigenvalue repulsion in this model. Actually, the decay $\sim \zeta^{-3}$ can be shown for any N_f and any masses, on the basis of (2.40). If we rescale the variable as $x \to x/\zeta$ in the numerator of (2.40), we get an additional overall factor ζ^{-4} while the rest of the integral tends to a well-defined large-ζ limit. Combined with ζ at the head of (2.40), the prefactor becomes ζ^{-3}.

In figure 5, $P_{N_f}(\zeta)$ for $N_f = 1$ and 2 are plotted and compared to the microscopic spectral density (2.30). P_{N_f} nicely fit the near-zero part of the spectral density. They tend to be more localized near the origin and represent a peak in the density when the masses are increased, as anticipated from the reduction to chGUE discussed in section 2.2.3.

We expect that the extension of our analysis to the k-th smallest eigenvalue distribution for general $k \in \mathbb{N}$ would be straightforward along the lines of [67].

3 Conclusions and outlook

In this paper we studied an unorthodox chiral random matrix model with a heavy tail. This model, which is an unquenched generalization of the model in [35], is a one-parameter

JHEP05(2016)166

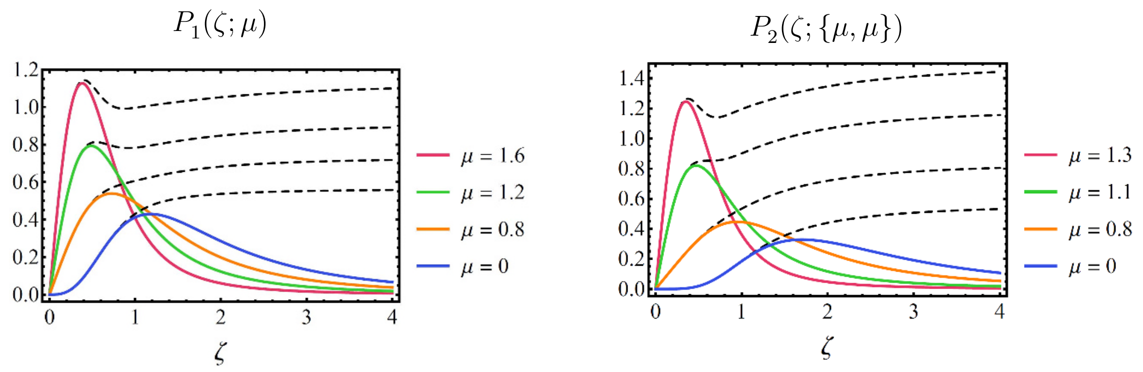

Figure 5. Smallest eigenvalue distribution (2.46) for $N_f = 1$ (left) and $N_f = 2$ (right) for varying masses. The microscopic spectral density for each mass is also shown for comparison (black dashed lines).

reweighting of the standard chGUE and can be solved exactly. We discussed potential relevance of this model to the Stern phase of QCD, where chiral condensate is zero but chiral symmetry is broken by higher-order condensates. We analytically obtained the microscopic spectral density and the smallest eigenvalue distribution in the large-N limit and discussed their dependence on the number of flavors and quark masses. Our model is not only useful as a conceptual toy model for the Stern phase but may also help a numerical evaluation of low-energy constants in future lattice simulations through fitting to the lattice Dirac spectrum.

There remain several issues that call for further investigation. While we only solved the model with unitary symmetry ($\beta = 2$), it would be technically straightforward to generalize it to $\beta = 1$ and 4; in fact, this has already been done for the case of $N_f = 0$ in [35]. One can also build a non-Hermitian extension of this model in the spirit of [10] by replacing the matrix X in (2.1) by a sum of two independent random matrices. Such extensions are of interest to study the Stern phase in QCD at finite quark density. Another unanswered question is whether the present model could be applied to QCD in three dimensions. To the best of our knowledge, there is so far no study of a Stern-like phase in $2+1$ dimensions. We wish to address some of these issues in future work.

Acknowledgments

This work was supported by the RIKEN iTHES project. The author thanks T. Wettig and J.J.M. Verbaarschot for valuable comments.

References

[1] T. Guhr, A. Müller-Groeling and H.A. Weidenmuller, *Random matrix theories in quantum physics: Common concepts*, Phys. Rept. **299** (1998) 189 [cond-mat/9707301] [INSPIRE].

[2] M.L. Mehta, *Random Matrices*, 3rd edition, Academic Press, Amsterdam (2004).

[3] G. Akemann, J. Baik and P. Di Francesco, *The Oxford Handbook of Random Matrix Theory*, Oxford University Press (2010).

[4] E.V. Shuryak and J.J.M. Verbaarschot, *Random matrix theory and spectral sum rules for the Dirac operator in QCD*, Nucl. Phys. **A 560** (1993) 306 [hep-th/9212088] [INSPIRE].

[5] J.J.M. Verbaarschot and I. Zahed, *Spectral density of the QCD Dirac operator near zero virtuality*, Phys. Rev. Lett. **70** (1993) 3852 [hep-th/9303012] [INSPIRE].

[6] J.J.M. Verbaarschot, *The Spectrum of the QCD Dirac operator and chiral random matrix theory: The Threefold way*, Phys. Rev. Lett. **72** (1994) 2531 [hep-th/9401059] [INSPIRE].

[7] T. Banks and A. Casher, *Chiral Symmetry Breaking in Confining Theories*, Nucl. Phys. **B 169** (1980) 103 [INSPIRE].

[8] M.A. Stephanov, *Random matrix model of QCD at finite density and the nature of the quenched limit*, Phys. Rev. Lett. **76** (1996) 4472 [hep-lat/9604003] [INSPIRE].

[9] A.M. Halasz, A.D. Jackson, R.E. Shrock, M.A. Stephanov and J.J.M. Verbaarschot, *On the phase diagram of QCD*, Phys. Rev. **D 58** (1998) 096007 [hep-ph/9804290] [INSPIRE].

[10] J.C. Osborn, *Universal results from an alternate random matrix model for QCD with a baryon chemical potential*, Phys. Rev. Lett. **93** (2004) 222001 [hep-th/0403131] [INSPIRE].

[11] G. Akemann, J.C. Osborn, K. Splittorff and J.J.M. Verbaarschot, *Unquenched QCD Dirac operator spectra at nonzero baryon chemical potential*, Nucl. Phys. **B 712** (2005) 287 [hep-th/0411030] [INSPIRE].

[12] G. Akemann, *The Complex Laguerre symplectic ensemble of non-Hermitian matrices*, Nucl. Phys. **B 730** (2005) 253 [hep-th/0507156] [INSPIRE].

[13] G. Akemann, M.J. Phillips and H.J. Sommers, *The Chiral Gaussian two-matrix ensemble of real asymmetric matrices*, J. Phys. **A 43** (2010) 085211 [arXiv:0911.1276] [INSPIRE].

[14] T. Kanazawa, T. Wettig and N. Yamamoto, *Chiral random matrix theory for two-color QCD at high density*, Phys. Rev. **D 81** (2010) 081701 [arXiv:0912.4999] [INSPIRE].

[15] G. Akemann, T. Kanazawa, M.J. Phillips and T. Wettig, *Random matrix theory of unquenched two-colour QCD with nonzero chemical potential*, JHEP **03** (2011) 066 [arXiv:1012.4461] [INSPIRE].

[16] P.H. Damgaard, U.M. Heller, K. Splittorff, B. Svetitsky and D. Toublan, *Extracting F_π from small lattices: Unquenched results*, Phys. Rev. **D 73** (2006) 074023 [hep-lat/0602030] [INSPIRE].

[17] JLQCD collaboration, H. Fukaya et al., *Two-flavor lattice QCD simulation in the ε-regime with exact chiral symmetry*, Phys. Rev. Lett. **98** (2007) 172001 [hep-lat/0702003] [INSPIRE].

[18] J.J.M. Verbaarschot, *Universal behavior in Dirac spectra*, hep-th/9710114 [INSPIRE] and exentend version of lectures given at the “Nato Advanced Study Institute” on “Confinement, Duality, and Non-Perturbative Aspects of QCD”, Cambridge 1997.

JHEP05(2016)166

[19] J.J.M. Verbaarschot and T. Wettig, *Random matrix theory and chiral symmetry in QCD*, Ann. Rev. Nucl. Part. Sci. **50** (2000) 343 [hep-ph/0003017] [INSPIRE].

[20] G. Akemann, *Matrix Models and QCD with Chemical Potential*, Int. J. Mod. Phys. **A 22** (2007) 1077 [hep-th/0701175] [INSPIRE].

[21] J.J.M. Verbaarschot, *Handbook Article on Applications of Random Matrix Theory to QCD*, to appear in *The Oxford Handbook of Random Matrix Theory*, G. Akemann, J. Baik and P. Di Francesco eds., Oxford University Press (2011), arXiv:0910.4134 [INSPIRE].

[22] T. Kanazawa, *Dirac Spectra in Dense QCD*, Springer Theses Vol. 124, Springer Japan (2013).

[23] P. Deift and D. Gioev, *Random matrix theory: invariant ensembles and universality*, vol. 18, American Mathematical Society (2009).

[24] Z. Burda and J. Jurkiewicz, *Heavy-tailed random matrices*, arXiv:0909.5228.

[25] P. Cizeau and J.P. Bouchaud, *Theory of Levy matrices*, Phys. Rev. **E 50** (1994) 1810 [INSPIRE].

[26] V. Freilikher, E. Kanzieper and I. Yurkevich, *Unitary random-matrix ensemble with governable level confinement*, Phys. Rev. **E 53** (1996) 2200 [cond-mat/9510001].

[27] K.A. Muttalib, Y. Chen, M.E.H. Ismail and V.N. Nicopoulos, *New family of unitary random matrices*, Phys. Rev. Lett. **71** (1993) 471 [INSPIRE].

[28] A.M. Garcia-Garcia, *Spectral properties of a generalized chiral Gaussian unitary ensemble*, Phys. Rev. **E 64** (2001) 066121 [cond-mat/0103043].

[29] F. Toscano, R.O. Vallejos and C. Tsallis, *Random matrix ensembles from nonextensive entropy*, Phys. Rev. **E 69** (2004) 066131 [cond-mat/0402215].

[30] A.C. Bertuola, O. Bohigas and M.P. Pato, *Family of generalized random matrix ensembles*, Phys. Rev. **E 70** (2004) 065102 [math-ph/0411033].

[31] A.Y. Abul-Magd, *Random matrix theory within superstatistics*, Phys. Rev. **E 72** (2005) 066114 [cond-mat/0510494].

[32] Z. Burda, A.T. Görlich and B. Waclaw, *Spectral properties of empirical covariance matrices for data with power-law tails*, Phys. Rev. **E 74** (2006) 041129 [physics/0603186] [INSPIRE].

[33] K.A. Muttalib and M.E.H. Ismail, *Power-law eigenvalue density, scaling, and critical random-matrix ensembles*, Phys. Rev. **E 76** (2007) 051105 [arXiv:0710.4527].

[34] O. Bohigas, J.X. de Carvalho and M.P. Pato, *Disordered ensembles of random matrices*, Phys. Rev. **E 77** (2008) 011122 [arXiv:0711.3719].

[35] G. Akemann and P. Vivo, *Power law deformation of Wishart-Laguerre ensembles of random matrices*, J. Stat. Mech. (2008) P09002 [arXiv:0806.1861] [INSPIRE].

[36] J. Choi and K.A. Muttalib, *Rotationally invariant family of Lévy like random matrix ensembles*, J. Phys. **A 42** (2009) 152001 [arXiv:0903.5266].

[37] A. Abul-Magd, G. Akemann and P. Vivo, *Superstatistical generalizations of Wishart-Laguerre ensembles of random matrices*, *J. Phys.* **A 42** (2010) 175207 [arXiv:0811.1992].

[38] T. Kanazawa, T. Wettig and N. Yamamoto, *Singular values of the Dirac operator in dense QCD-like theories*, JHEP **12** (2011) 007 [arXiv:1110.5858] [INSPIRE].

JHEP05(2016)166

[39] T. Kanazawa and T. Wettig, *Stressed Cooper pairing in QCD at high isospin density: effective Lagrangian and random matrix theory*, JHEP **10** (2014) 055 [arXiv:1406.6131] [INSPIRE].

[40] J. Stern, *Light quark masses and condensates in QCD*, Lect. Notes Phys. **513** (1998) 26 [hep-ph/9712438] [INSPIRE].

[41] J. Stern, *Two alternatives of spontaneous chiral symmetry breaking in QCD*, submitted to Phys. Rev. Lett. (1998) [hep-ph/9801282] [INSPIRE].

[42] R.F. Dashen, *Chiral* SU(3) × SU(3) *as a symmetry of the strong interactions*, Phys. Rev. **183** (1969) 1245 [INSPIRE].

[43] I.I. Kogan, A. Kovner and M.A. Shifman, *Chiral symmetry breaking without bilinear condensates, unbroken axial Z_N symmetry and exact QCD inequalities*, Phys. Rev. **D 59** (1999) 016001 [hep-ph/9807286] [INSPIRE].

[44] K. Rajagopal and F. Wilczek, *The Condensed matter physics of QCD*, hep-ph/0011333 [INSPIRE].

[45] M.G. Alford, A. Schmitt, K. Rajagopal and T. Schäfer, *Color superconductivity in dense quark matter*, Rev. Mod. Phys. **80** (2008) 1455 [arXiv:0709.4635] [INSPIRE].

[46] G. Baym, B.L. Friman and G. Grinstein, *Fluctuations and long range order in finite temperature pion condensates*, Nucl. Phys. **B 210** (1982) 193 [INSPIRE].

[47] Y. Hidaka, K. Kamikado, T. Kanazawa and T. Noumi, *Phonons, pions and quasi-long-range order in spatially modulated chiral condensates*, Phys. Rev. **D 92** (2015) 034003 [arXiv:1505.00848] [INSPIRE].

[48] S. Descotes-Genon and J. Stern, *Finite volume analysis of N_f-induced chiral phase transitions*, Phys. Rev. **D 62** (2000) 054011 [hep-ph/9912234] [INSPIRE].

[49] L. Girlanda, J. Stern and P. Talavera, *η' mass and chiral symmetry breaking at large N_c and N_f*, Phys. Rev. Lett. **86** (2001) 5858 [hep-ph/0103221] [INSPIRE].

[50] Y. Watanabe, K. Fukushima and T. Hatsuda, *Order parameters with higher dimensionful composite fields*, Prog. Theor. Phys. **111** (2004) 967 [hep-th/0312271] [INSPIRE].

[51] M. Harada, C. Sasaki and S. Takemoto, *Enhancement of quark number susceptibility with an alternative pattern of chiral symmetry breaking in dense matter*, Phys. Rev. **D 81** (2010) 016009 [arXiv:0908.1361] [INSPIRE].

[52] P. Adhikari, T.D. Cohen, R.R.M. Ayyagari and M.C. Strother, *On Chiral Symmetry Restoration at Finite Density in Large N_c QCD*, Phys. Rev. **C 83** (2011) 065201 [arXiv:1104.2236] [INSPIRE].

[53] T. Kanazawa, *Chiral symmetry breaking with no bilinear condensate revisited*, JHEP **10** (2015) 010 [arXiv:1507.06376] [INSPIRE].

[54] P.H. Damgaard, K. Splittorff and J.J.M. Verbaarschot, *Microscopic Spectrum of the Wilson Dirac Operator*, Phys. Rev. Lett. **105** (2010) 162002 [arXiv:1001.2937] [INSPIRE].

[55] J.C. Osborn, *Staggered chiral random matrix theory*, Phys. Rev. **D 83** (2011) 034505 [arXiv:1012.4837] [INSPIRE].

[56] G. Akemann, P.H. Damgaard, K. Splittorff and J.J.M. Verbaarschot, *Spectrum of the Wilson Dirac Operator at Finite Lattice Spacings*, Phys. Rev. **D 83** (2011) 085014 [arXiv:1012.0752] [INSPIRE].

JHEP05(2016)166

[57] A.M. Garcia-Garcia and J.J.M. Verbaarschot, *Chiral random matrix model for critical statistics*, *Nucl. Phys.* **B 586** (2000) 668 [hep-th/0003159] [INSPIRE].

[58] Y.V. Fyodorov, B. Khoruzhenko and H.J. Sommers, *Universality in the random matrix spectra in the regime of weak non-Hermiticity*, *Ann. Inst. Henri Poincare* **68** (1998) 449 [chao-dyn/9802025].

[59] N. Yamamoto and T. Kanazawa, *Dense QCD in a Finite Volume*, *Phys. Rev. Lett.* **103** (2009) 032001 [arXiv:0902.4533] [INSPIRE].

[60] R. Brower, P. Rossi and C.-I. Tan, *The External Field Problem for QCD*, *Nucl. Phys.* **B 190** (1981) 699 [INSPIRE].

[61] A.D. Jackson, M.K. Sener and J.J.M. Verbaarschot, *Finite volume partition functions and Itzykson-Zuber integrals*, *Phys. Lett.* **B 387** (1996) 355 [hep-th/9605183] [INSPIRE].

[62] A.B. Balantekin, *Character expansions, Itzykson-Zuber integrals and the QCD partition function*, *Phys. Rev.* **D 62** (2000) 085017 [hep-th/0007161] [INSPIRE].

[63] P.H. Damgaard and S.M. Nishigaki, *Universal spectral correlators and massive Dirac operators*, *Nucl. Phys.* **B 518** (1998) 495 [hep-th/9711023] [INSPIRE].

[64] T. Wilke, T. Guhr and T. Wettig, *The Microscopic spectrum of the QCD Dirac operator with finite quark masses*, *Phys. Rev.* **D 57** (1998) 6486 [hep-th/9711057] [INSPIRE].

[65] P.J. Forrester, *The spectrum edge of random matrix ensembles*, *Nucl. Phys.* **B 402** (1993) 709.

[66] S.M. Nishigaki, P.H. Damgaard and T. Wettig, *Smallest Dirac eigenvalue distribution from random matrix theory*, *Phys. Rev.* **D 58** (1998) 087704 [hep-th/9803007] [INSPIRE].

[67] P.H. Damgaard and S.M. Nishigaki, *Distribution of the k-th smallest Dirac operator eigenvalue*, *Phys. Rev.* **D 63** (2001) 045012 [hep-th/0006111] [INSPIRE].

JHEP05(2016)166