

$\mathcal{N} = 2$ supersymmetric Janus solutions and flows: from gauged supergravity to M theory

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ABSTRACT: We investigate a family of $SU(3) \times U(1) \times U(1)$ -invariant holographic flows and Janus solutions obtained from gauged $\mathcal{N} = 8$ supergravity in four dimensions. We give complete details of how to use the uplift formulae to obtain the corresponding solutions in M theory. While the flow solutions appear to be singular from the four-dimensional perspective, we find that the eleven-dimensional solutions are much better behaved and give rise to interesting new classes of compactification geometries that are smooth, up to orbifolds, in the infra-red limit. Our solutions involve new phases in which M2 branes polarize partially or even completely into M5 branes. We derive the eleven-dimensional supersymmetries and show that the eleven-dimensional equations of motion and BPS equations are indeed satisfied as a consequence of their four-dimensional counterparts. Apart from elucidating a whole new class of eleven-dimensional Janus and flow solutions, our work provides extensive and highly non-trivial tests of the recently-derived uplift formulae.

KEYWORDS: Flux compactifications, AdS-CFT Correspondence, M-Theory

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1 Introduction

Finding and understanding the infra-red structure of holographic RG flows of $\mathcal{N}=4$ Yang-Mills theory and of ABJM theory [1] remains an immensely rich but rather challenging subject that still has the capacity to surprise. In this context, gauged $\mathcal{N}=8$ supergravity in four and five-dimensions has proven to be a very powerful tool that continues to be extremely useful and yield interesting new results.

The catalog of physically interesting holographic solutions and flows that have been explicitly constructed in gauged supergravity is a very long one, whose early examples included the flows to highly non-trivial $\mathcal{N}=1$ supersymmetric “Leigh-Strassler” fixed points [2–6] and its ABJM analog ([7–9] and [10–15]), through examples of $\mathcal{N}=2$ Seiberg-Witten flows [16–18], to maximally supersymmetric flows [19–21]. There are many more examples, perhaps the most recent of which is the supersymmetric flow that we studied in [22], where the large- N theory on a stack of M2 branes flows to a new, “nearly conformal” supersymmetric theory in $(3+1)$ dimensions. One of our purposes here is to discuss these new families of flows and related Janus solutions in some detail. Another purpose of this paper is to highlight and explain some of the new techniques that were used in [22].

As has been noted in many places (see, for example, [22–26]), finding a holographic flow solution in lower dimensions, perhaps in some consistent truncation, does not often give direct insight into the underlying physics. More specifically, a holographic flow in the lower-dimensional theory may be singular and fields may flow to infinite values. It is only when these solutions are uplifted to M theory or IIB supergravity that one can give a proper interpretation of the singular behavior in terms of a distribution of branes and fluxes. In this way, singular low-dimensional solutions may actually encode very interesting physics in higher dimensions.

This, of course, raises the obvious question as to why one does not simply start in the higher-dimensional theory from the outset. The answer is straightforward: the lower-dimensional theory encodes fields much more simply and computably; complicated fluxes and metric deformations on internal manifolds become scalars described in terms of a potential. The supersymmetries may also involve these internal fluxes and geometry in non-trivial ways. The practical algorithms for solving supersymmetry variations directly in higher-dimensions therefore typically require the imposition of a high level of symmetry and supersymmetry. The power of using the low-dimensional theory and its potential structure is that one can handle solutions that have much lower levels of symmetry and supersymmetry. As will become evident, finding the solutions that we construct here directly in M theory is a truly daunting task, even when one knows exactly where to look.

The price of working in lower-dimensional gauged supergravity is that it describes a very restricted family of deformations. On the other hand they are some of the most interesting deformations since they are dual to marginal and relevant operators. What is surprising is that even after fifteen years since the first holographic flows in gauged supergravity [3, 27, 28], there are still interesting new physical flows to be found (like the one in [22]) and new Janus solutions that can be constructed explicitly in gauged supergravity (like those in [29]).

Since gauged supergravity continues to give us new very interesting, physical solutions, while their interpretation usually requires the “uplift” to M theory or IIB supergravity, it becomes ever more important to understand and develop the precise relationships between the gauged supergravity in low dimensions and the higher-dimensional supergravities. In particular, one wants to develop explicit uplift formulae that provide the exact M theory or IIB solution in terms of the gauged supergravity fields. There is also a vast literature on this subject and there has also been some remarkable progress on this in the last two or three years. For simplicity, we will only give a very brief review here and restrict our attention entirely to M theory and its relation to gauged $\mathcal{N}=8$ supergravity in four dimensions.

Gauged $\mathcal{N}=8$ supergravity in four dimensions was first constructed in [30] and there was a great deal of subsequent work that argued how this must be related to the S^7 -compactification of M theory. (For a review, see [31].) There was extensive discussion as to whether gauged supergravity was a consistent truncation or merely a low-energy effective field theory. The former is a much stronger statement in that it means that solving the equations of motion in gauged supergravity *guarantees* that one has also solved the equations of motion of the higher dimensional theory. Over the years it has become evident that the gauged theory is indeed a consistent truncation and formulae have emerged showing precisely how gauged supergravity encodes solutions to M theory.

One of the first general formula was given in [32] where it was shown how to compute the exact deformed metric on the S^7 in terms of all the supergravity scalars. This knowledge alone was immensely useful in finding uplifted solutions explicitly, see for example [8, 13, 14]. Exact formulae for fluxes proved to be a much greater challenge. Indeed, the formulae for the components of the 4-form field strength obtained as part of the original proof of consistent truncation of M theory on S^7 in [33, 34] were prohibitively difficult to use and also suffered from an ambiguity that would lead to some components having a wrong symmetry [35]. It is only recently that a new set of considerably more workable uplift Ansätze for the internal 3-form potential have been proposed and then extended to the other components of the flux [38–41]. However, explicit tests of those new formulae [37, 39, 42] were confined to uplifts of the simplest solutions of four-dimensional $\mathcal{N}=8$ supergravity, namely the AdS_4 solutions for the stationary points of the potential. Hence, it is important to perform non-trivial tests for solutions with varying scalar fields, such as holographic flows.

In looking at the holographic flows described in this paper and in [22], we first tried to find the uplift based entirely on knowing the internal metric through the formula of [32]. This turned out to be impossible and the flux uplift formula became an essential part of constructing the flow solution. Moreover, the Janus solutions are intrinsically even more complicated and we certainly could not have constructed them without knowing how to uplift the fluxes. We find that the new formula to uplift the fluxes [36–41] do indeed generate the exact solution. The only down-side is that they involve some heavy computations to arrive at a relatively simple result.

While our focus in this paper will be mainly on the details of how to construct the uplifts, one should not lose sight of the interesting physics of the solutions that we construct. As we described in [22], our flow solutions start from a UV fixed point of M2 branes that,

under a relevant perturbation, go to solutions sourced largely, or even entirely by M5 branes in the IR. Unlike many flows to the IR, these flows, when uplifted, have only mild orbifold singularities. Moreover, there is a special class of flows that go to pure M5 branes and can be interpreted as describing an “almost conformal” fixed point in $(3 + 1)$ dimensions. This was the focus of our earlier paper [22]. In this paper we will also look at Janus solutions that delve into the backgrounds described by the flows and so may be interpreted as describing interfaces between phases described by the holographic IR flows.

To date, much of the discussion of Janus solutions has been done directly in IIB supergravity [43–46] (see, however, [47, 48]) or M theory [49–53]. As we remarked above, such direct constructions usually require a high level of symmetry, or supersymmetry to make the computations feasible. In particular, until [29], there were very few $\frac{1}{4}$ -BPS (and no $\frac{1}{8}$ -BPS) Janus solutions in M theory known. These new Janus solutions were obtained in gauged supergravity and so we want to take one of the most non-trivial families of such flows and uplift to M theory so as to reveal the underlying geometric structure.

In section 2 we describe sector of gauged $\mathcal{N}=8$ supergravity upon which we will focus and describe the BPS flow and Janus solutions from the four-dimensional perspective. In section 3 we give the details of how this sector of gauged supergravity uplifts to M theory, while in section 4 we show how this uplifted solution solves the equations of motion in M theory. The supersymmetry structure of the solutions is studied from the eleven-dimensional perspective in section 5. In section 6 we return to studying the IR limits of our holographic flows and how they are related to distributions of M5 and M2 branes. We discuss general features of our solutions and how one might obtain more general families of solutions in section 7. Section 8 contains our concluding remarks. Our conventions and the tabulation of some of the more complicated formulae are given in the appendices.

2 The truncation and BPS equations in four dimensions

2.1 The truncation

In this section we summarize some explicit results for the truncation of four-dimensional, $\mathcal{N}=8$ supergravity [30] to the $SU(3)\times U(1)\times U(1)$ -invariant sector that we will need for the uplift to M theory in section 3. Our discussion here is based on [29] and [22]. The Lagrangian for the truncation can also be read-off from a more general $U(1)^4$ -invariant truncation in [54] and/or an $SU(3)$ -invariant truncation in [7] and [55, 56].

The $SU(3)\times U(1)\times U(1)\subset SO(8)$ symmetry group of the truncation is defined by its action on the supersymmetries, ϵ^i , of the $\mathcal{N}=8$ theory. We choose $SU(3)$ and the first $U(1)$ to act on the indices $i=1, \dots, 6$, while the second $U(1)$ on the indices $i=7, 8$. This corresponds to the branching

$$\mathbf{8}_v \quad \longrightarrow \quad (\mathbf{3}, 1, 0) + (\bar{\mathbf{3}}, -1, 0) + (\mathbf{1}, 0, 1) + (\mathbf{1}, 0, -1). \quad (2.1)$$

The resulting truncation is particularly simple since, as observed in [29], the commutant of the symmetry group in $E_{7(7)}$ consists of a single $SL(2, \mathbb{R})$. The invariant fields are: the graviton, $g_{\mu\nu}$, the gauge field, A_μ^α , for the two $U(1)$'s, a scalar, x , and a pseudoscalar, y .

As we will describe below, this may be viewed as the bosonic sector of $\mathcal{N}=2$ supergravity coupled to a vector multiplet.

The two non-compact generators of $\text{SL}(2, \mathbb{R})$ in the fundamental representation of $\text{E}_{7(7)}$ can be chosen as follows:

$$\mathbf{T}_s = \begin{pmatrix} 0 & \Phi_{IJKL}^+ \\ \Phi_{IJKL}^+ & 0 \end{pmatrix}, \quad \mathbf{T}_c = \begin{pmatrix} 0 & i\Phi_{IJKL}^- \\ -i\Phi_{IJKL}^- & 0 \end{pmatrix}, \quad (2.2)$$

where

$$\Phi_{IJKL}^\pm = 24(\delta_{IJKL}^{1234} + \delta_{IJKL}^{1256} \pm \delta_{IJKL}^{1278} + \delta_{IJKL}^{3456} \pm \delta_{IJKL}^{3478} \pm \delta_{IJKL}^{5678}), \quad (2.3)$$

are self-dual (+) and antiself-dual (-) $\text{SO}(8)$ tensors, respectively. Then the scalar ‘56-bein’ is

$$\mathcal{V} \equiv e^{\mathbf{V}} = \begin{pmatrix} u_{ij}^{IJ} & v_{ijIJ} \\ v^{ijIJ} & u^{ij}_{IJ} \end{pmatrix}, \quad \mathbf{V} = x\mathbf{T}_s + y\mathbf{T}_c, \quad (2.4)$$

where the scalar, $x \equiv \lambda \cos \zeta$, and the pseudoscalar, $y \equiv \lambda \sin \zeta$, parametrize the coset

$$\frac{\text{SL}(2, \mathbb{R})}{\text{SO}(2)}, \quad (2.5)$$

with the canonical complex coordinate, z , given by

$$z = \tanh \lambda e^{i\zeta}. \quad (2.6)$$

Given the explicit generators (2.2), it is easy to check that the exponential (2.4) reduces to a polynomial,

$$\mathcal{V} = a_0 + a_1 \mathbf{V} + a_2 \mathbf{V}^2 + a_3 \mathbf{V}^3, \quad (2.7)$$

where

$$a_0 = \frac{2 - 3|z|^2}{2(1 - |z|^2)^{3/2}}, \quad a_1 = \frac{6 - 7|z|^2}{6(1 - |z|^2)^{3/2}}, \quad a_2 = 3a_3 = \frac{1}{2(1 - |z|^2)^{3/2}}, \quad (2.8)$$

Note that the order of this polynomial coincides with the index of embedding of $\text{SL}(2, \mathbb{R})$ in $\text{E}_{7(7)}$.

Using the 56-bein (2.4), it is now straightforward to obtain the full bosonic action of the truncated theory [56]. In particular, we find that it is consistent to set the vector fields, A_μ^α , to zero. Then the resulting Lagrangian for the gravity coupled to the scalar fields is:¹

$$\begin{aligned} e^{-1} \mathcal{L} &= \frac{1}{2} R - 3 \frac{\partial_\mu z \partial^\mu \bar{z}}{(1 - |z|^2)^2} - 6g^2 \frac{1 + |z|^2}{1 - |z|^2} \\ &= \frac{1}{2} R - 3 \partial_\mu \lambda \partial^\mu \lambda - \frac{3}{4} \sinh^2(2\lambda) \partial_\mu \zeta \partial^\mu \zeta + 6g^2 \cosh(2\lambda). \end{aligned} \quad (2.9)$$

The Lagrangian (2.9) has no explicit dependence on the phase, ζ , and hence there is a conserved Noether current

$$\mathcal{J}_\mu = e \sinh^2(2\lambda) \partial_\mu \zeta, \quad (2.10)$$

¹See appendix B for additional details. This action and the potential (2.13) can be obtained from [54] by setting $\phi^{(1)} = \phi^{(2)} = \phi^{(3)} = 2\lambda$ and $\theta^{(1)} = \theta^{(2)} = \theta^{(3)} = \zeta$ and/or from [55, 56] by setting $\zeta_1 = \zeta_2 = 0$.

with the corresponding $U(1)_\zeta$ symmetry being simply a rotation between the scalar and the pseudoscalar.

It was shown in [29, 56] (see, also [54]) that by keeping the $SU(3)$ -invariant fermions, the truncation yields a $\mathcal{N}=2$ supergravity in four dimensions. Its R -symmetry is a combination of the two $U(1)$'s and, from the supersymmetry variations,

$$\delta\psi_\mu^i = 2D_\mu\epsilon^i + \sqrt{2}g A_a^{ij}\gamma_\mu\epsilon_j, \quad i, j = 7, 8, \quad (2.11)$$

the real superpotential, W , is given by an eigenvalue of the A_1 -tensor, $W = \sqrt{2}|A_1^{77}| = \sqrt{2}|A_1^{88}|$, see appendix B. Substituting the real fields, λ and ζ , in (B.4), we then find

$$W = \sqrt{2}\sqrt{\sinh^6\lambda + \cosh^6\lambda + 2\sinh^3\lambda\cosh^3\lambda\cos(3\zeta)}. \quad (2.12)$$

In terms of the superpotential, W , the potential

$$\mathcal{P} = -6\cosh(2\lambda), \quad (2.13)$$

is given by

$$\mathcal{P} = \frac{1}{3}\left[\left(\frac{\partial W}{\partial\lambda}\right)^2 + \frac{4}{\sinh^2(2\lambda)}\left(\frac{\partial W}{\partial\zeta}\right)^2\right] - 3W^2. \quad (2.14)$$

Note that unlike the potential, \mathcal{P} , the superpotential, W , is invariant only under a \mathbb{Z}_3 subgroup of $U(1)_\zeta$.

2.2 Domain wall Ansätze and BPS equations

In this paper we are interested in a special class of solutions corresponding to RG-flows and one-dimensional defects in the dual ABJM theory. Thus we take the metric given by a domain wall Ansatz

$$ds_{1,3}^2 = e^{2A(r)}ds_{1,2}^2 + dr^2, \quad (2.15)$$

and where the metric function, $A(r)$, and the scalar fields, $\lambda(r)$ and $\zeta(r)$, are functions of the radial coordinate, r , only. Furthermore, $ds_{1,2}^2$, is either a Minkowski metric (RG-flows) or a metric on AdS_3 of radius ℓ (Janus solutions),

$$ds_{1,2}^2 = e^{2y/\ell}(-dt^2 + dx^2) + dy^2. \quad (2.16)$$

Since, at least formally, the equations for the RG-flows can be obtained by taking the radius $\ell \rightarrow \infty$, throughout much of the discussion we will write only the more general formulae for the Janus solutions.

The equations of motion for the metric (2.15) and the scalar fields that follow from the Lagrangian (2.9) are

$$\begin{aligned} \lambda'' &= -3A'\lambda' + \frac{1}{4}\sinh(4\lambda)(\zeta')^2 - 2g^2\sinh(2\lambda), \\ \zeta'' &= -3A'\zeta' - 4\coth(2\lambda)\zeta'\lambda', \\ A'' &= -\frac{3}{2}(A')^2 - \frac{3}{2}(\lambda')^2 - \frac{3}{8}\sinh^2(2\lambda)(\zeta')^2 - \frac{e^{-2A}}{2\ell^2}, \end{aligned} \quad (2.17)$$

and

$$(A')^2 - (\lambda')^2 - \frac{1}{4} \sinh(2\lambda) (\zeta')^2 - 2g^2 \cosh(2\lambda) + \frac{e^{-2A}}{\ell^2} = 0, \quad (2.18)$$

where the last two equations are independent combinations of the Einstein equations.²

Imposing an unbroken supersymmetry along the flow, one obtains a first order system of the BPS equations. We refer the reader to [29] for further details and here only quote the final result:³

$$\lambda' = -\frac{1}{3} \left(\frac{A'}{W} \right) \frac{\partial W}{\partial \lambda} + \frac{2\kappa}{3} \left(\frac{e^{-A}}{\ell} \right) \frac{1}{\sinh(2\lambda)} \frac{1}{W} \frac{\partial W}{\partial \zeta}, \quad (2.19)$$

$$\zeta' = -\frac{4}{3} \left(\frac{A'}{W} \right) \frac{1}{\sinh^2(2\lambda)} \frac{\partial W}{\partial \zeta} - \frac{2\kappa}{3} \left(\frac{e^{-A}}{\ell} \right) \frac{1}{\sinh(2\lambda)} \frac{1}{W} \frac{\partial W}{\partial \lambda}, \quad (2.20)$$

together with

$$(A')^2 = g^2 W^2 - \frac{e^{-2A}}{\ell^2}. \quad (2.21)$$

The constant $\kappa = \pm 1$ is determined by the chirality of the unbroken supersymmetry, with $\mathcal{N} = (2, 0)$ for $\kappa = 1$ and $\mathcal{N} = (0, 2)$ for $\kappa = -1$. In the following we set $\kappa = 1$. Note that (2.21) is the same as (2.18) after one eliminates the derivatives of the scalar fields using (2.19) and (2.20). It is also straightforward to verify that the equations of motion (2.17) follow from the BPS equations.

Finally, the BPS equations for supersymmetric RG-flows

$$A' = \pm g W, \quad (2.22)$$

and

$$\lambda' = \mp \frac{g}{3} \frac{\partial W}{\partial \lambda}, \quad \zeta' = \mp \frac{4g}{3 \sinh^2(2\lambda)} \frac{\partial W}{\partial \zeta}, \quad (2.23)$$

are obtained from (2.19), (2.20) and (2.21) by taking the $\ell \rightarrow \infty$ limit. There is no constraint on the chirality of the unbroken $\mathcal{N} = 2$ supersymmetry.

2.3 Integrating the BPS equations

The Janus solutions to the BPS equations (2.19)–(2.21) have been studied in [29] where it was shown by a numerical analysis that there are three classes of solutions shown in figures 1 and 2: regular Janus solutions (shown in green) interpolating between two AdS_4 regions corresponding to the same $SO(8)$ stationary point of the potential (2.13) and singular solutions that diverge on either one side (shown in red) or both sides (shown in blue) of the flow. The Janus solutions are characterized by the presence of a special central point along a flow where the solution passes from one branch of (2.21) to another. This point is marked by a red dot and the corresponding values of the scalar fields are denoted by λ_c and ζ_c , respectively. The position of this point for a given flow determines the type of a solution, see figure 3 in [29]. In particular, for $\cos \zeta_c \neq -1$, all solutions are singular

²As a consequence of the Bianchi identities, the derivative of (2.18) follows from (2.17).

³Similar BPS equations for holographic domain walls with curved slices were written down in [47, 57, 58].

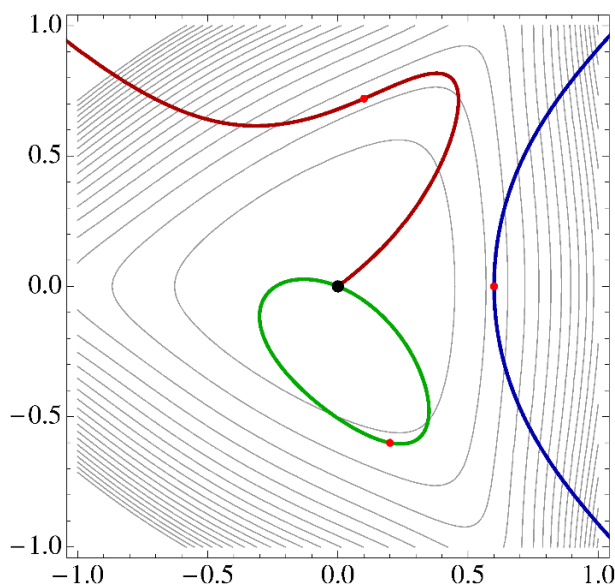


Figure 1. Typical flow trajectories for the Janus solutions to the BPS equations (2.19)–(2.21) in the $(\lambda \cos \zeta, \lambda \sin \zeta)$ -plane. The background contours are of the superpotential $W(\lambda, \zeta)$. A red dot denotes the “central point” of a flow at $(\lambda_c \cos \zeta_c, \lambda_c \sin \zeta_c)$ where $A' = 0$.

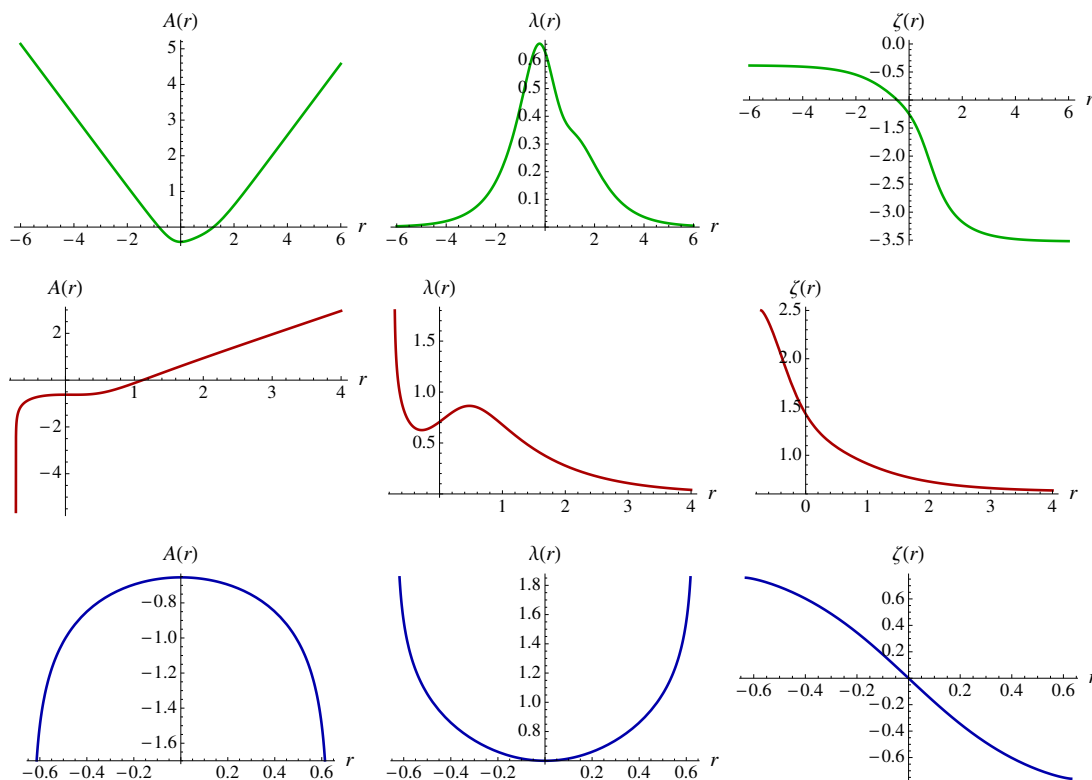


Figure 2. Typical profiles of the metric function, $A(r)$, and the scalar fields, $\lambda(r)$ and $\zeta(r)$, for the different types of flows in figure 1.

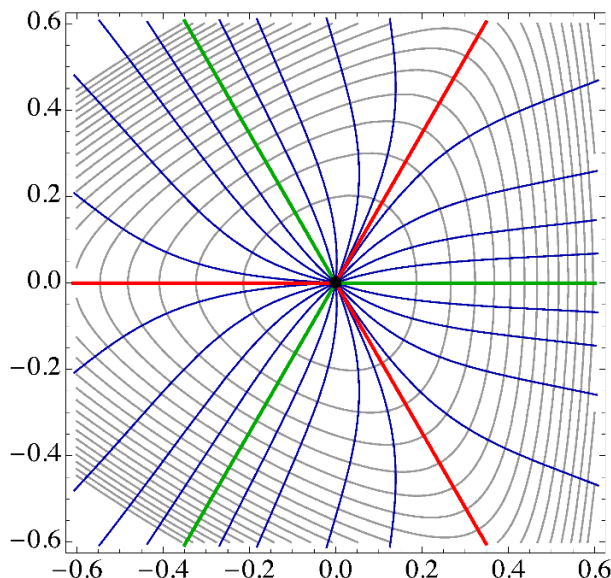


Figure 3. RG-flow trajectories in the $(\lambda \cos \zeta, \lambda \sin \zeta)$ -plane. The background contours are of the real superpotential $W(\lambda, \zeta)$. The ridge trajectories have constant ζ with $\cos 3\zeta = 1$ (green) and $\cos 3\zeta = -1$ (red), respectively.

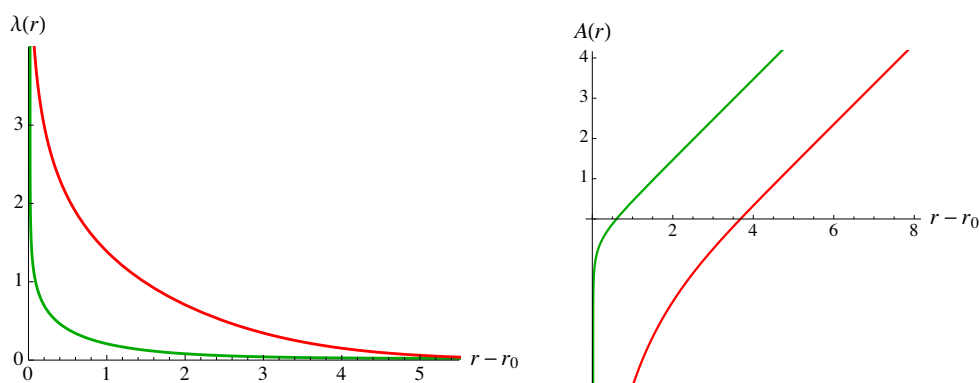


Figure 4. Ridge flows for $\cos 3\zeta = -1$ (red) and $\cos 3\zeta = 1$ (green) with $A_0 = 0$.

provided λ_c is large enough. It is only when $\cos \zeta_c = -1$ that all solutions are regular Janus solutions irrespective of the value of λ_c .

In addition, there are solutions akin to RG-flows, which asymptote to AdS_4 on one side and become singular on the other, while remaining on a single branch of (2.21). They can be thought of as a singular limit of Janus solutions where the central point is moved off to infinity. Simplest examples of such flows are obtained by taking constant $\zeta = \zeta_0$ with $\cos(3\zeta_0) \neq \pm 1$. Solving (2.19)–(2.21) for A' , λ' and A , and then imposing consistency between them, one is left with

$$\begin{aligned} \lambda' &= \mp \frac{g}{\sqrt{2}} \sinh(2\lambda) \sqrt{\cosh(2\lambda) + \cos(3\zeta_0) \sinh(2\lambda)}, \\ \frac{e^{-2A}}{\ell^2} &= \frac{g^2}{2} \frac{\sin^2(3\zeta_0) \sinh(2\lambda)}{\cos(3\zeta_0) + \coth(2\lambda)}. \end{aligned} \tag{2.24}$$

Choosing the top sign in (2.24), we can impose the AdS_4 boundary condition in the UV, that is $\lambda \rightarrow 0$ as $r \rightarrow \infty$, to integrate the first equation for $\lambda(r)$, and then solve the second equation for $A(r)$. The resulting solutions are similar to the ones in figure 4, which we will discuss shortly.

The situation simplifies considerably in the RG-flow limit where the scalar equations (2.23) do not involve the metric function, A , and can be solved first. Choosing the top sign in (2.22)–(2.23), which corresponds to the UV region at $r \rightarrow \infty$, one then finds flows shown in figures 3 and 4 [22].

In fact, as we have discussed in [22], generic solutions for the RG-flows can be determined analytically using two constants of motion: the general one

$$\mathcal{I}_1 = e^{3A} \sinh^2(2\lambda) \zeta', \tag{2.25}$$

valid for any ℓ and corresponding to the conserved current (2.10), and the second constant⁴

$$\begin{aligned} \mathcal{I}_2 &= \frac{W^2}{(\cosh 2\lambda + \cos \zeta \sinh 2\lambda)^3} \frac{\sin 3\zeta}{\sin^3 \zeta} \\ &= \frac{(4 \cos^2 \zeta - 1)}{2 \sin^2 \zeta} \frac{(3 + (\cosh 2\lambda - 2 \cos \zeta \sinh 2\lambda)^2)}{(\cosh 2\lambda + \cos \zeta \sinh 2\lambda)^2}, \end{aligned} \tag{2.26}$$

for the first order system (2.22)–(2.23). Using \mathcal{I}_1 and \mathcal{I}_2 , one can then determine A and ζ as a function of λ , which is sufficient given the reparametrization invariance for the coordinate along the flow.

However, this method of integration fails for the special flows with constant $\zeta = \zeta_0$. For an RG-flow, one must then have $\cos 3\zeta_0 = \pm 1$ (see, green and red ridge flows in figure 3). The resulting equations can be obtained from (2.24) by taking the limit,

$$\cos(3\zeta_0) \rightarrow \pm 1, \quad \ell \rightarrow \infty, \quad \ell g \sin(3\zeta_0) \rightarrow 2\sqrt{2} e^{-A_0}, \tag{2.27}$$

where A_0 is a constant. One can then integrate those equations directly to obtain

$$\operatorname{arccoth}(e^\lambda) \pm \arctan(e^\lambda) \mp \frac{\pi}{2} = \frac{g}{\sqrt{2}}(r - r_0), \tag{2.28}$$

and

$$A(r) = A_0 - \log(e^{4\lambda} - 1) + \begin{cases} 3\lambda & \text{for } \cos 3\zeta = +1 \\ \lambda & \text{for } \cos 3\zeta = -1 \end{cases} \tag{2.29}$$

where r_0 and A_0 are integration constants. The solutions to those ridge flows are shown in figure 4. Comparing with the trajectories in figure 3, we expect that the solutions with $\cos(3\zeta_0) = 1$ are representative of the generic RG-flows, while those with $\cos(3\zeta_0) = -1$ are special. This expectation is confirmed by the asymptotic expansions that we will now discuss.

⁴Given different types of solutions in figure 1, one would not expect to find such an additional constant of motion for the BPS equations (2.19)–(2.21) at finite ℓ .

2.4 Behavior at large λ

The holographic RG flows, governed by (2.22) and (2.23), have $\lambda \rightarrow \infty$ at some finite value of r . Generically, such solutions are dual to a massive flow toward some new infra-red limit. As was observed in [22], the RG flow solutions considered here can encode rich and interesting infra-red physics once one examines them in M theory. The Janus flows, governed by (2.19)–(2.21), can either form a loop starting and finishing at $\lambda = 0$ or start at $\lambda = 0$ and ultimately flow with $\lambda \rightarrow \infty$. There are also Janus solutions that begin and end with $\lambda \rightarrow \infty$. The Janus flows that involve large λ may be viewed as interfaces that explore the infra-red structure of the holographic flow solutions. We will therefore examine the limiting behaviors of these flows as $\lambda \rightarrow \infty$. In section 6 we will uplift these results to M theory to see more precisely how they may be interpreted in terms of M branes.

From the explicit solutions in figures 2 and 4, we see that the limit is characterized by

$$\lambda \rightarrow \infty, \quad \zeta \rightarrow \zeta_\infty, \quad A \rightarrow -\infty \quad \text{as } r \rightarrow r_0, \quad (2.30)$$

where ζ_∞ is a constant asymptotic angle for a given flow. This observation is confirmed by a more careful expansion of the BPS equations (2.19)–(2.23).

Expanding the superpotential (2.12) as $\lambda \rightarrow \infty$ for a generic ζ , we have

$$W \sim \frac{1}{4} \sqrt{1 + \cos(3\zeta)} e^{3\lambda} + O(e^{-\lambda}), \quad \cos(3\zeta) \neq -1, \quad (2.31)$$

while for the special ridge flows,

$$W \sim \frac{3}{2\sqrt{2}} e^\lambda + O(e^{-3\lambda}), \quad \cos(3\zeta) = -1. \quad (2.32)$$

The flow equations (2.23) and (2.22) simplify

$$\frac{dA}{d\lambda} = -3W \left(\frac{\partial W}{\partial \lambda} \right)^{-1} \sim \begin{cases} -3 & \text{for } \cos 3\zeta = -1 \\ -1 & \text{otherwise} \end{cases}. \quad (2.33)$$

$$\frac{d\zeta}{d\lambda} = \frac{4}{\sinh^2(2\lambda)} \frac{\partial W}{\partial \zeta} \left(\frac{\partial W}{\partial \lambda} \right)^{-1} \rightarrow 0. \quad (2.34)$$

The latter confirms the constancy of ζ at infinity while the former shows the rate of divergence of A depends upon that angle. This will translate into different physics once we uplift those flows to M theory.

3 The uplift

We have obtained the Lagrangian (2.9) and the BPS equations (2.19)–(2.23) by a consistent truncation of the bosonic Lagrangian and the supersymmetry variations of four-dimensional, $\mathcal{N}=8$ gauged supergravity. Since the latter theory is a consistent truncation of M theory on S^7 [33–35], any solution of the equations of motion for the Lagrangian (2.9) can be uplifted to a solution of the eleven-dimensional supergravity. Indeed, for a constant

axion, ζ , such an uplift has been already constructed in [54]. In the next two sections we verify the truncation for a nontrivial axion using explicit uplift formula for the metric [32] and the recently obtained uplift formulae for the flux [36–41].

Similar calculation verifying the new uplift Ansätze for the flux has been carried out recently for a special class of solutions of the four-dimensional, $\mathcal{N} = 8$ theory given by some of the stationary points of the potential: $\text{SO}(8)$, $\text{SO}(7)^\pm$, G_2 , $\text{SU}(4)^-$ [37, 39], and $\text{SO}(3) \times \text{SO}(3)$ [42], for which the four-dimensional space-time is AdS_4 and the scalar fields are constant. Our construction of the uplift is similar as in those references, which the reader should consult for any omitted background material.

3.1 $\text{SU}(3) \times \text{U}(1) \times \text{U}(1)$ invariants on S^7

The construction of an uplift inevitably leads to rather complicated formulae. Both to organize the calculation and to write down the result in a succinct form, it is convenient to express the internal components of the fields in terms of canonical $\text{SO}(7)$ tensors on S^7 that are associated with the $\text{E}_{7(7)}$ generators of the scalar fields in the truncation. To this end let us define:⁵

$$\xi_{mn} = -\frac{1}{16}\Phi_{IJKL}^+ K_m^{IJ} K_n^{KL}, \quad \xi_m = \frac{1}{16}\Phi_{IJKL}^+ K_{mn}^{IJ} K^{nKL}, \quad \xi = \overset{\circ}{g}{}^{mn}\xi_{mn}, \quad (3.1)$$

and

$$S_{mnp} = \frac{1}{16}\Phi_{IJKL}^- K_{mn}^{IJ} K_p^{KL}, \quad (3.2)$$

where Φ_{IJKL}^\pm are the $\text{SO}(8)$ tensors defined in (2.3) and

$$K^{mIJ} = i\bar{\eta}^I \overset{\circ}{\Gamma}^m \eta^J, \quad K_{mn}^{IJ} = \bar{\eta}^I \overset{\circ}{\Gamma}_{mn} \eta^J, \quad \overset{\circ}{D}_m K_n^{IJ} = m_7 K_{mn}^{IJ}, \quad (3.3)$$

are the $\text{SO}(8)$ Killing vectors (one-forms) and two-forms on the round S^7 given in terms of an orthonormal basis of Killing spinors, η^I ,

$$i\overset{\circ}{D}_m \eta^I = \frac{m_7}{2} \overset{\circ}{\Gamma}_m \eta^I, \quad \bar{\eta}^I \eta^J = \delta^{IJ}, \quad I, J = 1, \dots, 8. \quad (3.4)$$

The inverse radius of S^7 is denoted by $m_7 \equiv L^{-1}$ and $\Gamma^m = \overset{\circ}{e}_a{}^m \Gamma^a$. The circle indicates that $\overset{\circ}{e}^a$ is a siebenbein for the round metric on S^7 , $ds_{S^7}^2 = \overset{\circ}{g}{}_{mn} dy^m dy^n$ where $\overset{\circ}{g}{}_{mn} = \overset{\circ}{e}_m{}^a \overset{\circ}{e}_n{}^b \delta_{ab}$, and $\overset{\circ}{D}_m$ is the covariant derivative with respect to that metric. Unless indicated otherwise, all indices on the S^7 tensors are raised and lowered with the *round* metric, for example $K_m^{IJ} = \overset{\circ}{g}{}_{mn} K^{nIJ}$. The coordinates, y^m , on S^7 are for the moment arbitrary. However, one should note that ξ defined in (3.1) is a scalar harmonic on S^7 and may be thought of as providing a natural internal coordinate on the compactification manifold.

By construction, the tensors (3.1) and (3.2) are invariant, i.e. have vanishing Lie derivative, under the $\text{SU}(3) \times \text{U}(1) \times \text{U}(1) \subset \text{SO}(8)$ symmetry group of the truncation. In particular, the Killing vectors for the two $\text{U}(1)$'s,

$$v_m = \Omega_{IJ}^\alpha K_m^{IJ}, \quad w_m = \Omega_{IJ}^\beta K_m^{IJ}, \quad (3.5)$$

$$\Omega_{12}^\alpha = \Omega_{34}^\alpha = \Omega_{56}^\alpha = 1, \quad \Omega_{78}^\beta = 1, \quad (3.6)$$

⁵For a more extensive discussion of these tensors, see [42] and the original references therein.

provide additional invariant one-forms on S^7 . In the following we will show that the metric and the flux for the uplift can be simply written in terms of the round metric, \mathring{g}_{mn} , the one-forms

$$\xi_{(1)} \equiv \xi_m dy^m, \quad v_{(1)} \equiv v_m dy^m, \quad \omega_{(1)} = \omega_m dy^m, \quad (3.7)$$

the three-form,

$$S_{(3)} \equiv \frac{1}{6} S_{mnp} dy^m \wedge dy^n \wedge dy^p, \quad (3.8)$$

and the scalar, ξ .

3.2 The metric

The eleven-dimensional space-time for the uplifted solutions is a warped product, $\mathcal{M}_{1,3} \times \mathcal{M}_7$, with the metric

$$ds_{11}^2 = \Delta^{-1} ds_{1,3}^2 + ds_7^2, \quad (3.9)$$

where⁶ $ds_{1,3}^2 = \mathring{g}_{\mu\nu} dx^\mu dx^\nu$ is the metric in four dimensions for a particular solution at hand. The internal metric, $ds_7^2 = g_{mn} dy^m dy^n$, is determined by the celebrated formula for its densitized inverse [32]:

$$\Delta^{-1} g^{mn} = \frac{1}{8} K^{mIJ} K^{nKL} \left[(u^{MN}{}_{IJ} + v^{MNIJ}) (u_{MN}{}^{KL} + v_{MNKL}) \right], \quad (3.10)$$

from which the warp factor, Δ , can be calculated using

$$\Delta^{-9} = \det(\Delta^{-1} g^{mn} \mathring{g}_{np}). \quad (3.11)$$

While it is possible to express the densitized metric entirely using tensors (3.1) and (3.2) and their (contracted) products,⁷ the simplest expression is obtained by noting that the symmetric tensors resulting from such contractions can be rewritten using the round metric, \mathring{g}_{mn} , and bilinears in the one forms ξ_m , v_m and ω_m , as in the following examples:

$$\begin{aligned} \xi_{mn} &= \frac{1}{6} (3 + \xi) \mathring{g}_{mn} + \frac{1}{6(\xi - 3)} \xi_m \xi_n + \frac{3}{8(\xi - 3)} (v_m + \omega_m)(v_n + \omega_n), \\ S_{mpq} S_n{}^{pq} &= \frac{1}{4} (v_m - \omega_m)(v_n - \omega_n), \\ S_{mpr} S_n{}^r \xi^p \xi^q &= \frac{9}{4} v_m v_n + \frac{3}{4} (9 - 2\xi) \omega_m \omega_n + \frac{3\xi}{2} v_{(m} \omega_{n)}. \end{aligned} \quad (3.12)$$

After some algebra, we then find

$$\Delta^{-1} g^{mn} = c_1 \mathring{g}^{mn} + c_2 \xi^m \xi^n + c_3 v^m v^n + c_4 \omega^m \omega^n + c_5 v^{(m} \omega^{n)}, \quad (3.13)$$

⁶To distinguish the components of the four-dimensional metric (2.15) from the components of its eleven-dimensional counterpart along the four dimensions, we will denote the former by $\mathring{g}_{\mu\nu}$. Thus $g_{\mu\nu} = \Delta^{-1} \mathring{g}_{\mu\nu}$.

⁷See, for example a general discussion in [42].

where all the c_i can be expressed in terms of four-dimensional quantities and the scalar, ξ :

$$\begin{aligned}
 c_1 &= \cosh(2\lambda) - \frac{1}{6}(\xi + 3) \sinh(2\lambda) \cos(\zeta), \\
 c_2 &= \frac{1}{6(3 - \xi)} \sinh(2\lambda) \cos(\zeta), \\
 c_3 &= \frac{\sinh(2\lambda)}{32(\xi - 3)} \left[(\xi - 3) \cosh(4\lambda) \cos(\zeta) + (\xi - 3) \sinh(4\lambda) - (\xi + 9) \cos \zeta \right], \\
 c_4 &= \frac{\sinh(2\lambda)}{16(\xi - 3)} \left[(\xi - 3) \sinh^2(2\lambda) \cos(3\zeta) + \frac{1}{2}(\xi - 3) \sinh(4\lambda) - 6 \cos \zeta \right], \\
 c_5 &= \frac{\sinh(2\lambda)}{16(\xi - 3)} \left[(\xi - 3) \sinh(4\lambda) \cos(2\zeta) + (\xi - 3) \cosh(4\lambda) \cos \zeta - (\xi + 9) \cos \zeta \right].
 \end{aligned} \tag{3.14}$$

Note that the indices on the right hand side in (3.12) are raised using the round metric, \mathring{g}_{mn} , which is the convention followed throughout this section.

All that is needed now to invert the densitized metric (3.13) are contraction identities between the one-forms, which can be derived using the explicit form of the SO(8) tensors and properties of the Killing vectors summarized in [37, 42] and the references therein. We have

$$\begin{aligned}
 \xi^m \xi_m &= 27 - 6\xi - \xi^2, & v^m v_m &= 12 - \frac{8}{3}\xi, & \omega^m \omega_m &= 4, \\
 \xi^m v_m &= \xi^m \omega_m = 0, & v^m \omega_m &= -\frac{4}{3}\xi.
 \end{aligned} \tag{3.15}$$

It is then straightforward to check that

$$\Delta g_{mn} = g_1 \mathring{g}_{mn} + g_2 \xi_m \xi_n + g_3 (v_m v_n + \omega_m \omega_n) + g_4 v_{(m} \omega_{n)}, \tag{3.16}$$

where

$$\begin{aligned}
 g_1 &= \frac{6}{6 \cosh(2\lambda) - (\xi + 3) \sinh(2\lambda) \cos \zeta}, \\
 g_2 &= \frac{\mathcal{D}}{36(\xi - 3)} \cos \zeta [3 \sinh(4\lambda) - (\xi + 3) \sinh^2(2\lambda) \cos \zeta], \\
 g_3 &= \frac{\mathcal{D}}{16(\xi - 3)} [3 \sinh(4\lambda) \cos(\zeta) - \sinh^2(2\lambda)(\xi + 3 \cos(2\zeta))], \\
 g_4 &= \frac{\mathcal{D}}{8(\xi - 3)} [3 \sinh(4\lambda) \cos(\zeta) - \sinh^2(2\lambda)(\xi \cos(2\zeta) + 3)],
 \end{aligned} \tag{3.17}$$

and

$$\mathcal{D} = \frac{36}{[\sinh(2\lambda) \cos \zeta + \cosh(2\lambda)] [6 \cosh(2\lambda) - (\xi + 3) \sinh(2\lambda) \cos \zeta]^2}. \tag{3.18}$$

Using the contractions (3.15), one can also calculate the derivatives of the warp factor given by (3.11) with respect to λ and ζ . Then a simple integration yields

$$\Delta = \mathcal{D}^{1/3}, \tag{3.19}$$

with the overall normalization set by $\Delta = 1$ for the round sphere metric when $\lambda = 0$. Dividing out Δ in (3.16), yields the internal metric, g_{mn} , in terms of the SO(7) tensors associated with the truncation.

3.3 Internal coordinates and local expressions

In addition to the metric tensor, we will also need the corresponding orthonormal frames and those turn out to be rather cumbersome to write down using the invariant tensors (3.16). Also, the formulae like (3.16) tend to obscure the underlying geometry of the solution and its symmetry. To address both of these points, we will now choose a suitable set of coordinates, y^m , on the internal manifold. As usual, see for example [42] section 7.1, this can be done systematically as follows.

First, we embed S^7 into the ambient \mathbb{R}^8 as the surface

$$Y^A Y^A = m_7^{-1}, \tag{3.20}$$

such that the Killing vectors $K^{IJ} = K_m^{IJ} dy^m$ defined in (3.3) are related by triality to the familiar ones, that is

$$K^{IJ} = -\frac{m_7}{2} \Gamma_{AB}^{IJ} \mathcal{K}^{AB}, \quad \mathcal{K}^{AB} = -\frac{1}{8m_7} \Gamma_{AB}^{IJ} K^{IJ}, \tag{3.21}$$

where

$$\mathcal{K}^{AB} = Y^A dY^B - Y^B dY^A, \tag{3.22}$$

Similarly, we have

$$K_{(2)}^{IJ} \equiv \frac{1}{2} K_{mn}^{IJ} dy^m \wedge dy^n = \frac{1}{2} \Gamma_{AB}^{IJ} d\mathcal{K}^{AB}. \tag{3.23}$$

The action of the symmetry group $SU(3) \times U(1) \times U(1)$ in the ambient space is given by the branching⁸

$$\mathbf{8}_s \rightarrow \left(\mathbf{3}, -\frac{1}{2}, \frac{1}{2}\right) + \left(\bar{\mathbf{3}}, \frac{1}{2}, -\frac{1}{2}\right) + \left(\mathbf{1}, \frac{3}{2}, \frac{1}{2}\right) + \left(\mathbf{1}, -\frac{3}{2}, -\frac{1}{2}\right). \tag{3.24}$$

One can choose a representation of Γ -matrices such that the $SU(3)$ generators act in the subspace, Y^1, \dots, Y^6 , while the two $U(1)$ generators have 2×2 diagonal blocks. Then a convenient choice for the coordinates, $(y^m) = (\chi, \theta, \alpha_1, \alpha_2, \alpha_3, \psi, \phi)$, on S^7 , that makes the symmetry manifest is as follows:

$$\begin{aligned} Y^1 + i Y^2 &= m_7^{-1} \cos \chi \sin \theta \sin \frac{\alpha_1}{2} e^{\frac{i}{2}(\alpha_2 - \alpha_3)} e^{-i(\phi + \psi)}, \\ Y^3 + i Y^4 &= m_7^{-1} \cos \chi \sin \theta \cos \frac{\alpha_1}{2} e^{-\frac{i}{2}(\alpha_2 + \alpha_3)} e^{-i(\phi + \psi)}, \\ Y^5 + i Y^6 &= m_7^{-1} \cos \chi \cos \theta e^{-i(\phi + \psi)}, \\ Y^7 + i Y^8 &= m_7^{-1} \sin \chi e^{-i\phi}, \end{aligned} \tag{3.25}$$

⁸We follow here the usual convention that the supersymmetries, ϵ^i , transform in $\mathbf{8}_v$, while the ambient coordinates, Y^A , in $\mathbf{8}_s$ of $SO(8)$.

where $\alpha_1, \alpha_2, \alpha_3$ are the $SU(2)$ Euler angles, while the angles ψ and ϕ parametrize the $U(1) \times U(1)$ isometry.⁹ In this parametrization, the round metric on S^7 with unit radius is¹⁰

$$\begin{aligned}
 ds_{S^7}^2 &\equiv m_7^2 dY^A dY^A \\
 &= d\chi^2 + \cos^2 \chi \left[ds_{\mathbb{CP}^2}^2 + \sin^2 \chi \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 \right)^2 \right] \\
 &\quad + \left[d\phi + \cos^2 \chi \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 \right) \right]^2,
 \end{aligned} \tag{3.26}$$

where

$$ds_{\mathbb{CP}^2}^2 = d\theta^2 + \frac{1}{4} \sin^2 \theta (\sigma_1^2 + \sigma_2^2 + \cos^2 \theta \sigma_3^2), \tag{3.27}$$

is the metric on \mathbb{CP}^2 and σ_i are the $SU(2)$ -invariant forms. The first line in (3.26) is the metric on \mathbb{CP}^3 and the second line is the Hopf fiber. The $SU(3) \times U(1)_\psi$ symmetry acts transitively on \mathbb{CP}^2 and the ψ -fiber. Both the metric on \mathbb{CP}^2 and the one form $d\psi + \frac{1}{2} \sin^2 \theta$ are invariant.

Using (3.21) and (3.25), we can now express the invariants introduced in section 3.1 in terms of ambient and local coordinates. We find that the invariant function, ξ , is simply

$$\begin{aligned}
 \xi &= 3 - 12 m_7^2 [(Y^7)^2 + (Y^8)^2] = -9 + 12 m_7^2 [(Y^1)^2 + \dots + (Y^6)^2] \\
 &= 3(1 - 4 \sin^2 \chi),
 \end{aligned} \tag{3.28}$$

while the invariant one-forms are

$$\begin{aligned}
 \xi_{(1)} &= -12 m_7 (Y^7 dY^7 + Y^8 dY^8) = -6 m_7^{-1} \sin(2\chi) d\chi, \\
 v_{(1)} + \omega_{(1)} &= 8 m_7 (Y^7 dY^8 - Y^8 dY^7) = -8 m_7^{-1} \sin^2 \chi d\phi, \\
 v_{(1)} - 3\omega_{(1)} &= 8 m_7 (Y^1 dY^2 - Y^2 dY^1 + \dots + Y^5 dY^6 - Y^6 dY^5) \\
 &= -8 m_7^{-1} \cos^2 \chi \left(d\phi + d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 \right).
 \end{aligned} \tag{3.29}$$

We also have

$$\begin{aligned}
 S_{(3)} &\equiv \frac{1}{6} S_{mnp} dy^m \wedge dy^n \wedge dy^p = -\frac{1}{6} m_7 \Phi_{MNPQ}^+ Y^M dY^N \wedge dY^P \wedge dY^Q \\
 &= -m_7^{-3} J_{\mathbb{CP}^3} \wedge \vartheta_{S^7},
 \end{aligned} \tag{3.30}$$

where

$$J_{\mathbb{CP}^3} = \frac{1}{2} d\vartheta_{S^7}, \quad \vartheta_{S^7} = d\phi + \cos^2 \chi \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 \right), \tag{3.31}$$

are, respectively, the complex structure on \mathbb{CP}^3 and the corresponding Sasaki-Einstein one-form on S^7 .

⁹More precisely, the two $U(1)$ angles are $\phi + \psi/2$ and $-\phi - 3\psi/2$, respectively.
¹⁰All functions and forms in the ambient \mathbb{R}^8 are implicitly pulled-back onto S^7 using (3.20).

Finally, we substitute the invariants (3.28) and (3.29) into the warp factor (3.19) and the metric (3.16). To simplify expressions we define:

$$X_{\pm}(x) \equiv \cosh 2\lambda \pm \cos \zeta \sinh 2\lambda, \quad \Sigma(x, \chi) \equiv X_+ \sin^2 \chi + X_- \cos^2 \chi, \quad (3.32)$$

which are functions of the space-time coordinates, x^μ , and the internal coordinate, χ . Then the internal metric can be written as

$$ds_7^2 = m_7^{-2} \left(\frac{\Sigma}{X} \right)^{\frac{2}{3}} \left[d\chi^2 + \cos^2 \chi \frac{X}{\Sigma} \left(ds_{\mathbb{CP}^2}^2 + \sin^2 \chi \frac{X}{\Sigma} \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 + \frac{\Xi}{X} d\phi \right)^2 \right) + \frac{1}{\Sigma^2} \left(d\phi + \cos^2 \chi \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 \right) \right)^2 \right], \quad (3.33)$$

where to simplify the notation we set $X \equiv X_+$ and $\Xi \equiv X_+ - X_-$. The warp factor (3.19) is

$$\Delta = \frac{1}{X^{1/3} \Sigma^{2/3}}. \quad (3.34)$$

For $\lambda = 0$, we have $X_{\pm} = \Sigma = 1$ and the metric (3.33) reduces to the metric (3.26) on the sphere with radius m_7^{-1} . The deformation clearly preserves the $SU(3) \times U(1) \times U(1)$ symmetry as well as the metric along the Hopf fiber, which is now rescaled by Σ^{-2} with respect to the six-dimensional base. This suggests that there might be some deformed Kähler geometry still present in the background. We will return to this point below in section 6.

3.4 The transverse flux

It is rather remarkable that it took more than 25 years to obtain workable formulae for the four-form flux, $F_{(4)} = dA_{(3)}$. Indeed, while the general proof of the consistent truncation of eleven-dimensional supergravity on S^7 [33–35] yielded explicit formulae for $F_{(4)}$, those formulae were rather difficult if not impossible to use for all but the simplest stationary point solutions [35]. It is only recently that new Ansätze for various components of the four-form flux were found in [36–41] whose complexity is comparable to that of the metric Ansatz.

Starting with a domain wall solution in four-dimensions with a metric as in (2.15) and scalar fields depending only on the transverse coordinate, the corresponding four-form flux in eleven-dimensional supergravity can be decomposed into a sum of two terms

$$F_{(4)} = F_{(4)}^{\text{st}} + F_{(4)}^{\text{tr}}, \quad (3.35)$$

where $F_{(4)}^{\text{st}} = F_{(4,0)} + F_{(3,1)}$ is the “space-time” flux and $F_{(4)}^{\text{tr}} = F_{(0,4)} + F_{(1,3)}$ is the “transverse” flux. A label $(4 - p, p)$ indicates a $(4 - p)^{\text{th}}$ order form along $\mathcal{M}_{1,3}$ and a p^{th} order form along the internal manifold, \mathcal{M}_7 . Since by the Poincaré or conformal symmetry along the three-dimensional slices in $\mathcal{M}_{1,3}$ there can be no $(2, 2)$ -form in (3.35),¹¹ the Bianchi identity, $dF_{(4)} = 0$, implies that both $F_{(4)}^{\text{st}}$ and $F_{(4)}^{\text{tr}}$ must be closed. Hence $F_{(4)}^{\text{tr}} = dA_{(3)}^{\text{tr}}$, where $A_{(3)}^{\text{tr}}$ can have at most one “leg” along dr and thus can be always

¹¹Such terms must also vanish whenever the vector fields in four dimensions are set to zero [41].

gauge transformed into a 3-form with all three legs along the internal manifold \mathcal{M}_7 , that is $A_{(3)}^{\text{tr}} = \frac{1}{6} A_{mnp} dy^m \wedge dy^n \wedge dy^p$.

The components A_{mnp} are given by the new uplift Ansatz [36, 37], which, in our conventions, reads

$$\Delta^{-1} g^{pq} A_{mnp} = \frac{i}{16} K_{mn}^{IJ} K^{qKL} \left[(u^{MN}{}_{IJ} - v^{MNIJ}) (u_{MN}{}^{KL} + v_{MNKL}) \right]. \quad (3.36)$$

It is convenient to define a two-form $S_m \equiv \frac{1}{2} S_{mnp} dy^n \wedge dy^p$. Evaluating (3.36) in terms of invariants, we then find

$$\frac{1}{2} \Delta^{-1} g^{pq} A_{mnp} dy^m \wedge dy^n = (a_{11} v^q + a_{12} \omega^q) dv + (a_{21} v^q + a_{22} \omega^q) d\omega + a_3 S^q, \quad (3.37)$$

where the vector index on the right hand side is raised using the round metric and the coefficients are given by

$$\begin{aligned} a_{11} &= \frac{1}{64} m_7^{-1} \sinh^3(2\lambda) \sin \zeta, \\ a_{12} &= \frac{1}{64} m_7^{-1} \sinh^2(2\lambda) \sin \zeta [2 \cosh(2\lambda) \cos \zeta - \sinh(2\lambda)], \\ a_{21} &= -\frac{1}{64} m_7^{-1} \sinh^2(2\lambda) \sin \zeta [2 \cosh(2\lambda) \cos \zeta + \sinh(2\lambda)], \\ a_{22} &= -\frac{1}{64} m_7^{-1} \sin(3\zeta) \sinh^3(2\lambda), \\ a_3 &= -\frac{1}{2} \sin \zeta \sinh(2\lambda). \end{aligned} \quad (3.38)$$

Contracting with the densitized metric (3.16) and then using the contraction identities (3.15) together with

$$\begin{aligned} \xi^m S_m &= -\frac{3}{4} v_{(1)} \wedge \omega_{(1)}, \\ v^m S_m &= \frac{1}{12} m_7^{-1} (\xi - 6) dv_{(1)} + \frac{1}{4} m_7^{-1} d\omega_{(1)} - \frac{1}{6} \xi_{(1)} \wedge v_{(1)}, \\ \omega^m S_m &= \frac{1}{12} m_7^{-1} \xi dv_{(1)} - \frac{1}{4} m_7^{-1} d\omega_{(1)} - \frac{1}{6} \xi_{(1)} \wedge v_{(1)}, \end{aligned} \quad (3.39)$$

we find that the internal potential is simply given by

$$A_{(3)}^{\text{tr}} = \alpha_1 S_{(3)} + \alpha_2 \xi_{(1)} \wedge v_{(1)} \wedge \omega_{(1)}, \quad (3.40)$$

where

$$\alpha_1 = -\frac{1}{2\Sigma} \sin \zeta \sinh(2\lambda), \quad \alpha_2 = -\frac{1}{384 X \Sigma} \frac{\sin(2\zeta) \sinh^2(2\lambda)}{\sin^2 \chi}. \quad (3.41)$$

Rewriting (3.40) in local coordinates using (3.29) and (3.30) yields

$$A_{(3)}^{\text{tr}} = \frac{1}{2} m_7^{-3} \frac{\sin \zeta \sinh(2\lambda)}{\Sigma} \left[J_{\mathbb{C}\mathbb{P}^3} - \frac{1}{2} \sin(2\chi) \frac{\Xi}{X} d\chi \wedge d\phi \right] \wedge \vartheta_{S^7}. \quad (3.42)$$

Note that the $A_{(3)}^{\text{tr}}$ has only components along the internal manifold, \mathcal{M}_7 , so that its field strength, $F_{(4)}^{\text{tr}}$, can have at most one leg along the four-dimensional space-time.

3.5 The space-time flux

We now turn to the second part of the flux, $F_{(4)}^{\text{st}}$, which, as shown recently in [38, 39], can be determined from the uplift for the transverse dual potential, $A_{(6)}^{\text{tr}}$.

The starting point is the Maxwell equation (4.22) in eleven dimensions, which by setting $F_{(4)} = dA_{(3)}$ can be written locally as¹²

$$d(\star F_{(4)} + A_{(3)} \wedge F_{(4)}) = 0, \quad (3.43)$$

from which the dual potential, $A_{(6)}$, is defined by

$$dA_{(6)} = \star F_{(4)} + A_{(3)} \wedge F_{(4)}. \quad (3.44)$$

The space time flux, $F_{(4)}^{\text{st}}$, is determined by the transverse part of $A_{(6)}$, that is

$$F_{(4)}^{\text{st}} = -\star (dA_{(6)}^{\text{tr}} - A_{(3)}^{\text{tr}} \wedge F_{(4)}^{\text{tr}}), \quad (3.45)$$

where

$$A_{(6)}^{\text{tr}} = \frac{1}{16} T_{(6)} - 3m_7 \overset{\circ}{\zeta}_{(6)}. \quad (3.46)$$

The six-form, $T_{(6)} = \frac{1}{6!} T_{m_1 \dots m_6} dy^{m_1} \wedge \dots \wedge dy^{m_6}$, is given by the uplift Ansatz

$$T_{m_1 \dots m_6} = \Delta g_{p[m_1} K_{m_2 \dots m_6]}^{IJ} K^{pKL} (u_{MN}^{IJ} + v_{MN}^{IJ}) (u^{MN}{}_{KL} + v^{MN}{}_{KL}), \quad (3.47)$$

where $K_{m_1 \dots m_5}^{IJ} \equiv i \bar{\eta}^I \Gamma_{m_1 \dots m_5} \eta^J$, while $\overset{\circ}{\zeta}_{(6)}$ is the potential for the volume of the round S^7 ,

$$d\overset{\circ}{\zeta}_{(6)} \equiv \text{vol}_{S^7} = \frac{1}{8} m_7^{-7} \sin \chi \cos^5 \chi \sin^3 \theta \cos \theta d\chi \wedge d\theta \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \wedge d\psi \wedge d\phi. \quad (3.48)$$

Evaluating (3.47), we find

$$T_{(6)} = \frac{8}{3 + \xi - 6 \coth(2\lambda) \sec \zeta} \overset{\circ}{\star}_7 \xi_{(1)}, \quad (3.49)$$

where $\overset{\circ}{\star}_7$ is the dual on S^7 with respect to the round metric. In terms of the local coordinates,

$$T_{(6)} = 2m_7^{-6} \frac{\sin^2 \chi \cos^6 \chi \sin^3 \theta \cos \theta}{\cos(2\chi) - \coth(2\lambda) \sec \zeta} d\theta \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \wedge d\psi \wedge d\phi. \quad (3.50)$$

Then

$$\begin{aligned} dT_{(6)} = & -\frac{4m_7^{-1} \text{csch}^2(2\lambda) \sec^2 \zeta \sin(2\chi)}{(\coth(2\lambda) \sec \zeta - \cos(2\chi))^2} [4 \cos \zeta d\lambda - \sin \zeta \sinh(4\lambda) d\zeta] \wedge \iota_{\partial_\chi} \text{vol}_{S^7} \\ & + 8m_7 \frac{4 \coth(2\lambda) \sec \zeta (1 - 2 \cos(2\chi)) - 4 \cos(2\chi) + 3 \cos(4\chi) + 5}{(\coth(2\lambda) \sec \zeta - \cos(2\chi))^2} \text{vol}_{S^7}. \end{aligned} \quad (3.51)$$

From (3.42), we have

$$A_{(3)}^{\text{tr}} \wedge F_{(4)}^{\text{tr}} = -m_7 \frac{\sin^2 \zeta [(1 - 2 \cos(2\chi)) \cos \zeta + 3 \coth(2\lambda)]}{(\cos \zeta + \coth(2\lambda))(\coth(2\lambda) - \cos \zeta \cos(2\chi))^2} \text{vol}_{S^7}. \quad (3.52)$$

¹²See appendix A for definitions and properties of the various duals used in this section.

Substituting (3.51), (3.48) and (3.52) in (3.45), we get

$$F_{(4)}^{\text{st}} = \star \left[\frac{m_7^{-1} \sin(2\chi)}{(\cos \zeta \sinh(2\lambda) \cos(2\chi) - \cosh(2\lambda))^2} \left(\cos \zeta d\lambda - \frac{1}{4} \sin \zeta \sinh(4\lambda) d\zeta \right) \wedge \iota_{\partial_\chi} \text{vol}_{S^7} \right. \\ \left. - \frac{m_7}{X \Sigma^2} (\cos \zeta \sinh(2\lambda)(2 \cos(2\chi) - 1) - 3 \cosh(2\lambda)) \text{vol}_{S^7} \right], \quad (3.53)$$

where the dual is with respect to the full metric (3.9). Using identities (A.7) and (A.8) in appendix A and

$$\star \text{vol}_{S^7} = \Delta^{-1} \star \text{vol}_{M_7} = -\Delta^{-3} \text{vol}_{1,3}, \quad (3.54)$$

we find that the space time flux (3.53) is

$$F_{(4)}^{\text{st}} = -m_7^{-1} \sin(2\chi) \left(\cos \zeta \overset{\circ}{\star}_{1,3} d\lambda - \frac{1}{4} \sin \zeta \sinh(4\lambda) \overset{\circ}{\star}_{1,3} d\zeta \right) \wedge d\chi \\ + \frac{m_7 \Delta^{-3}}{X \Sigma^2} (\cos \zeta \sinh(2\lambda)(2 \cos(2\chi) - 1) - 3 \cosh(2\lambda)) \text{vol}_{1,3}. \quad (3.55)$$

For the flow solutions where the scalar fields depend only on the radial coordinate, r , we have $d\lambda = \lambda' dr$, $d\zeta = \zeta' dr$ and (3.55) evaluates to a very simple expression,

$$F_{(4)}^{\text{st}} = \frac{m_7}{3} e^{3A} \text{vol}_{1,2} \wedge (U dr + V d\chi), \quad (3.56)$$

where

$$U(r, \chi) = -3(1 - 2 \cos 2\chi) \sinh 2\lambda \cos \zeta - 9 \cosh 2\lambda, \\ V(r, \chi) = \frac{3}{4m_7^2} \sin 2\chi (4 \cos \zeta \lambda' - \sinh(4\lambda) \sin \zeta \zeta'), \quad (3.57)$$

and $\text{vol}_{1,2}$ is the volume along the $Min_{1,2}$ or AdS_3 slices.

It is straightforward to verify that $F_{(4)}^{\text{st}}$ given in (3.56) satisfies the Bianchi identity, $dF_{(4)}^{\text{st}} = 0$, when the four-dimensional fields, $A(r)$, $\lambda(r)$ and $\zeta(r)$, are on-shell, that is they satisfy the equations of motion (2.17) in four dimensions.

The calculation above illustrates the point we have raised before, namely, that a rather long and complicated derivation using uplift formulae yields a relatively simple final result. In fact, after we have completed this calculation a paper [41] appeared where a more direct Ansatz for the Freund-Rubin flux, namely the term in $F_{(4)}^{\text{st}}$ proportional to the volume of the four-dimensional space-time, is proposed. In the present context, the key observation is that the second term in U in (3.57) is the scalar potential (2.13) of the four-dimensional theory, while the first term is proportional to a derivative of the potential. This can be generalized to a more efficient uplift formula, which is summarized in appendix C.

3.6 A summary of the uplift

We conclude this section with a brief summary of the eleven-dimensional fields constructed in sections 3.2–3.5. While the formulae for the uplifted fields are valid for any field configuration in four dimensions, here we will specialize them to the four-dimensional flows we

are interested in. It turns out that the simplest form of the flux is obtained when we use suitable frames for the metric (3.9). We will also need those frames later in the proof of supersymmetry of the RG flows and Janus solutions in section 5.

Given (3.9) and (3.33), a natural choice for the frames, e^M , $M = 1, \dots, 11$, is to set

$$\begin{aligned}
 e^{1,2,3} &= X^{\frac{1}{6}} \Sigma^{\frac{1}{3}} e^A f^{1,2,3}, & e^4 &= X^{\frac{1}{6}} \Sigma^{\frac{1}{3}} dr, \\
 e^5 &= m_7 X^{-\frac{1}{3}} \Sigma^{\frac{1}{3}} d\chi, \\
 e^6 &= m_7 X^{\frac{1}{6}} \Sigma^{-\frac{1}{6}} \cos \chi d\theta, \\
 e^{7,8} &= \frac{m_7}{2} X^{\frac{1}{6}} \Sigma^{-\frac{1}{6}} \cos \chi \sin \theta \sigma_{1,2}, \\
 e^9 &= \frac{m_7}{2} X^{\frac{1}{6}} \Sigma^{-\frac{1}{6}} \cos \chi \sin \theta \cos \theta \sigma_3, \\
 e^{10} &= m_7 X^{\frac{2}{3}} \Sigma^{-\frac{2}{3}} \sin \chi \cos \chi \left(\left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 \right) + \frac{\Xi}{X} d\phi \right), \\
 e^{11} &= m_7 X^{-\frac{1}{3}} \Sigma^{-\frac{2}{3}} \left(d\phi + \cos^2 \chi \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 \right) \right),
 \end{aligned} \tag{3.58}$$

where f^i , $i = 1, 2, 3$, are the frames for the $Min_{1,2}$ or AdS_3 slices,

$$\begin{aligned}
 X(r) &= \cosh(2\lambda) + \cos \zeta \sinh(2\lambda), & \Xi(r) &= 2 \cos \zeta \sinh(2\lambda), \\
 \Sigma(r, \chi) &= \cosh(2\lambda) - \cos \zeta \sinh(2\lambda) \cos(2\chi).
 \end{aligned} \tag{3.59}$$

Then the transverse potential, $A_{(3)}^{\text{tr}}$, given in (3.42) becomes surprisingly simple,

$$A_{(3)}^{\text{tr}} = \frac{1}{2} p(r) (e^6 \wedge e^9 + e^7 \wedge e^8 - e^5 \wedge e^{10}) \wedge e^{11}, \tag{3.60}$$

where

$$p(r) = \sinh(2\lambda) \sin \zeta. \tag{3.61}$$

Note that the coefficient function, $p(r)$, depends only on the four-dimensional space time radial coordinate. All dependence in (3.60) on the internal geometry and coordinates enters only through the frames.

Finally, the space time flux is given in (3.56) and (3.57). This completes the construction of the uplift.

4 The equations of motion

In this section we verify explicitly that the metric and the four-form flux in the uplift satisfy the equations of motion of eleven-dimensional supergravity when the four-dimensional metric and the scalar fields satisfy the four-dimensional equations of motion (2.17)–(2.18).

4.1 Preliminaries

We start with some technical preliminaries that will help us simplify the algebra in the calculations that follow. The main idea is to work directly with the functions that appear

in the metric (3.33) and the flux (3.60), in particular, with $X(r)$ and $p(r)$ given in (3.59) and (3.61), respectively, rather than with the scalar fields, $\lambda(r)$ and $\zeta(r)$. To this end we use

$$\sin \zeta = -p \operatorname{csch}(2\lambda), \quad \cos \zeta = -\operatorname{csch}(2\lambda)(\cosh(2\lambda) - X), \quad (4.1)$$

and

$$\cosh(2\lambda) = \frac{1 + p^2 + X^2}{2X}, \quad (4.2)$$

to eliminate ζ and λ in terms of p and X . This converts complicated trigonometric expressions into rational functions of the new fields p and X that are typically easier to evaluate and simplify. In particular, the four-dimensional equations of motion (2.17)–(2.18) in the rationalized form are given by

$$\begin{aligned} \Delta'' &= -3A'\Delta' + \frac{1}{X} \left[(p^2 + 1)(X')^2 + X^2(p')^2 - 2pXp'X' \right] - 2g^2(p^2 + X^2 - 1) \\ p'' &= -3p'A' + \frac{p}{X^2} \left[(p^2 + 1)(X')^2 + X^2(p')^2 - 2pXp'X' \right] - 2g^2 \frac{p}{X}(p^2 + X^2 + 1), \\ A'' &= -\frac{3}{2}(A')^2 - \frac{3}{8X^2} \left[(p^2 + 1)(X')^2 + X^2(p')^2 - 2pXp'X' \right] \\ &\quad + \frac{3g^2}{2X} (1 + (p')^2 + (X')^2) - \frac{e^{-2A}}{2\ell^2}, \end{aligned} \quad (4.3)$$

and

$$(A')^2 - \frac{1}{4X^2} \left[(p^2 + 1)(X')^2 + X^2(p')^2 - 2pXp'X' \right] - \frac{g^2}{X}(p^2 + X^2 + 1) + \frac{e^{-2A}}{\ell^2} = 0. \quad (4.4)$$

Similarly, we find that the superpotential (2.12) is given by

$$W^2 = \frac{1}{8X} \left[9p^4 - 6p^2(X^2 - 3) + (X^2 + 3)^2 \right] \quad (4.5)$$

and the BPS equations (2.19) and (2.20) for the scalars become

$$\begin{aligned} X' &= -\frac{1}{4W^2} \left[9p^4 - 6p^2(X^2 - 1) + X^4 + 2X^2 - 3 \right] A' + \frac{e^{-A}}{\ell} \frac{p(3p^2 - X^2 + 3)}{W^2}, \\ p' &= -\frac{p}{4XW^2} \left[9p^4 - 6p^2(X^2 - 3) + X^4 - 2X^2 + 9 \right] A' \\ &\quad + \frac{e^{-A}}{\ell} \frac{3p^4 + 2p^2(X^2 + 3) - X^4 - 2X^2 + 3}{4XW^2}. \end{aligned} \quad (4.6)$$

As a consistency check one can verify once more that the first order equations (4.6) and (2.21) with W given in (4.5) imply the second order equations (4.3) and that (4.4) is equivalent to (2.21).

Finally, the other metric and the flux functions are:

$$\Sigma = \frac{1}{X} \left[\cos^2 \chi (1 + p^2) + \sin^2 \chi X \right], \quad \Xi = -\frac{1}{X} (p^2 - X^2 + 1), \quad (4.7)$$

and

$$\begin{aligned} U &= -\frac{6}{X} \cos^2 \chi (1 + p^2) + 3X (\cos(2\chi) - 2), \\ V &= \frac{3}{4m_7^2} \sin 2\chi \left[2(1 + p^2) \frac{X'}{X} - 2pp' \right]. \end{aligned} \tag{4.8}$$

This shows that indeed both the metric and the flux can almost entirely be written down using, up to overall factors, only rational functions of X and p , and their derivatives!

Finally, we will be often able to eliminate trigonometric functions of χ using

$$\cos(2\chi) = -\frac{p^2 + X^2 - 2\Sigma X + 1}{p^2 - X^2 + 1}, \tag{4.9}$$

which follows from (4.7).

4.2 The flux

The first place where using the rationalized parametrization becomes clearly advantageous is the calculation of the components, F_{MNPQ} , of the four-form flux, $F_{(4)}$. Indeed, for the space-time part of the flux given in (3.56) we simply have

$$\begin{aligned} F_{1234} &= \frac{m_7^{-1}}{\Sigma^{4/3} X^{2/3}} (2\Sigma + X), \\ F_{1235} &= \frac{\tan \chi}{X^{7/6} \Sigma^{4/3} \Xi} (\Sigma - X) \left[(p^2 + 1) X' - pXp' \right]. \end{aligned} \tag{4.10}$$

Turning to the transverse flux, $F_{(4)}^{\text{tr}} = dA_{(3)}^{\text{tr}}$, we note that the part of the three-form potential along $\mathbb{C}\mathbb{P}^2$ in (3.42) has the complex structure, $J_{\mathbb{C}\mathbb{P}^2}$, as a factor. Thus the corresponding components of the field strength must satisfy

$$F_{MN69} = F_{MN78}. \tag{4.11}$$

Modulo this identity, the non-vanishing components of the transverse part of the flux are:

$$\begin{aligned} F_{451011} &= \frac{pX' - Xp'}{2X^{7/6} \Sigma^{1/3}}, & F_{46911} &= \frac{\Sigma p' - p\Sigma'}{2X^{1/6} \Sigma^{4/3}}, \\ F_{56910} &= -\frac{m_7 p (\Sigma + X)}{X^{2/3} \Sigma^{4/3}}, & F_{6789} &= \frac{2m_7 p}{X^{2/3} \Sigma^{1/3}}, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} \Sigma' \equiv \frac{\partial \Sigma}{\partial r} &= \frac{1}{X^3 - (p^2 + 1)X} \left[2X (pXp' - (p^2 + 1)X') \right. \\ &\quad \left. + \Sigma (p^2 X' - 2pXp' + (X^2 + 1) X') \right]. \end{aligned} \tag{4.13}$$

Later we will also need

$$\frac{\partial \Sigma}{\partial \chi} = \sin(2\chi) \Xi. \tag{4.14}$$

It appears that the flux produced through the uplift formulae is rather special, in particular, we find that the following components

$$F_{4569} = F_{46910} = F_{56911} = 0, \tag{4.15}$$

accidentally vanish, that is not due to the underlying $SU(3) \times U(1) \times U(1)$ symmetry of the construction.

4.3 The Einstein equations

The Einstein equations of eleven-dimensional supergravity in our conventions¹³ are:

$$R_{MN} + g_{MN}R = \frac{1}{3}F_{MPQR}F_N{}^{PQR}. \quad (4.16)$$

We start by evaluating the components of the Ricci tensor, R_{MN} , in the basis of frames (3.58). The symmetries of the metric and the dependence of the scalar fields and the metric function in four dimensions on the radial coordinate only, imply that the non-vanishing components of the Ricci tensor can be at most the following ones:

$$\begin{aligned} R_{11} = -R_{22} = -R_{33}, \quad R_{44}, \quad R_{45} = R_{54}, \quad R_{55}, \\ R_{66} = R_{77} = R_{88} = R_{99}, \\ R_{1010}, \quad R_{1011} = R_{1110}, \quad R_{1111}. \end{aligned} \quad (4.17)$$

This agrees with the explicit result. Indeed, we find that after imposing the four-dimensional equations of motion (4.3)–(4.4) in the rational parametrization introduced above, the diagonal components of the Ricci tensor can be written in the form

$$R_{MM} = \mathcal{A}_M (X')^2 + \mathcal{B}_M (p')^2 + \mathcal{C}_M p' X' + \mathcal{D}_M, \quad (4.18)$$

where \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} are functions of p , X and χ (or, equivalently, Σ). In particular, we find that the cross-terms $A'X'$ and $A'p'$ are absent. Similarly, the off-diagonal components are of the form

$$R_{MN} = \mathcal{A}_{MN}X' + \mathcal{B}_{MN}p' + \mathcal{D}_{MN}, \quad M \neq N. \quad (4.19)$$

Explicit formulae for all non-vanishing coefficient functions are given in appendix D.

Evaluating the energy-momentum tensor on the right hand side in (4.16) is straightforward. We will forego the details and just look at one specific equation, the off-diagonal Einstein equation (4.16) with $M = 10$ and $N = 11$. On the one side, we have

$$R_{1011} = -2(g^2 - 2m_7^2) \tan \chi \frac{(\Sigma - X)}{X^{1/3}\Sigma^{5/3}}. \quad (4.20)$$

However, as one can see by inspection of the non-vanishing flux components, the other side must be zero. This verifies the relation between the four-dimensional coupling constant, g , and the inverse radius of the internal manifold, m_7 , [34]

$$g = \sqrt{2} m_7. \quad (4.21)$$

Given this relation, it is easy to check that all the remaining Einstein equations are satisfied as expected.

4.4 The Maxwell equations

The Maxwell equations are

$$d \star F_{(4)} + F_{(4)} \wedge F_{(4)} = 0. \quad (4.22)$$

¹³See appendix A.

For the flux (4.10)–(4.12), they yield seven independent equations: four first order and three second order.

The first order equations are along the components [1234569 11], [1234578 11], [12356789], and [456789 10 11], and all have the same structure as this last one

$$\frac{4m_7 p \tan \chi}{g^2 X^{17/6} \Sigma^{5/3} \Xi^2} (2m_7^2 - g^2) (p^2 - X^2 + 1) (\Sigma - X) (p^2 X' - p X p' + X') = 0, \quad (4.23)$$

namely, they come with an overall factor of $(g^2 - 2m_7^2)$.

The second order equations come from the components [1234569 10], [1234578 10] and [12346789] in (4.22). The first two equations are somewhat involved, but the last one is quite simple. There we find

$$\begin{aligned} 0 &= \frac{1}{2X^{4/3}\Sigma^{2/3}} \left[pX'' - Xp'' + 3A' (pX' - Xp') - 24m_7^2 \right] \\ &= \frac{2p(g^2 - 2m_7^2)}{X^{4/3}\Sigma^{2/3}}, \end{aligned} \quad (4.24)$$

where in going from the first to the second line we have used the four-dimensional equations of motion (4.3). Similarly, upon using (4.3), the other two equations reduce to the same expression modulo an overall factor of $X\Sigma$. Thus the Maxwell equations are satisfied if (4.21) holds.

To summarize, we have shown explicitly that the metric, g_{MN} , and the four-form flux, $F_{(4)}$, constructed using the uplift formulae in section 3 indeed satisfy the equations of motion of the eleven-dimensional supergravity when the scalar fields, $\lambda(r)$ and $\zeta(r)$, and the metric function, $A(r)$, are on-shell in four dimensions. It is important to note that to verify that we have used only the equations of motion in four dimensions (2.17)–(2.18) or, equivalently, (4.3)–(4.4), but not the BPS equations! This means that also non-supersymmetric solutions of the same type will uplift to solutions of M theory.

5 Supersymmetry

We now turn to the Janus and RG-flow solutions of the BPS equations (2.19)–(2.21) and (2.22)–(2.23), respectively, to demonstrate explicitly the $\mathcal{N} = (2, 0)$ and $\mathcal{N} = 2$ supersymmetry of the corresponding uplifts in M theory. This has been discussed already in some detail in [22], where we have argued that the $\mathcal{N} = 2$ supersymmetry of the RG flows is achieved by brane polarization and is naturally defined through projectors that reflect the underlying almost-complex structure and a dielectric projector much like those encountered in [19, 59–61]. The defect in the Janus solutions leads to additional chiral projector that is also present in four dimensions. The result for the RG-flows is then recovered by keeping both chiralities and taking the $\ell \rightarrow \infty$ limit.

5.1 Projector Ansätze

The BPS equations in eleven dimensions are obtained by setting the supersymmetry variations of the gravitinos to zero,

$$\delta\psi_M \equiv \partial_M \epsilon + \mathcal{M}_M \epsilon = 0, \quad (5.1)$$

where the algebraic operators, \mathcal{M}_M , are given by

$$\mathcal{M}_M \equiv \frac{1}{4} \omega_{MPQ} \Gamma^{PQ} + \frac{1}{144} \left(\Gamma_M^{NPQR} - 8\delta_M^N \Gamma^{PQR} \right) F_{MNPQ}. \quad (5.2)$$

The Killing spinors of unbroken supersymmetries are invariant under the Poincaré transformations in the tx -plane and are singlets of $SU(3)$ acting along \mathbb{CP}_2 . Hence ϵ does not depend on the coordinates t , x and θ , as well as the Euler angles, $\alpha_1, \dots, \alpha_3$. This means that the corresponding equations (5.1) are purely algebraic:¹⁴

$$\mathcal{M}_t \epsilon = \mathcal{M}_x \epsilon = 0, \quad \mathcal{M}_\theta \epsilon = \mathcal{M}_{\sigma_1} = \dots = \mathcal{M}_{\sigma_3} = 0. \quad (5.3)$$

Similarly, the dependence of ϵ on the $U(1) \times U(1)$ angles, ϕ and ψ ,

$$\frac{\partial \epsilon}{\partial \phi} = -\mathcal{M}_\phi \epsilon, \quad \frac{\partial \epsilon}{\partial \psi} = -\mathcal{M}_\psi \epsilon, \quad (5.4)$$

is determined by the charges, $q_\phi = 1$ and $q_\psi = 3/2$, respectively.

Let us now consider the first equation in (5.3), written in the form

$$\mathfrak{M} \epsilon = 0, \quad \mathfrak{M} \equiv \Gamma^1 \mathcal{M}_1. \quad (5.5)$$

The matrix \mathfrak{M} , expanded into the basis of Γ -matrices, is given by

$$\begin{aligned} \mathfrak{M} = & \frac{e^{-A}}{2\ell} \frac{1}{X^{1/6} \Sigma^{1/3}} \Gamma^3 + \frac{1}{12 X^{7/6} \Sigma^{4/3}} \left[2X \frac{\partial \Sigma}{\partial r} + \Sigma (X' + 6XA') \right] \Gamma^4 + \frac{m_7 X^{1/3}}{6 \Sigma^{4/3}} \frac{\partial \Sigma}{\partial \chi} \Gamma^5 \\ & + \frac{1}{3} (F_{1234} \Gamma^4 + F_{1235} \Gamma^5) \Gamma^{123} + \frac{1}{6} F_{451011} \Gamma^{451011} + \frac{1}{6} F_{6789} \Gamma^{6789} \\ & + \frac{1}{6} (F_{46911} \Gamma^{411} + F_{56910} \Gamma^{510}) (\Gamma^{69} + \Gamma^{78}). \end{aligned} \quad (5.6)$$

Together with the explicit formulae (3.56)–(4.12) for the flux, this gives us a homogenous system of linear equation for the thirty two components of ϵ .

It is clear that after substituting the expressions for the flux components (3.56)–(4.12) and expanding the derivatives of Σ , see (4.13) and (4.14), the operator \mathfrak{M} , as well as the other operators, \mathcal{M}_M , become quite complicated. Hence, before we proceed with the analytic calculation, we first explore numerically the space of solutions to (5.5). To do that, we first eliminate the derivatives X' , p' and A' using the BPS equations (4.6) and (2.21), and set $g = \sqrt{2} m_7$. Next we assign random values to the fields X , p , A , the angle χ , and the constants m_7 and ℓ upon which (5.5) becomes a purely numerical system that can be solved for the components of the Killing spinor, ϵ . Note that our numerical assignment amounts simply to choosing random initial conditions for the four-dimensional BPS equations and thus is not constrained in any way. Those numerical solutions yield us some information about the subspace of allowed Killing spinors, which confirms what one could also infer from an analysis in four dimensions and the $SU(3) \times U(1) \times U(1)$ symmetry.

¹⁴We will use the convention that the indices $M = 1, \dots, 11$ label components with respect to the frames (3.58), while $M = t, x, \dots, \psi$, or $M = \sigma_1, \dots, \sigma_3$, with respect to the local coordinates and/or forms.

More importantly, it allows us to short cut quite a bit of tedious analysis by fixing some of the signs in the projectors below that we would have to keep track of otherwise.

For finite ℓ , the space of numerical solutions is generically two-dimensional in agreement with $\mathcal{N}=(2,0)$ supersymmetry in four dimensions. The unbroken supersymmetries, ϵ , must thus satisfy four conditions

$$\Pi_0 \epsilon = \Pi_1 \epsilon = \Pi_2 \epsilon = \Pi_3 \epsilon = 0, \tag{5.7}$$

where Π_0, \dots, Π_3 are mutually commuting projectors. From the numerical analysis we also find that two of these projectors are constant. To conform with the conventions in [22], we will denote them by Π_1 and Π_3 . The first projector,

$$\Pi_1 \equiv \frac{1}{2} (1 + \Gamma^{6789}), \tag{5.8}$$

arises from the fact that the Killing spinor, ϵ , must be a singlet under the holonomy group, $SU(3)$, of $\mathbb{C}P_2$. It depends on the choice of orientation of $\mathbb{C}P_2$ defined by the frames e^6, \dots, e^9 . The second projector,

$$\Pi_3 \equiv \frac{1}{2} (1 - \Gamma^{12}), \tag{5.9}$$

is just an uplift of the corresponding chirality projector in four dimensions. In particular, choosing $\kappa = -1$ in (2.19) and (2.20) changes the sign in (5.9). Finally, we find that on the subspace of the Killing spinors satisfying (5.7),

$$\frac{\partial \epsilon}{\partial \phi} = -\Gamma^{69} \epsilon, \quad \frac{\partial \epsilon}{\partial \psi} = -\frac{3}{2} \Gamma^{69} \epsilon. \tag{5.10}$$

Together with (5.4), this gives us two additional algebraic equations, which as we will see simplifies the calculations significantly. We should also note that both projectors do not depend on the choice of the square root branch in (2.21) used to eliminate A' .

For the RG flows, taking the limit $\ell \rightarrow \infty$ eliminates the first term in (5.6). The space of solutions includes then both Γ^{12} -chiralities; the projector Π_3 is thus absent and we have a four-dimensional space of solutions corresponding to $\mathcal{N} = 2$ supersymmetry. We have shown in [22] that the remaining two commuting projectors in this limit are

$$\Pi_2^\infty = \frac{1}{2} \left[1 + (\cos \alpha \Gamma^5 - \sin \alpha \Gamma^4) \Gamma^{69} (\cos \omega \Gamma^{10} + \sin \omega \Gamma^{11}) \right], \tag{5.11}$$

and

$$\Pi_0^\infty = \frac{1}{2} \left[1 + \cos \beta \Gamma^{123} + \sin \beta (\cos \alpha \Gamma^4 + \sin \alpha \Gamma^5) \right], \tag{5.12}$$

where the angles α , β and ω are some functions of r and χ .¹⁵ In analogy with (5.8), the projector (5.11) can be associated with an extension of the complex structure of $\mathbb{C}P_2$ to an almost complex structure with extra pair of complex frames. Finally, (5.12) is the dielectric deformation of the standard M2-brane projector at $\beta = 0$.

¹⁵See, (4.8) and (4.5) in [22].

For the Janus solutions, the projectors (5.11) and (5.12) must be deformed to account for the defect, which gives rise to additional terms in the supersymmetry variations, such as the first term in (5.6). Including such terms in (5.11) and (5.12) leads to the following Ansatz for the projectors at finite ℓ :

$$\Pi_0 = \frac{1}{2} \left[1 + a_1 \Gamma^3 + a_2 \Gamma^4 + a_3 \Gamma^5 \right], \quad (5.13)$$

and

$$\Pi_2 = \frac{1}{2} \left[1 + (b_1 \Gamma^3 + b_2 \Gamma^4 + b_3 \Gamma^5) \Gamma^{69} (\cos \omega \Gamma^{10} + \sin \omega \Gamma^{11}) \right]. \quad (5.14)$$

Those two operators form a pair of commuting projectors provided the vectors $\mathbf{a} \equiv (a_1, a_2, a_3)$ and $\mathbf{b} \equiv (b_1, b_2, b_3)$ are orthonormal. Such a pair of vectors can be parametrized by three angles, α , β and γ :

$$\begin{aligned} a_1 &= \cos \beta \cos \gamma - \sin \alpha \sin \beta \sin \gamma, & a_2 &= \cos \alpha \sin \beta, \\ a_3 &= \sin \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma, \end{aligned} \quad (5.15)$$

and

$$b_1 = -\cos \alpha \sin \gamma, \quad b_2 = -\sin \alpha, \quad b_3 = \cos \alpha \cos \gamma. \quad (5.16)$$

Together with ω those angles are some functions of r and χ and will be determined by solving the supersymmetry variations. For $\gamma = 0$, the projectors Π_0 and Π_2 reduce to Π_0^∞ and Π_2^∞ , respectively. We can thus view the angle γ as the Janus deformation parameter which goes to zero in the RG-flow limit.

There is still certain redundancy in our description of the projectors (5.13) and (5.14). To see this, introduce a third vector, \mathbf{c} , so that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ are orthonormal and define $\mathbf{x} \cdot \Gamma \equiv (x_1 \Gamma^3 + x_2 \Gamma^4 + x_3 \Gamma^5)$. Observe that the product $(\mathbf{b} \cdot \Gamma)(\mathbf{c} \cdot \Gamma) \Gamma^{10} \Gamma^{11}$ commutes with all the projectors, Π_0, \dots, Π_3 and so preserves the space of supersymmetries. One is therefore free to rotate (5.13) and (5.14) using the action of $(\mathbf{b} \cdot \Gamma)(\mathbf{c} \cdot \Gamma) \Gamma^{10} \Gamma^{11}$ and this induces a simultaneous rotation $\omega \rightarrow \omega + \vartheta$ accompanied by a rotation of \mathbf{b} and \mathbf{c} by the angle ϑ . In the following we will use this freedom to simplify our calculations.

5.2 Supersymmetries for the Janus solutions

We will now calculate all the projectors and the Killing spinor, ϵ , by solving explicitly the BPS equations (5.1).

In principle, one should be able to determine all the projectors in (5.7), or equivalently solve for the angles α, β, γ and ω , directly from (5.5). The problem is that this effectively amounts to obtaining the individual projectors Π_0, \dots, Π_3 from a particular linear combination of products of these projectors. This, unsurprisingly, is not the best way to proceed. Instead, we will first solve algebraic equations that arise from judicious linear combinations of the variations (5.1) in which the flux terms either cancel completely or are simple.

The first such equation arises from the “magical combination” of variations

$$2 \Gamma^1 \delta \psi_1 + \Gamma^6 \delta \psi_6 + \Gamma^7 \delta \psi_7 + \Gamma^{10} \delta \psi_{10} + \Gamma^{11} \delta \psi_{11} = 0, \quad (5.17)$$

in which all flux terms cancel. After eliminating the derivatives with respect to the U(1) angles using (5.10) and modulo terms annihilated by Π_1 and Π_3 , it reads

$$[\mathfrak{A}_1 \Gamma^3 + \mathfrak{A}_2 \Gamma^4 + \mathfrak{A}_3 \Gamma^5 + \Gamma^{69} (\mathfrak{A}_4 \Gamma^{10} + \mathfrak{A}_5 \Gamma^{11})] \epsilon = 0, \quad (5.18)$$

where

$$\begin{aligned} \mathfrak{A}_1 &= \frac{1}{\Sigma^{1/3} X^{1/6}} \frac{e^{-A}}{\ell}, & \mathfrak{A}_2 &= \frac{A'}{\Sigma^{1/3} X^{1/6}}, & \mathfrak{A}_3 &= \frac{m_7 X^{1/3} (2 \cos(2\chi) - 1)}{\sin(2\chi) \Sigma^{1/3}}, \\ \mathfrak{A}_4 &= -\frac{m_7 X^{1/3} (\cos(2\chi) - 2)}{\sin(2\chi) \Sigma^{1/3}}, & \mathfrak{A}_5 &= \frac{m_7 X^{1/3} (X (\cos(2\chi) - 2) + 3 \Sigma)}{2 \cos^2 \chi \Sigma^{1/3}}. \end{aligned} \quad (5.19)$$

Iterating (5.18) one finds a single consistency condition

$$\mathfrak{A}_1^2 + \mathfrak{A}_2^2 + \mathfrak{A}_3^2 - \mathfrak{A}_4^2 - \mathfrak{A}_5^2 = 0, \quad (5.20)$$

which is satisfied by virtue of (2.21) and (4.21). This condition also means that, up to an invertible factor, (5.18) is in fact the projector (5.14) with

$$b_1 = \frac{\mathfrak{A}_1}{\mathfrak{A}}, \quad b_2 = \frac{\mathfrak{A}_2}{\mathfrak{A}}, \quad b_3 = \frac{\mathfrak{A}_3}{\mathfrak{A}}, \quad \cos \omega = \frac{\mathfrak{A}_4}{\mathfrak{A}}, \quad \sin \omega = \frac{\mathfrak{A}_5}{\mathfrak{A}}, \quad (5.21)$$

where

$$\mathfrak{A} \equiv (\mathfrak{A}_1^2 + \mathfrak{A}_2^2 + \mathfrak{A}_3^2)^{1/2} = (\mathfrak{A}_4^2 + \mathfrak{A}_5^2)^{1/2}. \quad (5.22)$$

Using (5.16) and (5.21), we then read off

$$\begin{aligned} \cos \alpha \cos \gamma &= -\frac{(2 \cos(2\chi) - 1) X^{1/2}}{\Omega^{1/2}}, & \cos \alpha \sin \gamma &= -\frac{a \sin(2\chi) e^{-A}}{\Omega^{1/2} \ell}, \\ \sin \alpha &= -\frac{a \sin(2\chi) A'}{\Omega^{1/2}}, \end{aligned} \quad (5.23)$$

and

$$\cos \omega = \frac{(\cos(2\chi) - 2) X^{1/2}}{\Omega^{1/2}}, \quad \sin \omega = \frac{\sin(2\chi) (3 p^2 - X^2 + 3)}{2 X^{1/2} \Omega^{1/2}}, \quad (5.24)$$

where

$$\Omega = (1 - 2 \cos(2\chi))^2 X + 2 \sin^2(2\chi) W^2. \quad (5.25)$$

Before proceeding we note that the rotation of the gamma matrices that define the projectors is equivalent to a rotation of the frames. In particular, the rotation by ω is equivalent to starting with the frames:

$$\hat{e}^{10} \equiv \cos \omega e^{10} + \sin \omega e^{11}, \quad \hat{e}^{11} \equiv -\sin \omega e^{10} + \cos \omega e^{11}. \quad (5.26)$$

Using (5.24) we find a rather simple result for one of these frames:

$$\hat{e}^{10} = m_7 X^{1/6} \Sigma^{1/3} \Omega^{-1/2} \left(d\phi + \frac{3}{2} \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 \right) \right). \quad (5.27)$$

Note that the mixing of ϕ and ψ does not involve functions of r and furthermore (5.10) implies that the supersymmetries only depend upon angles in precisely the combination $(\phi + \frac{3}{2}\psi)$. We will return to this observation later.

Continuing with the supersymmetry analysis, since the projectors (5.13) and (5.14) commute, we cannot obtain any information from (5.18) about the dielectric polarization angle, β . For that we turn to another magical combination,

$$\Gamma^1 \delta\psi_1 + \Gamma^7 \delta\psi_7 + \Gamma^8 \delta\psi_8 = 0, \quad (5.28)$$

which has no derivatives of ϵ and no terms with components of the internal flux. After imposing the constant projections, it reads

$$[\mathfrak{B}_1 + \mathfrak{B}_2 \Gamma^3 + \mathfrak{B}_3 \Gamma^4 + \mathfrak{B}_4 \Gamma^5 + \Gamma^{69} (\mathfrak{B}_5 \Gamma^{10} + \mathfrak{B}_6 \Gamma^{11})] \epsilon = 0, \quad (5.29)$$

where

$$\begin{aligned} \mathfrak{B}_1 &= \frac{m_7 p}{\Sigma^{3/2} X^{2/3}}, & \mathfrak{B}_2 &= \frac{1}{2 \Sigma^{1/3} X^{1/6}} \left[\frac{e^{-A}}{\ell} - \frac{pX' - p'X}{2X} \right], \\ \mathfrak{B}_3 &= \frac{2XA' + X'}{4 \Sigma^{3/2} X^{7/6}}, & \mathfrak{B}_4 &= -\mathfrak{B}_5 = -\frac{m_7 X^{1/3} \tan \chi}{\Sigma^{1/3}}, & \mathfrak{B}_6 &= \frac{m_7}{\Sigma^{1/3} X^{2/3}}. \end{aligned} \quad (5.30)$$

Note that the presence of the \mathfrak{B}_1 -term in (5.29), with an analogous term absent in (5.18), prevents (5.29) from being a projector. Still, by iteration one finds a consistency condition

$$\mathfrak{B}_1^2 - \mathfrak{B}_2^2 - \mathfrak{B}_3^2 - \mathfrak{B}_4^2 + \mathfrak{B}_5^2 + \mathfrak{B}_6^2 = 0, \quad (5.31)$$

which is indeed satisfied by virtue of (4.6) and (2.21).

Using the projectors (5.13) and (5.14) in (5.29), one is left with three independent products of Γ -matrices which yield the following equations:

$$\begin{aligned} \mathfrak{B}_1(\sin \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) + (\mathfrak{B}_5 \cos \omega + \mathfrak{B}_6 \sin \omega) \cos \alpha \cos \gamma \\ + (\mathfrak{B}_5 \sin \omega - \mathfrak{B}_6 \cos \omega) (\sin \alpha \cos \beta \cos \gamma - \sin \beta \sin \gamma) - \mathfrak{B}_4 = 0, \end{aligned} \quad (5.32)$$

$$\begin{aligned} \mathfrak{B}_2(\sin \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) + \mathfrak{B}_4(\sin \alpha \sin \beta \sin \gamma - \cos \beta \cos \gamma) \\ + (\mathfrak{B}_5 \cos \omega + \mathfrak{B}_6 \sin \omega) \cos \alpha \cos \beta + (\mathfrak{B}_5 \sin \omega - \mathfrak{B}_6 \cos \omega) \sin \alpha = 0, \end{aligned} \quad (5.33)$$

$$\begin{aligned} \mathfrak{B}_3(\sin \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) - \mathfrak{B}_4 \cos \alpha \sin \beta - (\mathfrak{B}_5 \sin \omega - \mathfrak{B}_6 \cos \omega) \cos \alpha \sin \gamma \\ + (\mathfrak{B}_5 \cos \omega + \mathfrak{B}_6 \sin \omega) (\sin \alpha \cos \beta \sin \gamma + \sin \beta \cos \gamma) = 0. \end{aligned} \quad (5.34)$$

However, only one of those equations is independent, which can be seen by solving one of them for $\tan(\beta/2)$ and then verifying that the other two are satisfied. Equivalently, one can solve the first two for $\cos \beta$ and $\sin \beta$ and then check that their squares indeed add up to one. Substituting the result into the third equations yields a consistency condition

$$\begin{aligned} \mathfrak{B}_2(\cos \beta \cos \gamma - \sin \alpha \sin \beta \sin \gamma) + \mathfrak{B}_3 \cos \alpha \sin \beta \\ + \mathfrak{B}_4(\sin \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) - \mathfrak{B}_1 = 0. \end{aligned} \quad (5.35)$$

This equation has a simple geometrical interpretation, namely that the (non-unit) vector

$$\mathbf{d} \equiv \left(\frac{\mathfrak{B}_2}{\mathfrak{B}_1}, \frac{\mathfrak{B}_3}{\mathfrak{B}_1}, \frac{\mathfrak{B}_4}{\mathfrak{B}_1} \right), \quad (5.36)$$

satisfies

$$\mathbf{a} \cdot \mathbf{d} = -1, \quad (5.37)$$

which is the consistency condition between the operator in (5.29) and the projector (5.13).

Similarly as for the equations of motion in section 4, the solution for $\cos \beta$ and $\sin \beta$ above can be simplified using rationalized BPS equations. After some algebra, we find the following result:

$$\cos \beta = \cos \gamma C_0 + \sin \gamma C_1, \quad \sin \beta = \cos \gamma S_0 + \sin \gamma S_1, \quad (5.38)$$

where

$$C_0 = -\frac{m_7^{-1}}{4 X^{3/2} \Sigma} \left[(X^2 + 3) (X^2 \sin^2 \chi + \cos^2 \chi) + p^2 (X^2 (\cos(2\chi) - 2) + 6 \cos^2 \chi) + 3 p^4 \cos^2 \chi \right] \frac{A'}{W^2}, \quad (5.39)$$

$$S_0 = \frac{p \Omega^{1/2} A'}{\sqrt{2} \Sigma W^2},$$

and

$$C_1 = \frac{p \sin(2\chi)}{8 \Sigma X W^2} \left[-X^4 + (16 \cos(2\chi) - 5(\cos(4\chi) + 3)) \csc^2(2\chi) X^2 + 6 p^2 (X^2 - 3) - 9 p^4 - 9 \right],$$

$$S_1 = -\frac{\sqrt{\Omega} \csc(2\chi)}{4 \Sigma X^{3/2} W^2} \left[X^4 \sin^2(\chi) + X^2 (p^2 - 1) (\cos(2\chi) - 2) + 3 (p^2 + 1)^2 \cos^2(\chi) \right]. \quad (5.40)$$

This completes the calculation of all the angles in the projectors Π_0 and Π_2 .

To determine explicitly the Killing spinors for unbroken supersymmetries, let us introduce rotations

$$\mathcal{R}_{ij}(x) = \cos x - \sin x \Gamma^{ij}, \quad i, j > 1, \quad (5.41)$$

and define

$$\mathcal{R}(\alpha, \beta, \gamma, \omega) = \mathcal{R}_{35}(\gamma/2) \mathcal{R}_{45}(\alpha/2) \mathcal{R}_{34}(\beta/2) \mathcal{R}_{1011}(\omega/2), \quad (5.42)$$

which commute with the projectors Π_1 and Π_3 . It is straightforward to check that the projectors (5.13) and (5.14) are then simply

$$\Pi_0 = \mathcal{R}(\alpha, \beta, \gamma, \omega) \Pi_0^{(0)} \mathcal{R}(\alpha, \beta, \gamma, \omega)^{-1}, \quad \Pi_2 = \mathcal{R}(\alpha, \beta, \gamma, \omega) \Pi_2^{(0)} \mathcal{R}(\alpha, \beta, \gamma, \omega)^{-1}, \quad (5.43)$$

where

$$\Pi_3^{(0)} = \frac{1}{2}(1 + \Gamma^3), \quad \Pi_4^{(0)} = \frac{1}{2}(1 + \Gamma^{56910}). \quad (5.44)$$

Thus any solution ϵ to (5.7) can be written as

$$\epsilon = \mathcal{R}(\alpha, \beta, \gamma, \omega) \tilde{\epsilon}, \tag{5.45}$$

where $\tilde{\epsilon}$ is in the kernel of the constant projectors (5.8), (5.9) and (5.44).

From the supersymmetry variations along the radial direction, y , we find

$$\frac{\partial \tilde{\epsilon}}{\partial y} = \frac{1}{2\ell} \tilde{\epsilon}, \tag{5.46}$$

which is the correct radial dependence for the Killing spinor along AdS_3 .

This leaves two variations along r and χ , which are solved as usual by setting

$$\tilde{\epsilon} = H_0^{1/2} \varepsilon, \tag{5.47}$$

where

$$H_0 = X^{1/6} \Sigma^{1/3} e^{A(r)}, \tag{5.48}$$

is the warp factor of the “time” frame, $e^1 = H_0 dt$, and ε is a constant spinor along the internal manifold and with the standard dependence along AdS_3 , which satisfies the same constant projections as $\tilde{\epsilon}$.

5.3 The RG-flow limit

The supersymmetry analysis simplifies significantly for the holographic flow solution. For this one simply imposes the projectors (5.8), (5.11) and (5.12) but does not impose a helicity projector like (5.13). We then find, taking the upper signs in (2.22) and (2.23):

$$\cos \alpha = \frac{(2 \cos(2\chi) - 1) X^{1/2}}{\Omega^{1/2}}, \quad \sin \alpha = - \frac{a \sin(2\chi) A'}{\Omega^{1/2}}, \tag{5.49}$$

$$\cos \omega = \frac{(\cos(2\chi) - 2) X^{1/2}}{\Omega^{1/2}}, \quad \sin \omega = \frac{\sin(2\chi)(3p^2 - X^2 + 3)}{2 X^{1/2} \Omega^{1/2}}, \tag{5.50}$$

and

$$\cos \beta = - \frac{1}{2\sqrt{2} W X^{3/2} \Sigma} \left[(X^2 + 3) (X^2 \sin^2 \chi + \cos^2 \chi) + p^2 (X^2 (\cos(2\chi) - 2) + 6 \cos^2 \chi) + 3p^4 \cos^2 \chi \right], \tag{5.51}$$

$$\sin \beta = \frac{p \Omega^{1/2}}{\sqrt{2} W \Sigma}.$$

The space-time components of the Maxwell fields also simplify and we obtain a seemingly standard relation for holographic flows:

$$h_0 = - \frac{1}{2} \cos \beta. \tag{5.52}$$

6 IR asymptotics in eleven dimensions

Having constructed the uplift in detail, we now examine the infra-red limits of the holographic RG flows described by (2.22) and (2.23) from the perspective of M theory. In an earlier paper [22] we focussed upon the special flow with $\zeta = \pi/3$ since this led to a very interesting new result. Here we will complete the asymptotic analysis for all flows.

First recall that ζ limits to a constant value as $\lambda \rightarrow +\infty$ and so the various warp factors behave as follows:

$$X \sim \frac{1}{2}(1+\cos\zeta)e^{2\lambda}, \quad \Xi \sim \cos\zeta e^{2\lambda}, \quad \Sigma \sim \frac{1}{2}e^{2\lambda}\widehat{\Sigma}, \quad \widehat{\Sigma} \equiv (1-\cos\zeta\cos 2\chi). \quad (6.1)$$

6.1 $\cos 3\zeta \neq -1$

For such a generic ζ one has

$$d\lambda \sim \mp \frac{g}{4} \sqrt{(1+\cos 3\zeta)} e^{3\lambda} dr, \quad e^A \sim R^2 e^{-\lambda}, \quad (6.2)$$

for some constant, R . Thus the warp factor for the branes and the corresponding frames are finite and smooth for $\zeta \neq 0, \pi$:

$$e^i \sim \frac{1}{\sqrt{2}} R^2 (1+\cos\zeta)^{\frac{1}{6}} \widehat{\Sigma}^{\frac{1}{3}} dx^i, \quad i = 1, 2, 3; \quad (6.3)$$

Thus the metric parallel to the branes is simply:

$$ds_3^2 = \sum_{i=1}^3 (e^i)^2 \sim (1+\cos\zeta)^{\frac{1}{3}} R^4 \widehat{\Sigma}^{\frac{2}{3}} (-dx_1^2 + dx_2^2 + dx_3^2). \quad (6.4)$$

From (3.58) and (6.1) one has

$$e^4 \sim \mp \frac{2\sqrt{2}}{g} \frac{(1+\cos\zeta)^{\frac{1}{6}} \widehat{\Sigma}^{\frac{1}{3}}}{(1+\cos 3\zeta)^{\frac{1}{2}}} e^{-2\lambda} d\lambda, \quad (6.5)$$

$$e^{11} \sim \frac{2m_7}{(1+\cos\zeta)^{\frac{1}{3}} \widehat{\Sigma}^{\frac{2}{3}}} e^{-2\lambda} \left(d\phi + \cos^2\chi \left(d\psi + \frac{1}{2} \sin^2\theta \sigma_3 \right) \right),$$

These are the only two frames to depend on λ in this limit.

The remaining frames limit to:

$$e^5 \sim m_7 \left(\frac{\widehat{\Sigma}}{1+\cos\zeta} \right)^{\frac{1}{3}} d\chi; \quad e^6 \sim m_7 \left(\frac{\widehat{\Sigma}}{1+\cos\zeta} \right)^{-\frac{1}{6}} \cos\chi d\theta; \quad (6.6)$$

$$e^7 \sim \frac{m_7}{2} \left(\frac{\widehat{\Sigma}}{1+\cos\zeta} \right)^{-\frac{1}{6}} \cos\chi \sin\theta \sigma_1; \quad (6.7)$$

$$e^8 \sim \frac{m_7}{2} \left(\frac{\widehat{\Sigma}}{1+\cos\zeta} \right)^{-\frac{1}{6}} \cos\chi \sin\theta \sigma_2; \quad (6.8)$$

$$e^9 \sim \frac{m_7}{2} \left(\frac{\widehat{\Sigma}}{1+\cos\zeta} \right)^{-\frac{1}{6}} \cos\chi \sin\theta \cos\theta \sigma_3; \quad (6.9)$$

$$e^{10} \sim m_7 \left(\frac{\widehat{\Sigma}}{1+\cos\zeta} \right)^{-\frac{2}{3}} \sin\chi \cos\chi \left(\left(d\psi + \frac{1}{2} \sin^2\theta \sigma_3 \right) + \frac{2\cos\zeta}{(1+\cos\zeta)} d\phi \right). \quad (6.10)$$

It is instructive to rewrite e^{11} in terms of the one-form appearing in e^{10} :

$$e^{11} \sim \frac{2 m_7 \widehat{\Sigma}^{\frac{1}{3}}}{(1 + \cos \zeta)^{\frac{4}{3}}} e^{-2\lambda} \left[d\phi + \frac{(1 + \cos \zeta) \cos^2 \chi}{\widehat{\Sigma}} \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 + \frac{2 \cos \zeta}{(1 + \cos \zeta)} d\phi \right) \right], \quad (6.11)$$

Then one has

$$\begin{aligned} ds_2^2 &= (e^4)^2 + (e^{11})^2 \\ &\sim \frac{m_7^2 (1 + \cos \zeta)^{\frac{1}{3}} \widehat{\Sigma}^{\frac{2}{3}}}{(1 + \cos 3\zeta)} \left[d\rho^2 + \rho^2 \frac{4(1 + \cos 3\zeta)}{(1 + \cos \zeta)^3} \times \right. \\ &\quad \left. \times \left(d\phi + \frac{(1 + \cos \zeta) \cos^2 \chi}{\widehat{\Sigma}} \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 + \frac{2 \cos \zeta}{(1 + \cos \zeta)} d\phi \right) \right)^2 \right], \end{aligned} \quad (6.12)$$

where $\rho \equiv e^{-2\lambda}$.

The remaining part of the metric is

$$\begin{aligned} ds_6^2 &= \sum_{j=5}^{10} (e^j)^2 \\ &\sim m_7^2 \left(\frac{\widehat{\Sigma}}{1 + \cos \zeta} \right)^{\frac{2}{3}} \left[d\chi^2 + \frac{(1 + \cos \zeta) \cos^2 \chi}{\widehat{\Sigma}} ds_{\mathbb{CP}_2}^2 \right. \\ &\quad \left. + \frac{(1 + \cos \zeta)^2}{\widehat{\Sigma}^2} \sin^2 \chi \cos^2 \chi \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 + \frac{2 \cos \zeta}{(1 + \cos \zeta)} d\phi \right) \right], \end{aligned} \quad (6.13)$$

The full eleven-dimensional metric limits to the sum of (6.4), (6.12) and (6.13).

Observe that (6.13) is conformally Kähler. That is, the metric

$$\begin{aligned} \widehat{ds}_6^2 &= \frac{\widehat{\Sigma}}{(1 + \cos \zeta)} d\chi^2 + \cos^2 \chi ds_{\mathbb{CP}_2}^2 \\ &\quad + \frac{(1 + \cos \zeta)}{\widehat{\Sigma}} \sin^2 \chi \cos^2 \chi \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 + \frac{2 \cos \zeta}{(1 + \cos \zeta)} d\phi \right), \end{aligned} \quad (6.14)$$

has a Kähler form:

$$\begin{aligned} \widehat{J} &\equiv -\sin \chi \cos \chi d\chi \wedge \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 + \frac{2 \cos \zeta}{(1 + \cos \zeta)} d\phi \right) + \cos^2 \chi J_{\mathbb{CP}_2} \\ &= d \left[\frac{1}{2} \cos^2 \chi \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 + \frac{2 \cos \zeta}{(1 + \cos \zeta)} d\phi \right) \right], \end{aligned} \quad (6.15)$$

where $J_{\mathbb{CP}_2}$ is the Kähler form on \mathbb{CP}_2 . Here we are, of course, taking ζ to be constant at its asymptotic value. One can also easily verify that as $\chi \rightarrow \pi/2$ this manifold is smooth, and is locally like the origin of \mathbb{R}^6 .

The only singular parts of the metric occur at $\rho = 0$ and at $\chi = 0$, where there are orbifold singularities in two different \mathbb{R}^2 planes in (6.12) and (6.13) respectively. As we will

discuss below, these loci represent the intersections of the various branes that are present in the infra-red limit.

The non-zero components of the Maxwell field are given by:

$$A_{(3)} \sim h_0(r, \chi) e^1 \wedge e^2 \wedge e^3 + \frac{1}{4} \sin \zeta (e^6 \wedge e^9 + e^7 \wedge e^8 - e^5 \wedge e^{10}) \wedge \hat{e}^{11}, \quad (6.16)$$

with

$$h_0 \sim \text{sign}(1 - 2 \cos \zeta) \frac{(\cos \zeta - \cos 2\chi)}{\widehat{\Sigma}}, \quad (6.17)$$

$$\hat{e}^{11} \equiv \frac{2 m_7}{(1 + \cos \zeta)^{\frac{1}{3}} \widehat{\Sigma}^{\frac{2}{3}}} \left(d\phi + \cos^2 \chi \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 \right) \right). \quad (6.18)$$

Thus $A_{(3)}$ has regular coordinate components. One might be concerned that the Maxwell tensor has a singular source at $\rho = 0$ because the e^{11} is vanishing. However, the frame components of the Maxwell tensor are, in fact, regular. The non-zero frame components in the compactified directions (including e^{11}) are:

$$F_{46911}, F_{47811} \sim - \frac{2 m_7 \text{sign}(1 - 2 \cos \zeta)}{\widehat{\Sigma}^{\frac{4}{3}}} (1 + \cos \zeta)^{\frac{1}{3}} (\cos 2\chi \cos \zeta - \sin^2 \chi) \tan \frac{1}{2} \zeta, \quad (6.19)$$

$$F_{451011} \sim - \frac{2 m_7 \text{sign}(1 - 2 \cos \zeta) \sin \zeta}{(1 + \cos \zeta)^{\frac{2}{3}} \widehat{\Sigma}^{\frac{1}{3}}}, \quad (6.20)$$

$$F_{56910}, F_{57810} \sim - \frac{2 m_7 (1 + \cos \zeta \sin^2 \chi) \sin \zeta}{(1 + \cos \zeta)^{\frac{2}{3}} \widehat{\Sigma}^{\frac{4}{3}}}, \quad F_{6789} \sim \frac{2 m_7 \sin \zeta}{(1 + \cos \zeta)^{\frac{2}{3}} \widehat{\Sigma}^{\frac{1}{3}}}. \quad (6.21)$$

It is also useful to note that the electric part of the Maxwell field is extremely simple

$$F_{(4)}^{\text{electric}} = dA_{(3)}^e, \quad A_{(3)}^e = -\frac{1}{2} R^6 \text{sign}(1 - 2 \cos \zeta) \cos \frac{1}{2} \zeta \cos 2\chi dx_1 \wedge dx_2 \wedge dx_3. \quad (6.22)$$

6.2 $\zeta = \pm\pi/3$

Here we simply take $\zeta = +\pi/3$ because the flow for $\zeta = -\pi/3$ simply involves reversing the sign of the internal components of the flux, $A_{(3)}$.

One now has rather different asymptotics:

$$\begin{aligned} d\lambda \sim \mp \frac{g}{2\sqrt{2}} e^\lambda dr, \quad e^A \sim R^2 e^{-3\lambda}, \quad (6.23) \\ ds_{11}^2 \sim 2^{-\frac{4}{3}} 3^{\frac{1}{3}} \widehat{\Sigma}^{\frac{2}{3}} \left[\frac{d\rho^2}{\rho^2} + \rho^2 R^2 (-dx_1^2 + dx_3^2 + dx_3^2) \right. \\ \left. + \frac{64}{27} \rho^2 \left(d\phi + \frac{3 \cos^2 \chi}{2 \widehat{\Sigma}} \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 + \frac{2}{3} d\phi \right) \right)^2 \right. \\ \left. + \frac{4}{3} m_7^{-2} \left(d\chi^2 + \frac{3}{2 \widehat{\Sigma}} \cos^2 \chi ds_{\mathbb{CP}_2}^2 \right. \right. \\ \left. \left. + \frac{9}{4 \widehat{\Sigma}^2} \sin^2 \chi \cos^2 \chi \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 + \frac{2}{3} d\phi \right) \right)^2 \right], \quad (6.24) \end{aligned}$$

where, as before,

$$\rho \equiv e^{-2\lambda}, \quad \widehat{\Sigma} \equiv \left(1 - \frac{1}{2} \cos 2\chi\right). \quad (6.25)$$

Note that the compact six-dimensional metric in (6.24) is simply the metric (6.13) specialized to $\zeta = \pi/3$ and is therefore also conformally Kähler.

Remarkably, for $\zeta = \pi/3$ many of the components of $F_{(4)}$ vanish in the infra-red and we find that this limiting Maxwell field is simply given by

$$F_{(4)} = dA_{(3)}^0, \quad A_{(3)}^0 = \frac{\sqrt{3} m_7^{-3}}{4 \widehat{\Sigma}} \cos^4 \chi J_{\mathbb{C}\mathbb{P}_2} \wedge \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 + \frac{2}{3} d\phi\right). \quad (6.26)$$

Note that the space-time components parametrized by h_0 vanish in this limit and that F is purely magnetic and lives entirely on the conformally Kähler six-manifold. Thus, for $\zeta = \pi/3$ there are only M5 branes in the infra-red: the M2 branes have dissolved completely.

6.3 The IR limit of the flows

The first and rather remarkable surprise is that the warp factor, $X^{\frac{1}{6}} \Sigma^{\frac{1}{3}} e^A$, in front of frames parallel to the M2-branes (3.58) is not singular in the infra-red for $\zeta \neq 0, \pi$. For $\zeta = 0, \pi$, this warp factor is expected to be singular because such a flow has no internal fluxes and the warp factor is then simply a power of the harmonic function describing M2 brane sources that have spread on the Coulomb branch. However (6.3) shows that there is no singularity for generic ζ and for $\zeta = \pm\pi/3$ equation (6.24) shows that this warp factor actually vanishes. Thus there are no strongly singular sources of M2 branes in the infra-red.

The second surprise is that the internal six-dimensional manifold goes to a *finite-sized* conformally Kähler, six-dimensional manifold and this manifold is smooth at $\chi = \pi/2$. Indeed, the only singularities are conical and occur at $\rho = 0$ where the U(1) fiber defined by e^{11} pinches off (see (6.12)) and at $\chi = 0$ where the U(1) fiber defined by e^{10} pinches off (see (6.13)). It is also evident from (6.17) and (6.19)–(6.21) that the core of this holographic flow is populated by finite, smooth electric (M2-brane) and magnetic (M5-brane) fluxes. Thus there is evidently brane polarization and a geometric transition in which the M2 branes partially dissolve into smooth M5-brane fluxes leaving a finite sized “bubble” in the form of a six-dimensional Kähler manifold. This is rather reminiscent of the kind of transition one finds in microstate geometries [62–64].

To understand the brane content in the infra-red in more detail it is perhaps easiest to examine the projectors that define the supersymmetries. These are given by (5.8), (5.11) and (5.12). Define the rotated frames

$$\Gamma^{\widehat{4}} \equiv \cos \alpha \Gamma^4 + \sin \alpha \Gamma^5, \quad \Gamma^{\widehat{5}} \equiv \cos \alpha \Gamma^5 - \sin \alpha \Gamma^4, \quad (6.27)$$

$$\Gamma^{\widehat{10}} \equiv \cos \omega \Gamma^{10} + \sin \omega \Gamma^{11}, \quad \Gamma^{\widehat{11}} \equiv \cos \omega \Gamma^{11} - \sin \omega \Gamma^{10}, \quad (6.28)$$

where $\alpha = \alpha(r, \chi)$ and $\omega = \omega(r, \chi)$ are functions that depend upon the flow. The details of these angles and how they flow are given in section 5.2 and may also be found in [22]. Given the other projectors and the fact that $\Gamma^{1\dots 11} = 1$, one can write (5.12) as

$$\Pi_0 \equiv \frac{1}{2}(1 + \cos \beta \Gamma^{123} + \sin \beta \Gamma^{\text{Int}}), \quad (6.29)$$

where Γ^{Int} is any one of the following

$$\Gamma^{12369\widehat{11}}, \quad \Gamma^{12378\widehat{11}}, \quad \Gamma^{1235\widehat{10}\widehat{11}}, \quad (6.30)$$

This means that the flow represents M2 branes polarizing into three sets of M5 branes that have $(3 + 1)$ common directions, those of the M2 branes and one compactified direction, defined by \hat{e}^{11} . This means that the directions transverse to the M5 branes are defined by \hat{e}^4, \hat{e}^{10} and four of the compact internal directions. Thus the brane wrapping is crucially determined by \hat{e}^{11} and hence by ω .

For $\cos 3\zeta \neq -1$ and $\lambda \rightarrow \infty$, one has:

$$\cos \beta = \frac{\cos \zeta - \cos \chi}{(1 - \cos \zeta \cos \chi)}, \quad \alpha = \omega = \frac{\pi}{2}. \quad (6.31)$$

Thus $\hat{e}^4 = e^5, \hat{e}^{11} = -e^{10}$ and so χ lies transverse to all the branes. Indeed, (6.31) shows that $\chi = 0$ involves only anti-M2 brane sources and so the conical singularity at this point is not altogether surprising. The locus $\rho \equiv e^{-2\lambda} = 0$ also defines the location of the residual M2 branes and of some of the M5 branes and thus another conical singularity is not surprising. All the M5 branes have a common direction along e^{10} , which is the Hopf fiber in the Kähler metric (6.14).

One rather interesting flow involves having $\zeta \rightarrow \pi/2$ at infinity. This does not mean that $\zeta = \pi/2$ all along the flow; indeed (2.26) takes the value $-\frac{1}{2}$ on such a flow and this implies that as $\lambda \rightarrow 0$ one must have $\zeta \rightarrow \arccos(\pm \frac{1}{\sqrt{3}})$. What makes this flow interesting is that $SU(4)$ symmetry is restored in the infra-red. In particular, the metric (6.14) becomes precisely that of $\mathbb{C}\mathbb{P}^3$

As described in [22], the situation is very different for $\cos 3\zeta \neq -1$. For $\zeta = \pi/3$ and $\lambda \rightarrow \infty$, one has:¹⁶

$$\cos \beta = 0, \quad \omega = 0, \quad (6.32)$$

and

$$\cos \alpha = \frac{2 \cos 2\chi - 1}{2 - \cos 2\chi}, \quad \sin \alpha = -\frac{\sqrt{3} \sin 2\chi}{2 - \cos 2\chi}. \quad (6.33)$$

We now have $\hat{e}^{11} = e^{11}$ and so the M5 branes wrap e^{11} while e^{10} remains transverse to the branes. More significantly, the M2-brane flux now vanishes entirely and all that remains is a very simple non-singular magnetic (M5-brane) flux (6.26). The limiting metric (6.24) is almost like that of $AdS_5 \times \mathcal{B}_6$ where \mathcal{B}_6 is the conformally Kähler metric. The five-dimensional manifold that we label as \widehat{AdS}_5 is AdS_5 in Poincaré form with one spatial direction compactified and fibered over \mathcal{B}_6 . Holographically it suggests that the IR phase is almost a CFT except that one spatial direction has been “put in a periodic” box of some fixed scale and that some interactions have been turned on so that this direction becomes non-trivially fibered. Thus the IR phase is almost a CFT fixed point.

Finally, we would like to note that all the fluxes and most, if not all, of the metric in the IR limit are purely functions of χ . This however, does not mean that these limits represent solutions to the equations of motion because the r (or λ) dependence is critical to giving finite terms that survive in the IR limit of the equations of motion.

¹⁶The solution for $\zeta = -\pi/3$ simply flips the signs of the internal fluxes and is completely equivalent.

7 Generalizations

There are several natural generalizations of the results presented here. The first and most obvious is to use a somewhat more general gauged supergravity Ansatz. Our Ansatz may be thought of as reducing to $\mathcal{N} = 2$ supergravity coupled to one vector and with a holomorphic superpotential, $\mathcal{V} = \sqrt{2}(1 + z^3)$ [22, 29]. This can easily be generalized to $\mathcal{N} = 2$ supergravity coupled to three vector multiplets while still remaining within gauged $\mathcal{N} = 8$ supergravity. This truncation was considered in [65] and the holomorphic superpotential becomes

$$\mathcal{V} = \sqrt{2}(1 + z_1 z_2 z_3), \tag{7.1}$$

where the z_i are the complex scalars of the three vector multiplets. Our results here may be thought of as the special case with the three vector multiplets set equal and, in particular, $z_1 = z_2 = z_3 = z$. As noted in [22], the uplift formulae will be far more complicated, but one expects that the infra-red limit will involve a more general Kähler manifold with a $U(1)^3$ symmetry. It may also have some non-trivial moduli in that the $\zeta = \pi/3$ condition may simply become a constraint on the overall phase of $z_1 z_2 z_3$. These moduli would probably be related to the three distinct sets of M5-brane fluxes on the Kähler manifold. We intend to investigate this further.

We have also attempted a much greater generalization in the spirit of [26, 60, 61]. One starts with the uplifted flow solution and introduces rotated frames that subsume the need for the rotation by α :

$$\hat{e}^4 \equiv \cos \alpha e^4 + \sin \alpha e^5, \quad \hat{e}^5 \equiv -\sin \alpha e^4 + \cos \alpha e^5. \tag{7.2}$$

Rather remarkably one can integrate these frames in our flow solution to find new variables, (u, v) , so that

$$\hat{e}^4 \sim du, \quad \hat{e}^5 \sim dv. \tag{7.3}$$

One can also find explicit expressions for these new coordinates:

$$u \equiv e^{2A} p(r) (2 \cos 2\chi - 1), \quad v \equiv e^{2A} \cos^3 \chi \sin \chi. \tag{7.4}$$

Combined with the observation implicit in the simple and very canonical form of \hat{e}^{10} given in (5.27), one finds that our flow solution has a some extra structure and, in particular, the pairing of v and the frame in \hat{e}^{10} in the supersymmetry projectors, along with the phase dependence of the supersymmetries is very suggestive of an underlying six-dimensional complex structure.

We therefore start with the (u, v) coordinates and their associated frames. We then take the metric and fluxes to have a completely general $SU(3) \times U(1) \times U(1)$ -invariant Ansatz involving arbitrary functions of (u, v) everywhere possible. We also assume that the frame

\hat{e}^{10} is universal and, in particular, take the metric Ansatz to be of the form:

$$\begin{aligned}
e^i &= H_0(u, v)^{\frac{1}{3}} f^i, & i = 1, \dots, 3, & & e^4 &= H_0(u, v)^{-\frac{1}{6}} H_1(u, v) du, \\
e^5 &= H_0(u, v)^{-\frac{1}{6}} H_2(u, v) dv, & & & e^6 &= H_0(u, v)^{-\frac{1}{6}} H_3(u, v) v d\theta, \\
e^7 &= \frac{1}{2} H_0(u, v)^{-\frac{1}{6}} H_3(u, v) v \sin \theta \sigma_1; & & & e^8 &= \frac{1}{2} H_0(u, v)^{-\frac{1}{6}} H_3(u, v) v \sin \theta \sigma_2, \\
e^9 &= \frac{1}{2} H_0(u, v)^{-\frac{1}{6}} H_3(u, v) v \sin \theta \cos \theta \sigma_3; & & & & (7.5) \\
e^{10} &= H_0(u, v)^{-\frac{1}{6}} H_4(u, v) \left(d\phi + \frac{3}{2} \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 \right) \right); \\
e^{11} &= H_0(u, v)^{-\frac{1}{6}} H_5(u, v) \left(d\phi + G(u, v) \left(d\phi + \frac{3}{2} \left(d\psi + \frac{1}{2} \sin^2 \theta \sigma_3 \right) \right) \right),
\end{aligned}$$

where the H_a and G are, *ab initio*, undetermined functions. For the supersymmetry projectors we take $\alpha = \omega = 0$ at the outset but retain $\beta = \beta(u, v)$. We also assume that the supersymmetries have the same ψ - and ϕ -dependence as in (5.10). Note that this implies that the supersymmetry only depends on the combination $\phi + \frac{3}{2}\psi$, which appears in e^{10} .

For the Maxwell field, we take the most general $SU(3) \times U(1) \times U(1)$ -invariant potential and choose a gauge in which all the components along e^4 vanish:

$$\begin{aligned}
A^{(3)} &= h_0 e^1 \wedge e^2 \wedge e^3 + p_0 e^5 \wedge e^{10} \wedge e^{11} + p_1 e^5 \wedge (e^6 \wedge e^9 + e^7 \wedge e^8) \\
&+ p_2 (e^6 \wedge e^9 + e^7 \wedge e^8) \wedge e^{10} + p_3 (e^6 \wedge e^9 + e^7 \wedge e^8) \wedge e^{11},
\end{aligned} \tag{7.6}$$

where the h_0 and the p 's are also functions of (u, v) . The whole point is that this Ansatz at least contains our uplifted flow solution and we want to use this more general structure to understand the underlying geometry and perhaps find more general supersymmetric solutions.

To that end, one solves the supersymmetry variations to fix as many functions as possible. In [26, 60, 61] the whole problem reduced to determining a single ‘‘master function’’ from which every other flux and metric function was derived. This master function itself satisfied a non-trivial, non-linear differential equation. In spite of the nice structure that we have discovered, the same kind of procedure applied here does not lead to such a simple reduction. We did, however, discover some rather general results that we will briefly summarize so as to give a flavor of what emerges.

First, we find that, *along the flow*, the metric functions H_1, H_2 and H_4 are all fixed algebraically in terms of H_3 and H_5 . Moreover, the v derivative of H_3 is simply related to H_3 and H_5 . These conditions combine to reveal that the non-compact, eight-dimensional metric transverse to the M2-branes, ds_8^2 , has six-dimensional foliations, defined by the $\mathbb{C}\mathbb{P}^2$, the coordinate v and the frame e^{10} , are necessarily a u -dependent family of Kähler metrics. By this we mean that there is indeed a Kähler form, J , on each leaf of the foliation but dJ is proportional to du : if u were held constant, dJ would indeed vanish. Thus the six-dimensional Kähler structure apparent in the IR actually descends from a family of such structures along the flow.

Secondly, we find that the flux parametrized by p_0 is necessarily pure gauge and so we can set $p_0 \equiv 0$ without loss of generality. The rest of the p 's satisfy a complicated system of equations that link them with β and the remaining metric functions. Ultimately one can reduce the system to show that all the unknown functions are determined in terms of *two* unknown functions of (u, v) , H_3 and a function that we will call F . The former satisfies an extremely complicated non-linear differential equation while F is a pre-potential in that its derivatives determine some of the functions in the Ansatz: the v -derivative of F gives the function G in (7.5) while the u -derivative of F gives $H_5^{-2} \cos \beta$. The remarkable fact is that F is *harmonic* (annihilated by the Laplacian) in a metric that is conformal to ds_8^2 . The Laplacian on ds_8^2 explicitly involves H_3 and so the harmonicity of F is far from simple to use in practice.

We have, of course, verified that the uplifted flows do indeed satisfy these conditions but so far have not been able to simplify and elucidate the general discussion to the degree that it is worthy of presentation in this paper. The important bottom-line though is that every generalization we have considered shows the same structure for the eight-manifold transverse to the M2 branes: it consists of six-dimensional Kähler manifolds foliated over an \mathbb{R}^2 base where one of the $U(1)$'s acts as an isometry and the Kähler potentials depend upon the radial coordinate in this \mathbb{R}^2 .

8 Conclusions

On a technical level, our results represent a highly non-trivial test of the recent results on uplifting gauged $\mathcal{N} = 8$ supergravity to M theory. Indeed, our discussion in section 7 illustrates just how difficult it would be to construct the very symmetric class of flows we consider directly within eleven dimensions.

More broadly, we have, once again, seen how apparently very singular “Flows to Hades” in gauged supergravity can encode some very interesting physical flows and Janus solutions when lifted to M theory. Results like this illustrate why it is very important to understand how gauged supergravities are encoded in higher dimensional theories.

The uplift formulae to eleven-dimensional supergravity have been well-studied and tested compared to the uplift of gauged $\mathcal{N} = 8$ supergravity in five dimensions to IIB supergravity in ten dimensions. One reason might be that for quite some time only uplift formulae for the internal metric and dilaton were known [16] and those were inferred by analogy to the M-theory result rather than proven directly. However, this has changed recently with various reformulations of type IIB supergravity and the resulting uplift formulae for all fields [66–68]. While there are quite a number of interesting physical examples of IIB flows, it is possible that there are others yet to be discovered because they look singular from the five-dimensional perspective. Given that the dual theory in $\mathcal{N} = 4$ Yang-Mills theory, it would be interesting to examine such flows using the new Ansätze that are now available.

The limitation in using gauged supergravity is that one is restricted to a relatively small family of fields from the higher-dimensional perspective. Fortunately the fields one has are quite a number of the simplest relevant and marginal perturbations and so one can still

probe interesting physics. The limitation is most sorely felt when one tries to probe details and subtleties of the families of IR fixed points and for this gauged supergravity is too blunt an instrument. On the other hand, the uplifts of gauged supergravity solutions can give invaluable insights into the geometric structures that underlie the more general classes of flow and thus enable broader, and perhaps more physically interesting solutions to be found. This was evident for $\frac{1}{2}$ -BPS flows [19–21] and so even though gauged supergravity sometimes does not describe the exact physics one wants, it can motivate and inform the search for physically interesting families of solutions.

In this spirit, we suspect that the families of flows and Janus solutions considered here should admit interesting generalizations. There are the generalizations within gauged supergravity as outlined at the beginning of section 7. However, there should be families that involve a six-dimensional Kähler manifold fibered over a two-dimensional base with a U(1) isometry. It will probably be very challenging to use merely this information to find general flow solutions. However, we saw here that the fluxes also took on a relatively simple form and if one could understand the geometry underlying this one should be able to move towards the general class of solutions.

From the physical perspective, the flows and Janus solutions we have constructed are very interesting in that they involve M2 branes polarizing and dissolving into *non-singular* (except for orbifolds) distributions of M5 branes and for one choice of parameter, flowing to a higher-dimensional, almost-conformal fixed point [22]. Indeed, it was shown in [22] that such flows exist for any choice of parameter if one does not insist on supersymmetry.

Apart from the interesting holographic interpretation of these flows, this kind of mechanism also underpins the microstate geometry program in which black holes are replaced with smooth, horizonless solitonic geometries. (For reviews, see [64, 69, 70].) The basic mechanism means that the black hole undergoes a phase transition (driven by the Chern-Simons interactions) in which the singular electric charge sources are replaced by smooth magnetic fluxes. In M theory, this is realized by M2 charges being replaced by M5 fluxes. There are vast families of supersymmetric examples of this in asymptotically flat backgrounds but so far there are no examples of this process (supersymmetric, or not) in asymptotically AdS_4 or AdS_5 space-times. It is thus extremely helpful to have an example of such a transition and perhaps use it to understand how such a phase transition might occur more generally in asymptotically AdS_4 or AdS_5 space-times.

Finally, there is the question of whether there are IIB analogs of the flows and Janus solutions studied here. These would probably be flows in which D3 branes polarized into families of intersecting D5 branes while preserving $\mathcal{N}=1$ supersymmetry in the field theory along the flow. Such solutions might even “flow up dimensions” to give a compactified higher-dimensional field theories in the infra-red.

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A Conventions

We use the same conventions as in [59] with the “mostly plus metric” and the eleven-dimensional equations of motion given by

$$R_{MN} + g_{MN}R = \frac{1}{3} F_{MPQR} F_N{}^{PQR}, \quad (\text{A.1})$$

$$\nabla_M F^{MNPQ} = -\frac{1}{576} \frac{1}{\sqrt{-g}} \epsilon^{NPQR_1\dots R_8} F_{R_1\dots R_4} F_{R_5\dots R_8}. \quad (\text{A.2})$$

The Maxwell equation can be rewritten in terms of forms as¹⁷

$$d \star F_{(4)} + F_{(4)} \wedge F_{(4)} = 0, \quad (\text{A.3})$$

where $\star \equiv \star_{1,10}$ is the Hodge dual in eleven dimensions.

In general, we define the Hodge dual of a k -form, ω , in d -dimensions by

$$(\star \omega)_{i_1\dots i_{d-k}} = \frac{1}{k!} \eta_{i_1\dots i_{d-k}}{}^{j_1\dots j_k} \omega_{j_1\dots j_k}, \quad (\text{A.4})$$

where

$$\eta^{i_1\dots i_d} \equiv \frac{1}{\sqrt{|g|}} \epsilon^{i_1\dots i_d}, \quad \epsilon^{i_1\dots i_d} = 1. \quad (\text{A.5})$$

Then

$$(\star \omega) \wedge \omega = \pm |\omega|^2 \text{vol}, \quad |\omega|^2 \equiv \frac{1}{k!} \omega_{i_1\dots i_k} \omega^{i_1\dots i_k}. \quad (\text{A.6})$$

with the $+$ sign is for a positive definite metric and the $-$ sign for a Minkowski signature mostly plus metric.

For a (p, q) -form $\Omega_{(p,q)}$ on $\mathcal{M}_{1,3} \times \mathcal{M}_7$ with the warped product metric (3.9), we have a convenient decomposition of the Hodge dual:

$$\star \Omega_{(p,q)} = (-1)^{p(7-q)} \star_{1,3} \star_7 \Omega_{(p,q)}, \quad (\text{A.7})$$

where $\star_{1,3}$ and \star_7 are, respectively, the dual on $\mathcal{M}_{1,3}$ with respect to the four-dimensional part of the metric, $g_{\mu\nu}$, and the dual on \mathcal{M}_7 with the internal metric, g_{mn} . Factoring out the warp factor, we have

$$\star_{1,3} \omega_{(p)} = \Delta^{p-2} \overset{\circ}{\star}_{1,3} \omega_{(p)}, \quad (\text{A.8})$$

where $\omega_{(p)}$ is a p -form on $\mathcal{M}_{1,3}$ and $\overset{\circ}{\star}_{1,3}$ is the dual with respect to $\overset{\circ}{g}_{\mu\nu}$.

B Reduced $E_{7(7)}$ tensors and the scalar action

Using the 56-bein (2.4) and with the $SO(8)$ gauge field set to zero, the $SU(8)$ composite gauge field of the $\mathcal{N}=8$ theory,

$$\mathcal{A}_\mu{}^{ijkl} \equiv \mathcal{A}_\mu{}^{[ijkl]} = -2\sqrt{2} \left(u^{ij}{}_{IJ} \partial_\mu v^{klIJ} - v^{ijIJ} \partial_\mu u^{kl}{}_{IJ} \right), \quad (\text{B.1})$$

¹⁷Note that our normalization of the four-form flux is according to the “old supergravity convention.”

has the following non-vanishing components:

$$A_\mu^{1234} = A_\mu^{1256} = A_\mu^{1278} = A_\mu^{3456} = A_\mu^{3478} = A_\mu^{5678} = -\frac{\sqrt{2}\partial_\mu\bar{z}}{1-|z|^2}. \quad (\text{B.2})$$

Hence the kinetic action of the scalar fields is

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{kin}} &\equiv -\frac{1}{96}\mathcal{A}_{\mu ijkl}\mathcal{A}_\mu^{ijkl} = -3\frac{\partial_\mu z\partial^\mu\bar{z}}{(1-|z|^2)^2} \\ &= -3\partial_\mu\lambda\partial^\mu\lambda - \frac{3}{4}\sinh^2(2\lambda)\partial_\mu\zeta\partial^\mu\zeta. \end{aligned} \quad (\text{B.3})$$

Similarly, for the A -tensors, $A_1^{ij} \equiv A_1^{ji}$ and $A_{2i}{}^{jkl} \equiv A_{2i}{}^{[jkl]}$, we have:

$$A_1^{11} = \dots = A_1^{66} = \frac{1+z\bar{z}^2}{(1-|z|^2)^{3/2}}, \quad A_1^{77} = A_1^{88} = \frac{1+z^3}{(1-|z|^2)^{3/2}}, \quad (\text{B.4})$$

and

$$\begin{aligned} A_{21}{}^{234} &= A_{21}{}^{256} = A_{23}{}^{124} = A_{23}{}^{456} = A_{25}{}^{126} = A_{25}{}^{346} = -\frac{(1+z)\bar{z}}{(1-|z|^2)^{3/2}}, \\ A_{22}{}^{134} &= A_{22}{}^{156} = A_{24}{}^{123} = A_{24}{}^{356} = A_{26}{}^{125} = A_{26}{}^{345} = \frac{(1+z)\bar{z}}{(1-|z|^2)^{3/2}}, \\ A_{21}{}^{278} &= A_{23}{}^{478} = A_{25}{}^{678} = -\frac{(1+z)z}{(1-|z|^2)^{3/2}}, \\ A_{22}{}^{178} &= A_{24}{}^{378} = A_{26}{}^{578} = \frac{(1+z)z}{(1-|z|^2)^{3/2}}, \\ A_{27}{}^{128} &= A_{27}{}^{348} = A_{27}{}^{568} = -\frac{z+\bar{z}^2}{(1-|z|^2)^{3/2}}, \\ A_{28}{}^{127} &= A_{28}{}^{347} = A_{28}{}^{567} = \frac{z+\bar{z}^2}{(1-|z|^2)^{3/2}}. \end{aligned} \quad (\text{B.5})$$

Then the scalar potential is

$$\mathcal{P} \equiv -\left(\frac{3}{4}|A_1{}^{ij}|^2 - \frac{1}{24}|A_{2i}{}^{jkl}|^2\right) = -\frac{6(1+|z|^2)}{1-|z|^2} = -6\cosh(2\lambda). \quad (\text{B.6})$$

C The Freund-Rubin flux

The calculation of the space-time part of the flux, $F_{(4)}^{\text{st}}$, using the method employed in section 3.5 is quite involved even for much simpler solutions such as uplifts of stationary points. For the latter solutions only the Freund-Rubin part of the space-time flux is present, so that

$$F_{(4)}^{\text{st}} = f_{\text{FR}}\mathring{\text{v}}\mathring{\text{o}}_{1,3}, \quad (\text{C.1})$$

is proportional to the volume of the four-dimensional space-time, $\mathcal{M}_{1,3}$, where the proportionality constant is determined universally by the scalar potential of the four-dimensional

theory [35]. This has been generalized recently in [41] to uplifts of arbitrary solutions by including corrections proportional to derivatives of the scalar potential. The new conjectured formula for the Freund-Rubin flux, f_{FR} , reads

$$f_{\text{FR}} = \frac{m_7}{2} \left[\mathcal{P} - \frac{1}{24} \left(Q^{ijkl} \widehat{\Sigma}_{ijkl} + \text{h.c.} \right) \right]. \quad (\text{C.2})$$

The Q^{ijkl} tensor is proportional to the first variation of the potential, \mathcal{P} , along the non-compact generators of $E_{7(7)}$ acting on the scalar coset, $E_{7(7)}/\text{SU}(8)$. It is given by [71]

$$Q^{ijkl} = \frac{3}{4} A_{2m} n^{[ij} A_{2n}^{kl]m} - A_1 m^{[i} A_{2m}^{jkl]}. \quad (\text{C.3})$$

The second tensor in (C.2) is a self-dual contraction

$$\widehat{\Sigma}_{ijkl} = (u_{ij}{}^{IJ} u_{kl}{}^{KL} - v_{ijIJ} v_{klKL}) K^{IJKL}, \quad (\text{C.4})$$

where

$$K^{IJKL} = \overset{\circ}{g}{}^{mn} K_m^{[IJ} K_n^{KL]}. \quad (\text{C.5})$$

Note that at a stationary point of the scalar potential, the Q -tensor becomes anti-self-dual [71] and hence the contraction terms in (C.2) vanish.

Specializing the contraction in (C.2) to the present solution we find

$$Q^{ijkl} \widehat{\Sigma}_{ijkl} + \text{h.c.} = -16 \xi \sinh(2\lambda) \cos \zeta. \quad (\text{C.6})$$

Then, using (2.13), (3.28) and (3.57), we obtain

$$f_{\text{FR}} = \frac{m_7}{2} \left[-6 \cosh(2\lambda) + 2(1 - 4 \sin^2 \chi) \sinh(2\lambda) \cos \zeta \right] = \frac{m_7}{3} U, \quad (\text{C.7})$$

which agrees with the calculation of the space-time flux in section 3.5.

D The Ricci tensor

The non-vanishing coefficients of the diagonal components of the Ricci tensor, R_{MM} , as defined in (4.18):

$$\begin{aligned} \mathcal{A}_1 &= \frac{1}{6X^{13/3}\Sigma^{8/3}\Xi^2} \left[(p^2 + 1) \Sigma X (p^2 + X^2 + 1) - 8(p^2 + 1)^2 X^2 \right. \\ &\quad \left. + \Sigma^2 \left(3(p^3 + p)^2 + 3p^2 X^4 - 2(3p^4 + 7p^2 + 4) X^2 \right) \right] \\ \mathcal{B}_1 &= \frac{X^2}{6X^{13/3}\Sigma^{8/3}\Xi^2} \left[-8p^2 X^2 + 16p^2 \Sigma X \right. \\ &\quad \left. + \Sigma^2 \left(3p^4 - 2p^2 (3X^2 + 1) + 3(X^2 - 1)^2 \right) \right], \\ \mathcal{C}_1 &= \frac{pX}{3X^{13/3}\Sigma^{8/3}\Xi^2} \left[-4\Sigma X (3p^2 + X^2 + 3) + 8(p^2 + 1) X^2 \right. \\ &\quad \left. + \Sigma^2 (-3p^4 + p^2 (6X^2 - 2) - 3X^4 + 10X^2 + 1) \right], \end{aligned}$$

$$\mathcal{D}_1 = \frac{2m_7^{-2}}{3X^{4/3}\Sigma^{8/3}} \left[\Sigma^2 (6X^2 - g^2 m_7^2 (3p^2 + 3X^2 + 4)) - 2\Sigma X (g^2 m_7^2 + 2p^2 + 2) - 2(p^2 + 1) X^2 \right]; \quad (\text{D.1})$$

$$\mathcal{A}_4 = \frac{1}{3X^{13/3}\Sigma^{8/3}\Xi^2} \left[2(p^2 + 1)^2 (3p^2 - 1) X^2 - 2(3p^4 + 2p^2 - 1) \Sigma X (p^2 + X^2 + 1) + \Sigma^2 (3(p^3 + p)^2 + 3p^2 X^4 - 2(p^2 + 1) X^2) \right],$$

$$\mathcal{B}_4 = \frac{1}{3X^{7/3}\Sigma^{8/3}\Xi^2} \left[-2p^2 \Sigma X (3p^2 + 3X^2 - 5) + 2(3p^2 - 1) p^2 X^2 + \Sigma^2 (3p^4 - 2p^2 + 3(X^2 - 1)^2) \right]$$

$$\mathcal{C}_4 = \frac{1}{3X^{10/3}\Sigma^{8/3}\Xi^2} \left[2\Sigma X (3p^2 (p^2 + X^2) + X^2 - 3) - 2(3p^4 + 2p^2 - 1) X^2 + \Sigma^2 (-3p^4 - 2p^2 - 3X^4 + 4X^2 + 1) \right],$$

$$\mathcal{D}_4 = \frac{m_7^{-2}}{3X^{4/3}\Sigma^{8/3}} \left[2\Sigma^2 (-3m_7^2 g^2 (p^2 + X^2) - 4m_7^2 g^2 + 6X^2) - 4\Sigma X (m_7^2 g^2 + 2p^2 + 2) - 4(p^2 + 1) X^2 \right]; \quad (\text{D.2})$$

$$\mathcal{A}_5 = -\frac{4}{3X^{10/3}\Sigma^{8/3}\Xi^2} (p^2 + 1) (\Sigma - X) (p^2 - \Sigma X + 1),$$

$$\mathcal{B}_5 = \frac{4p^2 X (\Sigma - X)^2}{3X^{10/3}\Sigma^{8/3}\Xi^2},$$

$$\mathcal{C}_5 = -\frac{4p}{3X^{10/3}\Sigma^{8/3}\Xi^2} (\Sigma - X) \left[\Sigma (p^2 + X^2 + 1) - 2(p^2 + 1) X \right],$$

$$\mathcal{D}_5 = \frac{2}{3\Sigma^{8/3} X^{4/3}} \left[-2\Sigma X (g^2 - 4m_7^2 (p^2 + 1)) + 2g^2 \Sigma^2 + m_7^2 (4p^2 + 1) X^2 \right]; \quad (\text{D.3})$$

$$\mathcal{A}_6 = \frac{2}{3X^{10/3}\Sigma^{8/3}\Xi^2} (p^2 + 1) (\Sigma - X) (p^2 - \Sigma X + 1),$$

$$\mathcal{B}_6 = -\frac{2p^2 (\Sigma - X)^2}{3\Sigma^{8/3} X^{7/3}\Xi^2},$$

$$\mathcal{C}_6 = \frac{2p}{3X^{10/3}\Sigma^{8/3}\Xi^2} (\Sigma - X) \left[\Sigma (p^2 + X^2 + 1) - 2(p^2 + 1) X \right],$$

$$\mathcal{D}_6 = \frac{2}{3X^{4/3}\Sigma^{8/3}} \left[\Sigma^2 (m_7^2 (9p^2 + 6) - g^2) + \Sigma X (g^2 + 2m_7^2 (p^2 + 1)) + m_7^2 (p^2 + 1) X^2 \right]; \quad (\text{D.4})$$

$$\mathcal{A}_{10} = \frac{2}{3X^{10/3}\Sigma^{8/3}\Xi^2} (3p^4 + 4p^2 + 1) (\Sigma - X) (p^2 - \Sigma X + 1),$$

$$\mathcal{B}_{10} = \frac{2p^2 X}{3X^{10/3}\Sigma^{8/3}\Xi^2} (\Sigma - X) \left[(3p^2 + 1) X + \Sigma (2 - 3X^2) \right],$$

$$\mathcal{C}_{10} = -\frac{4p}{3X^{10/3}\Sigma^{8/3}\Xi^2} (\Sigma - X) \left(\Sigma (-(3p^2 + 2) X^2 + p^2 + 1) + (3p^4 + 4p^2 + 1) X \right),$$

$$\mathcal{D}_{10} = -\frac{2}{3X^{4/3}\Sigma^{8/3}} \left[2\Sigma X (g^2 - 4m_7^2 (p^2 + 1)) - 2g^2 \Sigma^2 - m_7^2 (4p^2 + 1) X^2 \right]; \quad (\text{D.5})$$

$$\begin{aligned}
 \mathcal{A}_{11} &= \frac{2}{3X^{13/3}\Sigma^{8/3}\Xi^2} \left[(p^2 + 1)^2 (3p^2 - 1) X^2 - (3p^4 + 2p^2 - 1) \Sigma X (p^2 + X^2 + 1) \right. \\
 &\quad \left. + \frac{\Sigma^2}{2} \left(3(p^3 + p)^2 + 3p^2 X^4 - 2(p^2 + 1) X^2 \right) \right], \\
 \mathcal{B}_{11} &= \frac{1}{3X^{7/3}\Sigma^{8/3}\Xi^2} \left[2(3p^2 - 1) p^2 X^2 - 2p^2 \Sigma X (3p^2 + 3X^2 - 5) \right. \\
 &\quad \left. + \Sigma^2 \left(3p^4 - 2p^2 + 3(X^2 - 1)^2 \right) \right], \\
 \mathcal{C}_{11} &= \frac{2p}{3X^{10/3}\Sigma^{8/3}\Xi^2} \left[2\Sigma X (3p^2 (p^2 + X^2) + X^2 - 3) - 2(3p^4 + 2p^2 - 1) X^2 \right. \\
 &\quad \left. + \Sigma^2 (-3p^4 - 2p^2 - 3X^4 + 4X^2 + 1) \right], \\
 \mathcal{D}_{11} &= \frac{2}{3X^{4/3}\Sigma^{8/3}} \left[-\Sigma^2 (3g^2 p^2 - g^2 (3X^2 + 1) + 6m_7^2 (X^2 + 1)) \right. \\
 &\quad \left. - 4\Sigma X (m_7^2 (p^2 + 1) - g^2) - m_7^2 (2p^2 - 1) X^2 \right]. \tag{D.6}
 \end{aligned}$$

The non-vanishing coefficients of the off-diagonal components of the Ricci tensor, R_{MN} , $M \neq N$, as defined in (4.19):

$$\begin{aligned}
 \mathcal{A}_{45} &= -\frac{2m_7 \tan \chi}{X^{11/6}\Sigma^{8/3}\Xi} (p^2 + 1) (\Sigma - X)(2\Sigma + X), \\
 \mathcal{B}_{45} &= -\frac{2m_7 \tan \chi p}{X^{5/6}\Sigma^{8/3}\Xi} (X - \Sigma)(2\Sigma + X); \tag{D.7}
 \end{aligned}$$

$$\mathcal{D}_{1011} = -\frac{2(g^2 - 2m_7^2) \tan(\chi)(\Sigma - X)}{\Sigma^{5/3} X^{1/3}}. \tag{D.8}$$

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