On RG flow of τ_{RR} for supersymmetric field theories in three-dimensions

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ABSTRACT: The coefficient τ_{RR} of the two-point function of the superconformal $U(1)_R$ currents of $\mathcal{N} = 2$ SCFTs in three-dimensions is recently shown to be obtained by differentiating the partition function on a squashed three-sphere with respect to the squashing parameter. With this method, we compute the τ_{RR} for $\mathcal{N} = 2$ Wess-Zumino models and SQCD numerically for small number of flavors and analytically in the large number limit. We study the behavior of τ_{RR} under an RG flow by adding superpotentials to the theories. While the τ_{RR} decreases for the gauge theories, we find an $\mathcal{N} = 2$ Wess-Zumino model whose τ_{RR} increases along the RG flow. Since τ_{RR} is proportional to the coefficient C_T of the two-point correlation function of the stress-energy tensors for $\mathcal{N} = 2$ superconformal field theories, this rules out the possibility of C_T being a measure of the degrees of freedom which monotonically decreases along RG flows in three-dimensions.

KEYWORDS: Supersymmetric gauge theory, Field Theories in Lower Dimensions

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1 Introduction

Three-dimensional field theories have attracted renewed attentions since the development of localization of supersymmetric gauge theories on a three-sphere [1–3]. One of the intriguing applications of the localization technique is the F-theorem proposed in [4] based on the extremization principle of the partition function (Z-extremization) [2], stating that the free energy defined by the logarithm of the partition function on S^3, $F = -\log Z$, decreases along any renormalization group (RG) trajectory. For $\mathcal{N} = 2$ superconformal field theories, the free energy F is proved to be maximized in [5] that establishes the F-theorem as long as there is no accidental symmetry in the infrared (IR) fixed point of RG flows for the same reason as the a-maximization in four-dimensions [6]. The F-theorem was further conjectured to hold for any unitary field theory even without supersymmetry [7]. A proof is presented in [8] through the relation between F and the entanglement entropy of a circle for CFTs [9].[1] This is the analogue of the c-theorem in 2D [12] and the a-theorem in

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[1]Entanglement entropy of a circle and F are equivalent up to ultraviolet (UV) divergent terms. One can define a renormalized entanglement entropy that interpolates the free energies at the UV and IR fixed points [10], while it is not necessarily stationary against perturbation around the fixed point [11] different from the Zamolodchikov's c-function.

4D [13, 14] where the central charges are uniquely defined by the a-anomaly as a coefficient of the Euler characteristic in the trace of the stress-energy tensor.

It is, however, not obvious why the free energy on S^3 counts the number of degrees of freedom of three-dimensional field theories. One may well define a "thermal central charge" c_{Therm} as a coefficient of the thermal free energy $F_{\text{Therm}} \sim c_{\text{Therm}} T^3$ at finite temperature T. While c_{Therm} is equivalent to the central charge for CFT_2 up to a constant, it can not be a c-function in higher dimensions. Indeed, c_{Therm} increases along the RG flow from the critical $O(N)$ vector model to the Goldstone phase with spontaneously broken $O(N)$ symmetry in three-dimensions [15]. Another possibility is the use of the coefficient C_T of the two-point function of the stress-energy tensors as we will describe shortly.

The forms of the two-point functions of vector currents J_I^μ and stress-energy tensors $T_{\mu\nu}$ of conformal field theories are fixed by conformal symmetry and the conservation laws in d-dimensions (see e.g. [16])

$$\langle J_I^\mu(x) J_J^\nu(0) \rangle = C_{IJ} \frac{I_{\mu\nu}(x)}{x^{2(d-1)}} \,, \tag{1.1}$$

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = C_T \frac{I_{\mu\nu,\rho\sigma}(x)}{x^{2d}} \,, \tag{1.2}$$

where the Greek indices label the types of the currents and μ, ν are the spacetime indices running from 1 to d. The functions $I_{\mu\nu}$ and $I_{\mu\nu,\rho\sigma}$ are defined as follows:

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2} \,,$$

$$I_{\mu\nu,\rho\sigma}(x) = \frac{1}{2} \left(I_{\mu\nu}(x) I_{\rho\sigma}(x) + I_{\mu\rho}(x) I_{\nu\sigma}(x) \right) - \frac{\delta_{\mu\nu} \delta_{\rho\sigma}}{d} \,. \tag{1.3}$$

The correlation functions are characterized by the positive coefficients C_{IJ} and C_T for unitary field theories. C_T is the central charge c in two-dimensions, while it is not the a-, but the c-anomaly in four-dimensions [16] which does not necessarily decrease under RG flows [17, 18].

Less is known about C_T in three-dimensions, except for the $O(N)$ vector model in the large-N limit [19]. There are N independent free scalar fields in the UV fixed point, leading to $C_T^{\text{free}} = N$, while $N - 1$ fields contribute in the Goldstone phase, $C_T^{\text{Goldstone}} = N - 1$.[2] In between the RG flow, there is the critical $O(N)$ fixed point with $C_T^{\text{critical}} = N - \frac{40}{9\pi^2}$. As opposed to c_{Therm}, C_T decreases for the $O(N)$ vector model under the RG flow: $C_T^{\text{Goldstone}} < C_T^{\text{critical}} < C_T^{\text{free}}$. It follows from this observation that C_T is conjectured to be a measure of degrees of freedom in three-dimensions [20]. Although there are no other examples nor a proof for the conjectured C_T-theorem in field theories, holographic analysis of [21] implies its monotonicity along the RG flow in $d = 3$ (and of course $d = 2$) dimensions, assuming the null energy condition and the absence of ghosts of the gravity.

In this paper, we would like to test this conjecture with various examples. Especially, we will consider $\mathcal{N} = 2$ supersymmetric field theories in three-dimensions. These theories have the R-symmetry current associated to the superconformal $\text{U}(1)_R$ symmetry, and C_T

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[2]Here we normalized C_T so that one real free scalar field gives $C_T = 1$.

is proportional to the coefficient τ_{RR} of the two-point function of the R-symmetry currents

$$\langle J_R^\mu(x) J_R^\nu(0) \rangle = \frac{\tau_{RR}}{4\pi^2} \frac{I_{\mu\nu}(x)}{x^4} ,$$

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = \frac{3\,\tau_{RR}}{2\pi^2} \frac{I_{\mu\nu,\rho\sigma}(x)}{x^6} . \tag{1.4}$$

In general, the $U(1)_R$ symmetry can mix with other global $U(1)$ symmetries, and one needs to determine the correct R-charge at the IR fixed point to compute τ_{RR} or C_T. It can be implemented by minimizing τ_{RR} with respect to a trial R-charge as shown in [22]. This, however, is not practical because a trial τ_{RR} function has not been known for interacting field theories. A more practical way is to employ the Z-extremization [2, 5] where the partition function Z on S^3 as a function of trial R-charges is minimized. Moreover, τ_{RR} is able to be obtained by placing $\mathcal{N} = 2$ theories on a squashed three-sphere and taking the second derivative of the partition function with respect to the squashing parameter [23] as we will review in section 2. So to compute τ_{RR} of $\mathcal{N} = 2$ theories, we will carry out the following: 1) Compute the R-charge Δ by minimizing the sphere partition function Z as a function of Δ, 2) Substitute Δ obtained in 1) into the second derivative of the partition function given by eq. (2.2).

A class of the theories we will consider is $\mathcal{N} = 2$ supersymmetric QCD (SQCD) and Wess-Zumino models. In section 3, we start with calculating the τ_{RR} for the non-chiral $\mathcal{N} = 2$ SQCD with N_f flavors without superpotentials. The partition function is given by a multiple integral of hyperbolic gamma functions whose arguments depend on the matter contents of the theory. While the integral is to be performed analytically for some simple cases, it is generally beyond our ability. Nevertheless, we are able to obtain the few leading terms of Δ and τ_{RR} in the large-N_f limit. The value of Δ is extensively studied in the context of the F-theorem in [24, 25], and we will follow their method. Given the correct R-charge, a similar calculation leads us to τ_{RR} at the IR fixed point. We study the behavior of τ_{RR} by adding a superpotential to the theory that flows to an $\mathcal{N} = 3$ fixed point. Comparing the values at the fixed points with and without the superpotential, we will show τ_{RR} decreases along the RG flow. To tackle the small N_f region, we perform numerical computations of the multiple integral. They fit the analytic large-N_f results very well even for small N_f. We argue that our large-N_f results are valid in general and the C_T-theorem holds in these RG flows. The chiral case is also studied with the large number of flavors, followed by the same conclusion. More general class of theories is investigated and the details are presented in appendix B.

The τ_{RR} decreases for the gauge theories we have considered. However, it is not the case with an $\mathcal{N} = 2$ Wess-Zumino model as we will see in section 4. We will consider a theory of $N + 1$ chiral fields, X and $Z_i\,(i = 1, \ldots, N)$, interacting through a superpotential of the form $\mathcal{W} \sim X \sum_{i=1}^{N} Z_i^2$. We let it flow to another fixed point by decoupling the X field with the additional superpotential $\Delta\mathcal{W} \sim mX^2$. The resulting IR theory will have a quartic superpotential of Z_i, $\mathcal{W}_{IR} \sim (\sum_{i=1}^{N} Z_i^2)^2$. This model is reminiscent of the $O(N)$ vector model that disproves the "c_{Therm}"-theorem. We confirm that this flow is consistent with the F-theorem (the free energy on S^3 decreases). On the other hand, we find τ_{RR} increases along the RG flow and this model rules out the possibility of the C_T-theorem.

2 τ_{RR} in $\mathcal{N} = 2$ supersymmeric field theories

Three-dimensional $\mathcal{N} = 2$ superconformal field theories have U(1) R-symmetry. Since the R-symmetry current is in the same multiplet as the stress-energy tensor, their two-point functions are determined by the coefficient τ_{RR} defined by eq. (1.4). The normalization of τ_{RR} is fixed such that a free chiral multiplet has $\tau_{RR} = \frac{1}{4}$. In [23], it was shown that the flat-space two-point correlation functions (1.4) of $\mathcal{N} = 2$ SCFTs can be computed using localization on a squashed three-sphere S_b^3

$$ds^2 = \frac{1}{4} \left[d\theta^2 + \sin^2 \theta d\phi^2 + \omega^2 (d\psi + \cos \theta d\phi)^2 \right] , \qquad (2.1)$$

where the round sphere is recovered when $\omega = 1$. We parametrize ω by a squashing parameter b as $\omega = (b + b^{-1})/2$. The squashed three-sphere S_b^3 preserves SU(2) × U(1) symmetry and one can put $\mathcal{N} = 2$ theories on it with four supercharges [26, 27]. Given the partition function $Z_{S_b^3}$ and the free energy $F(b) = -\log Z_{S_b^3}$, τ_{RR} is obtained by taking the second derivative of $F(b)$ with respect to b [23]:

$$\tau_{RR} = \frac{2}{\pi^2} \operatorname{Re} \left. \frac{\partial^2 F}{\partial b^2} \right|_{b=1} . \qquad (2.2)$$

The above relation allows us to compute τ_{RR} of a given SCFT in terms of the partition function $F(b)$ and we can compare the values of τ_{RR} at the UV and IR fixed points.

2.1   Localization on a squashed three-sphere

The partition function of $\mathcal{N} = 2$ theories on S_b^3 are obtained via the localization [26, 27] in the following way:[3]

- The one-loop matter contribution to the partition function is given by

$$Z_{\text{matter}}^{\text{1-loop}} = \prod_I \prod_{\rho \in \mathcal{R}_I} \Gamma_h \left[\omega(\rho(\sigma) + i\Delta_I) \right] , \qquad (2.3)$$

  where $\Gamma_h[z] \equiv \Gamma_h(z; i\omega_1, i\omega_2)$ is the hyperbolic gamma function defined by eq. (A.1). Here, we choose the special values of the arguments $\omega_1 = b$, $\omega_2 = 1/b$, and $\omega = (\omega_1 + \omega_2)/2$. I labels the types of chiral multiplets and ρ is a weight in a representation $\mathcal{R}_I$. Δ_I is the R-charge of the scalar field in a chiral multiplet.

- The one-loop gauge contribution to the partition function and the path integral measure of the zero modes are combined to

$$Z_{\text{gauge}}^{\text{1-loop}} \cdot [d\sigma] = \frac{(2\pi\omega)^{\text{rank}\, G}}{|W| \text{Vol}(T)} \left(\prod_\alpha \Gamma_h \left[\omega\alpha(\sigma) \right] \right)^{-1} \cdot \prod_{i=1}^{\text{rank}\, G} d\sigma_i , \qquad (2.4)$$

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[3]In the original literatures [26, 27], the double sine functions are used instead of the hyperbolic gamma functions. The former is roughly the inverse of the latter. The latter is suitable for studying dualities [28, 29] that are recast as the identities between the integrals of the hyperbolic gamma functions [30].

where $|W|$ is the order of the Weyl group W of the group G, $\mathrm{Vol}(T)$ is the volume of the maximal torus T of G (e.g., $T = \mathrm{U}(1)^N$ for $G = \mathrm{U}(N)$), and α is a root of G. The factor $(2\pi\omega)^{\mathrm{rank}\,G}$ comes from the one-loop determinant of the gauge multiplets in the Cartan subalgebra. The ω dependence of this factor is important for calculations of τ_{RR}. See appendix B for the form of this one-loop determinant without gauge-fixing.

- The Chern-Simons term of level k and FI term of parameter ξ give a classical contribution

$$Z_{\mathrm{cl}} = \exp\left[-\pi i k \omega^2 \mathrm{Tr}(\sigma^2) - 2\pi i \xi \omega \mathrm{Tr}(\sigma)\right] \ . \tag{2.5}$$

We assume that the normalization of Tr is chosen such that the Chern-Simons level k is quantized to be an integer. In the case $G = \mathrm{U}(N)$, Tr is just the trace in the fundamental representation, $\mathrm{Tr} = \mathrm{Tr}_f$. Throughout this paper we will drop off the FI term.

We shall use these formulae to compute τ_{RR} for several examples below.

2.2 Calculation of τ_{RR}-coefficient

Wess-Zumino model: first, we consider a free chiral multiplet with the R-charge Δ. In this case, the one-loop partition function (2.3) gives the exact answer. The integral representation of the hyperbolic gamma function (A.1) is useful to see the b-dependence of the free energy

$$F_{\mathrm{chiral}}(b) = -\int_0^\infty \frac{dx}{2x}\left(\frac{\sinh(2(1-\Delta)\omega x)}{\sinh(bx)\sinh(b^{-1}x)} - \frac{2(1-\Delta)\omega}{x}\right) \ . \tag{2.6}$$

The τ_{RR} given by eq. (2.2) leads

$$\tau_{RR}(\Delta) = \frac{2}{\pi^2}\int_0^\infty dx\left[(1-\Delta)\left(\frac{1}{x^2} - \frac{\cosh(2x(1-\Delta))}{\sinh^2(x)}\right) + \frac{(\sinh(2x) - 2x)\sinh(2x(1-\Delta))}{2\sinh^4(x)}\right].$$

For a free chiral multiplet with $\Delta = 1/2$, a short computation yields $\tau_{RR} = \frac{1}{4}$. One can also check that $\tau_{RR}(\Delta)$ is not extremized at $\Delta = 1/2$. Therefore this example excludes the possibility that $\tau_{RR}(\Delta)$ defined by eq. (2.2) is extremized at the correct R-charges.

As a slightly nontrivial example, let us consider a Wess-Zumino model which consists of a chiral field X with a cubic superpotential,

$$\mathcal{W} = X^3 \ . \tag{2.7}$$

This model has an interacting IR fixed point with $\Delta_{\mathrm{IR}} = 2/3$ and $\tau_{RR}^{\mathrm{IR}} = \tau_{RR}(2/3) \simeq 0.182$. The UV limit is just a free chiral multiplet and hence $\tau_{RR}^{\mathrm{UV}} = 1/4$. In this model, τ_{RR} decreases along the RG flow, $\tau_{RR}^{\mathrm{IR}} < \tau_{RR}^{\mathrm{UV}}$.

Chern-Simons theory: we next consider $U(N)_k$ pure Chern-Simons theory whose partition function is

$$Z_{\text{CS}} = \frac{1}{N!} \int \prod_{i=1}^{N} \omega \, d\sigma_i \, e^{-\pi i k \omega^2 \sum_{i=1}^{N} \sigma_i^2} \prod_{\alpha \in \Delta_+} 4 \sinh\left(\pi b \omega \alpha(\sigma)\right) \sinh\left(\pi b^{-1} \omega \alpha(\sigma)\right) , \quad (2.8)$$

where we used the identity (A.2), and α runs over the positive roots of $U(N)$. We rewrite the partition function with the Weyl denominator formula and perform the multiple integrals to get (see e.g. [1])

$$Z_{\text{CS}} = \frac{\exp\left[\frac{\pi i}{6k} N(N^2 - 1)(1 - 2\omega^2)\right]}{k^{N/2}} \prod_{m=1}^{N-1} \left(2 \sin\left(\frac{\pi m}{k}\right)\right)^{N-m} . \quad (2.9)$$

Since the ω-(b-)dependence appears only in the imaginary part of the free energy, τ_{RR} vanishes for the CS theory, $\tau_{RR}|_{CS} = 0$. This is consistent with the fact that topological theories can not have non-zero C_T-coefficient since they do not couple to the background metric.[4]

Large-N gauge theories: in a certain large-N limit of $\mathcal{N} = 2$ SCFTs, the dependence on the squashing parameter b of the partition function becomes very simple [27] (see also [31])

$$F(b) = \frac{(b + b^{-1})^2}{4} F(1) , \quad (2.10)$$

where $F(1)$ is the round three-sphere partition function. Combining eqs. (2.2) and (2.10), we obtain the universal result for τ_{RR}

$$\tau_{RR} = \frac{4}{\pi^2} F(1) . \quad (2.11)$$

This is consistent with the holographic analysis in [32]. Since $C_T \propto \tau_{RR}$ in $\mathcal{N} = 2$ SCFTs, it follows that C_T is proportional to the round sphere partition function

$$C_T = \frac{6}{\pi^4} F(1) . \quad (2.12)$$

The F-theorem [4, 8] ensures $F(1)$ decreases under any RG flow. Then the above relation between C_T and $F(1)$ indicates $C_T^{\text{IR}} < C_T^{\text{UV}}$ for $\mathcal{N} = 2$ SCFTs in the large-N limit.

3 $\mathcal{N} = 2$ $U(N_c)$ gauge theory with flavors

Consider $\mathcal{N} = 2$ $U(N_c)$ Chern-Simons theory of level k coupled to N_f quarks Q_a and $\tilde{N}_f$ anti-quarks $\tilde{Q}_a$ in the fundamental and anti-fundamental representations of $U(N_c)$, respectively, without a superpotential. This theory enjoys $U(N_f) \times U(\tilde{N}_f)$ global symmetries acting on the quarks Q_a and the anti-quarks $\tilde{Q}_a$ as vector representations, that make all the R-charges of Q_a ($\tilde{Q}_a$) be equal. These symmetries leave us two R-charges $R[Q_a] = \Delta$

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[4]This result also gives a consistency check on the ω dependence of the overall factor of the partition function (2.4), which was neglected in [27].

and $R[\tilde{Q}_a] = \tilde{\Delta}$. Since the $\mathrm{U}(1)^2_F \subset \mathrm{U}(N_f) \times \mathrm{U}(\tilde{N}_f)$ flavor symmetry can mix with the R-symmetry, the correct values of the R-charges at the IR fixed point should be fixed by minimizing the partition function with respect to Δ and $\tilde{\Delta}$ [2, 5].

Following the rules described in section 2, one can write down the partition function of $\mathrm{U}(N_c)_k$ chiral SQCD with flavors on S^3_b

$$Z(b) = \frac{1}{N_c!} \int \prod_{i=1}^{N_c} d\sigma_i \, e^{-i\pi k \sum_{i=1}^{N_c} \sigma_i^2} \, Z_g(b,\sigma) \prod_{i=1}^{N_c} \Gamma_h[\sigma_i + i\omega\Delta]^{N_f} \, \Gamma_h[-\sigma_i + i\omega\Delta]^{\tilde{N}_f} , \quad (3.1)$$

where we rescaled the integration variable σ by ω and defined the function Z_g as

$$Z_g(b,\sigma) = \prod_{i<j} 4 \sinh\left(\pi b(\sigma_i - \sigma_j)\right) \sinh\left(\pi b^{-1}(\sigma_i - \sigma_j)\right) . \quad (3.2)$$

We will assume $N_f \geq \tilde{N}_f$ and introduce new parameters $\bar{N}_f$ and μ:

$$N_f = (1 + \mu)\bar{N}_f , \qquad \tilde{N}_f = (1 - \mu)\bar{N}_f , \qquad 0 \leq \mu \leq 1 . \quad (3.3)$$

We will find it convenient to use these parametrization in section 3.2 . Using eq. (2.2) and the integral representation of the hyperbolic gamma function (A.1), τ_{RR} of the SQCD becomes

$$\tau_{RR} = \mathrm{Re}\left[\frac{1}{N_c! \, Z(1)} \int \prod_{i=1}^{N_c} d\sigma_i \, e^{-i\pi k \sum_{i=1}^{N_c} \sigma_i^2} \, e^{N_f(\ell(1-\Delta+i\sigma)+\ell(1-\Delta-i\sigma))} \, Z_g(1,\sigma) \right.$$
$$\left. \cdot \left(g(\sigma) + \bar{N}_f \sum_{i=1}^{N_c} \left[(1+\mu)f(1-\Delta,\sigma_i) + (1-\mu)f(1-\tilde{\Delta},-\sigma_i) \right] \right) \right] , \quad (3.4)$$

where $\ell(z)$ is the Jafferis's ℓ-function [2] defined by eq. (A.4) and

$$f(z,\sigma) = \frac{2}{\pi^2} \int_0^\infty dx \left[\frac{z}{x^2} - \frac{z \cosh(2x(z+i\sigma))}{\sinh^2(x)} + \frac{(\sinh(2x) - 2x)\sinh(2x(z+i\sigma))}{2\sinh^4(x)} \right] ,$$
$$(3.5)$$

$$g(\sigma) = -\frac{2}{\pi^2} \sum_{i<j} \frac{(\pi(\sigma_i - \sigma_j))\sinh(2\pi(\sigma_i - \sigma_j)) - 2(\pi(\sigma_i - \sigma_j))^2}{\sinh^2(\pi(\sigma_i - \sigma_j))} . \quad (3.6)$$

To compute the value of τ_{RR} at the IR fixed point, we extremize the partition function with respect to the R-charge and determine Δ_{IR} and $\tilde{\Delta}_{\mathrm{IR}}$. In the case with $N_c = N_f = \tilde{N}_f = 1$, it is analytically computed to be $\Delta_{\mathrm{IR}} = \tilde{\Delta}_{\mathrm{IR}} = 1/3$ [2]. More generally, we have to rely on numerical computations to fix them [28].

Before discussing the details of the IR fixed points, let us comment on the UV limit of the theory. If we include the usual Yang-Mills action, that is irrelevant in the IR but relevant in the UV, the theory becomes free in the UV with $N_c(N_f + \tilde{N}_f)$ chiral fields and N_c^2 gauge supermultiplets. An $\mathcal{N} = 2$ gauge multiplet has the same propagating degrees of freedom as a chiral multiplet in three-dimensions,[5] so the UV theory may be considered

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[5]Here we neglect the topological degrees of freedom of gauge fields. Such topological degrees of freedom is irrelevant to the correlation function of the stress tensor, but contributes to F.

to be a theory of free $N_c(N_f + \tilde{N}_f + N_c)$ chiral multiplets. However, the stress tensor is not traceless unless we perform an appropriate improvement $T_{\mu\nu} \to T_{\mu\nu} + (\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\mathcal{O}$ which breaks shift symmetries of the scalars dual to gauge fields. The two-point function of the stress tensors takes the form of

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \frac{1}{64\pi^2}\left[(P_{\mu\rho}P_{\nu\sigma} + P_{\mu\sigma}P_{\nu\rho} - P_{\mu\nu}P_{\rho\sigma})\frac{\tau_{RR}}{x^2} + P_{\mu\nu}P_{\rho\sigma}\frac{\tau'_{RR}}{x^2}\right], \qquad (3.7)$$

where we have defined $P_{\mu\nu} = \delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu$. It is clear that τ_{RR} is invariant under the improvement of the stress tensor, while τ'_{RR} is not. We use this definition of τ_{RR} for the UV theory, resulting in

$$\tau_{RR}^{\text{UV}} = \frac{N_c(2\bar{N}_f + N_c)}{4} . \qquad (3.8)$$

3.1 Non-chiral theory

For the non-chiral theories with $\mu = 0$, there is an additional charge conjugation symmetry that exchanges the roles of quarks and anti-quarks. We have only one R-charge to vary: $\Delta = \tilde{\Delta}$.

3.1.1 Large-N_f limit

While the numerical computation of τ_{RR} is to be carried out without undue effort for small N_f as we shall see shortly in the next section, it becomes intractable for large N_f because the value of the partition function gets very small and we have to increase a precision of our numerics to obtain reliable values. In the latter case, however, the large-N_f expansion is a useful analytic method we can employ. The free energies of SQCD on a three-sphere are systematically studied in this limit in [24, 25].

Given an expansion of the IR R-charge around $N_f = \infty$

$$\Delta_{\mathcal{N}=2} = \frac{1}{2} + \frac{\Delta_1}{N_f} + \frac{\Delta_2}{N_f^2} + \cdots, \qquad (3.9)$$

one can minimize the partition function $Z(1)$ with respect to Δ_1 at some order of $1/N_f$ and obtains the value of Δ_1. Iterating this procedure, one finds Δ_2 and so on at a higher order of $1/N_f$. Using the partition function (3.1) on a round three-sphere ($b = 1$), we obtain the R-charge at the IR fixed point up to the order of $1/N_f^2$

$$\Delta_1 = -\frac{2N_c}{\pi^2(\kappa^2 + 1)},$$
$$\Delta_2 = \frac{2\left(\left(3\left(\pi^2 - 4\right)\kappa^2 - 5\pi^2 + 36\right)N_c^2 + \pi^2\left(-3\kappa^4 + 3\kappa^2 + 2\right)\right)}{3\pi^4\left(\kappa^2 + 1\right)^3}, \qquad (3.10)$$

where $\kappa = \frac{2k}{\pi N_f}$. More terms with $\kappa = 0$ are available in [25].

Once the IR R-charge is determined, we can proceed to compute τ_{RR} by substituting eq. (3.9) into eq. (3.4). Performing the similar procedure to the computation of the R-

charge leads

$$\tau_{RR}^{\mathcal{N}=2} = \frac{N_c N_f}{2} + \frac{(32 - 3\pi^2)N_c^2}{6\pi^2(\kappa^2 + 1)}$$
$$+ \frac{N_c}{18\pi^4 (\kappa^2 + 1)^3 N_f} \Big[2 \left(-3\pi^4 \kappa^4 + 3 \left(88 - 58\pi^2 + 3\pi^4 \right) \kappa^2 + 26\pi^2 - 504 \right) N_c^2$$
$$+ \pi^2 \left(12 \left(14 - \pi^2 \right) \kappa^4 + 9\pi^2 \kappa^2 + 9\pi^2 - 40 \right) \Big] + O(1/N_f^2) . \qquad (3.11)$$

The UV value of τ_{RR} is given by $\tau_{RR}^{\mathrm{UV}} = \frac{1}{4} N_c (2N_f + N_c)$ as discussed above and it is always greater than eq. (3.11) in the large-N_f limit.

We have considered the $\mathcal{N} = 2$ cases without superpotential, but we can introduce a superpotential $\lambda (QT^A \tilde{Q})^2$, where T^A are generators of the gauge group $\mathrm{U}(N_c)$. The theory flows to the $\mathcal{N} = 3$ fixed point with $\lambda = 2\pi/k$ in the IR limit. In this case the R-charge is given by $\Delta = 1/2$ and hence

$$\tau_{RR}^{\mathcal{N}=3} = \frac{N_c N_f}{2} + \frac{(8 - \pi^2)N_c^2}{2\pi^2(\kappa^2 + 1)}$$
$$+ \frac{N_c \left(-2 \left(\left(\pi^2 \left(\kappa^2 - 3 \right) + 40 \right) \kappa^2 + 8 \right) N_c^2 + \pi^2 \left(-4\kappa^4 + 3\kappa^2 + 3 \right) + 8 \left(6\kappa^4 + \kappa^2 - 1 \right) \right)}{6\pi^2 (\kappa^2 + 1)^3 N_f}$$
$$+ O(1/N_f^2) . \qquad (3.12)$$

This is always smaller than that for $\mathcal{N} = 2$ theories (3.11) as long as $N_f \gg 1$.

In summary, we consider the following RG flows between the fixed points,

$$\text{UV free theory} \quad \rightarrow \quad \mathcal{N} = 2 \text{ fixed point} \quad \rightarrow \quad \mathcal{N} = 3 \text{ fixed point} ,$$

and see the value of τ_{RR} decreases as

$$\tau_{RR}^{\mathrm{UV}} > \tau_{RR}^{\mathcal{N}=2} > \tau_{RR}^{\mathcal{N}=3} . \qquad (3.13)$$

3.1.2 Numerical computation for small N_f

We will study how τ_{RR} behaves for small N_f where the previous analysis may not be valid. There is no good approximation available, and we have to evaluate the integral given by eq. (3.4) in some way. To determine the correct IR R-charge, we numerically minimized the partition function (3.1) on a round sphere ($b = 1$) with respect to Δ. Then we performed the numerical integrations for τ_{RR} as well using Δ obtained by the Z-minimization. Some of the values are summarized in table 1.

Figure 1 shows a plot of τ_{RR} as a function of N_f for $N_c = 1, 2$ and $k = 0$.[6] The large-N_f approximation (3.11) and (3.12) gives the orange solid and blue dotted curves for

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[6]Note that the case with $N = N_c = N_f$ needs special attention. The $N = 1$ theory flows to the $\mathcal{N} = 3$ (or more precisely $\mathcal{N} = 4$ when $k = 0$) IR fixed point which can be described by a free (twisted) hypermultiplet, leading to $\tau_{RR}^{\mathcal{N}=3} = 2 \cdot \frac{1}{4} = 0.500$. For the $N = 2$ theory, it flows to the $\mathcal{N} = 2$ free theory where six chiral fields have canonical dimension 1/2 [33], consistent with $\tau_{RR}^{\mathcal{N}=2} = 6 \cdot \frac{1}{4} = 1.500$, while the partition function diverges at the $\mathcal{N} = 3$ (or $\mathcal{N} = 4$) fixed point. This may be related to the fact that this theory does not flow to a standard critical point [34] and an accidental symmetry may appear in the IR. We would like to thank I. Yaakov for suggesting this possibility to us.

N_f	1	2	3	4	5	6	7	8
$\Delta_{\mathcal{N}=2}$	1/3	0.4085	0.4369	0.4519	0.4611	0.4674	0.4719	0.4753
$\tau_{RR}^{\mathcal{N}=2}$	0.545	1.042	1.541	2.040	2.540	3.040	3.539	4.039
$\tau_{RR}^{\mathcal{N}=3}$	0.500	0.956	1.439	1.931	2.425	2.922	3.419	3.918

N_f	2	3	4	5	6	7	8
$\Delta_{\mathcal{N}=2}$	1/4	0.3417	0.3851	0.4101	0.4262	0.4375	0.4458
$\tau_{RR}^{\mathcal{N}=2}$	1.500	2.670	3.775	4.844	5.893	6.929	7.956
$\tau_{RR}^{\mathcal{N}=3}$	-	1.939	3.154	4.267	5.337	6.346	7.295

Table 1. The values of $\Delta_{\mathcal{N}=2}$, $\tau_{RR}^{\mathcal{N}=2}$ and $\tau_{RR}^{\mathcal{N}=3}$ for the non-chiral SQCD with $N_c = 1$ [Upper] and $N_c = 2$ [Lower].
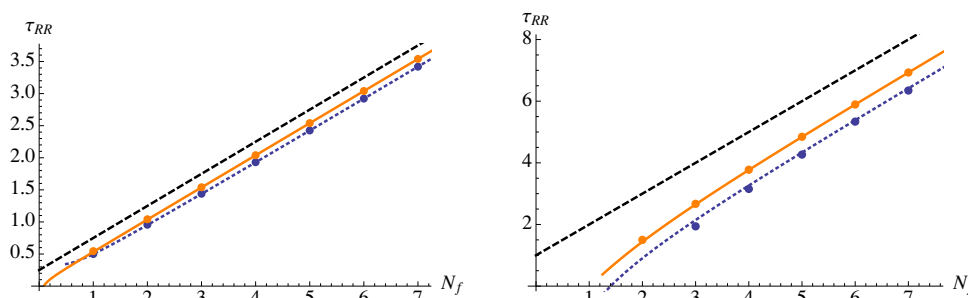
Figure 1. Plots of $\tau_{RR}^{\mathcal{N}=2}$ and $\tau_{RR}^{\mathcal{N}=3}$ as functions of N_f for the non-chiral U(N_c) SQCD of $N_c = 1$ [left] and $N_c = 2$ [right]. The solid orange and dotted blue curves are drawn using the large-N_f expansion at the $\mathcal{N} = 2$ and $\mathcal{N} = 3$ fixed points, respectively. The dashed black lines are the values at the UV fixed point. The orange and blue dots are computed numerically. They fit the large-N_f approximation very well even for small N_f.

the $\mathcal{N} = 2$ and $\mathcal{N} = 3$ fixed points, respectively. The orange and blue dots are plotted numerically. They fit the large-N_f approximation curves very well even for small N_f. The dashed black lines are for the UV fixed point where $\tau_{RR}^{\text{UV}} = N_c(2N_f + N_c)/4$. The black, orange and blue curves are ordered as in eq. (3.13). Our results here show the validity of the large-N_f analysis in the previous section even for small N_f. With these observations, we argue that the monotonicity of τ_{RR} given by eq. (3.13) holds for arbitrary N_c, N_f and k.

3.2 Chiral theory

We have treated the non-chiral case so far, but the previous discussion is easily generalized to the chiral theory. Now there are two independent R-charges Δ and $\tilde{\Delta}$ with respect to which the partition function is minimized. Expanding Δ and $\tilde{\Delta}$ in the large-$\bar{N}_f$ limit and extremizing the partition function on a round sphere term by term, we find [24]

$$\Delta = \frac{1}{2} - \frac{2(1+\mu)N_c}{\pi^2(1+\kappa^2)\bar{N}_f} + O(1/\bar{N}_f^2),$$
$$\tilde{\Delta} = \frac{1}{2} - \frac{2(1-\mu)N_c}{\pi^2(1+\kappa^2)\bar{N}_f} + O(1/\bar{N}_f^2). \tag{3.14}$$

Substituting them into eq. (3.4) leads τ_{RR} of $\mathcal{N} = 2$ chiral SQCD

$$\tau_{RR} = \frac{N_c \bar{N}_f}{2} + \frac{(32 - 3\pi^2)N_c^2}{6\pi^2(1 + \kappa^2)} + O(1/\bar{N}_f) . \tag{3.15}$$

This is smaller than the UV value given by eq. (3.8).

Introducing a superpotential[7] $\mathcal{W} \sim \lambda \sum_{a,b}(Q_a T^A \tilde{Q}_b)^2$ lets the theory flow to another fixed point where the R-charges have to satisfy the constraint $\Delta + \tilde{\Delta} = 1$. Extremizing the partition function under the constraint yields the R-charges [24]

$$\Delta = \frac{1}{2} - \frac{4\mu N_c}{\pi^2(1 + \kappa^2)\bar{N}_f} + O(1/\bar{N}_f^2) ,$$
$$\tilde{\Delta} = 1 - \Delta . \tag{3.16}$$

With these values, we obtain

$$\tau_{RR} = \frac{N_c \bar{N}_f}{2} + \frac{(8(3 + \mu^2) - 3\pi^2)N_c^2}{6\pi^2(1 + \kappa^2)} + O(1/\bar{N}_f) , \tag{3.17}$$

which is less than eq. (3.15) for any μ between 0 and 1. So even in the chiral theory, τ_{RR} decreases along RG flows.

3.3 More general theories

We can extend the large-N_f calculation to general gauge groups $G = \otimes_A G_A$ and general representations of matter fields $\mathcal{R}_I = \otimes_A \mathcal{R}_{I,A}$, where A labels each simple or U(1) gauge group G_A. Here we have no superpotential, and impose technical assumptions on U(1) charges given by eqs. (B.9) and (B.10) (the assumption eq. (B.9) is not satisfied in the chiral U(N) model discussed in the previous subsection). The details are described in appendix B and we sketch the results below. τ_{RR} is generally given by

$$\tau_{RR} = \frac{1}{4}\left[N_{\text{total}} + \frac{2(32 - 3\pi^2)}{3\pi^2}\sum_A \frac{\dim G_A}{1 + \kappa_A^2}\right] + O(1/N_f) , \tag{3.18}$$

where N_{total} is the total number of chiral matter fields (which is given by $N_{\text{total}} = \sum_I N_I \dim \mathcal{R}_I$ in the notation of appendix B), and κ_A is defined in eq. (B.16). This should be compared with the UV value

$$\tau_{RR}^{\text{UV}} = \frac{1}{4}\left[N_{\text{total}} + \sum_A \dim G_A\right] . \tag{3.19}$$

The leading contribution to τ_{RR} in the large-N_f limit is the same in the UV and IR, and the difference appears in the coefficient of $\dim G_A$ in the subleading term. One can see that τ_{RR} is also decreasing in this class of theories.

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[7]This superpotential respects $O(N_f) \times O(\tilde{N}_f)$ flavor symmetries which guarantee that the R-charges are the same among different flavors. One can also consider exactly marginal deformations of this superpotential.

4 RG flow with increasing τ_{RR}

We have considered the RG flow of the gauge theories where τ_{RR} monotonically decreases so far. In this section, however, a Wess-Zumino model we will discuss has an RG flow where τ_{RR} increases.

The model consists of $N + 1$ chiral fields denoted by X and Z_i ($i = 1, \cdots, N$), and we do not introduce any gauge field. We assume that Z_i are in the fundamental representation of a global $O(N)$ symmetry, that allows the following superpotential:

$$\mathcal{W} = X \sum_{i=1}^{N} (Z_i)^2 \ . \tag{4.1}$$

This model has an interacting fixed point due to the superpotential.

In the large-N limit, it is easy to obtain τ_{RR} analytically. First, the R-charges of Z_i and X, denoted as Δ_Z and Δ_X respectively, are obtained by extremizing $F = -N\ell(1 - \Delta_Z) - \ell(1 - \Delta_X)$ under the constraint $2\Delta_Z + \Delta_X = 2$. After a short calculation, we obtain

$$\Delta_Z = \frac{1}{2} + \frac{4}{\pi^2 N} + \frac{32}{\pi^4 N^2} + O(1/N^3) \,, \tag{4.2}$$

$$\Delta_X = 2(1 - \Delta_Z) \ . \tag{4.3}$$

Using these R-charges, the τ_{RR} and the free energy F are calculated in the same way as in the previous section

$$\tau_{RR}^{XZ} = \frac{N}{4} - \frac{4}{3\pi^2} + \left(\frac{68}{9\pi^2} - \frac{48}{\pi^4} \right) \frac{1}{N} + O(1/N^2) \,, \tag{4.4}$$

$$F^{XZ} = \frac{N}{2} \log 2 + \frac{4}{\pi^2 N} + O(1/N^2) \ . \tag{4.5}$$

These τ_{RR} and F are smaller than the value of the UV fixed point $\tau_{RR}^{\text{free}} = \frac{N+1}{4}$ and $F^{\text{free}} = \frac{N+1}{2} \log 2$ where one can neglect the superpotential and there are $N + 1$ free chiral fields.

Next, let us add a mass term to X,

$$\Delta \mathcal{W} = m X^2 \ . \tag{4.6}$$

Integrating out X leads to the theory of N chiral multiplets Z_i with the quadratic superpotential $\mathcal{W}_{\text{IR}} \sim (\sum_{i=1}^{N} (Z_i)^2)^2$. This superpotential is marginally irrelevant and hence the IR theory is just a free theory of N chiral multiplets. The R-charge of Z is $\Delta_Z = \frac{1}{2}$, and the τ_{RR} and F at the IR fixed point is equal to that of N free chiral multiplets:

$$\tau_{RR}^{Z} = \frac{N}{4} \,, \tag{4.7}$$

$$F^{Z} = \frac{N}{2} \log 2 \ . \tag{4.8}$$

Comparing eq. (4.4) and eq. (4.7), we find that τ_{RR} increases for sufficiently large-N under the RG flow, while the free energy F is decreasing as expected from the F-theorem.

N	1	2	3	4	5	6	7	8	9	10
Δ_Z	0.708	0.667	0.632	0.605	0.586	0.572	0.562	0.554	0.548	0.543
τ_{RR}	0.380	0.545	0.741	0.957	1.187	1.423	1.663	1.906	2.151	2.397
F	0.595	0.872	1.174	1.491	1.817	2.150	2.487	2.826	3.166	3.508

Table 2. The values of Δ_Z, τ_{RR} and F for the fixed points of the Wess-Zumino model with the superpotential (4.1).
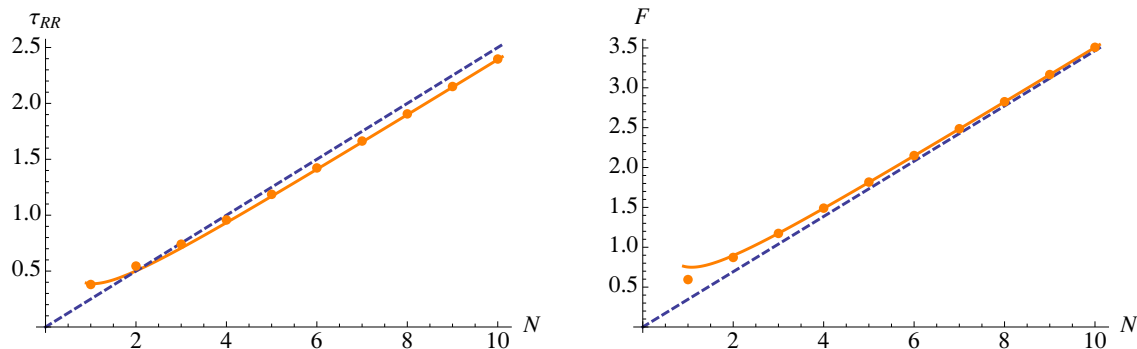
Figure 2. A plot of τ_{RR} as a function of N for the Wess-Zumino model (4.1) [left]. A plot of F for the same model [right]. The solid orange curves are calculated using the large-N expansion at the fixed point with eqs. (4.4) and (4.5). The orange dots are computed numerically. The dashed blue lines are the values at the IR free theory with eqs. (4.7) and (4.8).

Numerical results for small N in table 2 and figure 2 show that τ_{RR}^{XZ} is larger and smaller than τ_{RR}^{Z} for $N \leq 2$ and $N \geq 3$, respectively. On the other hand, the free energy F is always decreasing, consistent with the F-theorem. Therefore, we conclude that these are counter examples to the conjectured C_T-theorem.

Acknowledgments

A Hyperbolic gamma function

The hyperbolic gamma function is defined in the integral form [30]:

$$\Gamma_h\left(z; \xi_1, \xi_2\right) = \exp\left[i \int_0^\infty \frac{dx}{x}\left(\frac{z-\xi}{\xi_1\xi_2 x} - \frac{\sin(2x(z-\xi))}{2\sin(\xi_1 x)\sin(\xi_2 x)}\right)\right], \qquad (A.1)$$

for $z \in \mathbb{C}$ satisfying $0 < \mathrm{Im}(z) < \mathrm{Im}(2\xi)$ where $\xi = (\xi_1 + \xi_2)/2$. For brevity, we use a notation $\Gamma_h[z] \equiv \Gamma_h(z; \xi_1, \xi_2)$.

A useful identity for our purpose is

$$\frac{1}{\Gamma_h[z]\Gamma_h[-z]} = -4\sin\left(\frac{\pi z}{\xi_1}\right)\sin\left(\frac{\pi z}{\xi_2}\right) . \tag{A.2}$$

On a round sphere, i.e., $b = 1$, the hyperbolic gamma function is written in terms of Jafferis's ℓ-function [2]:

$$\Gamma_h(z; i, i) = e^{\ell(1+iz)} , \tag{A.3}$$

where

$$\ell(z) = -z\,\log(1 - e^{2\pi i z}) + \frac{i}{2}\left(\pi z^2 + \frac{\text{Li}_2(e^{2\pi i z})}{\pi}\right) - \frac{i\pi}{12} . \tag{A.4}$$

B Large-N_f expansion for a general class of theories

Let us consider an $\mathcal{N} = 2$ theory with gauge group $G = \otimes_A G_A$, where A labels each simple or U(1) gauge group G_A. We introduce chiral matter fields Φ_I labeled by I in an irreducible representation $\mathcal{R}_I = \otimes_A \mathcal{R}_{I,A}$ of the gauge group, and the R-charge of Φ_I will be denoted as $\Delta_I = \frac{1}{2} - a_I$. We introduce N_I flavors of Φ_I, and consider the limit where each N_I becomes large with the ratio N_I/N_J fixed. We call this limit a large-N_f limit, where N_f represents the order of flavor numbers N_I, i.e., $N_I \sim O(N_f)$.

The partition function of the theory on a squashed S^3 is given by

$$Z(b) = \int \prod_A [\mathcal{D}\sigma_A]_b \prod_I (Z_I(b))^{N_I} , \tag{B.1}$$

where we have defined

$$Z_I(b) = \prod_{\rho \in \mathcal{R}_I} \Gamma_h\left[\omega(\rho(\sigma) + i\Delta_I)\right] , \tag{B.2}$$

and

$$[\mathcal{D}\sigma_A]_b = \frac{1}{\text{Vol}(G_A)} \cdot d^{\dim G_A}\sigma_A \cdot \exp(-i\pi k_A \omega^2 \,\text{Tr}\,\sigma_A^2) \cdot \prod_{\alpha_A \in Ad(G_A)} (\alpha_A(\sigma_A)\Gamma_h\left[\omega\alpha_A(\sigma_A)\right])^{-1} . \tag{B.3}$$

Note that the integral $d^{\dim G_A}\sigma_A$ is over all the $\dim G$ components of σ_A (i.e., not only Cartan subalgebra), and the product $\prod_{\alpha \in Ad(G_A)}$ is taken over all the roots and vanishing weights of the adjoint representation. It is possible to reduce the integral $d^{\dim G_A}\sigma_A$ to an integral over the Cartan subalgebra by gauge-fixing as is in eq. (3.1), but we will find it more convenient to use the above expression in the following calculation.

B.1 *F*-maximization for theories without superpotential

For a round sphere $b = 1$, $Z_I(b)$ and $[\mathcal{D}\sigma_A]_b$ are simplified to

$$Z_I = \prod_{\rho \in \mathcal{R}_I} e^{\ell(1-\Delta_I+i\rho_I)}\,, \tag{B.4}$$

and

$$[\mathcal{D}\sigma_A] = \frac{(2\pi)^{\dim G_A}}{\mathrm{Vol}(G_A)} \cdot d^{\dim G_A}\sigma_A \cdot \exp(-i\pi k_A \mathrm{Tr}\,\sigma_A^2) \cdot \prod_{\alpha_A \in Ad(G_A)} \left(\frac{\sinh(\pi\alpha_A)}{\pi\alpha_A}\right)\,, \quad (B.5)$$

where we have used abbreviation $\rho_I = \rho_I(\sigma)$ and $\alpha_A = \alpha_A(\sigma_A)$ for brevity.

We use saddle point approximation below. To this end, we expand $\ell(1 - \Delta_I + i\rho_I)$ in a power series of ρ_I. From the definition of ρ_I, the following identity holds

$$\sum_{\rho \in \mathcal{R}_I} (\rho_I)^k = \mathrm{Tr}_{\mathcal{R}_I}\,\sigma^k\,. \tag{B.6}$$

It follows that we can rewrite the logarithm of eq. (B.4) as

$$\sum_{\rho \in \mathcal{R}_I} \ell(1 - \Delta_I + i\rho_I) = \sum_{k=0}^{\infty} \frac{i^k}{k!}\,\ell^{(k)}(1 - \Delta_I)\,\mathrm{Tr}_{\mathcal{R}_I}\,\sigma^k\,, \tag{B.7}$$

where $\ell^{(k)}(z)$ is the k-th derivative of $\ell(z)$. Similarly, we expand a part of the measure

$$\prod_{\alpha \in Ad} \left(\frac{\sinh(\pi\alpha_A)}{\pi\alpha_A}\right) = 1 + \frac{\pi^2}{6}\,\mathrm{Tr}_{Ad(G_A)}\,\sigma_A^2 + O(\sigma_A^4)\,. \tag{B.8}$$

We will make the following assumption for simplicity. If the gauge group G contains $U(1)$ (and $U(1)'$) gauge group(s), let q (q') be the $U(1)$ $(U(1)')$ charges of matter fields. Then, we assume[8]

$$\sum_{\text{all matters}} q = 0\,, \tag{B.9}$$

$$\sum_{\text{all matters}} qq' = 0\,. \tag{B.10}$$

See section 3.2 for an example which does not satisfy eq. (B.9).

The matrix σ in the representation $\mathcal{R}_I$ is

$$\sigma_{\mathcal{R}_I} = \sum_A 1_{\mathcal{R}_{I,1}} \otimes \cdots \otimes 1_{\mathcal{R}_{I,A-1}} \otimes \sigma_{\mathcal{R}_{I,A}} \otimes 1_{\mathcal{R}_{I,A+1}} \otimes \cdots\,, \tag{B.11}$$

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[8]In terms of Feynman diagrams, eq. (B.9) forbids 1-loop tadpole diagrams of a U(1) multiplet, while eq. (B.10) forbids 1-loop kinetic mixing diagrams between two U(1) multiplets.

where $1_{\mathcal{R}_{I,A}}$ is the unit matrix in the representation $\mathcal{R}_{I,A}$ and $\sigma_{\mathcal{R}_{I,A}}$ is the matrix σ_A in $\mathcal{R}_{I,A}$. The assumptions (B.9) and (B.10) leads to

$$\sum_I N_I \operatorname{Tr}_{\mathcal{R}_I} \sigma = 0 \,, \tag{B.12}$$

$$\begin{aligned}
\sum_I N_I \operatorname{Tr}_{\mathcal{R}_I} \sigma^2 &= \sum_I N_I \Bigg[\sum_A \frac{\dim \mathcal{R}_I}{\dim \mathcal{R}_{I,A}} \operatorname{Tr}_{\mathcal{R}_{I,A}} \sigma_A^2 \\
&\qquad + \sum_{A \neq B} \frac{\dim \mathcal{R}_I}{\dim \mathcal{R}_{I,A} \dim \mathcal{R}_{I,B}} (\operatorname{Tr}_{\mathcal{R}_{I,A}} \sigma_A)(\operatorname{Tr}_{\mathcal{R}_{I,B}} \sigma_B) \Bigg] \\
&= \sum_I N_I \sum_A \frac{\dim \mathcal{R}_I}{\dim \mathcal{R}_{I,A}} t_{\mathcal{R}_{I,A}} \operatorname{Tr} \sigma_A^2 \\
&\equiv \sum_I N_I \dim \mathcal{R}_I \sum_A \hat{t}_{\mathcal{R}_{I,A}} \operatorname{Tr} \sigma_A^2 \,. \tag{B.13}
\end{aligned}$$

Here $t_{\mathcal{R}_{I,A}}$ is the Dynkin index of the representation $\mathcal{R}_{I,A}$ defined as $\operatorname{Tr}_{\mathcal{R}_{I,A}}(T_a T_b) = t_{\mathcal{R}_{I,A}} \delta_{ab}$, where gauge group generators T_a are assumed to be normalized as $\operatorname{Tr}(T_a T_b) = \delta_{ab}$. We have also defined $\hat{t}_{\mathcal{R}_{I,A}} = t_{\mathcal{R}_{I,A}} / \dim \mathcal{R}_{I,A}$ for simplicity.

As we will see, $a_I = 1/2 - \Delta_I$ is of order $1/N_f$. Then, up to the first subleading corrections to the partition function, we obtain

$$\begin{aligned}
Z = \text{const.} \int \prod_A d^{\dim G_A} \sigma_A \cdot \exp &\Bigg[-\sum_A \left(\frac{\pi^2}{4} \sum_I \hat{t}_{\mathcal{R}_{I,A}} N_I \dim \mathcal{R}_I + i\pi k_A \right) \operatorname{Tr} \sigma_A^2 \Bigg] \\
\cdot \Bigg(1 + \frac{\pi^2}{4} \sum_I a_I^2 N_I \dim \mathcal{R}_I &- \pi^2 \sum_I \sum_A a_I N_I \dim \mathcal{R}_I \hat{t}_{\mathcal{R}_{I,A}} \operatorname{Tr} \sigma_A^2 \\
&+ (a_I\text{-independent terms}) + O(N_f^{-2}) \Bigg) \,, \tag{B.14}
\end{aligned}$$

where we have used $a_I \sim O(N_f^{-1})$ and $\sigma^k \sim O(N_f^{-k/2})$. We have also used $\ell'(1/2) = 0$, $\ell''(1/2) = \pi^2/2$ and $\ell'''(1/2) = 2\pi^2$.

The above integral is just a gaussian integral and can be done easily. From the F-maximization $\partial \Re[\log Z]/\partial a_I = 0$, we obtain

$$a_I = \frac{4}{\pi^2} \sum_A \frac{\dim G_A}{1 + \kappa_A^2} \left(\frac{\hat{t}_{\mathcal{R}_{I,A}}}{\sum_J \hat{t}_{\mathcal{R}_{J,A}} N_J \dim \mathcal{R}_J} \right) + O(N_f^{-2}) \,, \tag{B.15}$$

where we have defined

$$\kappa_A = \frac{4 k_A}{\pi \sum_J \hat{t}_{\mathcal{R}_{J,A}} N_J \dim \mathcal{R}_J} \,. \tag{B.16}$$

B.2 Computation of τ_{RR}

Next we discuss the computation of τ_{RR}. First, we define the expectation value of a function $f(\sigma)$ as

$$\langle f(\sigma) \rangle = \frac{\int \prod_A [\mathcal{D}\sigma_A] \prod_I (Z_{\mathcal{R}_I})^{N_I} f(\sigma)}{\int \prod_A [\mathcal{D}\sigma_A] \prod_I (Z_{\mathcal{R}_I})^{N_I}} \,. \tag{B.17}$$

Using this notation, τ_{RR} is given by

$$\tau_{RR} = \Re \left\langle \sum_I N_I f_{\mathcal{R}_I}(1 - \Delta_I, \sigma) + \sum_A g_A(\sigma) \right\rangle, \qquad \text{(B.18)}$$

where we have defined

$$f_{\mathcal{R}_I}(z, \sigma) = \frac{2}{\pi^2} \sum_{\rho \in \mathcal{R}_I} \int_0^\infty dx \left[z \left(\frac{1}{x^2} - \frac{\cosh(2x(z + i\rho_I))}{\sinh^2(x)} \right) \right.$$
$$\left. + \frac{(\sinh(2x) - 2x)\sinh(2x(z + i\rho_I))}{2\sinh^4(x)} \right], \qquad \text{(B.19)}$$

$$g_A(\sigma) = -\frac{1}{\pi^2} \sum_{\alpha_A \in Ad(G_A)} \frac{(\pi\alpha_A)\sinh(2\pi\alpha_A) - 2(\pi\alpha_A)^2}{\sinh^2(\pi\alpha_A)}. \qquad \text{(B.20)}$$

One can check that $\langle g_A(\sigma) \rangle$ starts from the order N_f^{-1} and we neglect it in this appendix. On the other hand, by expanding $f_{\mathcal{R}_I}(1 - \Delta_I, \sigma)$ in terms of a_I and σ, some computation yields

$$f_{\mathcal{R}_I}(1 - \Delta_I, \sigma) = \dim \mathcal{R}_I \left(\frac{1}{4} + \frac{1}{3}a_I \right) + \left(2 - \frac{\pi^2}{4} \right) \mathrm{Tr}_{\mathcal{R}_I} \sigma^2 + O(\sigma^4). \qquad \text{(B.21)}$$

In the saddle point approximation used in the previous subsection, we obtain

$$\left\langle \sum_I N_I \mathrm{Tr}_{\mathcal{R}_I} \sigma^2 \right\rangle = \frac{2}{\pi^2} \sum_A \frac{\dim G_A}{1 - i\kappa_A} + O(N_f^{-1}). \qquad \text{(B.22)}$$

Therefore, τ_{RR} is given by

$$\tau_{RR} = \frac{1}{4} \left[\sum_I \left(1 + \frac{4}{3}a_I \right) N_I \dim \mathcal{R}_I + \frac{2(8 - \pi^2)}{\pi^2} \sum_A \frac{\dim G_A}{1 + \kappa_A^2} \right] + O(N_f^{-1}). \qquad \text{(B.23)}$$

For theories without superpotential, the R-charges are determined in eq. (B.15). In this case, τ_{RR} is given by

$$\tau_{RR} = \frac{1}{4} \left[\sum_I N_I \dim \mathcal{R}_I + \frac{2(32 - 3\pi^2)}{3\pi^2} \sum_A \frac{\dim G_A}{1 + \kappa_A^2} \right] + O(N_f^{-1}). \qquad \text{(B.24)}$$

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