

## Gluing branes — II: flavour physics and string duality

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ABSTRACT: Recently we discussed new aspects of degenerate brane configurations, which can appear in the context of heterotic strings, perturbative type II, or  $M/F$ -theory. Here we continue our study of degenerate brane configurations, focussing on two applications. First we show how the notion of gluing can be viewed as a tool to engineer flavour structures in  $F$ -theory and type IIB, such as models with bulk matter and with Yukawa textures arising from the holomorphic zero mechanism. We find that there is in principle enough structure to solve some of the major flavour problems without generating exotics. In particular, we show how this addresses the  $\mu$ -problem, doublet/triplet splitting and proton decay. Secondly, we describe the Fourier-Mukai transform of heterotic monad constructions, which occur in the large volume limit of heterotic linear sigma model vacua. Degenerate structures again often appear. One may use this to explore strong coupling phenomena using heterotic/ $F$ -theory duality.

KEYWORDS: F-Theory, GUT, Superstring Vacua, String Duality

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**Contents**

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Flavour structure and degeneration	2
1.2	Strong coupling phenomena in $F$ -theory	4
<b>2</b>	<b>Bulk matter revisited</b>	<b>5</b>
2.1	SO(10)-models	5
2.1.1	Review of the problem of exotics	5
2.1.2	Turning on the gluing morphism	7
2.1.3	Building examples	9
2.1.4	Emerging flavour structures	13
2.2	$E_6$ -models	13
2.3	IIa/M-theory models	14
<b>3</b>	<b>Flavour structures</b>	<b>15</b>
3.1	Structures from D-terms	15
3.2	Textures from F-terms: holomorphic zeroes	16
3.3	A model of flavour, with Higgs fields in the bulk	19
3.3.1	Basic picture of the configuration	19
3.3.2	Consistency	21
3.3.3	Phenomenological properties	23
<b>4</b>	<b>The standard embedding</b>	<b>25</b>
4.1	General comments	25
4.2	The tangent bundle of a $K3$ surface	26
4.3	Derivation of the Fourier-Mukai transform	28
4.4	Calculation of the spectrum	31
<b>5</b>	<b>Linear sigma models</b>	<b>32</b>
5.1	General comments on bundles over elliptic fibrations	33
5.2	Jordan blocks	36
5.3	Review of linear sigma models	39
5.4	Monads over an elliptic three-fold	41
5.4.1	Summary of the algorithm	41
5.4.2	Derivation and example	42

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## 1 Introduction

The present paper is a continuation of our study of degenerate brane configurations. In [1] we pointed out the important role of the gluing morphism. In reference [2] we considered theoretical aspects more systematically, including and generalizing configurations studied in earlier work such as [3–11]. Amongst others, in reference [2] we discussed exact sequences associated to gluing operations, aspects of stability and the hermitian Yang-Mills-Higgs metric, and walls of marginal stability. In the present paper we focus on certain applications.

There were in fact two independent lines of inquiry which motivated us to take a closer look at degenerate configurations. The first motivation is to get a more systematic understanding of flavour structure in  $F$ -theory, or more generally models with intersecting branes. The second is to get a better understanding of certain string dualities. It will hopefully become clear that degenerate structures are ubiquitous and critical for the phenomenology of string compactifications. With the tools developed in part I, we have at least in principle everything we need to analyze them.

### 1.1 Flavour structure and degeneration

One of the main motivations for top-down models is the possibility to get some understanding of the origin of flavour structure. In particular, there are various hints that flavour should be generated near the GUT scale. So it is a natural question to ask what kind of flavour structures can naturally occur in Kaluza-Klein GUT models. In such models, the higher dimensional theory is rather constrained (in order for it to have a known UV completion), and much of the four-dimensional physics can be traced back to the geometry of the compactification. Flavour structure gets related to the geometric properties of wave functions in the extra dimensions.

Let us consider this question in  $F$ -theory. Here one grows four extra dimensions at the GUT scale. Thus we focus on an eight-dimensional gauge theory which is compactified on a four-manifold  $S$  down to four dimensions. Supersymmetric configurations are given by  $K_S$ -valued Higgs bundles on a complex manifold  $S$ . In generic  $SU(5)_{\text{GUT}}$  models, the gauge fields propagate in the bulk of  $S$  and the  $\mathbf{10}$  and  $\bar{\mathbf{5}}$  matter fields are confined to Riemann surfaces on  $S$ , called the matter curves.

Using a general equivalence between supersymmetric ALE fibrations, Higgs bundles and spectral covers [12, 13], such local models may be represented by a configuration of holomorphic 7-branes in an auxiliary Calabi-Yau three-fold, or as an ALE-fibration over  $S$  with flux. For  $SU(5)$  GUT models this ALE fibration is generically of the form:

$$y^2 = x^3 + b_0 z^5 + b_2 z^3 x + b_3 z^2 y + b_4 z x^2 + b_5 x y \quad (1.1)$$

where the  $b_i$  are complex polynomials on  $S$ . Matter in the  $\mathbf{10}$  or  $\bar{\mathbf{10}}$  is confined to the Riemann surface given by  $b_5 = 0$ , and matter in the  $\bar{\mathbf{5}}$  or  $\mathbf{5}$  is confined to  $b_0 b_5^2 - b_2 b_3 b_5 + b_3^2 b_4 = 0$ . We have explained elsewhere how to embed such models in a global compactification [12], effectively by using Tate's algorithm [14, 15] in reverse. However, in this paper we will be interested in aspects of flavour which must already be present in the local model.

With suitable Noether-Lefschetz fluxes, the moduli appearing in the  $b_i$  are stabilized at isolated critical points of the superpotential. We have estimated that one can construct at least  $10^{1000}$  models of this form with the spectrum of the MSSM.

Now a priori one might have thought that for generic values of the moduli there is no flavour structure whatsoever. It appears that the situation is actually better than that. The main point is that the degree (and hence the volume) of the matter curve for the  $\mathbf{5}$  and  $\bar{\mathbf{5}}$  is larger than the degree of the matter curve of the  $\mathbf{10}$ . Then one should expect that upon proper normalization of the kinetic terms, the  $\mathbf{10} \cdot \bar{\mathbf{5}} \cdot \bar{\mathbf{5}}$  down Yukawas are slightly suppressed compared to the  $\mathbf{10} \cdot \mathbf{10} \cdot \mathbf{5}$  up type Yukawas [16], by a factor

$$\lambda_d/\lambda_u \sim \sqrt{\text{deg}_{\mathbf{10}} / \text{deg}_{\bar{\mathbf{5}}}} \tag{1.2}$$

This goes clearly in the right direction and is much better than the situation in type IIb. Type IIb GUT models actually also exhibit a flavour structure, but it predicts that the top quark Yukawa coupling is exponentially suppressed with respect to the down coupling in the natural expansion parameter, the string coupling constant.

In any case, even if the above idea is correct, the hierarchy is not parametric (at least if we keep the degrees of the  $b_i$  fixed), and such a generic model does not seem to be able to naturally explain any additional flavour structures. It is clear that we need some extra structure, and quite a number of ideas have been explored in the literature.

The basic idea for getting extra structure is to degenerate the generic models in some way. In  $F$ -theory, the most obvious ingredients we can degenerate are the flavour branes, by varying the  $b_i$ . For instance for certain degenerations of the flavour branes one may get an extra light  $U(1)$  symmetry [17], which imposes selection rules on the Yukawa couplings. For other degenerations of the flavour branes, one may get matter in the bulk of a 7-brane, instead of on the intersection of two 7-branes, and again this implies extra flavour structure. Models with bulk matter were previously studied in [18, 19], but the results were not too encouraging.

Now when one considers degenerate configurations, one must be careful to include all the ingredients, as the rules are a little less obvious than for generic configurations. Indeed as found in [1], even the simplest possible flavour structures were not correctly understood: it turns out that the gluing data was missed, even though this appears very naturally in degenerations of more generic models.

The first half of the present paper focuses on flavour structures, using the improved understanding of degenerate configurations developed in part I. The main goal of these sections is to show that there is in principle enough structure to solve the major flavour problems, without generating exotics and destroying unification.

In particular, our improved understanding shows how to implement the holomorphic zero mechanism in intersecting brane configurations. The idea of using such holomorphic zeroes in  $F$ -theory (or brane configurations) was already discussed in [20, 21], but we believe the idea was not used to its full extent, and a few aspects had not been clear. In section 3.3 we will further show how one can simultaneously address the problems of  $R$ -parity, dimension five proton decay, the  $\mu$ -problem, doublet/triplet splitting and a

simple flavour hierarchy in  $F$ -theoretic GUTs (or the corresponding heterotic models). In particular, we will engineer a superpotential of the form

$$W = \mathbf{10}_m \mathbf{10}_m \mathbf{5}_h + \frac{\langle X \rangle}{M} \mathbf{10}_m \bar{\mathbf{5}}_m \bar{\mathbf{5}}_h + \frac{\langle X \rangle}{M^2} \mathbf{10}_m \mathbf{10}_m \mathbf{10}_m \bar{\mathbf{5}}_m \quad (1.3)$$

in which no  $R$ -parity violating terms appear.  $R$ -parity however is not preserved in the Kähler potential. Thus such models ultimately do predict some form of  $R$ -parity violation, which could have very interesting phenomenological consequences.

## 1.2 Strong coupling phenomena in $F$ -theory

$F$ -theory is only understood perturbatively as a large volume expansion. Clearly it would be of interest to get a better non-perturbative understanding, and explore the theory in other corners of the Kähler moduli space. The best available tool for this is heterotic/ $F$ -theory duality. Comparing BPS states yields the identification

$$\lambda_8 = V_{\mathbf{P}^1} \quad (1.4)$$

where  $\lambda_8$  is the eight-dimensional heterotic string coupling and  $V_{\mathbf{P}^1}$  is the volume of the base of the elliptically fibered  $K3$  on the  $F$ -theory side, measured in Planck units.  $F$ -theory is weakly coupled in the limit of large  $V_{\mathbf{P}^1}$ , and the heterotic string is weakly coupled in the limit of small  $V_{\mathbf{P}^1}$ .

One important technique for constructing heterotic vacua is the linear sigma model, which yields monad constructions in the geometric regime. In such models, the  $(0, 2)$  CFT is relatively well-understood and can be extrapolated to corners of the Kähler moduli space where curvatures are large. Such models also exhibit a number of interesting dualities. Now in order to map this to an  $F$ -theory model we need the associated spectral cover, and it turns out that the spectral cover for heterotic monad constructions is often degenerate [22]. (Our investigations actually indicate this is not the general situation, and the examples that were worked out were just too special.)

The apparent conclusion in the nineties was that the  $F$ -theory duals would be sick in some way. We would like to emphasize that this is not the case, although it is true that the  $11d$  supergravity description of  $F$ -theory can be problematic. As explained in [2], a degenerate cover gives rise to a smooth  $8d$  non-abelian gauge theory configuration provided a suitable stability condition is satisfied, and therefore many configurations with degenerate spectral covers make perfect sense in  $F$ -theory. However to understand the  $F$ -theory dual it is crucial that we obtain the spectral sheaf rather than the spectral cover, and the spectral sheaf for monad bundles has hitherto not been understood.

This gives us the second reason to revisit degenerate covers. Interesting enough, the degenerations that we study in the context of flavour also show up quite naturally as the Fourier-Mukai transform of standard heterotic constructions, such as the standard embedding and the construction of bundles by extension, even though they correspond to smooth solutions of the hermitian Yang-Mills equations. The fact that the standard embedding gives rise to such structures seems to us so fundamental, that we will spend a whole section examining a particular example. Recall also from part I that degenerate structures play

an important role in understanding walls of marginal stability. Altogether this serves to illustrate that many degenerate configurations are not only perfectly acceptable, they are in fact a critical aspect of the phenomenology of string compactifications, because they tend to imply additional structure.

It is clear that there is a lot of interesting work to be done comparing heterotic linear sigma model vacua with  $F$ -theory. In order to finish this paper in a finite amount of time however, we will only focus on establishing the technology. To this end, we will give a prescription for deriving the complete spectral data of monad bundles, extending previous work of [22, 23].

## 2 Bulk matter revisited

### 2.1 SO(10)-models

One of the main motivations for this project was to reconsider the issue of chiral matter in the bulk of a 7-brane in light of the gluing morphism. Bulk matter in  $F$ -theory models was originally discussed in [18, 19].

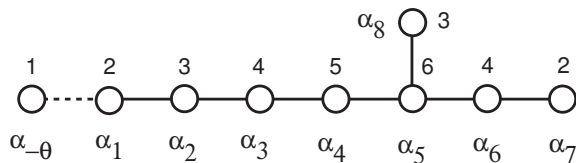
Apart from the interest in bulk matter and flavour structure, according to the analysis in [12] it is also much easier to decouple gravity and hidden sectors if additional sheets of the spectral cover coincide with  $S$ . Even for SO(10) models the constraints coming from the GUT divisor being contractible are already much weaker than for SU(5) models.

However if the gluing morphisms vanish, it turns out to be difficult to avoid exotics, and the constraints on the interactions can be too stringent. To exemplify this, in the first part of this section we would like to focus on SO(10) models with the breaking of the GUT group done by fluxes. For such models, it was argued by Beasley, Heckman and Vafa [24] that there are always exotics in the bulk. In retrospect, the argument assumed that the gluing morphism on  $C_1 \cap C_4$  vanishes.

In spite of the length of this section, our main point is very simple: once the gluing morphism is non-zero, modes in the bulk and on the matter curves are not independent. In the special class of examples that we will consider, where the gluing morphism can be turned off continuously, one can see explicitly how the bulk exotics can (and generically will) pair up with modes that are localized on a matter curve as we turn on the gluing. In the following discussion, we explain how this works in detail, and address some additional issues arising for degenerate configurations along the way.

#### 2.1.1 Review of the problem of exotics

Let us therefore reexamine SO(10) models. The terminology is perhaps slightly misleading, because we never have an unbroken  $4d$  SO(10) group. What we mean is that the  $F$ -theory geometry has an SO(10) singularity along the GUT cycle, with the remaining breaking due to non-trivial  $G$ -flux. In terms of the  $E_8$  gauge theory, this means that the spectral cover  $C_5$  for the  $SL(5, \mathbf{C})$  Higgs bundle which breaks  $E_8$  to  $SU(5)_{\text{GUT}}$  is reducible, i.e. we have  $C_5 = C_4 \cup C_1$  where  $C_1 = S$  and  $C_4$  is a non-trivial spectral cover for  $SU(4) \subset E_8$ . Such models can be analyzed in two steps, first breaking  $E_8$  to SO(10), where we use the



**Figure 1.** The extended  $E_8$  Dynkin diagram and Dynkin indices.

decomposition

$$248 = (\mathbf{1}, \mathbf{45}) + (\mathbf{15}, \mathbf{1}) + (\mathbf{4}, \mathbf{16}) + (\bar{\mathbf{4}}, \bar{\mathbf{16}}) + (\mathbf{6}, \mathbf{10}) \tag{2.1}$$

Even though the spectral cover is reducible, the spectral sheaf and hence the  $SL(5, \mathbf{C})$  Higgs bundle need not be reducible. The remaining breaking to  $SU(5)$  and further to  $SU(3) \times SU(2) \times U(1)$  is done by the spectral sheaf.

Ignoring the fluxes, such a configuration leaves two unbroken  $U(1)$ 's that commute with the Standard Model gauge group. We label the coroots of  $E_8$  by  $\omega_i$ , where  $\omega_i(\alpha_j) = \delta_{ij}$ . The structure group of our spectral cover is  $W_{A_3}$ , the Weyl group generated by the Weyl reflections associated to  $\{\alpha_{-\theta}, \alpha_1, \alpha_2\}$ , see figure 1. Written in terms of the  $\omega_i$ , we can take the unbroken  $U(1)$ 's to be

$$\omega_Y = \omega_4 - \frac{5}{6}\omega_5, \quad \omega_{B-L} = \omega_3 - \frac{4}{5}\omega_4 \tag{2.2}$$

We normalized  $B-L$  so that the right-handed neutrino has charge one. The matter curves of a generic  $SU(5)$  model split up in the following way:

$$\begin{aligned} S \cdot C_5 &= S \cdot S + \Sigma_{\mathbf{16}} = -c_1 + (2c_2 - t) = c_1 - t \\ S \cdot C_{10} &= \Sigma_{\mathbf{10}_v} + \Sigma_{\mathbf{16}} = (6c_1 - 2t) + (2c_1 - t) = 8c_1 - 3t \end{aligned} \tag{2.3}$$

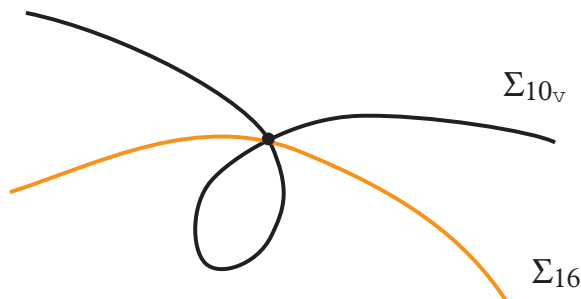
where  $c_1 = c_1(T_S)$  and  $t = c_1(N_S)$ . Thus the  $\mathbf{10}$  will split up into a  $\mathbf{10}_{-4/5}$  localized in the bulk and a  $\mathbf{10}_{1/5}$  localized on  $\Sigma_{\mathbf{16}}$ . Similarly the  $\bar{\mathbf{5}}$  splits up into a  $\bar{\mathbf{5}}_{2/5}$  localized on  $\Sigma_{\mathbf{10}_v}$  and a  $\bar{\mathbf{5}}_{-3/5}$  localized on  $\Sigma_{\mathbf{16}}$ . This of course fits neatly into  $SO(10)$  representations, as follows:

$$\begin{aligned} \Sigma_{\mathbf{16}} : \quad \mathbf{16} &= \mathbf{1}_1 + \mathbf{10}_{1/5} + \bar{\mathbf{5}}_{-3/5} \\ \Sigma_{\mathbf{10}_v} : \quad \mathbf{10} &= \bar{\mathbf{5}}_{2/5} + \mathbf{5}_{-2/5} \\ \text{bulk} : \quad \mathbf{45} &= \mathbf{24}_0 + \mathbf{1}_0 + \mathbf{10}_{-4/5} + \bar{\mathbf{10}}_{4/5} \end{aligned} \tag{2.4}$$

See figure 2. We will denote the singlets with charge one by  $N$  and the singlets with charge minus one by  $\bar{N}$ .

With these decompositions, we can look for exotic matter. We denote the restriction of the  $U(1)_Y$  line bundle to  $S$  by  $L_Y$ , and the restriction of the  $U(1)_{B-L}$  line bundle to  $S$  by  $L_{B-L}$ . The first part of the argument is the same as for  $SU(5)_{\text{GUT}}$  models with hypercharge flux. On a del Pezzo surface  $dP$ , for any line bundle  $M$  we have

$$\chi(dP, M) = 1 + \frac{1}{2}c_1(M)^2 - \frac{1}{2}c_1(M) \cdot c_1(K) \tag{2.5}$$



**Figure 2.** The curve on which the  $\bar{\mathbf{5}}$  or  $\mathbf{5}$  matter of an  $SU(5)$  GUT propagates has factorized into two pieces, but the modes on these two curves are not independent if the gluing morphism is non-zero. Similarly, modes in the  $\mathbf{10}$  or  $\bar{\mathbf{10}}$  of  $SU(5)$  seem to originate from the bulk or from a matter curve, but are not independent when the gluing morphism is non-zero.

We consider the off-diagonal modes  $(\mathbf{3}, \mathbf{2})_{5/6,0}$  from breaking the adjoint of  $SU(5)_{\text{GUT}}$  to  $SU(3)_c \times SU(2)_w \times U(1)_Y$ . These modes live purely in the bulk of  $S$ , so  $L_Y^{5/6}$  is integer quantized. The number of  $(\mathbf{3}, \mathbf{2})_{5/6,0}$  modes is counted by  $-\chi(dP, L_Y^{5/6})$ , and absence of the  $(\mathbf{3}, \mathbf{2})_{5/6,0}$  modes and their conjugates then requires that

$$\chi(dP, L_Y^{5/6}) + \chi(dP, L_Y^{-5/6}) = 2 \left( 1 + \frac{1}{2} c_1(L_Y^{5/6})^2 \right) = 0. \tag{2.6}$$

This implies the restrictions  $c_1(L_Y^{5/6}) \cdot c_1(K_S) = 0$  and  $c_1(L_Y^{5/6})^2 = -2$ , well-known from hypercharge flux breaking of  $SU(5)_{\text{GUT}}$  models [24, 25].

Now in addition consider the bulk modes coming from the  $\bar{\mathbf{10}}_{4/5}$ . Under  $SU(3)_c \times SU(2)_w \times U(1)_Y \times U(1)_{B-L}$  they split up as

$$\bar{\mathbf{10}}_{4/5} = (\bar{\mathbf{3}}, \mathbf{2})_{-1/6,4/5} + (\mathbf{1}, \mathbf{1})_{-1,4/5} + (\mathbf{3}, \mathbf{1})_{2/3,4/5} \tag{2.7}$$

It follows that the first Chern class of  $Q \equiv L_Y^{1/6} \otimes L_{B-L}^{-4/5}$  must also be integer quantized. Furthermore, absence of states in the  $\bar{\mathbf{10}}_{4/5}$  requires that

$$\begin{aligned} \chi(dP, Q^{-1} \otimes L_Y^{-5/6}) &= 0 \\ \chi(dP, Q^{-1}) &= 0 \\ \chi(dP, Q^{-1} \otimes L_Y^{5/6}) &= 0 \end{aligned} \tag{2.8}$$

By considering the linear combination (line 1 + line 3 - 2 × line 2), we find that  $c_1(L_Y^{5/6})^2 = 0$ , a contradiction [24]. The argument can even be extended to slightly more general configurations, because the same algebra works when  $Q^{-1}$  is a higher rank bundle.

### 2.1.2 Turning on the gluing morphism

The above argument implicitly assumes the vanishing of the gluing morphism, here the VEV of the field  $\bar{N}$ . When this VEV is non-zero, it is no longer true that the bulk modes are counted by (2.8). However the zero modes are still counted by infinitesimal



deformations of the Higgs bundle, and for the case at hand this specializes to the usual Ext groups known from generic models [12]. The only difference is that the support of the spectral sheaf has degenerated, and we need to take some more care in the computation.

As we discussed in section 2.7 of part I, generally the gluing morphism is only a meromorphic section of  $L_1^\vee \otimes L_2|_\Sigma$ , and in this case we need to use a sequence of the form

$$0 \rightarrow \mathcal{L} \rightarrow i_{1*}L_1 \oplus i_{2*}L_2 \rightarrow i_{\Sigma*}L_\Sigma \rightarrow 0 \tag{2.9}$$

to find the spectrum. Let us assume here that it is given by a holomorphic section, so that we can use the extension sequence

$$0 \rightarrow i_{2*}L_2(-\Sigma) \rightarrow \mathcal{L} \rightarrow i_{1*}L_1 \rightarrow 0 \tag{2.10}$$

instead. Although not the general case, it is already sufficient to show that the exotics can get lifted, and simplifies the calculations.

Then the correct way to count the zero modes proceeds as in section 2.7 of part I. As the gluing VEV is holomorphic, we may turn it off (while ignoring  $D$ -flatness), compute the modes in the bulk and on the matter curve, and then turn the gluing VEV back on. Then there are effectively two changes in the computation of exotics. First, the bulk spectrum is not necessarily computed by (2.8), because  $D$ -flatness is not satisfied when we turn off the gluing VEV. Secondly, candidate bulk modes may be lifted by a superpotential coupling between the modes in the bulk and the modes on the matter curve:

$$\overline{\mathbf{10}}_{4/5} \mathbf{10}_{1/5} \bar{N}_{-1} \tag{2.11}$$

when we turn on a VEV for the gluing morphism  $\bar{N}$ . (More precisely, we should decompose this coupling under the Standard Model gauge group, because the  $SU(5)_{\text{GUT}}$  is broken explicitly by the fluxes. We will see this in more detail below.)

Although plausible, one should not assume that the above superpotential coupling is automatically present. Indeed we will see examples where certain couplings are generically absent despite the fact that they are allowed by the gauge symmetries, and we will make good use of that to solve the mu-problem. Thus one of the main things to check below is that the above superpotential coupling is indeed present in the brane configuration.

Although our main interest here is in the  $\mathbf{10}$  and  $\overline{\mathbf{10}}$ , there is a similar story for the  $\mathbf{5}$  and  $\bar{\mathbf{5}}$  fields. Ignoring the hypercharge flux to simplify the discussion, these modes are computed by

$$\text{Ext}^p(\mathcal{O}_S, i_{10*}L_{C_{10}}) \tag{2.12}$$

as usual. The sheaf  $L_{C_{10}}$  is supported on a ten-fold covering  $C_{10}$ . The only difference with the generic case is that  $C_{10}$  happens to be reducible, and so we should apply the discussion in section 2.7 of part I to compute these zero modes. The cover  $C_{10}$  splits as a six-fold covering  $C_6$ , and a fourfold-covering  $C_4$  which we have already met above. Thus the  $\mathbf{5}$  and  $\bar{\mathbf{5}}$  fields are supported on two seemingly different matter curves, the  $\bar{\mathbf{5}}_{2/5}$  on  $S \cap C_6 = \Sigma_{\mathbf{10}_v}$ , and the  $\bar{\mathbf{5}}_{-3/5}$  on  $S \cap C_4 = \Sigma_{\mathbf{16}}$ . However with a non-vanishing gluing VEV, the zero modes on these two curves are not independent, and must be glued along the intersection

$\Sigma_{10_v} \cap \Sigma_{16}$ . Let us spell this out in some detail. In our case there is the additional complication that the curve  $\Sigma_{10_v}$  has a double point singularity at the intersection, and there is some intricate group theory involved. This is a known behaviour which is dealt with by lifting to the normalization. In the following we assume a simple intersection between smooth curves.

Then, abstractly we have a sheaf  $N$  on  $\Sigma = \Sigma_1 \cup \Sigma_2$ , restricting to  $N_1$  and  $N_2$  respectively, and we are interested in

$$H^0(\Sigma, N) \tag{2.13}$$

Using the long exact sequence, we find

$$\begin{aligned} 0 \rightarrow H^0(\Sigma_2, N_2(-p)) \rightarrow H^0(\Sigma, N) \rightarrow H^0(\Sigma_1, N_1) \\ \rightarrow H^1(\Sigma_2, N_2(-p)) \rightarrow H^1(\Sigma, N) \rightarrow H^1(\Sigma_1, N_1) \rightarrow 0 \end{aligned} \tag{2.14}$$

The intersection  $\Sigma_1 \cap \Sigma_2$  is a number of points, and the gluing morphism (restricted to  $\Sigma$ ) is a complex number for each intersection point  $p$ . For simplicity we assume there is only a single intersection point, the generalization being obvious. We also have a second long exact sequence on  $\Sigma_2$ :

$$\begin{aligned} 0 \rightarrow H^0(\Sigma_2, N_2(-p)) \rightarrow H^0(\Sigma_2, N_2) \rightarrow H^0(p, N_2|_p) \\ \rightarrow H^1(\Sigma_2, N_2(-p)) \rightarrow H^1(\Sigma_2, N_2) \rightarrow 0 \end{aligned} \tag{2.15}$$

Now the coboundary map in (2.14) representing the Yukawa coupling is given as follows. We take a section  $s_1 \in H^0(\Sigma_1, N_1)$ , restrict it to  $p$ , multiply by the gluing morphism (a complex number) to get a generator in  $H^0(p, N_2|_p)$ , and then compose with the coboundary map in (2.15). Then we see that if this map is zero, so that  $s_1$  is not lifted, then it defines a generator in  $H^0(p, N_2|_p)$  which can be extended to a section  $s_2 \in H^0(\Sigma_2, N_2)$ , the extension is unique up to sections which vanish at  $p$ , and  $s_1$  and  $s_2$  agree at  $p$  up to multiplication by the gluing morphism. From (2.14) we further see that the remaining generators of  $H^0(\Sigma, N)$  are sections  $s_2$  of  $N_2$  that vanish at  $p$ . Thus we derived from first principles the expected statement: generators of  $H^0(\Sigma, N)$  are given by pairs of sections  $(s_1, s_2)$  on  $\Sigma_1$  and  $\Sigma_2$  separately, which agree at the intersection  $p$  up to a complex number (the value of the gluing morphism). Thus the gluing condition will normally eliminate some candidate  $\mathbf{5}$  and  $\bar{\mathbf{5}}$  zero modes, and allows the zero modes to spread over the union of  $\Sigma_{10_v}$  and  $\Sigma_{16}$  if they do not vanish at the intersection.

The resulting wave functions for zero modes look a bit singular. This is an artefact of the purely holomorphic description that we used to construct them. When the Higgs bundle is stable, the actual solution of Hitchin's equations and the wave functions for the first order deformations should be smooth.

### 2.1.3 Building examples

In order to construct explicit examples, there are basically two ways to proceed. We could either degenerate a flat family of smooth models, or we could start with a reducible model and try to turn on the gluing VEV. Let us focus on the latter.

Let us list the data that we have to specify. We take the base to be a del Pezzo surface. On it, we start with a reducible configuration where the VEV of  $\bar{N}$  vanishes. Then the spectral cover really has six sheets: we have  $C_4$  for breaking  $E_8 \rightarrow \text{SO}(10)$ ,  $C_1 = S$  associated to  $\text{U}(1)_{B-L}$ , and another copy of  $S$  (let's call it  $C_0$ ) to accommodate non-zero  $L_Y$ . By abuse of notation, we write  $C_6 = C_0 \cup C_1 \cup C_4$  and  $C_0 \cup C_1 = 2S$  even though that is not quite correct scheme theoretically. We need to specify  $C_4$  and a line bundle  $L_4$  on it. On  $2S = C_0 \cup C_1$  we need to specify a sum of two line bundles:

$$L_Y^{-5/6} \oplus (L_Y^{1/6} \otimes L_{B-L}^{-4/5}) \equiv L_Y^{-5/6} \oplus Q \tag{2.16}$$

The field  $\bar{N}$  corresponds to a gluing morphism pointing from  $C_4$  to  $C_1$ . Turning it on corresponds to forming a new sheaf  $L_5$  which is the extension of  $i_{4*}L_4$  by  $i_{1*}Q$ , and supported on  $C_5 = C_4 \cup C_1$ . We will further have to specify a Kähler class and see if  $L_5$  is stable.

We can also view this as a heterotic model. Namely we start with a reducible rank six bundle of the form

$$V_6 = V_0 \oplus V_1 \oplus V_4 \tag{2.17}$$

where the  $V_i$  are the Fourier-Mukai transforms of the sheaves  $L_Y^{-5/6}$ ,  $Q$  and  $L_4$  above. Turning on  $\bar{N}$  means that we replace  $V_1 \oplus V_4$  by a non-trivial extension  $V_5$ , whose Fourier-Mukai transform is  $L_5$ . So the final configuration will be  $V_0 \oplus V_5$ . The original  $V_1 \oplus V_4$  will be unstable, but both  $V_0$  and  $V_5$  need to be slope stable of slope zero, so that the sum  $V_0 \oplus V_5$  is poly-stable.

The details for such constructions are explained in [12, 25]. The homology class of  $C_4$  is determined by choosing a class  $\eta \in H^2(S)$ , and the class of the matter curve in  $H^2(S)$  is then given by

$$[\Sigma_{16}] = \eta - 4c_1(S). \tag{2.18}$$

We can further specify  $L_4$  by finding a suitable algebraic representative for its first Chern class. In order to get an ample supply of such classes, it may be necessary to tune the complex structure moduli to get Noether-Lefschetz classes. There is enormous flexibility in this part of the construction so we will not detail it any further.

It is useful to note some simple consistency constraints. We assume that  $L_Y^{5/6}$  does not contribute to the net chirality, as happens under the usual condition that  $c_1(L_Y^{5/6})$  is topologically trivializable in the bulk of the compactification. Suppose that we have  $k_1$  generations of  $\mathbf{10}$  living on  $\Sigma_{16}$  and  $k_2$  generations of  $\mathbf{10}$  from the bulk, i.e. we have

$$\begin{aligned} \mathbf{10}_{1/5} : & \quad -\chi(V_4) = k_1 \\ \mathbf{10}_{-4/5} : & \quad -\chi(V_1) = k_2 \end{aligned} \tag{2.19}$$

Then we can deduce that

$$\begin{aligned} \bar{\mathbf{5}}_{2/5} : & \quad -\chi(\Lambda^2 V_4) = k_2 - k_3 \\ \bar{\mathbf{5}}_{-3/5} : & \quad -\chi(V_4 \otimes V_1) = k_1 + k_3 \\ \mathbf{1}_1 : & \quad -\chi(V_4 \otimes V_1^*) = k_1 - k_3 \end{aligned} \tag{2.20}$$

where  $k_3 = \int_{\Sigma_{16}} c_1(Q)$ . We can adjust  $c_1(Q)$  to get various numbers of  $\text{SU}(5)_{\text{GUT}}$  singlets charged under  $\text{U}(1)_{B-L}$ . Now we will assume there are three generations of  $\mathbf{10}_{1/5}$  on

$\Sigma_{16}$ . (Variations on this are possible.) If we further want three right-handed neutrinos  $N$ , and one  $\bar{N}$  (which will get eaten by the  $U(1)_{B-L}$ ), then it appears we should take  $c_1(Q) \cdot [\Sigma_{16}] = 1$ , although having additional right-handed neutrinos may be useful.

Getting both  $N$  and  $\bar{N}$  on the same curve requires that the genus  $g$  of the curve is at least equal to one, and that the net number of generations on the curve is between  $\pm(g - 1)$ . It is easy to show using Riemann-Roch that one cannot get chiral/anti-chiral pairs outside this range. Fortunately, it is very easy to make the genus of  $\Sigma_{16}$  large by making  $\eta$  moderately large.

Even inside this range, getting chiral/ant-chiral pairs is not generic and requires tuning complex structure moduli, i.e. it requires tuning a modulus  $U$  appearing in a Yukawa coupling  $UN\bar{N}$ . However  $\bar{N}$  gets a VEV in the final configuration we are interested in, so moduli such as  $U$  will be massive, and this is not unnatural. With one  $N/\bar{N}$ -pair, one modulus  $U$  pairs up with an  $N$  when  $\bar{N}$  gets a VEV, so we should actually take  $c_1(Q) \cdot [\Sigma_{16}] = 0$  if we want exactly three right-handed neutrinos.

A reducible configuration like above is typically unstable. Recall from the discussion above that we would like an expectation value for  $\bar{N}$ , to lift dangerous  $\bar{\mathbf{10}}$  bulk modes. This corresponds to a gluing morphism pointing from  $C_4$  to  $C_1$ . We would like to argue that the configuration obtained by turning on such a VEV can be made stable. The new rank five Higgs bundle  $E$ , given by the extension of  $p_{4*}L_4$  by  $Q$ , has

$$c_1(E) = c_1(p_{4*}L_4) + c_1(Q) = c_1(L_Y^{5/6}) \tag{2.21}$$

In order to avoid lifting  $U(1)_Y$  by closed string axions, the cohomology class  $c_1(L_Y^{5/6})$  should trivialize when we embed in a compact model, and so  $c_1(L_Y^{5/6})$  is orthogonal to the Kähler class. This topological condition also guarantees that there is no Fayet-Iliopoulos term for  $U(1)_Y$ . Then the slope of  $E$  is zero, and as discussed in section 3.2 of part I, there are two natural Higgs sub-bundles, one of these being  $Q$  (or of course any line bundle that maps into  $Q$ ). Proving stability is a hard issue, and we will not do so here. However for a configuration of the above type, there are only two natural necessary conditions: the slope of  $Q$  should be negative and the slope of  $Q(\Sigma_{16})$  should be positive. This can easily be arranged for suitable choices of  $Q$  and the Kähler class. We will choose an explicit class below. The Kähler class would eventually be determined by dynamical considerations, but that is beyond the scope of our simple example, and we fix it by hand.

It remains to check for exotic bulk matter. We will calculate the spectrum for  $\langle \bar{N} \rangle = 0$  and then check if the resulting fields may be lifted by a Yukawa coupling. The net number of  $\mathbf{10}_{-4/5}$ 's in the bulk is given by  $c_1(Q) \cdot c_1(K_S)$ . The degree two homology lattice of our del Pezzo surface is spanned by the hyperplane class  $H$  and the exceptional  $-1$ -classes  $E_i$  subject to the relations:

$$H \cdot H = 0, \quad H \cdot E_i = 0, \quad E_i \cdot E_j = -\delta_{ij} \tag{2.22}$$

We take

$$Q = \mathcal{O}(E_1 - E_2), \quad L_Y^{5/6} = \mathcal{O}(E_3 - E_4) \tag{2.23}$$

so there is no net matter in the bulk, and  $c_1(L_Y^{5/6})$  is trivializable. (Obviously variations on this theme are possible, for instance we could have four net generations on  $\Sigma_{16}$  and one net anti-generation in the bulk, and then lift the chiral/anti-chiral pairs using the gluing morphism.) Note that we can indeed make the slope of  $Q$  negative, e.g. by taking

$$J \sim -c_1(K) + \epsilon c_1(\mathcal{O}(H - E_2)), \tag{2.24}$$

where  $\epsilon$  is a real and positive number. The the slope of  $Q(\Sigma_{16})$  is given by

$$\mu(Q(\Sigma_{16})) = -\epsilon + J \cdot [\Sigma_{16}] \tag{2.25}$$

It is easy to make this positive as well as satisfy  $c_1(Q) \cdot [\Sigma_{16}] = 0$  by making  $\eta$  moderately large, e.g.  $\eta = 6c_1(S)$ . By exchanging  $E_1 \leftrightarrow E_2$  we could also reverse the signs, so that  $N$  would get a VEV instead of  $\bar{N}$ . At  $\epsilon = 0$  the reducible configuration is poly-stable, so  $\epsilon = 0$  corresponds to a wall of marginal stability.

At any rate, with these choices we find the following cohomology groups for bulk matter descending from the  $\overline{\mathbf{10}}_{4/5}$ :

$$\begin{aligned} (\bar{\mathbf{3}}, \mathbf{2})_{-1/6, 4/5} : & \quad H^0(Q^{-1} \otimes K) = 0 & \quad H^1(Q^{-1}) = 0 \\ (\mathbf{1}, \mathbf{1})_{-1, 4/5} : & \quad H^0(Q^{-1} \otimes L_Y^{-5/6} \otimes K) = 0 & \quad H^1(Q^{-1} \otimes L_Y^{-5/6}) = 1 \\ (\mathbf{3}, \mathbf{1})_{2/3, 4/5} : & \quad H^0(Q^{-1} \otimes L_Y^{5/6} \otimes K) = 0 & \quad H^1(Q^{-1} \otimes L_Y^{5/6}) = 1 \end{aligned} \tag{2.26}$$

Furthermore, since  $c_1(L_Y^{5/6}) \cdot c_1(K_S) = 0$ , we get the same number of net generations for all the fields descending from the  $\overline{\mathbf{10}}_{4/5}$ . We can also see this using that  $Q \rightarrow Q^{-1}$  is a diffeomorphism symmetry of the del Pezzo. Therefore, there is also one  $(\bar{\mathbf{3}}, \mathbf{1})_{-2/3, -4/5}$  bulk mode and one  $(\mathbf{1}, \mathbf{1})_{1, -4/5}$  bulk mode. These bulk modes cannot get lifted because that would require a VEV for  $N$  in the Yukawa coupling  $\mathbf{10}_{-4/5} \overline{\mathbf{10}}_{-1/5} N_1$ , and thus they will be part of the Standard Model fields. Geometrically all these bulk modes correspond to turning on off-diagonal components of the  $SO(10)$  gauge field on  $S$ .

Now we need to consider the Yukawa coupling. Geometrically this appears to correspond to the following: we take the generator  $s \in H^1(Q^{-1} \otimes L_Y^{\pm 5/6})$ , restrict it to  $\Sigma_{16}$ , and then multiply it by the gluing morphism  $f \in \text{Hom}_{\Sigma_{16}}(Q^{-1}, L_4^{-1} \otimes K_{C_4})$  corresponding to  $\bar{N}$ . This composition yields a map

$$H^1(Q^{-1} \otimes L_Y^{\pm 5/6}) \times \text{Hom}_{\Sigma_{16}}(Q^{-1}, L_4^{-1} \otimes K_{C_4}) \rightarrow H^1(\Sigma_{16}, L_4^{-1} \otimes L_Y^{\pm 5/6} \otimes K_{C_4}|_{\Sigma_{16}}) \tag{2.27}$$

If the image  $fs|_{\Sigma}$  is a non-zero, then the Yukawa coupling is non-vanishing and the dangerous bulk mode is lifted. Computing this composition requires writing down explicit generators and is therefore somewhat tedious, but it ends up in the right place: we know that  $H_{\Sigma}^1 = H^0(\Sigma_{16}, L_4 \otimes L_Y^{\mp 5/6} \otimes K_S|_{\Sigma_{16}})^{\vee}$  is non-vanishing and at least three-dimensional, as there are three chiral generations localized on  $\Sigma_{16}$ . Therefore generically we expect this lifting to take place. The end result then is that the chiral and anti-chiral fields get paired up, and we are left with a three-generation model with one right-handed lepton (in the  $(\mathbf{1}, \mathbf{1})_{1, -4/5}$ ) and one right-handed up type quark (in the  $(\bar{\mathbf{3}}, \mathbf{1})_{-2/3, -4/5}$ ) living in the bulk, and all the remaining Higgs and matter fields living on matter curves. The fact that one  $\mathbf{10}$  multiplet in the bulk is incomplete (with the remaining field in the multiplet living on a matter curve) can be seen as a remnant of the no-go theorem of [24].

### 2.1.4 Emerging flavour structures

In conclusion, just as in generic  $SU(5)_{\text{GUT}}$  models, we do not expect a general statement about problems with the spectrum, in particular there is no issue with exotic states in the low energy spectrum.

Then, what is the advantage of considering such degenerate models? The point is that precisely due to the extra structure, the situation is more interesting than in generic models, because such models can possess various flavour structures. Indeed let us summarize some of the structure we found in the above models.

We saw that for zero gluing VEV, the model has both chiral and anti-chiral fields in the bulk which can not get paired up, even though the mass term is allowed by all the gauge symmetries. The fields in the bulk can only pair up with fields on the matter curve after turning on the gluing VEV. The reason for this seems to be that we need a coupling  $\int_S \text{Tr}(A \wedge A \wedge \Phi)$  where  $(A, A, \Phi)$  are all bulk fields, but no suitable  $\Phi$  is available. Perhaps this selection rule can be rephrased in terms of a global symmetry. Furthermore we see that these bulk fields appear in incomplete multiplets. This provides an idea for simultaneously solving the  $\mu$ -problem and the doublet triplet splitting problem without using extra  $U(1)$  gauge symmetries, by putting the Higgs fields  $(H_u, H_d)$  in the bulk. We will consider a model of this type in section 3.3.

The kinetic terms of matter localized in the bulk of a 7-brane scale differently with the volume from kinetic terms of matter localized on a curve. Therefore in this model we get additional flavour structure after canonically normalizing the kinetic terms, see section 3.1.

There are also constraints on the holomorphic Yukawa couplings. We saw that a bulk mode cannot appear in an up-type  $\mathbf{10}_{-4/5} \mathbf{10}_{-4/5} \mathbf{5}$  Yukawa coupling, as it has negative  $U(1)_{B-L}$  charge and we cannot compensate it by multiplying with a suitable number of  $\langle \bar{N}_{-1} \rangle$  VEVs. Thus our model gives rise to texture zeroes. For more on the holomorphic couplings, see section 3.2.

## 2.2 $E_6$ -models

It is clear that our techniques can be used to construct a variety of new models with degenerate covers. We briefly consider  $E_6$  models, which leads one to study non-reduced covers. For simplicity we will take the final GUT group to be  $SU(5)_{\text{GUT}}$  and not break it further to the Standard Model. The commutant of  $E_6$  in  $E_8$  is  $SU(3)$ . Thus from the spectral cover perspective,  $E_6$  models are a special case of a  $3 + 2$  splitting of the  $SL(5, \mathbf{C})$  cover,  $C_5 = C_3 \cup C_2$ , with  $C_3$  a non-trivial covering and  $C_2 = 2S$ .

The apparent  $E_6$  gauge group is further broken to  $SU(5)$  by turning on a rank one sheaf  $L_5$  on  $C_5$ , restricting to rank one sheaves  $L_2$  on  $C_2$  and  $L_3$  on  $C_3$ , and a gluing morphism on the intersection. We denote  $C_3 \cap S$  by  $\Sigma_{27}$ , as it corresponds to the curve where chiral fields in the  $\mathbf{27}$  of  $E_6$  are localized. Labelling the matter content by representations of  $SU(5)_{\text{GUT}} \times SU(2) \times SU(3) \times U(1)_X$ , we see that the matter fields are distributed in the following way:

$$\begin{aligned}
 \Sigma_{27} : & \quad (\mathbf{10}, \mathbf{3}, \mathbf{1})_{2/5}, (\bar{\mathbf{5}}, \bar{\mathbf{3}}, \mathbf{1})_{4/5}, (\bar{\mathbf{5}}, \mathbf{3}, \mathbf{2})_{-1/5}, (\mathbf{1}, \mathbf{3}, \mathbf{2})_1 \\
 \text{bulk} : & \quad (\mathbf{10}, \mathbf{1}, \mathbf{2})_{-3/5}, (\bar{\mathbf{5}}, \mathbf{1}, \mathbf{1})_{-6/5}
 \end{aligned} \tag{2.28}$$

The main new feature compared to the previous subsection is that we have a rank one sheaf  $L_2$  supported on the non-reduced surface  $C_2 = 2S$ . As discussed in section 2.5 of part I,  $L_2$  could either be a rank two bundle on the underlying reduced scheme  $S$ , or a sheaf which is non-trivial on the first infinitesimal neighbourhood of  $S$  (i.e. a nilpotent Higgs VEV). Note further that  $C_2 \cap C_3 = 2\Sigma_{27}$ , where  $\Sigma_{27} = S \cap C_3$ , so the gluing morphism can also be a bit more intricate than in the previous example. Nevertheless the approach of building the spectral sheaf from more elementary pieces works equally well here.

Models of this type have various interesting flavour structures. Apart from possible texture zeroes associated to  $U(1)_X$ , we may also get texture zeroes associated to  $U(1)_{B-L}$  if  $L_2$  is built as the extension of two line bundles. There are also additional fields propagating in the bulk. For a more detailed discussion of flavour we refer to section 3.

### 2.3 IIA/M-theory models

We would briefly like to comment on the issue of bulk matter in IIA/M-theory models. In this context, local models are described by real Higgs bundles on a three-manifold  $Q_3$  [13]. We will only make some preliminary remarks about this interesting subject, which really requires a more thorough investigation.

The data in these models consists of a complex flat connection  $\mathbf{A}$  on a real three-manifold  $Q_3$ , whose hermitian part satisfies a harmonicity condition. The complex flat connection is allowed to have singularities, which correspond to non-compact flavour branes.

The background flat connection breaks some initial gauge group  $G'$  to a smaller gauge group  $G$ . In order to get chiral matter, we want an unbroken gauge group  $G$  on  $Q_3$  and an infinitesimal deformation  $\delta\mathbf{A}$  which transforms in a chiral representation of  $G$ . The wave functions of these modes depend on the values of the background higgs field, and are peaked when the higgs field vanishes. Thus to get a bulk mode, we want an enlarged gauge group  $G'$  to be ‘visible’ on most of  $Q_3$ .

In the usual intersecting brane configurations, we have precisely the opposite situation. The wave functions are confined to a small region around the zeroes of the background higgs field, where  $G'$  is unbroken, by a steep potential.

One may consider breaking  $G'$  to  $G$  on  $Q_3$  by a real flat connection. However in the conventional situation (no gluing VEVs), this flat connection must be defined on all of  $Q_3$ . This can be allowed when  $Q_3$  is negatively curved, but we are usually interested in models where  $Q_3$  is positively curved (e.g.  $Q_3 = S^3$  or  $Q_3 = S^3/\Gamma$  where  $\Gamma$  is a finite group). In such models  $h^1(Q_3) = 0$ , and any background flat connection continuously connected to the identity is actually gauge equivalent to zero. In fact even when  $Q_3$  is negatively curved, we still want  $h^1(Q_3) = 0$  to avoid massless adjoint fields. Discrete Wilson lines also do not help, as  $\delta\mathbf{A}$  should also be globally defined on  $Q_3$  in this situation and therefore will also be gauge equivalent to zero.

To avoid this, the basic idea is to consider spectral cover components coinciding with the zero section outside a codimension two subset  $\Delta$ , where one may have vertical components. Then one may have  $h^1(Q_3 \setminus \Delta) \neq 0$  or  $h^1(Q_3 \setminus \Delta, L) \neq 0$ , where  $L$  is a flat bundle on  $Q_3 \setminus \Delta$ . With a suitable parabolic structure along  $\Delta$ , this could yield a valid gauge theory

configuration, as discussed in section 2.5 of part I and also in section 4 on the standard embedding.

The reason for considering such configurations stems from simple phenomenological considerations. If chiral matter is localized at points of  $Q_3$ , then matter interactions (in particular Yukawa couplings) tend to be exponentially small. This can be seen in two equivalent ways. We can in principle solve for the exact wave-functions satisfying the linearized Hitchin's equations. They are approximately though not exactly gaussian, and their classical overlaps therefore tend to be small. Alternatively, since we are computing  $F$ -terms, we can use the freedom to scale up the Higgs field  $\phi$ , i.e. we consider the large angle approximation. In this approximation the overlap integrals are zero, but there are instanton corrections which generate the couplings, which again are small. With bulk matter however, the wave functions overlap classically and so Yukawa couplings would be present at tree level and therefore of order one.

### 3 Flavour structures

#### 3.1 Structures from D-terms

In generic  $F$ -theory models, the matter fields are localized on matter curves. We have seen in this paper that we can get matter in the bulk of the 7-brane by considering degenerate spectral covers. No-go theorems can be circumvented by using all available ingredients, in particular a non-zero gluing morphism. We have argued that turning off the gluing morphism typically requires an extra tuning of the Kähler moduli, and so is actually less generic. Thus we may equally well incorporate bulk matter in building realistic models. Later we will also see that certain well-known heterotic models have  $F$ -theory duals with bulk matter.

Models with bulk matter automatically have additional structure in the Yukawa couplings originating from the  $D$ -terms. The physical Yukawa couplings  $\hat{Y}$  can be expressed in terms of the holomorphic Yukawa couplings  $Y^0$  as

$$\hat{Y}_{\alpha\beta\gamma} = e^{\mathcal{K}/2} \frac{Y_{\alpha\beta\gamma}^0}{(K_\alpha K_\beta K_\gamma)^{1/2}} \tag{3.1}$$

where  $K_\alpha$  denotes the Kähler metric for a chiral field. Kinetic terms for modes localized in the 7-brane bulk and on a 7-brane intersection have different scaling behaviour with respect to Kähler moduli. Furthermore, bulk modes descending from  $A$  have different scaling than bulk modes descending from  $\Phi$ . According to [26] we have

$$K_A \sim \frac{1}{t}, \quad K_\Phi \sim 1, \quad K_I \sim \frac{1}{t^{1/2}} \tag{3.2}$$

where  $t$  is the volume of the four-cycle, and  $\mathcal{K}$  of course also depends on  $t$ . The subscripts  $(A, \Phi, I)$  denotes the origin of the chiral field as a bulk mode from  $A$  or  $\Phi$ , or from an intersection. Thus we automatically get flavour structure. For instance in the  $SO(10)$  model discussed earlier, we had one generation of  $\mathbf{10}$  in the bulk descending from  $A$  and all other matter and Higgs fields localized on curves. (This is not quite right because the  $\mathbf{10}$



multiplet was incomplete, but let us ignore that.) Then we find that in the large volume limit, one generation has Yukawa couplings which are parametrically larger than the other two, and the bottom is suppressed compared to the top for the heavy generation.

In fact we may also get models with bulk fields descending from  $\Phi$ , as we saw for  $E_6$  models. Thus we may contemplate the following scenario: we take two generations of  $\mathbf{10}$  to be ‘mostly’ bulk modes, one from  $A$  and one from  $\Phi$ . We say ‘mostly’ because as we have seen, when gluing VEVs are non-zero the distinction between bulk modes and modes on matter curves becomes a bit blurry. We take the remaining  $\mathbf{10}$  and all  $\bar{\mathbf{5}}_m$  matter to live on matter curves. We will also assume a Kähler potential with a large and a small size modulus. Then using equation (2.18) in [26] one would get a flavour model of the form

$$M_u \sim \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}, \quad M_d \sim \begin{pmatrix} \epsilon^3 & \epsilon^3 & \epsilon^3 \\ \epsilon^2 & \epsilon^2 & \epsilon^2 \\ \epsilon & \epsilon & \epsilon \end{pmatrix} \quad (3.3)$$

up to an overall factor of  $\epsilon^{1/2}$  which one could hopefully offset against the numerical pre-factor. This is known to be a decent approximation in the real world [27]. However this does not allow us to suppress proton decay, for which we will propose a different mechanism.

The above scaling behaviours should probably not be taken too literally. We have seen that we can degenerate a model with all matter localized on curves to a model with some matter in the bulk. Thus one should be able to smoothly interpolate between these scaling behaviours. Furthermore the actual wave functions can get quite complicated. The numerical approximation suggested in section 3.3 of part I may lead to a clearer picture.

### 3.2 Textures from F-terms: holomorphic zeroes

Proton decay is an important issue in string GUT constructions, and even in string MSSM constructions. The problem starts already with the Yukawa couplings, as  $R$ -parity is not automatic.

One of the most plausible solutions to this problem, though not the only one, is to realize  $R$ -parity as a remnant of an additional light  $U(1)_X$  symmetry. Such extra  $U(1)$  symmetries forbid the  $\mathbf{10}_m \cdot \bar{\mathbf{5}}_m \cdot \bar{\mathbf{5}}_m$  Yukawa coupling, as well as other couplings. The most well-known of these is  $U(1)_{B-L}$ .

As is well-known, there is some tension in such a scenario. The extra  $U(1)$  gauge boson must acquire a mass above the weak scale to avoid having been detected yet. But naively breaking the  $U(1)$  should reintroduce  $R$ -parity violating couplings, so the  $U(1)$  cannot be too heavy either. If the  $U(1)$  gets a mass through the Higgs mechanism, then it should be a few orders above the weak scale. Whether this is possible depends sensitively on the values of the soft terms, i.e. on the method of breaking and mediating supersymmetry, and leads to a conflict with the Majorana neutrino mass scenario. If instead the  $U(1)$  boson gets a mass through a Stückelberg coupling to Kähler moduli axions, then we still have a good  $U(1)$  symmetry in perturbation theory, violated only by  $M5$ -instantons, and one may imagine the  $U(1)$  boson to be heavier. But this has its own problems, for instance the Majorana mass term has to be induced by non-perturbative effects.

In this section, we consider the possibility that the extra  $U(1)_X$  symmetry is broken through the VEV of a chiral field, rather than through non-perturbative effects (which will still be present, but small). The naive expectation that all  $U(1)_X$  violating operators get generated is in general not correct in supersymmetric theories. Holomorphy may forbid certain couplings even when the  $U(1)_X$  is broken [28]. This mechanism leads to interesting textures which are essentially remnant selection rules of the broken  $U(1)_X$  symmetry. By an artful assignment of the charges (but still constrained by an embedding in  $E_8$ )<sup>1</sup> one can forbid problematic couplings while allowing for others, thereby addressing the tension one usually finds in such models.

Let us first consider engineering an extra  $U(1)_X$  symmetry. Unbroken symmetry generators correspond to cohomology classes  $\rho_\xi$  in  $\mathbb{H}^0(\mathcal{E}^\bullet)$ , i.e.  $\bar{\partial}$ -closed zero forms in  $\Omega^0(Ad(E_8))$  that commute with  $\Phi$ . Now for a generic Higgs bundle we would not get any such endomorphisms. This follows from the usual arguments: a generic spectral sheaf corresponds to a line bundle on a smooth and irreducible surface, and the corresponding Higgs bundle is stable. But stable Higgs bundles only have trivial endomorphisms, and so we should look for a poly-stable Higgs bundle instead.

To get a non-trivial endomorphism, we therefore should make the spectral cover reducible. However this condition is not sufficient to enforce a massless  $U(1)_X$ , as it does not guarantee that the Higgs bundle is reducible. For this, we need the rank of the map  $[\Phi, \cdot]$  to drop by one globally. As we have seen, the problem is that  $\Phi$  may have non-trivial Jordan structure at the intersection of the reducible pieces. To get a massless  $U(1)$ , we must also require that the Jordan structure or equivalently the gluing morphism is trivial.

As explained in part I, turning off the gluing morphisms corresponds to approaching a wall of marginal stability. Let us consider the effective field theory at such a wall, i.e. we consider the KK expansion around a reducible brane configuration with zero gluing VEV. As discussed, the resulting effective field theory is a version of the Fayet model. That is, in the minimal case we have a  $U(1)_X$  vector multiplet, a chiral field  $X$  charged under the  $U(1)_X$ , and a  $D$ -term potential

$$V_D = \frac{1}{2}(\zeta_X + q_X |X|^2)^2 \tag{3.4}$$

Turning on the Fayet-Iliopoulos parameter breaks the  $U(1)_X$  symmetry.

Now let us consider the effect of  $U(1)_X$  breaking on the Yukawa couplings. When the  $U(1)_X$  symmetries are explicitly broken by a non-zero expectation value for  $X$ , at first sight one might think there are no selection rules left on the couplings. However this is not correct. It is clearest if we only have a single  $U(1)_X$  charged field  $X$ , which we can take to have a positive sign for the charge. Let us consider general higher order couplings in the superpotential

$$W \sim W|_{X=0} + \Phi_\alpha \Phi_\beta \Phi_\gamma X + \dots \tag{3.5}$$

After turning on a VEV for  $X$ , we get various couplings violating the  $U(1)_X$  symmetry. However for the superpotential to be invariant, only holomorphic couplings that are nega-

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<sup>1</sup>It is not quite true that the  $U(1)_X$  must be embedded in  $E_8$ , but here we are only interested in the  $U(1)_X$  charges of fields localized near the GUT brane.

tively charged by an integer multiple of  $q_X$  can get generated in the effective theory. (For dimension five and higher one should be careful that the operator has a good  $\langle X \rangle \rightarrow 0$  limit, but otherwise a similar argument can be made.) Such couplings may still be generated non-perturbatively, but one may reasonably expect such corrections to be small. This leads to definite textures in the Yukawa couplings even after the  $U(1)_X$  symmetry is broken, as long as the effective Fayet-model is valid. This is the ‘holomorphic zero’ mechanism: holomorphy of the superpotential prevents operators with the wrong sign of the charge from being generated [28]. It has played an important role in proving non-renormalization theorems in supersymmetric theories and is clearly also present in string compactification, as has been observed in the context of the heterotic string and  $F$ -theory in for example [20, 21, 29–31].

Such textures tend to persist even if the low energy theory has fields with both signs of the  $U(1)_X$  charge,  $X_+$  and  $X_-$ . This is easily seen to be a consequence of the effective field theory, as noted for example in [21]. The symmetries allow for a superpotential coupling

$$W \simeq (X_+ X_-)^2 \tag{3.6}$$

and analogous higher order terms which have no reason to be absent. They will generically get generated from integrating out the KK modes. The  $F$ -term equations then forbid fields with both signs of the  $U(1)_X$  charges to get a VEV simultaneously. Consequently,  $U(1)_X$  textures with a definite sign will appear also in the more general case.

Let us briefly discuss in which range of parameters we can trust these textures. We have seen in section 3.2 of part I that the Fayet model itself can only be trusted for an infinitesimal Fayet-Iliopoulos parameter. This is expected from effective field theory, as the mass of the  $U(1)_X$  gauge boson should be parametrically small compared to the KK scale in order to trust the Fayet model. Even this may not be sufficient, it is the maximum that could be expected from the low energy effective field theory at the wall. Depending on the microscopic properties of the vacuum, new physics may come in below that scale. In fact we already know that the breakdown of the Fayet model happens strictly below the KK scale, because the VEV of  $X$  depends on the Fayet-Iliopoulos parameter  $\zeta_X$  which is a function of the Kähler moduli, and the Kähler moduli are dynamical modes stabilized below the KK scale.

Now let us simply fix the Kähler moduli by hand and ask what happens when we increase the Fayet-Iliopoulos parameter. When the mass of the  $U(1)_X$  gauge boson, which is proportional to  $\langle X \rangle$ , is not parametrically small compared to the KK scale, we are required to reexpand around the true vacuum configuration. The obvious configuration would correspond to deforming the brane configuration by a finite deformation whose tangent vector is the internal zero mode corresponding to the  $4d$  field  $X$ . In other words, we expand around a reducible brane configuration with non-zero gluing morphism. If this configuration is stable, then we can take this to be our new vacuum. But as we discussed in section 3.2 of part I, it is not guaranteed that this new configuration is stable, despite the fact that it seems to be suggested by the Fayet model, and in the generic case it would not be. If that is the case, then we have to further deform the reducible brane configuration to reach a true vacuum. Since the true vacuum is then generically obtained by turning on

modes with both signs of the  $U(1)_X$  charges (smoothing modes), the superpotential in the true vacuum would not satisfy  $U(1)_X$  selection rules.

At any rate, with these caveats we see that reducible configurations of branes or spectral covers can give rise to Yukawa textures. For a generic smooth spectral cover, we do not expect such Yukawa textures. By Fourier-Mukai transform, our picture for reducible brane configurations maps precisely to the construction of bundles by extension, well-known in the heterotic string. Suppose that we have a bundle  $F$  given by the extension

$$0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0 \tag{3.7}$$

Applying the Fourier-Mukai transform, we find

$$0 \rightarrow FM^1(E_1) \rightarrow FM^1(F) \rightarrow FM^1(E_2) \rightarrow 0 \tag{3.8}$$

In particular, the support of  $FM^1(F)$  is simply the union of the supports of  $FM^1(E_1)$  and  $FM^1(E_2)$ , i.e. a degenerate cover. The only difference between  $FM^1(F)$  and  $FM^1(E_1) \oplus FM^1(E_2)$  is the gluing morphism.

The holomorphic zeroes arising from abelian gauge symmetries are certainly not the only way to get  $F$ -term textures. For instance it could happen that we get holomorphic zeroes because the matter curves don't intersect. Or as we see in the next subsection, certain superpotential couplings could be zero because they involve bulk modes. However using abelian gauge symmetries is fairly natural in string compactification and we now have a much better understanding of the geometries that yield such textures. In the next section we will combine it with bulk modes to simultaneously address several phenomenological issues.

### 3.3 A model of flavour, with Higgs fields in the bulk

Now we would like to show how one can put the results of the previous sections together and simultaneously address the problems of  $R$ -parity, proton decay, the  $\mu$ -problem, and a crude flavour hierarchy in  $F$ -theory GUTs. To be fair, apart from the Higgses we will only consider the net amount of chiral matter. Nevertheless we do not expect issues with anti-generations, because the only unbroken gauge symmetry is the Standard Model gauge group, and everything that is not protected by index theorems should get lifted in a sufficiently generic model.

#### 3.3.1 Basic picture of the configuration

The idea is to consider a  $3 + 2$  split  $C_5 = C_3 \cup C_2$  of the  $SL(5, \mathbf{C})$  spectral cover, so that the effective theory below the KK scale has a certain extra  $U(1)_X$  symmetry (often called  $U(1)_{PQ}$  in the  $F$ -theory literature [32, 33]; see also [34–36] for global aspects). As originally emphasized in [17] in the  $F$ -theory context, this symmetry can be used to forbid dimension four and five proton decay and a  $\mu$ -term. Here we would like to argue that we largely preserve  $R$ -parity and suppressed proton decay while evading some of the problems when we break the  $U(1)_X$  with a gluing VEV. The model is similar in spirit to [20]. However, other than in previous work we will not use this  $U(1)_X$  symmetry to solve the  $\mu$ -problem.

In particular, this allows us to evade the problems with exotics discussed in [37, 38]. Instead we will use the observation in section 2.1 about the possibility of solving the  $\mu$ -problem and doublet/triplet splitting problem by putting the Higgses in the bulk. This will account for the Higgses being ‘different’ from ordinary matter in our scenario.

We will use an  $SL(3) \times SL(2)$  spectral cover to break the  $E_8$  to  $SU(6)$ , but it will be more convenient to label the fields by an  $SU(5)_{\text{GUT}} \times SU(3) \times SU(2) \times U(1)_X$  subgroup of  $E_8$ . The matter fields in the  $\mathbf{10}$  and  $\bar{\mathbf{5}}$  split up in the following way:

$$\begin{aligned} \mathbf{10} : & \quad (\mathbf{10}, \mathbf{1}, \mathbf{2})_{-3/5} + (\mathbf{10}, \mathbf{3}, \mathbf{1})_{2/5} \\ \bar{\mathbf{5}} : & \quad (\bar{\mathbf{5}}, \mathbf{1}, \mathbf{1})_{-6/5} + (\bar{\mathbf{5}}, \mathbf{3}, \mathbf{2})_{-1/5} + (\bar{\mathbf{5}}, \bar{\mathbf{3}}, \mathbf{1})_{4/5} \end{aligned} \tag{3.9}$$

We also have the  $U(1)$ -charged singlets:

$$(\mathbf{1}, \mathbf{3}, \mathbf{2})_1 + (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{2})_{-1} \tag{3.10}$$

We denote these singlets by  $X^+$  and  $X^-$ .

From the representations, it is easy to read off the distribution of matter in these models:

$$\begin{aligned} \Sigma_2 = C_2 \cap S : & \quad (\mathbf{10}, \mathbf{1}, \mathbf{2})_{-3/5} \\ \Sigma_3 = C_3 \cap S : & \quad (\mathbf{10}, \mathbf{3}, \mathbf{1})_{2/5}, \quad (\bar{\mathbf{5}}, \bar{\mathbf{3}}, \mathbf{1})_{4/5} \\ \Sigma_{32} = C_3 \cap C_2 : & \quad (\bar{\mathbf{5}}, \mathbf{3}, \mathbf{2})_{-1/5}, \quad (\mathbf{1}, \mathbf{3}, \mathbf{2})_1 \\ \text{bulk} : & \quad (\bar{\mathbf{5}}, \mathbf{1}, \mathbf{1})_{-6/5} \end{aligned} \tag{3.11}$$

The bulk modes can be understood as part of the adjoint of an  $SU(6)_{\text{GUT}}$  gauge group containing  $SU(5)_{\text{GUT}} \times U(1)_X$  that is left unbroken by our spectral cover. The fields on  $C_3 \cap C_2$  live in the fundamental  $\mathbf{6}$  of this  $SU(6)_{\text{GUT}}$ , the fields on  $C_3 \cap S$  live in the  $\Lambda^2 \mathbf{6} = \mathbf{15}$ , and the fields on  $C_2 \cap S$  live in the  $\Lambda^3 \mathbf{6} = \mathbf{20}_v$  of  $SU(6)_{\text{GUT}}$ . The corresponding ALE fibration will have  $SU(6)$  ALE singularities along  $S$  because the resultant  $R = b_0 b_5^2 - b_2 b_3 b_5 + b_3^2 b_4$  of a generic  $SU(5)_{\text{GUT}}$ -model vanishes identically, so we may call this an  $SU(6)$  model.

Let us make some more remarks on the matter curves. One may ask why the  $(\bar{\mathbf{5}}, \mathbf{3}, \mathbf{2})_{-1/5}$  is not localized on the zero section  $S$ , since it is charged under the  $SU(5) \times U(1)_X \subset SU(6)$  gauge fields localized on  $S$ . In fact we can also localize the computation of the spectrum of these fields on  $S$ . Suppose that the sheets of  $C_3$  are labelled by  $\{\lambda_1, \lambda_2, \lambda_3\}$  with  $\lambda_1 + \lambda_2 + \lambda_3 = 0$ , and the sheets of  $C_2$  are labelled by  $\{\lambda_4, \lambda_5\}$  with  $\lambda_4 + \lambda_5 = 0$ . We can define a six-fold spectral cover  $C_6$  whose sheets are labelled by  $\lambda_i + \lambda_j$ , where  $i = 1, 2, 3$  and  $j = 1, 2$ . Then the computation of the  $(\bar{\mathbf{5}}, \mathbf{3}, \mathbf{2})_{-1/5}$  can be localized on  $C_6 \cap S = \pi_* \Sigma_{32}$ . Explicitly, if  $C_3$  is given by  $c_0 \lambda^3 + c_2 \lambda + c_3 = 0$  and  $C_2$  is given by  $a_0 \lambda^2 + a_2 = 0$ , then  $\pi_* \Sigma_{32}$  is given by

$$0 = a_2^3 c_0^2 - 2a_0 a_2^2 c_0 c_2 + a_0^2 a_2 c_2^2 + a_0^3 c_3^2 \tag{3.12}$$

on  $S$ . This curve is singular however, and its normalization is  $\Sigma_{32}$ . Analogous remarks apply to the spectrum of  $(\mathbf{1}, \mathbf{3}, \mathbf{2})_1$ , which is charged under the  $U(1)_X$  gauge field localized on  $S$ .

Now we will consider a scenario where the Higgses live in the bulk, i.e.  $H_u$  and  $H_d$  should descend from  $(\bar{\mathbf{5}}, \mathbf{1}, \mathbf{1})_{6/5}$  and its conjugate. In order to get all  $\mathbf{10}_m \cdot \mathbf{10}_m \cdot \mathbf{5}_h$

couplings of order one, we see that all the  $\mathbf{10}$  matter comes from the  $(\mathbf{10}, \mathbf{1}, \mathbf{2})_{-3/5}$ . We further take  $\bar{\mathbf{5}}_m$  to descend from  $(\bar{\mathbf{5}}, \bar{\mathbf{3}}, \mathbf{1})_{4/5}$ . Clearly there are a number of variations on our scenario, and we invite the reader to make his or her own.

It is easy to see that with this matter content, the  $U(1)_X$  is anomalous. There is an additional contribution from the net number of  $X^\pm$  fields, but this is not enough to cancel both the  $\sum q$  and  $\sum q^3$  anomalies. This is not immediately deadly, because  $F$ -theory permits axions and a Green-Schwarz mechanism, although it may still imply constraints on the spectrum. (In  $M$ -theory models by contrast, such anomalies would apparently doom the model [13].) Furthermore we don't need this configuration to be stable because we still want to turn on a VEV for  $X^+$  and break the  $U(1)_X$ . We will argue below that this matter content is consistent with the index theorems, and so such models should exist. In particular this addresses the question of stringy anomaly cancellation, because this only depends on the net number of chiral fields with given charges. It would be more satisfactory to check that the actual matter content on the matter curves can be attained without anti-generations, but such calculations are more involved. As we explained in [12] however, in local  $F$ -theory models there is a landscape of solutions for the choice of flux on the flavour branes (consisting of Noether-Lefschetz fluxes), and no index theorem which protects pairs from lifting. So without further calculations, naturalness demands we should assume there are no anti-generations.

We will break the  $SU(6)$  symmetry by a flux for  $U(1)_X$  and a gluing VEV. The  $SU(5)_{\text{GUT}}$  symmetry will be further broken by hypercharge flux. As the Higgses live in the bulk and the matter fields live on curves, and we do not want to split the matter multiplets, we will take the hypercharge flux through the matter curves to be zero identically.

### 3.3.2 Consistency

Let us investigate if the spectrum can be arranged in this way. The reader who is not interested in these technicalities may skip to equation (3.25). It seems that  $H_u$  and  $H_d$  must generically pair up, because the mass term  $H_u H_d$  is a gauge singlet. This is why frequently configurations are considered where  $H_u$  and  $H_d$  are charged under the  $U(1)_X$  symmetry so as to forbid the  $\mu$ -term. But actually this expectation is not quite true for bulk fields, as we saw for the  $SO(10)$  models earlier. At least for vanishing gluing VEVs, the index theorems for bulk modes are stronger in the sense that they know about more than just the net amount of chiral matter. As remarked earlier, this presumably indicates the presence of a global symmetry under which bulk fields are charged. The fields in the bulk transform as

$$(\mathbf{2}, \mathbf{1})_{-1/2, -6/5} + (\mathbf{1}, \bar{\mathbf{3}})_{1/3, -6/5} \tag{3.13}$$

Thus we need to consider the rank two bundle on  $S$  given by

$$(L_Y^{-1/2} \otimes L_X^{-6/5}) \oplus (L_Y^{1/3} \otimes L_X^{-6/5}) \equiv (Q \otimes L_Y^{-5/6}) \oplus Q \tag{3.14}$$

We take  $S$  to be a del Pezzo surface, and denote the hyperplane class lifted from  $\mathbf{P}^2$  by  $H$  and the exceptional  $-1$ -curves by  $E_i$ . Taking for example

$$c_1(L_Y^{5/6}) = \mathcal{O}(E_3 - E_4), \quad c_1(Q) = \mathcal{O}(E_1 - E_2), \tag{3.15}$$

both may restrict trivially to the matter curves (which we still have to choose, we will do so below). Then we have  $H^1(S, Q^{\pm 1}) = 0$  and  $H^1(S, Q^{\pm 1} \otimes L_Y^{-5/6}) = 1$  as required. It follows that we simultaneously solve the  $\mu$ -problem and the doublet/triplet splitting problem this way.

We would now like to investigate the remaining fluxes and check that everything is consistent. We can do this by mapping to an elliptically fibered heterotic model  $\pi : Z \rightarrow B_2$  with  $B_2 = S$ , as the computations are slightly more straightforward that way. Here we have rank three bundle  $V$  and a rank two bundle  $W$ , both constructed from spectral covers  $(C_V, L_V)$  and  $(C_W, L_W)$ , such that

$$c_1(V) = \pi^* c_1(Q^{-1} \otimes L_Y^{5/6}), \quad c_1(W) = \pi^* c_1(Q) \tag{3.16}$$

That is, from the heterotic point of view we start with a rank six bundle of the form  $V_3 \oplus W_2 \oplus U_1$  where the subscripts indicate the rank. Eventually,  $V \oplus W$  will be replaced by a non-trivial rank five extension. Then we have the following table:

$$\begin{array}{ll}
 (\mathbf{10}, \mathbf{3}, \mathbf{1})_{2/5} & \chi(V) = 0 \\
 (\mathbf{10}, \mathbf{1}, \mathbf{2})_{-3/5} & \chi(W) = -3 \\
 (\bar{\mathbf{5}}, \bar{\mathbf{3}}, \mathbf{1})_{4/5} & \chi(\Lambda^2 V) = -3 \\
 (\bar{\mathbf{5}}, \mathbf{3}, \mathbf{2})_{-1/5} & \chi(V \otimes W) = 0 \\
 (\bar{\mathbf{5}}, \mathbf{1}, \mathbf{1})_{-6/5} & \chi(\Lambda^2 W) = \chi(Q) = 0 \\
 (\mathbf{1}, \mathbf{3}, \mathbf{2})_1 & \chi(V \otimes W^*) = N_X
 \end{array} \tag{3.17}$$

We don't need to specify the net number of singlets, as tuning to get vector-like  $X^\pm$  pairs is fine for our purposes. Such tuning moduli will be lifted after turning on the  $X^+$  VEV. We do need that  $(N_{X^-} - N_{X^+}) \geq -1$ , because otherwise there will be massless  $X^+$  fields remaining after turning on an  $X^+$  VEV. We also need the genus of  $C_2 \cap C_3$  to be larger than the net number of  $X^\pm$  generations, because otherwise the spectrum of  $X^\pm$  is purely chiral by Riemann-Roch and we cannot tune to get vector-like pairs. We will find below that actually  $(N_{X^-} - N_{X^+}) = 12$  for consistency. So the genus of  $C_2 \cap C_3$  must be such that  $g - 1 \geq 12$ . The matter curves are given by

$$[\Sigma_3] = \eta_V - 3c_1 \quad \text{and} \quad [\Sigma_2] = \eta_W - 2c_1 \in H^2(S) \tag{3.18}$$

where  $c_2(V) = \pi^* \eta_V$ ,  $c_2(W) = \pi^* \eta_W$  and  $c_1 = c_1(T_S)$ . The genus of  $C_2 \cap C_3$  grows quickly if we take  $\eta_V$  and  $\eta_W$  to be even moderately large.

Now let us consider the more refined information. Recall that in order to get zero net generations in the bulk, we had  $c_1(Q) \cdot c_1 = 0$ . We have [39]

$$\chi(\Lambda^2 V) = (N - 4)\chi(V) - \pi^*(\eta_V - 3c_1) \cdot \sigma_{B_2} \cdot c_1(V) \tag{3.19}$$

where  $\sigma_{B_2}$  is the section of the elliptic fibration. Therefore lines one and three of (3.17) are mutually consistent if

$$c_1(Q^{-1}) \cdot \eta_V = c_1(Q^{-1}) \cdot \Sigma_3 = 3. \tag{3.20}$$

Here we simplified using the fact that  $c_1(L_Y^{5/6})$  is orthogonal to  $c_1$  and to the matter curves. Furthermore,  $[\Sigma_3] = \eta_V - 3c_1$  must be effective. We can satisfy this for example by picking  $\eta_V \sim n_V c_1 + 3E_1$  with  $n_V \geq 4$ . We also find

$$\chi(V \otimes W) = \text{rk}(W)\chi(V) + \text{rk}(V)\chi(W) - c_1(V) \cdot \sigma \cdot \pi^* \eta_W - c_1(W) \cdot \sigma \cdot \pi^* \eta_V \quad (3.21)$$

and therefore lines one, two and four of (3.17) are mutually consistent if

$$c_1(Q^{-1}) \cdot \eta_W + c_1(Q) \cdot \eta_V = -9 \quad (3.22)$$

This determines the flux through the remaining matter curves, and the net number of  $X^\pm$  fields. We find that

$$c_1(Q^{-1}) \cdot \eta_W = c_1(Q^{-1}) \cdot C_2 = -6 \quad (3.23)$$

and again  $[\Sigma_2] = \eta_W - 2c_1$  must be effective. This can also be achieved, e.g. by picking  $\eta_W \sim n_W c_1 + 4E_1 - 2E_2$  with  $n_W \geq 4$ . More minimal solutions can probably be obtained by picking a slightly more complicated form for  $c_1(Q^{-1})$ . By comparing the first two lines with the last line of (3.17), and using the intersection numbers deduced above, we then predict there are twelve net  $X^-$  generations, four per generation of matter. And we can check that the second and fifth lines of (3.17) are mutually consistent, which also works. Thus, any remaining anomalies can be cancelled by a Green-Schwarz mechanism.

It remains to specify spectral line bundles on  $C_V$  and  $C_W$  so that  $\chi(V) = 0$  and  $\chi(W) = -3$ . Such constructions are explained in [12, 25] and the result is that by using the landscape of Noether-Lefschetz fluxes (and moderately large  $\eta$ ), one can get pretty much anything one wants.

### 3.3.3 Phenomenological properties

After these technicalities, let us now see what kind of structure we can find in the interactions in our scenario. The  $U(1)_X$  charge assignments we have chosen were as follows:

$$\mathbf{10}_m : -3/5, \quad \bar{\mathbf{5}}_m : 4/5, \quad \mathbf{5}_h : 6/5, \quad \bar{\mathbf{5}}_h : -6/5, \quad X^+ : 1 \quad (3.24)$$

Then the  $U(1)_X$  symmetry forbids the following couplings

$$\text{absent} : \bar{\mathbf{5}}_m \mathbf{5}_h, \quad \mathbf{10}_m \bar{\mathbf{5}}_m \bar{\mathbf{5}}_h, \quad \mathbf{10}_m \bar{\mathbf{5}}_m \bar{\mathbf{5}}_m, \quad \mathbf{10}_m \mathbf{10}_m \mathbf{10}_m \bar{\mathbf{5}} \quad (3.25)$$

As discussed above, although the  $\mathbf{5}_h \cdot \bar{\mathbf{5}}_h$  term is neutral, the  $\mu$ -problem and the doublet/triplet splitting problem have already been solved. We now turn on a large VEV for the gluing morphism  $X^+$ . This requires us to arrange the correct sign for the Fayet-Iliopoulos parameter  $\zeta_X$ , and we can do this exactly as in section 2.1 by taking the Kähler class to be of the form

$$J \sim -c_1(K) + \epsilon c_1(\mathcal{O}(H - E_2)). \quad (3.26)$$

As in section 2.1, actually proving stability would require a lengthy analysis which we will not attempt, but one can check the two natural necessary conditions. Then we generate



down type Yukawa couplings which are hierarchically suppressed compared to the up-type couplings:

$$\mathbf{10}_m \mathbf{10}_m \mathbf{5}_h + \frac{\langle X^+ \rangle}{M} \mathbf{10}_m \bar{\mathbf{5}}_m \bar{\mathbf{5}}_h \tag{3.27}$$

where the size of  $X^+$  is set by the Fayet-Iliopoulos parameter, and therefore by  $\epsilon$ . So we have a flavour hierarchy, admittedly very crude, but one can try to improve this e.g. by considering the Kähler potential. Note that whereas the down type couplings are localized at  $\Sigma_2 \cap \Sigma_3$ , the up-type couplings are not localized at points but along  $\Sigma_2$ . We can not get the couplings  $\bar{\mathbf{5}}_m \mathbf{5}_h$  or  $\mathbf{10}_m \bar{\mathbf{5}}_m \bar{\mathbf{5}}_m$ , because they are still forbidden by the remnant selection rules (i.e. the holomorphic zero mechanism). Thus,  $R$ -parity is still preserved at this level, even though the  $U(1)_X$  has been broken. There are bilinear  $R$ -parity violating terms in the Kähler potential, and they may affect neutrino oscillations, but this is known not to be a serious issue. The  $\mu$ -term is also not generated by a VEV for  $X^+$ .

We do generically get dimension five proton decay:

$$\frac{\langle X^+ \rangle}{M^2} \mathbf{10}_m \mathbf{10}_m \mathbf{10}_m \bar{\mathbf{5}}_m \tag{3.28}$$

but it is suppressed by  $\langle X^+ \rangle$ . This suppression is due to the suppression of the bottom Yukawa coupling (3.27), so by itself this doesn't quite solve the dimension five proton decay problem. One could consider addressing this problem by raising the masses of the squarks as in split SUSY models, but we will not go into that here. The corresponding operator with  $\bar{\mathbf{5}}_m \rightarrow \bar{\mathbf{5}}_h$  is suppressed by  $\langle X^+ \rangle^3$  and may effectively be ignored. It may be helpful to recall a simple relation between the  $U(1)$ -charges of certain couplings, which arises from

$$(\mathbf{10}_m \cdot \mathbf{10}_m \cdot \mathbf{5}_h)(\mathbf{10}_m \cdot \bar{\mathbf{5}}_m \cdot \bar{\mathbf{5}}_h) \simeq (\mathbf{10}_m \cdot \mathbf{10}_m \cdot \mathbf{10}_m \cdot \bar{\mathbf{5}}_m)(\mathbf{5}_h \cdot \bar{\mathbf{5}}_h) \tag{3.29}$$

at the level of  $U(1)$ -charges. Since in our scenario the dimension five proton decay operator is charged under a  $U(1)$ -symmetry and the mu-term is neutral, it follows that either the top Yukawa or the bottom Yukawa coupling (or both) must also be charged and vanish when  $\langle X^+ \rangle = 0$ . Of course this is what we found above.

Finally, there is the question of the neutrinos. One might think that it is straightforward to interpret the many  $X^-$  fields as the right-handed neutrinos  $N$  in our scenario. We even get Majorana neutrino masses:

$$\frac{\langle X^+ \rangle^2}{M^2} N^2 \tag{3.30}$$

However it is not that simple because with our assignments, Yukawa couplings of the form  $\bar{\mathbf{5}}_m \mathbf{5}_h X^-$  are perturbatively forbidden by the  $U(1)_X$  symmetry even after turning on a VEV for  $X^+$ . Alternatively, the Weinberg operator

$$LH_u LH_u \tag{3.31}$$

could be generated by integrating out Kaluza-Klein modes, but again this is forbidden by the  $U(1)_X$  symmetry. The only way to generate it in our scenario seems to be through

$M5$ -brane instantons, which can generate such terms when the  $U(1)_X$   $G$ -flux through the  $M5$ -worldvolume is non-vanishing [1]. Analogous neutrino scenarios were considered in [40, 41]. Our model is slightly different and yields an interesting twist on this, because Majorana masses are actually allowed. Generically these non-perturbative corrections give two contributions to the effective Weinberg operator, and one should take the dominant contribution. On the one hand it may generate the missing leptonic Yukawa couplings, which then give rise to the Weinberg operator after integrating out  $N$ . Since the Yukawa couplings are small, the Majorana mass term could be much lower than usual. On the other hand  $M5$ -instantons may generate the Weinberg operator directly as well. At least one can say that in both cases, the resulting neutrino masses are guaranteed to be small. This was much less clear in the scenario of [40, 41], where the Majorana mass term itself was generated by instantons.

We could consider several variations, such as allowing some of the bilinear  $LH_u$  terms, or allowing the  $\mathbf{10}_m \bar{\mathbf{5}}_m \bar{\mathbf{5}}_m$  coupling for the third generation, to get more spectacular signatures. We can also try to vary the complex structure so that one or two generations of  $\mathbf{10}$  approximate a bulk mode. Then as pointed out earlier, once the kinetic terms are properly normalized, one may get further flavour hierarchies due to suppression by a Kähler modulus.

Although the most dangerous sources of  $R$ -parity violation are avoided,  $R$ -parity is not conserved. This is the basic prediction of models of the above type (or indeed many models where  $R$ -parity is part of a  $U(1)$  gauge symmetry, since that gauge symmetry has to be broken and it is often hard to do so without also breaking  $R$ -parity). In principle one could preserve  $R$ -parity if we could find a field  $X$  with  $R$ -charge two to give a VEV to, but somehow in models of the above type we only seem to find fields with charge one.

$R$ -parity violation has many interesting phenomenological signatures. A summary of some of these, with emphasis on models similar to the above where the most dangerous  $\mathbf{10}_m \bar{\mathbf{5}}_m \bar{\mathbf{5}}_m$  terms are still forbidden, can be found in [21]. The simplest prediction of course is that the LSP is not stable and will decay. From a bottom-up perspective there is a bewildering number of  $R$ -parity violating scenarios one could consider, but as emphasized in [21], string compactification gives a theoretical framework for a class  $R$ -parity violating models using the holomorphic zero mechanism and embedding in  $E_8$ . Given the difficulty of engineering an exact  $R$ -parity, and the ubiquitousness of holomorphic zeroes due to abelian gauge symmetries in string compactifications, we should perhaps take this seriously as a possible testable prediction of string GUTs.

## 4 The standard embedding

### 4.1 General comments

Some of the oldest and simplest examples of heterotic compactifications are obtained by ‘embedding the spin connection in the gauge connection,’ i.e. by taking the non-trivial part of the bundle  $V$  for the left-movers to be the same as the tangent bundle  $TZ$  of the Calabi-Yau.

Now it is not hard to see that the standard embedding on an elliptic Calabi-Yau gives rise to degenerate spectral covers, with the structures that we have discussed extensively in part I and part II. On non-singular elliptic fibers  $E$ , we have the short exact sequence

$$0 \rightarrow T_E \rightarrow TZ|_E \rightarrow N_E \rightarrow 0 \tag{4.1}$$

For any fibration, we have that  $N_E$  is a trivial bundle, so when  $Z$  is a three-fold we have  $N_E \simeq \mathcal{O}_E \oplus \mathcal{O}_E$ . Furthermore away from the singular fibers, we have  $T_E = \mathcal{O}_E$ . Thus the spectral cover is supported on three copies of the zero section, plus possibly some vertical components at the singular fibers. Furthermore, the extension is generically non-trivial, so there is also a non-reduced structure or equivalently a nilpotent Higgs VEV along the zero section. And we expect (and will later show) that one gets non-trivial gluings along the intersection of the horizontal and vertical components.

Although  $T^3$ -fibrations are much trickier, the analogous argument yields a similar degenerate spectral cover picture for the dual  $M$ -theory models of standard embeddings. As we discussed in section 2.3, this type of structure is even more interesting for IIA/ $M$ -theory models than for IIB/ $F$ -theory models.

Thus the Fourier-Mukai transform of the tangent bundle automatically leads one to consider gluings, bulk matter and so forth. The fact that these issues already appear in the most prototypical of string compactifications seems to us pretty remarkable. It is all the more curious that such structures have been largely ignored in the study of models with branes.

In this section we consider the spectral data for the tangent bundle of an elliptically fibered  $K3$  surface in more detail. This bundle has previously been considered in [4, 22, 23, 42], and the spectral cover was known to be of a very degenerate form. We will summarize some of the findings of these papers, calculate the Fourier-Mukai transform of the tangent bundle, and then illustrate how to calculate the spectrum directly from the degenerate spectral cover.

## 4.2 The tangent bundle of a $K3$ surface

Let us first discuss the spectral data for the tangent bundle of a  $K3$  surface. Although one could use the general method for deriving the spectral data of a linear sigma model, in the present case there is a direct method which yields more insight. In general, for an  $SU(2)$  bundle with  $c_2 = k$  on  $K3$ , the homology class of the spectral cover is given by

$$[C] = 2[\sigma_B] + k[E] \tag{4.2}$$

and the spectral line bundle has degree  $2k - 6$ . To find the spectral cover for the tangent bundle, we proceed as in [4]. Restricting the tangent bundle to  $E$ , we have the exact sequence

$$0 \rightarrow T_E \rightarrow T_{K3}|_E \rightarrow N_E \rightarrow 0 \tag{4.3}$$

Now away from the singular fibers, we clearly we have  $T_E = N_E = \mathcal{O}_E$ , so the spectral cover is given by twice the zero section, plus perhaps some of the singular fibers. But we also know the homology class of  $C$ . Given that  $c_2 = 24$  we see that the full spectral

cover is given by twice the zero section together with the 24 vertical fibers. Since vertical components do not occur when  $E$  is regular, we identify the vertical fibers of the spectral cover with the twenty-four singular elliptic fibers.

We further want to know if the spectral sheaf on the zero section corresponds to a rank two bundle or a non-trivial sheaf on the first infinitesimal neighbourhood. This is the question whether the above sequence (4.3) splits. The short exact sequence (4.3) leads to the long exact sequence

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{O}_E) \rightarrow H^0(TK3|_E) \rightarrow H^0(\mathcal{O}_E) \\ \rightarrow H^1(\mathcal{O}_E) \rightarrow \dots \end{aligned} \tag{4.4}$$

and the sequence splits when the coboundary map vanishes. Now the coboundary map is also precisely the Kodaira-Spencer map. The Kodaira-Spencer map is the derivative of the period map  $\mathbf{P}_b^1 \rightarrow \mathbf{P}_\tau^1$ , so the zeroes correspond to the branch points of the period map. We can compute this number by a Riemann-Hurwitz calculation.

The period domain can be thought of as a Riemann sphere  $\mathbf{P}_\tau^1$  with three-special points, which we call  $w = 0, 1$  and  $\infty$ . At infinity the elliptic curve is nodal. At  $w = 0$  the elliptic curve has a  $\mathbf{Z}_4$  symmetry, and there is a  $\mathbf{Z}_4$  monodromy around this point. Finally at  $w = 1$  the elliptic curve has a  $\mathbf{Z}_6$  symmetry, and there is a  $\mathbf{Z}_6$  monodromy around this point.

The period map  $\mathbf{P}_B^1 \rightarrow \mathbf{P}_\tau^1$  has degree 24, as there are 24 singular fibers. Over  $w = 0$ , the 24 sheets meet in 12 pairs, and the  $\mathbf{Z}_4/\mathbf{Z}_2 = \mathbf{Z}_2$  monodromy interchanges the two sheets in each pair. (The  $\mathbf{Z}_2$  subgroup which corresponds to inversion on the elliptic curve acts trivially on these sheets.) Over  $w = 1$  the 24 sheets meet in eight triples, and the  $\mathbf{Z}_6/\mathbf{Z}_2 = \mathbf{Z}_3$  monodromy acts on the three sheets in each triple. The remaining ramification points are the ones whose number we want to calculate (see [43], proposition 3.3).

Now the Riemann-Hurwitz formula says that

$$\chi(\mathbf{P}_B^1) = -2 = \chi(\mathbf{P}_\tau^1) \times 24 + \text{Ram} \tag{4.5}$$

and we have

$$\text{Ram} = 12 \times (2 - 1) + 8 \times (3 - 1) + R \times (2 - 1) \tag{4.6}$$

where  $R$  denotes the remaining ramification points. So calculation shows that there are  $R = 18$  branch points, i.e. there are eighteen points on the base where we get  $\mathcal{O}_E \oplus \mathcal{O}_E$  instead of the non-trivial extension, which we denote by  $F_2$ .

In heterotic models with  $SU(2)$ -holonomy bundles, there are two interesting cohomology groups one could understand. Embedding the spin connection in the gauge connection yields a six dimensional theory with  $E_7$  gauge group,  $h^1(TK3) = 20$  half-hypermultiplets in the **56** of  $E_7$ , and  $2 \times 45 = 90$  moduli. Here we want to understand how the **56** matter fields are realized for the degenerate spectral cover dual to the tangent bundle, i.e. in the 7-brane picture.

The number of **56** matter multiplets is counted by  $H^1(TK3)$ , which just counts the number of complex structure deformations of  $K3$ . Using the Parseval theorem for the

Fourier-Mukai transform, which says that

$$\text{Ext}^p(V_1, V_2) = \text{Ext}^p(\text{FM}^1(V_1), \text{FM}^1(V_2)) \tag{4.7}$$

we find that

$$\dim \text{Ext}^1(i_{B*}\mathcal{O}_{\mathbf{P}^1}, \mathcal{L}) = 20 \tag{4.8}$$

Here we have switched to the convention where the Poincaré sheaf is normalized so that  $\text{FM}^1(\mathcal{O}_{K3}) = i_{B*}\mathcal{O}_{\mathbf{P}^1}$ , in order to avoid factors of the canonical bundle  $\mathcal{O}_{\mathbf{P}^1}(-2)$  from floating around. To understand these deformations, clearly we will only need the behaviour of the spectral cover near the zero section, so we may as well perform a normal cone degeneration and study the resulting spectral cover in the total space of  $\mathcal{O}(-2)_{\mathbf{P}^1}$  over the base  $\mathbf{P}^1$ . Let us temporarily assume that the spectral cover on which  $\mathcal{L}$  is supported is smooth and generic. A generic curve in the linear system  $2S + \pi^*\eta$ , with  $\eta$  consisting of twenty-four points on the base, is of the form

$$g_{24}s^2 + f_{20} = 0 \tag{4.9}$$

and intersects the zero section precisely twenty times. The Ext group (4.8) counts the gluing morphisms that one can turn on at each such intersection point, so from this point of view the number of multiplets in the **56** is twenty, precisely as expected.

Now we consider the degenerate cover corresponding to the tangent bundle. As we discussed above, the Higgs field must be generically nilpotent, but there are twenty-four vertical components and eighteen points where the Higgs field vanishes. So after normal cone degeneration, our Higgs field should be of the form

$$\Phi \sim \begin{pmatrix} 0 & f_{18}/g_{24} \\ 0 & 0 \end{pmatrix} \tag{4.10}$$

acting on  $E = \mathcal{O}(2) \oplus \mathcal{O}(-2)$ .

### 4.3 Derivation of the Fourier-Mukai transform

We want to understand  $\text{Ext}^1(i_*\mathcal{O}, \mathcal{L})$  (the number of **56s**) directly for the degenerate cover dual to the tangent bundle. Of course we already know that we are going to get twenty generators. But we would like to illustrate how to explicitly calculate with degenerate 7-brane configurations and identify the actual deformations corresponding to the matter fields. To do the calculation, we first need a more precise description of  $\mathcal{L}$ . We will try to break  $\mathcal{L}$  into several pieces.

Let us first try to find a global version of (4.3) which also holds at the singular fibers. Denote by  $T_\pi$  the tangent fibers to the projection map  $\pi : K3 \rightarrow \mathbf{P}_B^1$ . Then we claim that

$$0 \rightarrow T_\pi \rightarrow TK3 \xrightarrow{d\pi} \pi^*T_B \otimes I_{24} \rightarrow 0 \tag{4.11}$$

Here  $I_{24}$  is the ideal sheaf of the singular points on the 24 nodal fibers. This sequence is clearly correct generically, so the main thing to check is that it also works at the singular

points on the nodal fibers. We can do this by writing the map explicitly in local coordinates near these points. It is given by an equation

$$xy = t \tag{4.12}$$

where  $t$  is a local coordinate on the base. The projection map is given by  $\pi = xy$ . Note that the total space is smooth and that the singularity only appears by looking at the individual fiber  $\pi^{-1}(0)$ .

Now we can locally identify the tangent bundle of the  $K3$  with

$$TK3 = \mathbf{C}[x, y] \langle \partial_x, \partial_y \rangle \tag{4.13}$$

and  $d\pi$  is given by

$$d\pi : (\partial_x, \partial_y) \rightarrow (y\partial_t, x\partial_t) \tag{4.14}$$

Note that this map is not surjective at  $x = y = 0$ ; this is why we have the ideal sheaf  $I_{24}$  appearing on the right of (4.11). For subsequent use, we also note that  $T_\pi$  is generated by  $x\partial_x - y\partial_y$ .

We further claim that we have an injective map  $\pi^*T_B^* \rightarrow T_\pi$  by contracting with the Poisson structure  $\omega^{-1}$ . Again this is clear generically but we need to check what happens at the singular point on the nodal fibers. Using the equation above,  $\pi^*T_B^*$  is generated by  $dt$ , and the holomorphic  $(2, 0)$  form is given by  $\omega = dx \wedge dy$ . Thus we get the map

$$dt = xdy + ydx \xrightarrow{\omega^{-1}} x\partial_x - y\partial_y \tag{4.15}$$

As noted above  $T_\pi$  is generated by  $x\partial_x - y\partial_y$ , therefore contraction with  $\omega^{-1}$  extends to an isomorphism  $\pi^*T_B^* \cong T_\pi$  on the whole  $K3$ , even on the nodal fibers. Thus we have established a short exact sequence

$$0 \rightarrow \pi^*T_B^* \rightarrow TK3 \xrightarrow{d\pi} \pi^*T_B \otimes I_{24} \rightarrow 0 \tag{4.16}$$

where of course  $T_B^* = \mathcal{O}(-2)_{\mathbf{P}^1}$  and  $T_B = \mathcal{O}(2)_{\mathbf{P}^1}$

Now we can apply the Fourier-Mukai functor  $\mathbf{FM}^\bullet$  to this short exact sequence and deduce the structure of  $\mathcal{L} = \mathbf{FM}^1(TK3)$ . We get the long exact sequence

$$\begin{aligned} 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \\ \rightarrow i_{B*}T_B^* \rightarrow \mathcal{L} \rightarrow \mathcal{K} \rightarrow 0 \end{aligned} \tag{4.17}$$

Thus we see that  $\mathcal{L}$  is given by an extension of  $\mathcal{K}$  by  $i_{B*}T_B^*$ , where  $\mathcal{K} = \mathbf{FM}^1(\pi^*T_B \otimes I_{24})$  needs to be understood in more detail.

Let us denote the 24 singular points on the nodal fibers by  $p_i$ . Then we have another short exact sequence:

$$0 \rightarrow \pi^*T_B \otimes I_{24} \rightarrow \pi^*T_B \rightarrow \pi^*T_B|_{24p_i} \rightarrow 0 \tag{4.18}$$

Applying the Fourier-Mukai transform, away from the singular points we get

$$\begin{aligned} 0 \rightarrow 0 \rightarrow 0 \rightarrow \bigoplus_{i=1}^{24} \mathcal{O}_{E_i} \\ \rightarrow \mathcal{K} \rightarrow i_*T_B \rightarrow 0 \end{aligned} \tag{4.19}$$

where  $b_i = \pi(p_i)$ . In other words, away from the singular points,  $\mathcal{K}$  is just the extension of  $\mathcal{O}(2)$  on the base by 24 vertical components. We have not determined what the sheaves supported on the vertical components are exactly at the singular points  $p_i$ . In fact this does not really matter for the computation of  $\text{Ext}^1(i_*\mathcal{O}, \mathcal{L})$ , which is localized on the zero section, away from the  $p_i$ . It does matter for our calculation however if the extension above is non-trivial — that is, if the gluing morphism at the intersection of the horizontal and vertical components is non-zero. From our discussion in part I, we expect that the gluing morphism is non-zero, because turning off the gluing morphism is singular and would lead to a new branch (the small instanton transition). We will now check this explicitly.

The fiber of  $\mathcal{L}$  at  $b_i$  is given by

$$\text{FM}^1(TK3)|_{b_i} = H^1(E_i, TK3|_{E_i} \otimes \mathcal{O}(b_i - \sigma)|_{\sigma=b_i}) = H^1(E_i, TK3|_{E_i}) \quad (4.20)$$

where  $E_i$  is the fiber over  $b_i$ . The question is whether this is rank one (so that the gluing morphism is non-zero, and we get a line bundle) or whether this is rank two (so that the gluing morphism is zero).

To get  $H^1(E_i, TK3|_{E_i})$ , we can try to restrict our short exact sequence (4.16) to the singular fiber  $E_i$  and take cohomology. This doesn't immediately work because the restriction is not exact. We get

$$0 \rightarrow \ker(d\pi|_{E_i}) \rightarrow TK3|_{E_i} \rightarrow I_{p_i} \rightarrow 0 \quad (4.21)$$

but  $\ker(d\pi|_{E_i})$  is not quite the same as  $(\ker d\pi)|_{E_i} = \pi^*T_B^*|_{E_i} = \mathcal{O}_{E_i}$ . The failure again happens at the singular point, and we may use a local calculation to deduce  $\ker(d\pi|_{E_i})$ .

Using our previous model  $xy = t$ , we see that  $\ker(d\pi|_{E_i})$  is generated by

$$\ker(d\pi|_{t=0}) : \quad a\partial_x + b\partial_y \mid ay + bx = 0, \quad xy = 0 \quad (4.22)$$

Away from  $x = y = 0$  this is one-dimensional. But at  $x = y = 0$  this is two-dimensional. In other words, denoting the normalization map of the nodal  $\mathbf{P}^1$  by  $\nu$ , we have  $\ker(d\pi|_{E_i}) = \nu_*\mathcal{O}_{\mathbf{P}^1}(k)$  for some  $k$ . In fact, since we have the natural inclusion

$$\mathcal{O}_{E_i} = (\ker d\pi)|_{E_i} \hookrightarrow \ker(d\pi|_{E_i}) \quad (4.23)$$

we see that actually  $\ker(d\pi|_{E_i}) = \nu_*\mathcal{O}_{\mathbf{P}^1}(0)$ .

So we can now take cohomology of (4.21). In the associated long exact sequence we will encounter the cohomologies  $H^n(\nu_*\mathcal{O}_{\mathbf{P}^1}(0))$  and  $H^n(I_{p_i})$ . We can calculate  $H^n(\nu_*\mathcal{O}_{\mathbf{P}^1}(0))$  from the Leray sequence associated to the normalization map  $\nu$ . It is not hard to see that  $H^0(\nu_*\mathcal{O}_{\mathbf{P}^1}(0)) = 1$  and  $H^1(\nu_*\mathcal{O}_{\mathbf{P}^1}(0)) = 0$ . Similarly, using the short exact sequence

$$0 \rightarrow I_{p_i} \rightarrow \mathcal{O}_{E_i} \rightarrow \mathcal{O}_{p_i} \rightarrow 0 \quad (4.24)$$

we calculate that  $H^0(I_p) = 0$  and  $H^1(I_p) = 1$ . Then the long exact sequence associated to (4.21) gives us

$$\begin{aligned} 0 \rightarrow 1 \rightarrow H^0(TK3|_{E_i}) \rightarrow 0 \\ \rightarrow 0 \rightarrow H^1(TK3|_{E_i}) \rightarrow 1 \end{aligned} \quad (4.25)$$

Hence we find that  $H^1(TK3|_{E_i}) = 1$ . So  $\mathcal{L}|_{b_i}$  is a line bundle and the gluing morphism is non-zero, as promised.

To summarize, we found that  $\mathcal{L} = \text{FM}^1(TK3)$  can be built explicitly from two extension sequences, given in (4.17) and (4.19). Furthermore, we checked that the gluing morphisms appearing in (4.19) are non-zero. This matches well with our earlier expectations.

#### 4.4 Calculation of the spectrum

Given this explicit presentation, we can now proceed to find  $\text{Ext}^1(i_*\mathcal{O}, \mathcal{L})$ . First we compute  $\text{Ext}^1(i_*\mathcal{O}, \mathcal{K})$  using (4.19). We get the long exact sequence

$$\begin{aligned} 0 \rightarrow & 0 \rightarrow \text{Ext}^0(i_*\mathcal{O}, \mathcal{K}) \rightarrow \text{Ext}^0(i_*\mathcal{O}, i_*T_B) \\ & \rightarrow \oplus_{i=1}^{24} \text{Ext}^1(i_*\mathcal{O}, \mathcal{O}_{E_i}) \rightarrow \text{Ext}^1(i_*\mathcal{O}, \mathcal{K}) \rightarrow \text{Ext}^1(i_*\mathcal{O}, i_*T_B) \rightarrow 0 \end{aligned} \quad (4.26)$$

where we used that  $\text{Ext}^p(i_*\mathcal{O}, \mathcal{O}_{E_i}) = 0$  for  $p = 0$  and  $p = 2$ . Now  $\text{Ext}^0(i_*\mathcal{O}, i_*T_B) = H^0(\mathcal{O}_{\mathbf{P}^1}(2))$  which is three-dimensional, and  $\oplus_{i=1}^{24} \text{Ext}^1(i_*\mathcal{O}, \mathcal{O}_{E_i})$  consists of 24 gluing deformations. The coboundary map is given by taking a generator of  $H^0(\mathcal{O}_{\mathbf{P}^1}(2))$ , restricting to  $b_i$  and multiplying with the gluing VEV of  $\mathcal{K}$  at  $b_i$ . The gluing deformations in the image of this coboundary map can be removed by symmetries. Thus only  $24 - 3 = 21$  of the gluing deformations in  $\oplus_{i=1}^{24} \text{Ext}^1(i_*\mathcal{O}, \mathcal{O}_{E_i})$  are honest deformations that inject to  $\text{Ext}^1(i_*\mathcal{O}, \mathcal{K})$ , and we also get  $\text{Ext}^0(i_*\mathcal{O}, \mathcal{K}) = 0$ . We further have

$$\text{Ext}^1(i_*\mathcal{O}, i_*T_B) \cong H^1(\mathcal{O}_{\mathbf{P}^1}(2)) \oplus H^0(\mathcal{O}_{\mathbf{P}^1}(0)) \quad (4.27)$$

which yields zero non-abelian bundle deformations and one nilpotent deformation on the base. Thus altogether we find that  $\text{Ext}^1(i_*\mathcal{O}, \mathcal{K})$  consists of 22 first order deformations.

Finally we apply  $\text{Ext}$  to the exact sequence in (4.17). We find

$$\begin{aligned} 0 \rightarrow & 0 \rightarrow \text{Ext}^0(i_*\mathcal{O}, \mathcal{L}) \rightarrow 0 \\ & \rightarrow \text{Ext}^1(i_*\mathcal{O}, i_*T_B^*) \rightarrow \text{Ext}^1(i_*\mathcal{O}, \mathcal{L}) \rightarrow \text{Ext}^1(i_*\mathcal{O}, \mathcal{K}) \\ & \rightarrow \text{Ext}^2(i_*\mathcal{O}, i_*T_B^*) \rightarrow \text{Ext}^2(i_*\mathcal{O}, \mathcal{L}) \rightarrow 0 \end{aligned} \quad (4.28)$$

The result now depends on the rank of the coboundary map, which is given by composing with the extension class in  $\text{Ext}^1(\mathcal{K}, i_*T_B^*)$  that was used to build  $\mathcal{L}$ . This extension map is non-zero except at eighteen points on the base, and these points are separate from the intersections with the vertical fibers where  $\text{Ext}^1(i_*\mathcal{O}, \mathcal{K})$  are localized, so we expect that the rank of the coboundary map is maximal. Now we have

$$\text{Ext}^2(i_*\mathcal{O}, i_*T_B^*) \cong H^0(\mathcal{O}_{\mathbf{P}^1}(2))^* \quad (4.29)$$

which is three dimensional, and similarly  $\text{Ext}^1(i_*\mathcal{O}, i_*T_B^*)$  is one dimensional. This gives us that  $\text{Ext}^1(i_*\mathcal{O}, \mathcal{L})$  has dimension  $22 - 3 + 1 = 20$ , as required.

Since we are considering compactification on  $K3$ , we have extra supersymmetry, and our chiral fields in  $\text{Ext}^1(\mathcal{O}_S, \mathcal{L})$  get paired with chiral fields in  $\text{Ext}^1(\mathcal{L}, \mathcal{O}_S)$  into hypermultiplets. We have

$$\text{Ext}^1(\mathcal{L}, \mathcal{O}_S) \simeq H^1(T^*K3) \quad (4.30)$$



so these can be interpreted as deformations of the cotangent bundle. In the spectral cover picture these deformations are the duals of the ones found before.

The second cohomology group we could check is  $\text{Ext}^1(\mathcal{L}, \mathcal{L})$ , or equivalently the moduli of the spectral sheaf. The moduli space is hyperkähler and admits a fibration by tori. The base is given by deformations of the spectral cover, and the fiber corresponds to the Jacobian. The base and the fiber have the same complex dimension. For a generic  $\text{SU}(2)$  bundle with  $c_2 = k$ , the genus of the spectral curve (and hence the dimension of the Jacobian) is  $2k - 3$ , which yields 45 for  $k = 24$ , and the degree of the spectral sheaf is  $g - 3 = 42$ . The Mukai moduli space of sheaves on  $K3$  has only one component, so we should get the same dimension for our non-reduced spectral cover dual to the tangent bundle. This yields the quaternionic dimension

$$\dim H^1(\text{End}(TK3)) = \dim \text{Ext}^1(\mathcal{L}, \mathcal{L}) = 45 \tag{4.31}$$

Again we would like to understand these deformations explicitly for the degenerate cover. We could calculate this explicitly as above, but it is simpler to guess in this case. Using the embedding  $j : \mathbf{P}^1 \hookrightarrow C$  we get

$$(\text{moduli that keep } \mathbf{P}^1 \text{ fixed}) \rightarrow H^0(N_{C/K3}) \rightarrow H^0(\mathbf{P}^1, \mathcal{O}(20)) \tag{4.32}$$

The deformations on the right are obtained from pulling the normal bundle  $N_{C/K3}$  back to  $\mathbf{P}^1$ . Since the class of  $C$  has intersection number 20 with the  $\mathbf{P}^1$ , this gives us  $j^*N = \mathcal{O}(20)$ . These deformations correspond to smoothing deformations, and there are 21 of them. The moduli on the left correspond to moduli that leave the embedding  $\mathbf{P}^1 \hookrightarrow C$  fixed. They correspond to moving the 24 fibers. So in all we get 45 moduli for deforming the spectral cover  $C$ , as expected. Similarly we get one line bundle modulus from the line bundle on each of the 24 vertical fibers, and  $24 - 3 = 21$  independent gluing VEVs at the intersections.

It is interesting to compare this to the sheaf  $\mathcal{I}_{24}$  corresponding to 24 pointlike instantons [4]. Clearly on the generic elliptic fiber away from the 24 point-like instantons we have

$$\mathcal{I}_{24}|_E = \mathcal{O}_E \oplus \mathcal{O}_E \tag{4.33}$$

and since  $c_2 = 24$ , the spectral cover for  $\mathcal{I}_{24}$  is also given by twice the zero section and 24 elliptic fibers (though these need not coincide with the singular fibers). The difference with the tangent bundle is that this has an ordinary Higgs field VEV with only trivial Jordan structure, and zero gluing VEV at the intersection with the vertical fibers associates to the point-like instantons. This difference manifests itself in how the moduli are realized and in a jump in the number of matter fields. These computations are much simpler than for the tangent bundle and left as an exercise (partially done in [4]).

## 5 Linear sigma models

There are many realizations of Grand Unified Models in string theory. Although it is not quite precise, one could say that morally all these different realizations are dual to each other. Here we would like to focus on the connections between three realizations for which

currently the most powerful techniques are available:  $F$ -theory, large volume heterotic models, and heterotic Landau-Ginzburg orbifolds. As we briefly reviewed,  $F$ -theory is weakly coupled in the regime where the  $8d$  heterotic string coupling is large. Landau-Ginzburg orbifolds are special models which are valid in a regime where the heterotic string coupling and Kähler moduli are small. We can hope to learn about each of these realizations by comparing with the others.

These questions have some relevance for phenomenology. We do not have a non-perturbative formulation, and each weak coupling description is in principle only valid for infinitesimal values of the coupling — in  $F$ -theory, these couplings are inverse volumes in Planck units. An old argument of Dine and Seiberg [44] essentially guarantees that we cannot find a string vacuum with all moduli stabilized in perturbation theory. An exception to this argument would be if we could define a large  $N$  expansion, which requires an infinite number of vacua — this question has not yet been settled.<sup>2</sup> On the other hand, one could not simply disregard the evidence from perturbation theory. A better understanding of  $F$ -theory away from the large volume limit could help illuminate possible qualitative issues with vacua constructed using a perturbative expansion. Of course this goes both ways, and one can also learn about strong coupling behaviour in the heterotic string.

In order to set up such comparisons, we would like to be able to translate between the linear sigma model description of bundles, and the spectral cover description of bundles. The main purpose of this section is to find an explicit algorithm for producing the spectral sheaf associated to a monad on an elliptically fibered Calabi-Yau  $Z$ . Before we specialize to monads however, it will be useful to make some remarks that apply more generally.

### 5.1 General comments on bundles over elliptic fibrations

Given any bundle  $\tilde{V}$  on our elliptic Calabi-Yau  $Z$ , there is a natural ‘evaluation’ map

$$\Psi : \pi^* \pi_* \tilde{V} \rightarrow \tilde{V} \tag{5.1}$$

where  $\pi : Z \rightarrow B_2$  is the projection to the base. Let us describe this map. Given a (sufficiently small) open set  $U \subset Z$ , the local sections of  $\pi^* \pi_* \tilde{V}$  are given by

$$\Gamma(U, \pi^* \pi_* \tilde{V}) = \Gamma(\pi^{-1} \pi(U), \tilde{V}) \tag{5.2}$$

Since  $\pi^{-1} \pi(U)$  is obviously bigger than  $U$ , global sections of  $\tilde{V}$  over  $\pi^{-1} \pi(U)$  are clearly a subset of global sections of  $\tilde{V}$  over  $U$ . We can always restrict a global section over  $\pi^{-1} \pi(U)$  to get a global section over  $U$  (but not vice versa). This is the canonical map in (5.1).

In our applications,  $\tilde{V}$  will have some additional properties. We will be interested in the Fourier-Mukai transform of a bundle  $V$  with  $c_1(V) = 0$ . The restriction to the generic fiber  $E$  should be semi-stable and degree zero. We define

$$\tilde{V} \equiv V \otimes \mathcal{O}(\sigma_{B_2}) \tag{5.3}$$

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<sup>2</sup>An example of a runaway mode that is absent for zero coupling was studied in [45]. Also interesting in this regard is the warped deformed conifold. It exhibits a ‘Kähler’ mode that is parametrically light compared to the KK scale [46], even though compactification on the unwarped  $T^*S^3$  does not yield any such light modes, i.e. turning on the flux is not a small perturbation.

Given an elliptic fiber  $E$ , a generic  $V$  splits as a sum of  $r = \text{rank}(V)$  degree zero line bundles on  $E$ :

$$V_E \simeq \mathcal{O}_E(p_1 - p_\infty) \oplus \dots \oplus \mathcal{O}_E(p_r - p_\infty) \tag{5.4}$$

The points  $\{p_1, \dots, p_r\}$ , when varied over the base, sweep out the spectral cover of  $V$ , and  $p_\infty$  is the point that lies on the zero section. (We denoted it by  $p_\infty$  in order to clearly distinguish it from the  $p_i$ .) Then we have

$$\tilde{V}_E \simeq \mathcal{O}_E(p_1) \oplus \dots \oplus \mathcal{O}_E(p_r) \tag{5.5}$$

and

$$H^0(E, \tilde{V}_E) = (s_1, \dots, s_r) \quad H^1(E, \tilde{V}_E) = 0 \tag{5.6}$$

where  $s_i$  is the unique section of  $\mathcal{O}_E(p_i)$ , which vanishes at  $p_i$ .

Let us examine the map (5.1) on the fibers over a point  $p \in Z$ , and define  $E_p = \pi^{-1}\pi(p)$ . The fiber of  $\pi^*\pi_*\tilde{V}$  is

$$\pi^*\pi_*\tilde{V}_p = H^0(E_p, \tilde{V}_{E_p}) \tag{5.7}$$

This is spanned by the  $r$  sections  $(s_1, \dots, s_r)$ , and the map (5.1) evaluates them at  $p$ . The sections are linearly independent away from  $\{p_1, \dots, p_r\}$ , so we can represent  $\Psi$  as

$$\Psi_E \sim \begin{pmatrix} s_1(p) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & s_r(p) \end{pmatrix} : \mathcal{O}_E^r \rightarrow \tilde{V}_E \tag{5.8}$$

The spectral cover is given by the zero locus of the  $(s_1, \dots, s_r)$ , i.e. it is identified with the locus

$$\det(\Psi) = 0 \tag{5.9}$$

All this is of course completely analogous to the map  $\lambda I - \Phi$  that we encountered for Higgs bundles; after diagonalizing  $\lambda I - \Phi$  as in equation (2.7) of part I, the diagonal entries behave like the sections  $s_i$ . Indeed, Higgs bundles are not necessarily valued in a line bundle, but can take values in more general objects, like an elliptic curve. This was one of the main ideas in the adaptation of spectral cover methods for Higgs bundles to the heterotic string [47, 48].

In particular, if the point  $p$  does not lie on the spectral cover, then (5.1) yields an isomorphism on the fibers. But if the point  $p$  does lie on the spectral cover, then the map (5.1) has rank  $r - 1$ . As in equation (2.6) of part I for conventional Higgs bundles, it is natural to define a sheaf  $\mathcal{L}$  as the cokernel of  $\Psi$ :

$$0 \rightarrow \pi^*\pi_*\tilde{V} \rightarrow \tilde{V} \rightarrow \mathcal{L} \rightarrow 0 \tag{5.10}$$

By the observations above,  $\mathcal{L}$  is a rank one sheaf supported on the spectral cover. Given the generality of the construction, it must be essentially equal to the Fourier-Mukai transform of  $V$ . (After submission, we noticed that this claim was also made in [49].)

Let us briefly review some generalities about the Fourier-Mukai transform. A nice review is [50]. We define two projections,  $p_{1,2} : Z \times_S Z \rightarrow Z$  on the first and second factor respectively. We also define the Poincaré sheaf:

$$\mathcal{P} = \mathcal{O}_{Z \times_B Z}(\Delta - Z \times \sigma_{B_2} - \sigma_{B_2} \times Z) \otimes p_1^* \pi^* K_B^{-1} \tag{5.11}$$

Then the Fourier-Mukai transform of a sheaf  $V$  is defined to be

$$\text{FM}^\bullet(V) \equiv R^\bullet p_{2*}(p_1^* V \otimes \mathcal{P}) \tag{5.12}$$

It is strictly speaking a complex, but if  $V$  is reasonably well-behaved then this complex is non-zero in only one degree, and we simply get a coherent sheaf. We may also define the inverse transform

$$\widehat{\text{FM}}^i(V) \equiv R^{i-1} p_{2*}(p_1^* V \otimes \mathcal{P}') \tag{5.13}$$

where

$$\mathcal{P}' = \mathcal{P}^\vee \otimes p_1^* \pi^* K_{B_2}^{-1} \tag{5.14}$$

Since these definitions may look somewhat intimidating, let us explain some of the intuition by restricting to a given elliptic fiber  $E$ .

The Poincaré sheaf  $\mathcal{P}$  is the universal line bundle on  $E \times E^\vee$  such that the restriction to  $\sigma \in E^\vee$  gives the line bundle  $\mathcal{O}_E(\sigma - p_\infty)$  on  $E$ . This is the analogue of the factor  $e^{ikx}$  in the ordinary Fourier transform. Here  $E^\vee$  is the dual elliptic curve, which is isomorphic to  $E$  itself. Therefore on each  $E$ , the Fourier-Mukai transform is given by tensoring  $V_E$  with a flat line bundle  $L_\sigma = \mathcal{O}_E(p_\infty - \sigma)$ , where  $\sigma$  is a coordinate on  $E^\vee$ , and then taking the cohomology  $H^1(V_E \otimes L_\sigma)$ . In other words, as a line bundle it is defined by assigning to a point  $\sigma \in E$  the vector space

$$\sigma \rightarrow H^1(V_E \otimes L_\sigma) \tag{5.15}$$

Now let us go back to the situation where the restriction decomposes as  $\tilde{V}_E \cong L_1 \oplus \dots \oplus L_r$ , where each  $L_i$  is of degree one. We have  $L_i \simeq \mathcal{O}(p_i - p_\infty)$ , but we will not specify the isomorphism explicitly because it depends on a parameter which may vary over the base. Then we have

$$V_E \cong L_1(-p_\infty) \oplus \dots \oplus L_r(-p_\infty) \tag{5.16}$$

Since  $h^1(L_i(-\sigma)) = h^0(L_i(-\sigma))^\vee = 1$  if  $\sigma = p_i$ , and zero otherwise, we see that the Fourier-Mukai transform of such a bundle  $V$  is supported on  $\sigma = p_1, \dots, p_r$ , and the fibers of the dual sheaf at these points are given by  $H^1(L_i(-p_i))$  respectively.

Now let us show we recover the same fiberwise structure from the cokernel sequence (5.10). The spectral cover intersects  $E$  in the points  $p_1, \dots, p_r$ . Thus the fiber of  $\mathcal{L}$  is given by the fiber of  $L_i$  at  $p_i$ . This is clearly isomorphic to  $H^0(L_i|_{p_i})$ . From the long exact sequence associated to

$$0 \rightarrow L_i \otimes \mathcal{O}(-p_i) \rightarrow L_i \rightarrow L_i|_{p_i} \rightarrow 0 \tag{5.17}$$

we see that  $H^0(L_i|_{p_i})$  lifts to  $H^1(L_i \otimes \mathcal{O}(-p_i))$ . This is the same as what we got above from  $\text{FM}^1(V)|_E$ .

Thus if  $V$  is given by a holomorphic bundle on  $Z$  which is semi-stable on generic fibers, we expect that  $\text{FM}^1(V)$  and  $\mathcal{L}$  must agree up to tensoring with a line bundle pulled-back from the base. We will find the appropriate ‘normalization’ below. We have not proven this conjecture but one can do additional consistency checks. One such check will be done in section 5.2.

One way to find the ‘normalization’ is as follows. With the definition of  $\mathcal{P}$  as above, we have  $\text{FM}^1(\mathcal{O}_Z) = \sigma_{B_2*}K_{B_2}$  [50]. From (5.10) with  $V = \mathcal{O}_Z$  we get  $\mathcal{L} = \sigma_{B_2*}N_{B_2}$ , i.e. the normal bundle on  $B_2$ . Since  $Z$  is a Calabi-Yau three-fold, these are the same. So we have

$$\text{FM}^1(V) = \mathcal{L} \tag{5.18}$$

In some sense it would be more natural to twist the Poincaré sheaf by  $\pi^*K_B^{-1}$  so that  $\text{FM}^1(\mathcal{O}_Z) = \mathcal{O}_{B_2}$ , but at any rate this is a matter of convention.

Let us note two important and useful properties of FM. The first is the Parseval formula

$$\text{Ext}_X^i(F, G) = \text{Ext}_X^i(\text{FM}(F), \text{FM}(G)) \tag{5.19}$$

The second is that given a short exact sequence of sheaves

$$0 \rightarrow F \rightarrow G \rightarrow K \rightarrow 0 \tag{5.20}$$

we obtain the long exact sequence:

$$\rightarrow \text{FM}^{i-1}(K) \rightarrow \text{FM}^i(F) \rightarrow \text{FM}^i(G) \rightarrow \text{FM}^i(K) \rightarrow \text{FM}^{i+1}(F) \rightarrow \tag{5.21}$$

These properties will be useful in the following.

The conclusion of our discussion is that we can find the spectral sheaf if we can explicitly write the canonical evaluation map (5.1). So far our discussion was general. Now we will specialize to the case of monads.

Reference [22] provided an algorithm for writing down the spectral cover associated to a monad. This was based on the work of [42], which showed that the spectral cover corresponds to the zero locus of the determinant  $\det(\Psi) = 0$  of the map (5.1), as we also discussed above. We now want to extend this to write down the spectral sheaf. The algorithm of [22] actually provides a set of sections which generate  $(s_1, \dots, s_r)$ . By our previous discussion and making some adaptations of [22] we then also recover the spectral sheaf, constructed as the cokernel. We discuss this more explicitly in subsection 5.4.

## 5.2 Jordan blocks

In order for  $V$  to admit a spectral cover description, its restriction to the generic fiber  $E$  should be semi-stable and degree zero. If the restriction  $V|_E$  to an elliptic fiber  $E$  were unstable, then the Fourier-Mukai transform of  $V$  is supported on the whole fiber, so this should not happen generically. So far we seem to have assumed that any degree zero semi-stable bundle would decompose as a direct sum of line bundles:

$$V_E \simeq \mathcal{O}(p_1 - p_\infty) \oplus \dots \oplus \mathcal{O}(p_r - p_\infty) \tag{5.22}$$

Although this is generically the correct situation, it is not the most general possibility. There also exist semi-stable degree zero bundles on  $E$  that are not decomposable as a sum of line bundles. In fact, it will soon be clear that this situation occurs in codimension one on the base even for generic spectral covers, namely at the branch locus of the cover. In particular, the restriction of the hermitian Yang-Mills connection on  $V$  fails to be flat on these fibers.

The general classification of semi-stable bundles on  $T^2$  is due to Atiyah [51]. For each integer  $r$  there exist rank  $r$  bundles on  $E$  that are not decomposable. Consider the following extension sequence on an elliptic curve:

$$0 \rightarrow \mathcal{O}_E \rightarrow F_2 \rightarrow \mathcal{O}_E \rightarrow 0 \tag{5.23}$$

Since  $\text{Ext}^1(\mathcal{O}, \mathcal{O}) = H^1(\mathcal{O}) = \mathbf{C}$ , we have two possibilities: either  $F_2 \sim \mathcal{O}_E \oplus \mathcal{O}_E$ , or  $F_2$  is the unique non-trivial rank two extension. Similarly we can consider the unique non-trivial extensions on  $T^2$ :

$$0 \rightarrow F_{r-1} \rightarrow F_r \rightarrow \mathcal{O}_E \rightarrow 0 \tag{5.24}$$

The most general semi-stable bundle of slope zero is a sum of factors of the form  $F_r \otimes L$ , where  $L$  is a degree zero line bundle and we took  $F_1 = \mathcal{O}_E$ .

Since the spectral cover apparently exists for such more general bundles, let us try to see what it looks like. The bundle  $F_2$  is an extension of  $\mathcal{O}_E$  by itself, so the Fourier-Mukai transform of  $F_2$  will have the *same* support as  $\mathcal{O}_E \oplus \mathcal{O}_E$ . A similar statement obviously holds for  $F_r$  with  $r > 2$ . Thus the question arises how the spectral cover description distinguishes between  $F_2$  and  $\mathcal{O}_E \oplus \mathcal{O}_E$ . This has been previously explained in [4] (or even earlier in the math literature), and with our preparation in the previous sections it should not be hard to guess: the Fourier-Mukai transform of  $F_2$  has a nilpotent Higgs VEV, and the Fourier-Mukai transform of  $\mathcal{O}_E \oplus \mathcal{O}_E$  has vanishing Higgs VEV.

Let us check this more explicitly. Chasing through the definition of the Fourier-Mukai transform, we see that the fibers of the spectral sheaf over a point  $\lambda \in E$  are found as follows: we tensor  $F$  with a flat line bundle  $\mathcal{O}(\lambda - p_\infty)$ , and then take global sections. Clearly this does not yield  $\mathcal{O}_{p_\infty} \oplus \mathcal{O}_{p_\infty}$  but the non-trivial extension of  $\mathcal{O}_{p_\infty}$  by itself. We have seen this before in section 2.3 of part I: this corresponds to the structure sheaf of a fat point, or equivalently to a nilpotent Higgs VEV.

In the remainder of this subsection, we will show that the construction of the Fourier-Mukai transform as  $\text{coker}(\Psi)$  also reproduces this. The discussion is parallel with the discussion of section 2 of part I. For convenience we consider again the rank two case. We consider the twisted bundle

$$\tilde{V}_E = F_2 \otimes \mathcal{O}(p_\infty) \tag{5.25}$$

Now note that  $\tilde{V}_E$  is the extension of  $\mathcal{O}(p_\infty)$  by itself:

$$0 \rightarrow \mathcal{O}(p_\infty) \rightarrow F_2 \otimes \mathcal{O}(p_\infty) \rightarrow \mathcal{O}(p_\infty) \rightarrow 0 \tag{5.26}$$

In particular, from the long exact sequence we see that

$$H^0(\tilde{V}_E) = 2, \quad H^1(\tilde{V}_E) = 0 \tag{5.27}$$

Thus again  $H^0(\tilde{V}_E)$  is generated by two sections, just as in the decomposable case. More generally when  $\tilde{V}_E = F_r \otimes \mathcal{O}(p)$ , we have  $H^0(\tilde{V}_E) = r$  and  $H^1(\tilde{V}_E) = 0$ .

Just as before, we use the sections of  $H^0(\tilde{V}_E)$  to define a map

$$\Psi_E : \mathcal{O}_E \oplus \mathcal{O}_E \rightarrow \tilde{V}_E \tag{5.28}$$

and the spectral sheaf will be the cokernel of this map. We pick a local coordinate  $\lambda$  on  $E$  such that  $\lambda = 0$  corresponds to  $p_\infty$ . The two sections are linearly independent away from  $p_\infty$ , so the spectral sheaf will be localized at  $\lambda = 0$ , and to figure out the precise description we only need the form of the map near  $\lambda = 0$ .

Let us first consider the sections of  $F_2$ . From the exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow F_2 \rightarrow \mathcal{O} \rightarrow 0 \tag{5.29}$$

we get

$$0 \rightarrow H^0(\mathcal{O}_E) \rightarrow H^0(F_2) \rightarrow H^0(\mathcal{O}_E) \rightarrow H^1(\mathcal{O}_E) \tag{5.30}$$

The last map is the extension class, which is by definition non-zero, so we see that  $H^0(F_2)$  is one-dimensional. It has a unique section, obtained from  $\mathcal{O}_E$  by injecting it into  $F_2$ .

Now we come to the sections of  $\tilde{V}_E$ . The first section of  $\tilde{V}_E$  is inherited from  $F_2$ , i.e. we take the unique section of  $F_2$  and tensor it with a section of  $\mathcal{O}(p_\infty)$ . The unique section of  $\mathcal{O}_E$  is locally just given by 1, so up to a change of basis, locally we can always represent the section of  $F_2$  as  $(1, 0)$ . Furthermore the section of  $\mathcal{O}(p_\infty)$  can be represent as  $\lambda$ , so we can represent the first section of  $\tilde{V}_E$  as

$$s_1 = \lambda \cdot (1, 0) = (\lambda, 0) \tag{5.31}$$

The map from  $F_2$  to  $\mathcal{O}_E$  is given by projection on the second argument.

The second section of  $\tilde{V}_E$  maps to a section of the quotient in (5.26). In this case, the quotient is also  $\mathcal{O}(p_\infty)$ , and its section is given by  $\lambda$ , so the second section of  $\tilde{V}_E$  takes the form

$$s_2 = (*, \lambda) \tag{5.32}$$

We further know that this section cannot be inherited from  $F_2$ , so it is not proportional to  $\lambda$  (in particular it can not vanish at  $\lambda = 0$ ). This only leaves the following possibility:

$$s_2 = (1, \lambda) \tag{5.33}$$

We conclude that near  $\lambda = 0$ , we can represent  $\Psi_E$  as

$$\Psi_E = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \tag{5.34}$$

The spectral sheaf, restricted to  $E$ , is the cokernel of  $\Psi_E$ . As we have seen before, this is precisely the structure sheaf  $\mathcal{O}_{2p_\infty}$ .

Clearly we can generalize this to the higher rank versions. We have:

$$H^0(E, F_k \otimes \mathcal{O}(p_\infty)) = \langle s_1, \dots, s_k \rangle, \quad H^1(E, F_k \otimes \mathcal{O}(p_\infty)) = 0 \tag{5.35}$$

and the map  $\Psi_E$  near  $\lambda = 0$  consists of a rank  $k$  Jordan block:

$$\Psi_E \simeq \begin{pmatrix} \lambda & 1 & & & \\ 0 & \lambda & & & \\ & & \ddots & & \\ & & & \lambda & 1 \\ & & & 0 & \lambda \end{pmatrix} \quad (5.36)$$

The cokernel of  $\Psi_E$  is the structure sheaf of a fat point of length  $k$ ,  $\mathcal{O}_{kp_\infty}$ .

### 5.3 Review of linear sigma models

Linear sigma models are one of the prime methods for constructing exactly conformal  $(0, 2)$  CFTs, which can be used to build vacua for the heterotic string. These models exhibit many interesting properties. In geometric phases they correspond to bundles defined by a monad. A residue theorem of [52] shows that conformal invariance is not spoiled by world-sheet instantons. Let us briefly recall some basic aspects of such constructions. We refer to [53] for details.

We consider a two-dimensional  $U(1)$  gauge theory with  $(0, 2)$  supersymmetry. The right-moving fermionic coordinate is denoted by  $\theta$ . The main multiplets are  $(0, 2)$  chiral fields, defined by the condition

$$\bar{\mathcal{D}}_+ \Phi = 0 \quad (5.37)$$

and  $(0, 2)$  Fermi multiplets, defined by the condition

$$\bar{\mathcal{D}}_+ \Lambda = E \quad (5.38)$$

where

$$\bar{\mathcal{D}}_+ = -\frac{\partial}{\partial \bar{\theta}} + i\theta \mathcal{D}_{\bar{z}} \quad (5.39)$$

is the spinorial covariant derivative, and  $E$  is some (composite) chiral field. The  $U(1)$  charges of chiral fields are denoted by  $q$ , and charges of fermionic multiplets are denoted by  $\tilde{q}$ . Finally we have the  $(0, 2)$  vector multiplet, whose field strength will be denoted by  $\Upsilon$ .

The matter fields of our model are given by  $(0, 2)$  chiral superfields  $\Phi = \{X_i, P, \Sigma\}$  with  $U(1)$  charges  $q_i > 0$ ,  $q_P < 0$ , and  $q_\Sigma = 0$  respectively. We further have  $(0, 2)$  Fermi fields  $\Lambda = \{\Lambda_a, \Gamma\}$  with charges  $\tilde{q}_a > 0$  and  $\tilde{q}_\Gamma < 0$ . These charges are subject to the relations

$$\tilde{q}_\Gamma = -\sum q_i, \quad q_P = -\sum \tilde{q}_a. \quad (5.40)$$

The action is of the schematic form

$$\int d^2z d^2\theta \left[ \frac{1}{8e^2} \bar{\Upsilon} \Upsilon + \bar{\Phi} \bar{\mathcal{D}}_z \Phi + \bar{\Lambda} \Lambda \right] + \frac{t}{4} \int d^2z d\theta \Upsilon|_{\bar{\theta}=0} + \int d^2z d\theta W(\Phi, \Lambda)|_{\bar{\theta}=0} + \text{h.c.} \quad (5.41)$$

where we used  $\Phi$  and  $\Lambda$  to denote general chiral and Fermi fields, respectively. Here the parameter

$$t = ir + \frac{\vartheta}{2\pi} \quad (5.42)$$



contains the Fayet-Iliopoulos parameter and theta-angle. The integral over half of superspace is invariant if the integrand is chiral, i.e. annihilated by  $\bar{D}_+$ . We take the superpotential of the form

$$W(\Phi, \Lambda) = \Gamma G(X_i) + \Lambda_a P J^a(X_i) \quad (5.43)$$

In order for this to be gauge invariant,  $G$  is of degree  $-\tilde{q}_\gamma$  in the  $X_i$ . Similarly  $J^a$  is also chiral and of degree  $-q_p - \tilde{q}_a$  in the  $X_i$ , and in order to get a chiral integrand the  $J^a$  are constrained by a relation that we mention momentarily. We further take

$$\bar{D}_+ \Gamma = \Sigma P E_\Gamma(X_i), \quad \bar{D}_+ \Lambda_a = \Sigma E_a(X_i) \quad (5.44)$$

Then the requirement  $\bar{D}_+ W = 0$  implies that the  $E$ 's and  $J$ 's are subject to

$$E_a J^a = -E_\gamma G \quad (5.45)$$

The action (5.41) leads to a  $D$ -term potential of the form

$$V_D \simeq \frac{e^2}{2} \left( \sum q_i |x_i|^2 + \tilde{q}_\Gamma |p|^2 - r \right) + |G|^2 + |p|^2 \sum |J_a|^2 + |\sigma|^2 \left( \sum |E_a|^2 + |p E_\Gamma|^2 \right) \quad (5.46)$$

In the geometric regime  $r \gg 0$ , this flows to a non-linear sigma-model on the Calabi-Yau hypersurface  $G(x_i) = 0$  in  $\mathbf{WP}_{q_1, \dots, q_n}$ , and the surviving massless right-moving fermions  $\psi_{X_i}$  take value in the tangent bundle of the Calabi-Yau. We further have additional left-moving fermions  $\lambda_a$  together with some mass terms

$$\psi_P \lambda_a J^a(x) + \bar{\psi}_\Sigma \lambda_a \bar{E}_a \quad (5.47)$$

induced from the Yukawa couplings. The constraint coming from the second mass term can be formulated holomorphically by introducing fermionic gauge equivalences:

$$\lambda_a \sim \lambda_a + E_a s. \quad (5.48)$$

Then the surviving massless left-moving fermions live in a bundle  $V$  which is given by the cohomology of the monad

$$0 \rightarrow \mathcal{O} \xrightarrow{E_a} \bigoplus \mathcal{O}(\tilde{q}_a) \xrightarrow{J^a} \mathcal{O}(-q_P) \rightarrow 0 \quad (5.49)$$

This is a complex on the Calabi-Yau hypersurface  $G = 0$ , because  $E \cdot J = 0 \bmod G$ . In the special case where

$$E_i = q_i X_i, \quad E_\Gamma = -\tilde{q}_\Gamma, \quad J^i = \frac{\partial G}{\partial X_i} \quad (5.50)$$

this is the Euler sequence for the tangent bundle, and yields a left-right symmetric model. The above considerations may be easily generalized to multiple  $U(1)$ 's (allowing for multi-degrees), multiple  $\Gamma$ 's (allowing for complete intersections), and multiple  $\Sigma$ 's (allowing for extra fermionic gauge equivalences).

Not any monad defines a linear sigma model. We have already seen the conditions (5.40), which correspond to absence of anomalies for the right-moving  $U(1)_R$  symmetry and an additional left-moving global  $U(1)$  symmetry. Although these are global

symmetries, we need them to be non-anomalous in order to construct a string vacuum. Geometrically they can be interpreted as  $c_1(T) = c_1(V) = 0$ . To cancel the  $U(1)$  gauge anomaly, we get the further constraint  $\tilde{q}^2 - q^2 = 0$ , or more explicitly:

$$\sum \tilde{q}_i \tilde{q}_i + \tilde{q}_\gamma \tilde{q}_\gamma - \sum q_i q_i - q_p q_p = 0 \tag{5.51}$$

This is closely related, although in general not quite equivalent to the condition  $c_2(T) = c_2(V)$ .

### 5.4 Monads over an elliptic three-fold

In this section we would like to explain the algorithm for finding the spectral sheaf of a monad, generalizing the algorithm for finding the spectral cover in [22]. We assume of course that the monad bundle is semi-stable on generic elliptic fibers, because otherwise the Fourier-Mukai transform is not supported on a divisor. This is also reasonable because generic bundles with vanishing first Chern class will have this property.

#### 5.4.1 Summary of the algorithm

Let us state the strategy at the outset. We consider general complexes of the form

$$0 \rightarrow \mathcal{O}_Z^{\oplus q} \xrightarrow{E} \mathcal{H} \xrightarrow{J} \mathcal{N} \rightarrow 0 \tag{5.52}$$

The strategy will be to first study the bundle  $\mathcal{K}$  given by the kernel of the map  $J$ , without modding out by  $E$ . That is,  $\mathcal{K}$  is defined by the short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{H} \xrightarrow{J} \mathcal{N} \rightarrow 0 \tag{5.53}$$

Working with  $\mathcal{K}$  illustrates most of the important points, and adapting it for  $V = \ker(J)/\text{im}(E)$  is only a small modification of the procedure. Thus we will split the algorithm into two steps:

**Step i.** The bundle  $\mathcal{K}$  is an extension of  $V$  by  $\mathcal{O}_Z^{\oplus q}$ , where  $V$  is the bundle we are eventually interested in. It is also semi-stable and degree zero on the elliptic fibers if  $V$  is. Thus we can ask for the Fourier-Mukai transform of  $\mathcal{K}$ . We claim that the spectral sheaf of  $\mathcal{K}$  is given by the cohomology  $\ker(J)/\text{im}(A)$  of the following complex:

$$0 \rightarrow \pi^* \pi_* \tilde{\mathcal{K}} \xrightarrow{A} \tilde{\mathcal{H}} \xrightarrow{J} \tilde{\mathcal{N}} \rightarrow 0 \tag{5.54}$$

In particular, the support of  $\text{FM}^1(\mathcal{K})$  can be recovered from the determinant of this complex. The map  $A$  is the composition of the canonical map  $\pi^* \pi_* \tilde{\mathcal{K}} \rightarrow \tilde{\mathcal{K}}$  followed by the canonical inclusion  $\tilde{\mathcal{K}} \rightarrow \tilde{\mathcal{H}}$ .

**Step ii.** Once we have  $\text{FM}^1(\mathcal{K})$ , we can recover  $\text{FM}^1(V)$  from the short exact sequence

$$0 \rightarrow \text{FM}^1(\mathcal{O}_Z)^{\oplus q} \rightarrow \text{FM}^1(\mathcal{K}) \rightarrow \text{FM}^1(V) \rightarrow 0 \tag{5.55}$$

In particular, the support of  $\text{FM}^1(\mathcal{K})$  is simply the union of the support of  $\text{FM}^1(\mathcal{O}_Z)^{\oplus q}$ , which consists of  $q$  copies of the zero section  $\sigma_{B_2}$  of  $Z$ , and the support of  $\text{FM}^1(V)$ .

As before, recall that  $\tilde{V} \equiv V \otimes \mathcal{O}(\sigma)$  for any bundle  $V$ . Below we will explain these steps in more detail. We will illustrate the procedure with a concrete class of examples.

### 5.4.2 Derivation and example

For the class of examples we take some models that were considered in [9]. The Calabi-Yau is an elliptic fibration over the Hirzebruch surface  $\mathbf{F}_n$ , with  $n = 0, 1, 2$  so that  $Z$  is smooth. This can be embedded as a Weierstrass model in a  $\mathbf{WP}_{1,2,3}$  fibration over  $\mathbf{F}_n$ , so the linear sigma model will have three  $U(1)$  gauge fields. We will denote the homogeneous coordinates collectively by  $x_i$ ,  $i = 1, \dots, 7$ , the homogeneous coordinates for  $\mathbf{F}_n$  by  $\{u_0, u_1; v_0, v_1\}$ , and the homogenous coordinates for  $\mathbf{WP}_{1,2,3}$  by  $\{x, y, z\}$ . Then  $Z$  is given by a hypersurface equation

$$G \equiv -y^2 + x^3 + f(u, v)xz^4 + g(u, v)z^6 = 0 \tag{5.56}$$

Over this Calabi-Yau we consider the following monad:

$$0 \rightarrow \mathcal{O}_Z \xrightarrow{E} \mathcal{H} \xrightarrow{J} \mathcal{N} \rightarrow 0 \tag{5.57}$$

where we put

$$\mathcal{H} = \sum_i \mathcal{O}_Z \left( \sum_J n_{iJ} D_J \right), \quad \mathcal{N} = \mathcal{O}_Z \left( \sum_J m_J D_J \right) \tag{5.58}$$

and  $m_J = \sum_i n_{iJ}$ . This is precisely the type of monad one gets from gauged linear sigma models, cf. equation (5.49). We will take our bundle  $V_5$  to be a deformation of  $TZ \oplus \mathcal{O}_Z \oplus \mathcal{O}_Z$ . Then the number of left-moving fermionic fields  $\Lambda_a$ , which describe the bundle  $V_5$ , is the same as the number of bosonic fields  $X_i$  for the underlying toric manifold, and their  $U(1)^3$  charges are the same also:

$$n_{iJ} = \begin{pmatrix} 1 & 1 & n & 0 & 4 + 2n & 6 + 3n & 0 \\ 0 & 0 & 1 & 1 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 1 \end{pmatrix} \tag{5.59}$$

The divisors  $D_J$  are given by

$$D_1 = \{u_1 = 0\}, \quad D_2 = \{v_1 = 0\}, \quad D_3 = \{z = 0\} \tag{5.60}$$

Let us denote the base and fiber of the Hirzebruch by  $b$  and  $f$ , with  $b^2 = -n$ ,  $f^2 = 0$ ,  $b \cdot f = 1$ . Then  $D_1$  corresponds to  $\pi^*f$  and  $D_2$  corresponds to  $\pi^*b$ . The divisor  $D_3$  corresponds to the zero section  $\sigma_{B_2}$  of the elliptic fibration.

Let us first consider the number of generations in such a model. When  $J$  is given by the partial derivatives

$$J = (\partial G / \partial \vec{x})^T \tag{5.61}$$

then the bundle  $V$  is a non-trivial extension of  $TZ$  by  $\mathcal{O}_Z \oplus \mathcal{O}_Z$ . Since the Euler character is additive in an exact sequence, we have

$$\chi(V) = \chi(TZ) + 2\chi(\mathcal{O}_Z) = \chi(TZ) \tag{5.62}$$

Further, the net number of generations does not change under continuous deformations. Thus the net number of generations is given by the Euler characteristic of  $Z$ . Assuming that  $Z$  is a smooth Weierstrass model with one section, we have [54]

$$\chi(Z) = -60 c_1(S)^2 \tag{5.63}$$

Since  $c_1^2 = 8$  for all the Hirzebruch surfaces, we get 480 generations. Needless to say this is far from realistic, but our point here is only to illustrate the technology.

In our case,  $\mathcal{H}$  has rank seven and  $\mathcal{N}$  has rank one, so the kernel of  $J$  has rank six and will be denoted by  $\mathcal{K}_6$ . The bundle  $\mathcal{K}_6$  is defined by the short exact sequence:

$$0 \rightarrow \mathcal{K}_6 \rightarrow \mathcal{H} \xrightarrow{J} \mathcal{N} \rightarrow 0 \tag{5.64}$$

Now one may try to simplify the maps by using the freedom to make field redefinitions. Apart from coordinate redefinitions of the variety, we have additional field redefinitions of the form

$$\Lambda_a \rightarrow \Lambda_a + \sum p_{ab}(X_i)\Lambda_b \tag{5.65}$$

where  $p_{ab}$  is a matrix of polynomials of the appropriate degrees in the chiral fields  $X_i$ , whose bosonic components yield the coordinates  $x_i$  on the ambient variety. These field redefinitions correspond to bundle automorphisms  $\text{Ext}^0(\mathcal{H}, \mathcal{H})$ , i.e. symmetries of the rank 7 bundle  $\mathcal{H}$ . Under such automorphisms, we have

$$J^a \rightarrow J^a + p_{ba}(x_i)J^b \tag{5.66}$$

According to [9], generically we can set the first four entries of  $J$  equal to zero by field redefinitions. We were not able to reproduce this, but for illustrative purposes we take  $J$  to be of the following form:

$$J = \left( 0, 0, 0, 0, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z} + P_{8,8+4n}xz^3 \right) \tag{5.67}$$

where  $P_{8,8+4n}$  is a polynomial of bidegree  $(8, 8 + 4n)$  on  $\mathbf{F}_n$ . This family contains the essential features and keep the calculations simple enough to write out here.

As in section 5.1, we now twist the sequence (5.64) by  $\mathcal{O}(D_3)$ . As discussed, the Fourier-Mukai transform of  $\mathcal{K}_6$  is given by the short exact sequence

$$0 \rightarrow \pi^*\pi_*\tilde{\mathcal{K}}_6 \xrightarrow{\Psi_6} \tilde{\mathcal{K}}_6 \rightarrow \mathcal{L}_6 \rightarrow 0 \tag{5.68}$$

We have realized  $\tilde{\mathcal{K}}_6$  as the kernel of  $J$ , but we do not yet have  $\pi^*\pi_*\tilde{\mathcal{K}}_6$  or an explicit representative for  $\Psi_6$ . To get  $\pi^*\pi_*\tilde{\mathcal{K}}_6$ , we take cohomology along the elliptic fiber, that is we use (5.64) (twisted by  $\mathcal{O}(D_3)$ ) to get the long exact sequence

$$0 \rightarrow \pi^*\pi_*\tilde{\mathcal{K}}_6 \xrightarrow{i} \pi^*\pi_*\tilde{\mathcal{H}} \xrightarrow{J_*} \pi^*\pi_*\tilde{\mathcal{N}} \rightarrow 0 \tag{5.69}$$

where as usual, a tilde denotes twisting by  $\mathcal{O}(D_3)$ :

$$\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{O}(D_3) = \sum_i \mathcal{O}\left(\sum_J n'_{iJ}D_J\right), \quad \tilde{\mathcal{N}} = \mathcal{O}\left(\sum_J m'_J D_J\right) \tag{5.70}$$

and

$$n'_{iJ} = n_{iJ} + \delta_{J3}, \quad m'_J = m_J + \delta_{J3} \tag{5.71}$$

Our long exact sequence truncates after just three terms because  $H^1(\tilde{\mathcal{K}}_6|_{T^2})$  vanishes for stable bundles of positive degree on the elliptic fiber, as explained in section 5.1. Thus

we have realized  $\pi^*\pi_*\tilde{\mathcal{K}}_6$  as the kernel of  $J_*$ . Note that the kernel of  $J_*$  is  $13 - 7 = 6$  dimensional, the same as the rank of  $\mathcal{K}_6$  and  $\tilde{\mathcal{K}}_6$ , as expected by the general discussion in section 5.1.

Finally then, we need the map  $\Psi_6$ . Recall that this is the canonical evaluation map  $\pi^*\pi_*\tilde{\mathcal{K}}_6 \rightarrow \tilde{\mathcal{K}}_6$ . Now we have realized  $\pi^*\pi_*\tilde{\mathcal{K}}_6$  as the kernel of  $J_*$  in  $\pi^*\pi_*\tilde{\mathcal{H}}$  and we have realized  $\tilde{\mathcal{K}}_6$  as the kernel of  $J$  in  $\tilde{\mathcal{H}}$ , so what we need is the evaluation map from  $\pi^*\pi_*\tilde{\mathcal{H}}$  to  $\tilde{\mathcal{H}}$ .

To do this, we need to know the restriction of  $\mathcal{O}(D_J)$  to  $T^2$ .  $D_3$  intersects every  $T^2$  at a single distinguished point at infinity on the elliptic fiber, so we define  $\mathcal{O}(D_3)|_{T^2} = \mathcal{O}(1)$ .  $D_1$  and  $D_2$  are disjoint from  $T^2$  for almost all elliptic fibers, so at least on a dense set on the base we have that  $\mathcal{O}(D_1)|_{T^2}$  and  $\mathcal{O}(D_2)|_{T^2}$  are the trivial line bundle on  $T^2$  (denoted by  $\mathcal{O}(0)$ ).

It is not hard to derive the following identifications:

$$\begin{aligned} \pi_*\mathcal{O}_Z(a, b, 0) &= \mathcal{O}_S(a, b) \\ \pi_*\mathcal{O}_Z(a, b, k) &= \mathcal{O}_S(a, b) \otimes (\mathcal{O}_S \oplus K_S^2 \oplus K_S^3 \oplus \dots \oplus K_S^k) \end{aligned} \tag{5.72}$$

where the second line is for  $k \geq 1$ . We further have the canonical map

$$\pi^*\pi_*\mathcal{O}_Z(a, b, k) \rightarrow \mathcal{O}_Z(a, b, k) \tag{5.73}$$

which over an open subset is given by mapping local sections as

$$\text{ev} : (p_0, p_2, p_3, \dots, p_k) \rightarrow p_0z^k + p_2xz^{k-2} + \dots + p_kx^{(k-1)/2}y \tag{5.74}$$

for  $k$  odd, and the last term given by  $p_kx^{k/2}$  if  $k$  is even. Note that some of the  $p_k$  may not extend to global sections, which is why we have to work over an open subset. Equivalently, we can work with meromorphic sections.

Putting it all together, we find that the spectral sheaf  $\mathcal{L}_6$  is given by the cohomology of the following complex:

$$0 \rightarrow \pi^*\pi_*\tilde{\mathcal{K}}_6 \xrightarrow{A} \tilde{\mathcal{H}} \xrightarrow{J} \tilde{\mathcal{N}} \rightarrow 0 \tag{5.75}$$

Here  $A$  is given by the composition  $A = \text{ev}_{\tilde{\mathcal{H}}} \circ i$  where  $i$  is the inclusion in (5.69) and  $\text{ev}_{\tilde{\mathcal{H}}}$  is the canonical evaluation map  $\pi^*\pi_*\tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}$ . Thus to find  $\mathcal{L}_6$ , we see that everything eventually boils down to finding an explicit representative for  $A$ . It is induced from the evaluation map  $\text{ev}_{\tilde{\mathcal{H}}}$ , by restricting to the kernel of  $J_*$ . Moreover we can calculate  $\text{ev}_{\tilde{\mathcal{H}}}$  very explicitly because  $\tilde{\mathcal{H}}$  is just a sum of line bundles. This is a rather general result, and doesn't depend on the specific class of examples we have chosen.

In the case of our examples, we can choose local bases for  $\pi^*\pi_*\tilde{V}_6$  and  $\tilde{\mathcal{H}}$  so that the map  $A$  is represented as follows:

$$A = \begin{pmatrix} z & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial G}{\partial y} & Q_3 \\ 0 & 0 & 0 & 0 & -\frac{\partial G}{\partial x} & Q_4 \\ 0 & 0 & 0 & 0 & 0 & Q_2 \end{pmatrix} \tag{5.76}$$

with

$$\begin{aligned}
 Q_3 &= -3gP_8(4f - P_8)z^3 + (108g^2 + fP_8^2)xz \\
 Q_4 &= \frac{3}{2}(108g^2 + fP_8^2)yz \\
 Q_2 &= 2(27g^2 + f^2P_8)z^2 + 9g(4f - P_8)x.
 \end{aligned}
 \tag{5.77}$$

The matrix is  $7 \times 6$  as the source is  $\pi^*\pi_*\tilde{V}_6$  which has rank 6, and the target has rank 7.

The first four columns are very easy to understand. Clearly each of the first four line bundle of the form  $\pi^*\pi_*\mathcal{O}(a, b, 1) \cong \pi^*\mathcal{O}(a, b)$  in  $\tilde{\mathcal{H}}$  are in the kernel of  $J_*$ . We have

$$\text{Hom}(\pi^*\mathcal{O}(a, b), \mathcal{O}(a, b, 1)) = H^0(\mathcal{O}(0, 0, 1))
 \tag{5.78}$$

which is one dimensional and has a unique global section up to rescaling, given by  $z$ . So we can choose a basis so that the first four columns are as above.

The remaining two columns can not be written globally. We can write them over open subsets  $U$  of some covering of  $Z$ . We take  $U_i$  to be the complement of  $k_i(u, v) = 0$ , where  $k_i$  is a section of  $K_B^{-2}$ . We can cover  $Z$  by three such open sets.

As explained above, the spectral sheaf  $\mathcal{L}_6$  is recovered as  $\ker(J)/\text{Im}(A)$ . In particular, its support is given by  $\det(\Psi_6) = 0$  where

$$\vec{M}_A = \det(\Psi_6) \vec{J}
 \tag{5.79}$$

and  $\vec{M}_A$  is the vector of minors of  $A$ . Therefore, the equation of the spectral cover is given by

$$\det(\Psi_6) = z^4 Q_2.
 \tag{5.80}$$

The fact that the support of  $\mathcal{L}_6$  is reducible is hardly a surprise. The first four line bundles of  $\mathcal{H}$  are clearly in the kernel of  $J$ , and their transform is a sum of four line bundles supported on the zero section. The map  $J$  only acts non-trivially on the last three summands of  $\mathcal{H}$ , and its kernel on these three summands transforms to a sheaf  $\mathcal{L}_2$  supported on  $Q_2 = 0$ . The sheaf  $\mathcal{L}_6$  is the sum of the four line bundles on  $z = 0$  and  $\mathcal{L}_2$ .

Now we come to step two, where we want to also mod out by the image of  $E$  and recover  $V_5$ . That is, we now consider the short exact sequence

$$0 \rightarrow \mathcal{O}_Z \rightarrow \mathcal{K}_6 \rightarrow V_5 \rightarrow 0
 \tag{5.81}$$

Applying the Fourier-Mukai transform, we get a long exact sequence, which truncates to

$$0 \rightarrow \text{FM}^1(\mathcal{O}_Z) \rightarrow \text{FM}^1(V_6) \rightarrow \text{FM}^1(V_5) \rightarrow 0
 \tag{5.82}$$

We learn several things. First of all, the support of  $\text{FM}^1(\mathcal{K}_6)$  is the same as the union of the supports of  $\text{FM}^1(\mathcal{O}_Z)$  (which is just the zero section  $\sigma_B$ ) and  $\text{FM}^1(V_5)$ . Thus to find the spectral cover for  $V_5$ , we simply take the spectral cover we found for  $V_6$  and divide by  $z$  (the equation for the zero section). In particular, the extra steps in [22] can be skipped.

To find the spectral sheaf rather than just the support, we also need the map, which is given by  $\text{FM}^1(E)$ . In practice however, we are interested in the situation where  $V_5$  is

stable. A necessary condition for stability is that  $V_5$  has no sections. The role of  $E$  is to mod out by the sections of  $\ker(J)$ , i.e. the  $E$ 's are simply given by all the generators of  $H^0(\mathcal{K}_6) = \text{Hom}(\mathcal{O}_Z, \mathcal{K}_6)$ . The dual statement is simply that  $\text{FM}^1(E)$  is given by all the generators of  $\text{Hom}(\text{FM}^1(\mathcal{O}_Z), \text{FM}^1(\mathcal{K}_6))$ . So we do not have to translate  $E$  explicitly through the Fourier-Mukai transform.

In our toy example,  $V_5$  is actually not stable, but things are nevertheless very easy. The bundle  $\mathcal{K}_6$  is the direct sum of four degree zero line bundles pulled back from the base and a stable rank two bundle  $V_2$ , whose spectral cover is given by  $Q_2 = 0$ . Then  $E$  is necessarily of the form

$$E = (*, *, *, *, 0, 0, 0) \tag{5.83}$$

The transform  $\text{FM}^1(\mathcal{O}_Z)$  is supported on  $z = 0$  and so can only map into the part of the spectral sheaf that is supported at  $z^4 = 0$ . Furthermore, the  $E$ 's are then simply pull-backs of sections of line bundles on the base, which we may also call  $E$  (as they are given by the same expressions). Therefore we get a short exact sequence for a  $U(3)$  bundle  $U_3$  on  $B_2$ :

$$0 \rightarrow \mathcal{O}_{B_2} \xrightarrow{E} \mathcal{O}(1, 0)_{B_2}^2 \oplus \mathcal{O}(n, 1)_{B_2} \oplus \mathcal{O}(0, 1)_{B_2} \rightarrow U_3 \rightarrow 0 \tag{5.84}$$

The Fourier-Mukai transform of  $V_5$  is therefore the sum of  $\sigma_{B_2} U_3$  and  $\mathcal{L}_2$ .

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