

Conserved quantities and Virasoro algebra in new massive gravity

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ABSTRACT: Using the *off-shell* Noether current and potential we compute the entropy for the AdS black holes in new massive gravity. For the non-extremal BTZ black holes by implementing the so-called stretched horizon approach we reproduce the correct expression for the horizon entropy. For the extremal case, we adopt standard formalism in the AdS/CFT correspondence and reproduce the corresponding entropy by computing the central extension term on the asymptotic boundary of the near horizon geometry. We explicitly show the invariance of the angular momentum along the radial direction for extremal as well as non-extremal BTZ black holes in our model. Furthermore, we extend this invariance for the black holes in new massive gravity coupled with a scalar field, which correspond to the holographic renormalization group flow trajectory of the dual field theory. This provides another realization for the holographic c-theorem.

KEYWORDS: Models of Quantum Gravity, Black Holes, AdS-CFT Correspondence

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1 Introduction

Local symmetries play a crucial role in understanding the thermodynamical properties of black holes. It was shown that the Noether charges corresponding to the diffeomorphism symmetry of any generally covariant theory of gravity are related to the black hole entropy [1, 2]. This approach, when applied to the conventional Einstein-Hilbert action, reproduce the well known Bekenstein-Hawking entropy (BH) [3, 4]. However, the Wald’s formalism tells us little about the microscopic degrees of freedom responsible for the black hole entropy. A major step to understand the black hole entropy from the microscopic point of view was taken by Strominger and Vafa [5]. They showed that certain extremal black holes in string theory can be described, through a string duality map, by two dimensional conformal field theory (CFT). Then the entropy of this CFT was obtained by using the Cardy formula [6] and shown to be consistent with the standard BH entropy. In the related subsequent development [7], the entropy of the Banados-Teitelboim-Zanelli (BTZ) black holes [8] was also obtained through two dimensional CFT. This formalism relies on the remarkable observation that the algebra among the asymptotic symmetry generators is isomorphic to two copies of the Virasoro algebra with the central charge c [9]

$$[Q_m, Q_n] = (m - n)Q_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}. \tag{1.1}$$

This appearance of the Virasoro algebra of symmetry generators can be regarded as the predecessor of the AdS/CFT correspondence [10].

An alternative approach which uses only the near horizon properties of black holes was given by Carlip [11, 12]. In this ‘stretched horizon’ approach one begins by assuming the existence of an approximate null Killing vector. The location of the Killing horizon is

determined by the vanishing of the norm of that Killing vector. Under certain boundary conditions near the Killing horizon, it was shown that the Fourier modes of diffeomorphism generators realized by vector fields form subalgebra isomorphic to $\text{Diff}(S^1)$, which is also known as Witt algebra, as

$$[\xi_m, \xi_n]^a = -i(m - n)\xi_{m+n}^a. \tag{1.2}$$

The algebra among canonical conserved charges corresponding to the above generators is identical with a copy of the standard Virasoro algebra. This construction of the algebra indicates the existence of a certain two dimensional CFT, and allows us to read off the central charge of the underlying CFT. Then, the black hole entropy is reproduced by substituting the central charge and zero mode eigenvalue of the conserved charge in the Cardy formula. Based on this idea, several alternative methods have been proposed [13–24] to know more about the microscopic origin of the horizon entropy.

The above stretched horizon approach has been revisited by Majhi et al. [25] wherein the central charge and the horizon entropy is obtained by using the *off-shell* expressions for the Noether current and potential. One of the key features of this method is that the on-shell vanishing part of the Noether current is not essential for performing calculations in brackets among the Noether charges. In this sense the definition of bracket is more general. This formalism extends in a straightforward manner to Lanczos-Lovelock models of gravity. Recently, it has been shown that the same expressions for the Noether charges and horizon entropy can also be obtained by using either the Gibbons-Hawking surface term or the gravitational surface term [26, 27]. This analysis was based on the holographic property of the gravitational action functional [28]. Although Lanczos-Lovelock models generalize Einstein gravity to a great extent, they still come under the two derivative theories. It is known that a specific combination of Ricci scalar and Ricci tensor leads to interesting gravity models in three and four dimensions [29–31]. The higher curvature gravity in $2 + 1$ dimensions, the so-called new massive gravity (NMG), is originally introduced as the parity even counter part of topologically massive gravity [32]. NMG allows propagating massive gravitons and also incorporate various black hole solutions. Moreover, NMG has been shown to be consistent with the so-called holographic c-theorem [33–35]. It is therefore natural to extend the analysis of [25] to obtain the horizon entropy for black holes in genuinely higher derivative gravity such as NMG.

Another motivation to study $2 + 1$ dimensional Einstein gravity and NMG is as follows. For Einstein gravity, it was shown that the conserved quantities like mass and angular momentum for Kerr black hole geometries in arbitrary dimensions take the same values on the near horizon and at the asymptotic infinity [36, 37]. A generalization of this result for the asymptotically AdS black holes by using the improved surface integrals has been provided in [38, 39]. The improved surface integrals allow us to compute the conserved charges both at the horizon and the asymptotic infinity. In addition to this, it has also been shown that the angular momentum is invariant not only just at two asymptotic boundaries but also all along the entire radial direction. This procedure is valid for higher dimensional Kerr black holes in asymptotically flat as well as AdS geometries.

The rotating AdS black holes in $2 + 1$ dimensions (BTZ black holes) possess an extra feature. Since the asymptotic as well as the near horizon geometry correspond to AdS_3

space, this geometry may be viewed as the holographic renormalization group (RG) flow between the ultraviolet and the infrared CFT. This provides another realization of holographic c-theorem beyond domain wall solutions.¹ However, in the absence of any matter field this holographic RG flow becomes trivial. The next simplest step would be to study the rotating black holes in 2 + 1 dimensional Einstein gravity coupled to a scalar field. For the interpretation in the holographic RG flow, the extremal black hole solutions are the relevant one [40].² The deformed extremally rotating BTZ black holes in NMG coupled with a scalar are also discussed in [45]. It was shown that the family of extremally rotating hairy AdS black holes can be described by reduced-order equations of motions(EOM). These black holes may be regarded as the scalar-hairy deformation of extremally rotating BTZ black holes. However, the explicit derivation of the invariance of the angular momentum for these black holes has not been given in the literature. The detailed analysis of the invariance of the angular momentum for the Einstein as well as NMG by incorporating the scalar field would provide us fresh insights to understand RG flow in the dual CFT.

The purpose of the present work is to compute the horizon entropy from the point of view of the near horizon Virasoro algebra and to show explicitly the invariance of the angular momentum for the rotating BTZ black holes in NMG. We first consider the general expression for the *off-shell* Noether current and its potential [46, 47] for the NMG. Then, by integrating the Noether potential on the stretched horizon we obtain the expression for the Noether charge. Using the general definition of bracket given in [25], we construct the algebra among the Noether charges. It turns out that this algebra is isomorphic to a copy of the Virasoro algebra with a central extension. The zero mode eigenvalue and central charge are obtained by Fourier transforming the Noether charge and central extension term, respectively. Next, we evaluate these expressions for non extremal rotating BTZ black hole. Finally, using the Cardy formula we obtain the expression for the horizon entropy. We also sketch a method to calculate the entropy for extremal rotating BTZ black hole in NMG. For this case we will proceed in a slightly different way. We take the same expressions for the Noether current, potential and the bracket among the Noether charges mentioned earlier. However, we choose the diffeomorphisms different from the non-extremal case, which preserves the fall-off boundary conditions [48] at the asymptotic infinity of the near horizon extremal BTZ. We show that algebra among the Noether charges is isomorphic to the Virasoro algebra with the central extension. The central charge identified from this algebra is consistent with the one given in [49].

To show the angular momentum invariance for the rotating BTZ black holes along the radial direction, we adopt a specific quasi-local method for conserved charges, which is known as the Komar integrals. Here as well, we use the same Noether potential as in the computation for black hole entropy. We calculate the Noether potential corresponding to the rotational Killing vector for the BTZ black hole in the Einstein gravity and NMG. The angular momentum is obtained by integrating this expression over the surface with codimension two situated at $r_H \leq r \leq r_\infty$, where r_H is the position of the outer horizon.

¹The simplest examples of such holographic construction of RG flows are given by domain wall solutions interpolating two AdS spaces.

²For some specific choice of the scalar potential the non-extremal black hole solution is given in [8, 41–44].

In the Einstein case, our resulting expression for the angular momentum matches exactly with the corresponding one given in [38, 39]. By applying the same technique we show the invariance of the angular momentum for the rotating BTZ black holes in NMG. As mentioned earlier, the invariance of the angular momentum for the rotating BTZ black holes is related to the holographic RG flow. However, for the pure Einstein or NMG theories this RG flow is trivial. In order to gain further insights into the holographic RG flow, we consider extremally rotating BTZ solution for NMG coupled with a scalar field. Then, we compute the relevant Komar integrals [50] corresponding to the rotating Killing vector and show that the angular momentum is indeed invariant along the radial direction.

Apart from its relevance in understanding RG flow in the dual CFT's, the invariance of the angular momentum gives us an important relation between the infrared and ultraviolet entropies. According to the conventional AdS/CFT correspondence, the central charge of the dual ultraviolet CFT is always greater than that of the infrared CFT, which is coined as holographic c-theorem. Since the Cardy formula requires the conformal weights of dual states together with the central charge, this theorem is insufficient to identify which entropy of dual CFT corresponds to the BH entropy of those black holes. By establishing the angular momentum invariance of the hairy extremal black holes, we verify that the entropy of the dual infrared CFT is always less than or equal to the one of dual ultraviolet CFT and that the entropy of the infrared CFT gives the BH entropy of those black holes. This matching between the infrared dual CFT and the BH entropy of black holes is anticipated through the near horizon CFT approach [11]. However, we would like to emphasize that this near horizon CFT is not the same one with the infrared dual CFT used in holographic c-theorem. Nevertheless, by using conserved currents related to the generalized Komar potential, one can see that the same entropy of black holes can be reproduced in this way.

This paper is organized as follows. In the next section we summarize some basic facts about conserved currents for the local symmetry, for the sake of completeness, and then we consider the modified Noether current which was introduced as the off-shell conserved currents [46] and show that their meaning may be understood as the generalized Komar potential. In section three we obtain the entropy of black holes through the near horizon dual CFT by using the *off-shell* formalism of [25]. We conclude this section by providing a brief discussion on the entropy for extremally rotating BTZ black holes. In section four, using the generalized Komar potential, we obtain the quasi-local angular momentum of extremally rotating hairy AdS black holes on three dimensions and show its invariance along the radial direction. This verifies that the BH entropy of these black holes can be obtained by the infrared dual CFT. We summarize our results and discuss some future directions in the final section. Appendix A contains some useful formulae and definitions in the stretched horizon approach. A derivation of angular momentum invariance for the usual Einstein gravity is provided in appendix B.

2 Conserved currents and their potentials

In this section we review and summarize the Noether procedure for symmetry to fix our conventions. Since we are interested in the conserved charges and the entropy of black holes,

it is useful to collect some well-known results for conserved currents for local symmetry to clarify our presentation (See for reviews, [51–53]).

Let us consider the generic action, $I[\varphi]$, which enjoys local symmetry and contains various fields which are denoted collectively as φ^i . Some of these fields can be regarded as gauge fields and others as matter ones. Under the generic variation of the field $\varphi \rightarrow \varphi + \delta\varphi$, the action is varied as

$$\delta I[\varphi] = \int d^d x \sqrt{-g} \left[E_i(\varphi) \delta\varphi^i + \nabla_\mu \Theta^\mu(\varphi, \delta\varphi) \right], \quad (2.1)$$

where $E_i(\varphi) = 0$ denotes EOM for each field φ^i and $\Theta(\varphi, \delta\varphi)$ denotes the total surface term after integration by parts. The symmetry of the action is defined by the invariance of the action under the specific variation of fields $\varphi \rightarrow \varphi + \delta_\epsilon \varphi$. The invariance of the action under the symmetry can be written as

$$\delta_\epsilon I[\varphi] = \int d^d x \sqrt{-g} \nabla_a K^a(\varphi, \delta_\epsilon \varphi) = 0. \quad (2.2)$$

The standard Noether current is introduced as

$$\tilde{J}^a \equiv \Theta^a(\varphi, \delta_\epsilon \varphi) - K^a(\varphi, \delta_\epsilon \varphi). \quad (2.3)$$

By taking the generic variation as the symmetry in eq. (2.2), one can see that this current satisfies

$$\nabla_a \tilde{J}^a = -E_i(\varphi) \delta_\epsilon \varphi^i, \quad (2.4)$$

which tells us the on-shell conservation of the Noether current. For global symmetry this procedure leads to the well-defined conserved charges by integrating the current on the hypersurface. However, that is not the case for a local symmetry. The basic reason for the inadequacy of this procedure in local symmetries is the existence of the so-called Noether identities. In terms of local symmetry variation parameter $\epsilon = \epsilon(x)$, this identity can be written as the form of

$$-E_i(\varphi) \delta_\epsilon \varphi^i = \nabla_a S^a(E(\varphi), \delta_\epsilon \varphi), \quad (2.5)$$

where S^a denotes the on-shell vanishing current. Even for the the global symmetries, which may appear as the rigid limit of corresponding local symmetries, the current S^a can be introduced. However, for this case the corresponding Noether identities are somewhat trivial. The absence of gauge fields in the case of global symmetry tells us that there are missing equations for gauge fields among the equations of motion, $E(\varphi) = 0$. Because of these missing equations, S^a for the global symmetry does not need to vanish even at the the on-shell level.

Noether identities together with the Poincaré lemma enable us to express the Noether current in terms of the on-shell vanishing current S^a and a certain arbitrary anti-symmetric second rank tensor J^{ab} , which will be named as the *Noether potential*³

$$\tilde{J}^a = S^a + \nabla_b J^{ab}. \quad (2.6)$$

³This tensor, in the canonical terminology, is called as the *superpotential*. However, we have chosen the *potential* for this one since the same terminology is also used in the context of (fake) supersymmetry with the completely different meaning.

This expression shows that the Noether current for local symmetry becomes on-shell vanishing up to a certain ambiguity. Thus, it is unclear how to define conserved charges for local symmetry generically in terms of currents. To define conserved charges under the inherent ambiguity of the Noether current, one needs to adopt a certain prescription for choosing the appropriate current or potential.

One way to define conserved charges is to introduce the asymptotically conserved potential through the on-shell vanishing current by linearizing the fields and EOM on a given background. Then, by removing the ambiguity from the potential, one can define the conserved charges as their integrals of the *on-shell* potential corresponding to the ‘asymptotic Killing vectors’. More mathematically, the arbitrariness of the potential may be resolved in the language of the characteristic cohomology with the appropriate identification of asymptotic boundary behaviors [54]. The well-established Arnowitt-Deser-Misner (ADM) [55] and Abbott-Deser-Tekin (ADT) [56, 57] methods can be understood in this category.

There are also well known approaches [1, 2, 58] to define conserved charges by the Noether potential without the linearization, which belong to quasi-local category. For some specific cases, these quasi-local charges are shown to be consistent with the conserved charges under the linearized method by using the one parameter family of solutions when a certain integrability condition is satisfied [38]. This indicates that the conserved charges may be defined even at the non-linear level by the appropriate choice of the Noether potential.

While the conventional Noether current defined in eq. (2.3) is conserved *on-shell* only, it is possible to construct the current which would be conserved without using EOM, i.e. *off-shell* current. Indeed, by using the eq. (2.6), one can construct the corresponding *off-shell* Noether current J^a , as

$$J^a \equiv \tilde{J}^a - S^a = \nabla_a J^{ab}. \tag{2.7}$$

It turns out that this current satisfies the off-shell conservation law. Just like the case of the on-shell current, there exists a definite prescription to obtain the *off-shell* Noether current by the appropriate potential J^{ab} , directly from the Lagrangian of any diffeomorphism theory of gravity [46]. Since our derivation of the horizon entropy and angular momentum invariance uses the *off-shell* Noether current and its potential, we shall now review this approach.⁴

Consider a generally covariant Lagrangian built from metric and curvature tensor

$$I = \frac{1}{16\pi G} \int d^d x \sqrt{-g} \left[\mathcal{L}(g_{ab}, R_{abcd}) + \mathcal{L}_m(g_{ab}, \phi) \right]. \tag{2.8}$$

The variation of \mathcal{L} with respect to g^{ab} is given by

$$\delta(\mathcal{L}\sqrt{-g}) = \sqrt{-g}(\mathcal{G}_{ab}\delta g^{ab} + \nabla_a \mathcal{V}^a), \tag{2.9}$$

⁴It is also possible to extend the following analysis for the more general theories containing derivatives of curvature tensor. However, for the sake of simplicity we shall concentrate on the Lagrangians constructed from the metric and curvature tensor.

where

$$\begin{aligned} \mathcal{G}_{ab} &= P_a^{cde} R_{bcde} - 2\nabla^c \nabla^d P_{abcd} - \frac{1}{2} g_{ab} \mathcal{L}; & P^{abcd} &\equiv \frac{\partial \mathcal{L}}{\partial R_{abcd}}, \\ \mathcal{V}^a(\delta g) &= 2[P^{cbad} \nabla_b \delta g_{dc} - \delta g_{dc} \nabla_b P^{cabd}]. \end{aligned} \quad (2.10)$$

Here, \mathcal{G}^{ab} is the generalized Einstein tensor which satisfies Bianchi identity. Let us consider the variation in metric which is induced by the diffeomorphism

$$x^a \rightarrow x^a - \xi^a; \quad \delta g_{ab} = 2\nabla_{(a} \xi_{b)}. \quad (2.11)$$

Under the above transformation any scalar density changes as

$$\delta_\xi(\mathcal{L}\sqrt{-g}) = \sqrt{-g} \nabla_a(\mathcal{L}\xi^a). \quad (2.12)$$

This implies (using eq. (2.9)) the off-shell conservation law for the ‘modified’ Noether current

$$\nabla_a \left[2\mathcal{G}^{ab} \xi_b + \xi^a \mathcal{L} - \mathcal{V}^a(\xi) \right] = \nabla_a J^a = 0, \quad (2.13)$$

where

$$J^a = 2\mathcal{G}^{ab} \xi_b + \xi^a \mathcal{L} - \mathcal{V}^a(\xi). \quad (2.14)$$

Expressing the boundary term \mathcal{V}^a as a linear combination of δg_{ab} and $\delta \Gamma_{bc}^a$ [59, 60] and using the eq. (2.11), we can rewrite the above expression as

$$J^a = 2\nabla_b(P^{adcb} + P^{acdb})\nabla_c \xi_d + 2P^{abcd}\nabla_b \nabla_c \xi_d - 4\xi_d \nabla_b \nabla_c P^{abcd}. \quad (2.15)$$

There is an important point which we would like to emphasize. The usual expression for the Noether current consists of the last two terms of eq. (2.14). A comparison with eq. (2.7) clearly shows that $S^a = 2\mathcal{G}^{ab} \xi_b$. Consequently, this current is conserved only after using EOM. In contrast, the modified Noether current obtained above obeys the *off-shell* conservation law.

Next, we introduce the antisymmetric tensor J^{ab} (Noether potential) by the condition

$$J^a = \nabla_b J^{ab}. \quad (2.16)$$

Then, we can take

$$J^{ab} = 2P^{abcd}\nabla_c \xi_d - 4\xi_d (\nabla_c P^{abcd}). \quad (2.17)$$

It is worth mentioning that for Einstein gravity if the diffeomorphisms are Killing vectors then the Noether potential reduces to the well known Komar potential [50]. As we shall show later, for the higher curvature gravity like NMG, the above Noether potential can also be used to calculate the conserved quantities corresponding to the appropriate Killing vectors. In this sense, we call $J^{ab}[\xi_{Killing}]$ as the generalized Komar potential.

The conserved Noether charge corresponding to J^{ab} may be defined by

$$Q = \frac{1}{16\pi G} \int d\Sigma_{ab} J^{ab}. \quad (2.18)$$

Our convention for the area element for the subspace of codimension one and two is taken as $d\Sigma_a$ and $d\Sigma_{ab}$, respectively. These are defined such that the Stokes' theorem takes the following form

$$\int_{\mathcal{M}} d\Sigma_a J^a = \int_{\mathcal{M}} d\Sigma_a \nabla_b J^{ab} = \int_{\partial\mathcal{M}} d\Sigma_{ab} J^{ab}.$$

The variations of the current J^a and the measure $d\Sigma_a$ under an arbitrary diffeomorphism $x^a \rightarrow x^a - \xi'^a$ are given by

$$\delta_{\xi'} J^a = \xi'^b \nabla_b J^a - J^b \nabla_b \xi'^a, \quad \delta_{\xi'} d\Sigma_a = d\Sigma_a (\nabla_b \xi'^b), \quad (2.19)$$

which lead to the variation of the charge as

$$\delta_{\xi'} Q = \frac{1}{8\pi G} \int d\Sigma_{ab} \xi'^{[b} J^{a]}. \quad (2.20)$$

One can apply the above formalism even to gravity theory coupled with matters. To be specific, let us consider an interacting scalar field with two derivatives, whose Lagrangian is

$$\mathcal{L}_m(g_{ab}, \phi) = -\frac{1}{2} \partial_a \phi \partial^a \phi - V(\phi). \quad (2.21)$$

Its generic variation and diffeomorphism transformation are given by

$$\begin{aligned} \delta(\sqrt{-g} \mathcal{L}_m) &= \sqrt{-g} \left[\mathcal{E}_\phi \delta\phi + \nabla_a \mathcal{V}^a(\delta\phi) \right], \\ \delta_\xi(\sqrt{-g} \mathcal{L}_m) &= \sqrt{-g} (\xi^a \mathcal{L}_m), \end{aligned} \quad (2.22)$$

where $\delta_\xi \phi$ denotes the variation of the scalar field under diffeomorphism and so it is given by $\delta_\xi \phi = \mathcal{L}_\xi \phi = \xi^a \nabla_a \phi$. The additional contribution from a scalar field can be computed as

$$-2T^{ab} \xi_b + \xi^a \mathcal{L}_m - \mathcal{V}^a(\delta_\xi \phi) = 0. \quad (2.23)$$

As a result, the final expression for currents with a scalar field can be taken as the same form given earlier (2.14) without matter fields.

3 Killing horizon and entropy of black holes

In this section we shall compute the central extension term in the Virasoro algebra among the generators corresponding to the diffeomorphism symmetry for the NMG with a cosmological constant. Then, we implement the stretched horizon approach and calculate the horizon entropy for the non-extremal rotating BTZ black hole.

Let us consider the action for NMG with the cosmological constant [29, 30]

$$I = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[R + \frac{2}{\ell^2} + \frac{1}{m^2} \mathcal{K} \right], \quad (3.1)$$

where

$$\mathcal{K} \equiv R_{ab} R^{ab} - \frac{3}{8} R^2.$$

For this Lagrangian the corresponding P^{abcd} is given by

$$P^{abcd} = \frac{\partial \mathcal{L}}{\partial R_{abcd}} = g^{a[c} g^{d]b} + \frac{1}{m^2} \left(R^{a[c} g^{d]b} - R^{b[c} g^{d]a} - \frac{3}{4} R g^{a[c} g^{d]b} \right). \quad (3.2)$$

It is easy to see that $\nabla_a P^{abcd} \neq 0$ in general and therefore we should use the general expressions for the Noether current and potential given in (2.15) and (2.17), respectively. It is useful to compare this with the corresponding expressions in the usual Einstein or Lanczos-Lovelock gravity. In these cases, only second term in eq. (2.15) and first term in eq. (2.17) survive.

Our task is to compute the central extension term $C[\xi_1, \xi_2]$ for the algebra among the conserved Noether charges defined in eq. (2.18). The general form of $C[\xi_1, \xi_2]$ is given by

$$C(\xi_1, \xi_2) = [Q(\xi_1), Q(\xi_2)] - Q([\xi_1, \xi_2]), \quad (3.3)$$

where the *Lie* bracket $[\xi_1, \xi_2]$ is defined by

$$[\xi_1, \xi_2]^a \equiv \xi_1^b \nabla_b \xi_2^a - \xi_2^b \nabla_b \xi_1^a. \quad (3.4)$$

The essential part in the computation of the central extension term is to consider the *Lie* variation of the covariantly conserved Noether current under the diffeomorphism $x \rightarrow x - \xi_2$. Note that the current $J^a[\xi_1]$ is the consequence of the invariance of the original action under the diffeomorphism labeled by ξ_1 (see eq. (2.14)). The Lie variation of this current, $\delta_{\xi_2} J^a[\xi_1]$ induces the corresponding variation in the conserved Noether charge as

$$\delta_{\xi_2} Q[\xi_1] = \frac{1}{16\pi G} \int d\Sigma_{ab} \left(\xi_2^b J^a[\xi_1] - \xi_1^b J^a[\xi_2] \right). \quad (3.5)$$

This enable us to compute the bracket among the Noether charges [25]

$$[Q(\xi_1), Q(\xi_2)] \equiv \delta_{\xi_1} Q(\xi_2) - \delta_{\xi_2} Q(\xi_1) = 2 \int d\Sigma_{ab} \left(\xi_2^a J_1^b - \xi_1^a J_2^b \right). \quad (3.6)$$

where $J_1^a \equiv J^a[\xi_1]$. Note that the the definition for the bracket is general in the sense that it does not require explicit form for the Noether current and works well for any covariant theory of gravity.

Next, in order to obtain the horizon entropy we shall evaluate the algebra among the Noether charges, $Q([\xi_1, \xi_2])$ on the null surface. To this purpose we implement the stretched horizon approach given in [12]. In this approach one begins with an approximate Killing vector χ^a in the neighborhood of the boundary Σ of generic $d + 1$ dimensional Riemannian manifold. The local Killing horizon is defined by the condition $\chi^2 = 0$. This can be alternatively stated as

$$\chi^2 = \epsilon. \quad (3.7)$$

with ϵ being taken to be zero at the end. The vector orthogonal to the orbits of χ^a is given by

$$\rho_a = -\frac{1}{2\kappa} \nabla_a \chi^2. \quad (3.8)$$

Where κ is the surface gravity associated with the Killing vector field χ^a

$$\kappa^2 \equiv \lim_{\chi^2 \rightarrow 0} \left[\frac{\nabla_a \chi^2 \nabla^a \chi^2}{4\chi^2} \right]. \quad (3.9)$$

Consider a general diffeomorphism transformation

$$\xi^a = T\chi^a + R\rho^a, \quad (3.10)$$

with the condition that

$$\delta\chi^2 = 0; \quad \text{as} \quad \chi^2 \rightarrow 0. \quad (3.11)$$

This condition leads to the following relation between two arbitrary functions T and R :

$$R = \frac{1}{\kappa} \frac{\chi^2}{\rho^2} DT; \quad D \equiv \chi^a \nabla_a. \quad (3.12)$$

Now we compute the Noether potential J^{ab} for the diffeomorphism (3.10) on the stretched horizon. By exploiting the relation (3.12) we can express this diffeomorphism completely in terms of T . Then by using eq. (A.5) we get

$$J^{ab}(\xi) = - \left\{ P^{abcd} S_{cd} \left[2\kappa T - \frac{D^2 T}{\kappa} \right] + 4\nabla_c P^{abcd} \left(\chi_d T + \rho_d \frac{DT}{\kappa} \right) \right\}. \quad (3.13)$$

Integrating this expression over the null surface with the differential area element $d\Sigma_{ab}$ given by (A.7) and using the identity (A.9), we arrive at the expression for the conserved Noether charge

$$Q[\xi] = -\frac{1}{32\pi G} \int \sqrt{h} d^{d-2}x \left\{ P^{abcd} S_{ab} S_{cd} \left[2\kappa T - \frac{D^2 T}{\kappa} \right] + 4\nabla_c P^{abcd} S_{ab} \left(\chi_d T - \rho_d \frac{DT}{\kappa} \right) \right\}. \quad (3.14)$$

Note that the above expression contains the terms proportional to P^{abcd} as well as $\nabla_a P^{a\cdots}$. However, near the Killing horizon, $\nabla_a P^{a\cdots}$ term is of the higher order χ^2 compared to P^{abcd} term. Since the Killing horizon is defined by the limit $\chi^2 \rightarrow 0$, we can neglect the derivative term of P -tensor.⁵ This can be explicitly checked by Taylor expanding the P and ∇P terms near the Killing horizon. Similar kind of arguments were used earlier (see the eq. (19) of ref. [24]) to obtain the central charge for higher curvature gravity within the Carlip's on-shell approach. On using this fact, the expression for the Noether charge becomes

$$Q[\xi] = -\frac{1}{32\pi G} \int \sqrt{h} d^{d-2}x \left\{ P^{abcd} S_{ab} S_{cd} \left[2\kappa T - \frac{D^2 T}{\kappa} \right] \right\}. \quad (3.15)$$

We now evaluate the corresponding expression for $Q([\xi_1, \xi_2])$. First, we note that the Lie bracket for the diffeomorphisms (3.10) can be written as

$$[\xi_1, \xi_2]^a = \chi^a [T_1, T_2] - \frac{\rho^a}{\kappa} D[T_1, T_2]; \quad [T_1, T_2] = T_1 DT_2 - T_2 DT_1. \quad (3.16)$$

⁵For some specific cases, where the near horizon geometry is AdS (e.g. BTZ black hole) or the product of two maximally symmetric spaces like $AdS_2 \times \mathcal{M}^{n-2}$ or $Rindler \times \mathcal{M}^{n-2}$, ∇P vanishes identically.

Inserting this in eq. (3.15), we obtain the expression for $Q([\xi_1, \xi_2])$ as

$$Q[\{\xi_1, \xi_2\}] = -\frac{1}{32\pi G} \int \sqrt{h} d^{d-2} X \left\{ P^{abcd} S_{ab} S_{cd} \left[2\kappa (T_1 D T_2 - T_2 D T_1) - \frac{1}{\kappa} (D T_1 D^2 T_2 + T_1 D^3 T_2 - D T_2 D^2 T_1 - T_2 D^3 T_1) \right] \right\}. \quad (3.17)$$

Next, we perform the similar analysis on the right hand side of eq. (3.6). The explicit expression of the Noether current (2.15) for the diffeomorphism (3.10) is given by

$$J^a(\xi) = 2P^{abcd} \frac{\chi_c \rho_d \chi^b}{\chi^4} \left(2\kappa D T - \frac{D^3 T}{\kappa} \right) + 2\nabla_b P^{abcd} \frac{\chi_c \rho_d}{\chi^2} \left(2\kappa T - \frac{D^2 T}{\kappa} \right) - 4\nabla_b \left[\left(T \chi_d - \frac{D T}{\kappa} \rho_d \right) \nabla_c P^{abcd} \right]. \quad (3.18)$$

Substituting this in eq. (3.6) and using the identity (A.10), we can see

$$[Q(\xi_1), Q(\xi_2)] = \frac{1}{32\pi G} \int \sqrt{h} d^{d-2} x \left\{ P^{abcd} S_{ab} S_{cd} \left[\left(2\kappa D T_1 - \frac{D^3 T_1}{\kappa} \right) T_2 - (1 \leftrightarrow 2) \right] \right\}. \quad (3.19)$$

As before, we have neglected the terms proportional to ∇P and $\nabla \nabla P$. Combining this result with the eq. (3.17), the central extension term $C[\xi_1, \xi_2]$ finally becomes

$$C[\xi_1, \xi_2] = -\frac{1}{32\pi G} \int \sqrt{h} d^{d-2} X \left\{ P^{abcd} S_{ab} S_{cd} \frac{1}{\kappa} [D T_1 D^2 T_2 - D T_2 D^2 T_1] \right\}. \quad (3.20)$$

By rewriting the above expression in the Fourier variables

$$C(\xi_1, \xi_2) = \sum_{mn} f_{mn} C(\xi_m, \xi_n), \quad (3.21)$$

we obtain

$$C(\xi_m, \xi_n) = -\frac{1}{32\pi G} \int \sqrt{h} d^{d-2} x \left\{ P^{abcd} S_{ab} S_{cd} \frac{1}{\kappa} [D T_m D^2 T_n - D T_n D^2 T_m] \right\}. \quad (3.22)$$

We now apply this analysis for non-extremal rotating BTZ black hole given by the metric

$$ds^2 = L^2 \left[-\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} dt^2 + \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 \left(d\theta - \frac{r_+ r_-}{r^2} dt \right)^2 \right]. \quad (3.23)$$

For this case, the approximate Killing vector χ^a is given by

$$\chi^a = (1, 0, \Omega); \quad \chi_a = (g_{tt} + \Omega g_{t\theta}, 0, \Omega g_{\theta\theta} + g_{t\theta}), \quad (3.24)$$

where $\Omega = r_-/r_+$ is the angular velocity at the outer horizon. In the near horizon region given by $r = r_+ + \epsilon$, it is easy to see that the norm of the Killing vector χ_a behaves as

$$\chi^2 = -\epsilon \left[\frac{2(r_+^2 - r_-^2)}{r_+} L^2 \right]. \quad (3.25)$$

In order to obtain the central charge, we take the ansatz of T_n , appropriate for $\text{Diff}(S^1)$ algebra [21], as

$$T_n = \frac{1}{\alpha + \Omega} \exp[in(\alpha t + \theta + g(r))] . \quad (3.26)$$

Here, variables t and θ have periodicities $2\pi/\alpha$ and 2π respectively and $g(r)$ is a certain function regular at the Killing horizon. Inserting this in eq. (3.22), we arrive at

$$C[\xi_m, \xi_n] = \frac{\mathcal{A}}{8\pi G} \left[-im^3 \delta_{m+n,0} \frac{(\alpha + \Omega)}{\kappa} \right] , \quad (3.27)$$

where

$$\mathcal{A} \equiv -\frac{L}{2} \int r d\theta P^{abcd} S_{ab} S_{cd} . \quad (3.28)$$

Using the ansatz of T_n , one can see that the Fourier modes of the Noether charges are given by

$$Q(\xi_m) = \frac{\mathcal{A}}{8\pi G} \left[\delta_{m,0} \frac{\kappa}{(\alpha + \Omega)} \right] ; \quad Q([\xi_m, \xi_n]) = -i(m - n)Q(\xi_m) . \quad (3.29)$$

Using these expressions and the eq. (3.27) in the algebra (3.3), we identify the central charge c and the zero-mode eigenvalue of $Q(\xi_m)$ as

$$c = \frac{\alpha + \Omega}{\kappa} \frac{12}{8\pi G} \mathcal{A} ; \quad Q_0 = \frac{\kappa}{(\alpha + \Omega)} \frac{\mathcal{A}}{8\pi G} . \quad (3.30)$$

Through the Cardy formula, the entropy for non-extremal BTZ black holes is given by

$$S = 2\pi \sqrt{\frac{c\Delta_0}{6}} = 2\pi \sqrt{\frac{c(Q_0 - \frac{c}{24})}{6}} = \frac{\mathcal{A}}{4G} \left[2 - \left(\frac{\alpha + \Omega}{\kappa} \right)^2 \right]^{1/2} . \quad (3.31)$$

By setting $\alpha = \kappa - \Omega$, one can see that the above expression reduces to familiar Wald formula [1].

Using the P -tensor for the rotating BTZ black holes

$$P^{abcd} = g^{a[c} g^{d]b} \left[1 + \frac{1}{2m^2 L^2} \right] , \quad (3.32)$$

and the expression for S_{ab} in the eq. (3.28), we obtain

$$\mathcal{A} = 2\pi L r_+ \left[1 + \frac{1}{2m^2 L^2} \right] . \quad (3.33)$$

Hence, the corresponding entropy of the horizon (with the choice $\alpha = \kappa - \Omega$) becomes

$$S = \frac{\pi L r_+}{2G} \left[1 + \frac{1}{2m^2 L^2} \right] . \quad (3.34)$$

The first term in the above represents the entropy for rotating BTZ black hole in Einstein gravity, while the second term gives its correction due to the higher curvature terms in the Lagrangian (3.1). Our result for entropy is also consistent with the one given in [61].

We now briefly describe some important steps to compute the entropy for extremally rotating BTZ black holes by the *off-shell* expressions for Noether current and potential. For the extremal case, the ‘stretched horizon’ approach, given in [12] needs to be modified. An alternate way to deal with such geometries within the stretched horizon approach was presented in [67]. We shall not follow this approach here. Instead, we consider the near horizon limit of extremally rotating BTZ black hole in which case the resultant geometry is AdS_3 . We then compute the central extension term for the diffeomorphisms which preserve the asymptotic fall-off conditions [9, 68].

To begin with, let us take the near horizon extremal BTZ geometry in the form of

$$ds_{NH}^2 = \frac{L^2}{4} \left[-\rho^2 dt^2 + \frac{1}{\rho^2} d\rho^2 + 4r_H^2 \left(d\theta - \frac{\rho}{2r_H} dt \right)^2 \right]. \quad (3.35)$$

The relevant diffeomorphisms given in [48, 68], are

$$\xi_n = e^{in\theta} \left(\partial_\theta - in\rho\partial_\rho \right), \quad (3.36)$$

which preserve appropriate boundary conditions [9]. It can be easily checked that these diffeomorphisms satisfy $\text{Diff}(S^1)$ algebra (1.2). We now use the generic expression for the central term

$$C(\xi_1, \xi_2) = \frac{1}{8\pi G} \int d\Sigma_{ab} \left\{ \left[\left(2P^{bcde} \xi_2^a \nabla_c \nabla_d \xi_{1e} + \xi_2^a \left(2\nabla_c P^{bcde} \nabla_d \xi_{1e} - 4\nabla_d P^{bcde} \nabla_c \xi_{1e} \right) \right. \right. \right. \quad (3.37)$$

$$\left. \left. \left. - 4\xi_2^a \xi_{1e} \nabla_c \nabla_d P^{bcde} \right) - (1 \leftrightarrow 2) \right] - \frac{1}{2} \left[2P^{abcd} \nabla_c \{ \xi_1, \xi_2 \}_d - 4 \{ \xi_1, \xi_2 \}_d \nabla_c P^{abcd} \right] \right\}.$$

which can be obtained by using the eq. (3.3). It may be recalled that this central extension was derived using the off-shell expressions for Noether current (2.15) and potential (2.17). After performing a little algebra, we obtain

$$C(\xi_p, \xi_q) = -i \frac{L}{8G} \left[1 + \frac{1}{2m^2 L^2} \right] p(p^2 + 4r_H^2) \delta_{p+q,0}, \quad (3.38)$$

and hence the central charge becomes,

$$c_{IR} = \frac{3L}{2G} \left[1 + \frac{1}{2m^2 L^2} \right]. \quad (3.39)$$

The above expression for the central charge is also consistent with the corresponding one given in [49]. By exploiting the standard Cardy formula we can obtain the entropy for the infrared dual CFT as

$$S_{IR} = 2\pi \sqrt{\frac{c_{IR}}{6} (M_H L + J_H)} + 2\pi \sqrt{\frac{c_{IR}}{6} (M_H L - J_H)} = 2\pi \sqrt{\frac{c_{IR}}{3} J_H}, \quad (3.40)$$

where M_H and J_H represent the mass and angular momentum at the horizon, respectively. The second equality in the above expression comes from the fact that, at extremality, mass and angular momentum satisfy $M_H L = J_H$. As we shall see later, the quasi-local angular momentum J_H matches exactly with the total angular momentum J_∞ . This property is the consequence of the invariance of the angular momentum along the radial direction. In the next section we shall give the explicit proof for this invariance and show that the entropy (3.40) is indeed identical with the one computed earlier (3.34).

4 Angular momentum and extremal black holes

In this section we will compute the angular momentum for the BTZ black holes in NMG by using only the *off-shell* Noether potential (2.17). As will be shown later, the angular momentum remains invariant along the radial direction. Next, we show that hairy deformation of the extremal BTZ black holes also carry the same property. The point is that these extremal black hole solutions interpolate two AdS spaces, viz. the asymptotic AdS_3 and the near horizon AdS_3 and therefore can be regarded as a holographic realization of RG flows of a certain field theory. Our confirmation of the angular momentum invariance in these black hole solutions also verifies that the entropy of the infrared dual CFT is identical with the BH entropy.⁶

The action for NMG minimally coupled to a scalar field ϕ is

$$I = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[R + \frac{1}{m^2} \mathcal{K} - \frac{1}{2} \partial_a \phi \partial^a \phi - V(\phi) \right]. \quad (4.1)$$

The equations of motion for the metric and the scalar field are given by

$$\mathcal{E}_{ab} \equiv \mathcal{G}_{ab} - T_{ab} = 0, \quad \mathcal{E}_\phi \equiv \nabla^2 \phi - \partial_\phi V = 0, \quad (4.2)$$

where \mathcal{G}_{ab} denotes the generalized Einstein tensor and T_{ab} does the stress tensor for a scalar field as

$$\begin{aligned} \mathcal{G}_{ab} &= R_{ab} - \frac{1}{2} R g_{ab} + \frac{1}{2m^2} \mathcal{K}_{ab}, & T_{ab} &= \frac{1}{2} \partial_a \phi \partial_b \phi - \frac{1}{2} g_{ab} \left[\frac{1}{2} \partial_c \phi \partial^c \phi + V(\phi) \right], \\ \mathcal{K}_{ab} &\equiv g_{ab} \left(3R_{cd} R^{cd} - \frac{13}{8} R^2 \right) + \frac{9}{2} R R_{ab} - 8R_{ac} R_b^c + \frac{1}{2} \left(4\mathcal{D}^2 R_{ab} - \mathcal{D}_a \mathcal{D}_b R - g_{ab} \mathcal{D}^2 R \right). \end{aligned} \quad (4.3)$$

By applying the formalism given in section 2 to the present case, we obtain the quasi-local angular momentum corresponding to the rotational Killing vector on the domain \mathcal{B} of codimension two, as

$$J_{\mathcal{B}} \equiv Q_{\mathcal{B}}(\xi_R) = \frac{1}{16\pi G} \int_{\mathcal{B}} d\Sigma_{ab} J^{ab}(\xi_R) = \frac{1}{8G} \sqrt{-\det g} \left. J^{rt}(\xi_R) \right|_{\mathcal{B}}, \quad (4.4)$$

where $\det g$ denotes the determinant of the three-dimensional metric g_{ab} . Our convention for the normalization of the rotational Killing vector $\xi_R \equiv \frac{\partial}{\partial \theta}$ is chosen such as $\xi_R^2(r \rightarrow \infty) \rightarrow L^2 r^2$.

For the Killing vector ξ^μ of the metric, one can see that

$$\begin{aligned} J^a &= 2R^{ab} \xi_b - \frac{2}{m^2} R^{cd} \nabla_c \nabla_d \xi_b \\ &+ \frac{1}{m^2} \left[-2R^{acdb} R_{cd} - \frac{3}{2} R R^{ab} + 2\nabla^2 R^{ab} - \frac{1}{2} \nabla^a \nabla^b R - \frac{1}{2} g^{ab} \nabla^2 R \right] \xi_b. \end{aligned} \quad (4.5)$$

Let us consider rotating BTZ black holes (3.23) as an example to give some taste of our approach. By computing the Komar potential of BTZ black holes, one obtains

$$J^{rt}(\xi_R) = \frac{2r_+ r_-}{r L^2} \left[1 + \frac{1}{2m^2 L^2} \right], \quad (4.6)$$

⁶The relevant discussion for the Einstein gravity is provided in appendix B.

which leads to the quasi-local angular momentum as

$$J_{\mathcal{B}_r} = J_\infty = J_H = \frac{Lr_+r_-}{4G} \left[1 + \frac{1}{2m^2L^2} \right], \quad (4.7)$$

where $J_{\mathcal{B}_r}$, J_∞ and J_H denote the quasi-local angular momentum at the $r = \text{constant}$, the asymptotic infinity and the horizon, respectively. Note that this expression for the angular momentum at asymptotic infinity, J_∞ is consistent with the result for the angular momentum of BTZ black holes in other methods like ADT or boundary stress tensor [61–66]. In fact, this result shows us that the angular momentum is invariant along the radial direction as was argued generically to be the case for pure Einstein gravity in [36]. Now it is easy to see that by inserting J_H in the eq. (3.40) we can reproduce the entropy for extremally rotating BTZ black holes

$$S_{IR} = \frac{\pi Lr_+}{2G} \left[1 + \frac{1}{2m^2L^2} \right]. \quad (4.8)$$

It is interesting to note that our result agrees with the corresponding one obtained by using the central charge of ultraviolet CFT (c_{UV}) and the total angular momentum J_∞ [7, 61].

Now we consider the extremally rotating BTZ black holes in NMG coupled with a scalar field and show explicitly the invariance of the angular momentum along the radial direction. The most generic axi-symmetric metric of the rotating hairy black holes, deformed from BTZ ones, can be taken as

$$ds^2 = L^2 \left[-e^{2A(r)} dt^2 + e^{2B(r)} dr^2 + r^2 (d\theta + e^{C(r)} dt)^2 \right], \quad (4.9)$$

with the asymptotic AdS boundary condition for $r \rightarrow \infty$ as

$$e^{A(r)} \rightarrow r, \quad e^{B(r)} \rightarrow \frac{1}{r}, \quad e^{C(r)} \rightarrow \text{constant}, \quad \phi(r) \rightarrow \text{constant}. \quad (4.10)$$

To obtain the extremal hairy AdS black hole solutions in the case of NMG, the scalar potential can be taken in terms of the so-called superpotential $\mathcal{W}(\phi)$ as

$$V(\phi) = \frac{1}{2L^2} (\partial_\phi \mathcal{W})^2 \left[1 - \frac{1}{8m^2L^2} \mathcal{W}^2 \right]^2 - \frac{1}{2L^2} \mathcal{W}^2 \left[1 - \frac{1}{16m^2L^2} \mathcal{W}^2 \right], \quad (4.11)$$

which is motivated by the domain wall case [69–72] and can be explained by fake supersymmetry in the Einstein gravity limit [73]. As was shown in [45], extremal hairy deformed BTZ black holes satisfy some reduced EOM. By solving partially this reduced EOM, one can show that the metric functions A and B are determined in terms of a certain function Ψ and the superpotential \mathcal{W} as

$$rA'(r) - 1 = \frac{\Psi}{r} e^{B(r)}, \quad (4.12)$$

$$e^{-B(r)} = \frac{r}{2} \left[\mathcal{W} - \frac{\Psi}{r^2} \right], \quad (4.13)$$

where $'$ denotes the differentiation with respect to the radial coordinate r . The remaining part of the reduced EOM for the function Ψ is given by

$$\tilde{\Delta} \left[1 + \frac{1}{2m^2L^2} \right] = \frac{1}{m^2L^2} e^{-2B} (\ddot{\Psi} - \mathcal{W}\dot{\Psi}) + \left[1 + \frac{1}{8m^2L^2} (\mathcal{W}^2 - 4\dot{\mathcal{W}}) \right] \Psi, \quad (4.14)$$

where $\tilde{\Delta}$ is a certain integration constant and $\dot{}$ denotes the differentiation defined as $\dot{\Psi} \equiv e^{-B}\Psi'$. In fact, it turns out that the constant $\tilde{\Delta}$ is related to the horizon values of \mathcal{W} as

$$\tilde{\Delta} = r_H^2 \mathcal{W}(\phi_H) \left[1 + \frac{1}{8m^2 L^2} \mathcal{W}^2(\phi_H) \right] \left[1 + \frac{1}{8m^2 L^2} \mathcal{W}^2(\phi_\infty) \right]^{-1}. \quad (4.15)$$

Note that these solutions satisfy the extremality condition as $e^{-2B(r_H)} = \frac{d}{dr} e^{-2B(r_H)} = 0$. This is equivalent to the relation between the mass and the angular momentum: $M_\infty L = J_\infty$.

In the following we confine ourselves to the Brown-Henneaux boundary conditions [9]. These boundary conditions however are not the most general ones even for the usual Einstein gravity coupled to a scalar field [8, 41–43]. The asymptotic analysis in this case gives us the following expression for metric variables, the superpotential and the scalar field respectively

$$\begin{aligned} A(r) &= \ln r - \frac{\tilde{\Delta}}{2r^2} + \dots, & B(r) &= -\ln r + \frac{1}{2r^2} \left(\tilde{\Delta} - \frac{1}{2q} \tilde{\phi}_1^2 \right) + \dots, \\ \mathcal{W}(\phi(r)) &= 2 + \frac{\tilde{\phi}_1^2}{2qr^2} + \dots, & \phi(r) &= \phi_\infty + \frac{\tilde{\phi}_1}{r} + \dots, \end{aligned} \quad (4.16)$$

where q is defined by $q \equiv 1 - 1/2m^2 L^2$. By a straightforward near horizon analysis, one can see that

$$\begin{aligned} A(r) &= \tilde{s}_0 \mathcal{W}(\phi_H)(r - r_H) + \dots, & B(r) &= \frac{1}{\mathcal{W}(\phi_H)(r - r_H)} \dots, \\ \mathcal{W}(\phi(r)) &= \mathcal{W}(\phi_H) - \frac{1}{2} \mathcal{W}(\phi_H) \left[1 - \frac{1}{8m^2 L^2} \mathcal{W}^2(\phi_H) \right]^{-1} (\phi - \phi_H)^2 + \dots, \\ \phi(r) &= \phi_H + \tilde{g}_0 (r - r_H) + \dots, \end{aligned} \quad (4.17)$$

where \tilde{s}_0 and \tilde{g}_0 are certain non-vanishing constants. One can show that the near horizon geometry becomes the so-called self-dual orbifold of AdS_3 space whose metric can be written as

$$ds_{NH}^2 = \frac{\bar{L}^2}{4} \left[-\rho^2 dt^2 + \frac{1}{\rho^2} d\rho^2 + 4r_H^2 \left(d\theta - \frac{\rho}{2r_H} dt \right)^2 \right], \quad \bar{L} \equiv \frac{2L}{\mathcal{W}(\phi_H)}. \quad (4.18)$$

In this case, it turns out that the rt component of the potential is given by

$$\begin{aligned} J^{rt}(\xi_R) &= \frac{1}{L^2} e^{-2B-A} F \left[1 + \frac{1}{2m^2 L^2} e^{-2B} \right]'' - \frac{3}{4m^2 L^2} \frac{F}{r} (e^{-2B})' \\ &\quad + \frac{3}{2m^2 L^4} e^{-2B-A} F' (e^{-2B})' \\ &\quad + \frac{1}{8m^2 L^4} e^{-4B-A} \left[\frac{1}{r^2} F^3 - \frac{4}{r^2} F - \frac{12}{r} F F' + 8F'' \right], \\ F &\equiv rA'(r) - 1. \end{aligned} \quad (4.19)$$

By using the eq. (4.13) and the eq. (4.15), one can obtain a differential equation without the superpotential \mathcal{W} . By combining the resultant equation with the eq. (4.12), one can show

that the above quasi-local angular momentum is independent of the radial coordinate as

$$J_{\mathcal{B}_r}(\xi_R) = \frac{L}{8G} \tilde{\Delta} \left[1 + \frac{1}{2m^2 L^2} \right] = \frac{L}{8G} \tilde{\Delta} \left[1 + \frac{1}{8m^2 L^2} \mathcal{W}^2(\phi_\infty) \right]. \quad (4.20)$$

Note that the above expression of angular momentum at the asymptotic infinity for hairy deformed BTZ black holes is completely consistent with the previous results [64]. It is interesting to compute the quasi-local angular momentum directly on the near horizon geometry. The result is given by

$$J_H = \frac{L}{8G} r_H^2 \mathcal{W}(\phi_H) \left[1 + \frac{1}{8m^2 L^2} \mathcal{W}^2(\phi_H) \right].$$

Now, by using the relation between $\tilde{\Delta}$ and $\mathcal{W}(\phi_H)$ given in the eq. (4.15), one can verify that

$$J_{\mathcal{B}_r} = J_{r \rightarrow \infty} = J_H. \quad (4.21)$$

Some comments are in order. Firstly, the AdS radii on the asymptotic infinity and on the horizon are different but give the same expression for the angular momentum. In the Einstein gravity limit this result is consistent with the general argument given in the appendix B. Secondly, this result is also consistent with the one from the Brown-Henneaux method adopted in [40] for a specific example. Our result can be regarded as the generalization of this case to more generic extremal hairy black holes. Furthermore, we showed that the quasi-local angular momentum is invariant along whole RG trajectory not just at the two conformal points. Finally, one may note that the invariance of angular momentum for the extremal hairy deformed BTZ black holes is, more or less, connected with the reduced order EOM, which may have some implication for the entropy of black holes.

Now we study the relationship between the entropy of the deformed extremal BTZ black holes and holographic c -theorem. The central charge of the ultraviolet and the infrared dual CFT can be obtained through the dictionary of the AdS/CFT correspondence as

$$\begin{aligned} c_{UV} &= \frac{3L}{2G} \left[1 + \frac{1}{2m^2 L^2} \right] = \frac{3L}{G\mathcal{W}(\phi_\infty)} \left[1 + \frac{1}{8m^2 L^2} \mathcal{W}^2(\phi_\infty) \right], \\ c_{IR} &= \frac{3\bar{L}}{2G} \left[1 + \frac{1}{2m^2 \bar{L}^2} \right] = \frac{3L}{G\mathcal{W}(\phi_H)} \left[1 + \frac{1}{8m^2 L^2} \mathcal{W}^2(\phi_H) \right]. \end{aligned} \quad (4.22)$$

Though the central charge, c_{UV} , of ultraviolet dual CFT has been derived in various ways [74, 75], the central charge, c_{IR} of the infrared dual CFT has not done explicitly. Following the discussion given at the end of the previous section, it is straightforward to obtain the expression for c_{IR} for the extremal hairy BTZ black holes by using the near horizon geometry given in the eq. (4.18). This would provide a generalization of scalar-Einstein theory given in [40] to the NMG case. The holographic realization of the central charge function, which connects the above two central charges has been suggested in [45].

By using the relations between total conserved charges and conformal weights of the ultraviolet dual CFT

$$M_\infty = E_L + E_R, \quad J_\infty = L(E_L - E_R),$$

one can see that the entropy of the ultraviolet CFT computed from the Cardy formula is given by

$$S_{UV} = 2\pi\sqrt{\frac{c_L}{6}(M_\infty L + J_\infty)} + 2\pi\sqrt{\frac{c_R}{6}(M_\infty L - J_\infty)} = 2\pi\sqrt{\frac{c_{UV}}{3}J_\infty}, \quad (4.23)$$

where we have used the property of extremal black holes: $M_\infty L = J_\infty$. To apply the same relations for the infrared dual CFT on the near horizon self-dual orbifold of AdS_3 , one needs the quasi-local expressions for mass and angular momentum. Using our result for the angular momentum invariance, $J_\infty = J_H$ and the extremality condition for conserved charges $M_H \bar{L} = J_H$, it can be easily seen that the entropy of the dual CFT is related to the BH entropy as

$$S_{UV} \geq S_{IR} = S_{BH} = \frac{A_H}{4G} \left[1 + \frac{1}{2m^2 \bar{L}^2} \right], \quad A_H \equiv 2\pi \bar{L} r_H. \quad (4.24)$$

This result verifies our claim that the entropy of the infrared CFT is indeed the same as the usual black hole entropy.

5 Conclusion

In this work we have studied the entropy and the invariance of the angular momentum for the rotating BTZ black holes in NMG by using the *off-shell* expressions for the Noether current and potential. We have also showed the invariance of the angular momentum for extremally rotating scalar-hairy deformed BTZ black holes which can be interpreted as the RG flows for the dual field theory in the context of the AdS/CFT correspondence.

Firstly, we have computed the entropy for non-extremal rotating BTZ black hole by using the so-called stretched horizon approach. In this case, the entropy tensor P^{abcd} is not divergence free. Consequently, the generic expressions for *off-shell* Noether current, potential and conserved charge contain the terms proportional to P^{abcd} , ∇P and $\nabla \nabla P$. Simplification occurs when we evaluate these expressions in the vicinity of the Killing horizon. Near the Killing horizon, ∇P and $\nabla \nabla P$ terms are of the higher order in χ^2 and hence does not contribute to the Virasoro algebra. As a result, the final expression for the central term takes the same form with the one in Einstein or Lanczos-Lovelock gravity [25]. By Fourier-transforming this central extension term and using the ansatz for the scalar function T , we identified the central charge. Finally, by using this central charge and zero mode eigenvalue of the conserved Noether charges the black hole entropy is obtained.

We have also provided a brief derivation of the central charge for the rotating extremal BTZ case. In this case, we also used the *off-shell* expressions for the Noether current and potential as in the non-extremal case. However, we adopted the standard AdS/CFT dictionary to obtain the central charge, for which we have taken the diffeomorphisms preserving the fall-off boundary conditions at the asymptotic boundary of the near horizon extremal geometry. Our result for the central charge is in agreement with the one given in [7, 61] where the asymptotic boundary of whole extremal geometry is used.

Secondly, we have established the invariance of the angular momentum along the radial direction for black holes in our models. To this purpose, we have used the same expressions for Noether current or potential and verified our claims explicitly. In the context of the AdS/CFT correspondence conserved charges of three-dimensional AdS black holes are related to the conformal weights of states dual to the black holes. Therefore, the angular momentum invariance along the radial direction in this set-up indicates a certain RG flow behavior of scaling dimensions of dual operators. This invariance also plays a crucial role in the computation of the entropy. Our expression of the entropy for the extremally rotating BTZ black hole (see eq. (3.40)) contains quasi-local angular momentum J_H , which corresponds to conformal weight in the infrared CFT. On the other hand, the Cardy formula for ultraviolet case [7, 61] uses the asymptotic value for the angular momentum, J_∞ , as the conformal weight. The fact that the angular momentum is invariant along the radial direction allows us to match our result (4.8) with the corresponding one in the ultraviolet case. While this is somewhat anticipated by the stretched horizon approach to the black hole entropy, the quasi-local conserved charges need to be adopted to show this matching. Moreover, we have also shown the angular momentum invariance of the deformed extremally rotating BTZ black holes. In this case we verified that the corresponding entropy computed from the infrared CFT is less than the one from the ultraviolet case and it matches with the black hole entropy. Since our boundary conditions are rather restricted, it would be interesting to study hairy deformed rotating AdS black holes with more generic boundary conditions and investigate the angular momentum invariance in those cases.

The angular momentum invariance discussed here is restricted to some specific black hole solutions. It would be interesting to extend this analysis to the more generic case. This would enable us to connect the generalized Komar potential used here and the ADT/ADM potentials for the generic higher derivative gravity. We would like to investigate these issues in the near future.

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A Some formulae in stretched horizon approach

In this appendix we shall briefly state some results in Carlip's stretched horizon approach [12]. This will be useful in deriving the near horizon expressions for Noether current,

potential and charge. Detailed derivation of these results can be found in the appendix B of ref. [25].

The approximate Killing vector χ^a and the vector ρ^a which is normal to the Killing orbits satisfy

$$\frac{\rho^2}{\chi^2} = -1, \tag{A.1}$$

$$\nabla_a \chi_b = \frac{2\kappa}{\chi^2} \chi_{[a} \rho_{b]}, \tag{A.2}$$

$$\nabla_a \rho_b = \frac{\kappa}{\chi^2} (\chi_a \chi_b - \rho_a \rho_b). \tag{A.3}$$

For the diffeomorphisms given by eq. (3.10), we have

$$\nabla_a T = \frac{\chi_a}{\chi^2} DT, \tag{A.4}$$

$$\nabla_a \xi_b = \frac{1}{\chi^2} \left[DT \rho_a \rho_b + 2\kappa T \chi_{[a} \rho_{b]} - \frac{D^2 T}{\kappa} \chi_a \rho_b \right], \tag{A.5}$$

$$\nabla_a \nabla_b \xi_c = \frac{1}{\chi^4} \left[\chi_a \chi_b \rho_c \left(2\kappa DT - \frac{D^3 T}{\kappa} \right) - D^2 T \chi_a \chi_b \chi_c \right]. \tag{A.6}$$

These expressions are valid upto order χ^2 .

Next, we give the expression for $d - 2$ dimensional surface element $d\Sigma_{ab}$:

$$d\Sigma_{ab} = \sqrt{h} d^{d-2} x S_{ab}, \tag{A.7}$$

where h is the determinant of the metric h_{ab} on $d - 2$ dimensional surface and

$$S_{ab} = -\frac{2|\chi|}{\rho\chi^2} (\chi_{[a} \rho_{b]}). \tag{A.8}$$

Using the eq. (A.8) and the symmetries of P^{abcd} one can easily verify the following identities:

$$P^{abcd} \chi_c \rho_d = -\frac{2|\chi|}{\rho\chi^2} P^{abcd} S_{cd}, \tag{A.9}$$

$$\chi_c \chi_e \rho_d \rho_b P^{abcd} = \frac{\rho^2 \chi^2}{4} P^{abcd} S_{be} S_{cd}. \tag{A.10}$$

B Angular momentum invariance in Einstein gravity

In this appendix we show the angular momentum invariance in Einstein gravity minimally coupled with a scalar field. Note that our class of the hairy deformation of extremal BTZ black holes can be understood as the limit of the NMG case. In this case one can give more general argument for the angular momentum invariance.

Through the so-called fake supersymmetry formalism, one may take the scalar potential in this case as [73]

$$V(\phi) = \frac{1}{2L^2} (\partial_\phi \mathcal{W})^2 - \frac{1}{2L^2} \mathcal{W}^2, \tag{B.1}$$

and it turns out that the extremally rotating AdS black holes can be described by the following first order EOM

$$\phi' = -e^B \partial_\phi \mathcal{W}, \quad A' = e^B \mathcal{W} - \frac{1}{r}, \quad A' + B' = \frac{r}{2} \phi'^2, \quad (e^C)' = \pm \left(\frac{1}{r} e^A \right)', \quad (\text{B.2})$$

where $'$ denotes the derivative with respect to the radial coordinate r . Note that the last equation can be integrate as $e^{C(r)} = C_\pm \pm e^{A(r)}/r$. Since the sign in this equation is related to the rotation direction, we will take the upper sign for definiteness. And the integration constant C_+ is taken as $C_+ = 1$ to match with the BTZ black holes asymptotically.

By manipulating the first order EOM, one can show that the metric functions A and B are determined in terms of a certain constant Δ and the superpotential \mathcal{W} as

$$rA'(r) - 1 = \frac{\Delta}{r} e^{B(r)}, \quad e^{-B(r)} = \frac{r}{2} \left[\mathcal{W} - \frac{\Delta}{r^2} \right], \quad (\text{B.3})$$

which correspond to $\Psi = \Delta = \text{const.}$ in the NMG case.

In this case of Einstein gravity, it turns out that all the relevant quantities, e.g. the asymptotic and the near horizon expansion, the relation $\Delta = r_H^2 \mathcal{W}(\phi_H)$ and the Komar potential

$$J^{rt}(\xi_R) = \frac{1}{L^2} e^{-2B(r)-A(r)} \left[rA'(r) - 1 \right], \quad (\text{B.4})$$

can be understood as the limit $m^2 \rightarrow \infty$ in the NMG case.

By using the expression for the metric function A given in (B.3) and by noting that $\sqrt{-\det \bar{g}} = L^3 e^{A+B} r$, one can see that the quasi-local angular momentum is given by

$$J_B = \frac{L}{8G} \Delta = \frac{L}{8G} r_H^2 \mathcal{W}(\phi_H) = \frac{\bar{L}}{8G} \frac{r_H^2 \mathcal{W}^2(\phi_H)}{2}, \quad (\text{B.5})$$

where \mathcal{B}_r denotes the hypersurface of the constant radius r .

One may apply our formalism for non-extremal hairy black holes. For rotating hairy AdS black holes given in [44], one can check that the Komar potential leads to

$$J^{rt}(\xi_R) = \frac{6}{L^2} \frac{\omega B^2 (r + 2B)^3}{(r + B)^4 (1 - \omega^2)}. \quad (\text{B.6})$$

By using $\sqrt{-g} = L^3 (r + B)^4 / (r + 2B)^3$ for the metric of those black holes, one can see that

$$J_{\mathcal{B}_r} = J_\infty = J_H = \frac{3L}{4G} \frac{\omega B^2}{1 - \omega^2}, \quad (\text{B.7})$$

which is consistent with the result in [44] through the Hamiltonian formalism up to the convention-dependent normalization. While these hairy AdS black holes satisfy more generic boundary conditions than Brown-Henneaux ones, the Komar integrand gives us the consistent result with the one given in [44].

In Einstein gravity, the more general argument for this angular momentum invariance along the radial direction can be given as follows. When a hypersurface Σ has two

boundaries \mathcal{B}_1 and \mathcal{B}_2 , one can see that the quasi-local conserved charges for a Killing vector ξ at each boundary are related as

$$\int_{\mathcal{B}_1} d\Sigma_{ab} J^{ab}(\xi) = \int_{\mathcal{B}_2} d\Sigma_{ab} J^{ab}(\xi) + \int_{\Sigma} d\Sigma_a J^a(\xi). \quad (\text{B.8})$$

This expression shows us that, whenever the current, J^a vanishes on the hypersurface Σ , the quasi-local conserved charge is independent of its domain. Interestingly for Einstein gravity coupled with scalar fields, our current for a Killing vector ξ is simply given by

$$J^a(\xi) = 2P^{abcd}\nabla_b\nabla_c\xi_d = 2R^{\mu\nu}\xi_\nu. \quad (\text{B.9})$$

The angular momentum invariance follows from the fact that, for the rotational Killing vector ξ_R , the hypersurface Σ connecting two boundary $\mathcal{B}_{r=r_H}$ and $\mathcal{B}_{r=\infty}$ can be chosen such that it is orthogonal to the current, $J^a(\xi_R)$.

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