Published for SISSA by Springer

Received: *March 14, 2013*
Accepted: *April 15, 2013*
Published: *May 6, 2013*

Dual dynamics of three dimensional asymptotically flat Einstein gravity at null infinity

Glenn Barnich[a,1] **and Hernán A. González**[b]

[a] *Physique Théorique et Mathématique, Université Libre de Bruxelles and International Solvay Institutes, Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium*

[b] *Departamento de Física, P. Universidad Católica de Chile, Casilla 306, Santiago 22, Chile*

E-mail: gbarnich@ulb.ac.be, hdgonzal@uc.cl

Abstract: Starting from the Chern-Simons formulation, the two-dimensional dual theory for three-dimensional asymptotically flat Einstein gravity at null infinity is constructed. Solving the constraints together with suitable gauge fixing conditions gives in a first stage a chiral Wess-Zumino-Witten like model based on the Poincaré algebra in three dimensions. The next stage involves a Hamiltonian reduction to a BMS3 invariant Liouville theory. These results are connected to those originally derived in the anti-de Sitter case by rephrasing the latter in a suitable gauge before taking their flat-space limit.

Keywords: Gauge-gravity correspondence, Chern-Simons Theories, Classical Theories of Gravity, Conformal and W Symmetry

ArXiv ePrint: 1303.1075

[1] Research Director of the Fund for Scientific Research-FNRS Belgium.

doi:10.1007/JHEP05(2013)016

JHEP05(2013)016

Contents

1 Introduction

In the context of holographic approaches [1–3] to gravitational theories, $2+1$ dimensional models play a prominent role because there is detailed quantitative understanding both of the bulk theories and of their two dimensional dual.

For the particular case of three dimensional gravity, the Chern-Simons formulation [4, 5] can be used to good effect. Indeed, the dual theory on closed spatial sections is obtained simply by solving the constraints inside the Chern-Simons action [6–8] giving rise to a (chiral) Wess-Zumino-Witten model [9]. Unlike in most conformal field theory considerations, the relevant groups in applications to gravity are non-compact or non semi-simple, that is $SO(2,2)$ in the AdS case and $ISO(2,1)$ in the flat case. Furthermore, the spatial section is a plane and the choice of boundary conditions plays a crucial role in determining the dual theory.

In the AdS case, these questions have been addressed in [10] (see also [11–16] for related considerations). In particular, the chiral decomposition $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ allows one to apply standard techniques for semi-simple algebras in each sector. The first stage of the reduction then involves a formulation in terms of decoupled chiral models [17–20] that combine into a standard Wess-Zumino-Witten model in a well-understood way (see e.g. [21–23]). In a second stage, the gravitational boundary conditions allow for a further simplification by implementing a standard Hamiltonian reduction from the $SL(2,\mathbb{R})$ Wess-Zumino model to Liouville theory [24–26].

The main purpose of the present paper is to construct the dual theory for three dimensional asymptotically flat gravity at null infinity and to establish its connection with the AdS results. Apart from shedding light on details of holography in backgrounds that are not AdS, such a dual theory is liable to play a role as a toy model for cosmological scenarios (see e.g. [27] and references therein) due to the existence of time-dependent cosmological solutions in this context [28].

Not surprisingly, a detailed analysis of the Chern-Simons to Wess-Zumino-Witten reduction for the Poincaré algebra $\mathfrak{iso}(2,1)$ does exist [29]. We will however have to adjust the analysis to the case at hand. Indeed, for our purpose, it will be more convenient to work with the spinor rather than the vector representation of $\mathfrak{so}(2,1)$ in order to connect AdS and flat space results. Furthermore, the boundary conditions that have been used are not directly related to those of asymptotically flat spacetimes at null infinity. Implementing the appropriate boundary conditions modifies the resulting chiral Wess-Zumino-Witten model and is important in order to have as rich a dynamics in the flat as in the AdS case [30, 31] with a direct connection between the two asymptotic regimes [28] (see also [32]). In turn, this is crucial in order to repeat the semi-classical arguments for a microscopic explanation of the BTZ black hole entropy [33] of the corresponding asymptotically flat cosmological solutions [34, 35].

As for other non semi-simple algebras (see e.g. [36]), the chiral Wess-Zumino-Witten like model for $\mathfrak{iso}(2,1)$ admits a globally well-defined two-dimensional action. The central extension in the associated current algebra affects the brackets between rotation and translations generators. In this case, the Hamiltonian reduction of the model gives rise to a BMS3 invariant Liouville type theory that is discussed in more detail in the companion paper [37].

The paper is organized as follows. Instead of using asymptotic conditions, we consider instead a suitable gauge fixed form of the metric. This is not the more standard Fefferman-Graham form in the AdS case, but rather a BMS type gauge that allows for a parallel treatment of both the AdS and the flat case. After quickly reviewing the general solution to the three-dimensional Einstein's equations, we provide in section 2 explicit expressions for the associated dreibeins and spin connections.

Section 3 is devoted to constructing the associated group elements. The field corresponding to the Cartan generator of $\mathfrak{sl}(2,\mathbb{R})$ can then be related to a standard Liouville field in the AdS case and a BMS Liouville field in the flat case. In particular, the overall normalization that has been left unspecified in [37] can be fixed at this stage. The limit relating the group elements of the AdS to the flat case is provided and shows how the explicit time dependence emerges from this point of view.

JHEP05(2013)016

The remainder of the work consists in deriving the equations for the group elements on the level of action principles. In a first step in section 4, suitable boundary terms are added to the Chern-Simons action in order to make the variational problem well-defined for the gravitational solutions that we are interested in. In terms of vielbeins and spin connections, this step can be done in parallel for both AdS and flat space with an obvious limit.

In section 5 and the associated appendix, we first briefly recall results on the Chern-Simons to WZW reduction for the AdS case, in particular how the reduction gives rise in a first step to chiral $\mathfrak{sl}(2,\mathbb{R})$ WZW models. We then review the structure of these models from the point of view of constrained Hamiltonian systems, including their current algebras and classical conformal invariance. These steps can then be directly generalized to the flat case, where an appropriate chiral $\mathfrak{iso}(2,1)$ WZW model is constructed. Its general solution involves a linear time-dependence and the $\mathfrak{iso}(2,1)$ current algebra is constructed in terms of Dirac brackets. BMS3 invariance of the model is established in terms of the current algebra along standard lines. Finally, we show how to obtain the chiral $\mathfrak{iso}(2,1)$ WZW like model as a flat limit limit of two chiral $\mathfrak{sl}(2,\mathbb{R})$ models.

In the last section 6, the Hamiltonian reduction is implemented. In the AdS case, they reduce the chiral models $\mathfrak{sl}(2,\mathbb{R})$ WZW models to free chiral bosons that combine into Liouville theory in a standard way. In the flat case, a free first order action principle is obtained that is related to the BMS3 Liouville theory in a similar way.

In order to emphasize novel aspects, conventions, notations and intermediate formulae that are relevant only to follow the details of the computations are mostly relegated to the appendix.

In all the analysis, we have concentrated for simplicity on the boundary at future null infinity. This corresponds to analysing Chern-Simons theory with a spatial section that is a disk. In a more complete analysis, other boundaries, sources in the interior and holonomies can and should be taken into account by following the arguments in [8, 38, 39].

Obvious generalizations of the present work consist in including in the starting point Chern-Simons formulation the exotic term, i.e., the Chern-Simons terms for the spin-connection [5]. The inclusion of this term can be entirely captured through an extension of the invariant metric that does not affect equations of motion or constraints, but suitably modifies the current algebras. A related generalization consists in repeating the analysis for topologically massive gravity [40, 41].

We have limited ourselves to the classical theory, but it should obviously be interesting to consider quantum aspects of the $\mathfrak{iso}(2,1)$ chiral Wess-Zumino theory and investigate for instance to what extent the general analysis of [42, 43] applies. These questions will be addressed elsewhere.

2 BMS gauge, fall-off conditions and general solution

The BMS gauge consists in using the diffeomorphisms to put the metric in the form

$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}dudr + r^2(d\phi - Udu)^2, \tag{2.1}$$

in terms of three arbitrary functions β, V, U. Here r is a radial coordinate restricted to $r \in [\bar{r}, \infty)$, u is a null coordinate, while $\phi \in [0, 2\pi]$ is an angular coordinate. Associated dreibeins e^a such that $ds^2 = 2e^0e^1 + (e^2)^2$ can be chosen as

$$e^0 = \frac{1}{2}\left(e^{2\beta}\frac{V}{r} + r^2U^2\right)du - e^{2\beta}dr - r^2Ud\phi, \qquad e^1 = du, \qquad e^2 = rd\phi. \tag{2.2}$$

When imposing the fall-off conditions $\beta = o(1) = U$, the Einstein equations can be solved exactly. They imply in particular the stronger fall-off conditions

$$\frac{V}{r} = -\frac{r^2}{l^2} + O(1), \quad \beta = O(r^{-1}), \quad U = O(r^{-2}), \tag{2.3}$$

that can be used to complete the definitions of asymptotically anti-de Sitter or flat spacetimes in BMS gauge. In the flat case, the limit $l \to \infty$ is understood so that $\frac{V}{r} = O(1)$.

The exact solution is given by

$$ds^2 = \left(-\frac{r^2}{l^2} + \mathcal{M}\right)du^2 - 2dudr + 2\mathcal{N}dud\phi + r^2d\phi^2, \tag{2.4}$$

where $\partial_r\mathcal{M} = 0 = \partial_r\mathcal{N}$ and

$$\partial_u\mathcal{M} = \frac{2}{l^2}\partial_\phi\mathcal{N}, \quad 2\partial_u\mathcal{N} = \partial_\phi\mathcal{M}. \tag{2.5}$$

The general solution to these equations is

$$\mathcal{M} = 2(\Xi_{++} + \Xi_{--}), \quad \mathcal{N} = l(\Xi_{++} - \Xi_{--}), \tag{2.6}$$

with $\Xi_{\pm\pm} = \Xi_{\pm\pm}(x^\pm)$, in the AdS case and

$$\mathcal{M} = \Theta, \quad \mathcal{N} = \Xi + \frac{u}{2}\partial_\phi\Theta, \tag{2.7}$$

with $\Theta = \Theta(\phi)$ and $\Xi = \Xi(\phi)$ in the flat case. In terms of the arbitrary functions, the conserved charges in the AdS case associated to $\xi = Y^+\partial_+ + Y^-\partial_-$, $Y^\pm = Y^\pm(x^\pm)$, are given by

$$Q_{Y^\pm} = \frac{l}{8\pi G}\int_0^{2\pi} d\phi(Y^+\Xi_{++} + Y^-\Xi_{--}), \tag{2.8}$$

if normalized with respect to the $M = 0 = J$ BTZ black hole. In the flat case, they are associated to $\xi = (T + uY')\partial_u + Y\partial_\phi$, $T = T(\phi), Y = Y(\phi)$ and given by

$$Q_{T,Y} = \frac{1}{16\pi G}\int_0^{2\pi} d\phi(T\Theta + 2Y\Xi), \tag{2.9}$$

when normalized with respect to the null orbifold.

In the first order formalism, the equations of motion are

$$d\omega + \omega^2 + \frac{1}{l^2}e^2 = 0, \quad de + \omega e + e\omega = 0. \tag{2.10}$$

JHEP05(2013)016

Associated dreibeins and spin connections are given by

$$\begin{aligned} e^0 &= -\frac{1}{2}\left(\frac{r^2}{l^2}-\mathcal{M}\right)du - dr + \mathcal{N}d\phi, & e^1 &= du, & e^2 &= r d\phi, \\ \omega^0 &= \frac{\mathcal{N}}{l^2}du - \frac{1}{2}(\frac{r^2}{l^2}-\mathcal{M})d\phi, & \omega^1 &= d\phi, & \omega^2 &= \frac{r}{l^2}du. \end{aligned} \tag{2.11}$$

In particular, we note that for the gravitational solutions that we are interested in

$$\omega^a_\phi = e^a_u, \qquad \omega^a_u = \frac{1}{l^2}e^a_\phi, \qquad \omega^a_r = 0, \qquad \delta e^a_r = 0 = \partial_\phi e^a_r = \partial_u e^a_r. \tag{2.12}$$

Note that for flat space, these expressions simplify as all terms proportional to negative powers of l vanish.

The results for AdS and flat space can be related through a modified Penrose limit by translating the metric results of [28] to first order form. First, one introduces a dependence on a dimensionless parameter $\epsilon > 0$ in the arbitrary functions $\Xi_{\pm\pm}$ of the AdS results giving rise to ϵ dependent vielbeins and spin connections $e^{(\epsilon)}, \omega^{(\epsilon)}$. If, after the rescalings $(u, \phi, r; e^{(\epsilon)}, \omega^{(\epsilon)}) \to (\epsilon u, \epsilon r, \phi; \epsilon^{-1}e^{(\epsilon)}, \omega^{(\epsilon)})$, the limit is well-defined it can be shown to be a solution to the flat space equations. This is the case if the ϵ dependence[1] in $\Xi^{(\epsilon)}_{\pm\pm}$ is such that

$$\Xi^{(\epsilon)}_{\pm\pm}(x;\epsilon) = \frac{1}{4}\Theta(\pm x) \pm \frac{\epsilon}{2l}\Xi(\pm x) + O(\epsilon^2), \tag{2.13}$$

so that, when taking (2.6) into account,

$$\lim_{\epsilon\to 0}\mathcal{M}^{(\epsilon)}(\epsilon u, \phi) = \Theta(\phi), \qquad \lim_{\epsilon\to 0}\epsilon^{-1}\mathcal{N}^{(\epsilon)}(\epsilon u, \phi) = \Xi(\phi) + \frac{u}{2}\partial_\phi\Theta(\phi). \tag{2.14}$$

In the AdS case, the chiral Chern-Simons connections are given on-shell by

$$A^{\pm} = \begin{pmatrix} \frac{r}{2l}dx^{\pm} & \mp\frac{1}{\sqrt{2}}\left(\frac{dr}{l} + \left(\frac{r^2}{2l^2} - 2\Xi_{\pm\pm}\right)dx^{\pm}\right) \\ \pm\frac{1}{\sqrt{2}}dx^{\pm} & -\frac{r}{2l}dx^{\pm} \end{pmatrix}. \tag{2.15}$$

in matrix form. They satisfy

$$A^{\pm\alpha}_{\mp} = 0, \qquad A^{\pm 1}_r = 0 = A^{\pm 2}_r, \qquad \delta A^{\pm 0}_r = 0 = \partial_\mu A^{\pm 0}_r. \tag{2.16}$$

Let us briefly compare to the formulation in the more standard Fefferman-Graham gauge. In this case, the general solution is

$$ds^2 = \frac{l^2}{r^2}dr^2 - \left(r dx^+ - \frac{l^2}{r}\Xi_{--}dx^-\right)\left(r dx^- - \frac{l^2}{r}\Xi_{++}dx^+\right), \tag{2.17}$$

where now $x^{\pm} = \frac{t}{l} \pm \phi$ with a standard time-like coordinate t and a different radial coordinate r. Associated dreibeins e^a and spin connections ω^a are

$$\begin{aligned} e^0 &= -\frac{r}{\sqrt{2}}dx^- + \frac{l^2}{\sqrt{2}r}\Xi_{++}dx^+, & e^1 &= \frac{r}{\sqrt{2}}dx^+ - \frac{l^2}{\sqrt{2}r}\Xi_{--}dx^-, & e^2 &= \frac{l}{r}dr, \\ \omega^0 &= \frac{r}{\sqrt{2}l}dx^- + \frac{l}{\sqrt{2}r}\Xi_{++}dx^+, & \omega^1 &= \frac{r}{\sqrt{2}l}dx^+ + \frac{l}{\sqrt{2}r}\Xi_{--}dx^-, & \omega^2 &= 0. \end{aligned} \tag{2.18}$$

[1] When explicitly comparing to [28] one has to take into account the different normalization of the charges and the associated constant shifts of the functions $\Xi_{\pm\pm}, \mathcal{M}, \Theta$.

JHEP05(2013)016

In this gauge as well, conditions (2.12) hold. The corresponding chiral Chern Simons connections are

$$A^+ = \begin{pmatrix} \frac{dr}{2r} & \frac{l}{r}\Xi_{++}dx^+ \\ \frac{r}{l}dx^+ & -\frac{dr}{2r} \end{pmatrix}, \qquad A^- = \begin{pmatrix} -\frac{dr}{2r} & \frac{r}{l}dx^- \\ \frac{l}{r}\Xi_{--}dx^- & \frac{dr}{2r} \end{pmatrix}. \tag{2.19}$$

In particular, these connections satisfy (i) $A^+_- = 0 = A^-_+$ and (ii) $\partial_\pm A^{+1}_+ = 0 = \partial_\pm A^{-0}_-$, $A^{\pm 2}_\pm = 0$ for all values of r, and thus in particular asymptotically, respectively to leading order, which are the conditions used at spatial infinity in [10].

When comparing with (2.15), we see that (i) is valid in both gauges, while (ii) is changed to $\partial_+ A^{+0}_+ = 2\Xi'_{++}$, $\partial_- A^{-0}_- = -2\Xi'_{--}$, $\partial_\pm A^{+1}_+ = 0 = \partial_\pm A^{-1}_-$ and $A^{\pm 2}_\pm = 0$ in the BMS gauge.

3 On-shell group elements

It is instructive at this stage to exhibit the group elements that yield the flat connections discussed in the previous section. The general solution to $dA + A^2 = 0$ is locally given by $A = G^{-1}dG$.

In the AdS case, we deduce from $F^\pm = 0$ and $\partial_\pm A^\pm_r = 0 = \partial_\mp A^\pm_r$ that $A^\pm = G^{-1}_\pm dG_\pm$ where $G_\pm$ factorizes as $G_\pm = g_\pm(u,\phi)h_\pm(r)$.

In Fefferman-Graham gauge, the explicit form of the chiral connections then leads to

$$G_\pm = g_\pm e^{\pm\frac{1}{2}\ln\frac{r}{l}H}, \tag{3.1}$$

with

$$\begin{aligned} g_+^{-1}\partial_+ g_+ &= \Xi_{++}E_+ + E_-, & \partial_- g_+ &= 0, \\ g_-^{-1}\partial_- g_- &= E_+ + \Xi_{--}E_-, & \partial_+ g_- &= 0. \end{aligned} \tag{3.2}$$

When using the Gauss parametrization

$$g_\pm = e^{\sigma_\pm E_\mp} e^{\frac{1}{2}\mp\varphi_\pm H} e^{\tau_\pm E_\pm}, \tag{3.3}$$

this implies $\partial_\mp\sigma_\pm = \partial_\mp\tau_\pm = \partial_\mp\varphi_\pm = 0$ and

$$\partial_\pm\sigma_\pm = e^{\varphi_\pm}, \qquad \tau_\pm\partial_\pm\sigma_\pm = -\frac{1}{2}\partial_\pm e^{\varphi_\pm}, \qquad \partial_\pm\tau_\pm - \frac{1}{2}\tau_\pm\partial_\pm\varphi_\pm = \Xi_{\pm\pm}, \tag{3.4}$$

or, equivalently, in Riccatti form

$$\partial_\pm\tau_\pm + \tau_\pm^2 = \Xi_{\pm\pm}, \qquad \partial_\pm\varphi_\pm = -2\tau_\pm, \qquad \partial_\pm\sigma_\pm = e^{\varphi_\pm}. \tag{3.5}$$

When substituting the second in the first equation, one recognizes the characteristic expression for the energy-momentum tensor of a Liouville field,

$$\Xi_{\pm\pm} = \frac{1}{4}(\partial_\pm\varphi_\pm)^2 - \frac{1}{2}\partial_\pm^2\varphi_\pm = \partial_\pm\tau_\pm + \tau_\pm^2 = -\frac{1}{2}\{\sigma_\pm; x^\pm\}, \tag{3.6}$$

where $\{F;x\} = \frac{F'''}{F'} - \frac{3}{2}\frac{(F'')^2}{(F')^2} = (\ln F')'' - \frac{1}{2}((\ln F')')^2$ denotes the Schwarzian derivative. More precisely, consider a Liouville field φ_L with action

$$S = \int du d\phi \left(\pi\dot{\varphi}_L - \frac{1}{2}\pi^2 - \frac{1}{2l^2}{\varphi'_L}^2 - \frac{\mu}{2\gamma^2}e^{\gamma\varphi_L}\right). \tag{3.7}$$

It gives rise to the same relation (see e.g. equations (4.3), (4.5), (4.11) of [37] for details) provided that the first of (3.4) holds, that[2]

$$e^{\gamma\varphi_L} = \frac{16}{l^2\mu}\frac{\partial_+\sigma_+\partial_-\sigma_-}{(\sigma_+-\sigma_-)^2}. \tag{3.8}$$

and

$$\gamma^2 l^2 = 32\pi G. \tag{3.9}$$

For later use, let us point out that, off-shell, equation (3.8) together with an associated change of variables for the momenta,

$$\begin{aligned}\gamma\varphi_L &= \varphi_+ + \varphi_- - 2\ln(\sigma_+-\sigma_-) + \ln\frac{16}{l^2\mu},\\ \pi &= \frac{1}{\gamma l}\left(\varphi'_+ - \varphi'_- - 2\frac{\sigma'_+ + \sigma'_-}{\sigma_+-\sigma_-}\right),\end{aligned} \tag{3.10}$$

where $\sigma'_\pm = \pm e^{\varphi_\pm}$, is the change of variables that allows to write the Liouville action (3.7) in terms of decoupled chiral bosons,

$$S = \frac{1}{\gamma^2 l}\int du d\phi\left[\dot\varphi_+\varphi'_+ - \dot\varphi_-\varphi'_- - \frac{1}{l}(\varphi'_+)^2 - \frac{1}{l}(\varphi'_-)^2\right]. \tag{3.11}$$

In BMS gauge, one finds

$$G_\pm = f_\pm(u,\phi)e^{\mp\frac{r}{l}j_0}. \tag{3.12}$$

with

$$f_\pm^{-1}\partial_\pm f_\pm = \pm\sqrt{2}\Xi_{\pm\pm}E_+ \pm \frac{1}{\sqrt{2}}E_-, \quad \partial_\mp f_\pm = 0. \tag{3.13}$$

In this case, the parametrization

$$f_\pm = e^{\pm\frac{\sigma_\pm}{\sqrt{2}}E_-}e^{-\frac{1}{2}\varphi_\pm H}e^{\pm\sqrt{2}\tau_\pm E_+}, \tag{3.14}$$

leads to the same equations (3.4) or (3.5) as in Fefferman-Graham gauge.

Finally, in the flat case, we start by solving $d\omega + \frac{1}{2}\omega^2 = 0$. Since all r and u dependence drops out of (2.11) and

$$\omega = \frac{1}{2\sqrt{2}}\Theta d\phi E_+ + \frac{1}{\sqrt{2}}d\phi E_-, \tag{3.15}$$

we have $\omega = \Lambda^{-1}d\Lambda$ where $\Lambda = \Lambda(\phi)$. The parametrization

$$\Lambda = e^{\frac{\sigma}{\sqrt{2}}E_-}e^{-\frac{1}{2}\varphi H}e^{\sqrt{2}\tau E_+} \tag{3.16}$$

then leads to

$$\sigma' = e^\varphi, \quad \tau\sigma' = -\frac{1}{2}\varphi' e^\varphi, \quad \tau' - \frac{1}{2}\tau\varphi' = \frac{1}{4}\Theta, \tag{3.17}$$

or, again, in Riccatti form

$$\tau' + \tau^2 = \frac{1}{4}\Theta, \quad \varphi' = -2\tau, \quad \sigma' = e^\varphi. \tag{3.18}$$

[2]Note that σ_+, σ_- are denoted by A, B in section 4.1 of [37].

In this case, we have

$$\frac{1}{4}\Theta = \frac{1}{4}\varphi'^2 - \frac{1}{2}\varphi'' = \tau' + \tau^2 = -\frac{1}{2}\{\sigma;\phi\}. \tag{3.19}$$

The equation $de + \omega e + e\omega = 0$ is locally solved by $e = \Lambda^{-1} d a \Lambda$. With

$$e = \left[\frac{1}{2}\Theta du - dr + \left(\Xi + \frac{u}{2}\partial_\phi \Theta\right) d\phi\right] \frac{1}{\sqrt{2}} E_+ + \frac{1}{2} r d\phi H + du \frac{1}{\sqrt{2}} E_-, \tag{3.20}$$

the ansatz

$$\begin{aligned} a &= -\frac{r}{\sqrt{2}}\Lambda E_+ \Lambda^{-1} + u\partial_\phi \Lambda \Lambda^{-1} + \bar{a}(\phi), \\ \bar{a}(\phi) &= \frac{\eta}{\sqrt{2}} E_+ + \frac{\theta}{2} H + \frac{\zeta}{\sqrt{2}} E_-, \end{aligned} \tag{3.21}$$

then leads to the system

$$\eta' = e^{-\varphi}\Xi, \quad \theta' = -\sigma e^{-\varphi}\Xi, \quad \zeta' = -\frac{\sigma^2}{2} e^{-\varphi}\Xi, \tag{3.22}$$

which can be trivially integrated.

Let us compare with a BMS Liouville field [37] with action

$$S = \int du d\phi \left(\Pi\dot{\Phi} - \frac{1}{2}\Phi'^2 - \frac{\nu}{2\beta^2} e^{\beta\Phi}\right). \tag{3.23}$$

Taking into account the expression for the energy density of this field leads to the relation

$$e^{\beta\Phi} = \frac{4}{\nu}\left(\frac{\sigma'}{\sigma}\right)^2, \tag{3.24}$$

where the first of (3.17) holds, by using[3] the first of equations (4.29) and (4.30) of [37]. From (4.22) of [37] and (2.9), we then get

$$\beta^2 = 32\pi G. \tag{3.25}$$

Off-shell, the change of variables from Φ, Π to φ, ξ given by

$$\begin{aligned} \beta\Phi &= 2\varphi - 2\ln\sigma + \ln\frac{4}{\nu}, \\ \beta\Pi &= \xi' - (\ln\sigma)'\xi, \end{aligned} \tag{3.26}$$

where $\sigma' = e^\varphi$ maps the BMS Liouville action (3.23) to

$$S = \frac{2}{\beta^2}\int du d\phi \Big[\xi'\dot{\varphi} - \varphi'^2\Big]. \tag{3.27}$$

We are now in a position to discuss the flat limit. On the level of the Liouville action (3.7), let

$$\begin{aligned} \varphi_L &= \epsilon^{-\frac{1}{2}} l\Phi, \qquad \pi = \epsilon^{\frac{1}{2}} l^{-1}\Pi, \\ \gamma &= \epsilon^{\frac{1}{2}} l^{-1}\beta, \qquad \mu = l^{-2}\nu. \end{aligned} \tag{3.28}$$

[3] The function σ is denoted by B in section 4.3 of [37].

After rescaling $u \to \epsilon u$, the limit $\epsilon \to 0$ of (3.7) then gives rise to the BMS Liouville action (3.23).

On the level of solutions, consider the group element $f^{(\epsilon)}_{\pm}(x;\epsilon)$ determined through (3.13) where $\Xi_{\pm\pm}$ is replaced by $\Xi^{(\epsilon)}_{\pm\pm}$ with an expansion as in (2.13). A parametrization like in (3.14) with ϵ dependent fields $\sigma^{(\epsilon)}_{\pm}, \varphi^{(\epsilon)}_{\pm}, \tau^{(\epsilon)}_{\pm}$ now gives rise to equations (3.5), (3.6) in terms of ϵ dependent fields. Compatibility with (3.18), (3.19), (3.22) then leads to

$$\begin{aligned} \tau_{\pm}(x;\epsilon) &= \pm\tau(\pm x) \pm \epsilon\tau^{(1)}(\pm x) + O(\epsilon^2), & l\tau^{(1)} &= \frac{1}{2}e^{\varphi}\eta, \\ \varphi_{\pm}(x;\epsilon) &= \varphi(\pm x) + \epsilon\varphi^{(1)}(\pm x) + O(\epsilon^2), & l\varphi^{(1)} &= -\theta - \eta\sigma, \\ \sigma_{\pm}(x;\epsilon) &= \pm\sigma(\pm x) \pm \epsilon\sigma^{(1)}(\pm x) + O(\epsilon^2), & l\sigma^{(1)} &= \zeta - \theta\sigma - \frac{1}{2}\eta\sigma^2. \end{aligned} \tag{3.29}$$

For the chiral groups elements we find

$$f_{\pm}(x;\epsilon) = \left[\Lambda\left(\mathbf{1} + \epsilon\frac{b_{\pm}}{l}\right)\right](\pm x) + O(\epsilon^2), \tag{3.30}$$

with Λ parametrized as in (3.16) and

$$\frac{b_{\pm}}{l} = \frac{\sigma^{(1)}}{\sqrt{2}}e^{-\varphi}(E_- - \sqrt{2}\tau H - 2\tau^2 E_+) + \sqrt{2}\left(\tau^{(1)} - \tau\varphi^{(1)}\right)E_+ - \frac{1}{2}\varphi^{(1)}H. \tag{3.31}$$

This gives

$$f_{\pm}^{-1}(x;\epsilon)\partial f_{\pm}(x;\epsilon) = \pm\left[\Lambda^{-1}\partial\Lambda + \frac{\epsilon}{l}\left(\partial b_{\pm} + [\Lambda^{-1}\partial\Lambda, b_{\pm}]\right)\right](\pm x) + O(\epsilon^2), \tag{3.32}$$

According to the discussion on the flat limit in section 2, one defines

$$G^{\epsilon}_{\pm} = f_{\pm}\left(\frac{\epsilon u}{l} \pm \phi;\epsilon\right)e^{\mp\frac{\epsilon r}{l\sqrt{2}}E_+}, \tag{3.33}$$

and computes

$$\omega = \lim_{\epsilon\to 0}\frac{1}{2}\left({G^{\epsilon}_{+}}^{-1}dG^{\epsilon}_{+} + {G^{\epsilon}_{-}}^{-1}dG^{\epsilon}_{-}\right), \quad e = \lim_{\epsilon\to 0}\frac{l}{2\epsilon}\left({G^{\epsilon}_{+}}^{-1}dG^{\epsilon}_{+} - {G^{\epsilon}_{-}}^{-1}dG^{\epsilon}_{-}\right). \tag{3.34}$$

This reproduces the flat space results discussed previously, that is $\omega = \Lambda^{-1}d\Lambda$ with $\Lambda = \Lambda(\phi)$ and $e = \Lambda^{-1}da\Lambda$ with a given by

$$a = -\frac{r}{\sqrt{2}}\Lambda E_+\Lambda^{-1} + u\partial_{\phi}\Lambda\Lambda^{-1} + \frac{1}{2}\Lambda(b_+ + b_-)\Lambda^{-1}, \tag{3.35}$$

by taking into account that $\frac{1}{2}(b_+ + b_-) = \Lambda^{-1}\bar{a}\Lambda$.

4 Improved action principle

Neglecting again boundary terms and using $A = duA_u + \widetilde{A}$, $d = du\partial_u + \widetilde{d}$, the Chern-Simons action can directly be written in Hamiltonian form,

$$\begin{aligned} S[A] &= -\frac{k}{2\pi}\int dudrd\phi\left(\epsilon^{ij}\omega_{ai}\,\partial_u e^a_j - \mathcal{H}\right), \\ \gamma^a_e &= \frac{1}{2}\epsilon^{ij}[\partial_i e^a_j - \partial_j e^a_i + \epsilon^{abc}(\omega_{ib}e_{jc} - \omega_{jb}e_{ic})], \\ \gamma^a_\omega &= \frac{1}{2}\epsilon^{ij}\left[\partial_i\omega^a_j - \partial_j\omega^a_i + \epsilon^{abc}\left(\omega_{ib}\omega_{jc} + \frac{1}{l^2}e_{ib}e_{jc}\right)\right], \end{aligned} \tag{4.1}$$

where the Hamiltonian density is a combination of constraints, $\mathcal{H} = e_{ua}\gamma^a_\omega + \omega_{ua}\gamma^a_e$ and where $x^i = r, \phi$ with ϵ^{ij} determined by $\epsilon^{12} = -1$.

At this stage, we have to discuss boundary terms. On shell,

$$\delta\left(\epsilon^{ij}\omega_{ai}\,\partial_u e^a_j - \mathcal{H}\right) = -\partial_i\left(\epsilon^{ij}(e_{ua}\delta\omega^a_j + \omega_{ua}\delta e^a_j)\right) + \partial_u\left(\epsilon^{ij}\omega_{ai}\delta e^a_j\right). \tag{4.2}$$

It follows that

$$\begin{aligned}\delta\int dudrd\phi\left(\epsilon^{ij}\omega_{ai}\,\partial_u e^a_j - \mathcal{H}\right) = -\int dud\phi\Big[e_{ua}\delta\omega^a_\phi + \omega_{ua}\delta e^a_\phi\Big]^{r=\infty}_{r=\bar r} - \\ -\int dudr\Big[e_{ua}\delta\omega^a_r + \omega_{ua}\delta e^a_r\Big]^{\phi=2\pi}_{\phi=0} - \int drd\phi\Big[\omega_{ar}\delta e^a_\phi - \omega_{a\phi}\delta e^a_r\Big]^{u=u_f}_{u=u_i}.\end{aligned} \tag{4.3}$$

Assuming the fields to be single-valued on the circle, neglecting the inner boundary at $r = \bar r$ and taking into account conditions (2.12), solutions (2.11) respectively (2.18) provide a true extremum of the variational principle defined by

$$I[e,\omega] = -\frac{k}{2\pi}\int dudrd\phi\left(\omega_{a\phi}\dot e^a_r - \omega_{ar}\dot e^a_\phi - \mathcal{H}\right) - \frac{k}{4\pi}\int dud\phi\left[\omega_{a\phi}\omega^a_\phi + \frac{1}{l^2}e_{a\phi}e^a_\phi\right]^{r=\infty}. \tag{4.4}$$

5 Reduction to Wess-Zumino-Witten theory

In all cases, BMS gauge in AdS and flat space and in Fefferman-Graham gauge in AdS space, the on-shell vielbeins and spin connections satisfy in particular

$$\partial_\phi e_r = 0 = \partial_\phi \omega_r = 0, \tag{5.1}$$

which are a subset of conditions (2.12). In an off-shell formulation, these conditions can be taken as (partial) gauge fixing conditions. The reduced system is then simply obtained by solving the constraints with those gauge fixing conditions in the action.

Because of the form of the constraints, the analysis has to be done separately depending on whether there is a cosmological constant or not. In the next subsection we briefly review the results in the AdS case along the lines of [10] in order to better appreciate what happens in the flat case. Standard technical material can be found in the appendix.

5.1 AdS case

In the AdS case, one uses the chiral decomposition in terms of which the constraints split as

$$\widetilde{d}\widetilde{A}^{\pm} + (\widetilde{A}^{\pm})^2 = 0, \tag{5.2}$$

and the improved action is

$$I[e,\omega] = I^c[A^+] - I^c[A^-] - \frac{k}{4\pi}\int dud\phi\ \mathrm{Tr}\Big[(A^+_\phi)^2 + (A^-_\phi)^2\Big]^{r=\infty}, \tag{5.3}$$

where

$$I^c[A] = -\frac{kl}{4\pi}\int dudrd\phi\ \mathrm{Tr}\Big(A_\phi\dot A_r - A_r\dot A_\phi\Big) - \frac{kl}{4\pi}\int du\ \mathrm{Tr}\Big(A_u\big(\widetilde{d}\widetilde{A} + \widetilde{A}^2\big)\Big). \tag{5.4}$$

The general solution to the constraints is locally given by $\widetilde{A}^{\pm} = G_{\pm}^{-1}\tilde{d}G_{\pm}$. Taking into account in addition the gauge fixing conditions, which can be rewritten as $\partial_\phi A_r^{\pm} = 0$, the general solution factorizes,

$$G_{\pm} = g_{\pm}(u,\phi)h_{\pm}(r,u), \tag{5.5}$$

in terms of group elements $g_{\pm}, h_{\pm}$. At finite $r = \bar{r}$ fixed, we can assume without loss of generality that $\dot{h}_{\pm}(\bar{r},u) = 0$ by absorbing $h_{\pm}(\bar{r},u)$ into $g_{\pm}(u,\phi)$. We will assume that this condition also holds for $\bar{r} = \infty$,

$$\dot{h}_{\pm}(\infty,u) = 0, \tag{5.6}$$

as it indeed does for the gravitational solutions of interest.

When inserting such a solution of the constraints into the improved action, one gets the sum of two decoupled chiral Wess-Zumino-Witten models, $I[e,\omega] = I_+[g_+] + I_-[g_-]$,

$$I_{\pm}[g_{\pm}] = \pm\frac{k}{2\pi}\int du d\phi \mathrm{Tr}\Big[(g_{\pm}^{-1}\partial_+ g_{\pm} - g_{\pm}^{-1}\partial_- g_{\pm})g_{\pm}^{-1}\partial_{\mp}g_{\pm}\Big] \pm \frac{kl}{2\pi}\Gamma[G_{\pm}], \tag{5.7}$$

where

$$\Gamma[G] = \frac{1}{3!}\int \mathrm{Tr}(G^{-1}dG)^3. \tag{5.8}$$

One can then follow [10] and define

$$G = G_+^{-1}G_-, \quad g = g_+^{-1}g_-, \quad \pi = -g_-^{-1}g_+' g_+^{-1}g_- - g_-^{-1}g_-', \tag{5.9}$$

in terms of which

$$I[e,\omega] = \frac{kl}{2\pi}\int du d\phi\ \mathrm{Tr}\left[\frac{1}{2}\pi g^{-1}\dot{g} - \frac{1}{4l}\big(\pi^2 + (g^{-1}g')^2\big)\right] - \frac{kl}{2\pi}\Gamma[G]. \tag{5.10}$$

After elimination of the momenta, one gets the (non-chiral) WZW action at the boundary $r = \infty$,

$$I[g] = -\frac{kl^2}{8\pi}\int du d\phi\ \mathrm{Tr}\Big[\eta^{\mu\nu}g^{-1}\partial_\mu g g^{-1}\partial_\nu g\Big] - \frac{kl}{2\pi}\Gamma[G]. \tag{5.11}$$

The equations of motions of the non-chiral theory are $\partial_+(g^{-1}\partial_- g) = 0$, with general solution $g = k_+(x^+)k_-(x^-)$. It then follows directly from the Polyakov-Wiegmann identities that the WZW action is invariant under $g \to \Theta_+(x^+)g\Theta_-^{-1}(x^-)$. The conserved Noether currents for the associated infinitesimal transformations $\delta_{\theta_\pm}g = \theta_+ g - g\theta_-$ are

$$J^+_{\theta_\pm} = -\frac{k}{\pi}\mathrm{Tr}\Big[\theta_- g^{-1}\partial_- g\Big], \quad J^-_{\theta_\pm} = \frac{k}{\pi}\mathrm{Tr}\Big[\theta_+\partial_+ g g^{-1}\Big], \tag{5.12}$$

with time components

$$J^0_{\theta\pm} = 2\mathrm{Tr}[\theta_\pm I_\pm], \quad I_+ = \frac{kl}{4\pi}\partial_+ g g^{-1}, \quad I_- = -\frac{kl}{4\pi}g^{-1}\partial_- g. \tag{5.13}$$

As briefly recalled in the appendix, in the Hamiltonian formulation their Poisson bracket algebra consists of two commuting copies of an $\mathfrak{sl}(2,\mathbb{R})$ current algebra,

$$\begin{aligned}
&\{I_a^{\pm}(\phi), I_b^{\pm}(\phi')\} = \epsilon_{ab}{}^c I_c^{\pm}(\phi)\delta(\phi-\phi') \pm \frac{kl}{4\pi}\eta_{ab}\partial_\phi\delta(\phi-\phi'),\\
&\{I_a^{+}(\phi), I_b^{-}(\phi')\} = 0.
\end{aligned} \tag{5.14}$$

Alternatively, in order to better compare with the flat case, one can concentrate on the two chiral copies. The equations of motion are $\partial_\mp(g_\pm^{-1}g'_\pm) = 0$ which implies $g_\pm = h_\pm(u)k_\pm(x^\pm)$. The analog of the Polyakov-Wiegmann identities for the chiral case imply invariance of the chiral theories $I_\pm[g_\pm]$ under $g_\pm \to g_\pm\Theta_\pm^{-1}(x^\pm)$. The conserved Noether currents for the associated infinitesimal transformations $\delta_{\theta_\pm}g_\pm = -g_\pm\theta_\pm$ are given by

$$\begin{aligned} J^+_{\theta_+} &= 0, & J^-_{\theta_+} &= -\frac{k}{\pi}\mathrm{Tr}\Big[\theta_+ g_+^{-1}g'_+\Big], \\ J^+_{\theta_-} &= \frac{k}{\pi}\mathrm{Tr}\Big[\theta_- g_-^{-1}g'_-\Big], & J^-_{\theta^-} &= 0. \end{aligned} \tag{5.15}$$

The time-components of these currents can be written as

$$J^0_{\theta_\pm} = 2\mathrm{Tr}[\theta_\pm I^\pm], \quad I^\pm = \mp\frac{kl}{4\pi}g_\pm^{-1}g'_\pm. \tag{5.16}$$

They agree on-shell with the current components (5.13) of the non chiral WZW theory, which justifies the same notation. In this case, it is their Dirac bracket algebra that forms the $\mathfrak{sl}(2,\mathbb{R})$ current algebra given in (5.14). This is shown in the appendix along the lines of [20]. The chiral models are more complicated than the non-chiral theory in the sense that their Hamiltonian formulation involves constraints, the zero modes of which are first class and generate the arbitrary functions of time in the general solution to the equations of motion, while all other modes are second class.

Let us briefly recall how classical conformal invariance is expressed on the level of the Hamiltonian formulation of the chiral models. On the constraint surface, the Hamiltonian and momentum densities can be written as

$$\begin{aligned} \mathcal{H}^\pm &= \frac{1}{2}\left[\frac{1}{l^2}\frac{8\pi}{k}(\pi^B_\pm)_a(\pi^B_\pm)^a + \frac{k}{8\pi}(g_\pm^{-1}g'_\pm)_a(g_\pm^{-1}g'_\pm)^a\right] \approx \frac{2\pi}{kl^2}I^\pm_a I^a_\pm, \\ \mathcal{P}^\pm &= -(\pi^B_\pm)^a(g_\pm^{-1}g'_\pm)_a \approx \pm\frac{2\pi}{kl}I^\pm_a I^a_\pm. \end{aligned} \tag{5.17}$$

By using the Dirac bracket version of (5.14), one gets by direct computation,

$$\begin{aligned} \{\mathcal{H}^\pm(\phi),\mathcal{H}^\pm(\phi')\}^* &= \frac{1}{l^2}(\mathcal{P}^\pm(\phi)+\mathcal{P}^\pm(\phi'))\partial_\phi\delta(\phi-\phi'), \\ \{\mathcal{H}^\pm(\phi),\mathcal{P}^\pm(\phi')\}^* &= (\mathcal{H}^\pm(\phi)+\mathcal{H}^\pm(\phi'))\partial_\phi\delta(\phi-\phi'), \\ \{\mathcal{P}^\pm(\phi),\mathcal{P}^\pm(\phi')\}^* &= (\mathcal{P}^\pm(\phi)+\mathcal{P}^\pm(\phi'))\partial_\phi\delta(\phi-\phi'). \end{aligned} \tag{5.18}$$

For the energy-momentum tensor components $T^\pm_{uu} = \mathcal{H}^\pm = \frac{1}{l^2}T^\pm_{\phi\phi}$, $T^\pm_{u\phi} = \mathcal{P}^\pm = T^\pm_{\phi u}$, the components in light-cone coordinates

$$\begin{aligned} T^\pm_{++} &= \frac{l^2}{2}\left(\mathcal{H}^\pm + \frac{1}{l}\mathcal{P}^\pm\right) \approx \delta^\pm_+\frac{2\pi}{k}I^+_a I^a_+, \\ T^\pm_{--} &= \frac{l^2}{2}\left(\mathcal{H}^\pm - \frac{1}{l}\mathcal{P}^\pm\right) \approx \delta^\pm_-\frac{2\pi}{k}I^-_a I^a_-, \\ T^\pm_{+-} &= 0 = T^\pm_{-+}, \end{aligned} \tag{5.19}$$

satisfy

$$\{T^\pm_{\pm\pm}(\phi),T^\pm_{\pm\pm}(\phi')\}^* = \pm l(T^\pm_{\pm\pm}(\phi)+T^\pm_{\pm\pm}(\phi'))\delta'(\phi-\phi'), \tag{5.20}$$

and all other brackets vanishing.

5.2 Flat case

In this case, taking into account that $\partial_\phi \omega_r = 0$, the general solution to $\gamma_\omega = 0$ is given by $\widetilde{\omega} = \Lambda^{-1}\widetilde{d}\Lambda$ where $\Lambda = \lambda(u,\phi)\mu(r,u)$, and λ, μ are group elements. The general solution to the remaining constraint $\gamma_e = 0$, which is equivalent to $\widetilde{d}\widetilde{e} + \widetilde{\omega}\widetilde{e} + \widetilde{e}\widetilde{\omega} = 0$, together with the gauge fixing condition $\partial_\phi e_r = 0$, is given by

$$\widetilde{e} = \Lambda^{-1}\widetilde{d}a\Lambda, \quad a = \alpha + \lambda\beta\lambda^{-1}, \tag{5.21}$$

where $\alpha = \alpha(u,\phi)$, $\beta = \beta(u,r)$, which gives explicitly

$$\begin{aligned} \omega_r &= \mu^{-1}\partial_r\mu, & \omega_\phi &= \mu^{-1}\lambda^{-1}\lambda'\mu, \\ e_r &= \mu^{-1}\partial_r\beta\mu, & e_\phi &= \mu^{-1}\big(\lambda^{-1}\alpha'\lambda + [\lambda^{-1}\lambda', \beta]\big)\mu. \end{aligned} \tag{5.22}$$

Inserting this solution into the improved action (4.4) with $l \to \infty$ gives

$$I[e,\omega] = \frac{k}{\pi}\int du d\phi \, \mathrm{Tr}\left[\Lambda'\Lambda^{-1}\dot{a} - \frac{1}{2}(\Lambda^{-1}\Lambda')^2\right] + \frac{k}{\pi}\Gamma[\Lambda, a], \tag{5.23}$$

with

$$\Gamma[\Lambda, a] = \int \mathrm{Tr}\left(d\Lambda\Lambda^{-1}d\Lambda\Lambda^{-1}da\right). \tag{5.24}$$

As for the non semi-simple Lie algebra considered in [36], this Wess-Zumino term for the Poincaré algebra $\mathfrak{iso}(2,1)$ is exact,

$$\mathrm{Tr}\left(d\Lambda\Lambda^{-1}d\Lambda\Lambda^{-1}da\right) = d\Big[\mathrm{Tr}\big(d\Lambda\Lambda^{-1}da\big)\Big], \tag{5.25}$$

so that, when concentrating on the boundary at $r = \infty$,

$$\Gamma[\Lambda, a] = \int du d\phi \, \mathrm{Tr}\left[\dot{\Lambda}\Lambda^{-1}a' - \Lambda'\Lambda^{-1}\dot{a}\right]. \tag{5.26}$$

Furthermore, when using the decompositions of Λ and a,

$$\begin{aligned} \mathrm{Tr}\Big[\dot{\Lambda}\Lambda^{-1}a'\Big] = \mathrm{Tr}\Big[&\dot{\lambda}\lambda^{-1}\alpha' + \dot{\mu}\mu^{-1}(\lambda^{-1}\alpha'\lambda + \lambda^{-1}\lambda'\beta - \beta\lambda^{-1}\lambda') \\ &+ \lambda^{-1}\lambda'\dot{\beta} - \partial_u(\lambda^{-1}\lambda'\beta) + \partial_\phi(\lambda^{-1}\lambda'\dot{\beta})\Big]. \end{aligned} \tag{5.27}$$

For the gravitational solutions of interest we have again

$$\dot{\mu}(\infty, u) = 0 = \dot{\beta}(\infty, u), \tag{5.28}$$

so that only the first term survives and

$$I[\lambda, \alpha] = \frac{k}{\pi}\int du d\phi \; \mathrm{Tr}\left[\dot{\lambda}\lambda^{-1}\alpha' - \frac{1}{2}(\lambda'\lambda^{-1})^2\right]. \tag{5.29}$$

This action differs from the WZW action for flat gravity proposed in [29] by the potential energy term, which originates from the boundary term in (4.4).

The equations of motion are

$$(\dot{\lambda}\lambda^{-1})' = 0, \quad D_u^{-\dot{\lambda}\lambda^{-1}}\alpha' = (\lambda'\lambda^{-1})', \tag{5.30}$$

These equations are equivalent to the conservation laws $\partial_\mu \widetilde{J}^\mu = 0$, $\partial_\mu P^\mu = 0$ where

$$\widetilde{J}^0 = \lambda^{-1}\alpha'\lambda, \quad \widetilde{J}^1 = -\lambda^{-1}\lambda', \quad P^0 = \lambda'\lambda^{-1}, \quad P^1 = 0. \tag{5.31}$$

The general solution of the first equation is $\lambda = \mu(u)\nu(\phi)$. After defining $\alpha = \mu\gamma\mu^{-1}$ the second equation reads $\dot{\gamma}' = (\nu'\nu^{-1})'$ with general solution $\gamma = \rho(\phi) + \delta(u) + u\nu'\nu^{-1}$.

Solution space is invariant under $\lambda \to \lambda\Theta^{-1}(\phi)$, $\alpha \to \alpha - u\lambda\Theta^{-1}\Theta'\lambda^{-1}$ and also under $\lambda \to \Xi(u)\lambda$, $\alpha \to \Xi\alpha\Xi^{-1}$. The infinitesimal version of the former, $\delta_\theta\lambda = -\lambda\theta$, $\delta_\theta\alpha = -u\lambda\theta'\lambda^{-1}$, leave action (5.29) invariant and the associated Noether currents are now $\widetilde{J}^0_\theta = -\frac{k}{\pi}\mathrm{Tr}[u\theta'\lambda^{-1}\lambda' + \theta\lambda^{-1}\alpha'\lambda]$, $\widetilde{J}^1_\theta = \frac{k}{\pi}\mathrm{Tr}[\theta\lambda^{-1}\lambda']$. For Noether currents, the physically meaningful quantity is the equivalence class $[J^\mu]$, where $J^\mu \sim J^\mu + t^\mu + \partial_\nu k^{[\mu\nu]}$ with $t^\mu \approx 0$. Choosing $k^{[\mu\nu]} = \frac{k}{\pi}u\epsilon^{\mu\nu}\mathrm{Tr}(\theta\lambda^{-1}\lambda')$ and $t^\mu = \frac{k}{\pi}\delta^\mu_1\mathrm{Tr}\big[\theta(\partial_0(\lambda^{-1}\lambda')\big]$, an equivalent representative for the Noether current is

$$J^0_\theta = 2\mathrm{Tr}[\theta J], \quad J = -\frac{k}{2\pi}\big[\lambda^{-1}\alpha'\lambda - u(\lambda^{-1}\lambda')'\big], \quad J^1_\theta = 0. \tag{5.32}$$

The action is furthermore invariant under $\lambda \to \lambda$ and $\alpha \to \alpha + \lambda\Sigma(\phi)\lambda^{-1}$. Associated infinitesimal transformations are $\delta_\sigma\lambda = 0, \delta_\sigma\alpha = \lambda\sigma\lambda^{-1}$ with Noether currents

$$P^0_\sigma = 2\mathrm{Tr}[\sigma P], \quad P = \frac{k}{2\pi}\lambda^{-1}\lambda', \quad P^1_\sigma = 0. \tag{5.33}$$

As shown in the appendix, in the Hamiltonian formulation, the Dirac brackets of their time components satisfy the $\mathfrak{iso}(2,1)$ current algebra

$$\begin{aligned} &\{P_a(\phi), P_b(\phi')\}^* = 0, \\ &\{J_a(\phi), P_b(\phi')\}^* = \epsilon_{ab}{}^c P_c(\phi)\delta(\phi-\phi') - \frac{k}{2\pi}\eta_{ab}\partial_\phi\delta(\phi-\phi'), \\ &\{J_a(\phi), J_b(\phi')\}^* = \epsilon_{ab}{}^c J_c(\phi)\delta(\phi-\phi'). \end{aligned} \tag{5.34}$$

Let us now discuss BMS3 invariance of the model. Using the Hamiltonian analysis that has been done in the appendix, it follows that, on the constraint surface, the Hamiltonian and momentum densities are given by

$$\mathcal{H} \approx \frac{\pi}{k}P^a P_a, \quad \mathcal{P} \approx -\frac{2\pi}{k}J^a P_a. \tag{5.35}$$

When using the current algebra (5.34), we find

$$\begin{aligned} &\{\mathcal{H}(\phi), \mathcal{H}(\phi')\}^* = 0, \\ &\{\mathcal{H}(\phi), \mathcal{P}(\phi')\}^* = (\mathcal{H}(\phi) + \mathcal{H}(\phi'))\partial_\phi\delta(\phi-\phi'), \\ &\{\mathcal{P}(\phi), \mathcal{P}(\phi')\}^* = (\mathcal{P}(\phi) + \mathcal{P}(\phi'))\partial_\phi\delta(\phi-\phi'). \end{aligned} \tag{5.36}$$

This is the form BMS3 invariance takes in the Hamiltonian framework. Indeed, in terms of modes $P_m = \int_0^{2\pi} d\phi\, e^{im\phi}\mathcal{H}$, $J_m = \int_0^{2\pi} d\phi\, e^{im\phi}\mathcal{P}$, one finds

$$i\{P_m, P_n\} = 0, \quad i\{J_m, P_n\} = (m-n)P_{m+n}, \quad i\{J_m, J_n\} = (m-n)J_{m+n}. \tag{5.37}$$

When translating to the Lagrangian level, the transformations

$$-\delta_\xi = \{\cdot, Q_\xi\}, \quad Q_\xi = \int_0^{2\pi} d\phi(\mathcal{H}T + \mathcal{P}Y), \tag{5.38}$$

where $T = T(\phi)$, $Y = Y(\phi)$, are expressed through

$$-\delta_\xi \lambda = Y\lambda', \quad -\delta_\xi \alpha = f\lambda'\lambda^{-1} + Y\alpha', \quad f = T + uY'. \tag{5.39}$$

It can then readily be checked that they leave action (5.29) invariant, $\delta_\xi \mathcal{L} = \partial_\mu k^\mu_\xi$ and that the energy-momentum tensor $-j^\mu_\xi \equiv T^\mu{}_\nu \xi^\nu = k^\mu_\xi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \lambda}\delta_\xi \lambda + \frac{\partial \mathcal{L}}{\partial \partial_\mu \alpha}\delta_\xi \alpha$ reads

$$T^u{}_\nu \xi^\nu = -\frac{k}{\pi}\mathrm{Tr}\left[\frac{1}{2}(\lambda'\lambda^{-1})^2 f + \lambda'\lambda^{-1}\alpha' Y\right], \quad T^\phi{}_\nu \xi^\nu = \frac{k}{2\pi}\mathrm{Tr}\big[(\lambda'\lambda^{-1})^2 Y\big]. \tag{5.40}$$

Agreement with the Hamiltonian analysis follows by using $\widetilde{T^\mu}{}_\nu \xi^\nu = T^\mu{}_\nu \xi^\nu + \partial_\rho k^{[\rho\mu]}_\xi$, with $k^{[u\phi]}_\xi = -\frac{k}{2\pi}\mathrm{Tr}\big[u(\lambda'\lambda^{-1})^2 Y\big]$, so that

$$\widetilde{T^\mu}{}_\nu \xi^\nu = \delta^\mu_0 \frac{k}{\pi}\mathrm{Tr}\left[-\frac{1}{2}(\lambda^{-1}\lambda')^2 T - \lambda'\lambda^{-1}\alpha' Y + u(\lambda^{-1}\lambda')(\lambda^{-1}\lambda')' Y\right]. \tag{5.41}$$

The chiral WZW theory for flat space (5.29) can be understood as a flat limit of the sum of chiral $\mathfrak{sl}(2,\mathbf{R})$ WZW theories described by $I_\pm$ in (5.7). In order to take the limit in terms of a dimensionless parameter we replace in $I_\pm$ the cosmological radius l by $l^\epsilon = \epsilon^{-1} l$ and $G_\pm$ by $G^\epsilon_\pm$ involving an explicit ϵ dependence. If we assume $G^\epsilon_\pm = \Lambda(\mathbf{1} \pm \epsilon \frac{b_\pm}{l}) + O(\epsilon^2)$ with $a = \frac{1}{2}\Lambda(b_+ + b_-)\Lambda^{-1}$, we have $\lim_{\epsilon\to 0} \frac{kl}{2\pi\epsilon}(\Gamma[G^\epsilon_+] - \Gamma[G^\epsilon_-]) = \frac{k}{2\pi}\Gamma[\Lambda, a]$, while for the two dimensional term one gets $\frac{k}{2\pi}\mathrm{Tr}(\Lambda'\Lambda^{-1}\dot{a} + \dot{\Lambda}\Lambda^{-1}a')$ as $\epsilon \to 0$. Summing up both contributions and using (5.26) gives the result.

6 Reduction to Liouville

6.1 AdS case

Let us discuss the reduction at the level of the chiral WZW actions.

In Fefferman-Graham gauge, one can read from equations (3.2) that $(g^{-1}_\pm \partial_\pm g_\pm)^\mp = 1$ where the superscript denotes the component along the Lie algebra element E_- respectively E_+. Using in addition $\partial_\pm g_\mp = 0$, this implies the conditions $(g^{-1}_+ g'_+)^- = 1$ while $(g^{-1}_- g'_-)^+ = 1$, which correspond to fixing some of the chiral conserved current components in (5.15).

In terms of the parametrization

$$g_\pm = e^{\sigma_\pm E_\mp} e^{\mp\frac{1}{2}\varphi_\pm H} e^{\tau_\pm E_\pm}, \tag{6.1}$$

actions (5.7) read

$$I_{\pm}[g_{\pm}] = \pm\frac{k}{4\pi}\int du d\phi\big[\varphi'_{\pm}\partial_{\mp}\varphi_{\pm} - 4e^{-\varphi_{\pm}}\sigma'_{\pm}\partial_{\mp}\tau_{\pm}\big], \tag{6.2}$$

while the reduction conditions become $e^{-\varphi_{\pm}}\sigma'_{\pm} = \pm 1$. Up to boundary terms, the reduced actions are the ones for chiral bosons,

$$I^R_{\pm} = \pm\frac{k}{4\pi}\int du d\phi(\varphi'_{\pm}\partial_{\mp}\varphi_{\pm}). \tag{6.3}$$

In the BMS gauge, we use the parametrization

$$g_{\pm} = e^{\pm\frac{\sigma_{\pm}}{\sqrt{2}}E_-}e^{-\frac{1}{2}\varphi_{\pm}H}e^{\pm\sqrt{2}\tau_{\pm}E_+}, \tag{6.4}$$

which, when inserted into actions (5.7) gives the same actions (6.2). The reduction conditions $(f_{\pm}^{-1}f'_{\pm})^- = \frac{1}{\sqrt{2}}$ then become again $e^{-\varphi_{\pm}}\sigma'_{\pm} = \pm 1$ and give rise to the same reduced actions (6.3) as in the previous gauge.

As shown in section 3, the sum of the chiral boson actions

$$I^R_{+} + I^R_{-} = \frac{kl}{8\pi}\int du d\phi\left(\dot{\varphi}_+\varphi'_+ - \frac{1}{l}(\varphi'_+)^2 - \dot{\varphi}_-\varphi'_- - \frac{1}{l}(\varphi'_-)^2\right), \tag{6.5}$$

can be written in Liouville form (3.7) by using transformation (3.10).

The reduction of the chiral models to Liouville can also be discussed in terms of the modified Sugawara construction. By allowed redefinitions of the currents associated to conformal transformations as discussed in the previous section, equivalent representatives for the time components can be chosen as $\widetilde{T}^{\pm}_{\pm\pm} = T^{\pm}_{\pm\pm} + \mu^a_{\pm}\partial_{\phi}I^{\pm}_a$. Note that in this case, the spatial parts of the currents have to be modified accordingly. On the surface defined by the reduction constraints, the only representatives that commute with the first class reduction constraints $I^+_0 = -\frac{kl}{4\pi}\sqrt{2}$, $I^-_1 = \frac{kl}{4\pi}\sqrt{2}$ (FG gauge), respectively $I^{\pm}_0 = \mp\frac{kl}{4\pi}$ (BMS gauge) are $\widetilde{T}^{\pm}_{\pm\pm} \approx T^{\pm}_{\pm\pm} \pm l\partial_{\phi}I^{\pm}_2$. Being first class, these are observables of the reduced theory and their Dirac brackets in Liouville theory coincide, on the constraint surface, with their brackets in the chiral WZW models, which are explicitly given by

$$\{\widetilde{T}^{\pm}_{\pm\pm}(\phi), \widetilde{T}^{\pm}_{\pm\pm}(\phi')\}^* = \pm l(\widetilde{T}^{\pm}_{\pm\pm}(\phi) + \widetilde{T}^{\pm}_{\pm\pm}(\phi'))\partial_{\phi}\delta(\phi-\phi') \mp \frac{kl^3}{4\pi}\partial^3_{\phi}\delta(\phi-\phi'). \tag{6.6}$$

In terms of modes $L^{\pm}_m = \frac{1}{l}\int_0^{2\pi} d\phi e^{\pm im\phi}\widetilde{T}^{\pm}_{\pm\pm}$, this gives the standard Dirac bracket algebra

$$\begin{aligned} &i\{L^{\pm}_n, L^{\pm}_n\}^* = (m-n)L^{\pm}_{m+n} + \frac{c}{12}m^3\delta^0_{m+n}, \quad c = 6kl = \frac{3l}{2G}, \\ &i\{L^{\pm}_n, L^{\mp}_n\}^* = 0. \end{aligned} \tag{6.7}$$

6.2 Flat case

From equation (3.15) and (3.20), the reduction conditions are

$$(\lambda^{-1}\lambda')^- \approx \frac{1}{\sqrt{2}} = \frac{2\pi}{k}P^-, \quad J^- \approx 0 \Rightarrow (\lambda^{-1}\alpha'\lambda)^- \approx 0. \tag{6.8}$$

In terms of the parametrization,

$$\lambda = e^{\frac{1}{\sqrt{2}}\sigma E_-} e^{-\frac{1}{2}\varphi H} e^{\sqrt{2}\tau E_+}, \quad \alpha = \frac{\eta}{\sqrt{2}} E_+ + \frac{\theta}{2} H + \frac{\zeta}{\sqrt{2}} E_-, \tag{6.9}$$

the flat chiral WZW action (5.29) reads

$$I[\varphi,\sigma,\tau,\eta,\theta,\zeta] = \frac{k}{2\pi}\int du d\phi \Big[-(\theta' + \sigma\eta')\dot{\varphi} + \eta'\dot{\sigma} - \\ -(\eta'\sigma^2 + 2\theta'\sigma - 2\zeta')e^{-\varphi}\dot{\tau} - \frac{1}{2}\varphi'^2 - 2\sigma'\tau' e^{-\varphi}\Big], \tag{6.10}$$

while the reduction conditions become

$$\sigma' e^{-\varphi} \approx 1, \quad \eta'\sigma^2 + 2\theta'\sigma - 2\zeta' \approx 0. \tag{6.11}$$

Using integration by parts and neglecting all boundary terms, the reduced action can be written as

$$I = \frac{k}{4\pi}\int du d\phi \Big[\xi'\dot{\varphi} - \varphi'^2\Big], \tag{6.12}$$

where $\xi = -2(\theta + \sigma\eta)$. This is the centrally extended BMS_3 invariant action in the form (3.27). Again, as shown in section 3, it is related to the Liouville-like form (3.23) through the transformation (3.26).

The analog of the modified Sugawara construction for the flat case is as follows. The time-components of the currents associated to BMS_3 transformations may be redefined as $\widetilde{\mathcal{H}} = \mathcal{H} + \mu^a\partial_\phi P_a + \nu^a \partial_\phi J_a$ and $\widetilde{\mathcal{P}} = \mathcal{P} + \rho^a \partial_\phi P_a + \sigma^a \partial_\phi J_a$. Note that in this case, the spatial parts of the currents do not need to be modified since $\partial_0 P \approx 0 \approx \partial_0 J$. On the surface defined by the reduction constraints, the only representatives that commute with the first class reduction constraints $J_0 = 0$, $P_0 = \frac{k}{2\pi}$ are $\widetilde{\mathcal{H}} \approx \mathcal{H} + \partial_\phi P_2$ and $\widetilde{\mathcal{P}} \approx \mathcal{P} - \partial_\phi J_2$. Being first class, these are observables of the reduced theory and their Dirac brackets in BMS Liouville theory coincide, on the constraint surface, with their brackets in the chiral $\mathfrak{iso}(2,1)$ WZW like model, which are given by

$$\begin{aligned} &\{\widetilde{\mathcal{H}}(\phi), \widetilde{\mathcal{H}}(\phi')\}^* = 0, \\ &\{\widetilde{\mathcal{H}}(\phi), \widetilde{\mathcal{P}}(\phi')\}^* = (\widetilde{\mathcal{H}}(\phi) + \widetilde{\mathcal{H}}(\phi'))\partial_\phi \delta(\phi - \phi') - \frac{k}{2\pi}\partial_\phi^3 \delta(\phi - \phi'), \\ &\{\widetilde{\mathcal{P}}(\phi), \widetilde{\mathcal{P}}(\phi')\}^* = (\widetilde{\mathcal{P}}(\phi) + \widetilde{\mathcal{P}}(\phi'))\partial_\phi \delta(\phi - \phi'). \end{aligned} \tag{6.13}$$

In terms of modes, $P_m = \int_0^{2\pi} d\phi e^{im\phi}\widetilde{\mathcal{H}}$, $J_m = \int_0^{2\pi} d\phi e^{im\phi}\widetilde{\mathcal{P}}$, this gives the centrally extended BMS3 algebra,

$$\begin{aligned} &i\{P_m, P_n\}^* = 0, \\ &i\{J_m, P_n\}^* = (m-n)P_{m+n} + \frac{c_2}{12}m^3\delta^0_{m+n}, \qquad c_2 = 12k = \frac{3}{G}, \\ &i\{J_m, J_n\}^* = (m-n)J_{m+n} + \frac{c_1}{12}m^3\delta^0_{m+n}, \qquad c_1 = 0. \end{aligned} \tag{6.14}$$

JHEP05(2013)016

7 A comment on zero modes

As already stressed in [38], the change of variables (5.9) is not well-defined in the zero mode sector. As a consequence, the equivalence of the sum of the two chiral models with the non-chiral theory does not hold in this sector. The same applies to the transformation (3.10) used in order to relate the sum of two chiral bosons with Liouville theory, and also to the transformation (3.26) that relates a free chiral boson like action to BMS Liouville theory.

It then follows that asymptotically AdS or flat gravity is, strictly speaking, not equivalent to (BMS) Liouville theory, but rather to the respective chiral models.

Acknowledgments

The authors thank A. Gomberoff for collaboration during the early stages of this work and C. Troessaert for useful discussions. This work is supported in part by the Fund for Scientific Research-FNRS (Belgium), by IISN-Belgium, by "Communauté française de Belgique - Actions de Recherche Concertées" and by Fondecyt Projects No. 1085322 and No. 1090753. H.G. thanks Conicyt for financial support.

A Chern-Simons formulation of gravity

Let $A = -1, 0, 1, 2$ and $a = 0, 1, 2$ and consider the flat metric $\eta_{AB} = \text{diag}(-1-1, 1, 1)$. In terms of the (anti-hermitian) generators $P_a = \frac{1}{l} J_{-1a}, J_{ab}$, the $\mathfrak{so}(2,2)$ algebra reads

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad [J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \qquad [P_a, P_b] = \frac{1}{l^2} J_{ab}\,. \tag{A.1}$$

The three dimensional Poincaré algebra $\mathfrak{iso}(2,1)$is obtained by keeping the generators P_a, J_{ab} fixed and taking the limit of l to infinity.

Let $\epsilon_{012} = 1$ and take $\eta_{ab} = \text{diag}(-1, 1, 1)$ and its inverse to lower and raise tangent space indices $a, b, c, \ldots$. In terms of $J^a = -\frac{1}{2}\epsilon^{abc} J_{bc} \iff J_{ab} = \epsilon_{abc} J^c$, the algebra reads

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, P_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = \frac{1}{l^2}\epsilon_{abc} J^c. \tag{A.2}$$

When neglecting boundary terms, the gravitational action in terms of dreibeins $e^a = e^a_\mu dx^\mu$ and spin connection $\omega = \frac{1}{2}\omega^{ab}_\mu J_{ab} dx^\mu = \omega^a_\mu J_a dx^\mu$ can be written as

$$S[e,\omega] = \frac{1}{16\pi G}\int d^3x\, e(e^\mu_a e^\nu_b R^{ab}_{\mu\nu} - 2\Lambda) = -\frac{1}{8\pi G}\int \left(e_a R^a - \frac{\Lambda}{6}\epsilon_{abc} e^a e^b e^c \right), \tag{A.3}$$

with $\frac{1}{2} R^{ab} J_{ab} = d\omega + \omega^2 = R^a J_a$. We always omit the wedge product and have chosen the orientation for the integration of 3-forms according to $d^3x = dr du d\phi$ so that the boundary Wess-Zumino-Witten actions come with the standard sign. The latter action is equivalent to the Chern-Simons action

$$S[A] = -\frac{k}{4\pi}\int \langle A, dA + \frac{2}{3}A^2 \rangle, \tag{A.4}$$

where $A = \omega^a J_a + e^a P_a$, $\langle J_a, P_b\rangle = \eta_{ab}$, $\langle J_a, J_b\rangle = 0 = \langle P_a, P_b\rangle$ and

$$k = \frac{1}{4G}, \quad \Lambda = -\frac{1}{l^2}. \tag{A.5}$$

In order to adapt the problem to our gauge choice, we now use light-cone coordinates in tangent space by introducing two null vectors,

$$e^\mu_a e_{\mu b} = \eta_{ab}, \qquad \eta_{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.6}$$

In the fundamental representation of $\mathfrak{sl}(2,\mathbb{R})$, generators satisfying $j_a j_b = \frac{1}{2}\epsilon_{abc} j^c + \frac{1}{4}\eta_{ab}\mathbf{1}$, $\mathrm{Tr}(j_a j_b) = \frac{1}{2}\eta_{ab}$, $[j_a, j_b] = \epsilon_{abc} j^c$ are given by

$$j_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad j_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad j_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.7}$$

In terms of $e = e^a_\mu dx^\mu j_a$ and $\omega = \omega^a_\mu dx^\mu j_a$, with $\omega^{ab} = \epsilon^{abc}\omega_c$, the explicit form of the equations of motion, the zero curvature condition $F \equiv dA + A^2 = 0$ is (2.10).

The chiral decomposition $J^\pm_a = \frac{1}{2}(J_a \pm l P_a)$, $A^\pm = A^{\pm a} J^\pm_a$, $A^{a\pm} = \omega^a \pm \frac{1}{l} e^a$ disentangles the algebra in terms of $\mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$ and allows one to write the gravitational action (A.3) as the difference of two Chern-Simons terms,

$$S[A^+, A^-] = -\frac{lk}{8\pi}\int \eta_{ab}\left(A^{+a}\left[dA^+ + \frac{2}{3}(A^+)^2\right]^b - A^{-a}\left[dA^- + \frac{2}{3}(A^-)^2\right]^b\right). \tag{A.8}$$

It is only well-defined for non-zero cosmological constant, while all previous considerations have a straightforward flat space limit $l \to \infty$ for which $\mathfrak{so}(2,2)$ contracts to $\mathfrak{iso}(2,1)$. Action (A.8) can be written in matrix form as

$$S[A^+, A^-] = \frac{l}{2}(S_{CS}[A^+] - S_{CS}[A^-]), \; S_{CS}[A] = -\frac{k}{2\pi}\int \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right). \tag{A.9}$$

We will use coordinates r, u, ϕ and, in the AdS case, $x^\pm = \frac{u}{l} \pm \phi$ with $2\partial_\pm = l\partial_u \pm \partial_\phi$. Note that the redefinition $l_{-1} = \sqrt{2} j_0$, $l_1 = -\sqrt{2} j_1$, $l_0 = j_2$ gives $[l_m, l_n] = (m-n) l_{m+n}$ for $m, n = -1, 0, 1$ and $E_+ = \sqrt{2} j_0$, $E_- = \sqrt{2} j_1$, $H = 2 j_2$ gives $[E_+, E_-] = H$, $[H, E_+] = 2E_+$, $[H, E_-] = -2E_-$ and

$$\begin{aligned} &e^{[xE_+,\cdot]}E_- = E_- + xH - x^2 E_+, && e^{[xE_+,\cdot]}H = H - 2xE_+, \\ &e^{[yE_-,\cdot]}E_+ = E_+ - yH - y^2 E_-, && e^{[yE_-,\cdot]}H = H + 2yE_-, \\ &e^{[\frac{1}{2}zH,\cdot]}E_+ = e^z E_+, && e^{[\frac{1}{2}zH,\cdot]}E_- = e^{-z} E_-. \end{aligned} \tag{A.10}$$

B Wess-Zumino-Witten theories

B.1 Generalites

For the two dimensional Wess-Zumino-Witten theories, we will use coordinates u, ϕ with $\eta_{\mu\nu} = \mathrm{diag}(-1, l^2)$ and $\epsilon^{\mu\nu}$ determined by $\epsilon^{01} = 1$. In the light-cone basis, we have $\eta^{+-} = -\frac{2}{l^2} = \eta^{-+}$ and $\eta^{++} = 0 = \eta^{--}$, while $\epsilon^{+-} = -\frac{2}{l}$.

JHEP05(2013)016

For factorized group elements $G = g(\phi, u)h(r, u)$ satisfying $\dot{h}(\infty, u) = 0$, we have $\partial_\pm G G^{-1} = -(\partial_\pm g h + \frac{l}{2} g\dot{h})h^{-1}g^{-1} = -\partial_\pm g g^{-1}$. With only the boundary at $r = \infty$ one has for the Wess-Zumino-Witten term $\Gamma[G]$ defined in (5.8)

$$\delta\Gamma[G] = \frac{1}{2}\int du d\phi \mathrm{Tr}(\epsilon^{\mu\nu}\delta g g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}), \tag{B.1}$$

with $\epsilon^{01} = 1$ and where we have assumed that $\delta G = \delta g h$. Furthermore

$$\Gamma[G_1^{-1}G_2] = \Gamma[G_1^{-1}] + \Gamma[G_2] + \frac{1}{l}\int du d\phi \mathrm{Tr}[\partial_- g_1 g_1^{-1}\partial_+ g_2 g_2^{-1} - \partial_+ g_1 g_1^{-1}\partial_- g_2 g_2^{-1}], \tag{B.2}$$

while the WZW action defined in (5.11) satisfies the Polyakov-Wiegmann identities

$$I[g^{-1}h] = I[g^{-1}] + I[h] - \frac{k}{\pi}\int du d\phi \mathrm{Tr}\Big[\partial_- G G^{-1}\partial_+ H H^{-1}\Big]^{r\to\infty}, \tag{B.3}$$

Under the same assumptions, for the chiral Wess-Zumino-Witten theories defined in (5.7) we get instead,

$$\begin{aligned} I_\pm[g_\pm^{-1}h_\pm] = I_\pm[g_\pm^{-1}] + I_\pm[h_\pm] \mp \\ \mp \frac{k}{\pi}\int du d\phi \mathrm{Tr}\Big[(\partial_+ G_\pm G_\pm^{-1} - \partial_- G_\pm G_\pm^{-1})\partial_\mp H_\pm H_\pm^{-1}\Big]^{r\to\infty}. \end{aligned} \tag{B.4}$$

B.2 Group elements and Poisson brackets

Introduce local coordinates ζ^a on the group manifold. We have

$$\begin{aligned} g^{-1}dg = \theta^a j_a, \quad \theta^a = M^a{}_b(\zeta)d\zeta^b, \quad d\theta^a = -\frac{1}{2}\epsilon^a_{bc}\theta^b\theta^c, \\ dg g^{-1} = \kappa^a j_a, \quad \kappa^a = N^a{}_b(\zeta)d\zeta^b, \quad d\kappa^a = \frac{1}{2}\epsilon^a_{bc}\kappa^b\kappa^c, \end{aligned} \tag{B.5}$$

where

$$\begin{aligned} N^a{}_b = K^a{}_c M^c{}_b, \qquad K^a{}_c = 2\mathrm{Tr}(j^a g j_c g^{-1}), \qquad K^{-1ac} = 2\mathrm{Tr}(j^a g^{-1} j^c g) = K^{ca}, \\ \epsilon_{cd}{}^e (K^{-1})^c{}_a (K^{-1})^d{}_b = \epsilon_{ab}{}^f (K^{-1})^e{}_f, \\ \epsilon^{ab}{}_e (K^{-1})^c{}_a (K^{-1})^d{}_b = \epsilon^{cd}{}_f (K^{-1})^f{}_e. \end{aligned} \tag{B.6}$$

Locally,

$$\Gamma[G] = \frac{1}{2}\int du d\phi\, \epsilon^{\mu\nu}B_{ab}\partial_\mu\zeta^a\partial_\nu\zeta^b, \qquad B_{ab}(\zeta) = -B_{ba}(\zeta). \tag{B.7}$$

Let $H_{abc} = \partial_a B_{bc} +$ (cyclic a, b, c). From the variation of Γ, one has

$$\begin{aligned} H_{abc} &= \frac{1}{2}\mathrm{Tr}\left(g^{-1}\frac{\partial g}{\partial\zeta^a}\left[g^{-1}\frac{\partial g}{\partial\zeta^b}, g^{-1}\frac{\partial g}{\partial\zeta^c}\right]\right) = \frac{1}{4}M^g{}_a M^e{}_b M^f{}_c \epsilon_{gef} \\ &= \frac{1}{2}\mathrm{Tr}\left(\frac{\partial g}{\partial\zeta^a}g^{-1}\left[\frac{\partial g}{\partial\zeta^b}g^{-1}, \frac{\partial g}{\partial\zeta^c}g^{-1}\right]\right) = \frac{1}{4}N^g{}_a N^e{}_b N^f{}_c \epsilon_{gef}. \end{aligned} \tag{B.8}$$

Consider the canonical momenta $\eta = \eta_a j^a$, $\{\zeta^a(\phi), \eta_b(\phi')\} = \delta^a_b\delta(\phi - \phi')$ and define $\pi = -\eta_b(M^{-1})^b{}_a j^a$, the Poisson brackets are

$$\begin{aligned} \{g(\phi), \pi_a(\phi')\} &= -g(\phi)j_a\delta(\phi - \phi'), \\ \{\pi_a(\phi), \pi_b(\phi')\} &= \epsilon_{ab}{}^c\pi_c(\phi)\delta(\phi - \phi'), \\ \{(g^{-1}g')_a(\phi), \pi_b(\phi')\} &= \epsilon_{ab}{}^c(g^{-1}g')_c(\phi)\delta(\phi - \phi') - \eta_{ab}\partial_\phi\delta(\phi - \phi'). \end{aligned} \tag{B.9}$$

JHEP05(2013)016

Similarly, with $\rho = \eta_b (N^{-1})^b{}_a j^a$,

$$\begin{aligned}
\{g(\phi), \rho_a(\phi')\} &= j_a g(\phi)\delta(\phi - \phi'),\\
\{\rho_a(\phi), \rho_b(\phi')\} &= \epsilon_{ab}{}^c \rho_c(\phi)\delta(\phi - \phi'),\\
\{(g' g^{-1})_a(\phi), \rho_b(\phi')\} &= \epsilon_{ab}{}^c (g' g^{-1})_c(\phi)\delta(\phi - \phi') + \eta_{ab}\partial_\phi \delta(\phi - \phi'),\\
\{\pi_a(\phi), \rho_b(\phi')\} &= 0.
\end{aligned} \tag{B.10}$$

B.3 Current algebra of the non-chiral WZW theory

In local coordinates, the Lagrangian density for the non-chiral WZW action is given by

$$\frac{16\pi}{kl^2}\mathcal{L} = M_{ab} M^a{}_e \left(\dot\zeta^b \dot\zeta^e - \frac{1}{l^2}\zeta^{b\prime}\zeta^{e\prime}\right) - \frac{8}{l} B_{ab}\dot\zeta^a \zeta^{b\prime}. \tag{B.11}$$

The relation between canonical momenta η_g and velocities is

$$\eta_c \approx \frac{\partial \mathcal{L}}{\partial(\partial_0 \zeta^c)} = \frac{kl^2}{8\pi}\left(M^a{}_c (g^{-1}\dot g)_a - \frac{4}{l} B_{cb}\zeta^{b\prime}\right). \tag{B.12}$$

Defining $v_c = \eta_c + \frac{kl}{2\pi} B_{cb}\zeta^{b\prime}$, we have

$$\{v_a(\phi), v_b(\phi')\} = -\frac{kl}{2\pi} H_{abc}\zeta^{c\prime}(\phi)\delta(\phi - \phi'). \tag{B.13}$$

The Hamiltonian is

$$H = \frac{4\pi}{kl^2}(M^{-1})^a{}_b (M^{-1})^{cb} v_a v_c + \frac{k}{16\pi} M_{ba} M^b{}_c \zeta^{a\prime}\zeta^{c\prime}. \tag{B.14}$$

In terms of the improved momenta $\pi^B = -v_b (M^{-1})^b{}_a j^a \approx -\frac{kl^2}{8\pi} g^{-1}\dot g$, the Poisson brackets are the same as in (B.9) with π replaced by π^B, except for

$$\{\pi^B_a(\phi), \pi^B_b(\phi')\} = \epsilon_{ab}{}^c \left(\pi^B_c - \frac{kl}{8\pi}(g^{-1}g')_c\right)(\phi)\delta(\phi - \phi'), \tag{B.15}$$

while the first order action principle can be written as

$$I_H = -\int du d\phi \,\mathrm{Tr}\left[2\pi^B g^{-1}\dot g + \frac{k}{8\pi}(g^{-1}g')^2 + \frac{8\pi}{kl^2}(\pi^B)^2\right] - \frac{kl}{2\pi}\Gamma[G]. \tag{B.16}$$

The current components I_- of (5.13) are given on-shell by $I_- \approx \pi^B + \frac{kl}{8\pi} g^{-1} g'$ so that $\int d\phi' 2\mathrm{Tr}[I_- \theta_-]$ is the canonical generators of the symmetry transformation $\delta g = -g\theta_-$. Evaluating the Poisson brackets then leads to (5.14) for I_-.

Defining $\rho^B = v_b (N^{-1})^b{}_a j^a$. Since in (B.11) and (B.12) one can replace $M^a{}_b$ by $N^a{}_b$, we have $\rho^B \approx \frac{kl^2}{8\pi}\dot g g^{-1}$. The Poisson bracket are as in (B.10) except for

$$\{\rho^B_a(\phi), \rho^B_b(\phi')\} = \epsilon_{ab}{}^c \left(\rho^B_c - \frac{kl}{8\pi}(g' g^{-1})_c\right)(\phi)\delta(\phi - \phi'). \tag{B.17}$$

On shell I_+ of (5.13) is now given by $I_+ \approx \rho^B + \frac{kl}{8\pi} g' g^{-1}$ so that $\int d\phi' 2\mathrm{Tr}[I_+ \theta_+]$ is the canonical generators of the symmetry transformation $\delta g = \theta_+ g$. Evaluating the Poisson brackets then leads to (5.14) for I_+. In the same way, one then establishes that the left and right current components have vanishing Poisson brackets.

JHEP05(2013)016

B.4 Current algebra of the chiral models

Since a different parametrization for the right and left group elements will be useful, we will use $\zeta^a_\pm$ in what follows. The local Lagrangian densities are

$$\frac{8\pi}{kl}\mathcal{L}^\pm = \pm(g_\pm^{-1}g_\pm')_a(g_\pm^{-1}\dot{g}_\pm)^a - \frac{1}{l}(g_\pm^{-1}g_\pm')_a(g_\pm^{-1}g_\pm')^a \pm 4B^\pm_{ab}\dot{\zeta}^a_\pm\zeta^{b\prime}_\pm. \tag{B.18}$$

The canonical momenta are related to the velocities through

$$\eta^\pm_a \approx \frac{\partial\mathcal{L}^\pm}{\partial\partial_0\zeta^a_\pm} \approx \pm\frac{kl}{8\pi}\Big((g_\pm^{-1}g_\pm')_b M^b_{\pm\alpha} + 4B^\pm_{ab}\zeta^{b\prime}_\pm\Big). \tag{B.19}$$

Defining

$$v^\pm_a = \eta^\pm_a \mp \frac{kl}{2\pi}B^\pm_{ab}\zeta^{b\prime}_\pm, \tag{B.20}$$

we now have

$$\{v^\pm_a(\phi), v^\pm_b(\phi')\} = \pm\frac{kl}{2\pi}H^\pm_{abc}\zeta^{c\prime}_\pm(\phi)\delta(\phi-\phi'), \tag{B.21}$$

with Poisson brackets of variables associated to different chiral copies all commuting. In terms of the improved momenta $\pi^{B\pm} = -v^\pm_b(M^{-1})^b_{\pm a}j^a$, the Poisson brackets are as in (B.9) for each copy except for

$$\{\pi^{B\pm}_a(\phi), \pi^{B\pm}_b(\phi')\} = \epsilon_{ab}{}^c\left(\pi^{B\pm}_c \pm \frac{kl}{8\pi}(g_\pm^{-1}g_\pm')_c\right)(\phi)\delta(\phi-\phi'). \tag{B.22}$$

The primary constraints can be written as

$$\phi_\pm = \pi^B_\pm \pm \frac{kl}{8\pi}g_\pm^{-1}g_\pm' \approx 0. \tag{B.23}$$

Consider then

$$I_\pm = \pi^B_\pm \mp \frac{kl}{8\pi}g_\pm^{-1}g_\pm'. \tag{B.24}$$

They agree on the constraint surface with the time components of the conserved currents (5.16), $I_\pm \approx \mp\frac{kl}{4\pi}g_\pm^{-1}g_\pm'$. Furthermore, $\int d\phi'\, 2\mathrm{Tr}[I_\pm\theta_\pm]$ are the canonical generators of the symmetry transformations $\delta_{\theta_\pm}g_\pm = -g_\pm\theta_\pm$. The components of $I_\pm$ satisfy the current algebra (5.14) in the standard Poisson bracket and have weakly vanishing Poisson brackets with the constraints, i.e., they are first class,

$$\{I^\pm_a(\phi), \phi^\pm_b(\phi')\} = \epsilon_{ab}{}^c\phi^\pm_c(\phi)\delta(\phi-\phi'). \tag{B.25}$$

This proves the result on the Dirac brackets of the chiral currents.

Defining $\rho^B_\pm = v^\pm_b(N_\pm^{-1})^b{}_a j^a$, the Poisson bracket are as in (B.10) except for

$$\{\rho^{B\pm}_a(\phi), \rho^{B\pm}_b(\phi')\} = \epsilon_{ab}{}^c\left(\rho^{B\pm}_c \pm \frac{kl}{8\pi}(g_\pm' g_\pm^{-1})_c\right)(\phi)\delta(\phi-\phi'). \tag{B.26}$$

Since $p^\pm_a \approx \frac{\partial\mathcal{L}^\pm}{\partial\partial_0\zeta^a_\pm} = \pm\frac{kl}{8\pi}\Big((g_\pm' g_\pm^{-1})_b N^b_{\pm a} + 4B^\pm_{ab}\zeta^{b\prime}_\pm\Big)$, the primary constraints can be written as

$$\psi_\pm = \rho^B_\pm \mp \frac{kl}{8\pi}g_\pm' g_\pm^{-1} \approx 0. \tag{B.27}$$

They satisfy the current algebra

$$\{\psi_a^\pm(\phi), \psi_b^\pm(\phi')\} = \epsilon_{ab}{}^c \psi_c^\pm(\phi)\delta(\phi - \phi') \mp \frac{kl}{4\pi}\eta_{ab}\partial_\phi \delta(\phi - \phi'). \tag{B.28}$$

In terms of this representation of the constraints,

$$\{I_a^\pm(\phi), \psi_b^\pm(\phi')\} = 0. \tag{B.29}$$

It follows from (B.28) that the zero modes $\Psi_a^\pm = \int_0^{2\pi} d\phi\ \psi_a^\pm$ are first class constraints that generate the arbitrary function of u in the general solution to the equations of motion, while all other modes are second class constraints.

The chiral Hamiltonians are

$$\begin{aligned} H^\pm &= \frac{k}{4\pi}\int d\phi\, \mathrm{Tr}\,[g'_\pm g_\pm^{-1} g'_\pm g_\pm^{-1}] + \int d\phi\, 2\mathrm{Tr}\,[u_\pm \psi_\pm] \\ &= \frac{k}{4\pi}\int d\phi\, \mathrm{Tr}\,[g_\pm^{-1} g'_\pm g_\pm^{-1} g'_\pm] + \int d\phi\, 2\mathrm{Tr}\,[v_\pm \phi_\pm], \end{aligned} \tag{B.30}$$

where $u_\pm = u^a_\pm j_a, v_\pm = v^a_\pm j_a$ contain the Lagrange multipliers. Taking the Poisson bracket with the primary constraints shows that there are no secondary ones.

B.5 Current algebra of the flat model

Locally,

$$\frac{2\pi}{k}\mathcal{L} = \dot{\zeta}^b N^a{}_b \alpha'_a - \frac{1}{2} N^a{}_b N_{ac} \zeta^{b\prime} \zeta^{c\prime}, \tag{B.31}$$

and so, if η_a, ω_a are the momenta conjugate to ζ^a, α^a, the primary constraints are

$$\eta_a \approx \frac{k}{2\pi} N^b{}_a \alpha'_b, \quad \omega_a \approx 0. \tag{B.32}$$

The primary constraints can be written as

$$\psi = \rho - \frac{k}{2\pi}\alpha' \approx 0, \quad \omega \approx 0. \tag{B.33}$$

Up to zero modes, they are second class since their algebra is

$$\begin{aligned} &\{\psi_a(\phi), \psi_b(\phi')\} = \epsilon_{ab}{}^c \rho_c \delta(\phi - \phi'), \\ &\{\psi_a(\phi), \omega_b(\phi')\} = -\frac{k}{2\pi}\eta_{ab}\partial_\phi \delta(\phi - \phi'), \\ &\{\omega_a(\phi), \omega_b(\phi')\} = 0. \end{aligned} \tag{B.34}$$

Consider

$$\begin{aligned} P &= \lambda^{-1}\omega\lambda + \frac{k}{2\pi}\lambda^{-1}\lambda', \\ J &= -\lambda^{-1}\rho\lambda + uP' = \pi + uP'. \end{aligned} \tag{B.35}$$

On the constraint surface, they agree with the time components of the Noether currents. Furthermore, $\int d\phi' 2\mathrm{Tr}[P\sigma]$, $\int d\phi' 2\mathrm{Tr}[J\theta]$ are the canonical generators of the infinitesimal symmetry transformations, $\delta_\sigma \lambda = 0$, $\delta_\sigma \alpha = \lambda\sigma\lambda^{-1}$ and $\delta_\theta \lambda = -\lambda\theta$, $\delta_\theta \alpha = -u\lambda\theta'\lambda^{-1}$.

They have weakly vanishing Poisson brackets with the constraints,

$$\begin{aligned} \{P(\phi), \psi_b(\phi')\} &= \lambda^{-1}[\omega, j_b]\lambda\delta(\phi - \phi'), & \{P(\phi), \omega_b(\phi')\} &= 0, \\ \{J(\phi), \psi_b(\phi')\} &= u(\lambda^{-1}[\omega, j_b]\lambda\delta(\phi - \phi'))', & \{J(\phi), \omega_b(\phi')\} &= 0, \end{aligned} \tag{B.36}$$

and by direct computation, one finds that their Poisson brackets, and thus also their Dirac brackets, form the $\mathfrak{iso}(2,1)$ current algebra given in (5.34).

The Hamiltonian of the model is

$$H = \frac{k}{2\pi}\int d\phi \mathrm{Tr}[\lambda'\lambda^{-1}\lambda'\lambda^{-1}] + \int d\phi 2\mathrm{Tr}[u\psi + v\omega]. \tag{B.37}$$

Again, there are no secondary constraints.

References

[1] G. 't Hooft, *Dimensional reduction in quantum gravity*, gr-qc/9310026 [INSPIRE].

[2] L. Susskind, *The world as a hologram*, J. Math. Phys. **36** (1995) 6377 [hep-th/9409089] [INSPIRE].

[3] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200] [INSPIRE].

[4] A. Achucarro and P. Townsend, *A Chern-Simons Action for three-dimensional anti-de Sitter supergravity theories*, *Phys. Lett.* **B 180** (1986) 89 [INSPIRE].

[5] E. Witten, *(2+1)-Dimensional Gravity as an Exactly Soluble System*, *Nucl. Phys.* **B 311** (1988) 46 [INSPIRE].

[6] E. Witten, *Quantum Field Theory and the Jones Polynomial*, *Commun. Math. Phys.* **121** (1989) 351 [INSPIRE].

[7] G.W. Moore and N. Seiberg, *Taming the conformal zoo*, *Phys. Lett.* **B 220** (1989) 422 [INSPIRE].

[8] S. Elitzur, G.W. Moore, A. Schwimmer and N. Seiberg, *Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory*, *Nucl. Phys.* **B 326** (1989) 108 [INSPIRE].

[9] E. Witten, *Nonabelian bosonization in two-dimensions*, *Commun. Math. Phys.* **92** (1984) 455 [INSPIRE].

[10] O. Coussaert, M. Henneaux and P. van Driel, *The asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant*, *Class. Quant. Grav.* **12** (1995) 2961 [gr-qc/9506019] [INSPIRE].

[11] M. Bañados, *Global charges in Chern-Simons field theory and the (2+1) black hole*, *Phys. Rev.* **D 52** (1996) 5816 [hep-th/9405171] [INSPIRE].

[12] S. Carlip, *The statistical mechanics of the (2+1)-dimensional black hole*, *Phys. Rev.* **D 51** (1995) 632 [gr-qc/9409052] [INSPIRE].

[13] S. Carlip, *Quantum gravity in 2+1 dimensions*, Cambridge University Press, (1998).

[14] M. Bañados, T. Brotz and M.E. Ortiz, *Boundary dynamics and the statistical mechanics of the (2+1)-dimensional black hole*, *Nucl. Phys.* **B 545** (1999) 340 [hep-th/9802076] [INSPIRE].

JHEP05(2013)016

[15] S. Carlip, *What we don't know about BTZ black hole entropy*, *Class. Quant. Grav.* **15** (1998) 3609 [hep-th/9806026] [INSPIRE].

[16] M. Bañados, *Three-dimensional quantum geometry and black holes*, in *Trends in Theoretical Physics II*, H. Falomir, R.E. Gamboa Saravi and F.A. Schaposnik eds., volume 484 of *American Institute of Physics Conference Series*, pg. 147 [hep-th/9901148] [INSPIRE].

[17] R. Floreanini and R. Jackiw, *Selfdual Fields as Charge Density Solitons*, *Phys. Rev. Lett.* **59** (1987) 1873 [INSPIRE].

[18] M. Bernstein and J. Sonnenschein, *A comment on the quantization of chiral bosons*, *Phys. Rev. Lett.* **60** (1988) 1772 [INSPIRE].

[19] J. Sonnenschein, *Chiral bosons*, *Nucl. Phys.* **B 309** (1988) 752 [INSPIRE].

[20] P. Salomonson, B. Skagerstam and A. Stern, *Canonical quantization of chiral bosons*, *Phys. Rev. Lett.* **62** (1989) 1817 [INSPIRE].

[21] M. Stone, *How to make a bosonized Dirac fermion from two bosonized Weyl fermions*, ILL-TH-89-23 NSF-ITP-89-97 (1989).

[22] M.-f. Chu, P. Goddard, I. Halliday, D.I. Olive and A. Schwimmer, *Quantization of the Wess-Zumino-Witten model on a circle*, *Phys. Lett.* **B 266** (1991) 71 [INSPIRE].

[23] L. Caneschi and M. Lysiansky, *Chiral quantization of the WZW* SU(N) *model*, *Nucl. Phys.* **B 505** (1997) 701 [hep-th/9605099] [INSPIRE].

[24] P. Forgacs, A. Wipf, J. Balog, L. Feher and L. O'Raifeartaigh, *Liouville and Toda Theories as Conformally Reduced WZNW Theories*, *Phys. Lett.* **B 227** (1989) 214 [INSPIRE].

[25] A. Alekseev and S.L. Shatashvili, *Path Integral Quantization of the Coadjoint Orbits of the Virasoro Group and* 2*D Gravity*, *Nucl. Phys.* **B 323** (1989) 719 [INSPIRE].

[26] M. Bershadsky and H. Ooguri, *Hidden* SL*(n) Symmetry in Conformal Field Theories*, *Commun. Math. Phys.* **126** (1989) 49 [INSPIRE].

[27] L. Cornalba and M.S. Costa, *Time dependent orbifolds and string cosmology*, *Fortsch. Phys.* **52** (2004) 145 [hep-th/0310099] [INSPIRE].

[28] G. Barnich, A. Gomberoff and H.A. Gonzalez, *The flat limit of three dimensional asymptotically anti-de Sitter spacetimes*, *Phys. Rev.* **D 86** (2012) 024020 [arXiv:1204.3288] [INSPIRE].

[29] P. Salomonson, B. Skagerstam and A. Stern, ISO*(2,1) chiral models and quantum gravity in (2+1)-dimensions*, *Nucl. Phys.* **B 347** (1990) 769 [INSPIRE].

[30] G. Barnich and G. Compere, *Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions*, *Class. Quant. Grav.* **24** (2007) F15 *Erratum ibid.* **24** (2007) 3139 [gr-qc/0610130] [INSPIRE].

[31] G. Barnich and C. Troessaert, *Aspects of the BMS/CFT correspondence*, *JHEP* **05** (2010) 062 [arXiv:1001.1541] [INSPIRE].

[32] A. Bagchi and R. Fareghbal, *BMS/GCA Redux: Towards Flatspace Holography from Non-Relativistic Symmetries*, *JHEP* **10** (2012) 092 [arXiv:1203.5795] [INSPIRE].

[33] A. Strominger, *Black hole entropy from near horizon microstates*, *JHEP* **02** (1998) 009 [hep-th/9712251] [INSPIRE].

JHEP05(2013)016

[34] G. Barnich, *Entropy of three-dimensional asymptotically flat cosmological solutions*, *JHEP* **10** (2012) 095 [arXiv:1208.4371] [INSPIRE].

[35] A. Bagchi, S. Detournay, R. Fareghbal and J. Simon, *Holography of 3d Flat Cosmological Horizons*, arXiv:1208.4372 [INSPIRE].

[36] C.R. Nappi and E. Witten, *A WZW model based on a nonsemisimple group*, *Phys. Rev. Lett.* **71** (1993) 3751 [hep-th/9310112] [INSPIRE].

[37] G. Barnich, A. Gomberoff and H.A. Gonzalez, *BMS_3 invariant two dimensional field theories as flat limit of Liouville*, arXiv:1210.0731 [INSPIRE].

[38] M. Henneaux, L. Maoz and A. Schwimmer, *Asymptotic dynamics and asymptotic symmetries of three-dimensional extended AdS supergravity*, *Annals Phys.* **282** (2000) 31 [hep-th/9910013] [INSPIRE].

[39] M. Rooman and P. Spindel, *Holonomies, anomalies and the Fefferman-Graham ambiguity in AdS_3 gravity*, *Nucl. Phys.* **B 594** (2001) 329 [hep-th/0008147] [INSPIRE].

[40] S. Deser, R. Jackiw and S. Templeton, *Topologically Massive Gauge Theories*, *Annals Phys.* **140** (1982) 372 [*Erratum ibid.* **185** (1988) 406] [INSPIRE].

[41] S. Deser, R. Jackiw and S. Templeton, *Three-Dimensional Massive Gauge Theories*, *Phys. Rev. Lett.* **48** (1982) 975 [INSPIRE].

[42] J.M. Figueroa-O'Farrill and S. Stanciu, *Nonsemisimple Sugawara constructions*, *Phys. Lett.* **B 327** (1994) 40 [hep-th/9402035] [INSPIRE].

[43] J.M. Figueroa-O'Farrill and S. Stanciu, *Nonreductive WZW models and their CFTs*, *Nucl. Phys.* **B 458** (1996) 137 [hep-th/9506151] [INSPIRE].