

Generalized Whittaker states for instanton counting with fundamental hypermultiplets

Hiroaki Kanno^a and Masato Taki^b

^aGraduate School of Mathematics and
Kobayashi-Maskawa Institute for the Origin of Particles and the Universe (KMI),
Nagoya University,
Nagoya, 464-8602, Japan

^bYukawa Institute for Theoretical Physics, Kyoto University,
Kyoto 606-8502, Japan

E-mail: kanno@math.nagoya-u.ac.jp, taki@yukawa.kyoto-u.ac.jp

ABSTRACT: M-theoretic construction of $\mathcal{N} = 2$ gauge theories implies that the instanton partition function is expressed as the scalar product of coherent states (Whittaker states) in the Verma module of an appropriate two dimensional conformal field theory. We present the characterizing conditions for such states that give the partition function with fundamental hypermultiplets for SU(3) theory and SU(2) theory with a surface operator. We find the states are no longer the coherent states in the strict sense but we can characterize them in terms of a few annihilation operators of lower levels combined with the zero mode (Cartan part) of the Virasoro algebra L_0 or the $\mathfrak{sl}(2)$ current algebra J_0^0 .

KEYWORDS: Supersymmetric gauge theory, Conformal and W Symmetry, Supersymmetry and Duality, Solitons Monopoles and Instantons

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1 Introduction

The M5-brane is one of the fundamental dynamical objects in M-theory. Its dynamics is still mysterious, though the low energy effective dynamics is supposed to be governed by six-dimensional $\mathcal{N} = (2, 0)$ superconformal field theory. More precisely we have ADE classification of $\mathcal{N} = (2, 0)$ theory in six dimensions and the world-volume theory of $N + 1$ M5-branes gives the theory of A_N type. A crucial problem is that such six-dimensional

$\mathcal{N} = (2, 0)$ theory does not allow any Lagrangian description. But one may consider a compactification to supersymmetric gauge theories in lower dimensions by wrapping multiple M5-branes on a suitable manifold. For example, the compactification on S^1 gives five-dimensional $\mathcal{N} = 1$ Yang-Mills theory.

The (twisted) compactification of the six-dimensional $\mathcal{N} = (2, 0)$ theory on a Riemann surface C gives rise to diverse $\mathcal{N} = 2$ theories in four dimensions [1, 2]. Such $\mathcal{N} = 2$ theories are therefore labeled by the Riemann surfaces, on which we can define conformal field theory (CFT). It is proposed the instanton partition function of the $\mathcal{N} = 2$ superconformal theory agrees with the conformal block of an appropriate CFT on C [3, 4]. In this correspondence, the choice of $\mathcal{N} = 2$ theory determines the chiral algebra of CFT, which controls the conformal block. Furthermore, for non-conformal (asymptotically free) cases it is expected that the partition function is expressed as the scalar product (or the norm) of appropriate states in the Verma module [5, 6]. This construction gives us a generalization of the conformal block in the sense that it has irregular singularities due to “irregular vertex insertions”. For example, for $SU(2)$ gauge theory which corresponds to the Virasoro algebra with generators L_n , the instanton (Nekrasov) partition function of the pure Yang-Mills theory [7] is given by

$$Z_{SU(2)}^{(N_f=0)} = \langle G_0 | G_0 \rangle, \tag{1.1}$$

where we introduce a state $|G_0\rangle$ in the Virasoro Verma module with the conformal weight Δ and the central charge c by the conditions

$$L_1 |G_0\rangle = \Lambda^2 |G_0\rangle, \quad L_2 |G_0\rangle = 0. \tag{1.2}$$

The state $|G_0\rangle$ is called Gaiotto state and expanded in the dynamical scale parameter Λ of the pure $SU(2)$ Yang-Mills theory.

In this paper we will consider the conditions that characterize such a state for asymptotically free $\mathcal{N} = 2$ theories with fundamental matter hypermultiplets. This problem is largely motivated by the fact that the conditions (1.2) have indeed the following M-theoretical origin. The pure $SU(2)$ Yang-Mills theory is the world-volume theory of two M5-branes on a Riemann sphere \mathbb{CP}^1 with defects at $z = 0, \infty$. Notice that these two defects correspond to the boundaries of D4-branes terminated on two parallel NS5-branes in the Hanany-Witten construction [8]. The low-energy dynamics can be collected into the spectral curve $\langle \det(xdz - \Phi) \rangle = 0$ for a one-form field Φ on \mathbb{CP}^1 ;

$$x^2 (dz)^2 = \left(\frac{\Lambda^2}{z} + 2u + \Lambda^2 z \right) \left(\frac{dz}{z} \right)^2 =: \phi_2(z), \tag{1.3}$$

which gives the Seiberg-Witten curve. This curve is specified by a quadratic differential $\langle \text{Tr } \Phi^2 \rangle = \phi_2(z)$ with poles at the insertion loci of defects. The point is that this quadratic differential is translated into the energy-momentum tensor $T(z) = T_{zz}(dz)^2$ of CFT as $\langle G_0 | T(z) | G_0 \rangle \sim \phi_2(z)$. By using the mode-expansion $T_{zz} = \sum L_n z^{-2-n}$ and the proposed dictionary $u \sim \Delta$ of [3], we can encode the information (1.3) of the defects into the conditions (1.2) for the state which describes the singularity at $z = 0, \infty$.

For pure Yang-Mills theories without matter hypermultiplets this story has been generalized to $SU(N)$ theory and those with a surface operator [9]–[26]. In particular in [18] the conditions for the state whose norm gives the instanton partition function are proposed for $SU(N)$ theory with a surface operator of general type. All these states are characterized as coherent states of generalized W algebra, since they are defined as a simultaneous eigenstate of the annihilation operators for the highest weight state of the Verma module. In mathematics such states are often called Whittaker states. It is a natural question if the characterization of the state as a Whittaker state is valid for asymptotically free theories with matter hypermultiplets. As far as we know, this problem is worked out only for $SU(2)$ theory, where one can introduce another Gaiotto state $|G_1, m\rangle$ that satisfies

$$L_1|G_1, m\rangle = (\epsilon_+ - 2m)\Lambda|G_1, m\rangle, \quad L_2|G_1, m\rangle = -\Lambda^2|G_1, m\rangle, \quad (1.4)$$

where m is the mass parameter and $\epsilon_+ := \epsilon_1 + \epsilon_2$ is the sum of the Ω -background parameters, or the equivariant parameters of the toric action on \mathbb{C}^2 . Then the partition function of $N_f = 1, 2$ theory is given by

$$Z_{SU(2)}^{(N_f=1)} = \langle G_0 | G_1, m \rangle, \quad Z_{SU(2)}^{(N_f=2)} = \langle G_1, m_1 | G_1, m_2 \rangle, \quad (1.5)$$

respectively.¹ These proposals are proved in [27, 28]. Unfortunately their proof works only for the Virasoro conformal block. In this paper we will generalize the definition of states like (1.4) into two directions. One is to make the rank of the gauge group higher and the other is to introduce a surface operator. Thus, we consider $SU(3)$ theory and $SU(2)$ theory with a surface operator. For $SU(3)$ theory the chiral algebra on CFT side is W_3 algebra with generators L_n and W_n [4], while it is the untwisted affine algebra $A_1^{(1)}$ or $\mathfrak{sl}(2)$ current algebra with generators $J_n^{\pm, 0}$ for the latter [11]. For the $SU(3)$ theory with $N_f = 1$ we find the following conditions;

$$L_1|G_1, m\rangle = i \frac{\Lambda_{SU(3)}^2}{\epsilon_1 \epsilon_2} |G_1, m\rangle, \quad (1.6)$$

$$W_1|G_1, m\rangle = \sqrt{\frac{27}{4\epsilon_1 \epsilon_2 + 15\epsilon_+^2}} \frac{(2m - \epsilon_+) \Lambda_{SU(3)}^2}{2\epsilon_1 \epsilon_2} |G_1, m\rangle, \quad (1.7)$$

and $|G_1, m\rangle$ is annihilated by $L_{n \geq 2}$ and $W_{n \geq 2}$. Thus, as in the case of $SU(2)$ theory, the state $|G_1, m\rangle$ is still a coherent state of W_3 algebra, even if we add a fundamental matter. However, for $SU(2)$ theory with a surface operator, we obtain

$$(J_0^+ + \sqrt{x} J_0^0) |G_1, m\rangle = \frac{\sqrt{x}}{2\epsilon_1} (\epsilon_1 + 2m) |G_1, m\rangle, \quad (1.8)$$

$$J_1^0 |G_1, m\rangle = \frac{\sqrt{z}}{2\epsilon_1} |G_1, m\rangle, \quad (1.9)$$

$$J_1^- |G_1, m\rangle = \frac{\sqrt{z}}{\epsilon_1 \sqrt{x}} |G_1, m\rangle, \quad (1.10)$$

¹Precisely speaking the second equality is valid up to a contribution from $U(1)$ theory.

where z, x are the parameters of topological expansion of the partition function. In general a coherent state $|\Psi\rangle$ must have zero eigenvalue for a generator that can be expressed as a commutator of two annihilation operators. For the Virasoro algebra $L_{n \geq 3}$ are such generators. We note that there appears the zero mode J_0^0 in (1.8), which implies the state $|G_1, m\rangle$ has non vanishing eigenvalue for $J_1^0 \sim [J_0^+, J_1^-]$. Thus it is not a Whittaker state in the genuine sense, since the coherent condition for J_0^+ involves the zero mode of the current J^0 . In this paper we will call it generalized Whittaker state. On the Verma module of the $\mathfrak{sl}(2)$ current algebra the action of J_0^0 can be written as $J_0^0 = j - \sqrt{x} \frac{\partial}{\partial \sqrt{x}}$, where j is the eigenvalue of the primary state and the Euler derivative with respect to \sqrt{x} counts the total $SU(2)$ spin. Due to the dictionary $j = -\frac{1}{2} + \frac{a}{\epsilon_1}$ by [11], the condition for generalized Whittaker state involves the Coulomb moduli parameter a . It is quite interesting that the same features also appear in the definition of the state $|G_2, m_1, m_2\rangle$ for the $SU(3)$ theory with $N_f = 2$. The state is an eigenstate of a linear combination of W_1 and the zero mode L_0 of the Virasoro subalgebra. The L_0 part introduces the dependence on the Coulomb moduli $a_{1,2}$ of $SU(3)$ theory. Finally the state has non-vanishing eigenvalues for W_2 and W_3 which are commutators of lower generators.

From the viewpoint of M theory the existence of such generalized Whittaker states is understood as follows; the S^1 compactification of the six-dimensional $\mathcal{N} = (2, 0)$ theory gives a five-dimensional Yang-Mills theory on $\mathbb{R}^4 \times \mathbb{R}_t$ with the maximal supersymmetry. If we consider the large volume limit of the space-like part; $\text{Vol}(\mathbb{R}^4) \gg 1$, we obtain a supersymmetric quantum mechanics on the instanton moduli space with the wavefunction $\Psi(t) : \mathbb{R}_t \rightarrow \mathcal{H}(\mathcal{M}_{\text{inst}})$. Let us further assume the time direction is a segment of length ℓ and put boundary conditions at $t = 0$ and $t = \ell$ which break half of the supersymmetry. Then the partition function of our system is

$$Z_{\text{SQM}} = \langle \Psi_f | e^{-\ell H} | \Psi_i \rangle, \tag{1.11}$$

where $\Psi_{i,f}$ is in the subspace of BPS states $\mathcal{H}_{\text{BPS}} \subset \mathcal{H}(\mathcal{M}_{\text{inst}})$ that correspond to the boundary conditions at $t = 0, \ell$. We can identify Z_{SQM} with the Nekrasov partition function, where $e^{-\ell H}$ gives rise to the parameter of instanton expansion by $\Lambda^{2N_c - N_f} = e^{-\ell}$. The point here is that an appropriate generalized W algebra which is obtained by the Hamiltonian reduction of the $\mathfrak{sl}(n)$ current algebra, acts (at least) on the BPS subspace \mathcal{H}_{BPS} and consequently it can be identified with the Verma module of the generalized W algebra.² The Hamiltonian of the quantum mechanics then acts on \mathcal{H}_{BPS} as the Virasoro zero-mode L_0 , so that the instanton expansion is just the level expansion on the CFT side. The corresponding CFT is obtained by taking an opposite limit;³ $\text{Vol}(\mathbb{R}^4) \ll 1$, and we expect we can find the states in the Verma module which correspond to $\Psi_{i,f}$. These are our generalized Whittaker states. Note that the world sheet of this CFT is a cylinder and if S^1 shrinks it becomes a sphere with two punctures. We thus recover the sphere that appeared above in connection with $SU(2)$ Seiberg-Witten curve. We expect such a curve would be

²It is not necessarily irreducible.

³This ‘‘compactification of \mathbb{R}^4 ’’ is materialized by introducing Ω -deformation [7] which localizes the field configurations to the origin.

related to the spectral curve of the Hitchin system in general. It is desirable to understand the conditions derived in this paper from the viewpoint of the Hitchin system.

This article is organized as follows; In section 2, we assume that the state $|G_1, m\rangle$ is a simultaneous eigenstate of L_1 and W_1 and derive the conditions (1.6) and (1.7) by comparing the one instanton partition function with $N_f = 1, 2$ and the level one Shapovalov matrix of W_3 algebra. Then we do the same for the state $|G_2, m_1, m_2\rangle$ with two mass parameters and the partition function with $N_f = 3, 4$. It turns out that the level one “eigenvalue” of W_1 for $|G_2, m_1, m_2\rangle$ depends on the Coulomb moduli $a_{1,2}$ of $SU(3)$ theory. In section 3 we test the level one results in section 2 by computing the decoupling limit from the superconformal theory with $N_f = 6$. We find that at higher levels the Coulomb moduli dependence of the W_1 -eigenvalue should be promoted to the action of the Virasoro zero mode L_0 and consequently the state $|G_2, m_1, m_2\rangle$ cannot be a W_1 -eigenstate. In section 4 we understand our results from the viewpoint of the Seiberg-Witten curve. We show that the phenomena found in section 3 take place in general for $SU(N)$ theory with $N_f = N - 1$. In section 5 we derive the conditions (1.8)–(1.10) from the decoupling limit of the correspondence of the instanton partition function with a surface operator and the affine conformal block of $A_1^{(1)}$ current algebra proposed by Alday-Tachikawa. Some of the technical details are collected in appendices.

2 One instanton computation of $SU(3)$ theory

The AGT relation [3] between four-dimensional $\mathcal{N} = 2$ $SU(2)$ gauge theories and two-dimensional Virasoro CFT is generalized by Wyllard to $SU(N)$ gauge theories [4]. The instanton partition functions are then related with the conformal blocks for the non-linear conformal algebra W_N . W_2 -algebra is precisely the Virasoro symmetry of the original proposal. In this section we study $SU(3)$ theories in which case we know the explicit form of the commutation relations of the corresponding W_3 -algebra. Especially we focus on the $SU(3)$ theory with $N_f = 1, 2$ flavors and characterize the $N_f = 1$ Whittaker state as a simultaneous eigenstate of lowest annihilation operators. As we will see in section 4 the case $N_f = N - 1$ requires a redefinition of the W_N currents and we should take care of it in our definition of the coherent state.

2.1 W_3 -algebra and the level one Shapovalov matrix

W_N -algebra is a kind of generalization of the Virasoro algebra. This algebra is generated by the energy momentum tensor $T(z)$ and higher spin currents $W^{(s)}(z)$ $s = 3, 4, \dots, N$. This is the conformal symmetry of the Toda CFT of type A_{N-1} , as the Virasoro symmetry controls the Liouville CFT. In general, it is very hard to write down the explicit algebra of these currents with central extension. Fortunately, the explicit form of the W_3 -algebra, which is of our interest here, is known. We first review the basic facts on W_3 algebra mainly following [9] and fix our notations. The W_3 -algebra consists of the energy-momentum tensor $T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n$ and the spin-three current $W(z) = \sum_{n \in \mathbb{Z}} z^{-n-3} W_n$.

The commutation relations among the modes L_n, W_n are

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \quad (2.1)$$

$$[L_n, W_m] = (2n - m)W_{n+m}, \quad (2.2)$$

$$[W_n, W_m] = \frac{9}{2} \left[(n - m) \left(\frac{(n + m + 2)(n + m + 3)}{15} - \frac{(n + 2)(m + 2)}{6} \right) L_{n+m} + \frac{16}{22 + 5c}(n - m)\Lambda_{n+m} + \frac{c}{3 \cdot 5!}n(n^2 - 1)(n^2 - 4)\delta_{n+m,0} \right], \quad (2.3)$$

where Λ_n is a composite operator

$$\Lambda_n := \sum_{m \in \mathbb{Z}} : L_m L_{-m} : + \frac{x_n}{5} L_n, \quad (2.4)$$

with $x_{2\ell} = (1 - \ell)(1 + \ell)$, $x_{2\ell+1} = (1 - \ell)(12 + \ell)$. We introduce Q to parametrize the central charge $c = 2(1 - 12Q^2)$.

Let us construct the highest weight representation of the conformal algebra. The resulting representation space is called the Verma module of the algebra. The highest weight state $|\Delta(\vec{\alpha})\rangle$ of W_3 -algebra is labeled by the Toda momenta $\vec{\alpha} = (\alpha, \beta)$ and satisfies

$$L_0|\Delta(\vec{\alpha})\rangle = \Delta(\vec{\alpha})|\Delta(\vec{\alpha})\rangle, \quad W_0|\Delta(\vec{\alpha})\rangle = w(\vec{\alpha})|\Delta(\vec{\alpha})\rangle, \quad (2.5)$$

where the eigenvalues are

$$\Delta(\vec{\alpha}) := \alpha^2 + \beta^2 - Q^2, \quad (2.6)$$

$$w(\vec{\alpha}) := \sqrt{\frac{4}{4 - 15Q^2}}\alpha(\alpha^2 - 3\beta^2). \quad (2.7)$$

This highest weight state vanishes when we act the annihilation operators $L_{n>0}$ and $W_{n>0}$. The descendants in the Verma module, which are generated by acting the creation operators, are labeled by a pair of the Young diagrams $\vec{Y} = \{Y_L, Y_W\}$ as $L_{-Y_L}W_{-Y_W}|\Delta(\vec{\alpha})\rangle$. Here we adopt the notation $L_{-Y} = L_{-Y_d}L_{-Y_{d-1}} \cdots L_{-Y_1}$. With this basis of the Verma module, the Shapovalov matrix at level N is defined by the following Gram matrix

$$Q_{\Delta}^{(N)}(\vec{Y}; \vec{Y}) := \langle \Delta(\vec{\alpha}) | W_{Y_W} L_{Y_L} L_{-\vec{Y}_L} W_{-\vec{Y}_W} | \Delta(\vec{\alpha}) \rangle. \quad (2.8)$$

Since the matrix element $\langle \Delta(\vec{\alpha}) | W_{Y_W} L_{Y_L} L_{-\vec{Y}_L} W_{-\vec{Y}_W} | \Delta(\vec{\alpha}) \rangle$ is nonzero only for $|\vec{Y}| = |\vec{Y}| = N$, the full Shapovalov matrix Q_{Δ} is block-diagonal with the blocks $Q_{\Delta}^{(N)}$. In the following let us denote the components of level one matrix and its inverse as

$$Q_{\Delta}^{(1)} = \begin{pmatrix} Q_{LL} & Q_{LW} \\ Q_{WL} & Q_{WW} \end{pmatrix}, \quad (Q_{\Delta}^{(1)})^{-1} = \begin{pmatrix} R_{LL} & R_{LW} \\ R_{WL} & R_{WW} \end{pmatrix}, \quad (2.9)$$

where the indices mean $L = \{\square, \phi\}$ and $W = \{\phi, \square\}$. They are explicitly given by [9]

$$Q_{\Delta}^{(1)} = \begin{pmatrix} 2\Delta & 3w \\ 3w & \frac{9D\Delta}{2} \end{pmatrix}, \quad (Q_{\Delta}^{(1)})^{-1} = \frac{1}{9(D\Delta^2 - w^2)} \begin{pmatrix} \frac{9D\Delta}{2} & -3w \\ -3w & 2\Delta \end{pmatrix}, \quad (2.10)$$

where

$$D(\Delta) = \frac{4\Delta}{4 - 15Q^2} + \frac{3Q^2}{4 - 15Q^2}. \quad (2.11)$$

The level- N Kac determinant is given by the determinant of the Shapovalov matrix $\det Q_{\Delta}^{(N)}$. At level one, the Kac determinant can be factorized as it should be;

$$D\Delta^2 - w^2 = \frac{4}{4 - 15Q^2} \left(\beta^2 - \frac{Q^2}{4} \right) ((\beta - Q)^2 - 3\alpha^2)((\beta + Q)^2 - 3\alpha^2). \quad (2.12)$$

If we use the following identification of parameters of CFT and SYM side⁴ [9],

$$\alpha = \frac{\sqrt{3}}{2\sqrt{-\epsilon_1\epsilon_2}}(a_1 + a_2), \quad \beta = \frac{1}{2\sqrt{-\epsilon_1\epsilon_2}}(-a_1 + a_2), \quad Q = \frac{\epsilon_1 + \epsilon_2}{\sqrt{-\epsilon_1\epsilon_2}}, \quad (2.13)$$

we have

$$D\Delta^2 - w^2 = \frac{-(a_{12} - \epsilon_+)(a_{12} + \epsilon_+)(a_{23} - \epsilon_+)(a_{23} + \epsilon_+)(a_{31} - \epsilon_+)(a_{31} + \epsilon_+)}{(\epsilon_1\epsilon_2)^2(4\epsilon_1\epsilon_2 + 15\epsilon_+^2)}, \quad (2.14)$$

where $a_{ij} := a_i - a_j$ and $\epsilon_+ := \epsilon_1 + \epsilon_2$. Notice that this is proportional to the denominator of the one instanton part of SU(3) Nekrasov partition function. We use this property to determine Whittaker states up to one instanton. We also have

$$\Delta = \frac{a_1^2 + a_1a_2 + a_2^2 - \epsilon_+^2}{-\epsilon_1\epsilon_2}, \quad w = \frac{3\sqrt{3}}{\sqrt{4\epsilon_1\epsilon_2 + 15\epsilon_+^2}} \frac{a_1a_2(a_1 + a_2)}{-i\epsilon_1\epsilon_2}. \quad (2.15)$$

As we will see these factors appear in the numerator of the partition function.

2.2 Whittaker state of W_3 algebra

Let us consider a Whittaker vector of W_3 algebra in the Verma module over the highest weight state $|\Delta(\vec{\alpha})\rangle$. Our final goal is to construct such states whose scalar product gives the instanton partition function

$$\sum_{k=0}^{\infty} Z_{\text{SU}(3),k}^{(N_f)} = \langle G_{N_f-n} | G_n \rangle, \quad \text{for } 0 \leq N_f - n, n \leq 2, \quad (2.16)$$

where k labels the instanton number. See appendix A for details of the instanton partition function. Actually, the states in the right hand side of (2.16) belong to the SU(3) theory with n fundamentals and $N_f - n$ anti-fundamentals in our convention. Since the instanton number on the gauge theory side corresponds to the level in the Verma module [5], among the annihilation operators $L_{n>0}, W_{n>0}$ the relevant ones for the one instanton partition function are L_1 and W_1 . Hence we introduce a simultaneous eigenstate;

$$L_1|q_L, q_W\rangle = q_L|q_L, q_W\rangle, \quad W_1|q_L, q_W\rangle = q_W|q_L, q_W\rangle, \quad (2.17)$$

and expand it in the Verma module as

$$|q_L, q_W\rangle = |\Delta(\vec{\alpha})\rangle + c_L L_{-1}|\Delta(\vec{\alpha})\rangle + c_W W_{-1}|\Delta(\vec{\alpha})\rangle + \dots \quad (2.18)$$

⁴We have rescaled by $\sqrt{-\epsilon_1\epsilon_2}$ to make the parameters on CFT side dimensionless.

Looking at $\langle \Delta(\vec{\alpha})|L_{-1}|q_L, q_W\rangle$ and $\langle \Delta(\vec{\alpha})|W_{-1}|q_L, q_W\rangle$, we find the equations that determine the coefficients $c_{L,W}$:

$$\begin{aligned} q_L &= Q_{LL}c_L + Q_{LW}c_W, \\ q_W &= Q_{WL}c_L + Q_{WW}c_W. \end{aligned} \quad (2.19)$$

Hence the norm up to level one is

$$\langle q'_L, q'_W|q_L, q_W\rangle = 1 + q'_L R_{LL}q_L + q'_L R_{LW}q_W + q'_W R_{WL}q_L + q'_W R_{WW}q_W + \dots \quad (2.20)$$

On the other hand the one instanton part of the SU(3) Nekrasov function takes the following form;

$$Z_{\text{SU}(3),k=1}^{(N_f)}(a_1, a_2, m_i; \epsilon_1, \epsilon_2) = \Lambda^{6-N_f} \frac{z^{(N_f)}(a_1, a_2, m_i; \epsilon_1, \epsilon_2)}{D(a_1, a_2; \epsilon_1, \epsilon_2)}. \quad (2.21)$$

While the numerator $z^{(N_f)}$ depends on the number N_f of matter hypermultiplets, the denominator is independent of matter contents and given by

$$\begin{aligned} D &= \epsilon_1 \epsilon_2 (a_{12} - \epsilon_+) (a_{12} + \epsilon_+) (a_{23} - \epsilon_+) (a_{23} + \epsilon_+) (a_{31} - \epsilon_+) (a_{31} + \epsilon_+) \\ &= (-\epsilon_1 \epsilon_2)^3 (4\epsilon_1 \epsilon_2 + 15\epsilon_+^2) (D\Delta^2 - w^2). \end{aligned} \quad (2.22)$$

If we have a matter in the fundamental representation with mass m (see appendix A for our convention of matter contribution to the instanton counting),

$$\begin{aligned} z^{(1)} &= -6 \left(m - \frac{\epsilon_+}{2} \right) (a_1^2 + a_1 a_2 + a_2^2 - \epsilon_+^2) + 9 a_1 a_2 (a_1 + a_2) \\ &= \epsilon_1 \epsilon_2 \left[27 \left(m - \frac{\epsilon_+}{2} \right) R_{WW} + i \sqrt{27(4\epsilon_1 \epsilon_2 + 15\epsilon_+^2)} R_{WL} \right] (D\Delta^2 - w^2). \end{aligned} \quad (2.23)$$

Here we use (2.15) to get the second equality. Hence

$$Z_{\text{SU}(3),k=1}^{(1)} = \frac{\Lambda^5}{(\epsilon_1 \epsilon_2)^2 (4\epsilon_1 \epsilon_2 + 15\epsilon_+^2)} \left[27 \left(\frac{\epsilon_+}{2} - m \right) R_{WW} - i \sqrt{27(4\epsilon_1 \epsilon_2 + 15\epsilon_+^2)} R_{WL} \right], \quad (2.24)$$

is the one instanton partition function in the CFT language. Comparing it with

$$\langle 0, q_0|q_L, q_W\rangle = 1 \pm \sqrt{\frac{27}{4\epsilon_1 \epsilon_2 + 15\epsilon_+^2}} \frac{\Lambda^3}{\epsilon_1 \epsilon_2} (R_{WL}q_L + R_{WW}q_W) + \dots, \quad (2.25)$$

we find

$$q_L = \mp i \frac{\Lambda^2}{\epsilon_1 \epsilon_2}, \quad q_W = \pm \sqrt{\frac{27}{4\epsilon_1 \epsilon_2 + 15\epsilon_+^2}} \frac{(\frac{\epsilon_+}{2} - m)\Lambda^2}{\epsilon_1 \epsilon_2}, \quad (2.26)$$

where we take $q_0 := \pm \sqrt{\frac{27}{4\epsilon_1 \epsilon_2 + 15\epsilon_+^2}} \frac{\Lambda^3}{\epsilon_1 \epsilon_2}$ for the Whittaker state of the pure SU(3) theory [10]. The $N_f = 1$ Whittaker state is then $|G_1, m\rangle = |q_L, q_W\rangle$. For anti-fundamental matter by replacing $m \rightarrow \epsilon_+ - m$ we have

$$\widetilde{q}_L = q_L, \quad \widetilde{q}_W = \pm \sqrt{\frac{27}{4\epsilon_1 \epsilon_2 + 15\epsilon_+^2}} \frac{(m - \frac{\epsilon_+}{2})\Lambda^2}{\epsilon_1 \epsilon_2}. \quad (2.27)$$

From the viewpoint of the decoupling construction to be discussed in the next section, it is natural to associate the Whittaker ket-vector $\langle G_1, m| = \langle q_L, \widetilde{q}_W|$ with anti-fundamental matter.

2.3 $N_f = 2$ theory

When $N_f = 2$ we have to consider two cases; in the symmetric realization one is fundamental and the other is anti-fundamental. In the asymmetric realization both are in the fundamental representation. These two cases correspond to $n = 1$ and $n = 0, 2$ of (2.16) respectively. The numerator of the one instanton partition function is

$$z_A^{(2)} = 6(a_1^2 + a_1 a_2 + a_2^2 - \epsilon_+^2) \left(m_1 - \frac{\epsilon_+}{2}\right) \left(m_2 - \frac{\epsilon_+}{2}\right) - 9a_1 a_2 (a_1 + a_2)(m_1 + m_2 - \epsilon_+) + 2(a_1^2 + a_1 a_2 + a_2^2)^2 - \frac{5}{2}(a_1^2 + a_1 a_2 + a_2^2)\epsilon_+^2 + \frac{1}{2}\epsilon_+^4, \quad (2.28)$$

for the asymmetric choice. By substituting $m_2 \rightarrow \epsilon_+ - m_2$ we obtain

$$z_S^{(2)} = -6(a_1^2 + a_1 a_2 + a_2^2 - \epsilon_+^2) \left(m_1 - \frac{\epsilon_+}{2}\right) \left(m_2 - \frac{\epsilon_+}{2}\right) - 9a_1 a_2 (a_1 + a_2)(m_1 - m_2) + 2(a_1^2 + a_1 a_2 + a_2^2)^2 - \frac{5}{2}(a_1^2 + a_1 a_2 + a_2^2)\epsilon_+^2 + \frac{1}{2}\epsilon_+^4, \quad (2.29)$$

for the symmetric choice. Using (2.15), we can rewrite the first line of the equation into a CFT quantity. The second line also has the following simple expression;

$$(D\Delta^2 - w^2)R_{LL} = \frac{1}{2(4 - 15Q^2)}(4(\alpha^2 + \beta^2)^2 - 5Q^2(\alpha^2 + \beta^2) + Q^4) \quad (2.30)$$

$$= \frac{2(a_1^4 + a_2^4 + 2a_1^3 a_2 + 2a_1 a_2^3 + 3a_1^2 a_2^2) - \frac{5\epsilon_+^2}{2}(a_1^2 + a_2^2 + a_1 a_2) + \frac{1}{2}\epsilon_+^4}{\epsilon_1 \epsilon_2 (4\epsilon_1 \epsilon_2 + 15\epsilon_+^2)}.$$

Hence, for symmetric realization, the one instanton partition function is

$$Z_{\text{SU}(3), k=1}^{(2)} = \left[\frac{-27}{4\epsilon_1 \epsilon_2 + 15\epsilon_+^2} \left(m_1 - \frac{\epsilon_+}{2}\right) \left(m_2 - \frac{\epsilon_+}{2}\right) R_{WW} + i\sqrt{\frac{27}{4\epsilon_1 \epsilon_2 + 15\epsilon_+^2}}(m_1 - m_2)R_{LW} - R_{LL} \right] \frac{\Lambda^4}{(\epsilon_1 \epsilon_2)^2} \quad (2.31)$$

$$= \widetilde{q}_W(m_2)R_{WW}q_W(m_1) + \widetilde{q}_W(m_2)W_{WL}q_L + q_L W_{LW}q_W(m_1) + q_L R_{LL}q_L,$$

and we have

$$1 + Z_{\text{SU}(3), k=1}^{(2)} + \mathcal{O}(\Lambda^8) = \langle q_L, \widetilde{q}_W(m_2) | q_L, q_W(m_1) \rangle \quad (2.32)$$

at one instanton level. For the anti-symmetric realization we should have

$$Z_{\text{SU}(3), k=1}^{(2)} = \langle 0, q_0 | q_L^{(2)}, q_W^{(2)} \rangle |_{\text{level 1}} = \pm \frac{\epsilon_1 \epsilon_2 \Lambda^3}{D(a_1, a_2; \epsilon_1, \epsilon_2)} \times \left(9a_1 a_2 (a_1 + a_2) q_L^{(2)} + 6(a_1^2 + a_1 a_2 + a_2^2 - \epsilon_+^2) \sqrt{\frac{4\epsilon_1 \epsilon_2 + 15\epsilon_+^2}{27}} q_W^{(2)} \right), \quad (2.33)$$

which implies

$$q_L^{(2)}(m_1, m_2) = \mp(m_1 + m_2 - \epsilon_+) \frac{\Lambda}{\epsilon_1 \epsilon_2}, \quad (2.34)$$

$$q_W^{(2)}(m_1, m_2) = \pm \left[\left(m_1 - \frac{\epsilon_+}{2}\right) \left(m_2 - \frac{\epsilon_+}{2}\right) + \frac{1}{3}(a_1^2 + a_1 a_2 + a_2^2) - \frac{1}{12}\epsilon_+^2 \right] \sqrt{\frac{27}{4\epsilon_1 \epsilon_2 + 15\epsilon_+^2}} \frac{\Lambda}{\epsilon_1 \epsilon_2}. \quad (2.35)$$

Note that $q_W^{(2)}$ depends on the Coulomb moduli $a_{1,2}$. The $N_f = 2$ Whittaker state up to level one is therefore $|G_2\rangle_{0+1} = |q_L^{(2)}, q_W^{(2)}\rangle$. By substituting $m_i \rightarrow \epsilon_+ - m_i$ we obtain the eigenvalues for the anti-fundamental matters;

$$\widetilde{q}_L^{(2)}(m_1, m_2) = -q_L^{(2)}(m_1, m_2), \quad \widetilde{q}_W^{(2)}(m_1, m_2) = q_W^{(2)}(m_1, m_2). \quad (2.36)$$

${}_{0+1}\langle G_2| = \langle \widetilde{q}_L^{(2)}, \widetilde{q}_W^{(2)}|$ is the Whittaker state for two anti-fundamental matters. In the next section we will see that $|q_L^{(2)}, q_W^{(2)}\rangle$ cannot agree with the true Whittaker-like state $|G_2, m_1, m_2\rangle$. The discrepancy appears beyond level one. The genuine state $|G_2, m_1, m_2\rangle$ is actually not an eigenstate of W_1 , but we can characterize it as a typical example of generalized Whittaker state.

We can check that the one instanton partition function with $N_f = 3, 4$ can be reproduced from the Whittaker states we have obtained. Namely

$$1 + Z_{\text{SU}(3), k=1}^{(3)} + \mathcal{O}(\Lambda^6) = \langle q_L, \widetilde{q}_W(m_3) | q_L^{(2)}(m_1, m_2), q_W^{(2)}(m_1, m_2) \rangle, \quad (2.37)$$

$$1 + Z_{\text{SU}(3), k=1}^{(4)} - \frac{1}{3} \frac{\Lambda^2}{\epsilon_1 \epsilon_2} + \mathcal{O}(\Lambda^4) = \langle \widetilde{q}_L^{(2)}(m_3, m_4), \widetilde{q}_W^{(2)}(m_3, m_4) | q_L^{(2)}(m_1, m_2), q_W^{(2)}(m_1, m_2) \rangle. \quad (2.38)$$

When $N_f = 4$, we observe a shift $-\frac{1}{3} \frac{\Lambda^2}{\epsilon_1 \epsilon_2}$, which arises from a remnant of the U(1) contribution $\exp\left(-\frac{1}{3} \frac{\Lambda^2}{\epsilon_1 \epsilon_2}\right)$.

2.4 Comment on $N_f = 0$ theory

The decoupling limit $m \rightarrow \infty$ and $m\Lambda^2 \rightarrow \Lambda^3$ of (2.24) leads to the Whittaker state $|G_0\rangle := |0, q_0\rangle$ for the pure SU(3) Yang-Mills theory. This is an eigenstate of W_1

$$W_1 |G_0\rangle = \pm \sqrt{\frac{27}{4\epsilon_1 \epsilon_2 + 15\epsilon_+^2}} \frac{\Lambda^3}{\epsilon_1 \epsilon_2} |G_0\rangle, \quad (2.39)$$

and annihilated by $L_{n \geq 1}$ and $W_{n \geq 2}$. The solution to the condition is given by [10] as

$$|G_0\rangle = \sum_{\vec{Y}} \left(\pm \sqrt{\frac{27}{4\epsilon_1 \epsilon_2 + 15\epsilon_+^2}} \frac{\Lambda^3}{\epsilon_1 \epsilon_2} \right)^n Q_{\Delta(\vec{\alpha})}^{-1}(\Phi, [1^n]; \vec{Y}) | \vec{Y} \rangle. \quad (2.40)$$

In the next section we will generalize this expression for the Whittaker states with $N_f = 1, 2$ flavors. The expression (2.40) also appeared in [23], however the pre-factor $\sqrt{27/4 + 15(\epsilon_+^2/\epsilon_1 \epsilon_2)}$ was missing in the eigenvalue. This is because they use another normalization of the generators W_n . Their normalization is natural in the free-field construction of W -algebra and this convention would be useful to extend our argument to generic gauge groups.

3 Decoupling limit and generalized Whittaker state

In the last section we assumed the existence of a Whittaker state of W_3 algebra and determined simultaneous eigenvalues of L_1 and W_1 by comparing the scalar product with

the one instanton partition function of $SU(3)$ theory with $N_f \leq 4$. In this section we will investigate the agreement at higher levels or instanton numbers by identifying the Whittaker state in the decoupling limit from the superconformal theory with $N_f = 6$, where the AGT-W relation tells that the Nekrasov partition function agrees with the conformal block of the Toda theory of A_2 type.⁵ From the commutation relations of W_3 algebra one may argue that additional non-vanishing eigenvalue is allowed only for L_2 even at higher levels, since $L_{n \geq 3}$ and $W_{m \geq 2}$ are written as (multiple) commutators of L_1, L_2 and W_1 . We will see this is the case for $N_f = 1$ (actually the eigenvalue of L_2 vanishes). However, it turns out that the state for $N_f = 2$ should have non-vanishing eigenvalues of W_2 and W_3 . In this sense the $N_f = 2$ Gaiotto-like state in $SU(3)$ theory cannot be a genuine Whittaker state and we will call it generalized Whittaker state. An ingenious mechanism to give non-zero eigenvalues of W_2 and W_3 will be made clear by studying W_1 action on the state. In the next section we discuss the origin of non-vanishing eigenvalues of W_2 and W_3 from the viewpoint of Seiberg-Witten curve.

3.1 Decoupling limit of W_3 conformal block

We consider the $SU(3)$ theory with $N_f = 6$ flavors. Following [29], we take six hypermultiplets in anti-fundamental representation.⁶ In this case the AGT-W relation is a correspondence between this superconformal $SU(3)$ theory and the A_2 -type Toda CFT on the Riemann sphere with four punctures. Two A_2 -Toda momenta $\vec{\alpha}_{1,2} = (\alpha_{1,2}, \beta_{1,2})$ and three mass parameters $\mu_{1,2,3}$ for the anti-fundamentals are related through the dictionary of Wyllard that generalizes the AGT relation [3];

$$\begin{aligned} \mu_1 &= -\frac{1}{\sqrt{3}}\alpha_1 + \frac{Q}{2} + \frac{2}{\sqrt{3}}\alpha_2, \\ \mu_2 &= -\frac{1}{\sqrt{3}}\alpha_1 + \frac{Q}{2} - \frac{1}{\sqrt{3}}\alpha_2 - \beta_2, \\ \mu_3 &= -\frac{1}{\sqrt{3}}\alpha_1 + \frac{Q}{2} - \frac{1}{\sqrt{3}}\alpha_2 + \beta_2. \end{aligned} \tag{3.1}$$

Notice that in this section all the gauge theory parameters are dimensionless by scaling out their overall mass scale with $\sqrt{-\epsilon_1\epsilon_2}$ as $\epsilon_+/\sqrt{-\epsilon_1\epsilon_2} = Q$. In (3.1) we choose $\vec{\alpha}_1$ to be the simple puncture of [2], and the corresponding primary state will be of semi-null type [4] with momentum $\beta_1 = -Q/2$. See [4, 9] for details. The anti-fundamental masses $\mu_{4,5,6}$ are given by the remaining Toda momenta $\vec{\alpha}_{3,4}$ through a similar relation:

$$\begin{aligned} \mu_4 &= \frac{1}{\sqrt{3}}\alpha_3 + \frac{Q}{2} + \frac{2}{\sqrt{3}}\alpha_4, \\ \mu_5 &= \frac{1}{\sqrt{3}}\alpha_3 + \frac{Q}{2} - \frac{1}{\sqrt{3}}\alpha_4 - \beta_4, \\ \mu_6 &= \frac{1}{\sqrt{3}}\alpha_3 + \frac{Q}{2} - \frac{1}{\sqrt{3}}\alpha_4 + \beta_4. \end{aligned} \tag{3.2}$$

⁵Note that the agreement is still a conjecture and we assume it in this section.

⁶This is just to make our argument parallel to that of [29]. We can easily get the answer for fundamental representation by replacing mass μ with $\epsilon_+ - \mu$.

Now $\vec{\alpha}_3$ corresponds to the simple puncture. The proposal of AGT-W is that the instanton partition function is precisely equal to the W_3 conformal block of the sphere with four punctures $\vec{\alpha}_{1,\dots,4}$:

$$Z_{\text{SU}(3)}^{N_f=6}(a_1, a_2, \mu_i, q; \epsilon_1, \epsilon_2) = Z_{\text{U}(1)} \sum q^{|\vec{Y}|} \langle V_3 V_4 V_{\vec{Y}, \vec{\alpha}} \rangle Q_{\Delta(\vec{\alpha})}^{-1}(\vec{Y}; \vec{Y}') \langle V_{\vec{Y}', \vec{\alpha}} | V_1 V_2 \rangle, \quad (3.3)$$

where $Q_{\Delta(\vec{\alpha})}^{-1}$ denotes the inverse of the Shapovalov matrix. Here the building blocks $\langle V V V_{\vec{Y}, \vec{\alpha}} \rangle$ and $\langle V_{\vec{Y}, \vec{\alpha}} | V V \rangle$ are the spherical three point conformal blocks which include a descendant field insertion. See [29] for more details.

Let us consider the decoupling limit of a single anti-fundamental hypermultiplet μ_1 and three anti-fundamentals $\mu_{4,5,6}$. This is done by taking infinitely-massive limit of these matter fields. To get a non-empty theory after this limit, we also have to scale the gauge coupling constant of the theory correctly. Thus the decoupling limit to $N_f = 2$ theory is

$$\mu_{1,4,5,6} \rightarrow \infty, \quad \mu_{2,3} : \text{fixed}, \quad \text{with} \quad q \mu_1 \mu_4 \mu_5 \mu_6 \rightarrow (\Lambda_{N_f=2})^4, \quad (3.4)$$

where $q = e^{2\pi i \tau}$ is the coupling constant of $N_f = 6$ theory. $\Lambda_{N_f=2}$ is the dynamical scale of the gauge theory with two flavors. We can translate this procedure into a scaling limit of the conformal block through the dictionary (3.1) and (3.2). Since the full-decoupling limit $\mu_{4,5,6} \rightarrow \infty$ of the anti-fundamentals was discussed in detail in [10], we focus on the decoupling limit of the hypermultiplet $\mu_1 \rightarrow \infty$. In the language of Toda CFT, this limit means

$$C := \alpha_1 + \alpha_2 = \frac{\sqrt{3}}{2} (Q - (\mu_2 + \mu_3)) : \text{fixed}, \quad (3.5)$$

$$2\beta_2 = -\mu_2 + \mu_3 : \text{fixed}, \quad (3.6)$$

$$A := \alpha_1 \rightarrow \infty. \quad (3.7)$$

From (2.6) and (2.7) the conformal dimensions for the external vertex operators are given by the Toda momenta as

$$\Delta_1 = A^2 - \frac{3}{4}Q^2, \quad \Delta_2 = (C - A)^2 + \beta_2^2 - Q^2, \quad (3.8)$$

$$w_1 = \sqrt{\kappa} A \left(A^2 - \frac{3}{4}Q^2 \right), \quad w_2 = \sqrt{\kappa} (C - A) \left((C - A)^2 - 3\beta_2^2 \right), \quad (3.9)$$

where $\sqrt{\kappa} := \sqrt{\frac{4}{4-15Q^2}}$. From these expressions we can determine asymptotic values of the three point functions $\langle V V V \rangle$ in the decoupling limit $A \rightarrow \infty$.

The key ingredient in our computation is the recursion relations for the three point conformal blocks which were developed by Russian group [29]:

$$\langle L_{-n} V_{\vec{Y}, \vec{\alpha}} | V_1(1) V_2(0) \rangle = (\Delta_{\vec{Y}, \vec{\alpha}} + n\Delta_1 - \Delta_2) \langle V_{\vec{Y}, \vec{\alpha}} | V_1(1) V_2(0) \rangle, \quad (3.10)$$

$$\begin{aligned} \langle W_{-n} V_{\vec{Y}, \vec{\alpha}} | V_1(1) V_2(0) \rangle &= \langle W_0 V_{\vec{Y}, \vec{\alpha}} | V_1(1) V_2(0) \rangle + \left(\frac{n(n+3)w_1}{2} - w_2 \right) \langle V_{\vec{Y}, \vec{\alpha}} | V_1(1) V_2(0) \rangle \\ &\quad + n \langle V_{\vec{Y}, \vec{\alpha}} | (W_{-1} V_1)(1) V_2(0) \rangle, \end{aligned} \quad (3.11)$$

where $\vec{Y}, \vec{\alpha}$ is the label for the descendants. Since we choose the external state V_1 as a semi-null state, the action of W_{-1} on the primary is given by $W_{-1}V_1 = \frac{3w_1}{2\Delta_1}L_{-1}V_1$ and we obtain

$$\begin{aligned} \langle V_{\vec{Y}, \vec{\alpha}} | (W_{-1}V_1)(1)V_2(0) \rangle &= \frac{3w_1}{2\Delta_1} \langle V_{\vec{Y}, \vec{\alpha}} | (L_{-1}V_1)(1)V_2(0) \rangle \\ &= \frac{3w_1}{2\Delta_1} (\Delta_{\vec{Y}, \vec{\alpha}} - \Delta_1 - \Delta_2) \langle V_{\vec{Y}, \vec{\alpha}} | V_1(1)V_2(0) \rangle. \end{aligned} \quad (3.12)$$

Our current interest is in the behavior of these three point functions in the limit $\mu_1 \rightarrow \infty$. On the CFT side, this is the heavy momenta limit $\alpha_{1,2} \rightarrow \pm\infty$ with fixed C . Since the external momenta are very large $\Delta_{1,2}, w_{1,2} \gg \Delta_{\vec{Y}, \vec{\alpha}}$, the asymptotic behavior of these three point function is then given by

$$\langle L_{-n}V_{\vec{Y}, \vec{\alpha}} | V_1(1)V_2(0) \rangle \sim (n\Delta_1 - \Delta_2) \langle V_{\vec{Y}, \vec{\alpha}} | V_1(1)V_2(0) \rangle, \quad (3.13)$$

$$\begin{aligned} \langle W_{-n}V_{\vec{Y}, \vec{\alpha}} | V_1(1)V_2(0) \rangle &\sim \left(\frac{n(n+3)w_1}{2} - w_2 - n \frac{3w_1}{2\Delta_1} (-\Delta_{\vec{Y}, \vec{\alpha}} + \Delta_1 + \Delta_2) \right) \\ &\times \langle V_{\vec{Y}, \vec{\alpha}} | V_1(1)V_2(0) \rangle. \end{aligned} \quad (3.14)$$

In the following, we show that the dominant contributions in the decoupling limit come from $L_{-1,-2}$ and $W_{-3,-2,-1}$. The asymptotic values of the following factors in (3.13) and (3.14) are crucial in our analysis:

$$f_n = (n\Delta_1 - \Delta_2), \quad g_n = \left(\frac{n(n+3)w_1}{2} - w_2 - n \frac{3w_1}{2\Delta_1} (-\Delta_{\vec{Y}, \vec{\alpha}} + \Delta_1 + \Delta_2) \right). \quad (3.15)$$

It is easy to see the following behavior of f_n in the limit;

$$f_1 \sim 2CA, \quad f_2 \sim A^2, \quad f_{n(\geq 3)} < \mathcal{O}(A^3). \quad (3.16)$$

Among the Virasoro descendants $L_{-Y_L}V_{Y_W, \vec{\alpha}}$ with a fixed level $\ell = |Y_L|$, the special one $L_{-2}^r L_{-1}^s V_{Y_W, \vec{\alpha}}$ therefore gives the dominant contribution $A^{2r+s} = A^\ell$ to the conformal block in the decoupling limit. Similarly the behavior of the W -descendants is controlled by

$$\begin{aligned} g_1 &\sim \sqrt{\kappa} \left(\frac{3}{2}C^2 - \frac{9}{2}\beta_2^2 + \frac{9}{8}Q^2 + \frac{3}{2}(\Delta_{\vec{\alpha}} + |\vec{Y}|) \right) A, \quad g_2 \sim 3C\sqrt{\kappa}A^2, \\ g_3 &\sim \sqrt{\kappa}A^3, \quad g_{n(\geq 4)} < \mathcal{O}(A^4), \end{aligned} \quad (3.17)$$

and then the descendants $W_{-3}^p W_{-2}^q W_{-1}^t V_{\vec{\alpha}}$ give the dominant contribution to the three point functions for a fixed $|Y_W|$. By combining there results, we find the following descendants dominate in the level n :

$$\langle L_{-2}^r L_{-1}^s W_{-3}^p W_{-2}^q W_{-1}^{n-3p-2(q+r)-s} V_{\vec{\alpha}} | V_1 V_2 \rangle \sim \text{const. } A^n. \quad (3.18)$$

These are the three point functions for the descendants labeled by $Y_L = [2^r \cdot 1^s]$, $Y_W = [3^p \cdot 2^q \cdot 1^{n-3p-2(q+r)-s}]$. Therefore only the contributions from these indices survive in the scaling limit of the conformal block. The decoupling limit of the anti-fundamental

hypermultiplets $\mu_{4,5,6} \rightarrow \infty$ implies the following asymptotic value of the remaining three point function [10]:

$$\langle V_3 V_4 V_{\vec{Y}', \vec{\alpha}} \rangle \sim (\mu_4 \mu_5 \mu_6)^n \frac{(\sqrt{3})^{3n}}{(\sqrt{4-15Q^2})^n} \delta_{\vec{Y}', \vec{\Phi}, [1^n]}. \quad (3.19)$$

This dominant term comes from the special descendant $Y'_L = \Phi$, $Y'_W = [1^n]$. In this way we find that the irregular conformal block for $N_f = 2$ takes the form⁷

$$\begin{aligned} \mathcal{B}^{(N_f=2)} &:= \lim \sum q^{|\vec{Y}'|} \langle V_3 V_4 V_{\vec{Y}', \vec{\alpha}} \rangle Q_{\Delta(\vec{\alpha})}^{-1}(\vec{Y}'; \vec{Y}) \langle V_{\vec{Y}', \vec{\alpha}} | V_1 V_2 \rangle \\ &= \sum_{\substack{n,p,q,r,s \geq 0 \\ n \geq 3p+2q+2r+s}} (-\Lambda_{N_f=2}^4)^n \frac{2^{-n+4p+2q+2r+s} (\sqrt{3})^{6n-12p-5q-8r-3s}}{(\sqrt{4-15Q^2})^{2n-2p-q-2r-s}} \\ &\quad \times q_L(\mu_2, \mu_3)^{s+q} \prod_{\ell=0}^{n-3p-2q-2r-s-1} \left(q_W(\mu_2, \mu_3) + \frac{2\ell}{3} \right) \\ &\quad \times Q_{\Delta(\vec{\alpha})}^{-1}([2^r \cdot 1^s], [3^p \cdot 2^q \cdot 1^{n-3p-2(q+r)-s}]; \vec{\Phi}, [1^n]), \end{aligned} \quad (3.20)$$

where

$$q_L(\mu_2, \mu_3) := Q - \mu_2 - \mu_3, \quad (3.21)$$

$$q_W(\mu_2, \mu_3) := 2\mu_2\mu_3 - Q(\mu_2 + \mu_3) + Q^2 + \frac{2}{3}(a_1^2 + a_1 a_2 + a_2^2 - Q^2). \quad (3.22)$$

Up to overall normalization these are nothing but $q_{L,W}^{(2)}$ in the previous section. Notice that we use $\mu_1 \sim -\sqrt{3}A$ to take the decoupling limit. The above formula is an explicit expression of the irregular conformal block for $N_f = 2$ theory. The level-one part of the irregular conformal block is

$$\begin{aligned} \mathcal{B}_1^{(N_f=2)} &= -\frac{27\Lambda^4}{2(4-15Q^2)} q_W(\mu_2, \mu_3) Q_{\Delta(\vec{\alpha})}^{-1}(\vec{\Phi}, [1]; \vec{\Phi}, [1]) \\ &\quad - \frac{3\sqrt{3}\Lambda^4}{\sqrt{4-15Q^2}} q_L(\mu_2, \mu_3) Q_{\Delta(\vec{\alpha})}^{-1}([1], \vec{\Phi}; \vec{\Phi}, [1]). \end{aligned} \quad (3.23)$$

In appendix B, we show that this function actually reproduces the one-instanton partition function of SU(3) gauge theory with $N_f = 2$ flavors.

3.2 $N_f \leq 2$ Whittaker states

Recall that the irregular conformal block for the SU(3) theory with $N_f = 2$ anti-fundamental flavors is the scalar product of the Whittaker states for $N_f = 2$ and $N_f = 0$;

$$\mathcal{B}^{(N_f=2)} = \langle G_2, m_1, m_2 | G_0 \rangle. \quad (3.24)$$

⁷The U(1) factor $Z_{U(1)}$ gives a trivial contribution in the decoupling limit.

The expression for the irregular conformal block (3.20) therefore tells the following Gaiotto-like states for W_3 -algebra:

$$|G_0\rangle = \sum_{\vec{Y}} (i\Lambda^3)^n \left(\frac{3\sqrt{3}}{\sqrt{4-15Q^2}} \right)^n Q_{\Delta(\vec{\alpha})}^{-1}(\Phi, [1^n]; \vec{Y}) \cdot |\vec{Y}\rangle, \quad (3.25)$$

$$\begin{aligned} |G_2, m_1, m_2\rangle &= \sum_{\substack{\vec{Y}, p, q, r, s \geq 0 \\ n \geq 3p+2q+2r+s}} (i\Lambda)^n \frac{2^{-n+4p+2q+2r+s} (\sqrt{3})^{3n-12p-5q-8r-3s}}{(\sqrt{4-15Q^2})^{n-2p-q-2r-s}} \\ &\times (q_L(m_1, m_2))^{s+q} \left(\frac{2}{3} \right)^{n-3p-2q-2r-s} \left(\frac{3q_W(m_1, m_2)}{2} \right)_{n-3p-2q-2r-s} \\ &\times Q_{\Delta(\vec{\alpha})}^{-1}([2^r \cdot 1^s], [3^p \cdot 2^q \cdot 1^{n-3p-2(q+r)-s}]; \vec{Y}) \cdot |\vec{Y}\rangle, \end{aligned} \quad (3.26)$$

where $(x)_n := x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol. There is an ambiguity of overall sign \pm in front of Λ in the definition of Whittaker states and in this section we choose $+$ sign for simplicity. From (3.26) we obtain the following conditions for the $N_f = 2$ Whittaker state $|G_2, m_1, m_2\rangle (=: |G_2\rangle)$ for anti-fundamentals;

$$L_1|G_2\rangle = i\Lambda(Q - (m_1 + m_2))|G_2\rangle, \quad (3.27)$$

$$L_2|G_2\rangle = \frac{(i\Lambda)^2}{3}|G_2\rangle, \quad (3.28)$$

$$(W_1 - wi\Lambda L_0)|G_2\rangle = \frac{3wi\Lambda}{2}(2m_1m_2 - Q(m_1 + m_2) + Q^2)|G_2\rangle, \quad (3.29)$$

$$W_2|G_2\rangle = w(i\Lambda)^2(Q - (m_1 + m_2))|G_2\rangle, \quad (3.30)$$

$$W_3|G_2\rangle = \frac{2w(i\Lambda)^3}{9}|G_2\rangle, \quad (3.31)$$

where

$$w := \frac{\sqrt{3}}{\sqrt{4-15Q^2}} \quad (3.32)$$

and $|G_2\rangle$ is annihilated by $L_{n \geq 3}$ and $W_{m \geq 4}$. Change the masses $m_{1,2}$ into $\epsilon_+ - m_{1,2}$ to get the result for fundamental hypermultiplets. See appendix C for a derivation of these equations. Actually as we will see below the last two conditions for $W_{2,3}$ follow from the first three conditions. Note that, up to one instanton, the conditions (3.27) and (3.29) agree with (2.34) and (2.35), respectively. To see this, let us recall that the Whittaker state is the following superposition of the states in the Verma module

$$|G_2\rangle = \sum_{n=0}^{\infty} \Lambda^n |n, \Delta\rangle, \quad (3.33)$$

where $|n, \Delta\rangle$ is explicitly given by (3.26). The action of L_0 involves therefore the following Euler derivative with respect to the dynamical scale

$$L_0|G_2\rangle = \left(\Delta + \Lambda \frac{\partial}{\partial \Lambda} \right) |G_2\rangle, \quad (3.34)$$

together with the eigenvalue Δ of L_0 on the primary state $|0, \Delta\rangle$. This Δ -term is the origin of the Coulomb moduli dependence of the ‘‘eigenvalue’’ of the level one Whittaker state $|q_L^{(2)}, q_W^{(2)}\rangle$:

$$W_1|G_2\rangle = \frac{3\sqrt{3}i\Lambda}{2\sqrt{4-15Q^2}} \times \left(2m_1m_1 - Q(m_1 + m_2) + Q^2 + \frac{2}{3}(a_1^2 + a_1a_2 + a_2^2 - Q^2) + \frac{2\Lambda}{3} \frac{\partial}{\partial\Lambda} \right) |G_2\rangle. \tag{3.35}$$

The derivative term of the right hand side vanishes for one instanton expansion of the relation $\frac{\partial}{\partial\Lambda}|G_2\rangle_0 = 0$. Notice that the relation (3.29) has the derivative term, or the level counting operator L_0 , and this state is therefore not a usual Whittaker state in the strict sense. This is the reason why we introduce the notion of generalized Whittaker state. The appearance of such an Euler differential is not a mere accident, and we will encounter a quite similar state in section 5, where $SU(2)$ theory with a surface operator is studied. Note that L_0 in (3.29) plays an important role for the existence of a simultaneous eigenstate of $L_{1,2}$ and $W_{2,3}$. At first sight, it seems that the eigenvalues of $W_{2,3}$ should be zero because of the commutation relation $(2n - 3)W_n = [L_{n-1}, W_1]$. However $|G_2\rangle$ is not a genuine eigenstate of W_1 , and so the emerging L_0 term gives the following contribution for $n > 1$

$$W_n|G_2\rangle = \frac{\sqrt{3}i\Lambda}{(2n-3)\sqrt{4-15Q^2}} [L_{n-1}, L_0]|G_2\rangle = \frac{(n-1)\sqrt{3}i\Lambda}{(2n-3)\sqrt{4-15Q^2}} L_{n-1}|G_2\rangle, \tag{3.36}$$

which leads to non-zero eigenvalue for $n = 2, 3$. Therefore the generalized Whittaker state involving L_0 term can be a simultaneous eigenstate of $L_{1,2}$ and $W_{2,3}$.

Let us consider the further decoupling limit of the Whittaker state (3.26) to $N_f = 1$. By applying the limit $m_2 \rightarrow \infty$, $\Lambda_{N_f=2}m_2 \rightarrow (\Lambda_{N_f=1})^2$, the state (3.26) reduces to

$$|G_1, m\rangle = \sum_{\vec{Y}, 0 \leq s \leq n} (i\Lambda_{N_f=1}^2)^n \frac{2^{-n+s}(\sqrt{3})^{3n-3s}}{(\sqrt{4-15Q^2})^{n-s}} (-1)^s (2m - Q)^{n-s} \times Q_{\Delta(\vec{\alpha})}^{-1}([1^s], [1^{n-s}]; \vec{Y}) \cdot |\vec{Y}\rangle \tag{3.37}$$

Thus, in the case of $N_f = 1$ the eigenvalues of L_1 and W_1 , which have been fixed at level one in section 2, completely characterize the Whittaker state. The coherent condition for $N_f = 1$ no longer involves the Euler derivative and $|G_1, m\rangle$ is therefore a conventional Whittaker state for L_1 and W_1 . In addition, the eigenvalues are independent of the Coulomb moduli parameters. Finally the $N_f = 0$ limit $m \rightarrow \infty$, $(\Lambda_{N_f=1})^2m \rightarrow (\Lambda_{N_f=0})^3$ for the state (3.37) reproduces the Whittaker state (3.25) for the pure super Yang-Mills theory. The definition (3.25) is therefore consistent with that of the partner (3.26) in the decomposition of $\mathcal{B}^{(N_f=2)}$.

4 Whittaker states from the Seiberg-Witten curve

As we have seen in the last section $N_f = 2$ Whittaker state of $SU(3)$ theory has non-vanishing eigenvalues of W_2 and W_3 . In this section by looking at the Seiberg-Witten curve we argue that when $N_f = N - 1$ we have to redefine the W_N currents to remove the $U(1)$ current. The non-vanishing eigenvalues of higher modes come from a contribution from the $U(1)$ part.

The Seiberg-Witten curve of $SU(N)$ theory with N_f fundamental matter is [30, 31]

$$P_N(\lambda) = \Lambda^N z + \frac{\Lambda^{N-N_f} Q_{N_f}(\lambda)}{z}, \tag{4.1}$$

where

$$P_N(\lambda) := \lambda^N - \sum_{k=0}^{N-2} u_{N-k} \lambda^k, \quad Q_{N_f}(z) := \prod_{\ell=1}^{N_f} (\lambda + m_\ell). \tag{4.2}$$

If we substitute $\lambda = xz$, we obtain

$$x^N = \sum_{k=0}^{N-2} u_{N-k} x^k z^{k-N} + \Lambda^N z^{1-N} + \Lambda^{N-N_f} z^{N_f-N-1} \prod_{\ell=1}^{N_f} \left(x + \frac{m_\ell}{z}\right), \tag{4.3}$$

which may be compared with the Gaiotto curve

$$x^N = \sum_{n=2}^N \phi_n(z) x^{N-n}. \tag{4.4}$$

When $N_f = 0$, our curve is

$$x^N = u_2 x^{N-2} z^{-2} + \dots + u_{N-1} x z^{1-N} + u_N z^{-N} + \Lambda^N z^{1-N} + \Lambda^N z^{-N-1}. \tag{4.5}$$

Thus we find

$$\phi_2(z) = u_2 z^{-2}, \dots, \phi_{N-1}(z) = u_{N-1} z^{1-N}, \tag{4.6}$$

and

$$\phi_N(z) = u_N z^{-N} + \Lambda^N z^{1-N} + \Lambda^N z^{-N-1}. \tag{4.7}$$

Following Gaiotto [3, 5], we want to identify $\phi_n(z)$ with $\langle G_0 | W^{(n)}(z) | G_0 \rangle / \langle G_0 | G_0 \rangle$ in the limit $\epsilon_{1,2} \rightarrow 0$, where $|G_0\rangle$ is a Whittaker state in the Verma module. The mode expansion of the spin n current is $W^{(n)}(z) = \sum_{m \in \mathbb{Z}} z^{-m-n} W_m^{(n)}$. Hence we should have

$$\frac{\langle G_0 | W_0^{(\ell)} | G_0 \rangle}{\langle G_0 | G_0 \rangle} = u_\ell(\epsilon_1, \epsilon_2), \quad W_n^{(\ell)} | G_0 \rangle = 0, \quad 2 \leq \ell \leq N-1, n > 0, \tag{4.8}$$

and⁸

$$\frac{\langle G_0 | W_0^{(N)} | G_0 \rangle}{\langle G_0 | G_0 \rangle} = u_N(\epsilon_1, \epsilon_2), \quad W_1^{(N)} | G_0 \rangle = \Lambda^N | G_0 \rangle, \tag{4.9}$$

⁸Notice that the second condition of (4.9) implies $\langle G_0 | W_1^{(\ell)} = \Lambda^N \langle G_0 |$ since the conjugation is now defined as $W_n^\dagger = W_{-n}$.

where $u_\ell(\epsilon_1, \epsilon_2) = u_\ell + \mathcal{O}(\epsilon)$ are the “quantum corrected” Coulomb moduli in the presence of the Ω background. Actually the conditions $\langle G_0 | W_0^{(\ell)} | G_0 \rangle / \langle G_0 | G_0 \rangle = u_\ell(\epsilon_1, \epsilon_2)$ in (4.8) for the zero-modes are achieved by the fact that the Whittaker state $|G_0\rangle$ belongs to the Verma module with the primary state $W_0^{(\ell)}|\vec{\alpha}\rangle = w_0^{(\ell)}(\alpha)|\vec{\alpha}\rangle$ and the dictionary of the AGT-W relation. Note that the Whittaker state is a superposition of vectors in different levels of the Verma module and it is not an eigenstate of the zero-modes.

When $N_f = N - 1$ there appears an x^{N-1} term in (4.3). One can eliminate it by an appropriate shift of $x \rightarrow x + c$. Since the linear term describes the center-of-mass degrees of freedom in the brane construction, this completing square is just the decoupling of the overall U(1) subgroup of U(N) gauge group. Hence it is natural to introduce an additional U(1) current $W^{(1)}$ in our W algebra. Since the coefficient of the x^{N-1} term takes a universal form Λz^{-2} , the condition concerning the U(1) current is actually independent of N ;

$$W_1^{(1)}|G_{N-1}\rangle = \Lambda|G_{N-1}\rangle, \quad W_n^{(1)}|G_{N-1}\rangle = 0, \quad n > 1. \quad (4.10)$$

Let us see how it works for SU(2) case where the curve is⁹

$$x^2 = \Lambda z^{-2}x + m\Lambda z^{-3} + uz^{-2} + \Lambda^2 z^{-1}. \quad (4.11)$$

The condition for the Gaiotto-Whittaker state $|G_1\rangle$ for $N_f = 1$ is

$$\frac{\langle G_0 | L_0 | G_1 \rangle}{\langle G_0 | G_1 \rangle} = u, \quad L_1 | G_1 \rangle = m\Lambda | G_1 \rangle. \quad (4.12)$$

and

$$J_1 | G_1 \rangle = \Lambda | G_1 \rangle, \quad (4.13)$$

from (4.10). Now let us modify the original Virasoro by

$$\tilde{L}(z) = L(z) + \alpha : J(z)J(z) : . \quad (4.14)$$

Then the condition for the Gaiotto-Whittaker state in terms of the new Virasoro generator is

$$\tilde{L}_1 | G_1 \rangle = m\Lambda | G_1 \rangle, \quad \tilde{L}_2 | G_1 \rangle = \alpha\Lambda^2 | G_1 \rangle, \quad (4.15)$$

where we have assumed $J_0 | G_1 \rangle = 0$. We recover the original form proposed by Gaiotto [5]. Recall that $\hat{x}^2 = \hat{x}\hat{J}(z) + \hat{L}(z)$ describes the “quantum Seiberg-Witten curve” [3] because the expectation value of this CFT operator gives the classical curve of the corresponding gauge theory. The modification of the energy momentum tensor (4.14) is thus equivalent to the completing square of the curve $(\hat{x} + \hat{J}(z)/2)^2 = (\hat{x} + \hat{J}(z)/2)\hat{J}(z) + \hat{L}(z)$. Hence, after decoupling the center-of-mass degrees of freedom, the modification (4.14) is induced and the Whittaker state becomes a simultaneous eigenstate of $L_{1,2}$.

Let us do the same with $N = 3$ and $N_f = 2$. The curve is

$$x^3 = \Lambda z^{-2}x^2 + (uz^{-2} + (m_1 + m_2)\Lambda z^{-3})x + vz^{-3} + \Lambda^3 z^{-2} + m_1 m_2 \Lambda z^{-4}. \quad (4.16)$$

⁹The mass term in the eigenvalue is actually corrected by ϵ_+ dependent term as we saw in the previous section. We can fix it by comparing the state with one instanton partition function. In this section, we neglect ϵ_+ shift of mass parameters.

The condition for the Whittaker state $|G_2\rangle$ for $N_f = 2$ is therefore

$$\frac{\langle G_0|L_0|G_2\rangle}{\langle G_0|G_2\rangle} = u(\epsilon), \quad L_1|G_2\rangle = (m_1 + m_2)\Lambda|G_2\rangle, \quad (4.17)$$

$$\frac{\langle G_0|W_0|G_2\rangle}{\langle G_0|G_2\rangle} = v(\epsilon), \quad W_1|G_2\rangle = m_1m_2\Lambda|G_2\rangle, \quad (4.18)$$

and again we put

$$J_1|G_2\rangle = \Lambda|G_2\rangle. \quad (4.19)$$

Now as before let us consider the following redefinition of the currents;

$$\tilde{L}(z) = L(z) + \alpha : J(z)J(z) :, \quad \tilde{W}(z) = W(z) + \beta : L(z)J(z) : + \gamma : J(z)J(z)J(z) :. \quad (4.20)$$

Assuming $J_0|G_2\rangle = 0$, we find the condition for $|G_2\rangle$ in terms of the new generators

$$\tilde{L}_1|G_2\rangle = (m_1 + m_2)\Lambda|G_2\rangle, \quad \tilde{L}_2|G_2\rangle = \alpha\Lambda^2|G_2\rangle, \quad (4.21)$$

$$\tilde{W}_1|G_2\rangle = (m_1m_2\Lambda + \beta\Lambda\tilde{L}_0)|G_2\rangle, \quad (4.22)$$

$$\tilde{W}_2|G_2\rangle = \beta(m_1 + m_2)\Lambda^2|G_2\rangle, \quad \tilde{W}_3|G_2\rangle = \gamma\Lambda^3|G_2\rangle. \quad (4.23)$$

The eigenvalue of \tilde{W}_3 must be $\gamma = 2\alpha\beta/3$ so that \tilde{L}_n and \tilde{W}_m form the closed W_3 -algebra again. In section 3, we observed that when $N_f = 2$ the action of W_1 on the Whittaker state involves the Virasoro zero mode which leads to the moduli parameter $a_{1,2}$ dependence. We see this fact comes from the remnant of $U(1)$ current. The parameters α, β, γ can be fixed by comparing it with the Nekrasov partition function, or by estimating the decoupling limit from the superconformal theory as we have done in the last section. These parameters should also be fixed by completing cube of the original curve which eliminates the quadratic term in x . This is because the completed curve must be consistent with the original one including the $U(1)$ degrees of freedom. We should emphasize that there are “quantum corrections” by the Ω background ϵ_1 and ϵ_2 . In principle, these ϵ dependences would be fixed by comparing the expectation values of currents $W^{(n)}$ with those of gauge theory operators $\text{tr } \Phi^n$ in the presence of Ω background.

5 $SU(2)$ theory with a surface operator

In this section we consider the instanton partition function in the presence of the surface operator which gives a codimension two defect. Such an operator is realized by imposing the boundary condition in performing the path integral as

$$A_\mu dx^\mu \sim \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_N) id\theta, \quad (5.1)$$

near the insertion locus ($z = re^{i\theta} = 0$) of the defect [32]. We can then define the instanton partition function in the presence of surface operator by integrating over the field configurations with this boundary condition. In general there are several types of the surface operator according to the breaking pattern of the gauge symmetry on the defect. The remaining gauge symmetry is a subgroup of $U(N)$ given as the commutant of $\text{diag}(\alpha_1, \alpha_2, \dots, \alpha_N)$

which is classified by the partitions of N . In the following we only consider $U(2)$ or $SU(2)$ theory where we have a unique surface operator corresponding to $U(2) \rightarrow U(1)^2$. When we introduce the surface operator, the moduli space is labeled by a new topological number (monopole, or vortex number) in addition to the instanton number. Hence the instanton partition function is expanded in two parameters x and z .

According to [33] the surface operator in $\mathcal{N} = 2$ gauge theory corresponds to the degenerate primary field $\Phi_{1,2}(x)$ that has the conformal weight $(b + b^{-1})^2/4 - (b/2 + b^{-1})^2$ and is in the Virasoro Verma module with the central charge $c = 1 + 6(b + b^{-1})$. The point is that $\Phi_{1,2}(x)$ satisfies the null state condition $(b^2 L_{-1}^2 + L_{-2})\Phi_{1,2} = 0$. Using the Gaiotto states $|G_0\rangle$ and $|G_1, m\rangle$ that are defined by the same conditions in the $SU(2)$ theory without the surface operator, we have

$$Z_{SU(2)}^{(S), N_f=0} = \frac{\langle G_0, - | \Phi_{1,2}(x) | G_0, + \rangle}{\langle G_0 | G_0 \rangle}, \tag{5.2}$$

$$Z_{SU(2)}^{(S), N_f=1} = \frac{\langle G_1, -, m | \Phi_{1,2}(x) | G_0, + \rangle}{\langle G_1, m | G_0 \rangle}, \tag{5.3}$$

$$Z_{SU(2)}^{(S), N_f=2} = \frac{\langle G_1, -, m_1 | \Phi_{1,2}(x) | G_1, +, m_2 \rangle}{\langle G_1, m_1 | G_1, m_2 \rangle}, \tag{5.4}$$

where $Z_{SU(2)}^{(S), N_f}$ denotes the instanton partition function with the surface operator. Actually this description through the Virasoro degenerate field is believed to correspond to the surface operator of simple type [33]. Note that since there is a $\Phi_{1,2}(x)$ operator insertion in the numerator, the states $|G_0, \pm\rangle$ and $|G_1, \pm, m\rangle$ should be in the Verma module over the primary field with the conformal weight $\Delta(a \pm \frac{1}{4b}) := (b + b^{-1})^2/4 - (a \pm 1/(4b))^2$. In this approach the mode expansion parameter x of the degenerate field $\Phi_{1,2}(x)$ gives the monopole expansion parameter. On the other hand it was shown in [11] that the same partition function can be obtained from the conformal block of the affine $\mathfrak{sl}(2)$ algebra. This affine CFT describes the surface operator of full type [11]. It is this second approach that we will take in this section. Since the surface operator of simple type and of full type coincide for $SU(2)$ theories, we expect both descriptions give the same result. As we will see in the next subsection the monopole expansion parameter x comes from the $SU(2)$ spin variable that is carried by the primary fields.

5.1 Review of pure Yang-Mills case

As argued by Alday-Tachikawa the $SU(2)$ instanton partition function with a surface operator is related to affine $\mathfrak{sl}(2)$ conformal blocks [11]. For instance, the relation for the superconformal gauge theory with $N_f = 4$ flavors takes the form

$$Z_{SU(2)}^{(S), N_f=4} = Z_{U(1)}^{(S)} \langle V_{j_1} | V_{j_2}(1, 1) \mathbf{1}_j \mathcal{K}(x, z) V_{j_3}(x, z) | V_{j_4} \rangle, \tag{5.5}$$

where V_j is the vertex operator with spin j . $\mathbf{1}_j$ is the projection operator on the Verma module spanned by V_j , and $\mathcal{K}(x, z)$ is the Alday-Tachikawa K -operator introduced in [11].

The basic dictionary between gauge theory side and CFT side is

$$\begin{aligned}
 j_1 &= -\frac{\epsilon_+ + \mu_1 - \mu_2}{2\epsilon_1}, & j_2 &= -\frac{\epsilon_+ + \epsilon_1 + \mu_1 + \mu_2}{2\epsilon_1}, \\
 j_3 &= -\frac{\epsilon_+ + \epsilon_1 - \mu'_1 - \mu'_2}{2\epsilon_1}, & j_4 &= -\frac{\epsilon_+ + \mu'_1 - \mu'_2}{2\epsilon_1}, \\
 j &= -\frac{1}{2} + \frac{a_1}{\epsilon_1}, & k &= -2 - \frac{\epsilon_2}{\epsilon_1},
 \end{aligned}
 \tag{5.6}$$

where $a_1 = -a_2 = a$ as usual, and k is the level of the affine algebra. $\mu_{1,2}$ and $\mu'_{1,2}$ are the mass parameters for fundamental and anti-fundamental matters. The point in [11] is that the conformal symmetry which controls the corresponding gauge theory will change if we introduce a surface operator on the gauge theory side. By introducing a surface operator, the Virasoro algebra is replaced by the (untwisted) affine $\mathfrak{sl}(2)$ algebra with the commutation relations;

$$[J_n^0, J_m^0] = \frac{k}{2}n\delta_{n+m,0}, \tag{5.7}$$

$$[J_n^0, J_m^\pm] = \pm J_{n+m}^\pm, \tag{5.8}$$

$$[J_n^+, J_m^-] = 2J_{n+m}^0 + kn\delta_{n+m,0}. \tag{5.9}$$

We can construct the Verma module of the affine $\mathfrak{sl}(2)$ algebra as the highest weight representation of it. The highest weight state $|j\rangle$ satisfies

$$J_0^0|j\rangle = j|j\rangle, \quad J_{1+n}^-|j\rangle = J_{1+n}^0|j\rangle = J_n^+|j\rangle = 0, \quad (n \geq 0). \tag{5.10}$$

Recall that the instanton partition function of an asymptotically free gauge theory is believed to be equal to the norm of the corresponding Whittaker state. As we will see soon, this correspondence will be extended to theories with surface operators. For the pure $SU(2)$ Yang-Mills case the Whittaker state is explicitly constructed in [12] (see also Braverman-Etingov [13, 14]) as follows;

$$J_0^+|x, z; j\rangle = \sqrt{x}|x, z; j\rangle, \quad J_1^-|x, z; j\rangle = \sqrt{\frac{z}{x}}|x, z; j\rangle, \tag{5.11}$$

where z and x are identified with the parameters that count instanton and monopole number, respectively, on the gauge theory side. In terms of the inverse of the Shapovalov matrix $Q_j(\mathbf{n}', \mathbf{A}'; \mathbf{n}, \mathbf{A})$ of the Verma module on $|j\rangle$, the Whittaker state is expanded as

$$|x, z; j\rangle = \sum_{n,p} \sum_{\mathbf{n}, \mathbf{A}} x^{n/2} \left(\frac{z}{x}\right)^{p/2} Q_j^{-1}(n, p; \mathbf{n}, \mathbf{A}) |\mathbf{n}, \mathbf{A}\rangle, \tag{5.12}$$

where

$$|\mathbf{n}, \mathbf{A}\rangle := J_{-n_1}^{A_1} \cdots J_{-n_\ell}^{A_\ell} |j\rangle \tag{5.13}$$

is a basis of the Verma module¹⁰ and

$$|n, p\rangle := (J_{-1}^+)^p (J_0^-)^n |j\rangle \tag{5.14}$$

¹⁰The ordering is $J_{-n}^{A_n}$ sits left of $J_{-n'}^{A'}$, for $n > n'$ or $n = n', A < A'$.

is a special state with level p and spin $p - n + j$. On the gauge theory side p counts the instanton number and $n - p$ counts the monopole number. The norm of the Whittaker state $|x, z; j\rangle$ is computed as follows;

$$\begin{aligned}
 & \langle x, z; j | x, z; j \rangle \\
 &= \sum_{n', p'} \sum_{\mathbf{n}', \mathbf{A}'} \sum_{n, p} \sum_{\mathbf{n}, \mathbf{A}} x^{(n+n')/2} \left(\frac{z}{x}\right)^{(p+p')/2} Q_j^{-1}(n', p'; \mathbf{n}', \mathbf{A}') Q_j^{-1}(n, p; \mathbf{n}, \mathbf{A}) Q_j(\mathbf{n}', \mathbf{A}'; \mathbf{n}, \mathbf{A}) \\
 &= \sum_{n', p'} \sum_{n, p} x^{(n+n')/2} \left(\frac{z}{x}\right)^{(p+p')/2} Q_j^{-1}(n, p; n', p') = \sum_{n, p} x^n \left(\frac{z}{x}\right)^p Q_j^{-1}(n, p; n, p). \quad (5.15)
 \end{aligned}$$

Thus we should compare $(n, p; n, p)$ component of Q_j^{-1} with the partition function in the sector with instanton number p and monopole number $n - p$. In [12], it is checked for special values of instanton and monopole numbers that the irregular conformal block of the Whittaker state gives the instanton partition function with a surface operator

$$Z_{\text{SU}(2)}^{(S), N_f=0} = \langle x, z; j | x, z; j \rangle. \quad (5.16)$$

We will extend this relation to SU(2) theory with fundamental flavor. As a biproduct, we find that the proposal of [12] follows precisely from the decoupling limit of Alday-Tachikawa conjecture.

5.2 Decoupling limit of affine conformal block

As we see in section 3 for SU(3)- W_3 case, the decoupling limit of fundamental hypermultiplets leads to the Whittaker states for asymptotically free gauge theories. This idea also works for the gauge theories with a surface operator. In this section, we construct the Whittaker states by applying the decoupling limit to the correspondence (5.5) between the instanton partition function with a surface operator and the affine conformal block. The four point conformal block in question is

$$\begin{aligned}
 \mathcal{B} &= x^{-j+j_3+j_4} z^{\Delta-\Delta_3-\Delta_4} \\
 &\times \sum_{\mathbf{n}, \mathbf{A}} \langle j_1 | V_{j_2}(1, 1) | \mathbf{n}, \mathbf{A}; j \rangle Q_j^{-1}(\mathbf{n}, \mathbf{A}; \mathbf{n}', \mathbf{A}') \langle \mathbf{n}', \mathbf{A}'; j | \mathcal{K}(x, z) V_{j_3}(x, z) | j_4 \rangle.
 \end{aligned}$$

The vertex operator associated with the primary state satisfies

$$[J_{-n}^A, V_j(x, z)] = z^{-n} D^A V_j(x, z), \quad (5.17)$$

where the representation of $\mathfrak{sl}(2)$ on the primary state is defined by

$$D^- = \partial_x, \quad D^0 = -x\partial_x + j, \quad D^+ = 2jx - x^2\partial_x. \quad (5.18)$$

We introduce the following three point spherical conformal blocks for later consideration;

$$\rho(j_1; j_2; \mathbf{n}, \mathbf{A}, j_3 | x, z) := \langle j_1 | V_{j_2}(x, z) | \mathbf{n}, \mathbf{A}; j_3 \rangle, \quad (5.19)$$

$$\rho^\mathcal{K}(\mathbf{n}, \mathbf{A}, j_1; j_2; j_3 | x, z) := \langle \mathbf{n}, \mathbf{A}; j_1 | \mathcal{K}(x, z) V_{j_2}(x, z) | j_3 \rangle. \quad (5.20)$$

These three point blocks are the basic building blocks of the above affine four point function \mathcal{B} .

We move on to explicit computation of the four point function in asymptotically free gauge theory case. In order to compute the decoupling limit, let us derive the explicit expression of the three point function. Since the action of the operator J_0^0 is given by

$$\begin{aligned} \langle j_1 | [J_0^0, V_{j_2}(x, z)] | \mathbf{n}, \mathbf{A}; j_3 \rangle \\ = (\langle j_1 | J_0^0 V_{j_2}(x, z) | \mathbf{n}, \mathbf{A}; j_3 \rangle - \langle j_1 | V_{j_2}(x, z) J_0^0 | \mathbf{n}, \mathbf{A}; j_3 \rangle), \end{aligned} \quad (5.21)$$

the three point conformal block satisfies the differential equation which determines the x -dependence

$$x \partial_x \rho(j_1; j_2; \mathbf{n}, \mathbf{A}, j_3 | x, z) = (-j_1 + j_2 + j_3 + Q) \rho(j_1; j_2; \mathbf{n}, \mathbf{A}, j_3 | x, z). \quad (5.22)$$

The total charge is defined by $Q := \sum_i A_i$. This equation and the action of L_0 give the following x - and z -dependence of the block

$$\rho(j_1; j_2; \mathbf{n}, \mathbf{A}, j_3 | x, z) = x^{-j_1+j_2+j_3+Q} z^{\Delta_1-\Delta_2-\Delta_3-N} \rho(j_1; j_2; \mathbf{n}, \mathbf{A}, j_3). \quad (5.23)$$

Here we introduce $\rho(j_1; j_2; \mathbf{n}, \mathbf{A}, j_3) := \rho(j_1; j_2; \mathbf{n}, \mathbf{A}, j_3 | 1, 1)$.

The following is the derivation of the $\rho(j_1; j_2; \mathbf{n}, \mathbf{A}, j_3 | 1, 1)$ part of the three point block. In order to determine an explicit form of the function, we employ the recursive structure for the labels of the descendants

$$\begin{aligned} \langle j_1 | V_{j_2}(x, z) J_{-n}^A | \mathbf{n}, \mathbf{A}; j_3 \rangle &= -\langle j_1 | [J_{-n}^A, V_{j_2}(x, z)] | \mathbf{n}, \mathbf{A}; j_3 \rangle + \langle j_1 | J_{-n}^A V_{j_2}(x, z) | \mathbf{n}, \mathbf{A}; j_3 \rangle \\ &= -z^{-n} d^A \langle j_1 | V_{j_2}(x, z) | \mathbf{n}, \mathbf{A}; j_3 \rangle, \end{aligned} \quad (5.24)$$

where the coefficients are given by

$$d^- = x^{-1}(-j_1 + j_2 + j_3 + Q), \quad d^0 = j_1 - j_3 - Q, \quad d^+ = x(j_1 + j_2 - j_3 - Q). \quad (5.25)$$

When normalizing the three point function as $\rho(j_1; j_2; j_3) = 1$, the recursive relation leads to the following explicit expression of the three point conformal block:

$$\rho(j_1; j_2; \mathbf{n}, \mathbf{A}, j) = (-j_1 - j_2 + j - c)_a (-j_1 + j - c)^b (j_1 - j_2 - j)_c. \quad (5.26)$$

Here a, b and c are the number of J^+, J^0 and J^- in the descendant $|\mathbf{n}, \mathbf{A}; j\rangle$ respectively.¹¹ Notice that we can rewrite the arguments by using gauge theory parameters:

$$\begin{aligned} j_1 - j_2 - j &= 1 + \frac{\mu_2 - a_1}{\epsilon_1}, \\ j - j_1 &= \frac{\epsilon_2 + 2a_1 + \mu_1 - \mu_2}{2\epsilon_1}, \\ j - j_1 - j_2 &= \frac{\epsilon_+ + a_1 + \mu_1}{\epsilon_1}. \end{aligned} \quad (5.27)$$

¹¹Recall $[*]_n$ denotes the Pochhammer symbol.

Let us proceed to the computation of the decoupling limit of a single hypermultiplet μ_1 while keeping μ_2 finite. In the decoupling limit $\mu_1 \rightarrow \infty$, the three point block behaves

$$\rho(j_1; j_2; \mathbf{n}, \mathbf{A}, j) \sim \left(\frac{\mu_1}{\epsilon_1}\right)^{a+b} \frac{1}{2^b} \left(1 + \frac{\mu_2 - a_1}{\epsilon_1}\right)_c. \quad (5.28)$$

For a fixed charge $Q = a - c$ and level $N = \sum_{i,q} n_i^q$, the dominant contribution in the limit comes from the descendants with $n_i^+ = n_i^0 = 1, n_i^- = 0$. These terms behave as $(\mu_1)^N$, and then the conformal block $\mathcal{B} = \sum \rho Q^{-1} \rho^{\mathcal{K}}$ becomes

$$\begin{aligned} \mathcal{B} \sim & \sum_{N=0}^{\infty} \sum_{a=0}^N \sum_{Q=-\infty}^a \left(\frac{\sqrt{z}\mu_1}{\epsilon_1}\right)^N (\sqrt{x})^{-Q} \frac{1}{2^{N-a}} \left(1 + \frac{\mu_2 - a_1}{\epsilon_1}\right)_{a-Q} \\ & \times \sum_{\mathbf{n}, \mathbf{A}} Q_j^{-1} \left(\underbrace{(1, \dots, 1)}_a; \underbrace{1, \dots, 1}_{N-a}; \underbrace{0, \dots, 0}_{a-Q} \right), (+, \dots, +; 0 \dots, 0; - \dots, -); \mathbf{n}, \mathbf{A} \rho^{\mathcal{K}}(\mathbf{n}, \mathbf{A}). \end{aligned} \quad (5.29)$$

The irregular conformal block takes the form $\sum \langle G_1, m | \mathbf{n}, \mathbf{A} \rangle Q^{-1}(\mathbf{n}, \mathbf{A}; \mathbf{n}', \mathbf{A}') \rho^{\mathcal{K}}(\mathbf{n}', \mathbf{A}')$, so that the decoupling limit $\mu \rightarrow \infty, \mu\sqrt{z} \rightarrow \sqrt{z}$ with fixed x implies the following formula for the Whittaker-like state with one flavor

$$\begin{aligned} |G_1, m\rangle = & \sum_{N=0}^{\infty} \sum_{a=0}^N \sum_{Q=-\infty}^a \sum_{\mathbf{n}, \mathbf{A}} \left(\frac{\sqrt{z}}{\epsilon_1}\right)^N (\sqrt{x})^{-Q} \frac{1}{2^{N-a}} \left(1 + \frac{m - a_1}{\epsilon_1}\right)_{a-Q} \\ & \times Q_j^{-1} \left(\underbrace{(1, \dots, 1)}_a; \underbrace{1, \dots, 1}_{N-a}; \underbrace{0, \dots, 0}_{a-Q} \right), (+, \dots, +; 0 \dots, 0; - \dots, -); \mathbf{n}, \mathbf{A} | \mathbf{n}, \mathbf{A}; j \rangle. \end{aligned} \quad (5.30)$$

We derive the coherent condition for the state in the following. Actually this state is not genuine Whittaker state, and it turn out to be of generalized type.

Notice that by applying an additional decoupling limit $m \rightarrow \infty$, the generalized Whittaker state $|G, m\rangle$ reproduces the proposal of [12] for pure SU(2) Yang-Mills theory:

$$\begin{aligned} |G_0\rangle = & \sum_{N=0}^{\infty} \sum_{Q=-\infty}^N \sum_{\mathbf{n}, \mathbf{A}} \left(\frac{\sqrt{z}}{\epsilon_1}\right)^N (\sqrt{x})^{-Q} \\ & \times Q_j^{-1} \left(\underbrace{(1, \dots, 1)}_N; \underbrace{0, \dots, 0}_{N-Q} \right), (+, \dots, +; 0 \dots, 0); \mathbf{n}, \mathbf{A} | \mathbf{n}, \mathbf{A}; j \rangle. \end{aligned} \quad (5.31)$$

This result is exactly equal to (5.12).

5.3 Generalized Whittaker state from the decoupling

From the computation of a decoupling limit in the last section, the Whittaker-like state with a matter hypermultiplet is identified with (5.30). From the Shapovalov matrix of level zero, we find the level zero part of the state is

$$|G_1, m\rangle_0 = \sum_{n=0}^{\infty} (\sqrt{x})^n \left(1 + \frac{m-a}{\epsilon_1}\right)_n Q_j^{-1}((J_0^-)^n; \mathbf{n}, \mathbf{A}) | \mathbf{n}, \mathbf{A}; j \rangle \quad (5.32)$$

$$= \sum_{n=0}^{\infty} \frac{(-\sqrt{x})^n}{n!} \frac{\left(1 + \frac{m-a}{\epsilon_1}\right)_n}{\left(1 - \frac{2a}{\epsilon_1}\right)_n} (J_0^-)^n |j\rangle, \quad (5.33)$$

which means

$$\langle j|(J_0^+)^n|G_1, m\rangle = (\sqrt{x})^n \left(1 + \frac{m-a}{\epsilon_1}\right)_n. \quad (5.34)$$

Hence $|G_1, m\rangle$ cannot be an eigenstate of J_0^+ .

Since any state in the Verma module is expanded as

$$|\Psi\rangle = \sum_{\mathbf{n}', \mathbf{A}'} \sum_{\mathbf{n}, \mathbf{A}} |\mathbf{n}', \mathbf{A}'; j\rangle Q_j^{-1}(\mathbf{n}', \mathbf{A}'; \mathbf{n}, \mathbf{A}) \langle \mathbf{n}, \mathbf{A}; j | \Psi \rangle, \quad (5.35)$$

at each level N , $|G_1, m\rangle$ is orthogonal to the subspace spanned by those states other than $(J_{-1}^+)^a (J_{-1}^0)^{N-a} (J_0^-)^{a+Q} |j\rangle$. Namely

$$\langle \mathbf{n}, \mathbf{A}; j | G_1, m \rangle = 0, \quad (5.36)$$

unless $\langle \mathbf{n}, \mathbf{A}; j | = \langle j | (J_0^+)^{\ell} (J_1^0)^m (J_1^-)^n$ for which

$$\langle j | (J_0^+)^{\ell} (J_1^0)^m (J_1^-)^n | G_1, m \rangle = \left(\frac{\sqrt{z}}{\epsilon_1}\right)^{m+n} (\sqrt{x})^{\ell-n} \frac{1}{2^m} \left(1 + \frac{m-a}{\epsilon_1}\right)_{\ell}. \quad (5.37)$$

Hence among the annihilation operators J_n^A , $J_n^A |G_1, m\rangle$ is non-vanishing only for $J_n^A = J_0^+, J_1^0, J_1^-$. To evaluate the action of these three operators on $|G_1, m\rangle$, let us look at

$$\begin{aligned} \langle j | (J_0^+)^{\ell} (J_1^0)^m (J_1^-)^n J_0^+ | G_1, m \rangle &= -2n \langle j | (J_0^+)^{\ell} (J_1^0)^{m+1} (J_1^-)^{n-1} | G_1, m \rangle \\ &\quad + \langle j | (J_0^+)^{\ell+1} (J_1^0)^m (J_1^-)^n | G_1, m \rangle \\ &= \sqrt{x} \left(\ell - n + 1 + \frac{m-a}{\epsilon_1}\right) \langle j | (J_0^+)^{\ell} (J_1^0)^m (J_1^-)^n | G_1, m \rangle, \end{aligned} \quad (5.38)$$

$$\begin{aligned} \langle j | (J_0^+)^{\ell} (J_1^0)^m (J_1^-)^n J_1^0 | G_1, m \rangle &= \langle j | (J_0^+)^{\ell} (J_1^0)^{m+1} (J_1^-)^n | G_1, m \rangle \\ &= \left(\frac{\sqrt{z}}{2\epsilon_1}\right) \langle j | (J_0^+)^{\ell} (J_1^0)^m (J_1^-)^n | G_1, m \rangle, \end{aligned} \quad (5.39)$$

$$\begin{aligned} \langle j | (J_0^+)^{\ell} (J_1^0)^m (J_1^-)^n J_1^- | G_1, m \rangle &= \langle j | (J_0^+)^{\ell} (J_1^0)^m (J_1^-)^{n+1} | G_1, m \rangle \\ &= \left(\frac{\sqrt{z}}{\epsilon_1 \sqrt{x}}\right) \langle j | (J_0^+)^{\ell} (J_1^0)^m (J_1^-)^n | G_1, m \rangle. \end{aligned} \quad (5.40)$$

By using (5.36) we can easily see $\langle \mathbf{n}, \mathbf{A}; j | J_n^A | G_1, m \rangle = 0$ unless $\langle \mathbf{n}, \mathbf{A}; j | = \langle j | (J_0^+)^{\ell} (J_1^0)^m (J_1^-)^n$ for $J_n^A = J_0^+, J_1^0, J_1^-$. Then, if $\langle j | (J_0^+)^{\ell} (J_1^0)^m (J_1^-)^n J_n^A | G_1, m \rangle = C \langle j | (J_0^+)^{\ell} (J_1^0)^m (J_1^-)^n | G_1, m \rangle$ is satisfied, the Whittaker state is precisely the eigenstate of J_n^A :

$$\begin{aligned} J_n^A |G_1, m\rangle &= \sum |\mathbf{n}', \mathbf{A}'; j\rangle Q_j^{-1}(\mathbf{n}', \mathbf{A}'; \mathbf{n}, \mathbf{A}) \langle \mathbf{n}, \mathbf{A}; j | J_n^A | G_1, m \rangle \\ &= C \sum |\mathbf{n}', \mathbf{A}'; j\rangle Q_j^{-1}(\mathbf{n}', \mathbf{A}'; \mathbf{n}, \mathbf{A}) \langle \mathbf{n}, \mathbf{A}; j | G_1, m \rangle \\ &= C |G_1, m\rangle. \end{aligned} \quad (5.41)$$

This idea also works for the case when C is a differential operator acting on $|G_1, m\rangle$. Hence, the relations (5.38)–(5.40) leads to the following relations;

$$J_0^+ |G_1, m\rangle = \left[x \frac{\partial}{\partial(\sqrt{x})} + \sqrt{x} \left(1 + \frac{m-a}{\epsilon_1} \right) \right] |G_1, m\rangle, \quad (5.42)$$

$$J_1^0 |G_1, m\rangle = \frac{\sqrt{z}}{2\epsilon_1} |G_1, m\rangle, \quad (5.43)$$

$$J_1^- |G_1, m\rangle = \frac{\sqrt{z}}{\epsilon_1 \sqrt{x}} |G_1, m\rangle. \quad (5.44)$$

Compared with the pure Yang-Mills case,¹² the crucial differences are a non-vanishing eigenvalue of J_1^0 and the differential term in J_0^+ . In fact they are not unrelated, because $J_1^0 \sim 1/2[J_0^+, J_1^-]$. A similar condition involving differential operators for the Viroso CFT appeared recently in [24]. Note that our Whittaker-like condition takes a very special form of generalization involving the zero mode of J^0 . The Euler differential with the Coulomb moduli dependent term in (5.42) act as J_0^0

$$J_0^0 |G_1, m\rangle = \left(j - \sqrt{x} \frac{\partial}{\partial \sqrt{x}} \right) |G_1, m\rangle, \quad (5.45)$$

since the eigenvalue of J_0^0 on a descendant is $j + \sum A_i$. We can therefore rewrite the first condition (5.42) into the following form

$$J_0^+ |G_1, m\rangle = \sqrt{x} \left(\frac{1}{2} + \frac{m}{\epsilon_1} - J_0^0 \right) |G_1, m\rangle. \quad (5.46)$$

In the right hand side, we have the zero-mode contribution J_0^0 as the case of the W_3 algebra, and therefore $|G_1, m\rangle$ is also of our generalized type introduced in section 3. The appearance of the zero-mode enables us to impose the eigenstate condition of J_1^0 in addition to $J_1^- |G_1, m\rangle \propto |G_1, m\rangle$:

$$J_1^0 |G_1, m\rangle = \frac{1}{2} [J_0^+, J_1^-] |G_1, m\rangle = -\frac{\sqrt{x}}{2} [J_0^0, J_1^-] |G_1, m\rangle = \frac{\sqrt{x}}{2} J_1^- |G_1, m\rangle. \quad (5.47)$$

This is the very same phenomenon that we saw in section 3, where L_0 makes it possible for the state to be a simultaneous eigenstate of some annihilation operators. These examples suggest that we have to introduce the generalized Whittaker states of conformal algebras for studying AGT-W correspondence in more generic set-up with matter fields.

We can check the level zero part of the Whittaker state (5.33) satisfies (5.42) explicitly. From

$$J_0^+ (J_0^-)^n |j\rangle = n(2j - n + 1) (J_0^-)^{n-1} |j\rangle = n \left(\frac{2a}{\epsilon_1} - n \right) |j\rangle \quad (5.48)$$

¹²Note that the rescaling $\sqrt{z} \rightarrow \sqrt{z}/\epsilon_1$ makes the parameter z dimensionless.

we obtain

$$\begin{aligned}
 J_0^+ |G_1, m\rangle_0 &= \sqrt{x} \sum_{n=1}^{\infty} \frac{(-\sqrt{x})^{n-1}}{(n-1)!} \frac{\left(1 + \frac{m-a}{\epsilon_1}\right)_n}{\left(1 - \frac{2a}{\epsilon_1}\right)_{n-1}} (J_0^-)^{n-1} |j\rangle \\
 &= \sqrt{x} \left(1 + \frac{m-a}{\epsilon_1}\right) \sum_{n=1}^{\infty} \frac{(-\sqrt{x})^{n-1}}{(n-1)!} \frac{\left(1 + \frac{m-a}{\epsilon_1}\right)_{n-1}}{\left(1 - \frac{2a}{\epsilon_1}\right)_{n-1}} (J_0^-)^{n-1} |j\rangle \\
 &\quad + \sqrt{x} \sum_{n=2}^{\infty} \frac{(-\sqrt{x})^{n-1}}{(n-2)!} \frac{\left(1 + \frac{m-a}{\epsilon_1}\right)_{n-1}}{\left(1 - \frac{2a}{\epsilon_1}\right)_{n-1}} (J_0^-)^{n-1} |j\rangle.
 \end{aligned} \tag{5.49}$$

On the other hand

$$\begin{aligned}
 x \frac{\partial}{\partial \sqrt{x}} |G_1, m\rangle &= -x \sum_{n=1}^{\infty} \frac{(-\sqrt{x})^{n-1}}{(n-1)!} \frac{\left(1 + \frac{m-a}{\epsilon_1}\right)_n}{\left(1 - \frac{2a}{\epsilon_1}\right)_n} (J_0^-)^n |j\rangle \\
 &= \sqrt{x} \sum_{n=1}^{\infty} \frac{(-\sqrt{x})^n}{(n-1)!} \frac{\left(1 + \frac{m-a}{\epsilon_1}\right)_n}{\left(1 - \frac{2a}{\epsilon_1}\right)_n} (J_0^-)^n |j\rangle.
 \end{aligned} \tag{5.50}$$

This shows that (5.33) satisfies (5.42).

Suppose we have the following states at level one;¹³

$$|\Psi_0\rangle_1 = \sum_{n=-1}^{\infty} a_n \sum_{k=1}^3 Q_{1k}^{-1} |k; n, j\rangle + \sum_{n=0}^{\infty} \tilde{a}_n \sum_{k=1}^3 Q_{2k}^{-1} |k; n, j\rangle, \tag{5.51}$$

$$|\Psi_1\rangle_1 = \sum_{n=-1}^{\infty} b_n \sum_{k=1}^3 Q_{1k}^{-1} |k; n, j\rangle + \sum_{n=0}^{\infty} \tilde{b}_n \sum_{k=1}^3 Q_{2k}^{-1} |k; n, j\rangle, \tag{5.52}$$

then the scalar product of them is

$$\langle \Psi_0 | \Psi_1 \rangle_1 = \sum_{n=-1}^{\infty} a_n b_n Q_{11}^{-1} + \sum_{n=0}^{\infty} (a_n \tilde{b}_n + b_n \tilde{a}_n) Q_{12}^{-1} + \sum_{n=0}^{\infty} \tilde{a}_n \tilde{b}_n Q_{22}^{-1}. \tag{5.53}$$

For pure Yang-Mills, we have $a_n = \left(\frac{\sqrt{z}}{\epsilon_1}\right) (\sqrt{x})^n$, $\tilde{a}_n = 0$ and for $N_f = 1$ we take

$$b_n = \left(\frac{\sqrt{z}}{\epsilon_1}\right) (\sqrt{x})^n \left(1 + \frac{m-a}{\epsilon_1}\right)_{n+1}, \quad \tilde{b}_n = \left(\frac{\sqrt{z}}{\epsilon_1}\right) \frac{(\sqrt{x})^n}{2} \left(1 + \frac{m-a}{\epsilon_1}\right)_n. \tag{5.54}$$

We expect one instanton part of the partition function agrees with

$$\begin{aligned}
 \langle G_0 | G_1, m \rangle_1 &= \frac{z}{\epsilon_1^2} \left[\sum_{n=-1}^{\infty} x^n \left(1 + \frac{m-a}{\epsilon_1}\right)_{n+1} Q_{11}^{-1} + \frac{1}{2} \sum_{n=0}^{\infty} x^n \left(1 + \frac{m-a}{\epsilon_1}\right)_n Q_{12}^{-1} \right] \\
 &= \frac{z}{x \epsilon_1^2} \frac{1}{k-2j} + \frac{z}{\epsilon_1^2} \frac{k \left(1 + \frac{m-a}{\epsilon_1}\right) - 2j}{2j(k+2)(k-2j)} + O(x).
 \end{aligned} \tag{5.55}$$

¹³See appendix D for the notations of a basis of level one states. We have suppressed here the dependence of Q_{ij}^{-1} on the monopole number n .

Using the “dictionary” (5.6) for k and j , we see

$$\langle G_0|G_1, m\rangle_1 = \frac{z}{-x\epsilon_1} \frac{1}{2a + \epsilon_1 + \epsilon_2} + \frac{z}{-\epsilon_1} \frac{m(2\epsilon_1 + \epsilon_2) - a\epsilon_2 + \epsilon_1^2 + \epsilon_1\epsilon_2}{\epsilon_2(2a - \epsilon_1)(2a + \epsilon_1 + \epsilon_2)} + O(x). \quad (5.56)$$

After a redefinition

$$a \rightarrow -a - \frac{\epsilon_2}{2}, \quad m \rightarrow -m + \frac{\epsilon_2}{2}, \quad (5.57)$$

we find

$$\langle G_0|G_1, m\rangle_1 = \frac{z}{x\epsilon_1} \frac{1}{2a - \epsilon_1} - \frac{z}{\epsilon_1} \frac{-m(2\epsilon_1 + \epsilon_2) + a\epsilon_2 + (\epsilon_1 + \epsilon_2)^2}{\epsilon_2(2a - \epsilon_1)(2a + \epsilon_1 + \epsilon_2)} + O(x). \quad (5.58)$$

which agrees with the computation in [15], where we keep $2M_2 \equiv m$ finite in the decoupling limit. We summarize the result of this decoupling limit in appendix E.

For general $n \geq 1$ the coefficient of x^n is

$$\left(1 + \frac{m - a}{\epsilon_1}\right)_n \left[\left(n + 1 + \frac{m - a}{\epsilon_1}\right) Q_{11}^{-1}(n) + \frac{Q_{12}^{-1}(n)}{2} \right], \quad (5.59)$$

where the components $Q_{ij}^{-1}(n)$ of the inverse of the level one Shapovalov matrix are given in appendix D. After the redefinition (5.57) we can check an exact agreement of (5.59) with the result of the same decoupling limit in appendix E.

6 Conclusion and discussion

For $\mathcal{N} = 2$ superconformal gauge theories in four dimensions, the AGT-W conjecture claims the equivalence between the instanton partition functions and the conformal blocks of corresponding CFT. Pure Yang-Mills theory also has a natural formulation in terms of CFT: the partition functions are given by the norms of certain coherent states of the conformal algebra of the CFT. Such a special state in the Verma module is called the Whittaker state. Between these two extreme cases, there exists a series of the asymptotically free gauge theories with hypermultiplets. So far, the CFT formulation of these series was studied only for $SU(2)$ gauge theory without a surface operator. We therefore tried to make the generic feature of the 4d/2d correspondence clear by studying asymptotically free gauge theories with fundamental flavors.

Our strategy we adopted in this paper is as follows; we start with assuming the AGT-W relation for superconformal cases $N_f = 2N_c$. Then the decoupling limit leads to irregular conformal blocks which is expected to be equal to the instanton partition functions of asymptotically free gauge theories. We regard this conformal block as the norm of a certain state in the Verma module. For pure Yang-Mills theories, this is just the Gaiotto-Whittaker state and our results provide a generalization to the case with fundamental matter. We have worked out the conditions that our state satisfies and introduced the notion of the generalized Whittaker states. Namely for the $SU(3)$ gauge theory with $N_f = 2$, we found a little non-familiar relation:

$$(W_1 - wi\Lambda L_0)|G_2, m_1, m_2\rangle = \frac{3wi\Lambda}{2} (2m_1m_1 - Q(m_1 + m_2) + Q^2)|G_2, m_1, m_2\rangle.$$

Similarly, for $SU(2)$ gauge theory with a surface operator, $N_f = 1$ flavor is encoded into the state $|G_1, m\rangle$ that is characterized by the conditions with the zero mode J_0^0 of the $\mathfrak{sl}(2)$ current algebra. These two examples suggest the generality of the existence of the generalized Whittaker states in CFT description for gauge theories with flavors.

In section 5, we saw that the scalar product of the generalized Whittaker states $\langle G_0|G_1\rangle$ reproduces the instanton partition function for $SU(2)$ theory with $N_f = 1$ flavor. The irregular conformal block $\langle G_0|G_0\rangle$ also gives that for pure Yang-Mills. Note that an explicit combinatorial formula for the same partition function $Z_{SU(2)}^{(S), N_f=0}$ of the Yang-Mills theory was proposed in [34]. Though this combinatorial formula is essentially based on another description (5.2) of surface operators which realizes them as the degenerates field of Virasoro CFT, we can see in the lower order of the instanton expansion this approach implies the same answer as the affine Whittaker state gives. It would be interesting to study such an equivalence between affine Whittaker states and Virasoro Gaiotto states with $\Phi_{1,2}$ for the theory with flavors and extend the combinatorial formula to these cases.

We may be able to define the generalized Whittaker states for more generic chiral algebra of CFT such as W -algebra. As argued in section 4 the M-theoretic construction of $SU(N)$ theory suggests that the Whittaker-like state for $N_f = N - 1$ flavors would be of the generalized type. Let us consider a simultaneous eigenstate for W_N algebra: $L_1|G\rangle = \ell_1|G\rangle$, $L_2|G\rangle = \ell_2|G\rangle$. Since the spin- s current satisfies $[L_n, W_m^{(s)}] = ((s - 1)n - m)W_{n+m}^{(s)}$, we can impose the following conditions:

$$W_1^{(s)}|G\rangle = (w_1 + w_0 L_0)|G\rangle \implies W_2^{(s)}|G\rangle = \frac{\ell_1 w_0}{s - 2}|G\rangle, \quad W_3^{(s)}|G\rangle = \frac{2\ell_2 w_0}{2s - 3}|G\rangle. \quad (6.1)$$

Actually, all conditions must be consistent with the commutation relations of modes of currents, which are nonlinear and very complicated. It will be very cumbersome to write down the full series of the consistent conditions, but we expect it is doable in principle. They will provide the irregular conformal blocks which coincide with instanton partition functions for more generic $SU(N)$ gauge theories. We also expect such generalized Whittaker states exist for BCDEFG gauge groups [23] and it is an interesting challenge to carry out these extensions.

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A Building block of the Nekrasov partition function

The instanton partition function introduced by Nekrasov [7] is an integral of certain volume form over the instanton moduli space. In the strict sense this is defined as an equivariant

integral, and we can evaluate it with the localization formula. The resulting generic formula for $SU(N)^p$ linear quiver gauge theory takes the form¹⁴

$$\begin{aligned}
 Z_{SU(N)^p, \text{lin. quiver}} = & \sum_{\vec{Y}_{1,2,\dots,p}} \prod_{i=1}^p \Lambda_i^{(2N-N_{f,i})|\vec{Y}_i|} \prod_{\vec{f}=1}^{\vec{F}} z_{\text{antifun}}(\vec{a}_1, \vec{Y}_1 : \tilde{m}_{\vec{f}}) z_{\text{vec}}(\vec{a}_1, \vec{Y}_1) \quad (\text{A.1}) \\
 & \times z_{\text{bif}}(\vec{a}_1, \vec{Y}_1; \vec{a}_2, \vec{Y}_2; \mu_1) z_{\text{vec}}(\vec{a}_2, \vec{Y}_2) z_{\text{bif}}(\vec{a}_2, \vec{Y}_2; \vec{a}_3, \vec{Y}_3; \mu_2) z_{\text{vec}}(\vec{a}_3, \vec{Y}_3) \\
 & \times \cdots z_{\text{bif}}(\vec{a}_{p-1}, \vec{Y}_{p-1}; \vec{a}_p, \vec{Y}_p; \mu_{p-1}) z_{\text{vec}}(\vec{a}_p, \vec{Y}_p) \prod_{f=1}^F z_{\text{fun}}(\vec{a}_p, \vec{Y}_p : m_f).
 \end{aligned}$$

The weight factor z_* , which is a combinatorial rational function of gauge theory parameters, represents the contribution from the vector or hyper multiplet labeled by $*$. Since Λ_i is the dynamical scale for i -th gauge group, $|\vec{Y}_i|$ is the instanton number of the gauge group factor. The k_i -instanton partition function is therefore defined by summing the Young diagrams with fixing the number of boxes as $|\vec{Y}_i| = k_i$. We use the following weight factor in the localization formula of the Nekrasov function;

The contribution of a bifundamental matter multiplet is

$$\begin{aligned}
 z_{\text{bif}}(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; m) = & \prod_{\alpha, \beta=1}^N \prod_{(i,j) \in Y_\alpha} (a_\alpha - b_\beta - m + \epsilon_1(-W_{\beta,j}^t + i) + \epsilon_2(Y_{\alpha,i} - j + 1)) \\
 & \times \prod_{(i,j) \in W_\beta} (a_\alpha - b_\beta - m + \epsilon_1(Y_{\alpha,j}^t - i + 1) + \epsilon_2(-W_{\beta,i} + j)), \quad (\text{A.2})
 \end{aligned}$$

The contributions of an adjoint matter and a vector matter multiplet are related to $z_{\text{bif}}(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; m)$ by

$$z_{\text{adj}}(\vec{a}, \vec{Y}; m) = z_{\text{bif}}(\vec{a}, \vec{Y}; \vec{a}, \vec{Y}; m), \quad z_{\text{vec}}(\vec{a}, \vec{Y}) = \frac{1}{z_{\text{adj}}(\vec{a}, \vec{Y}; 0)}. \quad (\text{A.3})$$

Finally a fundamental matter and an anti-fundamental matter contribute

$$z_{\text{fun}}(\vec{a}, \vec{Y} : m) = \prod_{\alpha=1}^N \prod_{(i,j) \in Y_\alpha} (a_\alpha - m + \epsilon_1 i + \epsilon_2 j), \quad z_{\text{anti}}(\vec{a}, \vec{Y} : m) = z_{\text{fun}}(\vec{a}, \vec{Y} : -m + \epsilon_1 + \epsilon_2). \quad (\text{A.4})$$

B Explicit 1-instanton check of Wyllard conjecture

In section 3, we apply the decoupling limit to the derivation of the Whittaler states. The decoupling limit is also useful to simplify the check of Wyllard's proposal because the original set-up is too complicated to verify by explicit computation.

¹⁴ $N_{f,i}$ is the flavor number for i -th gauge group. For conformal gauge group $2N = N_{f,i}$, we introduce the gauge coupling constant in compensation for vanishing dynamical scale as $\Lambda_i^{2N-N_{f,i}} \rightarrow q_i = e^{2\pi i \tau_i}$.

At one instanton level, the Wylalrd conjecture suggests the relation between the 1-instanton partition function $Z_{k=1}$ for the SU(3) superconformal SQCD and the level-1 W_3 conformal block \mathcal{B}_1

$$\mathcal{B}_1^{N_f=6} = Z_{\text{SU}(3),k=1}^{N_f=6} + \nu, \quad (\text{B.1})$$

for the following U(1) factor [9]

$$Z_{\text{U}(1)} = (1 - q)^{-\nu}, \quad \nu = \frac{1}{4}(\sqrt{3}Q - 2\alpha_1)(\sqrt{3}Q + 2\alpha_3). \quad (\text{B.2})$$

In the 2-flavor limit

$$\mu_1\mu_4\mu_5\mu_6q \rightarrow \mu_1\mu_4\mu_5\Lambda_{N_f=5} \rightarrow \mu_1\mu_4\Lambda_{N_f=4}^2 \rightarrow \mu_1\Lambda_{N_f=3}^3 \rightarrow \Lambda_{N_f=2}^4, \quad (\text{B.3})$$

the U(1) factor disappears, and we expect the following relation for the 1-instanton sector of SQCD with two anti-fundamental flavors

$$\lim_{2\text{-flavor}} \mathcal{B}_1^{N_f=6} = Z_{\text{SU}(3),k=1}^{N_f=2}. \quad (\text{B.4})$$

This relation must hold if the Wyllard conjecture is true. Let us prove it.

B.1 1-instanton Nekrasov partition function

The one-instanton partition function of the SU(3) gauge theory with two anti-fundamentals is

$$Z_{\text{SU}(3),k=1}^{N_f=2}(\epsilon_{1,2}) = \Lambda_{N_f=2}^4 \sum_{i=1}^3 \frac{(a_i + \mu_1)(a_i + \mu_2)}{\prod_{j \neq i} a_{ij}(a_{ij} + \epsilon_+)}, \quad (\text{B.5})$$

where $a_{ij} = a_i - a_j$ for $i, j = 1, 2, 3$. For the self-dual Ω -background $\epsilon_+(= Q) = 0$, the partitin function becomes very simple

$$\begin{aligned} Z_{\text{SU}(3),k=1}^{N_f=2}(\epsilon_+ = 0) & \\ &= \frac{\Lambda_{N_f=2}^4}{(a_{12}a_{23}a_{31})^2} \left[(a_1 + a_2)^4 + a_1^4 + a_2^4 - 9(\mu_2 + \mu_3)a_1a_2a_3 + 6\mu_2\mu_3(a_1^2 + a_1a_2 + a_2^2) \right]. \end{aligned} \quad (\text{B.6})$$

We can easily check that this result recovered from the CFT side only by hand.

B.2 Level-1 irregular conformal block

The level-1 conformal block is

$$\begin{aligned} \mathcal{B}_1^{N_f=6} &= q(\Delta + \Delta_1 - \Delta_2, w + 2w_1 - w_2 + \frac{3w_1}{2\Delta_1}(\Delta - \Delta_1 - \Delta_2)) (Q_{\Delta(\vec{\alpha})}^{(1)})^{-1} \\ &\quad \times \left(\begin{matrix} \Delta + \Delta_3 - \Delta_4 \\ w + w_3 + w_4 - \frac{3w_3}{2\Delta_3}(\Delta + \Delta_3 - \Delta_4) \end{matrix} \right). \end{aligned} \quad (\text{B.7})$$

Let us take the 3-flavor limit $\mu_{4,5,6} \rightarrow \infty$ first. By using the result in [10], we get

$$\mathcal{B}_1^{N_f=3} = \Lambda_{N_f=3}^3 \left(\Delta + \Delta_1 - \Delta_2, w + 2w_1 - w_2 + \frac{3w_1}{2\Delta_1}(\Delta - \Delta_1 - \Delta_2) \right) \times (Q_{\Delta(\vec{\alpha})}^{(1)})^{-1} \begin{pmatrix} 0 \\ \frac{3\sqrt{3}}{\sqrt{4-15Q^2}} \end{pmatrix}. \quad (\text{B.8})$$

Then by applying the 2-flavor limit $\lim \mu_1 \Lambda_{N_f=3}^3 = -\lim \sqrt{3} A \Lambda_{N_f=3}^3 = \Lambda_{N_f=2}^4$, we get

$$\begin{aligned} \mathcal{B}_1^{N_f=2} &= \frac{-3}{\sqrt{4-15Q^2}} \Lambda_{N_f=2}^4 \frac{1}{\det Q_{\Delta(\vec{\alpha})}^{(1)}} \\ &\times \left(2C, \frac{9\sqrt{k}}{4}(2\mu_2\mu_3 - Q(\mu_2 + \mu_3) + Q^2 + \frac{2}{3}(a_1^2 + a_1a_2 + a_2^2 - Q^2)) \right) \begin{pmatrix} -3w \\ 2\Delta \end{pmatrix} \\ &= \Lambda_{N_f=2}^4 \frac{1}{(a_{12} + Q)(a_{12} - Q)(a_{23} + Q)(a_{23} - Q)(a_{31} + Q)(a_{31} - Q)} \\ &\times \left[(6\mu_2\mu_3 - 3Q(\mu_2 + \mu_3) + 3Q^2 + 2(a_1^2 + a_1a_2 + a_2^2 - Q^2))(a_1^2 + a_1a_2 + a_2^2 - Q^2) \right. \\ &\quad \left. + 9(\mu_2 + \mu_3 - Q)a_1a_2(a_1 + a_2) \right]. \end{aligned} \quad (\text{B.9})$$

Of course this result is consistent with the generic formula we get in section 3. By setting $Q = 0$ we recover the gauge theory result (B.6). We can also check the generic case $Q \neq 0$.

C Conditions for $|G_2, m_1, m_2\rangle$

The following arguments are quite parallel to those in section 5.3. The point is that the formula (3.26) in section 3 implies the following non-vanishing inner product

$$\begin{aligned} &\langle \Delta(\vec{\alpha}) | W_1^t W_2^q W_3^p L_1^s L_2^r | G_2, m_1, m_2 \rangle \\ &= (i\Lambda)^{3p+2q+2r+s+t} \frac{2^p (\sqrt{3})^{-3p+q-2r+t}}{(\sqrt{4-15Q^2})^{p+q+t}} (q_L(m_1, m_2))^{s+q} \left(\frac{3}{2} q_W(m_1, m_2) \right)_t, \end{aligned} \quad (\text{C.1})$$

where

$$q_L(m_1, m_2) := Q - m_1 - m_2, \quad (\text{C.2})$$

$$q_W(m_1, m_2) := 2m_1m_2 - Q(m_1 + m_2) + Q^2 + \frac{2}{3}(a_1^2 + a_1a_2 + a_2^2 - Q^2), \quad (\text{C.3})$$

and $|G_2, m_1, m_2\rangle$ is orthogonal to other states in the Verma module. By this orthogonality and the commutation relation of W_3 algebra, we can compute the result of the insertion of W_2 as follows;

$$\begin{aligned} &\langle \Delta(\vec{\alpha}) | W_1^t W_2^q W_3^p L_1^s L_2^r W_2 | G_2, m_1, m_2 \rangle = \langle \Delta(\vec{\alpha}) | W_1^t W_2^{q+1} W_3^p L_1^s L_2^r | G_2, m_1, m_2 \rangle \\ &= \frac{\sqrt{3}(i\Lambda)^2}{\sqrt{4-15Q^2}} q_L(m_1, m_2) \langle \Delta(\vec{\alpha}) | W_1^t W_2^q W_3^p L_1^s L_2^r | G_2, m_1, m_2 \rangle, \end{aligned} \quad (\text{C.4})$$

where the commutation relation $[L_1, W_2] = (2 \cdot 1 - 2)W_{1+2} = 0$ is crucial. We can read the eigenvalue of W_2 on the state $|G_2, m_1, m_2\rangle$ from this relation. Similarly we compute $L_{1,2}$ and W_3 insertions to obtain their eigenvalues. The computation of W_1 insertion is special in the sense that the contributions from the commutation relations survive;

$$\begin{aligned} & \langle \Delta(\vec{\alpha}) | W_1^t W_2^q W_3^p L_1^s L_2^r W_1 | G_2, m_1, m_2 \rangle \\ &= \frac{\sqrt{3}i\Lambda}{\sqrt{4-15Q^2}} \left(\frac{3}{2} q_W(m_1, m_2) + t \right) \langle \Delta(\vec{\alpha}) | W_1^t W_2^q W_3^p L_1^s L_2^r | G_2, m_1, m_2 \rangle + \dots \end{aligned} \quad (\text{C.5})$$

where the additional contributions \dots come from the commutation relations

$$[L_2, W_1] = 3W_3, \quad [L_1, W_1] = W_2, \quad [L_1, [L_1, W_1]] = 0, \quad (\text{C.6})$$

$$[W_3, W_1] \sim \frac{2 \cdot 9}{4-15Q^2} (L_2)^2, \quad [W_2, W_1] \sim \frac{9}{4-15Q^2} 2L_1 L_2. \quad (\text{C.7})$$

Note that the t -dependent term in (C.5) comes from the Pochhammer product in (C.1). Using (C.1), we can check the contribution from the commutation relation $[X_n, W_1]$ for $X = L, W$ is always proportional to $\langle \Delta(\vec{\alpha}) | W_1^t W_2^q W_3^p L_1^s L_2^r | G_2, m_1, m_2 \rangle$ and the coefficient is $\frac{n\sqrt{3}i\Lambda}{\sqrt{4-15Q^2}}$ times the number of X_n between $|\Delta(\vec{\alpha})\rangle$ and $|G_2, m_1, m_2\rangle$. For example, the contribution from $[W_2, W_1]$ is

$$\begin{aligned} & \frac{18q}{4-15Q^2} \langle \Delta(\vec{\alpha}) | W_1^t W_2^{q-1} W_3^p L_1^{s+1} L_2^{r+1} | G_2, m_1, m_2 \rangle \\ &= \frac{2q\sqrt{3}i\Lambda}{\sqrt{4-15Q^2}} \langle \Delta(\vec{\alpha}) | W_1^t W_2^q W_3^p L_1^s L_2^r | G_2, m_1, m_2 \rangle. \end{aligned} \quad (\text{C.8})$$

Hence, combined with the t dependent term in (C.5), the sum of the additional contributions is neatly expressed by the action of the Euler derivative

$$\frac{\sqrt{3}i\Lambda}{\sqrt{4-15Q^2}} \Lambda \frac{\partial}{\partial \Lambda}, \quad (\text{C.9})$$

that counts the level of the state in the Verma module. We further note that the last term of $q_W(m_1, m_2)$ is proportional to the eigenvalue of L_0 on the primary state. Together with this part the Euler derivative gives the action of the Virasoro zero mode L_0 .

D Level one Shapovalov matrix of $SU(2)$ current algebra

We summarize the data of the Shapovalov matrix at level one which corresponds to one instanton sector with monopole number $\mathbf{m} = n - 1 \geq -1$. At level one, we have three states with spin $1 - n + j$,

$$|1\rangle = J_{-1}^+(J_0^-)^n |j\rangle, \quad |2\rangle = J_{-1}^0 (J_0^-)^{n-1} |j\rangle, \quad |3\rangle = J_{-1}^-(J_0^-)^{n-2} |j\rangle. \quad (\text{D.1})$$

Using commutation relations, we obtain the Shapovalov matrix. When $n \geq 2$ the Shapovalov matrix is

$$Q(n) = \begin{pmatrix} (k-2j+2n)M(n) & M(n) & 0 \\ M(n) & \frac{k}{2}M(n-1) & -M(n-1) \\ 0 & -M(n-1) & (k+2j-2(n-2))M(n-2) \end{pmatrix}, \quad (\text{D.2})$$

with

$$M(n) := n!(-1)^n(-2j)_n, \tag{D.3}$$

where $(X)_n := X(X+1)\cdots(X+n-1)$. The determinant of $Q(n)$ factorizes as follows;

$$\det Q(n) = \frac{1}{2}(k+2)(2j+k+2)(k-2j)M(n-2)M(n-1)M(n). \tag{D.4}$$

When $n = 0, 1$ the Shapovalov matrix is reduced to one by one, or two by two matrix.

$$Q(0) = (k-2j), \quad Q(1) = \begin{pmatrix} 2j(k-2j+2) & 2j \\ & 2j & \frac{k}{2} \end{pmatrix}. \tag{D.5}$$

The inverse of the Shapovalov matrix for $n = 0, 1$ is simple. The components of the inverse of the Shapovalov matrix for $n \geq 2$ are

$$Q(n)_{11}^{-1} = \frac{2n^2 - 2(k+3+2j)n + 2j(k+2) + (k+2)^2}{(k+2)(2j+2+k)(k-2j)M(n)}, \tag{D.6}$$

$$Q(n)_{12}^{-1} = \frac{-2(k+2j-2n+4)}{(k+2)(2j+2+k)(k-2j)M(n-1)}, \tag{D.7}$$

$$Q(n)_{13}^{-1} = \frac{-2}{(k+2)(2j+k+2)(k-2j)M(n-2)}, \tag{D.8}$$

$$Q(n)_{22}^{-1} = \frac{-2(-k+2j-2n)(k+2j-2n+4)}{(k+2)(2j+k+2)(k-2j)M(n-1)}, \tag{D.9}$$

$$Q(n)_{23}^{-1} = \frac{-2(-k+2j-2n)}{(k+2)(2j+k+2)(k-2j)M(n-2)}, \tag{D.10}$$

$$Q(n)_{33}^{-1} = \frac{2n^2 - 2(1-k+2j)n - 2kj + k^2}{(k+2)(2j+k+2)(k-2j)M(n-2)}. \tag{D.11}$$

E Partition function of SU(2) theory with a surface operator

We here quote the result presented in section 8 of [15], where we computed the partition function of superconformal ($N_f = 4$) SU(2) theory with a surface operator at one instanton and arbitrary monopole number by localization. As is common in SU(2) theories the fixed points of the torus action on the moduli space are labeled by a pair of Young diagrams (partitions) $\vec{\lambda} = (\lambda_1, \lambda_2)$. We identify the instanton number k and the monopole number \mathfrak{m} at each fixed point by:

$$k = k_1, \quad \mathfrak{m} = k_2 - k_1, \tag{E.1}$$

where k_1 and k_2 are given by

$$k_1(\vec{\lambda}) = \sum_{n \geq 1} \lambda_{1,2n-1} + \sum_{n \geq 1} \lambda_{2,2n}, \quad k_2(\vec{\lambda}) = \sum_{n \geq 1} \lambda_{1,2n} + \sum_{n \geq 1} \lambda_{2,2n-1}. \tag{E.2}$$

Thus in one instanton sector with $k_1 = 1$, $k_2 = \mathfrak{m} + 1$, there are four choices for $\vec{\lambda}$ as follows:

$$\begin{aligned} (A) \quad \vec{\lambda}_{\mathfrak{m}A} &= ((1), (\mathfrak{m}+1)), & \mathfrak{m} &\geq -1, & (B) \quad \vec{\lambda}_{\mathfrak{m}B} &= (\emptyset, (\mathfrak{m}+1, 1)), & \mathfrak{m} &\geq 0, \\ (C) \quad \vec{\lambda}_{\mathfrak{m}C} &= (\emptyset, (\mathfrak{m}, 1, 1)), & \mathfrak{m} &\geq 1, & (D) \quad \vec{\lambda}_{\mathfrak{m}D} &= ((1, 1), (\mathfrak{m})), & \mathfrak{m} &\geq 0. \end{aligned} \tag{E.3}$$

Taking a decoupling limit

$$M_1, M_3, M_4 \rightarrow \infty, \quad \tilde{z} = 4M_1M_3x, \quad \tilde{x} = -2M_4y, \quad (\text{E.4})$$

in the result of section 8 of [15], one obtains the following contributions to the partition function for $N_f = 1$ theory at one instanton

$$\begin{aligned} & Z_1^{(A)}(a, M, \epsilon_1, \epsilon_2; \tilde{z}, \tilde{x}) \\ &= \sum_{\mathbf{m}=-1}^{\infty} \tilde{z} \tilde{x}^{\mathbf{m}+1} \frac{-\prod_{k=1}^{\mathbf{m}+1} (a - 2M + k\epsilon_1 + \epsilon_2)}{(2a + \mathbf{m}\epsilon_1)\epsilon_1^{\mathbf{m}+2}(\mathbf{m} + 1)! \prod_{k=0}^{\mathbf{m}} (2a + k\epsilon_1 + \epsilon_2)}, \\ & Z_1^{(B)}(a, M, \epsilon_1, \epsilon_2; \tilde{z}, \tilde{x}) \\ &= \sum_{\mathbf{m}=0}^{\infty} \tilde{z} \tilde{x}^{\mathbf{m}+1} \frac{\prod_{k=1}^{\mathbf{m}+1} (a - 2M + k\epsilon_1 + \epsilon_2)}{(\mathbf{m}\epsilon_1 - \epsilon_2)\epsilon_1^{\mathbf{m}+1}\mathbf{m}! \prod_{k=0}^{\mathbf{m}+1} (2a + k\epsilon_1 + \epsilon_2)}, \\ & Z_1^{(C)}(a, M, \epsilon_1, \epsilon_2; \tilde{z}, \tilde{x}) \\ &= \sum_{\mathbf{m}=1}^{\infty} \tilde{z} \tilde{x}^{\mathbf{m}+1} \frac{(a - 2M + \epsilon_1 + 2\epsilon_2) \prod_{k=1}^{\mathbf{m}} (a - 2M + k\epsilon_1 + \epsilon_2)}{(2a + \epsilon_1 + 2\epsilon_2)(-\mathbf{m}\epsilon_1 + \epsilon_2)\epsilon_2\epsilon_1^{\mathbf{m}}(\mathbf{m} - 1)! \prod_{k=0}^{\mathbf{m}} (2a + k\epsilon_1 + \epsilon_2)}, \\ & Z_1^{(D)}(a, M, \epsilon_1, \epsilon_2; \tilde{z}, \tilde{x}) \\ &= \sum_{\mathbf{m}=0}^{\infty} \tilde{z} \tilde{x}^{\mathbf{m}+1} \frac{(a + 2M - \epsilon_1 - \epsilon_2) \prod_{k=1}^{\mathbf{m}} (a - 2M + k\epsilon_1 + \epsilon_2)}{(2a - \epsilon_1)(2a + \mathbf{m}\epsilon_1)\epsilon_2\epsilon_1^{\mathbf{m}+1}\mathbf{m}! \prod_{k=0}^{\mathbf{m}-1} (2a + k\epsilon_1 + \epsilon_2)}. \end{aligned} \quad (\text{E.5})$$

The one instanton partition function $Z_{\text{SU}(2),k=1}^{(S),N_f=1}$ is the sum of the above contributions and a formal power series in x . Note that the number of the fixed points is reduced for lower monopole numbers $\mathbf{m} = -1, 0$.

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