On the scalar spectrum of the $Y^{p,q}$ manifolds

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Abstract: The spectra of supergravity modes in anti de Sitter (AdS) space on a five-sphere endowed with the round metric (which is the simplest 5d Sasaki-Einstein space) has been studied in detail in the past. However for the more general class of cohomogeneity one Sasaki-Einstein metrics on $S^2 \times S^3$, given by the $Y^{p,q}$ class, a complete study of the spectra has not been attempted. Earlier studies on scalar spectrum were restricted to only the first few eigenstates. In this paper we take a step in this direction by analysing the full scalar spectrum on these spaces. However it turns out that finding the exact solution of the corresponding eigenvalue problem in closed form is not feasible since the computation of the eigenvalues of the Laplacian boils down to the analysis of a one-dimensional operator of Heun type, whose spectrum cannot be computed in closed form. However, despite this analytical obstacle, we manage to get both lower and upper bounds on the eigenvalues of the scalar spectrum by comparing the eigenvalue problem with a simpler, solvable system. We also briefly touch upon various other new avenues such as non-commutative and dipole deformations as well as possible non-conformal extensions of these models.

Contents

1 Introduction

The gravity dual of $\mathcal{N} = 1$ CFT has been studied earlier from many different perspectives starting with [1] where the associated CFT, endowed with a simple product gauge group and a simple quartic superpotential, appeared from N D3-branes placed at the tip of a conifold geometry. One way to change the gauge group and the superpotential structure is to change the underlying conifold geometry itself by either an orbifolding or an orientifolding action. A subsequent T-duality, mapping these actions to either the *interval* [2, 3] or the *brane-box* models [4, 5], then gives us simple ways to analyse the underlying $\mathcal{N} = 1$ CFTs.

An alternative way to change the gauge group and the superpotential structure is to change the Calabi-Yau condition of the conifold itself, namely, change the Kähler class

and the complex structures so as to put different Ricci flat metrics on the conifold. Since there are infinite ways of doing it, there would exist infinite variations of the conifold that are all Calabi-Yau manifolds. All of these would lead to gravity duals of the form $AdS_5 \times Y^{p,q}$ where $Y^{p,q}$ are the so-called Sasaki-Einstein manifolds. These ideas, including the underlying gauge/gravity duality, were developed few years ago in [6–9].

In this paper we study spectrum of Sasaki-Einstein manifold $Y^{p,q}$, using spectral-theoretic methods, continuing the work of [10]. More precisely, we study the Laplacian operator of a $Y^{p,q}$ manifold, associated to its scalar spectrum, using the framework laid out in [10]. The authors of [10] analyzed the Cauchy problem, and presented a Fourier-type decomposition for the eigenfunction. In order to use spectral-theoretic methods, they used the Friedrichs extension of the Laplacian operator to rule out logarithmic singularities. This way a self-adjoint extension of unbounded symmetric operator could be determined. Our starting point, in this paper, is to use this operator to study its eigenmodes.

The *lowest* eigenmodes of the Laplacian were first studied in [11] for $Y^{p,q}$, wherein they also tried to construct an AdS/CFT dictionary. This work was followed by [12] where they studied the lowest eigenmodes for more generic manifolds like the $L^{a,b,c}$ examples. An important progress in [11] was the realization that the Laplacian operator could be expressed in terms of a Heun type operator, whose lowest modes are easily computable. However, for higher modes not much progress has been made in the literature. Even numerical studies do not look simple. In [13], the spectrum is studied numerically for S^5 case, which is the simplest Sasaki-Einstein manifold in 5d, but an equivalent work for the $Y^{p,q}$ case is still lacking.

In this paper, we will use mathematical tools developed in analysis and spectral theory, to address the question of finding all the eigenmodes. However, as it will turn out, finding the exact solution of the eigenvalue problem in closed form does not seem feasible since the computation of the eigenvalues of the Laplacian boils down to the analysis of a one-dimensional differential operator (of Heun type), which has *four* regular singular points. What we will do, therefore, is to find *bounds* for the eigenvalues of the operator, which will allow us to approximate the conformal dimensions of the dual CFTs. In our approach, we will get results in two different regimes, which match in their overlap. The first is for highly excited modes (by focussing on the leading terms in mode number k), and the second is for small a, where $0 < a < 1$ parametrizes $Y^{p,q}$ geometry by implicitly parametrizing (p,q). The parameter a is determined by p, q, and $a \ll 1$ is equivalent to $q \ll p$.

Our work uses some techniques in analysis and spectral theory that may not be too familiar to some readers in the physics community. Additionally, a more physical motivation to study Sasaki-Einstein manifolds is never spelled out in the literature, although detailed mathematical reviews exist. Therefore in the following sub-sections we will introduce two concepts to the reader. First will be the motivation to study Sasaki-Einstein manifolds in general; and the second will be the minimal mathematical background necessary to sketch the mathematical techniques used in this paper.

1.1 Motivation to study Sasaki-Einstein manifolds

Here we will introduce and motivate the study of Sasaki-Einstein manifolds, mostly summarizing the extensive reviews in [14, 15] in a language slightly more appropriate for

the physicists. A Sasaki-Einstein manifold is both Sasakian and Einstein, and it is an odd-dimensional cousin of Kähler-Einstein manifold, and sandwiched between two Kähler-Einstein manifolds of one dimension lower and higher respectively [14]. Einstein condition of Sasaki/Kähler manifold is inherited between lower and higher dimensions (see [16] for examples for $d = 4, 5, 6$).

Two mathematical facts about these manifolds are readily available: Cone over Sasaki-Einstein is Kähler-Einstein manifold with one dimension higher; and Kähler geometry is a symplectic geometry, while Sasakian is a contact geometry. Extra physical motivation comes from the fact that in Hamiltonian mechanics phase space with n momentum and n position forms a $2n$ dimensional symplectic manifold. By adding one more direction, i.e the time evolution, we get $2n + 1$ dimensional contact manifold. Contact geometry (therefore also Sasakian geometry) is just as important as symplectic geometry for physicists, as one can see in [17] for example. Extensive details on contact geometry are given in the handbook [18] for the enthusiastic readers to dwell upon.

Another property of these manifolds is associated to their Reeb vectors. If, for example, the Reeb vector fields have compact orbits forming circles and if the U(1) actions are free (not free, resp.), then the Sasakian manifolds are regular (quasi-regular, resp.). If, on the other hand, the orbits are non-compact, the Sasakian manifolds are irregular.

In contrast to the fact that there are abundant four and six-dimensional Kähler-Einstein manifolds, until recently there were only two known Sasaki-Einstein manifolds in 5d, namely the S^5 and $T^{1,1}$. Using the M-theory solution of [6], the authors of [7] found examples of new 5d Sasaki-Einstein metrics $Y^{p,q}$ on $S^2 \times S^3$. These examples contain both quasi-regular and regular cases, and the corresponding CFT duals have rational and irrational central charges respectively [8, 9, 19]. These Sasaki-Einstein metrics are also critical points of volume functional [14, 20]. Note that a bigger group of 5d Sasaki-Einstein manifolds namely the L^{abc} manifolds contain the $Y^{p,q}$ manifolds as a subset. The metric for this bigger class of manifolds were constructed from Kerr Black hole solutions by taking some scaling limits (see [21] for more details).

Finally, another motivation to study these manifolds will be to build brane constructions in flat space which are T-duals to the $Y^{p,q}$ geometries (much like the one for the $T^{1,1}$ case in [2, 3]). This will not only help us to analyze the corresponding gauge theories but will also provide new brane constructions in string theory.

1.2 A brief sketch of the mathematical techniques for the physicists

After having discussed the physical motivations to study the $Y^{p,q}$ manifolds, let us summarise the key mathematical concepts that we will be using throughout the paper, i.e the concept of unbounded operators and Friedrichs extensions. For completeness we will also give a brief discussion of the Sturm-Liouville theory.

1.2.1 Quantum mechanical observables and unbounded operators

As in many parts of quantum mechanics, unbounded self-adjoint operators will play an important role in this paper. For the benefit of the reader, we will briefly review some basic ideas that we will touch upon later. We recall that a linear operator L between two

normed vector spaces X and Y is *bounded*, or continuous, if the ratio of the norm of Lv to that of v remains bounded. We will be mainly interested in the case when $X = Y$ is a Hilbert space, usually some L^2 space. It is well known that linear operators between finite-dimensional vector spaces are always bounded.

A bounded linear operator is self-adjoint if and only if it is symmetric (i.e., Hermitian). For unbounded operator, this is not the case: there are examples of unbounded symmetric operators which are not self-adjoint, due to subtleties regarding the domain of the operator. Since self-adjointness (and *not* mere symmetry) is key for the validity of the spectral theorem, for the purposes of quantum mechanics it is often crucial to ensure that a given unbounded operator is self-adjoint with a given domain of definition. We remark as well that the domain of an unbounded operator can never be the whole Hilbert space, but only dense in it. (Incidentally, let us remark that observables in quantum mechanics, including the free Hamiltonian in $\mathbb{R}^d$, the Coulomb Hamiltonian, and the position, momentum and angular momentum operators, are unbounded, self-adjoint operators, and this was the motivation for von Neumann and M. Stone's original work in this area.)

The need to have bona fide self-adjoint operators leads to the theory of self-adjoint extensions. Given a symmetric operator densely defined in a Hilbert space, it does not necessarily admit a self-adjoint extension, and even when it does, this extension does not need to be unique, and deciding which extension is physically relevant is nontrivial. Fortunately for us, in this work all the self-adjoint extensions we shall need are of Friedrichs type, which is the preferred, time-honored way to define self-adjoint extensions of lower-bounded operators. For our purposes, it is enough to know that the Friedrichs extension is a standard procedure to derive a self-adjoint operator, densely defined in an L^2 space, from an operator L whose action in the set of test functions is lower bounded (that is, $\langle Lv, v \rangle \geqslant -C\|v\|^2$ for all $v \in C_0^\infty$). The idea of this method is that L can be used to define a stronger norm (in quantum mechanics, typically of Sobolev type) which allows to complete the minimal domain of L to get a larger domain in which the operator is self-adjoint. This extension is widely used in physics; for example, it is the usual way to define operators with Dirichlet boundary conditions.

1.2.2 Sturm-Liouville theory

A one-dimensional Sturm-Liouville operator of second-order is of the form

$$Lf(t) = \frac{1}{W(t)}\left[-\frac{d}{dt}\left(P(t)\frac{df(t)}{dt} \right) + Q(t)f(t) \right], \tag{1.1}$$

with W, P nonnegative functions in an interval of the real line (a, b). The importance of Sturm-Liouville operators is that it is a class of symmetric operators for which we have a lot of information about their self-adjointness and spectra. In particular, and depending on the properties of the functions W, P, Q that define the operator and its domain of definition, sometimes we have formulas for the essential spectrum of the operator or for the asymptotic value of its eigenvalues. We will make use of some of them in forthcoming sections but, as technical conditions are sometimes hard to express without a concrete

example in view, we will refrain from stating them at this point. We recall that special functions, such as Bessel functions or Laguerre polynomials, are often defined as solutions to the eigenvalue problem of a Sturm-Liouville operator.

1.3 Organization of the paper

The paper is organized in the following way. In section 2 we review the basics of $Y^{p,q}$ geometry. Section 3 studies the scalar spectrum of $Y^{p,q}$ geometry by analysing the solution of the Laplacian operator. In subsection 3.1, we present how the spectrum of Type IIB on $AdS_5 \times Y^{p,q}$ is related to the scalar spectrum of $Y^{p,q}$ geometry. In subsection 3.2 scalar modes in $Y^{p,q}$ are studied, by separating the variables in the wave function for the scalar modes. Behaviour of the eigenvalues for highly excited modes is studied in subsection 3.2.1 using Sturm-Liouville theory. In subsection 3.2.2, we compare Laplacian operator with simpler solvable operators in order to give upper and lower bounds for the all eigenvalues, which works best for $a \ll 1$ or equivalently $q \ll p$. Section 4 discusses examples of various other modes and analyze cases that may take us beyond the scalar spectra of IIB. Subsection 4.1 studies possible type IIA brane realisation, and subsection 4.2 discusses non-commutative and dipole deformations. One may note that in this section (and also the next) we will not address the spectra of the theory. To analyse the spectra we would not only need to go beyond the scalar fields, but would also require exact eigenvalues of the KK modes for all spin-states of the theory — a calculation that will be relegated for future works. In section 5 we go beyond the conformal cases to study new non-conformal duals that may arise from possible geometric transitions. Earlier results in this direction were more along the lines of cascading theories of Klebanov-Strassler type. To study geometric transitions for our case, we need both the resolved and the deformed cone over the $Y^{p,q}$ manifolds. In subsection 5.1, we review the metric for the cone $Y^{p,q}$ after resolution and then discuss the possibility of generating deformed cones over $Y^{p,q}$ base. We briefly argue why these deformations may not give rise to Kähler or complex manifolds. In subsection 5.2 we discuss the first step of geometric transitions, namely, constructions of the backgrounds with wrapped D5 branes on the resolved cones over the $Y^{p,q}$ manifolds. In subsection 5.3 we discuss the actual process of geometric transitions briefly and point our possible issues that may make the underlying calculations highly non-trivial. Finally in 6 we conclude by pointing out various future directions. In appendix A, eigenvalues of the differential operator S (Laplacian) are discussed, mostly borrowing some results from [10].

2 $Y^{p,q}$ geometry

The $Y^{p,q}$ metrics are Sasaki-Einstein and therefore a cone over them is Calabi-Yau. We start with the local metric

$$ds^2 = \frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1-cy}{2f(y)}dy^2 + \frac{f(y)}{9(a-y^2)}(d\psi - \cos\theta d\phi)^2$$
$$+ \frac{2(a-y^2)}{1-cy}\left[d\alpha + \frac{ac - 2y + y^2c}{6(a-y^2)}(d\psi - \cos\theta d\phi)\right]^2 \tag{2.1}$$

where $f(y) = 2cy^3 - 3y^2 + a$. As in [7] one can show that $Ric = 4g$ for all values of a and c therefore satisfying Einstein condition. For $c = 0$ and $a = 3$ the metric is exactly the local form of the standard metric on $T^{1,1}$. For $c \neq 0$ one can always rescale y ($y \to y/c$, and also $a \to a/c^2$, $f \to f/c^2$, etc) to set $c = 1$ which we will take in the following.

It is obvious that the first two terms give the metric of an S^2 for a fixed y, if the periodicity of θ and ϕ are π and 2π respectively. To study the (y, ψ) space one first requires

$$1 - y > 0, \quad a - y^2 > 0$$
$$f = a - 3y^2 + 2y^3 \geqslant 0. \tag{2.2}$$

In order for y to have solutions a must satisfy $0 < a < 1$. The negative solution of $f = 0$ and the smallest positive solution are denoted by y_- and y_+ respectively. Then y needs to take values between $y_- < y < y_+$, (so that all the terms in the metric come with positive sign). When $a = 1$ the metric (2.1) is the local round metric of S^5. If ψ has the period of 2π then (y, ψ) is topologically a 2-sphere.[1]

In order to have a compact manifold one takes the period of α to be $2\pi l$. Then $l^{-1}A$, where A is the last term in the second line of (2.1), becomes a connection on a U(1) bundle over $S^2 \times S^2$ which puts constraints on A. In general such U(1) bundles are completely specified topologically by the gluing on the equator of the two S^2 cycles, C_1 and C_2. These are measured by the corresponding Chern numbers in $H^2(S^2, \mathbb{Z}) = \mathbb{Z}$ which will be labeled as p and q. The Chern numbers are given by the integrals of $l^{-1}A/2\pi$ over C_1 and C_2, namely:

$$p = \frac{1}{2\pi l} \int_{C_1} A = \frac{y_- - y_+}{6y_- y_+}, \qquad q = \frac{1}{2\pi l} \int_{C_2} A = \frac{(y_- - y_+)^2}{9y_- y_+} \tag{2.3}$$

From their ratio $\frac{p}{q} = \frac{3}{2(y_+ - y_-)}$, it follows

$$a = \frac{1}{2} - \frac{p^2 - 3q^2}{4p^3}\sqrt{4p^2 - 3q^2}, \qquad l = \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}} \tag{2.4}$$

Metric (2.1) can be written in a canonical way if one makes the coordinate change

$$\alpha = -\beta/6 - c\psi'/6, \qquad \psi = \psi' \tag{2.5}$$

to (2.1). This converts (2.1) to the following metric:

$$ds^2 = \frac{1 - cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1 - cy}{2f(y)}dy^2 + \frac{f(y)}{18(1 - cy)}(d\beta + c\cos\theta d\phi)^2$$
$$+ \frac{1}{9}(d\psi - \cos\theta d\phi + y(d\beta + c\cos\theta d\phi))^2. \tag{2.6}$$

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[1]The range of y is taken to be $[y_-, y_+]$. This ensures that w (defined in (3.11)) is strictly positive in this interval and $r \geqslant 0$, vanishing only at the endpoints $y_\pm$. If we identify ψ periodically, the part of g_B (g_B is only defined in [10] but not in this paper) given by

$$\frac{1 - cy}{2f(y)} \, dy^2 + \frac{f(y)}{9(a - y^2)} \, d\psi^2$$

describes a circle fibered over the interval (y_-, y_+), the size of the circle shrinking to zero at the endpoints. Remarkably, the (y, ψ) fibers are free of conical singularities if the period of ψ is 2π, in which case the circles collapse smoothly and the (y, ψ) fibers are diffeomorphic to a 2-sphere.

The Killing vector

$$\frac{\partial}{\partial\psi'} = \frac{\partial}{\partial\psi} - \frac{1}{6}\frac{\partial}{\partial\alpha} \tag{2.7}$$

is globally well defined. For a generic value of a its orbit is not closed, in which case the Sasaki-Einstein metric is irregular. It is quasi-regular, if and only if $4p^2 - 3q^2 = m^2, m \in \mathbb{Z}$.

3 The spectrum of the $Y^{p,q}$ manifolds

After our brief discussion of the geometry of the $Y^{p,q}$ manifolds, let us come to the main analysis of paper: the study of scalar spectrum of these manifolds. We will start by analysing the solution of the Laplacian operator arising from the Fourier decomposition of functions as discussed earlier in [10]. However, as it will turn out, finding the exact solution of the eigenvalue problem in closed form does not seem feasible since the computation of the eigenvalues of the Laplacian boils down to the analysis of a one-dimensional differential operator (which we call S) of Heun type, which has *four* regular singular points. What we will do, therefore, is to find bounds for the eigenvalues of S, which will allow us to approximate the conformal dimensions of the theory. In subsection 4, we will study some examples of these modes and discuss cases that may take us beyond the scalar spectra.

3.1 Harmonic expansion on $Y^{p,q}$

We will follow the argument in [22] which gives the spectrum of Type IIB on $AdS_5 \times T^{1,1}$. The background solution in Type IIB is

$$ds^2 = \frac{r^2}{R^2}(-dx_0^2 + dx_i^2) + \frac{R^2}{r^2}dr^2 + R^2 ds_{Y^{p,q}}^2 \tag{3.1}$$

with the self-dual 5-form flux $F_5 = (1 + *)dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge d\left(\frac{r^4}{R^4}\right)$.

When Kaluza-Klein reducing this solution to AdS_5, we first have to compute the fluctuations of the 10-dimensional fields. The fluctuation of the gravitational fields are parametrized as

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} - \frac{1}{3}g_{\mu\nu}h_a^a, \quad \tilde{g}_{\mu a} = h_{\mu a}, \quad \tilde{g}_{ab} = g_{ab} + h_{ab} \tag{3.2}$$

where μ, ν denote the AdS_5 space time while a, b denote the internal space, and g denotes the background metric while h is the fluctuation.

Now we expand the fields $h_{\mu\nu}$, $h_{\mu a}$, h_{ab} and h_a^a into a complete set of harmonic functions on $Y^{p,q}$. With the de Donder and Lorentz-type gauge conditions $D^a h_{(ab)} = 0$ and $D^a h_{a\mu} = 0$ we have the following expansions:[2]

$$h_{\mu\nu}(x,y) = \sum_{\{\lambda\}} H_{\mu\nu}^{\{\lambda\}}(x)Y^{\{\lambda\}}(y), \qquad h_a^a(x,y) = \sum_{\{\lambda\}} \pi^{\{\lambda\}}(x)Y^{\{\lambda\}}(y)$$

$$h_{a\mu}(x,y) = \sum_{\{\lambda\}} B_\mu^{\{\lambda\}}(x)Y_a^{\{\lambda\}}(y), \qquad h_{(ab)}(x,y) = \sum_{\{\lambda\}} \phi^{\{\lambda\}}(x)Y_{(ab)}^{\{\lambda\}}(y) \tag{3.3}$$

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[2](x,y) denote coordinates of the AdS_5 and $Y^{p,q}$ spaces respectively and therefore should not be confused with the y coordinates that we will be using to write the metric etc of the $Y^{p,q}$ spaces.

where $[\lambda] \equiv [\lambda_1, \cdots, \lambda_{[5/2]}]$ denotes the SO(5) representation. Similarly with the gauge condition $D^a A_{a\mu} = 0$ and $D^a A_{ab} = 0$ we can expand the type IIB complex zero and the two-forms, B and A_{mn} respectively, as

$$A_{\mu\nu}(x,y) = \sum_{\{\lambda\}} a_{\mu\nu}^{\{\lambda\}}(x) Y^{\{\lambda\}}(y), \qquad A_{a\mu}(x,y) = \sum_{\{\lambda\}} a_\mu^{\{\lambda\}}(x) Y_a^{\{\lambda\}}(y)$$

$$A_{ab}(x,y) = \sum_{\{\lambda\}} a^{\{\lambda\}}(x) Y_{[ab]}^{\{\lambda\}}(y), \qquad B(x,y) = \sum_{\{\lambda\}} B^{\{\lambda\}}(x) Y^{\{\lambda\}}(y) \qquad (3.4)$$

For the four-form flux we can do the same thing by imposing the conditions $D^a a_{abcd} = 0$, $D^a a_{abc\mu} = 0$, $D^a a_{ab\mu\nu} = 0$ and $D^a a_{a\mu\nu\gamma} = 0$,

$$a_{abcd} = \sum_{\{\lambda\}} b^{\{\lambda\}}(x) Y_{abcd}^{\{\lambda\}}(y), \qquad a_{abc\mu} = \sum_{\{\lambda\}} b_\mu^{\{\lambda\}}(x) Y_{abc}^{\{\lambda\}}(y)$$

$$a_{ab\mu\nu} = \sum_{\{\lambda\}} b_{\mu\nu}^{\{\lambda\}}(x) Y_{ab}^{\{\lambda\}}(y), \qquad a_{a\mu\nu\gamma} = \sum_{\{\lambda\}} b_{\mu\nu\gamma}^{\{\lambda\}}(x) Y_a^{\{\lambda\}}(y)$$

$$a_{\mu\nu\gamma\rho} = \sum_{\{\lambda\}} b_{\mu\nu\gamma\rho}^{\{\lambda\}}(x) Y^{\{\lambda\}}(y) \qquad (3.5)$$

Notice that $Y^{p,q}$ is topologically $S^2 \times S^3$, the same as $T^{1,1}$, so we can argue similarly as in [22] to simplify the expansion,

$$a_{abcd} = \sum_{\{\lambda\}} b^{\{\lambda\}}(x) \epsilon_{abcd}^e D_e Y^{\{\lambda\}}(y) \qquad (3.6)$$

The full linearized equation of motion can be found in [23]. In this paper we are only interested in scalar harmonics which means that we are only looking at the following modes in AdS_5, coming from first line of (3.3), (3.4), and the last line of (3.5):

$$h_{\mu\nu}(x,y) = \sum_{\{\lambda\}} H_{\mu\nu}^{\{\lambda\}}(x) Y^{\{\lambda\}}(y), \qquad h_a^a(x,y) = \sum_{\{\lambda\}} \pi^{\{\lambda\}}(x) Y^{\{\lambda\}}(y)$$

$$A_{\mu\nu}^{(i)}(x,y) = \sum_{\{\lambda,i\}} a_{\mu\nu}^{\{\lambda\}}(x) Y^{\{\lambda\}}(y), \qquad B^{(j)}(x,y) = \sum_{\{\lambda\}} B^{\{\lambda,j\}}(x) Y^{\{\lambda\}}(y)$$

$$a_{\mu\nu\gamma\rho}(x,y) = \sum_{\{\lambda\}} b_{\mu\nu\gamma\rho}^{\{\lambda\}}(x) Y^{\{\lambda\}}(y) \qquad (3.7)$$

where $A_{\mu\nu}^{(i)}(x,y)$ would be the NS and RR two-forms respectively and $B^{(j)}(x,y)$, where $i,j = 1,2$, would be the axion and the dilaton respectively. The other two quantities π and b that appear respectively from the expansion of h_a^a in (3.3) and from the expansion of a_{abcd} in (3.6), are related to the metric and the four-form respectively. Therefore taking all these into account, we are left with the following equations:

$$(\Box_x + \boxtimes_y) H_{\mu\nu}^{\{\lambda\}} = 0$$
$$(\Box_x + \boxtimes_y) B^{\{\lambda,j\}} = 0$$

$$(\text{Max} + \boxtimes_y)a_{\mu\nu}^{\{\lambda,i\}} + \frac{2i}{R}\epsilon_{\mu\nu}{}^{\sigma\tau\gamma}\partial_\sigma a_{\tau\gamma}^{\{\lambda,i\}} = 0$$

$$\Box_x \begin{pmatrix} \pi^{\{\lambda\}} \\ b^{\{\lambda\}} \end{pmatrix} + \begin{pmatrix} \boxtimes_y - 32R^{-2} & 80R^{-1}\boxtimes_y \\ -\frac{4}{5}R^{-1} & \boxtimes_y \end{pmatrix} \begin{pmatrix} \pi^{\{\lambda\}} \\ b^{\{\lambda\}} \end{pmatrix} = 0 \qquad (3.8)$$

where Max denotes the Maxwell operator and $\Box_x$, $\boxtimes_y$ are the kinetic operators in the AdS_5 space time and $Y^{p,q}$ spaces respectively. In our case the latter is exactly given by the action of the covariant Laplacian operator on the corresponding SO(5) representation,[3] which can be formally written as

$$\boxtimes_y \equiv \frac{\Box_y Y^{\{\lambda\}}}{Y^{\{\lambda\}}} \qquad (3.9)$$

Our next step then is to analyze the eigenvalues of the Laplacian operator in order to find the mass spectrum for these fields.

3.2 Scalar modes in $Y^{p,q}$

As we discussed in detail in the above subsection, our goal is to compute the eigenvalues λ_n of the Laplacian in the manifold $Y^{p,q}$. These eigenvalues enter the scalar wave equation on $AdS_5 \times Y^{p,q}$ as masses, so that the conformal dimensions of the associated fields at infinity (i.e for the CFT dual) are given by Witten's formula [24]:

$$\Delta_k = 2 + \sqrt{4 + \lambda_k}\,.$$

It is well known that the Laplacian on $Y^{p,q}$, which we denote by $\Box_y$, defines a nonnegative, self-adjoint operator whose domain is the Sobolev space $H^2(Y^{p,q})$ of square-integrable functions with square-integrable second derivatives. The Laplacian is given in local coordinates as [10]:[4]

$$\Box_y \equiv g^{ij}\nabla_i\nabla_j = \frac{1}{\rho(y)}\frac{\partial}{\partial y}\rho(y)\,w(y)\,r(y)\frac{\partial}{\partial y} + \frac{1}{w(y)}\frac{\partial^2}{\partial\alpha^2} + \frac{9}{r(y)}\left(\frac{\partial}{\partial\psi} - h(y)\frac{\partial}{\partial\alpha}\right)^2$$
$$+ \frac{6}{1-y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\phi} + \cos\theta\frac{\partial}{\partial\psi}\right)^2\right] \qquad (3.10)$$

where the various coefficients appearing above can be identified from (2.1) after rescaling to set $c = 1$, i.e.,

$$w(y) \equiv \frac{2(a - y^2)}{1 - y}, \quad r(y) \equiv \frac{2y^3 - 3y^2 + a}{a - y^2}, \quad h(y) \equiv \frac{y^2 - 2y + a}{6(a - y^2)}, \quad \rho(y) \equiv \frac{1 - y}{18} \quad (3.11)$$

The scalar mode $\Phi(y, \theta, \phi, \psi, \alpha)$ in the internal space now takes the following wavefunctional form that was derived in [10]:

$$\Phi = u(y)v(\theta)e^{i(n\phi + 2m\psi + l\sigma\alpha/\tau)} \qquad (3.12)$$

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[3]For more details on the Maxwell and the Laplacian operator see [22, 23].

[4]Note that y denotes different things on l.h.s. and r.h.s. of (3.10). See footnote 2.

which means that the Laplacian satisfies:

$$\Box_y \Phi = [S_{nmlj} u(y)] \, v(\theta) e^{i(n\phi + 2m\psi + l\sigma\alpha/\tau)} \tag{3.13}$$

where we saw in [10] that the analysis of the eigenvalues of the Laplacian on $Y^{p,q}$ is reduced to that of (the Friedrichs extension of) the one-dimensional operators

$$
\begin{aligned}
S_{nmlj} &\equiv -\frac{1}{\rho(y)}\frac{\partial}{\partial y}\rho(y)\,w(y)\,r(y)\,\frac{\partial}{\partial y} + \frac{1}{w(y)}\left(\frac{\sigma l}{\tau}\right)^2 + \frac{9}{r(y)}\left(2m - h(y)\,\frac{\sigma l}{\tau}\right)^2 + \frac{6\Lambda_{nmj}}{1-y}\,, \\
&= -\frac{2}{1-y}\frac{\partial}{\partial y}(a - 3y^2 + 2y^3)\frac{\partial}{\partial y} + \frac{\gamma^2(1-y)}{4(a-y^2)} + \frac{6\Lambda_{nmj}}{1-y} \\
&\qquad\qquad + \frac{9(a-y^2)}{a-3y^2+2y^3}\left(2m - \frac{\gamma(a-2y+y^2)}{6(a-y^2)}\right)^2,
\end{aligned}
\tag{3.14}
$$

densely defined on $L^2((y_-, y_+), \rho\, dy)$. We refer to the aforementioned paper for more detailed discussions on the derivation of the above formula.[5] We have set $\gamma \equiv \sigma l/\tau$, and the function $v(\theta)$ defined in (3.12) satisfies the eigenvalue equation that comes from the angular direction θ as:

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \left(\frac{n+2m\cos\theta}{\sin\theta}\right)^2\right] v_{nmj} = -\Lambda_{nmj} v_{nmj}\,. \tag{3.15}$$

The eigenvalues Λ_{nmj} are given by the explicit formula:

$$\Lambda_{nmj} \equiv 2\Big[2j(j+1) + \big(|n+2m| + |n-2m|\big)(2j+1) + |n+2m||n-2m| + 2m^2 + n^2\Big]. \tag{3.16}$$

In what follows we will drop the indices when there is no risk of confusion.

Before going on, it is worth recalling that the integers n, m, l that label the operators S arise from the (quite subtle) Fourier decomposition of functions we discussed in [10] and given above in (3.12), while the label j (also an integer) was obtained by explicitly solving an auxiliary eigenvalue problem associated with the geometry of the sphere bundles (which had three regular singular points). However, as we have already mentioned, there is little hope of solving the eigenvalue problem for S in closed form, since the spectral problem for the operator S is governed by a Heun differential equation. What we will do, therefore, is to obtain some estimates for the eigenvalues of S that will allow us to approximate the conformal dimensions of the corresponding CFT.

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[5]The approach taken in [10] exploits the separability of the AdS$^5 \times Y^{p,q}$ metrics to compute the eigenfunctions of the Laplace operator in $Y^{p,q}$ in quasi closed form, by expressing them in terms of the eigenfunctions of the Friedrichs extension of a single second-order ordinary differential operator with four regular singular points. The subtle geometry of the spaces $Y^{p,q}$ introduces additional complications in the analysis, since the 'angular' variables in which the metric of $Y^{p,q}$ separates are not defined globally. In order to circumvent this problem the steps taken in [10] is to start by constructing a Fourier-type decomposition of the space of square-integrable functions on $Y^{p,q}$ adapted to the global structure of the manifold and to the action of the Laplacian. Once the eigenfunctions of the Laplacian in $Y^{p,q}$ have been computed, the analysis of the Klein-Gordon equation in AdS$^5 \times Y^{p,q}$ can be reduced to that of a family of linear hyperbolic equations in anti-de Sitter space. In [10] a detailed discussion of the existence and uniqueness of causal propagators for these equations using Ishibashi and Wald's spectral-theoretic approach to wave equations on static space-times based on [25–27] were presented. Note that for our purpose, this presents several advantages over the classical method of Riesz transforms, since the latter method only yields local solutions to the Cauchy problem in the case in which the underlying space-time is not globally hyperbolic [28].

3.2.1 Behaviour of large eigenvalues (highly excited modes)

In this subsubsection, we give an asymptotically exact result for large energies (highly excited modes) of the operator $S \equiv S_{nmlj}$. The basic idea is that, if we label the eigenfunctions of this operator by an integer $k = 1, 2, \ldots$, the kth eigenvalue is very close to a constant multiple of k^2 for large k. To put it in a different way, the eigenvalues tend to those of an infinite well, the width of the well determined by the functions P, Q, W that define the Sturm-Liouville operator S. Very informally, the justification would be that at high energies the leading terms are the derivatives; this kind of asymptotic results are usually proved using pseudo-differential operators.

The first observation is that, without any further assumptions, we have an asymptotic formula (for large k, for highly excited modes) for the eigenvalues of S namely: the eigenvalues $\lambda_k \equiv \lambda_k(n, m, l, j)$ of S are asymptotically given by following expansion[6]

$$\lambda_k = C_0 k^2 + o(k^2) \,, \tag{3.17}$$

where the constant

$$C_0 \;\equiv\; 2\pi^2 \left[\int_{y_-}^{y_+} \left(\frac{1 - y}{a - 3y^2 + 2y^3} \right)^{1/2} dy \right]^{-2} \tag{3.18}$$

depends on the geometry of the manifold (that is, on p and q) through $y_\pm$ but not on the Fourier modes n, m, l, j. (So that only the error term knows about these indices.)

The above statement is a consequence of general results in the theory of singular Sturm-Liouville operators. Indeed, it suffices to note that S is a lower-bounded one-dimensional self-adjoint operator, so it follows from [29, section 10.8] that (3.17) holds true with

$$C_0 \;\equiv\; \left(\frac{1}{\pi} \int_{y_-}^{y_+} \left(w(y) \, r(y) \right)^{-1/2} dy \right) . \tag{3.19}$$

An easy computation shows that this integral takes the above form, which can in turn be expressed in terms of elliptic functions.

Before ending this subsection, let us make the following remark. Weyl's law[7] ensures that, when the eigenvalues of all the one-dimensional operators corresponding to the various

---

[6]A word of caution about the notation: The error term is $o(k^2)$, and not $\mathcal{O}(k^2)$. The respective notations mean different things, and $o(k^2) \ll \mathcal{O}(k^2)$ for large k. The notation $f(k) = o(g(k))$ means that $\lim_{k \to \infty} \frac{f(k)}{g(k)} = 0$. On the other hand the notation $f(k) = \mathcal{O}(g(k))$ means that that there exists a positive constant C such that for k sufficiently large $|f(k)| \leqslant C|g(k)|$. Simply, $\mathcal{O}(k^n)$ means that a term scaling like k^n in the proper limit of k, as familiar to physicists. Here $k \to \infty$ is appropriate, and later in (3.41) $a \to 0$ is so. The two notions are different and in particular the $o(k^2)$ notation above indicates that the error term grows slower than quadratically in k. If it had a power-law behaviour, it would be $o(k^2) = \mathcal{O}(k^{2-\epsilon})$ with $\epsilon > 0$. In some sense $\mathcal{O}$ is used when we know the power scaling of a term, and o is when we know only the upper bound of the scaling.

[7]The Weyl law states that the first term in the asymptotic expansion for the k-th eigenvalue λ_k of the Laplacian on an n-dimensional compact Riemannian manifold is:

$$\lambda_k \;=\; C_n k^{2/n} / (\text{Vol } M)^{2/n} + o(k^{2/n})$$

as $k \to \infty$. This was proved by Weyl in [30]. The second term was conjectured by Weyl in 1913 [31] and proved only in 1980 by Ivrii [32].

Fourier modes are taken into account, the eigenvalues of the Laplace operator on $Y^{p,q}$ (let's call them $\widetilde{\lambda}_k$) obey the asymptotic law

$$\widetilde{\lambda}_k = (2\pi)^2 \left(\frac{5k}{|\mathbb{S}^4|\text{Vol}(Y^{p,q})} \right)^{2/5} + o(k^{2/5}) \tag{3.20}$$

where $|\mathbb{S}^4|$ denotes the volume of the unit 4-sphere and the volume of the manifold being given by [7]

$$\text{Vol}(Y^{p,q}) = \frac{\pi^2 q^2 [2p + (4p^2 - 3q^2)^{1/2}]}{3p^2[3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}]} \, . \tag{3.21}$$

Eq. (3.17) provides a somewhat more tangible way of presenting this asymptotic result in the sense that the asymptotics is separated into families labeled by additional "quantum numbers". A straightforward but tedious computation shows that, of course, when degeneracies are taken into account, the asymptotics (3.17) can be summed with respect to the additional "quantum numbers" to obtain (3.20).

Let us elaborate this a little bit more. We have seen that the analysis of the eigenvalues of the Laplacian in $Y^{p,q}$ can be reduced to that of the eigenvalues of a family of one-dimensional operators $S = S_{nmlj}$. These operators are labeled by three integers n, m, l and a nonnegative integer j. Notice that if any of the quantum numbers n, m or l is nonzero ("higher Fourier modes"), all the eigenvalues of the Laplacian corresponding to these quantum numbers are necessarily degenerate, as mapping (n, m, l) to $(-n, -m, -l)$ leaves the eigenvalue equation invariant. A convenient way of understanding the behavior of the eigenvalues if the Laplacian in geometric terms is the Weyl's law. For this, let's denote by $\widetilde{\lambda}_k$ the k-th lowest eigenvalue of the Laplacian in $Y^{p,q}$, where each eigenvalue is repeated according to its multiplicity. Obviously, for each k there are "quantum numbers" (n, m, l, j) such that $\widetilde{\lambda}_k = \lambda_{k'}(n, m, l, j)$ for some k'.[8] Weyl's law then ensures that the asymptotic distribution of the eigenvalues $\widetilde{\lambda}_k$ of the Laplacian is related to the volume of the manifold through the relation (3.20).

3.2.2 Bounds for the eigenvalues for small a

In the previous subsubsection we obtained an asymptotic formula for the eigenvalues, which is asymptotically exact for large energies. It does not provide any information on low-lying eigenvalues, however. So our goal in this subsubsection is to provide some estimates for the whole spectrum in an appropriate regime. This regime will be the case when the parameter a is small; as we will see, then we can obtain two-sided bounds for the eigenvalues that provide an adequate control of the energies.

The technique we apply here is that, using the fact that a is small $0 < a < 1$, we can Taylor expand the Laplacian operator in terms of small a and drop higher orders of a (as in (3.25)). Obviously this works the best if a is very small, or equivalently when $q \ll p$, but even moderately small a, it is a valid Taylor expansion. Instead of trying to obtain

---

[8]It is worth emphasizing that one cannot explicitly compute the degeneracy of the eigenvalues, as there could be *non-geometric degeneracies* in the sense that $\lambda_{k_0(n_0,m_0,l_0,j_0)} = \lambda_{k_1(n_1,m_1,l_1,j_1)}$ for some pair of indices *not* related by a symmetry of the equation.

the spectrum of the original Laplacian operator, we use another operator (3.30) whose spectrum is exactly known as in (3.31). With an appropriate constant C which does not depend on the parameters of the equation, we can compute the upper and lower bounds of the eigenvalues of Laplacian.[9]

Before passing to the actual derivation of the bounds, let us discuss the meaning of the smallness of a. It should be noticed that this is in fact a geometric hypothesis on the manifold. In order to see this, let us recall the connection between the parameter a and the integers p, q that controlled the geometry of the bundle. In [7, section 3] it is explained that the relationship between p, q and the endpoints $y_\pm$ is that

$$y_+ - y_- = \frac{3q}{2p} \tag{3.22}$$

The idea now is that it can be easily seen that for any value of the latter quotient we can find an a for which (3.22) is satisfied; indeed, a can be chosen as

$$a = \frac{3}{4}\left[1 - \frac{3q}{2p} - \left(1 - \frac{1}{3}\left(\frac{3q}{2p}\right)^2\right)^{1/2}\right]^2 - \frac{1}{4}\left[1 - \frac{3q}{2p} - \left(1 - \frac{1}{3}\left(\frac{3q}{2p}\right)^2\right)^{1/2}\right]^3 \tag{3.23}$$

Hence it is not hard to see that $a \ll 1$ is equivalent to $q \ll p$, so this condition translates immediately as a condition on the geometry of the bundles. It this case,

$$a = \frac{27q^2}{16p^2} + \mathcal{O}(q^3/p^3) \tag{3.24}$$

A closer look at the subsection on rational roots in [7] reveals that there is also an infinite number of solutions with rational roots and arbitrarily small values of a (recall that in this case the Sasaki-Einstein structure is quasi-regular.)

The idea now is that, for very small a, the operator $-S$ should be very similar to the one we obtain by dropping higher powers of a (e.g. in the Taylor expansion of the coefficients), namely

$$-2\frac{\partial}{\partial y}(a - 3y^2)\frac{\partial}{\partial y} + \frac{\gamma^2}{2(a - y^2)} + 6\Lambda + \frac{18(a - y^2)}{a - 3y^2}\left(m + \frac{\gamma y}{6(a - y^2)}\right)^2 \tag{3.25}$$

This expression defines a self-adjoint operator on $L^2(-(a/3)^{1/2}, (a/3)^{1/2})$ via its Friedrichs extension (notice we still have too many singular points to solve the eigenvalue equation for S). It is convenient to make things independent of a by rescaling. For future convenience, we introduce the variable $t \equiv a^{-1/2}y$ and, noticing that

$$\gamma = \sigma l q (3a)^{1/2}\left(1 + \mathcal{O}(a)\right) \tag{3.26}$$

we set $\bar{\gamma} \equiv a^{-1/2}\gamma$ (observe that $\bar{\gamma}$ still depends on a, although it tends to a well-defined nonzero limit as $a \to 0$). Here and in what follows, by $\mathcal{O}(a)$ we will denote quantities

---

[9]An interesting question is how small C can be, because if C becomes large, the bound is very loose. Furthermore, by comparing with the known low-lying scalar spectrum, we may learn something useful about C. The former and the latter points will be addressed in footnotes 10 and 19 respectively.

bounded by a constant (independent of any labels and of the geometry) times a, and whose derivatives satisfy analogous bounds (i.e., behave like symbols with respect to these bounds). We are thus led to consider (the Friedrichs extension of) the operator

$$T \equiv -\frac{\partial}{\partial t}P(t)\frac{\partial}{\partial t} + Q(t) \tag{3.27}$$

in $L^2(I)$, with $I \equiv (-3^{-1/2}, 3^{-1/2})$ and

$$
\begin{aligned}
P(t) &\equiv 2(1 - 3t^2)\,, \\
Q(t) &\equiv \frac{\bar{\gamma}^2}{2(1 - t^2)} + 6\Lambda + \frac{18(1 - t^2)}{1 - 3t^2}\left(m + \frac{\bar{\gamma}t}{6(1 - t^2)}\right)^2\,.
\end{aligned}
$$

In order to relate the spectral properties of S (as an unbounded self-adjoint operator on $L^2((y_-, y_+), \rho\, dy)$ to those of T (on the space $L^2(I)$ with the standard Lebesgue measure dt), it is convenient to start by relating these two L^2 spaces. An obvious way to do so is through the following a-dependent change of variables:

$$t \equiv -\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\frac{\int_{y_-}^y \rho(y')\,dy'}{\int_{y_-}^{y_+} \rho(y')\,dy'} \equiv \mathcal{T}_a(y)\,. \tag{3.28}$$

This induces a unitary transformation $L^2((y_-, y_+), \rho\, dy) \to L^2(I, dt)$, which transforms S into the Sturm-Liouville operator of the form:

$$\widetilde{S} \equiv -\frac{\partial}{\partial t}\widetilde{P}(t)\frac{\partial}{\partial t} + \widetilde{Q}(t)\,. \tag{3.29}$$

To derive the bounds, we start with the following observation: the spectrum of the auxiliary operator

$$T_\mu \equiv -\frac{\partial}{\partial t}P(t)\frac{\partial}{\partial t} + \frac{\mu}{1 - 3t^2} \tag{3.30}$$

on $L^2(I)$, as a function of the parameter μ, is given by

$$\ell_k(\mu) \equiv \frac{3}{2}\left(1 + \sqrt{\frac{8\mu}{3}} + 2k\right)^2 - \frac{3}{2} \tag{3.31}$$

The proof of the above statement can be argued using a straightforward computation. To start, observe that it suffices to see that the exponents of the equation $T_\mu f = -\ell f$ are

$$\pm\sqrt{\frac{\mu}{24}}, \qquad \frac{1}{2}\left(1 \pm \sqrt{1 + \frac{2\lambda}{3}}\right) \tag{3.32}$$

at (0 and at 1) and at ∞ respectively. The eigenvalues then arise as the necessary condition for

$$(1 - 3t^2)^{-(\mu/24)^{1/2}}f(t) \tag{3.33}$$

to be a polynomial in t, thus proving the required statement.

After developing the necessary mathematical preliminaries, we are now ready to compute bounds for the eigenvalues of S (which coincide with those of $\tilde{S}$, by definition). Notice that we cannot obtain bounds using a relative compactness argument, as any perturbation of the function P will lead to corrections that are not relatively compact with respect to the original operator (because they have the same number of derivatives as the initial operator). What we can do is to exploit monotonicity using the following two observations. The first observation is that there is a constant C, which does not depend on the parameters of the equation, such that the following bounds for $\widetilde{P}(t)$ hold for all $t \in I$:

$$(1 - Ca)P(t) \leqslant \widetilde{P}(t) \leqslant (1 + Ca)P(t). \tag{3.34}$$

This inequality is obvious in view of the formula (3.28) for the map $y \mapsto t$, and simply asserts (roughly speaking) that the map does not alter the singularities too much.

Our second observation is somewhat similar to the first one in the sense that we again claim that there is a constant C, which does not depend on the parameters of the equation, such that the following bounds for $\widetilde{Q}(t)$ hold for all $t \in I$:

$$\widetilde{Q}(t) \geqslant (1 - Ca)\left(\frac{\mu_-}{1 - 3t^2} + \frac{1 + \bar{\gamma}^2}{2} + 6\Lambda - Ca\right),$$
$$\widetilde{Q}(t) \leqslant (1 + Ca)\left(\frac{\mu_+}{1 - 3t^2} + \frac{3(1 + \bar{\gamma}^2)}{4} + 6\Lambda + Ca\right), \tag{3.35}$$

where μ_+ and μ_- are defined in the following way:

$$\mu_- \equiv 12 \max\left\{0, m - \frac{\bar{\gamma}}{4\sqrt{3}}\right\}^2, \qquad \mu_+: \equiv 18\left(m + \frac{\bar{\gamma}}{4\sqrt{3}}\right)^2. \tag{3.36}$$

The proof of the above two inequalities are a straightforward consequence of the fact that

$$(1 - Ca)(Q(t) - Ca) \leqslant \widetilde{Q}(t) \leqslant (1 + Ca)(Q(t) + Ca). \tag{3.37}$$

(One might wonder why we included an additive error Ca here and not in the estimate for $\tilde{P}$. The reason is that $\tilde{P}$ does not vanish in the interval I, and this is enough for us to control the error via a multiplicative constant.)

It is standard that if we take nicely behaved functions $P_j(t)$ and $Q_j(t)$ on I, with $j = 1, 2$ and $P_j(t) > 0$, and suppose that $P_1(t) \geqslant P_2(t)$ and $Q_1(t) \geqslant Q_2(t)$ (resp. $P_1(t) \leqslant P_2(t)$ and $Q_1(t) \leqslant Q_2(t)$), then the k-th eigenvalue of (the Friedrichs extension of) the operator $-\frac{d}{dt}P_1(t)\frac{d}{dt} + Q_1(t)$ is larger or equal (resp. smaller or equal) than those of $-\frac{d}{dt}P_2(t)\frac{d}{dt} + Q_2(t)$. Hence it is elementary to derive the bounds

$$\Lambda_k^{[-]} \leqslant \lambda_k \leqslant \Lambda_k^{[+]} \tag{3.38}$$

where

$$\Lambda_k^{[-]} = (1 - Ca)\left(\ell_k(\mu_-) + \frac{1 + \bar{\gamma}^2}{2} + 6\Lambda - Ca\right),$$
$$\Lambda_k^{[+]} = (1 + Ca)\left(\ell_k(\mu_+) + \frac{3(1 + \bar{\gamma}^2)}{4} + 6\Lambda + Ca\right), \tag{3.39}$$

from the inequalities (3.34) and (3.35), the formula for the eigenvalues $\ell_k(\mu)$ of the auxiliary operator T_μ derived in (3.31) and elementary inequalities in I such as

$$1 \leqslant (1 - t^2)^{-1} \leqslant 3/2 \,. \tag{3.40}$$

The bounds (3.38), in which C stands for an a-independent constant and $\ell_k(\mu)$ is given by (3.31), constitute the main result of this subsection.[10]

As a remark, notice that the above bounds also ensure that the eigenvalues have the asymptotic behavior

$$\lambda_k = 6(1 + \mathcal{O}(a))k^2 + \mathcal{O}(k) \,. \tag{3.41}$$

This is precisely the growth rate computed in (3.17), since it is easy to see that the constant

$$C_0 \equiv 2\pi^2 \left[\int_{y_-}^{y_+} \left(\frac{1 - y}{a - 3y^2 + 2y^3} \right)^{1/2} dy \right]^{-2} \tag{3.42}$$

entering Weyl's law (3.18) tends to 6 as the constant a tends to 0.

4 Examples of scalar and other modes

Now that we have discussed the spectrum of scalar modes in the internal $Y^{p,q}$ space, it is time to study some examples of these modes. However before moving ahead we should point out that in this section (and also the next) we will *not* address the spectra of the theory. To analyse the spectra (for example along the lines of [33–35]) we would not only need to go beyond the scalar fields, but would also require exact eigenvalues of the KK modes for all spin-states of the theory — a calculation that will be relegated for future works. The advances that we made in the previous section is a good starting point and we will benefit from further development. At this point we will suffice ourselves by studying some basics aspects of scalar and other modes from supergravity perspective in this section. In the next section we will discuss possible non-conformal extensions of our model. Again the emphasis therein would be to study the supergravity background and not the matching of spectra.

The simplest examples of scalar and other modes that appear for our case are from the decomposition of the 2-forms in (3.4). These decompositions lead to two possible theories on the boundary where we define the CFTs.

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[10]One might worry about the strength of our bounds. For example a question would be whether the bounds could be loose if constant such as C is large. To answer this we first note that the constants do not arise exactly from a power series expansion, but rather as the Taylor formula with estimates for the remainder (which is essentially the mean value theorem). Therefore, the constant C can be explicitly computed as the (sum of the) supremum (for t and a between certain values) of the derivative of some functions appearing in P or Q with respect to the parameter a. For this reason, the behavior of this constant is controlled, and can be computed explicitly. For example, a rough computation reveals that the constant C can be chosen to be of order 10 when a is smaller than 10^{-n-1}, so the relative error is at most of order 10^{-n}. Since $a = (q/p)^2$ up to some inessential factors, it is enough that $q/p < 10^{-(n+1)/2}$. These estimates can be refined easily.

- Non-commutative geometry: Let us consider the NS B field with both components along the boundary, i.e we can switch on $B_{ij}(x)$ where $i,j = 1,2,3$ and x_μ specify coordinates in AdS_5 space, leading to non-commutative geometry in the dual gauge theory. For example, a B-field component of the form $B_{ij}(r)$, with r being the radial direction in the AdS_5 space, would be able to generate non-commutative theory on the boundary. Clearly this mode is a scalar mode in the internal $Y^{p,q}$ space.

- Dipole theory: This time we consider the NS B-field which has one component along the boundary and the other component either along the radial r direction or along the internal $Y^{p,q}$ directions. Consider first a component of the NS B field of the form B_{ir}. However if this component is only a function of x^μ, then we can make a gauge transformation to rotate the NS B field components along the boundary which in turn will convert the boundary theory to a non-commutative theory. The other alternative is to make it gauge equivalent to zero for the B field component of the form $B_{ir}(r)$. Thus the only non-trivial cases appear to be of the form $B_{ir}(y), B_{ia}(x,y)$ and they both lead to the dipole theories. However none of these are scalar modes in the internal $Y^{p,q}$. The special case where the NS B field is of the form $B_{ia}(x,y)$ fits in with our decomposition (3.4), and leads to a simple vector decomposition of the boundary theory.

Thus the simplest scalar mode leading to noncommutativity can be specified by a 2-form θ^{ij} such that the commutator of the coordinates on the boundary theory is $[x^i, x^j] = i\theta^{ij}$. The parameter θ^{ij} has dimensions -2. At low-energies, noncommutative super Yang-Mills theory (NCSYM) can be described by augmenting the action with:

$$\int \theta^{ij} \mathcal{O}_{ij}(x) d^4x, \tag{4.1}$$

where $\mathcal{O}_{ij}$ is an operator of dimension 6 in the superconformal SYM on a commutative space. In the conventions such that the SYM Lagrangian is:

$$\mathcal{L}_{\mathrm{SYM}} = \mathrm{tr}\left[\frac{1}{2g^2}\sum_{I=1}^{6}\partial_i\phi^I\partial^i\phi^I + \frac{1}{4g^2}F_{ij}F^{ij} + \frac{1}{2g^2}\sum_{I<J}\left[\phi^I,\phi^J\right]^2\right] + \text{fermions}, \tag{4.2}$$

the bosonic part of the operator $\mathcal{O}_{ij}$ can be written as:

$$\mathrm{tr}\left[\frac{1}{2g^2}F_{jk}F^{kl}F_{li} - \frac{1}{2g^2}F_{ij}F^{kl}F_{kl} + \frac{1}{g^2}F_{ik}\sum_{I=1}^{6}\partial_j\phi^I\partial^k\phi^I - \frac{1}{4g^2}F_{ij}\sum_{I=1}^{6}\partial_k\phi^I\partial_k\phi^I\right]. \tag{4.3}$$

Here, g is the SYM coupling constant, F_{ij} is the U(N) field-strength, and ϕ^I ($I = 1\ldots 6$) are the scalars.

For the second case we expect the boundary theory to be deformed by an operator of the form $\mathcal{O}_i$. The deformation by $L^i\mathcal{O}_i$ (where L^i is a constant vector) is the low-energy expansion of a nonlocal field-theory, the so-called dipole-theory, described in [36–38].

Furthermore, as discussed in [36] (see also [37–39]), the bosonic part of the SYM operator $\mathcal{O}_i$ can be calculated by changing to local variables (see [36] for more details). We

can write it in $\mathcal{N} = 1$ superfield notation as [36]:

$$\mathcal{O}_i = \frac{i}{g_{YM}^2} \int d^2\theta \epsilon^{ab} \text{tr} \left[\sigma_i^{\alpha\dot{\alpha}} W_\alpha \Phi_a D_{\dot{\alpha}} \Phi_b + \Phi \Phi_a D_i \Phi_b \right] + \text{c.c.} \tag{4.4}$$

Here, we denote the $\mathcal{N} = 1$ chiral field as Φ and the $\mathcal{N} = 1$ vector-multiplet with the field-strength W_α. The original $\mathcal{N} = 2$ hypermultiplet is now written in terms of the two $\mathcal{N} = 1$ chiral multiplets Φ_a ($a = 1, 2$). Finally, $\sigma_i^{\alpha\dot{\alpha}}$ are Pauli matrices. As expected, the operator $\mathcal{O}_i$ has conformal dimension 5.

4.1 Possible type IIA brane realisation

In the following we will discuss these backgrounds in somewhat more details by switching on appropriate B fields. This is slightly different from allowing the B field as a *fluctuation*. A non-trivial background B field will change the geometry in some particular way which would reflect the corresponding backreactions. To analyse the corresponding backreactions we have to study the scenario directly from N D3-branes probing the geometry given by a cone over the $Y^{p,q}$ spaces. This starting point in fact has many intriguing possibilities in addition to the ones related to generating non-local field theories. One of the possibilities is to see whether a brane realisation of the form [3] in type IIA can also be made for our case. We will therefore start by analysing this interesting possibility first and then go for the non-local theories.

To study D3-branes at the tip of a cone over the $Y^{p,q}$ manifolds, we will assume the usual ansatz for the D3-brane metric given in terms of a harmonic function H which is typically a function of r and the $Y^{p,q}$ coordinates. Let us therefore take the following metric ansatz:

$$ds_{\text{IIB}}^2 = H^{-1/2} ds_{0123}^2 + H^{1/2}(dr^2 + r^2 dM_5^2), \tag{4.5}$$

where dM_5^2 is the same in eq. (2.1) and $F_5 = (1 + *)d\beta_0 \wedge dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$ with $H = 1 + \frac{r_0^4}{r^4} \equiv \beta_0$. We also assume the dilaton is zero. As in [3], the internal metric has three isometries along the α, ψ and ϕ directions. We first do a T-duality along α direction. The metric becomes

$$ds_{\text{IIA}}^2 = H^{-\frac{1}{2}} ds_{0123}^2 + H^{\frac{1}{2}} \left\{ dr^2 + r^2 \left[\frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1-cy}{2f(y)}dy^2 \right. \right.$$

$$\left. + \frac{f(y)}{9(a-y^2)}(d\psi^2 - \cos\theta d\phi)^2 + \frac{(1-cy)}{2Hr^4(a-y^2)}d\alpha^2 \right] \bigg\}$$

$$= H^{-\frac{1}{2}} \left[dx_{0123}^2 + \frac{1-cy}{2r^2(a-y^2)}d\alpha^2 \right] + H^{\frac{1}{2}} \left\{ dr^2 + r^2 \left[\frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) \right. \right.$$

$$\left. + \frac{1-cy}{2f(y)}dy^2 + \frac{f(y)}{9(a-y^2)}(d\psi - \cos\theta d\phi)^2 \right] \bigg\}, \tag{4.6}$$

with the following two components of the B-fields:

$$B_{\alpha\psi} = \frac{ac - 2y + y^2 c}{6(a - y^2)}, \qquad B_{\alpha\phi} = -\frac{ac - 2y + y^2 c}{6(a - y^2)} \cos\theta, \tag{4.7}$$

and the original D3 branes become D4 branes. The existence of the two B-fields might indicate the possibility of two NS5 branes, provided $H_{NS} = dB$ is a source term and the integral of H_{NS} over a three-cycle is an integer. The first one is harder to determine because the knowledge of the global behavior of the two B-field components is lacking, although the metric that we are dealing with is global. This is because we *delocalized* along the α direction to make the harmonic function H independent of that direction so that T-duality rules of [40] could be implemented. This is of course a slight oversimplification as this works well for some purposes, but not others. The harmonic function should be taken to be a function of α as well, and then one may T-dualise the background using the technique illustrated in [41]. Under such a T-duality both the B-field components will pick up dependences on H as well. We will discuss more on this a little later.

For the second case, one may do better by converting the three-forms to two-forms and integrating over two-cycles. This can be easily achieved by making a U-duality transformation of the form $T_\alpha S T_3$ where S denotes a S-duality transformation and T_m denotes a T-duality along x^m direction. Thus making a T-duality along x_3 direction we get the following metric in type IIB theory:

$$ds^2 = H^{-\frac{1}{2}}dx_{012}^2 + H^{\frac{1}{2}}\left\{ dx_3^2 + dr^2 + r^2\left[\frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1-cy}{2f(y)}dy^2 \right.\right.$$
$$\left.\left. + \frac{f(y)}{9(a-y^2)}(d\psi^2 - \cos\theta d\phi)^2 + \frac{(1-cy)}{2Hr^4(a-y^2)}d\alpha^2 \right] \right\}. \tag{4.8}$$

Under this T-duality the D4 branes become D3 branes but extending along x^0, x^1, x^2 and α directions. However the B-fields remain unchanged. If these B-fields are coming from some source NS5-branes, then the NS5-branes would not change under the T-duality.

Let us now do the S-duality under which the NS B-fields become RR B-fields and the metric gets an overall factor from the dilaton field $\sqrt{\frac{2r^2(a-y^2)}{1-cy}}$ while the D3 branes remain the same. When we T-dualise this background along α direction, the metric becomes

$$ds^2 = \mathcal{H}^{-\frac{1}{2}}dx_{012}^2 + H\mathcal{H}^{-\frac{1}{2}}\left\{ dx_3^2 + dr^2 + r^2\left[\frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1-cy}{2f(y)}dy^2 \right.\right.$$
$$\left.\left. + \frac{f(y)}{9(a-y^2)}(d\psi^2 - \cos\theta d\phi)^2 + d\alpha^2 \right] \right\}, \tag{4.9}$$

and the RR three-form fields become the type IIA gauge fields. We have also defined $\mathcal{H} = \frac{H}{2r^2}\frac{1-cy}{a-y^2}$ as our modified harmonic function. If these gauge fields are sourced by D6 branes then they are the ones that come from the type IIB D5 branes. The D3 branes on the other hand become D2 branes. Lifting this configuration to M-theory the eleventh direction has the required local ALE fibration with M2 branes at a point on the four-fold.

The above set of manipulation is suggestive of NS5 branes in the original type IIA configuration provided the gauge field EOM has a source term. Thus if we write the local type IIA gauge field over a patch as:

$$A = A_\psi d\psi + A_\phi d\phi \equiv \frac{ac - 2y + y^2 c}{6a - 6y^2}\Big[\mathcal{F}_1(H)d\psi - \mathcal{F}_2(H)\cos\theta\, d\phi \Big], \tag{4.10}$$

where we have inserted the correction from the harmonic function as $\mathcal{F}_{1,2}(H)$, then there exists a global field strength $F = dA$. Now if it satisfies the two conditions mentioned earlier, namely

$$d * F = \text{sources}, \qquad \int_{S^2} F = \text{integer}, \qquad (4.11)$$

then this would not only help us to identify the NS5 branes in the original type IIA set-up, but also help us to *count* the number of the NS5 branes. Such a source term in (4.11) may not be too difficult to see from our analysis if we take (4.10) seriously. The l.h.s. of (4.11) will involve terms like $d * d\mathcal{F}_1(H)$ and $d * d\mathcal{F}_2(H)$. Since $\square H = *d * dH$ lead to source terms in the supergravity solution, it should be no surprise if the above two terms in (4.11) coming from $\mathcal{F}_{1,2}(H)$ lead to D6 brane source terms in our model.

The above analysis is definitely suggestive of this scenario, although the precise orientations of the NS5 branes are not clear to us at this stage. Furthermore there is the subtlety pointed out in [42] which we might have to consider too. Note also that from (4.6) the D4 branes are wrapped along a non-trivial S_α^1 cycle. More details on this will be relegated to future works.

Before we end this subsection, we would like to point out another scenario related to the type IIB metric (2.6). As has been described earlier, (2.6) is related to (2.1) by a series of coordinate transformations. Interestingly the metric (2.6) is closely related to the conifold metric if one makes the following substitutions in (2.6):

$$c = 0, \qquad a = 3, \qquad y = -\cos\theta_2, \qquad \beta = \phi_2, \qquad \theta = \theta_1, \qquad \phi = \phi_1 \qquad (4.12)$$

where β was defined in (2.5). So a natural question to ask would be what happens if one makes a T-duality along the ψ direction. It is of course well known that, in the limit (4.12), a T-duality along ψ direction leads to an orthogonal (not necessarily intersecting) NS5 branes configuration [3]. If we now make a T-duality along ψ direction, the metric that we get in type IIA side is the following:

$$ds^2 = H^{-1/2} \left[dx_{0123}^2 + \frac{18(1 - cy)}{r^2 W} d\psi^2 \right] + H^{1/2} \left[dr^2 + r^2 \left(\frac{1 - cy}{6} (d\theta^2 + \sin^2\theta d\phi^2) \right. \right.$$
$$\left. \left. + \frac{1 - cy}{2f} dy^2 + \frac{4f}{W} d\alpha^2 \right) \right], \qquad (4.13)$$

where $W = 3c^2 y^2 - 6cy + 2 + ac^2$. Interestingly, we find the metric has the simpler form without cross-terms at all. This is again reminiscent of [3]. We also find two NS B fields whose components are given as:

$$B_{\psi\alpha} = \frac{6(ac - 2y + cy^2)}{W}, \qquad B_{\psi\phi} = -\cos\theta. \qquad (4.14)$$

The absence of a cross-term is not a big surprise because we can rewrite (2.6) in a suggestive way using the coordinates (4.12) and taking (c, a) away from the conifold value

$(0,3)$. The metric (2.6) becomes:

$$ds^2 = a_1(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \left[a_2 \sin^2\theta_2 d\theta_2^2 + a_3 \left(d\phi_2 + c\,\cos\theta_1 d\phi_1\right)^2\right]$$
$$+ \frac{1}{9}\left[d\psi + (1 + c\,\cos\theta_2)\cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2\right]^2, \qquad (4.15)$$

where a_1, a_2 and a_3 are given by the following expressions:

$$a_1 = \frac{1 + c\,\cos\theta_2}{6}, \quad a_2 = \frac{1}{2} \cdot \frac{1 + c\,\cos\theta_2}{a - 3\cos^2\theta_2 - 2c\,\cos^3\theta_2}, \quad a_3 = \frac{1}{18} \cdot \frac{a - 3\cos^2\theta_2 - 2c\cos^3\theta_2}{1 + c\,\cos\theta_2}. \quad (4.16)$$

A T-duality along ψ direction will give us the configuration that we discussed above (using non-canonical coordinates).[11] To see what (4.13) and (4.14) imply, let us again go to the limit where $c = 0$ and $a = 3$. In this limit[12] we recover the exact brane picture of type IIA discussed in [3]. This may mean that we have some NS5 branes along the (θ, ϕ) directions and some NS5 branes along (α, y) directions (or in a more canonical language, we have a set of NS5 branes along (θ_1, ϕ_1) directions and another set of NS5 branes along (θ_2, ϕ_2) directions). These two set of NS5 branes are locally orthogonal to each other, so as to preserve $\mathcal{N} = 1$ supersymmetry. The $d\psi$ fibration structure in (4.15) also tells us that there are two local B-fields in type IIA side that would T-dualise to give us the required background (4.15). The N type IIB D3-branes become N of D4 branes along ψ direction suspended between these NS5 branes.

Unfortunately the $c \neq 0$ scenario is not quite the same as the simpler $(c, a) = (0, 3)$ scenario. In particular[13] at $y = y_1$ and $y = y_2$ the metric (4.13) develops conical singularities, in other words now y and α no longer form a sphere. This can be easily seen by taking the limit $y \to y_i$ where $i = 1, 2$. In this limit we can write the metric along the y and α directions as:

$$\frac{1 - y_i}{f_i'(y - y_i)} dy^2 + \frac{4 f_i'(y - y_i)}{W_i} d\alpha^2. \qquad (4.17)$$

This is not quite the metric of a 2-sphere. To see this more clearly, let us define a quantity R in the following way:

$$R \equiv 2\sqrt{\frac{(1 - y_i)(x - y_i)}{f_i'}}. \qquad (4.18)$$

Using this defination we can rewrite the metric (4.17) in a bit more suggestive way:

$$dR^2 + \frac{f_i'^2 R^2}{(1 - y_i)W_i} d\alpha^2. \qquad (4.19)$$

---

[11]Note however that (4.13) and the T-dual of (4.15) may look different because in (4.13) one cannot substitute the coordinate transformation directly as the coordinates of (4.13) are the T-dual coordinates of (2.1). Thus a simple substitution of $\alpha = -\frac{1}{6}(\phi_2 + c\psi)$ in (4.13) cannot be done.

[12]For all other purposes we set $c = 1$.

[13]We will henceforth use only the non-canonical coordinates by choice. An equivalent construction could be easily done with the canonical coordinates (4.12).

Clearly the above metric becomes the metric of a 2-sphere only when α is periodic with a period of $L \equiv 2\pi\sqrt{(1-x_i)W_i}/f_i'$. However recall that instead α has a period of $l \neq L$. This means we will always have two conical singularities at $y = y_i$.

Let us now prove that there are no other singularities in this metric. Notice that other singularities can happen only at $W = 0$, which has two roots:

$$y_{\pm} \;=\; 1 \pm \sqrt{\frac{1-a}{3}}. \tag{4.20}$$

Since $y_+ > 1$, it is clear that y_+ is already out of the range of y, while it is not so obvious for $0 < y_- < 1$. To see the range of y_-, we substitute y_- into f to get:

$$f(y_-) \;=\; -\frac{2}{3\sqrt{3}}(1-a)\sqrt{1-a} \;<\; 0, \tag{4.21}$$

which means $y_- > y_2$ and therefore it is also out of the range. Therefore there are no other singularities in this metric.

The above picture gives us an indication how the brane dual could be constructed although the actual details are much harder to present than our previous construction. It is also true that the delocalization effects are again present in the harmonic function but this time, thanks to the canonical representation of the metric (4.15), a direct mapping to the intersecting brane configuration for $c = 0, a = 3$ gives us a hope that similar brane dual description does exist for generic cases (although at this stage one may need to consider the subtleties pointed out in [42]). The interesting thing however is that a T-duality along α also seems to lead to a similar configuration provided of course (4.11) holds. This shouldn't be a surprise because α and ψ are related by a linear coordinate transformation for $c \neq 0$.

4.2 Non-commutative and dipole deformations

The above T-duality arguments give us a way to study the underlying $\mathcal{N} = 1$ gauge theory from two different point of views: one directly from N D3 branes at the tip of the cone in type IIB theory, and other from N D4 branes in a configuration of two orthogonal set of NS5 branes in type IIA theory; although for the latter case the precise orientations of the two NS5 branes still need to be determined.

The non-commutative and the dipole deformations could also be studied from these two viewpoints. However in this paper we will not consider the type IIA brane interpretations of these deformations. Here we will suffice with only the type IIB description and a fuller picture will be elaborated in a forthcoming work.

Our starting point is the well known observation that once we have a solution we can use TsT to deform it into various different solutions, where T is a T-duality transformation and s is a shift.

Given the background metric (4.5) with D3 branes we have three kinds of deformations:

- T-dualise along one space direction say x_3 then shift along another space direction say x_2 mixing (x_2, x_3) and then T-dualise back along x_3 direction.

- T-dualise along x_3 and then shift[14] along one of the internal directions that are isometries of the background namely along α, ϕ or ψ and then T-dualise back along x_3 direction.

- T-dualise, shift and then T-dualise along internal directions.

The first of these operations would lead to the non-commutative gauge theory on the D3 branes whose details we discussed earlier. For the other two cases, the set of operations may lead to non-local dipole theories on the D3 branes.

In this paper we only study the first kind of deformation, whose advantage is that the internal metric remains unchanged so our scalar modes analysis in $Y^{p,q}$ is still valid. Of course this still doesn't help us to get the exact matching of spectra as we pointed out earlier. Therefore, in the following, we will briefly spell out the supergravity background. For the rest two kinds of deformations our analysis generally cannot be applied as the internal metric will change quite a bit. We will leave a detailed analysis of dipole deformations for future works.

For the non-commutative case, the starting point would be the choice of the shift after a T-duality along the x_3 direction. We choose the shift to be

$$x_2 \mapsto \frac{x_2}{\cos\theta} + x_3 \, \sin\theta, \quad x_3 \mapsto x_3 \, \cos\theta. \tag{4.22}$$

After the series of duality transformations the background can be easily determined to take the following form:

$$ds^2 = \frac{1}{\sqrt{H}} \Big[-dx_0^2 + dx_1^2 + J(dx_2^2 + dx_3^2) \Big] + \sqrt{H}(dr^2 + d\mathcal{M}_5^2), \tag{4.23}$$

which clearly tells us that the internal $Y^{p,q}$ space do not change, but the Lorentz invariance along the x^2 and x^3 direction is broken as one would have expected. The metric has the same form as in [43] and the gauge theory on D3 branes should be non-commutative in x_2 and x_3 directions. The non-commutativity parameter, which is the B_{23} field, and the Lorentz breaking term J, in (4.23), are defined in the following way:

$$J \;=\; \frac{H}{\sin^2\theta + H\cos^2\theta}, \qquad B_{23} \;=\; \frac{\tan\theta}{\sin^2\theta + H\cos^2\theta}. \tag{4.24}$$

This completes our discussion of the conformal models related to the $Y^{p,q}$ spaces. In the following section we will discuss the non-conformal extensions of the above models. We will specifically concentrate on the possibility of geometric transitions in these models.

5 Non-conformal duals and geometric transitions

The non-conformal duals to the $Y^{p,q}$ spaces, along the lines of the cascading model of [44], have already been addressed in the literature (see for example [45, 46] etc). The UV gauge

---

[14]Again mixing x_3 with one of the internal directions.
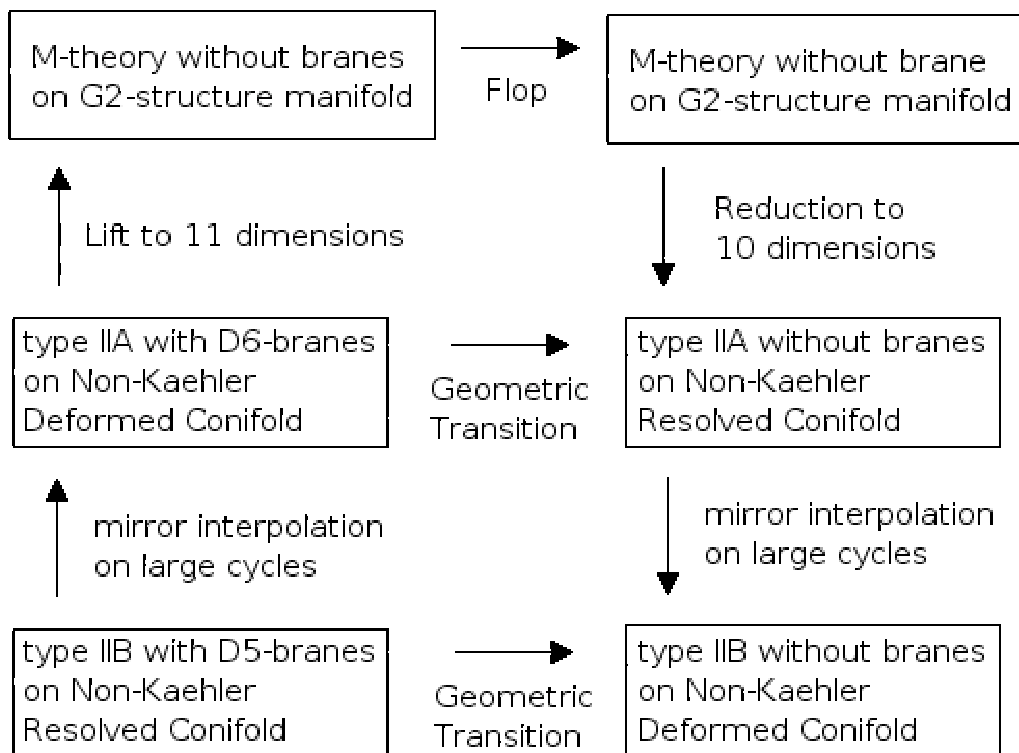
Figure 1. The duality map to generate the full geometric transitions in the supersymmetric global set-up of type IIA and type IIB theories.

groups for $Y^{p,1}$ and $Y^{p,p-1}$ are respectively given in equations (75) and (87) of [45]. For both the cases the IR gauge group is:

$$SU(M) \times SU(2M) \times ... \times SU(2pM) \tag{5.1}$$

where M denotes the number of D5 branes wrapping the two-cycles of $Y^{p,1}$ and $Y^{p,p-1}$ spaces. Such a gauge group is more complicated than the simple picture that we had for [44] and therefore the *far* IR picture could be more involved: there could be non-trivial baryonic branches. This story has not yet been fully clarified, and therefore it gives hope that the brane picture that we developed here may help us to study the far IR picture in more details.[15] We will however not pursue the cascading story anymore here. Instead we go to a slightly different direction that may provide us with an alternative way to study the far IR physics of these models [48, 49].

Our starting point would be to ask whether the far IR physics of the non-conformal set-up could be likened to the geometric transtion story [50] that we developed in the series of papers starting with [51] and culminating with [49]. For the geometric transition picture to hold, we need few essential ingredients:

---

[15]For example the brane picture developed for the $T^{1,1}$ case in [47] clearly showed how the far IR physics for cascading theory could be understood. We expect similar story to unfold here too.

- Resolution and deformation for the cone over $Y^{p,q}$. These resolved and deformed spaces are not required to be Calabi-Yau spaces, but they should have at least SU(3) structures (in the presence of branes and fluxes) so that supersymmetric models could be constructed.

- Supersymmetric configurations with D5 branes wrapped on two-cyles of the resolved $Y^{p,q}$ and D6 branes wrapped on three-cycles of the deformed $Y^{p,q}$ including supersymmetric configurations *without* branes but with fluxes. Again the *overall* pictures for both cases should preserve SU(3) structures.

- Two kinds of G_2 structure manifolds should exist in M-theory. One, the lift of the deformed $Y^{p,q}$ space with wrapped D6 branes in type IIA, and two, the lift of the resolved $Y^{p,q}$ space with fluxes but without branes again in type IIA. Additionally these two G_2 structure manifolds should be related by a flop transition, similar to the one constructed for the $T^{1,1}$ case in [52].

If all the three ingredients discussed above are present then one would be able to describe geometric transition using the resolved and the deformed $Y^{p,q}$ manifolds via the duality map given in figure 1. In the following we will describe a possible realisation of these scenarios. Our starting point would be the resolution and the deformation of the cones over $Y^{p,q}$ manifolds, which lie in the heart of these scenarios.

5.1 Resolution and deformation of the cones over $Y^{p,q}$

A natural question is whether there can be resolutions for the cone over $Y^{p,q}$ as the resolved conifold. The answer is in the affirmative and the metric on the resolved cone over $Y^{p,q}$ was obtained explicitly in [53, 54] and [55]. The metric is,

$$ds^2_{RS} = \frac{(1-y)(1-x)}{3}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{(y-x)(1-y)}{h(y)}dy^2 + \frac{(x-y)(1-x)}{f(x)}dx^2$$
$$+\frac{f(x)}{9(1-x)(x-y)}\Big[d\psi - \cos\theta d\phi + y(d\beta + \cos\theta d\phi)\Big]^2$$
$$+\frac{h(y)}{9(1-x)(y-x)}\Big[d\psi - \cos\theta d\phi + x(d\beta + \cos\theta d\phi)\Big]^2, \tag{5.2}$$

where $f(y) = 2y^3 - 3y^2 + a$ and $h(x) = 2x^3 - 3x^2 + b$. We will also take the sechsbein e^a to be the ones given in eq (2.8) of [48] with appropriate redefinations of the variables therein.

As explained in the subsection 2, $y_- < y < y_+$. One can take x to be non-compact and denote two consecutive roots of $h(x)$ by x_- and x_+. We focus on the case where the resolution is obtained by blowing up a CP^1, referred to as small partial resolutions in [55]. For this type of resolution we have $x_- = y_-$ which requires $a = b$. Thus we get,

$$-\infty < x < y_-, \quad y_- < y < y_+, \quad a = b. \tag{5.3}$$

If one takes $x = -r^2/2$ and expand the metric (5.2) in the large r it becomes $ds^2_{RS} \to dr^2 + r^2 ds^2$ where ds^2 is exactly (2.6), so it is a cone over $Y^{p,q}$.

Having got the resolution of the cone over $Y^{p,q}$, we now want to study the deformation of the cone over $Y^{p,q}$, which should be a mirror of the resolved cone over $Y^{p,q}$. Strominger, Yau, and Zaslow conjectured that the mirror manifold can be obtained by three T-dualities [56]. There are three isometric directions ψ, β and ϕ, so we will first do T-dualities along these directions. The metric we get after three T-dualities is:

$$
\begin{aligned}
ds^2_{\text{SYZ}} = {} & \frac{(1-y)(1-x)}{3}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{(y-x)(1-y)}{h(y)}dy^2 + \frac{(x-y)(1-x)}{f(x)}dx^2 \\
& + \frac{f(x)h(y)(x-y)\cos\theta}{9(f(x)y(1-y)^2) - h(y)x(1-x)^2}\left(d\psi + \frac{f(x)y^2(1-y) - h(y)x^2(1-x)}{f(x)y(1-y) - h(y)x(1-x)}d\beta + \frac{d\phi}{\cos\theta}\right) \\
& + \frac{9h(y)(1-x)}{f(x)h(y)(x-y)\cos^2\theta}\Big[(1-x)\cos\theta d\beta + xd\phi\Big]^2 \\
& + \frac{9f(x)(1-y)}{f(x)h(y)(y-x)\cos^2\theta}\Big[(1-y)\cos\theta d\beta + yd\phi\Big]^2.
\end{aligned}
\tag{5.4}
$$

The above metric however cannot be the full answer as T-dualities *à la* [56] require us to take the base to be very large. In [49] (see also [51]) we saw that making the base large actually *mixes* the isometry directions, leading eventually to the generation of additional cross-terms missing from the metric obtained by making naive T-dualities. Thus the actual mirror metric will have cross-terms in addition to what we already have in (5.4).

The complete picture is rather involved as the recipe for making the base bigger using coordinate transformations *à la* [49] is not readily available now. However despite this obstacle, one thing is clear from the analysis of [49]: the resultant metric will not be a Kähler manifold, in fact, it may not even be a complex manifold. This is consistent with the result of [57, 58] (see also [59] where certain obstructions to the existence of Sasaki-Einstein metrics on this manifold is shown). It will also be interesting to compare our result with the one got in [60].

5.2 D5 branes on the resolved $Y^{p,q}$ manifold

The technical obstacle that we encountered in the previous subsection doesn't prohibit us to write the metric of N D5 branes wrapped on the two-cycle of the resolved cone over $Y^{p,q}$ manifold. Recently the NS5 brane picture has been studied in [48]. The analysis of [48] is similar in spirit to the one discussed in [49], both the analyses being motivated by the work of [61]. The complete background for N D5 branes wrapped on the resolution two-cycle is given by:

$$
F_3 = h \cosh\beta\, e^{-2\phi} * d\left(e^{2\phi}J\right), \qquad\qquad H_3 = -hF_0^2\sinh\beta\, e^{-2\phi}d\left(e^{2\phi}J\right)
$$

$$
F_5 = -\frac{1}{4}(1+*)dA_0 \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3
\tag{5.5}
$$

$$
ds^2 = F_0 ds^2_{0123} + \sum_{a=1}^{6} f_a e^{2a}, \qquad\qquad \phi = \log F_0 + \frac{1}{2}\log h,
$$

where e^a are the sechsbein defined in [48] and J is the fundamental form associated with the internal metric. The above background is supersymmetric by construction and since

the RR three-form F_3 is not closed, it represents precisely the IR configuration of wrapped D5-branes on warped non-Kähler resolved $Y^{p,q}$ manifold. The two warp factors (h, F_0) as well as the coefficients f_a in the internal metric are all functions of (r, y, x) which, in turn, preserve the three isometries of the internal space. Notice also that the background has a non-trivial dilaton, with the internal space being a non-Kähler resolved cone over $Y^{p,q}$. The form of the background (5.5) is similar to the one that we had in [49] except now the internal space is different. This is of course expected if one had to preserve $\mathcal{N} = 1$ supersymmetry. The five-form, which is switched on to preserve the susy, has the form F_5 in (5.5) with:

$$
\begin{aligned}
A_0 &= \frac{\cosh \beta \, \sinh \beta (1 - e^{-2\phi} h^{-2} F_0^{-4})}{e^{2\phi} h^{-2} F_0^{-4} \cosh^2 \beta - \sinh^2 \beta} \\
&= (F_0^2 - 1) \tanh \beta \left[1 + \left(\frac{1 - F_0^2}{F_0^2} \right) \operatorname{sech}^2 \beta + \left(\frac{1 - F_0^2}{F_0^2} \right)^2 \operatorname{sech}^4 \beta \right].
\end{aligned}
\tag{5.6}
$$

Let us now make a few observations. The parameter β that we have in the background is in general constant and could take any value. This means that there is a class of allowed backgrounds satisfying the supersymmetry condition. Imagine also that we define a six-dimensional internal space in the following way:

$$
ds_6^2 = \left(\frac{N F_0 \cosh^2 \beta}{1 + F_0^2 \sinh^2 \beta} \right) \sum_{a=1}^{6} f_a e^{2a},
\tag{5.7}
$$

then one could easily argue that there are a series of dualities[16] that would convert the following background

$$
ds^2 = ds_{0123}^2 + N ds_6^2, \qquad H_{\mathrm{NS}} = e^{-2\Phi} * d \left(e^{2\Phi} J \right), \qquad \Phi = -\phi
\tag{5.8}
$$

to the one given earlier in (5.5). The above background (5.8) is of course the one studied in [48]. Although this is no big surprise, but it is satisfying to see that our picture can be made consistent with both [48] as well as [49].

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[16] Starting with the background (5.8), we perform a S-duality that transforms the NS three-form to RR three-form F_3 and converts the dilaton Φ to ϕ without changing the metric in the Einstein frame. We now make three T-dualities along the spacetime directions $x^{1,2,3}$ that takes us to type IIA theory. Observe that this is *not* the mirror construction. We then lift the type IIA configuration to M-theory and perform a boost (with a parameter β) along the eleventh direction. This boost is crucial in generating D0-brane *gauge* charges in M-theory. A dimensional reduction back to IIA theory does exactly what we wanted: it generates the necessary number of D0-brane charges from the boost, without breaking the underlying supersymmetry of the system. Finally, once we have the IIA configuration, we go back to type IIB by performing the three T-dualities along $x^{1,2,3}$ directions. From the D0-brane charges, we get back our three-brane charges namely the five-form. The duality cycle also gives us NS three-form H_3 as well as the expected RR three-form F_3. Therefore the final configuration is exactly what we required for IR geometric transition: wrapped D5s with necessary sources on a non-Kähler globally defined resolved $Y^{p,q}$ background (5.5). Also as expected, the background preserves supersymmetry and therefore should be our starting point. One may also note that the thee-forms that we get in (5.5) satisfy

$$
\cosh \beta \, H_3 \; + \; F_0^2 \, \sinh \beta * F_3 \; = \; 0
$$

which is the modified ISD (imaginary self-duality) condition. For more details, see [49, 61].

5.3 Toward geometric transitions for $Y^{p,q}$ manifolds

Once we have the background (5.5) and (5.6) we should be able to use directly the duality cycle shown in figure 1. This however will turn out to be more subtle than the story that we developed in [49]. But before we go about elucidating the issues, let us clarify certain things about *generalized* SYZ. The original work of SYZ [56] is based on two facts: (a) all Calabi-Yau manifolds can be written in terms of a T^3 fibration over a base $\mathcal{B}$, and (b) in the limit where $\mathcal{B}$ is much larger than the T^3 fiber, mirror of the given CY manifold is given by three simultaneous T-dualities along the T^3 fiber directions.

For our case, the starting manifold (5.5) is not a CY manifold but instead is a six-dimensional manifold with an SU(3) structure and torsion H_3. For this case there does exist a generalisation of the SYZ technique: it is again given by three T-dualities along the T^3 fiber [62–64]. The difference now is that we cannot claim that *all* SU(3) structure manifolds can be expressed in terms of T^3 fibrations over some base manifolds (although [63–65] has discussed more generic cases by applying *local* T-dualities). This generalization of the SYZ technique is called the generalized mirror rule.[17]

Our method now would be to use the generalized SYZ technique to go to the type IIA mirror manifold with wrapped D6-branes. Unfortunately now there are two subtleties that make the analysis much more non-trivial than the one that we had in [49]. The first one is already been discussed earlier: we don't know exactly what kind of coordinate transformations we should do to make the base bigger than the T^3 fiber. Recall that in [49], out of infinite possible coordinate transformations available, we could find a class of transformations that can not only make the base bigger but also lead us to the right mirror manifold. The main reason why we could find that particular class of transformations earlier was solely based on the fact that we *knew* the existence of a deformed conifold solution. This privileged information, unfortunately, is not available to us now.

The second issue is even more non-trivial. Looking at the background (5.5) and from $H_3 = dB_{\mathrm{NS}}$, we see that the B_{NS} fields will have components that are *parallel* to the directions of the T^3 fiber. T-dualities with B_{NS} fields along the directions of duality lead to non-geometric manifolds! Therefore the type IIA dual manifold will most likely be a non-geometric space which in turn means that the duality cycle depicted in figure 1 cannot be very straightforward.[18]

Existence of non-geometric space, however, does not mean that there is no underlying gauge/gravity duality. In fact in the geometric transition set-up there were already indications, even for the simplest resolved conifold case, that the full gauge/gravity duality will involve non-geometric manifolds [67], although we argued in [49] that there is

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[17]For more details as to why the generalized mirror rule would lead to another SU(3) structure manifold that is the *mirror* of the original manifold is discussed in [63, 64].

[18]There is a third subtlety that has to do with the size of the T^3 fiber in the mirror manifold. If the size of the fiber is small i.e of $\mathcal{O}(\alpha')$, then supergravity description may not be possible, and one might have to go to a Gepner type sigma model description. For the model studied in [49, 51] this was not an issue because we could study a class of manifolds parametrised by choice of warp factors that not only satisfy EOMs but also lie in subspaces, where sugra descriptions are valid, on both sides of figure 3 in [66]. These subspaces are related by geometric transitions. For generic choices of the warp factors in [49, 66], it would be interesting to see if the subspaces could incorporate the $Y^{p,q}$ manifolds.

small configuration space of fluxes where we expect the duality to be captured by purely geometric manifolds. The question now is whether such a scenario, with only geometric spaces, could be realised for the present case also. We will leave this for future work.

6 Conclusion and open question

In this paper we studied the scalar spectrum of $Y^{p,q}$ manifold. Earlier works in this direction [11, 12] mostly studied the lowest eigenmodes of the scalar Laplacian, as finding the exact eigenmodes in closed form for the full tower of states is practically impossible. The main difficulty lies in the existence of four regular singular points for an operator of Heun type that the scalar Laplacian can be reduced to. Despite this problem we have managed to find both upper and lower bounds for all the eigenmodes λ_k of the scalar Laplacian. Our result can be expressed as:

$$\Lambda_k^{[-]} \;\leqslant\; \lambda_k \;\leqslant\; \Lambda_k^{[+]} \tag{6.1}$$

where $k = 1, 2, \ldots$ and $\Lambda_k^{[\pm]}$ are given in (3.39). We also show that asymptotically, i.e for large k, the eigenmodes grow quadratically as in (3.41). Note that this is the opposite regime of the spectrum of [11], where they give exact lowest eigenvalues. Our analysis presented here works best for $a \ll 1$ or equivalently $q \ll p$. By comparing against the known low-lying scalar spectrum for example those from [11], we may learn something useful about the bound. Furthermore, our natural guess would be that the spectra contain all the BPS and non-BPS states. As we saw earlier in (3.41), for large k, the masses are proportional to k and are therefore additive to leading order. However for small k we don't have the precise behavior and therefore cannot pin-point their BPS or non-BPS nature. In fact, as we wrote above, we expect both these states to show up. More details on this will be discussed elsewhere. It will be also interesting to study the implications of the consistent massive truncation along the line of [68–71] where they also focus on the lowest massive modes.

The absence of solution in closed form signifies the possibility of introducing numerical techniques to solve the problem. This is along the lines of [13] where a numerical study was done for the simplest Sasaki-Einstein manifold, namely the S^5. Unfortunately the success of the S^5 case doesn't necessarily guarantee the same for the $Y^{p,q}$ case, again precisely due to the fact that the corresponding Heun equation has four regular singular points. Therefore it seems at this stage, unless we know how to tackle these singularities, a closed form solution via numerical analysis looks unfeasible. Any progress in this direction will be a productive boost for completing the duality dictionary for this case.[19] For example computing the super-conformal index along the lines of [72], or even going beyond the supergravity modes à la [73].

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[19]In [11] the authors used AdS/CFT correspondence to map the Reeb Killing vector Q_R, $\hat{K} = J(J+1)$ and pN_α to the R-symmetry, SU(2) spin and U(1) flavor charge on the field theory side respectively. They compared states with quantum numbers (Q_R, J, N_α) and chiral operators with the same charges under these symmetries. With some chosen values of Q_R, J, pN_α they found the eigenvalues for the scalar Laplacian of $Y^{p,q}$ and from there claimed that the ground states satisfy the BPS condition. We find that the eigenvalues N_ψ and pN_α are not always integers, and their results are covered in our spectrum with some specified values of m, n, l, j. This can also be used to determine the range of the constant C .

Another line of thought that we followed in this paper is the non-conformal extensions of the conformal examples. These non-conformal models are closer to the geometric transition models of [49–51] and therefore would require the existence of the corresponding deformed cones over the $Y^{p,q}$ manifolds. The deformed cones over $Y^{p,q}$ manifolds, unfortunately, couldn't be Calabi-Yau manifolds [57–59] so the underlying picture cannot be as simple as the ones studied in [49, 51]. Our analysis, however, reveals that the gravity duals might not even be geometric manifolds, so that the obstructions pointed out in [57–59] could be circumvented. Although no concrete examples exist at this stage, the above approach is a hopeful avenue to realise non-conformal duals. Additionally a success in this direction would also be a good test for the generalized mirror symmetry that relate two manifolds with SU(3) structures (i.e manifolds with intrinsic torsions and H_3 fluxes).

Clearly what we opened up here is just the tip of an iceberg, and happily there are more questions than answers right now. In future works we will address some of these issues in more details.

Acknowledgments

A Eigenvalues of the differential operator S

To make this paper self-contained, here we will borrow two lemmas proved in [10].

Let us consider the differential operator

$$S_{ml}(\Lambda) := \frac{1}{\rho(y)} \frac{\partial}{\partial y} \rho(y)\, w(y)\, r(y)\, \frac{\partial}{\partial y} - \frac{1}{w(y)} \left(\frac{\sigma l}{\tau}\right)^2 - \frac{9}{r(y)} \left(2m - h(y)\, \frac{\sigma l}{\tau}\right)^2 - \frac{6\Lambda}{1-y}\,, \quad (A.1)$$

arising from (3.14), which depends on a real parameter $\Lambda \geq 0$. It is clear that we cannot hope to express the solutions of the formal eigenvalue equation

$$S_{ml}(\Lambda)\, w = -\lambda w \qquad (A.2)$$

in closed form using special functions because (A.2) is a Fuchsian differential equation with *four* regular singular points, located at the three roots of the cubic $a - 3y^2 + 2y^3 = 0$, at 1, at $\pm a^{1/2}$ and at infinity.[20] However, the information contained in the following lemma will suffice for our purposes.

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[20]An ordinary differential equation whose only singular points, including the point at infinity, are regular singular points is called a Fuchsian ordinary differential equation.

Lemma 1. *For all $\Lambda \geq 0$, the differential operator* (A.1) *defines a nonnegative self-adjoint operator in $L^2((y_-, y_+), \rho(y)\,dy)$, which we also denote by $S_{ml}(\Lambda)$, whose domain consists of the functions $w \in \mathcal{AC}^1((y_-, y_+))$ such that $S_{ml}(\Lambda)w \in L^2((y_-, y_+))$ and*

$$\lim_{y \searrow y_-} y\,w'(y) = 0 \ \ \text{if } m = (2p-q)\sigma l/4 \quad \text{and} \quad \lim_{y \nearrow y_+} y\,w'(y) = 0 \ \ \text{if } m = -q\sigma l/4\,.$$

Its spectrum consists of a decreasing sequence of eigenvalues $(-\ell_{mlk}(\Lambda))_{k\in\mathbb{N}} \searrow -\infty$ of multiplicity one whose associated normalized eigenfunctions $w_{mlk}(\Lambda)$ are $\mathcal{O}((y_+ - y)^{|m+q\sigma l/4|})$ as $y \nearrow y_+$ and $\mathcal{O}((y - y_-)^{|m+(q-2p)\sigma l/4|})$ as $y \searrow y_-$.

Proof. Let y_ϵ be one of the endpoints of the interval (y_-, y_+) and set $\zeta := y - y_\epsilon$. An easy computation shows that

$$a - 3y^2 + 2y^3 = -6y_\epsilon(1 - y_\epsilon)\,\zeta + \mathcal{O}(\zeta^2)\,, \qquad r(y) = -\frac{\zeta}{3y_\epsilon} + \mathcal{O}(\zeta^2)$$

as $y \to y_\epsilon$, which shows that the differential equation (A.2) can be asymptotically written as

$$-\left(12y_\epsilon\,\zeta + \mathcal{O}(\zeta^2)\right)\widetilde{w}''(\zeta) - \left(12y_\epsilon + \mathcal{O}(\zeta)\right)\widetilde{w}'(\zeta) + \left[\frac{3y_\epsilon}{\zeta}\left(2m - h(y_\epsilon)\frac{\sigma l}{\tau}\right)^2 + \mathcal{O}(1)\right]\widetilde{w}(\zeta) = 0\,,$$

with $\widetilde{w}(\zeta) := w(\zeta + y_\epsilon)$ standing for the expression of the function $w(y)$ in the new variable ζ. $\qquad\square$

It then follows that the characteristic exponents of the equation (A.2) at y_ϵ are $\pm\nu_\epsilon$, with $\nu_\epsilon := |m - h(y_\epsilon)\sigma l/(2\tau)|$. Using

$$\frac{h(y_+) - h(y_-)}{2\,h(y_+)} = \frac{p}{q}\,, \qquad \tau \equiv -2\,h(y_+)/q\,, \qquad \sigma := \mathrm{lcm}\{2, pq, 2p-q\}\,. \tag{A.3}$$

one can immediately derive the more manageable formula

$$\nu_+ = \big|m + q\sigma l/4\big|\,, \qquad \nu_- = \big|m + (q-2p)\sigma l/4\big|\,. \tag{A.4}$$

Let us now consider the following lemma.

Lemma 2. *Let $\mathcal{Y}$ be the complex Hilbert space*

$$\mathcal{Y} := \left\{ (u_{nml})_{n,m,l\in\mathbb{Z}} : u_{nml} \in L^2\big((y_-, y_+), \rho(y)\,dy\big) \otimes L^2\big((0,\pi), \sin\theta\,d\theta\big) \right\}\,,$$

endowed with the norm

$$\left\| (\Phi_{nml} \otimes \Theta_{nml})_{n,m,l\in\mathbb{Z}} \right\|_{\mathcal{Y}}^2 := \sum_{n,m,l\in\mathbb{Z}} \left(\int_{y_-}^{y_+} \big|\Phi_{nml}(y)\big|^2 \rho(y)\,dy \right) \left(\int_0^{2\pi} \big|\Theta_{nml}(\theta)\big|^2 \sin\theta\,d\theta \right)\,,$$

and with τ and σ defined as in (A.3). *Then the map defined by*

$$\mathcal{Y} \ni (\Phi_{nml} \otimes \Theta_{nml})_{n,m,l\in\mathbb{Z}} \mapsto \sum_{n,m,l\in\mathbb{Z}} \Phi_{nml}(y)\,\Theta_{nml}(\theta)\,\frac{e^{i(n\phi + 2m\psi + \sigma l\alpha/\tau)}}{(2\pi)^{3/2}} \in L^2(Y^{p,q})\,, \tag{A.5}$$

defines an isomorphism between $\mathcal{Y}$ and $L^2(Y^{p,q})$.

Since σ is even by the above lemma, it stems from the latter equation that $2\,\nu_\epsilon$ is a nonnegative integer. Therefore, it is standard that the symmetric operator defined by (A.1) on $C_0^\infty((y_-, y_+))$ is in the limit point case at y_ϵ if and only if $\nu_\epsilon \neq 0$. If $\nu_+\nu_- \neq 0$, the latter operator is then essentially self-adjoint on $C_0^\infty((y_-, y_+))$, and has a unique self-adjoint extension of domain [74]

$$\mathcal{D} := \left\{ w \in \mathcal{AC}^1((y_-, y_+)) : S_{ml}(\Lambda)\,w \in L^2((y_-, y_+)) \right\} \dots$$

When $\nu_+\nu_- = 0$, the above symmetric operator is not essentially self-adjoint. In this case, in order to rule out logarithmic singularities we shall choose its Friedrichs extension [75], whose domain consists of the functions $w \in \mathcal{D}$ such that

$$\lim_{y\searrow y_-} y\,w'(y) = 0 \ \text{ if } \nu_- = 0 \quad \text{ and } \quad \lim_{y\nearrow y_+} y\,w'(y) = 0 \ \text{ if } \nu_+ = 0\,,$$

It is well known [74] that $S_{ml}(\Lambda)$ is then a nonnegative operator with compact resolvent and that its eigenvalues are nondegenerate.

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