Published for SISSA by Springer

Received: *March 16, 2011*
Accepted: *May 5, 2011*
Published: *May 13, 2011*

JHEP05(2011)066

Towards extending the Chern-Simons couplings at order $O(\alpha'^2)$

Mohammad R. Garousi and Mozhgan Mir

Department of Physics, Ferdowsi University of Mashhad,
P.O. Box 1436, Mashhad, Iran

E-mail: garousi@ferdowsi.um.ac.ir, mo_mi757@stu-mail.um.ac.ir

Abstract: Using the compatibility of the anomalous Chern-Simons couplings on D$_p$-branes with the linear T-duality and with the antisymmetric B-field gauge transformations, some couplings have been recently found for $C^{(p-3)}$ at order $O(\alpha'^2)$. We examine these couplings with the S-matrix element of one RR and two antisymmetric B-field vertex operators. We find that the S-matrix element reproduces these couplings as well as some other couplings. Each of them is invariant under the linear T-duality and the B-field gauge transformations.

Keywords: Superstrings and Heterotic Strings, D-branes, String Duality

ArXiv ePrint: 1102.5510

doi:10.1007/JHEP05(2011)066

Contents

1 Introduction and results

The dynamics of the D-branes of type II superstring theories is well-approximated by the effective world-volume field theories which consist of the sum of Dirac-Born-Infeld (DBI) and Chern-Simons (CS) actions. The DBI action which describes the dynamics of the brane in the presence of the NSNS background fields at order $\mathcal{O}(\alpha'^0)$ is given by [1, 2]

$$S_{\rm DBI} = -T_p \int d^{p+1}x\, e^{-\phi}\sqrt{-\det\left(G_{ab} + B_{ab}\right)} \tag{1.1}$$

where G_{ab} and B_{ab} are the pulled back of the bulk fields $G_{\mu\nu}$ and $B_{\mu\nu}$ onto the world-volume of D-brane.[1] The abelian gauge field can be added to the action as $B_{ab} \to B_{ab} + 2\pi\alpha' f_{ab}$. The curvature corrections to this action has been found in [3] by requiring the consistency of the effective action with the $O(\alpha'^2)$ terms of the corresponding disk-level scattering amplitude [4, 5]. The couplings of non-constant dilaton and B-field at the order $O(\alpha'^2)$ have been found in [6] by requiring the consistency of the curvature couplings with the standard rules of linear T-duality transformations, and by the scattering amplitude.[2]

The CS part which describes the coupling of D-branes to the RR potential at order $\mathcal{O}(\alpha'^0)$ is given by [10, 11]

$$S_{\rm CS} = T_p \int_{M^{p+1}} e^B C \tag{1.2}$$

[1]Our index conversion is that the Greek letters $(\mu, \nu, \cdots)$ are the indices of the space-time coordinates, the Latin letters $(a, d, c, \cdots)$ are the world-volume indices and the letters $(i, j, k, \cdots)$ are the normal bundle indices.

[2]The couplings of non-constant gauge field strength f_{ab} to the DBI and CS actions have been considered in [7–9].

where M^{p+1} represents the world volume of the D$_p$-brane, C is the sum over all RR potential forms, and the multiplication rule is the wedge product. The abelian gauge field can be added to the action as $B \to B + 2\pi\alpha' f$. The curvature correction to this action has been found in [12–14] by requiring that the chiral anomaly on the world volume of intersecting D-branes (I-brane) cancels with the anomalous variation of the CS action. The curvature couplings at order $O(\alpha'^2)$ in static gauge are

$$\frac{\pi^2\alpha'^2 T_p}{2!2!(p-3)!}\int d^{p+1}x\epsilon^{a_0\cdots a_p}C^{(p-3)}_{a_4\cdots a_{p-4}}\left[R_{a_0a_1}{}^{ab}R_{a_2a_3ba} - R_{a_0a_1}{}^{ij}R_{a_2a_3ji}\right] \tag{1.3}$$

They have been confirmed by the S-matrix calculation in [15–17]. The above couplings have been extended in [18, 19] to include the B-field at the order $O(\alpha'^2)$ by requiring them to be consistent with the linear T-duality transformations. The new B-field couplings, however, are not invariant under the B-field gauge transformation. Adding some other B-field couplings which are themselves invariant under the linear T-duality, one can write the resulting couplings in a gauge invariant form [19]. The gauge invariant couplings are

$$\frac{\pi^2\alpha'^2 T_p}{2!2!(p-3)!}\int d^{p+1}x\epsilon^{a_0\cdots a_p}C^{(p-3)}_{a_4\cdots a_{p-4}}\left[\frac{1}{2}H_{a_0a_1a,i}H_{a_2a_3}{}^{a,i} - \frac{1}{2}H_{a_0a_1i,a}H_{a_2a_3}{}^{i,a}\right] \tag{1.4}$$

where commas denote partial differentiation. Unlike the gravity couplings, the B-field couplings (1.4) are not invariant under the RR gauge transformation so one may expect that there should be some other couplings as well. Since there are no gravity couplings for $C^{(p-3)}$ other than those given by (1.3), we are not allowed to have any other T-duality invariant couplings which include both gravity and B-fields. However, it is consistent to have couplings which involve only B-fields. In this paper, we will show that the S-matrix element of one RR and two B-field vertex operators produce the couplings (1.4) as well as some other T-duality invariant couplings.

The scattering amplitude of one RR potential $C^{(p-3)}$ and two gravitons at order $O(\alpha'^2)$ has no massless open string or closed string pole. It has only contact terms given by (1.3) [15–17] which are invariant under the RR gauge transformation. On the other hand, as we will see the scattering amplitude of one RR potential $C^{(p-3)}$ and two B-fields at order $O(\alpha'^2)$ has both open and closed string poles, as well as some contact terms. We will show that the sum of all these contributions at any order of α' has the RR gauge symmetry. We are interested in this paper in the massless open string poles and the contact terms of the amplitude which dictate the appropriate couplings on D-branes.

It has been shown in [20] that the CS action should also include couplings which involve linear NSNS field. These couplings have been found by studying the S-matrix element of one RR and one NSNS vertex operators at order $O(\alpha'^2)$ [4]. These couplings for $F^{(p)}$ are [20]

$$\frac{\pi^2\alpha'^2 T_p}{2!(p-1)!}\int d^{p+1}x\,\epsilon^{a_0\cdots a_p}\left(F^{(p)}_{ia_2\cdots a_p,a}H_{a_0a_1}{}^{a,i} - F^{(p)}_{aa_2\cdots a_p,i}H_{a_0a_1}{}^{i,a}\right)$$

They can be written in terms of the RR potential as

$$-\frac{\pi^2\alpha'^2 T_p}{(p-1)!}\int d^{p+1}x\,\epsilon^{a_0\cdots a_p}\left[\frac{p-1}{3!}C^{(p-1)}_{ia_3\cdots a_p}H_{a_0a_1a_2}{}^{,ia}{}_a + \frac{1}{2!}C^{(p-1)}_{a_2\cdots a_p,i}(2H_{a_0a_1}{}^{a,i}{}_a - H_{a_0a_1}{}^{i,a}{}_a)\right] \tag{1.5}$$

They are invariant under the linear T-duality transformations. The consistency of these couplings with the standard nonlinear T-duality [21–27] requires some nonlinear couplings for $C^{(p-3)}$. However, because of the appearance of the transverse index in the RR potential, the nonlinear T-duality transformations of the RR field in the absence of D-branes, i.e.,$\widetilde{C}^{(n)}_{\mu\nu\cdots\alpha\beta} = C^{(n+1)}_{\mu\nu\cdots\alpha\beta y} + nC^{(n-1)}_{[\mu\nu\cdots\alpha}B_{\beta]y} + \cdots$, produces some couplings which break the B-field gauge symmetry. On the other hand, since the contracted indices i, a in the above equation are derivative indices, the nonlinear terms are invariant under linear T-duality at the level of two B-fields [19]. So it is consistent with T-duality to remove the terms which break the gauge symmetry. Consider then the following part of the nonlinear T-duality transformation of the RR potential:

$$\begin{aligned}\widetilde{C}^{(p-1)}_{ia_3\cdots a_p} &= C^{(p)}_{ia_3\cdots a_p y} + (p-2)C^{(p-2)}_{i[a_3\cdots a_{p-1}}B_{a_p]y} + \cdots\\ \widetilde{C}^{(p-1)}_{a_2\cdots a_p,i} &= C^{(p)}_{a_2\cdots a_p y,i} + (p-1)\partial_i C^{(p-2)}_{[a_2\cdots a_{p-1}}B_{a_p]y} + \cdots\end{aligned} \tag{1.6}$$

where dots represent higher nonlinear terms. Following [20], one finds that the consistency of the couplings (1.5) with the above T-duality requires the following couplings for $C^{(p-3)}$:

$$\begin{aligned}-\pi^2\alpha'^2 T_p\int d^{p+1}x\,\epsilon^{a_0\cdots a_p}\Big[&\frac{1}{3!2!(p-4)!}(B_{a_3a_4}+2\pi\alpha' f_{a_3a_4})C^{(p-3)}_{ia_5\cdots a_p}H_{a_0a_1a_2}{}^{,ia}{}_a \\ &+\frac{1}{2!2!(p-3)!}(B_{a_2a_3}+2\pi\alpha' f_{a_2a_3})C^{(p-3)}_{a_4\cdots a_p,i}(2H_{a_0a_1}{}^{a,i}{}_a - H_{a_0a_1}{}^{i,a}{}_a)\Big]\end{aligned} \tag{1.7}$$

where we have also used the replacement $B \to B + 2\pi\alpha' f$ to make the couplings gauge invariant. The above couplings are invariant under the linear T-duality transformation at the level of two B-fields. They produce some massless open string poles and contact terms which can be combined into massless poles written in terms of field strength H. The massless pole corresponding to the couplings in the first line is reproduced by the disk level S-matrix element of one RR and two B-fields vertex operators in which the RR potential carries one transverse index [28]. We will show that the massless poles corresponding to the couplings in the second line are reproduced by the S-matrix element in which the RR potential carries only world volume indices.

The S-matrix element still reproduces some other T-duality invariant couplings which are given by

$$\begin{aligned}\frac{\pi^2\alpha'^2T_p}{(p-3)!}\int d^{p+1}x\,\epsilon^{a_0a_1\cdots a_p}C^{(p-3)}_{a_4\cdots a_p}\Big[&\frac{1}{2!2!}H^{aa_0a_1}{}_{,ab}H^{ba_2a_3}+\frac{1}{3!}H^{a_0a_1a_2}{}_{,ai}H^{iaa_3} \\ &+\frac{1}{2!2!}H^{a_0a_1a,i}{}_aH^{a_2a_3}{}_i+\frac{1}{3!}H^{a_0a_1a_2}{}_{,a}H^{aba_3}{}_{,b}+\frac{1}{3!}H^{a_0a_1a_2}{}_{,i}H^{iaa_3}{}_{,a} \\ &-\frac{1}{3!}H^{a_0a_1a_2,a}{}_{ab}(B^{ba_3}+2\pi\alpha' f^{ba_3})-\frac{1}{2!2!}H^{a_0a_1b,a}{}_a(B^{a_2a_3}+2\pi\alpha' f^{a_2a_3})_{,b}\Big]\end{aligned} \tag{1.8}$$

Note that the contracted indices i, a, b are derivative indices, hence, the above couplings are all invariant under the linear T-duality transformation at the level of two B-fields. As in (1.7), we will see that the terms which include $(B+2\pi\alpha' f)$ produce massless open string poles. Our limitation to calculate the triple integrals that appear in the S-matrix element,

JHEP05(2011)066

does not allow us to calculate the coefficient of all such terms. However, there is no such limitation for calculating the contact terms.

An outline of the paper is as follows: In section 2.1 we examine the calculation of the S-matrix element of one RR and two NSNS vertex operators in superstring theory. We perform the calculation in full details for the RR potential $C^{(p-3)}$ which has only world volume indices and expand the amplitude at low energy. In section 2.2, using the couplings in (1.4), (1.7) and (1.8), we calculate the massless open string poles and the contact terms for the scattering amplitude of one RR scalar and two B-fields. We show that they are reproduced exactly by string theory amplitude at order $O(\alpha'^2)$.

2 Scattering amplitude

A powerful method for finding the low energy field theory of the string theory is to compare the scattering amplitudes of the field theory with the corresponding amplitudes in the string theory expanded at low energy. The disk level scattering amplitude of one RR and two NSNS vertex operators, at low energy, produces both massless open string and closed string poles as well as some contact terms. The closed string poles dictate the supergravity couplings in the bulk and the couplings of one RR and one NSNS states on the brane. On the other hand, the open string poles and the contact terms dictate the couplings of one RR and two B-fields on the brane in which we are interested in this paper. We shall show that the couplings in (1.4), (1.7) and (1.8) are produced by the scattering amplitude at low energy. In the next section, we calculate the string theory amplitude.

2.1 String theory amplitude

The scattering amplitude of one RR and two NSNS states has been studied in [18, 28] for a particular class of terms in the amplitude to confirm some part of the couplings resulting from the consistency of the CS action (1.3) with the linear T-duality, and the couplings (1.5) with nonlinear T-duality. In this paper, however, we are interested in finding all couplings that string theory produces for the RR potential $C^{(p-3)}$ which carries only the world volume indices.

In string theory, the tree level scattering amplitude of one RR and two NSNS states on the world-volume of a D_p-brane is given by the correlation function of their corresponding vertex operators on the disk. Since the background charge of the world-sheet with topology of a disk is $Q_\phi = 2$ one has to choose the vertex operators in the appropriate pictures to produce the compensating charge $Q_\phi = -2$. One may choose the RR vertex operator in $(-1/2, -1/2)$ picture, and one of the NSNS vertex operators in $(-1, 0)$ and the other one in $(0, 0)$. However, in this picture the symmetry between the two NSNS is not manifest from the very beginning. After performing the correlators, one has to make more effort to rewrite the final result in a symmetric form. Alternatively, one can choose the RR vertex operator in $(-1/2, -3/2)$ picture [29] and the two NSNS vertex operators in $(0, 0)$ picture. In this form the symmetry of the NSNS states is manifest from the beginning. We prefer to do the calculation in the latter form. We will show that the final result, after using some identities, are independent of the choice of the picture.

The scattering amplitude is given by the following correlation function:

$$\mathcal{A} \sim < V_{RR}^{(-1/2,-3/2)}(\varepsilon_1^{(n)}, p_1) V_{NSNS}^{(0,0)}(\varepsilon_2, p_2) V_{NSNS}^{(0,0)}(\varepsilon_3, p_3) > \tag{2.1}$$

Using the doubling trick [4], the vertex operators are given by[3]

$$\begin{aligned}
V_{RR}^{(-1/2,-3/2)} &= (P_- H_{1(n)} M_p)^{AB} \int d^2 z_1 : e^{-\phi(z_1)/2} S_A(z_1) e^{ip_1\cdot X} : e^{-3\phi(\bar{z}_1)/2} S_B(\bar{z}_1) e^{ip_1\cdot D\cdot X} : \\
V_{NSNS}^{(0,0)} &= (\varepsilon_2\cdot D)_{\mu_3\mu_4} \int d^2 z_2 : (\partial X^{\mu_3} + ip_2\cdot\psi\psi^{\mu_3}) e^{ip_2\cdot X} : (\partial X^{\mu_4} + ip_2\cdot D\cdot\psi\psi^{\mu_4}) e^{ip_2\cdot D\cdot X} : \\
V_{NSNS}^{(0,0)} &= (\varepsilon_3\cdot D)_{\mu_5\mu_6} \int d^2 z_3 : (\partial X^{\mu_5} + ip_3\cdot\psi\psi^{\mu_5}) e^{ip_3\cdot X} : (\partial X^{\mu_6} + ip_3\cdot D\cdot\psi\psi^{\mu_6}) e^{ip_3\cdot D\cdot X} :
\end{aligned}$$

where the indices $A, B, \cdots$ are the Dirac spinor indices and $P_- = \frac{1}{2}(1 - \gamma_{11})$ is the chiral projection operator which makes the calculation of the gamma matrices to be with the full 32×32 Dirac matrices of the ten dimensions. The matrix D^{μ}_{ν} is diagonal with $+1$ in the world volume directions and -1 in the transverse directions, and

$$\begin{aligned}
H_{1(n)} &= \frac{1}{n!} \varepsilon_{1\mu_1\cdots\mu_n} \gamma^{\mu_1} \cdots \gamma^{\mu_n} \\
M_p &= \frac{\pm 1}{(p+1)!} \epsilon_{a_0\cdots a_p} \gamma^{a_0} \cdots \gamma^{a_p}
\end{aligned} \tag{2.2}$$

where ϵ is the volume $(p+1)$-form of the D_p-brane. The polarization of the RR field is given by $\varepsilon_1^{(n)}$ and the polarizations of the B-fields are given by ε_2, ε_3. The on-shell conditions are

$$p_i\cdot p_i = p_i^{\mu}(\varepsilon_i)_{\mu\cdots} = 0, \text{ for } i = 1, 2, 3 \tag{2.3}$$

It is useful to write the matrix $D_{\mu\nu}$ and the flat metric $\eta_{\mu\nu}$ in terms of the two projection operators $N_{\mu\nu}$ and $V_{\mu\nu}$, i.e.,

$$\begin{aligned}
\eta_{\mu\nu} &= V_{\mu\nu} + N_{\mu\nu} \\
D_{\mu\nu} &= V_{\mu\nu} - N_{\mu\nu}
\end{aligned} \tag{2.4}$$

The components of vectors projected into each of these subspaces N and V or η and D are independent objects. If 1 in the chiral projection P_- produces couplings for $C^{(n)}$, then the γ_{11} produces the couplings for $C^{(10-n)}$. Hence, we consider 1 in the chiral projection and extend the result to all RR potentials.

Choosing the above integral form of the vertex operators, one has to also divide the amplitude (2.1) by the volume of SL$(2, R)$ group which is the conformal symmetry of the upper half z-plane. We will remove this factor after preforming the correlators. Moreover, the overall factor of the amplitude (2.1) may be fixed by comparing the final result with field theory.

[3]Our conversions set $\alpha' = 2$ in the string theory calculations.

Using the standard world-sheet propagators, one can calculate the correlators in (2.1). The amplitude (2.1) can be written as [28]

$$\begin{aligned}\mathcal{A} \sim & \frac{1}{2}(H_{1(n)}M_p)^{AB}(\varepsilon_2\!\cdot\! D)_{\mu_3\mu_4}(\varepsilon_3\!\cdot\! D)_{\mu_5\mu_6}\int d^2z_1 d^2z_2 d^2z_3\,(z_1-\bar{z}_1)^{-3/4}\\ & \times(b_1+b_2+\cdots+b_{10})_{AB}^{\mu_3\mu_4\mu_5\mu_6}\delta^{p+1}(p_1^a+p_2^a+p_3^a)+(2\leftrightarrow 3)\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
(b_1)_{AB}^{\mu_3\mu_4\mu_5\mu_6} &= <:S_A(z_1):S_B(\bar{z}_1):> g_1^{\mu_3\mu_4\mu_5\mu_6}\\
(b_2)_{AB}^{\mu_3\mu_4\mu_5\mu_6} &= 2(ip_2)_{\beta_3} <:S_A:S_B:\psi^{\beta_3}\psi^{\mu_3}:> g_2^{\mu_4\mu_5\mu_6}\\
(b_3)_{AB}^{\mu_3\mu_4\mu_5\mu_6} &= 2(ip_2\!\cdot\! D)_{\beta_4} <:S_A:S_B:\psi^{\beta_4}\psi^{\mu_4}:> g_3^{\mu_3\mu_5\mu_6}\\
(b_4)_{AB}^{\mu_3\mu_4\mu_5\mu_6} &= 2(ip_2)_{\beta_3}(ip_2\!\cdot\! D)_{\beta_4} <:S_A:S_B:\psi^{\beta_3}\psi^{\mu_3}:\psi^{\beta_4}\psi^{\mu_4}:> g_4^{\mu_5\mu_6}\\
(b_5)_{AB}^{\mu_3\mu_4\mu_5\mu_6} &= (ip_2)_{\beta_3}(ip_3)_{\beta_5} <:S_A:S_B:\psi^{\beta_3}\psi^{\mu_3}:\psi^{\beta_5}\psi^{\mu_5}:> g_5^{\mu_4\mu_6}\\
(b_6)_{AB}^{\mu_3\mu_4\mu_5\mu_6} &= 2(ip_2)_{\beta_3}(ip_3\!\cdot\! D)_{\beta_6} <:S_A:S_B:\psi^{\beta_3}\psi^{\mu_3}:\psi^{\beta_6}\psi^{\mu_6}:> g_6^{\mu_4\mu_5}\\
(b_7)_{AB}^{\mu_3\mu_4\mu_5\mu_6} &= (ip_2\!\cdot\! D)_{\beta_4}(ip_3\!\cdot\! D)_{\beta_6} <:S_A:S_B:\psi^{\beta_4}\psi^{\mu_4}:\psi^{\beta_6}\psi^{\mu_6}:> g_7^{\mu_3\mu_5}\\
(b_8)_{AB}^{\mu_3\mu_4\mu_5\mu_6} &= 2(ip_2)_{\beta_3}(ip_2\!\cdot\! D)_{\beta_4}(ip_3)_{\beta_5} <:S_A:S_B:\psi^{\beta_3}\psi^{\mu_3}:\psi^{\beta_4}\psi^{\mu_4}:\psi^{\beta_5}\psi^{\mu_5}:> g_8^{\mu_6}\\
(b_9)_{AB}^{\mu_3\mu_4\mu_5\mu_6} &= 2(ip_2)_{\beta_3}(ip_2\!\cdot\! D)_{\beta_4}(ip_3\!\cdot\! D)_{\beta_6} <:S_A:S_B:\psi^{\beta_3}\psi^{\mu_3}:\psi^{\beta_4}\psi^{\mu_4}:\psi^{\beta_6}\psi^{\mu_6}:> g_9^{\mu_5}\\
(b_{10})_{AB}^{\mu_3\mu_4\mu_5\mu_6} &= (ip_2)_{\beta_3}(ip_2\!\cdot\! D)_{\beta_4}(ip_3)_{\beta_5}(ip_3\!\cdot\! D)_{\beta_6}\\
&\quad\times <:S_A:S_B:\psi^{\beta_3}\psi^{\mu_3}:\psi^{\beta_4}\psi^{\mu_4}:\psi^{\beta_5}\psi^{\mu_5}:\psi^{\beta_6}\psi^{\mu_6}:> g_{10}
\end{aligned} \tag{2.6}$$

where g's are the correlators of X's which can easily be performed using the standard world-sheet propagators, and the correlator of ψ can be calculated using the Wick-like rule [28, 30].

Combining the gamma matrices coming from the Wick-like rule with the gamma matrices in (2.5), one finds the following trace [28]:

$$\begin{aligned}T(n,p,m) &= (H_{1(n)}M_p)^{AB}(\gamma^{\alpha_1\cdots\alpha_m}C^{-1})_{AB}A_{[\alpha_1\cdots\alpha_m]}\\ &= \frac{1}{n!(p+1)!}\varepsilon_{1\nu_1\cdots\nu_n}\epsilon_{a_0\cdots a_p}A_{[\alpha_1\cdots\alpha_m]}\mathrm{Tr}(\gamma^{\nu_1}\cdots\gamma^{\nu_n}\gamma^{a_0}\cdots\gamma^{a_p}\gamma^{\alpha_1\cdots\alpha_m})\end{aligned} \tag{2.7}$$

where $A_{[\alpha_1\cdots\alpha_m]}$ is an antisymmetric combination of the momenta and/or the polarizations of the NSNS states. The trace (2.7) can be evaluated for specific values of n. One can verify that the amplitude is non-zero only for $n=p-3$, $n=p-1$, $n=p+1$, $n=p+3$, $n=p+5$. We are interested in the case

$$n = p-3 \tag{2.8}$$

The case $n=p+5$ will be studied in the appendix B. In above case, the trace relation (2.7) gives non-zero result only for $m\geq 4$. One immediately concludes that b_1, b_2 and b_3 in (2.6) have no contribution to the amplitude. The cases that the RR field carries transverse indices are studied in [28]. We consider here the case that the RR potential carries only world volume indices. The gamma matrices in (2.7) corresponding to $\varepsilon_1^{(p-3)}$ must be contracted with the gamma matrices corresponding to the world volume form, otherwise they both

JHEP05(2011)066

would contract with the gamma matrices corresponding to $A_{[\alpha_1\cdots\alpha_m]}$ which gives zero result because of the appearance of repeated world volume indices in $A_{[\alpha_1\cdots\alpha_m]}$. In the following we consider the RR scalar field. The result can easily be extended to the RR n-form by contracting its indices with the world volume form.

For the RR scalar $n=0$, and from the relation (2.8) one gets $p=3$. The trace (2.7) is non-zero only for $m=4$. It becomes

$$T(0,3,4) = 32\epsilon^{a_0\cdots a_5}A_{[a_0\cdots a_3]} \tag{2.9}$$

where 32 is the trace of the 32×32 identity matrix. Since all indices of $A_{[a_0\cdots a_3]}$ are world volume, one finds b_4 has no contribution to the amplitude. The ψ correlators in b_{10}, b_9, b_8, b_7, b_6, b_5 have non-zero contributions to the amplitude (2.1). The X correlator in b_{10} is

$$\begin{aligned} g_{10} &= |z_{12}|^{2p_1\cdot p_2}|z_{13}|^{2p_1\cdot p_3}|z_{23}|^{2p_2\cdot p_3}|z_{1\bar2}|^{2p_1\cdot D\cdot p_2}|z_{1\bar3}|^{2p_1\cdot D\cdot p_3}|z_{2\bar3}|^{2p_2\cdot D\cdot p_3} \\ &\quad\times(z_{1\bar1})^{p_1\cdot D\cdot p_1}(z_{2\bar2})^{p_2\cdot D\cdot p_2}(z_{3\bar3})^{p_3\cdot D\cdot p_3}(i)^{p_1\cdot D\cdot p_1+p_2\cdot D\cdot p_2+p_3\cdot D\cdot p_3} \equiv K \end{aligned} \tag{2.10}$$

where $z_{ij}=z_i-z_j$ and $z_{i\bar j}=z_i-\bar z_j$. We have added the phase factor to make it real. The above function appears in all other X correlators in (2.6). The ψ correlators in b_{10} gives 24×12 terms which result from different Wick-like contractions. The contractions which end up with having p_2 and $p_2\!\cdot\! D$ or p_3 and $p_3\!\cdot\! D$ in $A_{[a_0\cdots a_3]}$ give zero result.

The X correlators in b_8, b_9 are [28]

$$\begin{aligned} g_8^{\mu_6} &= \frac{iK}{z_{3\bar3}}\left(\frac{p_1^{\mu_6}z_{31}}{z_{1\bar3}}+\frac{p_2^{\mu_6}z_{32}}{z_{2\bar3}}+\frac{(p_1\cdot D)^{\mu_6}z_{3\bar1}}{z_{\bar1\bar3}}+\frac{(p_2\cdot D)^{\mu_6}z_{3\bar2}}{z_{\bar2\bar3}}\right) \\ g_9^{\mu_5} &= \frac{iK}{z_{\bar33}}\left(\frac{p_1^{\mu_5}z_{\bar31}}{z_{13}}+\frac{p_2^{\mu_5}z_{\bar32}}{z_{23}}+\frac{(p_1\cdot D)^{\mu_5}z_{\bar3\bar1}}{z_{\bar13}}+\frac{(p_2\cdot D)^{\mu_5}z_{\bar3\bar2}}{z_{\bar23}}\right) \end{aligned}$$

The ψ correlators in each of b_8, b_9 gives 12 terms which result from different Wick-like contractions.

The X correlators in b_5, b_6, b_7 are

$$\begin{aligned} g_5^{\mu_4\mu_6} &= -\frac{\eta^{\mu_4\mu_6}K}{z_{\bar2\bar3}^2}-\frac{K}{z_{2\bar2}z_{3\bar3}}\left(\frac{p_1^{\mu_4}z_{21}}{z_{1\bar2}}+\frac{p_3^{\mu_4}z_{23}}{z_{3\bar2}}+\frac{(p_1\cdot D)^{\mu_4}z_{2\bar1}}{z_{\bar1\bar2}}+\frac{(p_3\cdot D)^{\mu_4}z_{2\bar3}}{z_{\bar3\bar2}}\right) \\ &\quad\times\left(\frac{p_1^{\mu_6}z_{31}}{z_{1\bar3}}+\frac{p_2^{\mu_6}z_{32}}{z_{2\bar3}}+\frac{(p_1\cdot D)^{\mu_6}z_{3\bar1}}{z_{\bar1\bar3}}+\frac{(p_2\cdot D)^{\mu_6}z_{3\bar2}}{z_{\bar2\bar3}}\right) \\ g_6^{\mu_4\mu_5} &= -\frac{\eta^{\mu_4\mu_5}K}{z_{\bar23}^2}-\frac{K}{z_{2\bar2}z_{\bar33}}\left(\frac{p_1^{\mu_4}z_{21}}{z_{1\bar2}}+\frac{p_3^{\mu_4}z_{23}}{z_{3\bar2}}+\frac{(p_1\cdot D)^{\mu_4}z_{2\bar1}}{z_{\bar1\bar2}}+\frac{(p_3\cdot D)^{\mu_4}z_{2\bar3}}{z_{\bar3\bar2}}\right) \\ &\quad\times\left(\frac{p_1^{\mu_5}z_{\bar31}}{z_{13}}+\frac{p_2^{\mu_5}z_{\bar32}}{z_{23}}+\frac{(p_1\cdot D)^{\mu_5}z_{\bar3\bar1}}{z_{\bar13}}+\frac{(p_2\cdot D)^{\mu_5}z_{\bar3\bar2}}{z_{\bar23}}\right) \\ g_7^{\mu_3\mu_5} &= -\frac{\eta^{\mu_3\mu_5}K}{z_{23}^2}-\frac{K}{z_{\bar22}z_{\bar33}}\left(\frac{p_1^{\mu_3}z_{\bar21}}{z_{12}}+\frac{p_3^{\mu_3}z_{\bar23}}{z_{32}}+\frac{(p_1\cdot D)^{\mu_3}z_{\bar2\bar1}}{z_{\bar12}}+\frac{(p_3\cdot D)^{\mu_3}z_{\bar2\bar3}}{z_{\bar32}}\right) \\ &\quad\times\left(\frac{p_1^{\mu_5}z_{\bar31}}{z_{13}}+\frac{p_2^{\mu_5}z_{\bar32}}{z_{23}}+\frac{(p_1\cdot D)^{\mu_5}z_{\bar3\bar1}}{z_{\bar13}}+\frac{(p_2\cdot D)^{\mu_5}z_{\bar3\bar2}}{z_{\bar23}}\right) \end{aligned} \tag{2.11}$$

The ψ correlators in each of them gives one term. Examining the transformation of the above X-correlators and the correlators of ψ's in Wick-like rule, one can easily verify that the amplitude (2.5) is invariant under the SL$(2,R)$ transformation. So one can map the results to disk with unit radius. That is, one can use the following replacement [28]:

$$\begin{aligned}
z_{i\bar{j}} &\to -(1-z_i\bar{z}_j)\\
z_{ij} &\to z_{ij}\\
z_{\bar{i}\bar{j}} &\to -z_{\bar{i}\bar{j}}\\
z_{\bar{i}j} &\to (1-\bar{z}_i z_j)
\end{aligned}$$

Obviously the result is still SL$(2,R)$ invariant. To fix this symmetry, we then set [15]

$$z_1 = 0, \text{ and } z_2 = \bar{z}_2 = r_2 \tag{2.12}$$

Under this fixing the measure in (2.5) changes as

$$d^2z_1 d^2z_2 d^2z_3 \to r_2 dr_2\, r_3 dr_3\, d\theta,\quad 0<r_2,r_3<1,\ 0<\theta<2\pi \tag{2.13}$$

where we have chosen the polar coordinate $z_3 = r_3 e^{i\theta}$, and K changes as

$$\begin{aligned}
\widetilde{K}=&\ {r_2}^{2p_1\cdot p_2}\ {r_3}^{2p_1\cdot p_3}(1-{r_2}^2)^{p_2\cdot D\cdot p_2}(1-{r_3}^2)^{p_3\cdot D\cdot p_3}\\
&\times|r_2-r_3e^{i\theta}|^{2p_2\cdot p_3}|1-r_2r_3e^{i\theta}|^{2p_2\cdot D\cdot p_3}
\end{aligned} \tag{2.14}$$

The first terms in (2.11) produce structures in which the polarization tensors contract with each other. Let us consider these terms. They appear in the amplitude as

$$\begin{aligned}
\mathcal{A}_5(\varepsilon_2\!\cdot\!\varepsilon_3^T) &\sim -4\epsilon_{a_0\cdots a_3}p_2^{a_0}p_3^{a_1}(\varepsilon_2\!\cdot\!\varepsilon_3^T)^{a_2a_3}\int d^2z_1d^2z_2d^2z_3\frac{K}{z_{\bar{2}\bar{3}}^2 z_{21}z_{2\bar{1}}z_{31}z_{3\bar{1}}}\\
\mathcal{A}_6(\varepsilon_2\!\cdot\!D\!\cdot\!\varepsilon_3) &\sim -8\epsilon_{a_0\cdots a_3}p_2^{a_0}p_3^{a_1}(\varepsilon_2\!\cdot\!D\!\cdot\!\varepsilon_3)^{a_2a_3}\int d^2z_1d^2z_2d^2z_3\frac{K}{z_{\bar{2}3}^2 z_{21}z_{2\bar{1}}z_{\bar{3}1}z_{3\bar{1}}}\\
\mathcal{A}_7(\varepsilon_2^T\!\cdot\!\varepsilon_3) &\sim -4\epsilon_{a_0\cdots a_3}p_2^{a_0}p_3^{a_1}(\varepsilon_2^T\!\cdot\!\varepsilon_3)^{a_2a_3}\int d^2z_1d^2z_2d^2z_3\frac{K}{z_{23}^2 z_{\bar{2}1}z_{2\bar{1}}z_{\bar{3}1}z_{3\bar{1}}}
\end{aligned} \tag{2.15}$$

where we have written the sub-amplitudes corresponding to b_i in (2.5) as $\mathcal{A}_i$. The above sub-amplitudes are zero when one polarization is symmetric and the other one is antisymmetric.

There are other sub-amplitudes which have also terms in which the polarization tensor contract with each other. The amplitudes (2.15) have second order poles, e.g.,$1/(z_{\bar{2}\bar{3}})^2$ in $\mathcal{A}_5$. They indicates that there is a tachyon propagating in the amplitude. This undesirable feather appears when one uses the vertex operator in 0-picture [31]. However, the whole amplitude (2.5) has no tachyon which means all the tachyons in the sub-amplitudes must be canceled among them. So one may keep the second order poles in the sub-amplitudes and the tachyons would be canceled finally using the properties of the functions that appear in the final amplitude, e.g.,in the four point function one has $(p_2{\cdot}p_3-1)\Gamma(p_2{\cdot}p_3-1)=\Gamma(p_2{\cdot}p_3)$ where $\Gamma(p_2\!\cdot\!p_3-1)$ has tachyon pole whereas $\Gamma(p_2\!\cdot\!p_3)$ has massless pole.

Alternatively, one can show that the second order poles appear in the whole amplitude as derivative of first order poles. Then using by part integration, one can write them in terms of first order poles. In this case, one needs to use a by part integration to remove the tachyon. Mapping the above amplitudes to unit disk and fixing the SL$(2,R)$ symmetry as (2.12), one can write (2.15), after a by part integration, as

$$\begin{aligned}
\mathcal{A}_5(\varepsilon_2\!\cdot\!\varepsilon_3^T) &\sim -4\epsilon_{a_0\cdots a_3}p_2^{a_0}p_3^{a_1}(\varepsilon_2\!\cdot\!\varepsilon_3^T)^{a_2a_3}\int dr_2dr_3d\theta\frac{\frac{\partial\widetilde{K}}{\partial i\theta}}{r_3(r_2-r_3e^{-i\theta})}\\
\mathcal{A}_6(\varepsilon_2\!\cdot\! D\!\cdot\!\varepsilon_3) &\sim 8\epsilon_{a_0\cdots a_3}p_2^{a_0}p_3^{a_1}(\varepsilon_2\!\cdot\! D\!\cdot\!\varepsilon_3)^{a_2a_3}\int dr_2dr_3d\theta\frac{\frac{\partial\widetilde{K}}{\partial i\theta}}{r_3(r_2-r_3e^{i\theta})}\\
\mathcal{A}_7(\varepsilon_2^T\!\cdot\!\varepsilon_3) &\sim 4\epsilon_{a_0\cdots a_3}p_2^{a_0}p_3^{a_1}(\varepsilon_2^T\!\cdot\!\varepsilon_3)^{a_2a_3}\int dr_2dr_3d\theta\frac{\frac{\partial\widetilde{K}}{\partial i\theta}}{r_3(r_2-r_3e^{i\theta})}
\end{aligned}\tag{2.16}$$

where

$$\frac{\partial\widetilde{K}}{\partial i\theta} = r_2r_3(e^{i\theta}-e^{-i\theta})\left(\frac{p_2\!\cdot\! p_3}{|r_2-r_3e^{i\theta}|^2}+\frac{p_2\!\cdot\! D\!\cdot\! p_3}{|1-r_2r_3e^{i\theta}|^2}\right)\widetilde{K}\tag{2.17}$$

Note that the sub-amplitudes (2.15) have two momenta whereas the sub-amplitudes (2.16) have four momenta, as all the other structures in the amplitude (2.5).

Since there is no conservation of momentum in the transverse directions in (2.5), the terms in which p_1^i contracts with each of the polarizations, i.e.,$p_1\!\cdot\! N\!\cdot\!\varepsilon_3$, $p_1\!\cdot\! N\!\cdot\!\varepsilon_2$, as well as $p_2\!\cdot\! N\!\cdot\!\varepsilon_3$, $p_3\!\cdot\! N\!\cdot\!\varepsilon_2$ are independent structures. For the other terms we use the conservation of momentum along the brane, i.e.,

$$(p_1+p_2+p_3)\!\cdot\! V_\mu = 0\tag{2.18}$$

to write p_1^a in terms of p_2^a and p_3^a.

When one tensor is symmetric and the other one is antisymmetric the result is zero. The result for two symmetric tensors is

$$\begin{aligned}
A \sim \epsilon_{a_0a_1a_2a_3}p_2^{a_0}p_3^{a_1}\Big[&p_2\!\cdot\! N\!\cdot\! p_3(\varepsilon_2\!\cdot\! N\!\cdot\!\varepsilon_3)^{a_2a_3}+p_2\!\cdot\! V\!\cdot\! p_3(\varepsilon_2\!\cdot\! V\!\cdot\!\varepsilon_3)^{a_2a_3}\\
&-p_3\!\cdot\! N\!\cdot\!\varepsilon_2^{a_2}\,p_2\!\cdot\! N\!\cdot\!\varepsilon_3^{a_3}-p_3\!\cdot\! V\!\cdot\!\varepsilon_2^{a_2}\,p_2\!\cdot\! V\!\cdot\!\varepsilon_3^{a_3}\Big]\mathcal{J}
\end{aligned}\tag{2.19}$$

where $\mathcal{J}$ is

$$\mathcal{J} = -4\int_0^1 dr_2\int_0^1 dr_3\, r_2r_3\int_0^{2\pi}d\theta\frac{\sin^2(\theta)\widetilde{K}}{|1-r_2r_3e^{i\theta}|^2|r_2-r_3e^{-i\theta}|^2}$$

This is the result that has been found in [15]. It is shown in [15] that the above integral has only contact term at low energy, i.e.,

$$\mathcal{J} = -\frac{\pi^3}{3}+\cdots\tag{2.20}$$

where dots represent terms with two and more momenta which correspond to the amplitude at order $O(\alpha'^3)$ in which we are not interested. It has been shown in [15] that the above contact terms reproduce the gravity couplings in (1.3).

JHEP05(2011)066

The indices of the graviton polarization tensors in the second line of (2.19) are contracted with the world volume form or with the momentum. This indicates that these terms are invariant under linear T-duality when the Killing coordinate is an index of the RR potential [19]. When the Killing coordinate is an index of the graviton polarization tensor, T-duality relates them to the higher RR form [19] in which we are not interested in this paper. On the other hand, one of the indices of the graviton polarization tensors in the first line of (2.19) are contracted with each other. This indicates that the terms in the first line are not invariant under T-duality [19]. Hence, there must be antisymmetric tensor couplings as well to make them invariant.

However, the amplitude for two antisymmetric tensors has much more terms than those that are needed to make the gravity couplings to be invariant under linear T-duality. The result is

$$
\begin{aligned}
\mathcal{A} \sim \frac{1}{2}\epsilon_{a_0a_1a_2a_3}\varepsilon_3^{a_2a_3}\Big(& p_2^{a_0}p_3^{a_1}p_3\!\cdot\!V\!\cdot\!\varepsilon_2\!\cdot\!V\!\cdot\!p_2\mathcal{J}_1-\frac{1}{2}p_2^{a_0}p_3^{a_1}p_3\!\cdot\!V\!\cdot\!\varepsilon_2\!\cdot\!N\!\cdot\!p_1\mathcal{I}_3 \\
&+p_2^{a_0}p_3^{a_1}p_2\!\cdot\!V\!\cdot\!\varepsilon_2\!\cdot\!N\!\cdot\!p_3\mathcal{J}_2-p_2^{a_0}p_3^{a_1}p_3\!\cdot\!V\!\cdot\!\varepsilon_2\!\cdot\!N\!\cdot\!p_3\mathcal{J}_0+\frac{1}{2}p_2^{a_0}p_3^{a_1}p_3\!\cdot\!N\!\cdot\!\varepsilon_2\!\cdot\!N\!\cdot\!p_1\mathcal{I}_2 \\
&-2p_2^{a_0}p_3\!\cdot\!V\!\cdot\!p_3\,p_2\!\cdot\!V\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_3+\frac{1}{2}p_2^{a_0}p_3\!\cdot\!V\!\cdot\!p_3\,p_3\!\cdot\!V\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_4+\frac{1}{2}p_3^{a_0}p_2\!\cdot\!V\!\cdot\!p_2\,p_3\!\cdot\!V\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_1 \\
&+p_2^{a_0}p_3\!\cdot\!V\!\cdot\!p_3\,p_1\!\cdot\!N\!\cdot\!\varepsilon_2^{a_1}\mathcal{I}_4-\frac{1}{2}p_2^{a_0}p_3\!\cdot\!V\!\cdot\!p_3\,p_3\!\cdot\!N\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_{12}-\frac{1}{2}p_3^{a_0}p_2\!\cdot\!V\!\cdot\!p_2\,p_3\!\cdot\!N\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_2 \\
&+\frac{1}{2}p_3^{a_0}p_2\!\cdot\!V\!\cdot\!p_3\,p_1\!\cdot\!N\!\cdot\!\varepsilon_2^{a_1}\mathcal{I}_3-\frac{1}{2}p_3^{a_0}p_2\!\cdot\!N\!\cdot\!p_3\,p_1\!\cdot\!N\!\cdot\!\varepsilon_2^{a_1}\mathcal{I}_2-p_3^{a_0}p_2\!\cdot\!V\!\cdot\!p_3\,p_2\!\cdot\!V\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_1 \\
&+p_3^{a_0}p_2\!\cdot\!N\!\cdot\!p_3\,p_2\!\cdot\!V\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_2-\frac{1}{2}p_2^{a_0}p_2\!\cdot\!V\!\cdot\!p_3\,p_3\!\cdot\!V\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_6-\frac{1}{2}p_3^{a_0}p_2\!\cdot\!V\!\cdot\!p_3\,p_3\!\cdot\!V\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_6 \\
&-p_2^{a_0}p_2\!\cdot\!N\!\cdot\!p_3\,p_3\!\cdot\!V\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_7-p_3^{a_0}p_2\!\cdot\!N\!\cdot\!p_3\,p_3\!\cdot\!V\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_8-p_2^{a_0}p_2\!\cdot\!V\!\cdot\!p_3\,p_3\!\cdot\!N\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_9 \\
&-p_3^{a_0}p_2\!\cdot\!V\!\cdot\!p_3\,p_3\!\cdot\!N\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_{10}-\frac{1}{2}p_2^{a_0}p_2\!\cdot\!N\!\cdot\!p_3\,p_3\!\cdot\!N\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_6-\frac{1}{2}p_3^{a_0}p_2\!\cdot\!N\!\cdot\!p_3\,p_3\!\cdot\!N\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_6 \\
&+\frac{1}{8}(p_2\!\cdot\!V\!\cdot\!p_3)^2\,\varepsilon_2^{a_0a_1}\mathcal{J}_6+\frac{1}{8}(p_2\!\cdot\!N\!\cdot\!p_3)^2\,\varepsilon_2^{a_0a_1}\mathcal{J}_6+\frac{1}{4}p_2\!\cdot\!N\!\cdot\!p_3\,p_2\!\cdot\!V\!\cdot\!p_3\,\varepsilon_2^{a_0a_1}\mathcal{J}_{11} \\
&+\frac{1}{4}p_2\!\cdot\!V\!\cdot\!p_2\,p_3\!\cdot\!V\!\cdot\!p_3\,\varepsilon_2^{a_0a_1}\mathcal{J}_3\Big) \\
&+\frac{1}{2}\epsilon_{a_0a_1a_2a_3}p_2^{a_0}p_3^{a_1}\Big(-p_1\!\cdot\!N\!\cdot\!\varepsilon_2^{a_2}\,p_2\!\cdot\!V\!\cdot\!\varepsilon_3^{a_3}\mathcal{I}_3-p_1\!\cdot\!N\!\cdot\!\varepsilon_2^{a_2}\,p_1\!\cdot\!N\!\cdot\!\varepsilon_3^{a_3}\mathcal{I}_1 \\
&+p_1\!\cdot\!N\!\cdot\!\varepsilon_2^{a_2}\,p_2\!\cdot\!N\!\cdot\!\varepsilon_3^{a_3}\mathcal{I}_2+2p_3\!\cdot\!V\!\cdot\!\varepsilon_2^{a_2}\,p_2\!\cdot\!N\!\cdot\!\varepsilon_3^{a_3}\mathcal{J}_5+4p_1\!\cdot\!N\!\cdot\!\varepsilon_2^{a_2}\,p_3\!\cdot\!V\!\cdot\!\varepsilon_3^{a_3}\mathcal{I}_4 \\
&-2p_2\!\cdot\!V\!\cdot\!\varepsilon_2^{a_2}\,p_2\!\cdot\!N\!\cdot\!\varepsilon_3^{a_3}\mathcal{J}_2+2p_2\!\cdot\!V\!\cdot\!\varepsilon_2^{a_2}\,p_2\!\cdot\!V\!\cdot\!\varepsilon_3^{a_3}\mathcal{J}_1-4p_2\!\cdot\!V\!\cdot\!\varepsilon_2^{a_2}\,p_3\!\cdot\!V\!\cdot\!\varepsilon_3^{a_3}\mathcal{J}_3 \\
&+p_2\!\cdot\!N\!\cdot\!p_3(\varepsilon_2\!\cdot\!V\!\cdot\!\varepsilon_3)^{a_2a_3}\mathcal{J}-p_2\!\cdot\!V\!\cdot\!p_3(\varepsilon_2\!\cdot\!N\!\cdot\!\varepsilon_3)^{a_2a_3}\mathcal{J}\Big)+(2\leftrightarrow 3)
\end{aligned}
\tag{2.21}
$$

The indices of the polarization tensors in all terms except the terms in the last line are contracted with the world volume form or with the momentum. Hence they all are invariant under linear T-duality. The terms in the last line combines with the terms in the first line of (2.19) to make a T-dual combination [18, 19]. However, the above amplitude has more couplings than those have been found in [18, 19] by requiring the consistency of the graviton couplings in the Chern-Simons action with linear T-duality. The new couplings which are invariant under linear T-duality can be found by studying the integrals that appear in the amplitude.

The integral $\mathcal{J}$ is the one which appears also in the graviton amplitude, and the integrals $\mathcal{I}_1, \cdots, \mathcal{I}_4$ in (2.21) are those which appear also in the scattering amplitude considered in [28] in which the RR potential carries both transverse and world volume indices. These integrals are

$$\begin{aligned}
\mathcal{I}_1 &= -\int_0^1 dr_2 \int_0^1 dr_3 \frac{1}{r_2 r_3} \int_0^{2\pi} d\theta \widetilde{K} \\
\mathcal{I}_2 &= 2\int_0^1 dr_2 \int_0^1 dr_3 \frac{(1-r_2^2)}{r_2} \int_0^{2\pi} d\theta \frac{[r_3(1+r_2^2) - r_2(1+r_3^2)\cos(\theta)]\widetilde{K}}{|1-r_2 r_3 e^{i\theta}|^2 |r_2 - r_3 e^{-i\theta}|^2} \\
\mathcal{I}_4 &= -\int_0^1 dr_2 \int_0^1 dr_3 \frac{(1+r_3^2)}{r_2 r_3 (1-r_3^2)} \int_0^{2\pi} d\theta \widetilde{K}
\end{aligned}$$

and $\mathcal{I}_3(p_1,p_2,p_3) = \mathcal{I}_2(p_1,p_3,p_2)$. The other integrals are

$$\begin{aligned}
\mathcal{J}_0 &= \int_0^1 dr_2 \int_0^1 dr_3 \frac{1}{r_2 r_3} \int_0^{2\pi} d\theta \frac{[(1-r_2^2 r_3^2)(r_2^2 - r_3^2) - 4r_2^2 r_3^2 \sin^2(\theta)]\widetilde{K}}{|1-r_2 r_3 e^{i\theta}|^2 |r_2 - r_3 e^{-i\theta}|^2} \\
\mathcal{J}_1 &= 2\int_0^1 dr_2 \int_0^1 dr_3 \frac{(1+r_2^2)(1-r_3^2)}{r_3(1-r_2^2)} \int_0^{2\pi} d\theta \frac{[r_2(1+r_3^2) - r_3(1+r_2^2)\cos(\theta)]\widetilde{K}}{|1-r_2 r_3 e^{i\theta}|^2 |r_2 - r_3 e^{-i\theta}|^2} \\
\mathcal{J}_2 &= 2\int_0^1 dr_2 \int_0^1 dr_3 \frac{(1+r_2^2)}{r_2} \int_0^{2\pi} d\theta \frac{[r_3(1+r_2^2) - r_2(1+r_3^2)\cos(\theta)]\widetilde{K}}{|1-r_2 r_3 e^{i\theta}|^2 |r_2 - r_3 e^{-i\theta}|^2} \\
\mathcal{J}_3 &= -\int_0^1 dr_2 \int_0^1 dr_3 \frac{(1+r_3^2)(1+r_2^2)}{r_2 r_3 (1-r_3^2)(1-r_2^2)} \int_0^{2\pi} d\theta \widetilde{K} \\
\mathcal{J}_5 &= \int_0^1 dr_2 \int_0^1 dr_3 \frac{(1-r_2^2 r_3^2)(r_2^2 - r_3^2)}{r_2 r_3} \int_0^{2\pi} d\theta \frac{\widetilde{K}}{|1-r_2 r_3 e^{i\theta}|^2 |r_2 - r_3 e^{-i\theta}|^2} \\
\mathcal{J}_6 &= -2\int_0^1 dr_2 \int_0^1 dr_3 (1-r_2^2)(1-r_3^2) \int_0^{2\pi} d\theta \frac{\cos(\theta)\widetilde{K}}{|1-r_2 r_3 e^{i\theta}|^2 |r_2 - r_3 e^{-i\theta}|^2} \\
\mathcal{J}_7 &= -\int_0^1 dr_2 \int_0^1 dr_3 \frac{r_2}{r_3} \int_0^{2\pi} d\theta \frac{[(1-r_3^2)^2 + 4r_3^2 \sin^2(\theta)]\widetilde{K}}{|1-r_2 r_3 e^{i\theta}|^2 |r_2 - r_3 e^{-i\theta}|^2} \\
\mathcal{J}_9 &= -\int_0^1 dr_2 \int_0^1 dr_3 \frac{(1-r_2^2)^2 r_3}{r_2} \int_0^{2\pi} d\theta \frac{\widetilde{K}}{|1-r_2 r_3 e^{i\theta}|^2 |r_2 - r_3 e^{-i\theta}|^2} \\
\mathcal{J}_{11} &= -\int_0^1 dr_2 \int_0^1 dr_3 \frac{1}{r_2 r_3} \int_0^{2\pi} d\theta \frac{[(1+r_2^2 r_3^2)(r_2^2 + r_3^2) - 4r_2^2 r_3^2 + 4r_2^2 r_3^2 \sin^2(\theta)]\widetilde{K}}{|1-r_2 r_3 e^{i\theta}|^2 |r_2 - r_3 e^{-i\theta}|^2}
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{J}_4(p_1,p_2,p_3) = \mathcal{J}_1(p_1,p_3,p_2)\,;\ \mathcal{J}_{12}(p_1,p_2,p_3) = \mathcal{J}_2(p_1,p_3,p_2) \\
&\mathcal{J}_8(p_1,p_2,p_3) = \mathcal{J}_7(p_1,p_3,p_2)\,;\ \mathcal{J}_{10}(p_1,p_2,p_3) = \mathcal{J}_9(p_1,p_3,p_2)
\end{aligned} \tag{2.22}$$

It is shown in [28] that the integrals $\mathcal{I}_1, \cdots, \mathcal{I}_4$ have no contact terms. However, as we will see later some of the integrals $\mathcal{J}_0, \cdots, \mathcal{J}_{11}$ have contact terms.

The amplitude (2.21) should satisfy the Ward identities associated with the RR field and with the B-fields. If one could perform the integrals explicitly, then one would be able to check these identities explicitly. Alternatively, by demanding the amplitude (2.21) to satisfy these Ward identities, one would be able to find some relations between the integrals. Checking these relations explicitly would confirm the amplitude satisfies the Ward identities. We use the latter method in this paper.

The relations between the integrals may be used to write the amplitude (2.21) either in terms of RR field strength, F, or in terms of field strength of the B-field, H. Since the relations between the integrals involve only Mandelstam variables $p_2\cdot V\cdot p_2$, $p_3\cdot V\cdot p_3$, $\cdots$, we expect the terms in the amplitude (2.21) which have no Mandelstam variables can easily be written in terms of H. For example, the first term in (2.21) can be written as $H_3^{a_1a_2a_3}H_2^{a_0ab}p_{2a}p_{3b}/3$. This term includes the first term in (2.21) and some extra terms which are proportional to the Mandelstam variables. The contribution of all such terms should be canceled in the amplitude after using the relation between the integrals. We will see that in this way one is able to write the amplitude in terms of H.

Before finding the relations between the integrals, we reduce the number of integrals involved in the amplitude (2.21). One observes that the integrals $\mathcal{J}_0$, $\mathcal{J}_7$, $\mathcal{J}_8$, $\mathcal{J}_{11}$ include $\sin(\theta)^2$. This part of the integrals is exactly $\mathcal{J}$. Separating this part, one can rewrite the amplitude in the following form:

$$
\begin{aligned}
\mathcal{A}\sim&\frac{1}{2}\epsilon_{a_0a_1a_2a_3}\bigg[\varepsilon_3^{a_2a_3}\bigg(-p_2^{a_0}p_3^{a_1}p_3\!\cdot\! V\!\cdot\!\varepsilon_2\!\cdot\! N\!\cdot\! p_3-p_2^{a_0}p_2\!\cdot\! N\!\cdot\! p_3\,p_3\!\cdot\! V\!\cdot\!\varepsilon_2^{a_1}\\
&-p_3^{a_0}p_2\!\cdot\! N\!\cdot\! p_3\,p_3\!\cdot\! V\!\cdot\!\varepsilon_2^{a_1}+\frac{1}{4}p_2\!\cdot\! N\!\cdot\! p_3\,p_2\!\cdot\! V\!\cdot\! p_3\,\varepsilon_2^{a_0a_1}\bigg)\\
&+p_2^{a_0}p_3^{a_1}\Big(p_2\!\cdot\! N\!\cdot\! p_3(\varepsilon_2\!\cdot\! V\!\cdot\!\varepsilon_3)^{a_2a_3}-p_2\!\cdot\! V\!\cdot\! p_3(\varepsilon_2\!\cdot\! N\!\cdot\!\varepsilon_3)^{a_2a_3}\Big)\bigg]\mathcal{J}\\
&+\frac{1}{2}\epsilon_{a_0a_1a_2a_3}\varepsilon_3^{a_2a_3}\bigg[p_2^{a_0}p_3^{a_1}p_3\!\cdot\! V\!\cdot\!\varepsilon_2\!\cdot\! V\!\cdot\! p_2\mathcal{J}_1-\frac{1}{2}p_2^{a_0}p_3^{a_1}p_3\!\cdot\! V\!\cdot\!\varepsilon_2\!\cdot\! N\!\cdot\! p_1\mathcal{I}_3\\
&+p_2^{a_0}p_3^{a_1}p_2\!\cdot\! V\!\cdot\!\varepsilon_2\!\cdot\! N\!\cdot\! p_3\mathcal{J}_2-p_2^{a_0}p_3^{a_1}p_3\!\cdot\! V\!\cdot\!\varepsilon_2\!\cdot\! N\!\cdot\! p_3\mathcal{J}_5+\frac{1}{2}p_2^{a_0}p_3^{a_1}p_3\!\cdot\! N\!\cdot\!\varepsilon_2\!\cdot\! N\!\cdot\! p_1\mathcal{I}_2\\
&-2p_2^{a_0}p_3\!\cdot\! V\!\cdot\! p_3\,p_2\!\cdot\! V\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_3+\frac{1}{2}p_2^{a_0}p_3\!\cdot\! V\!\cdot\! p_3\,p_3\!\cdot\! V\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_4+\frac{1}{2}p_3^{a_0}p_2\!\cdot\! V\!\cdot\! p_2\,p_3\!\cdot\! V\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_1\\
&+p_2^{a_0}p_3\!\cdot\! V\!\cdot\! p_3\,p_1\!\cdot\! N\!\cdot\!\varepsilon_2^{a_1}\mathcal{I}_4-\frac{1}{2}p_2^{a_0}p_3\!\cdot\! V\!\cdot\! p_3\,p_3\!\cdot\! N\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_{12}-\frac{1}{2}p_3^{a_0}p_2\!\cdot\! V\!\cdot\! p_2\,p_3\!\cdot\! N\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_2\\
&+\frac{1}{2}p_3^{a_0}p_2\!\cdot\! V\!\cdot\! p_3\,p_1\!\cdot\! N\!\cdot\!\varepsilon_2^{a_1}\mathcal{I}_3-\frac{1}{2}p_3^{a_0}p_2\!\cdot\! N\!\cdot\! p_3\,p_1\!\cdot\! N\!\cdot\!\varepsilon_2^{a_1}\mathcal{I}_2-p_3^{a_0}p_2\!\cdot\! V\!\cdot\! p_3\,p_2\!\cdot\! V\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_1\\
&+p_3^{a_0}p_2\!\cdot\! N\!\cdot\! p_3\,p_2\!\cdot\! V\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_2+\frac{1}{4}p_2^{a_0}p_2\!\cdot\! p_3\,p_3\!\cdot\! D\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_5-\frac{1}{4}p_3^{a_0}p_2\!\cdot\! p_3\,p_3\!\cdot\! D\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_5\\
&+\frac{1}{4}p_2^{a_0}p_2\!\cdot\! p_3\,p_3\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_{13}+\frac{1}{4}p_3^{a_0}p_2\!\cdot\! p_3\,p_3\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_{13}-\frac{1}{4}p_2^{a_0}p_2\!\cdot\! D\!\cdot\! p_3\,p_3\!\cdot\! D\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_{14}\\
&-\frac{1}{4}p_3^{a_0}p_2\!\cdot\! D\!\cdot\! p_3\,p_3\!\cdot\! D\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_{14}-\frac{1}{4}p_2^{a_0}p_2\!\cdot\! D\!\cdot\! p_3\,p_3\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_5+\frac{1}{4}p_3^{a_0}p_2\!\cdot\! D\!\cdot\! p_3\,p_3\!\cdot\!\varepsilon_2^{a_1}\mathcal{J}_5\\
&-\frac{1}{16}(p_2\!\cdot\! p_3)^2\,\varepsilon_2^{a_0a_1}\mathcal{J}_{13}+\frac{1}{16}(p_2\!\cdot\! D\!\cdot\! p_3)^2\,\varepsilon_2^{a_0a_1}\mathcal{J}_{14}+\frac{1}{4}p_2\!\cdot\! V\!\cdot\! p_2\,p_3\!\cdot\! V\!\cdot\! p_3\,\varepsilon_2^{a_0a_1}\mathcal{J}_3\bigg]\\
&+\frac{1}{2}\epsilon_{a_0a_1a_2a_3}p_2^{a_0}p_3^{a_1}\Big[-p_1\!\cdot\! N\!\cdot\!\varepsilon_2^{a_2}\,p_2\!\cdot\! V\!\cdot\!\varepsilon_3^{a_3}\mathcal{I}_3-p_1\!\cdot\! N\!\cdot\!\varepsilon_2^{a_2}\,p_1\!\cdot\! N\!\cdot\!\varepsilon_3^{a_3}\mathcal{I}_1\\
&+p_1\!\cdot\! N\!\cdot\!\varepsilon_2^{a_2}\,p_2\!\cdot\! N\!\cdot\!\varepsilon_3^{a_3}\mathcal{I}_2+2p_3\!\cdot\! V\!\cdot\!\varepsilon_2^{a_2}\,p_2\!\cdot\! N\!\cdot\!\varepsilon_3^{a_3}\mathcal{J}_5+4p_1\!\cdot\! N\!\cdot\!\varepsilon_2^{a_2}\,p_3\!\cdot\! V\!\cdot\!\varepsilon_3^{a_3}\mathcal{I}_4\\
&-2p_2\!\cdot\! V\!\cdot\!\varepsilon_2^{a_2}\,p_2\!\cdot\! N\!\cdot\!\varepsilon_3^{a_3}\mathcal{J}_2+2p_2\!\cdot\! V\!\cdot\!\varepsilon_2^{a_2}\,p_2\!\cdot\! V\!\cdot\!\varepsilon_3^{a_3}\mathcal{J}_1-4p_2\!\cdot\! V\!\cdot\!\varepsilon_2^{a_2}\,p_3\!\cdot\! V\!\cdot\!\varepsilon_3^{a_3}\mathcal{J}_3\Big]+(2\leftrightarrow 3)
\end{aligned}
\tag{2.23}
$$

where

$$\mathcal{J}_{15}(p_1,p_2,p_3)=\mathcal{J}_5(p_1,p_3,p_2)=-\mathcal{J}_5(p_1,p_2,p_3)$$

and

$$\begin{aligned}\mathcal{J}_{13}=&\int_0^1 dr_2\int_0^1 dr_3\frac{1}{r_2r_3}\int_0^{2\pi}d\theta\frac{[(1+r_2^2r_3^2)(r_2^2+r_3^2)-4r_2^2r_3^2+2r_2r_3(1-r_2^2)(1-r_3^2)\cos(\theta)]\widetilde{K}}{|1-r_2r_3e^{i\theta}|^2|r_2-r_3e^{-i\theta}|^2}\\ \mathcal{J}_{14}=&\int_0^1 dr_2\int_0^1 dr_3\frac{1}{r_2r_3}\int_0^{2\pi}d\theta\frac{[(1+r_2^2r_3^2)(r_2^2+r_3^2)-4r_2^2r_3^2-2r_2r_3(1-r_2^2)(1-r_3^2)\cos(\theta)]\widetilde{K}}{|1-r_2r_3e^{i\theta}|^2|r_2-r_3e^{-i\theta}|^2}\end{aligned}$$

Instead of integrals $\mathcal{J}_0$, $\mathcal{J}_6$, $\mathcal{J}_7$, $\mathcal{J}_9$, $\mathcal{J}_{11}$ which appear in (2.21), the amplitude (2.23) has the integrals $\mathcal{J}_{13}$, $\mathcal{J}_{14}$. As we will see, these integrals are easier to perform explicitly.

We now find the relations between the integrals. The amplitude (2.23) must satisfy the Ward identity corresponding to polarizations ε_2 and ε_3. Imposing these conditions, one finds some relations between the integrals. The Ward identity for ε_2 gives the following relations:

$$\begin{aligned}-2p_1\!\cdot\! N\!\cdot\! p_2\mathcal{I}_1+2p_2\!\cdot\! V\!\cdot\! p_2\mathcal{I}_7+p_2\!\cdot\! N\!\cdot\! p_3\mathcal{I}_3-p_2\!\cdot\! V\!\cdot\! p_3\mathcal{I}_2=0 \quad (2.24)\\ -2\mathcal{I}_2p_1\!\cdot\! N\!\cdot\! p_2+(\mathcal{J}_{13}-\mathcal{J}_{14})p_2\!\cdot\! N\!\cdot\! p_3+2\mathcal{J}_2p_2\!\cdot\! V\!\cdot\! p_2\\ +(-4\mathcal{J}+\mathcal{J}_{13}+\mathcal{J}_{14}-2\mathcal{J}_5)p_2\!\cdot\! V\!\cdot\! p_3=0\\ 2\mathcal{I}_3p_1\!\cdot\! N\!\cdot\! p_2-2\mathcal{J}_1p_2\!\cdot\! V\!\cdot\! p_2+(\mathcal{J}_{13}-\mathcal{J}_{14})p_2\!\cdot\! V\!\cdot\! p_3\\ +(\mathcal{J}_{13}+\mathcal{J}_{14}+2\mathcal{J}_5)p_2\!\cdot\! N\!\cdot\! p_3=0\\ -2\mathcal{I}_4p_1\!\cdot\! N\!\cdot\! p_2+\mathcal{J}_{12}p_2\!\cdot\! N\!\cdot\! p_3+2\mathcal{J}_3p_2\!\cdot\! V\!\cdot\! p_2-\mathcal{J}_4p_2\!\cdot\! V\!\cdot\! p_3=0\\ (-\mathcal{J}_{13}+\mathcal{J}_{14})\left((p_2\!\cdot\! N\!\cdot\! p_3)^2+(p_2\!\cdot\! V\!\cdot\! p_3)^2\right)+2p_1\!\cdot\! N\!\cdot\! p_2\left(\mathcal{I}_2p_2\!\cdot\! N\!\cdot\! p_3-\mathcal{I}_3p_2\!\cdot\! V\!\cdot\! p_3\right)\\ +2p_2\!\cdot\! V\!\cdot\! p_2(p_2\!\cdot\! V\!\cdot\! p_3\mathcal{J}_1-p_2\!\cdot\! N\!\cdot\! p_3\mathcal{J}_2)-2(-2\mathcal{J}+\mathcal{J}_{13}+\mathcal{J}_{14})p_2\!\cdot\! V\!\cdot\! p_3\,p_2\!\cdot\! N\!\cdot\! p_3=0\end{aligned}$$

where $\mathcal{I}_7(p_1,p_2,p_3)=\mathcal{I}_4(p_1,p_3,p_2)$. The relation in the first line has been appeared in the amplitude considered in [28]. From the $(2\leftrightarrow 3)$ part of the amplitude (2.23), one finds the following relations:

$$\begin{aligned}-2p_1\!\cdot\! N\!\cdot\! p_3\mathcal{I}_1+2p_3\!\cdot\! V\!\cdot\! p_3\mathcal{I}_4+p_2\!\cdot\! N\!\cdot\! p_3\mathcal{I}_2-p_2\!\cdot\! V\!\cdot\! p_3\mathcal{I}_3=0 \quad (2.25)\\ -2\mathcal{I}_3p_1\!\cdot\! N\!\cdot\! p_3+(\mathcal{J}_{13}-\mathcal{J}_{14})p_2\!\cdot\! N\!\cdot\! p_3+2\mathcal{J}_{16}p_3\!\cdot\! V\!\cdot\! p_3\\ +(-4\mathcal{J}+\mathcal{J}_{13}+\mathcal{J}_{14}+2\mathcal{J}_5)p_2\!\cdot\! V\!\cdot\! p_3=0\\ 2\mathcal{I}_2p_1\!\cdot\! N\!\cdot\! p_3-2\mathcal{J}_4p_3\!\cdot\! V\!\cdot\! p_3+(\mathcal{J}_{13}-\mathcal{J}_{14})p_2\!\cdot\! V\!\cdot\! p_3\\ +(\mathcal{J}_{13}+\mathcal{J}_{14}-2\mathcal{J}_5)p_2\!\cdot\! N\!\cdot\! p_3=0\\ -2\mathcal{I}_7p_1\!\cdot\! N\!\cdot\! p_3+\mathcal{J}_2p_2\!\cdot\! N\!\cdot\! p_3+2\mathcal{J}_3p_3\!\cdot\! V\!\cdot\! p_3-\mathcal{J}_1p_2\!\cdot\! V\!\cdot\! p_3=0\\ (-\mathcal{J}_{13}+\mathcal{J}_{14})\left((p_2\!\cdot\! N\!\cdot\! p_3)^2+(p_2\!\cdot\! V\!\cdot\! p_3)^2\right)+2p_1\!\cdot\! N\!\cdot\! p_3\left(\mathcal{I}_3p_2\!\cdot\! N\!\cdot\! p_3-\mathcal{I}_2p_2\!\cdot\! V\!\cdot\! p_3\right)\\ +2p_3\!\cdot\! V\!\cdot\! p_3(p_2\!\cdot\! V\!\cdot\! p_3\mathcal{J}_4-p_2\!\cdot\! N\!\cdot\! p_3\mathcal{J}_{12})-2(-2\mathcal{J}+\mathcal{J}_{13}+\mathcal{J}_{14})p_2\!\cdot\! V\!\cdot\! p_3\,p_2\!\cdot\! N\!\cdot\! p_3=0\end{aligned}$$

The relation in the first line has been also appeared in the amplitude considered in [28]. If the explicit form of the integrals were known, then one could verify the above relations explicitly. We will verify the above relations for a special case in which the integrals can be calculated explicitly.

An indirect check of the above relations is that using them one can write the amplitude in terms of RR field strength which can then be checked with the S-matrix element in $(-1/2,-1/2)$-picture in which the RR vertex operator is in terms of field strength F. Using the above relations, one can write (2.23) as

$$\begin{aligned}
\mathcal{A} \sim\; & \frac{1}{2}p_2^{a_0}p_3^{a_1}\bigg[\Big(p_2\!\cdot\! N\!\cdot\! p_3(\varepsilon_2\!\cdot\! V\!\cdot\!\varepsilon_3)^{a_2a_3} - p_2\!\cdot\! V\!\cdot\! p_3(\varepsilon_2\!\cdot\! N\!\cdot\!\varepsilon_3)^{a_2a_3} - \varepsilon_3^{a_2a_3}p_3\!\cdot\! V\!\cdot\!\varepsilon_2\!\cdot\! N\!\cdot\! p_3\Big)\mathcal{J}\\
&+\varepsilon_3^{a_2a_3}\Big(p_3\!\cdot\! V\!\cdot\!\varepsilon_2\!\cdot\! V\!\cdot\! p_2\mathcal{J}_1 - \frac{1}{2}p_3\!\cdot\! V\!\cdot\!\varepsilon_2\!\cdot\! N\!\cdot\! p_1\mathcal{I}_3 + p_2\!\cdot\! V\!\cdot\!\varepsilon_2\!\cdot\! N\!\cdot\! p_3\mathcal{J}_2 - p_3\!\cdot\! V\!\cdot\!\varepsilon_2\!\cdot\! N\!\cdot\! p_3\mathcal{J}_5\\
&+\frac{1}{2}p_3\!\cdot\! N\!\cdot\!\varepsilon_2\!\cdot\! N\!\cdot\! p_1\mathcal{I}_2\Big) - p_1\!\cdot\! N\!\cdot\!\varepsilon_2^{a_2}\, p_2\!\cdot\! V\!\cdot\!\varepsilon_3^{a_3}\mathcal{I}_3 - p_1\!\cdot\! N\!\cdot\!\varepsilon_2^{a_2}\, p_1\!\cdot\! N\!\cdot\!\varepsilon_3^{a_3}\mathcal{I}_1\\
&+p_1\!\cdot\! N\!\cdot\!\varepsilon_2^{a_2}\, p_2\!\cdot\! N\!\cdot\!\varepsilon_3^{a_3}\mathcal{I}_2 + 2p_3\!\cdot\! V\!\cdot\!\varepsilon_2^{a_2}\, p_2\!\cdot\! N\!\cdot\!\varepsilon_3^{a_3}\mathcal{J}_5 + 4p_1\!\cdot\! N\!\cdot\!\varepsilon_2^{a_2}\, p_3\!\cdot\! V\!\cdot\!\varepsilon_3^{a_3}\mathcal{I}_4\\
&-2p_2\!\cdot\! V\!\cdot\!\varepsilon_2^{a_2}\, p_2\!\cdot\! N\!\cdot\!\varepsilon_3^{a_3}\mathcal{J}_2 + 2p_2\!\cdot\! V\!\cdot\!\varepsilon_2^{a_2}\, p_2\!\cdot\! V\!\cdot\!\varepsilon_3^{a_3}\mathcal{J}_1 - 4p_2\!\cdot\! V\!\cdot\!\varepsilon_2^{a_2}\, p_3\!\cdot\! V\!\cdot\!\varepsilon_3^{a_3}\mathcal{J}_3\bigg]\epsilon_{a_0a_1a_2a_3}\\
&+\frac{1}{2}p_1^{a_0}\bigg[\frac{1}{4}p_3\!\cdot\! V\!\cdot\!\varepsilon_2^{a_2}\varepsilon_3^{a_1a_3}\,(2p_2\!\cdot\! V\!\cdot\! p_2\mathcal{J}_1 + 2p_3\!\cdot\! V\!\cdot\! p_3\mathcal{J}_4 - 4p_2\!\cdot\! N\!\cdot\! p_3\mathcal{J})\\
&+\frac{1}{4}p_3\!\cdot\! N\!\cdot\!\varepsilon_2^{a_2}\varepsilon_3^{a_1a_3}\,\Big(\,2(\mathcal{J}_{13}-\mathcal{J}_{14})p_2\!\cdot\! N\!\cdot\! p_3 + (-4\mathcal{J}+\mathcal{J}_{13}+\mathcal{J}_{14})p_2\!\cdot\! V\!\cdot\! p_3\Big)\\
&-2p_2\!\cdot\! V\!\cdot\!\varepsilon_2^{a_2}\varepsilon_3^{a_1a_3}p_3\!\cdot\! V\!\cdot\! p_3\mathcal{J}_3 + p_1\!\cdot\! N\!\cdot\!\varepsilon_2^{a_2}\varepsilon_3^{a_1a_3}p_3\!\cdot\! V\!\cdot\! p_3\mathcal{I}_4\bigg]\epsilon_{a_0a_1a_2a_3}\\
&+\frac{1}{2}(p_1)_i\bigg[\frac{1}{2}p_3\!\cdot\! V\!\cdot\!\varepsilon_2^{a_2}\varepsilon_3^{a_1a_3}(p_2^ip_2^{a_0}\mathcal{I}_3 + p_3^ip_3^{a_0}\mathcal{I}_2) + \frac{1}{2}p_3\!\cdot\! N\!\cdot\!\varepsilon_2^{a_2}\varepsilon_3^{a_1a_3}(p_3^ip_2^{a_0}\mathcal{I}_3 + p_2^ip_3^{a_0}\mathcal{I}_2)\\
&-2p_3^ip_3^{a_0}p_2\!\cdot\! V\!\cdot\!\varepsilon_2^{a_2}\varepsilon_3^{a_1a_3}\mathcal{I}_7 + p_3^ip_3^{a_0}p_1\!\cdot\! N\!\cdot\!\varepsilon_2^{a_2}\varepsilon_3^{a_1a_3}\mathcal{I}_1\bigg]\epsilon_{a_0a_1a_2a_3} + (2\leftrightarrow 3)\\
&+2\varepsilon_2^{a_0a_1}\varepsilon_3^{a_2a_3}(p_1)_i\bigg[p_2^ip_3\!\cdot\! V\!\cdot\! p_3\mathcal{I}_4 + \frac{1}{2}p_3^ip_2\!\cdot\! V\!\cdot\! p_3\mathcal{I}_2 - \frac{1}{2}p_3^ip_2\!\cdot\! N\!\cdot\! p_3\mathcal{I}_3\bigg]\epsilon_{a_0a_1a_2a_3}
\end{aligned}\tag{2.26}$$

As we mentioned before, the result for the scattering amplitude can easily be extended to the arbitrary RR potential by replacing $\epsilon_{a_0a_1a_2a_3}$ with $\epsilon_{a_0\cdots a_p}\varepsilon_1^{a_4\cdots a_p}/(p-4)!$ where $\varepsilon_1^{a_4\cdots a_p}$ is the RR polarization tensor. The couplings in the last three lines above are consistent with the couplings found in [28] for the RR potential with one transverse index. They can be combined to be written in terms of RR field strength $F_{ia_4\ldots a_p}$. This part of amplitude has been checked explicitly in [28] by the evaluation of the S-matrix element in $(-1/2,-1/2)$-picture. The other couplings can easily be written in terms of $F_{a_0a_4\cdots a_p}$. We confirmed them by evaluating the S-matrix element in $(-1/2,-1/2)$-picture.

Having written the amplitude in terms of the RR field strength, one observes that the amplitude at each order of α' enjoys the RR gauge symmetry, as expected. In particular, this symmetry appear in the amplitude at order $O(\alpha'^2)$ which, as we will see, includes contact terms, massless open and closed string poles.

We now try to write the amplitude (2.23) in terms of H. As we mentioned before, strategy for doing this step if to look at the amplitude (2.21) and find terms which are not proportional to the Mandelstam variables and write them in terms of H and some extra

JHEP05(2011)066

terms which are proportional to the Mandelstam variables. Then using the relations (2.24) and (2.25), one should simplify the terms which are proportional to the Mandelstam variables. The remaining terms which are proportional to the Mandelstam variables, should then be either zero or be written in terms of H. Doing this, one finds the following result:

$$\begin{aligned}\mathcal{A}\sim\Big[&(p_2)_a H_2^{aa_0a_1}(p_3)_b H_3^{ba_2a_3}\mathcal{J}_3-\frac{1}{2}(p_2)_a(p_2)_b H_2^{aa_0a_1}H_3^{ba_2a_3}\mathcal{J}_1+\frac{1}{2}(p_2)_a(p_2)_i H_2^{aa_0a_1}H_3^{ia_2a_3}\mathcal{J}_2\\&-(p_1)_i(p_2)_a H_2^{aa_0a_1}H_3^{ia_2a_3}\mathcal{I}_7-\frac{1}{2}(p_2)_i H_2^{aa_0a_1}(p_3)_a H_3^{ia_2a_3}\mathcal{J}_5-\frac{1}{4}(p_1)_i(p_2)_j H_2^{ia_0a_1}H_3^{ja_2a_3}\mathcal{I}_2\\&+\frac{1}{4}(p_1)_i(p_1)_j H_2^{ia_0a_1}H_3^{ja_2a_3}\mathcal{I}_1+\frac{1}{4}(p_1)_i(p_2)_a H_2^{ia_0a_1}H_3^{aa_2a_3}\mathcal{I}_3-\frac{1}{3}(p_2)_a H_2^{a_0a_1a_2}(p_3)_b H_3^{aba_3}\mathcal{J}_4\\&-\frac{1}{6}(p_1)_i(p_2)_a H_2^{a_0a_1a_2}H_3^{iaa_3}\mathcal{I}_2+\frac{1}{3}(p_2)_i H_2^{a_0a_1a_2}(p_3)_a H_3^{iaa_3}\mathcal{J}_{12}\\&+\frac{1}{6}(p_1)_i(p_2)_j H_2^{a_0a_1a_2}H_3^{ija_3}\mathcal{I}_3-\frac{1}{3}(p_2)_i(p_2)_a H_2^{a_0a_1a_2}H_3^{iaa_3}(-\mathcal{J}_5+\mathcal{J})\\&+\frac{1}{4}(p_2\!\cdot\! N\!\cdot\! p_3)H_2^{aa_0a_1}H_3^{aa_2a_3}\mathcal{J}-\frac{1}{4}(p_2\!\cdot\! V\!\cdot\! p_3)H_2^{ia_0a_1}H_3^{ia_2a_3}\mathcal{J}\Big]\frac{1}{2}\epsilon_{a_0a_1a_2a_3}+[2\leftrightarrow 3]\end{aligned}\tag{2.27}$$

The terms in the last line are the only terms which are proportional to the Mandelstam variables. This is our final result for the string theory scattering amplitude. We will fix the normalization of the amplitude in section 2.2 by comparing the above amplitude at low energy with the corresponding field theory. The contracted indices in the terms in the last line are not momentum indices, so they are not invariant under the linear T-duality. They combine with the corresponding terms in the gravity amplitude (2.19) to produce a T-dual amplitude. All other terms are invariant under the linear T-duality. Hence, the combination of (2.27) and (2.19) is invariant under the linear T-duality when the Killing coordinate is an index of the RR potential. In other cases, one should add the amplitude for higher RR potential in which we are not interested in this paper.

Note that the above amplitude is not in terms of RR field strength. Hence, the amplitude can be written either in terms of H or in terms of the RR field strength. This indicates that the field theory couplings which are invariant under the B-field gauge transformations, are not invariant under the RR gauge transformation. However, as we mentioned before, the combination of the field theory couplings and the massless open and closed string poles at each order of α' is invariant under the RR gauge transformation.

2.1.1 Low energy limit

To find the low energy limit of the string theory amplitude (2.27), we are now trying to evaluate the integrals. It is hard to evaluate the integrals for the general case, so we concentrate on the special kinematic setup [28, 32]. Examining the Feynman diagrams involved, one can easily verify that the amplitude considered in this paper has no massless pole in the p_2p_3-channel. Moreover, there is no closed or open string channel corresponding to the Mandelstam variable $p_2\!\cdot\! D\!\cdot\! p_3$, hence, we restrict the Mandelstam variables to

$$p_2\!\cdot\! D\!\cdot\! p_3=0,\text{ and } p_2\!\cdot\! p_3=0\tag{2.28}$$

Even though the amplitude has no massless pole in $p_2\cdot p_3$-channel, the integrals which appear with the coefficient $p_2\!\cdot\! p_3$ in the amplitude may have massless pole in $p_2\!\cdot\! p_3$, so one can not set to zero the terms which are proportional to $p_2{\cdot}p_3$. The integrals $\mathcal{I}_2$, $\mathcal{I}_4$, $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, $\mathcal{J}_5$ and $\mathcal{J}_{14}$ which appear in the amplitude (2.23) have no pole in $p_2{\cdot}p_3$. The reason for this is that if one uses the constraint (2.28) the result of integrals would be finite. However, the the result of integral $\mathcal{J}_{13}$ under the constraint (2.28) is infinite, hence, it has massless pole in $p_2{\cdot}p_3$.

Using the Maple, one can easily preform the θ-integral in the integrals $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_4$, $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, $\mathcal{J}_5$ which appear in the amplitude (2.27), for the constraint (2.28). The result is

$$\begin{aligned}
\mathcal{I}_1 &= -2\pi\int_0^1 dr_3\int_0^1 dr_2\frac{\widetilde{K}'}{r_2r_3}\\
\mathcal{I}_2 &= 4\pi\int_0^1 dr_3\int_0^{r_3} dr_2\frac{\widetilde{K}'}{r_2r_3}\\
\mathcal{I}_4 &= -2\pi\int_0^1 dr_2\int_0^1 dr_3\frac{(1+r_3^2)\widetilde{K}'}{r_2r_3(1-r_3^2)}\\
\mathcal{J}_1 &= 4\pi\int_0^1 dr_2\int_0^{r_2} dr_3\frac{(1+r_2^2)\widetilde{K}'}{r_2r_3(1-r_2^2)}\\
\mathcal{J}_2 &= 4\pi\int_0^1 dr_3\int_0^{r_3} dr_2\frac{(1+r_2^2)\widetilde{K}'}{r_2r_3(1-r_2^2)}\\
\mathcal{J}_3 &= -2\pi\int_0^1 dr_2\int_0^1 dr_3\frac{(1+r_3^2)(1+r_2^2)\widetilde{K}'}{r_2r_3(1-r_3^2)(1-r_2^2)}\\
\mathcal{J}_5 &= 2\pi\int_0^1 dr_2\int_0^{r_2} dr_3\frac{(1+r_3^2)\widetilde{K}'}{r_2r_3(1-r_3^2)} - (2\leftrightarrow 3)
\end{aligned}\tag{2.29}$$

where $\widetilde{K}'$ is the value of $\widetilde{K}$ in the constraint (2.28), i.e.,

$$\widetilde{K}' = {r_2}^{2p_1\cdot p_2}\ {r_3}^{2p_1\cdot p_3}(1-{r_2}^2)^{p_2\cdot D\cdot p_2}(1-{r_3}^2)^{p_3\cdot D\cdot p_3}\tag{2.30}$$

Using the definition of beta function

$$\int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta)\tag{2.31}$$

The radial integral in $\mathcal{I}_1$, $\mathcal{I}_4$ and $\mathcal{J}_3$ becomes

$$\begin{aligned}
\mathcal{I}_1 &= -\frac{\pi}{2}B(s,1+p)B(t,1+q)\\
\mathcal{I}_4 &= -\frac{\pi}{2}\frac{(2t+q)}{q}B(s,1+p)B(t,1+q)\\
\mathcal{J}_3 &= -\frac{\pi}{2}\frac{(2s+p)}{p}\frac{(2t+q)}{q}B(s,1+p)B(t,1+q)
\end{aligned}\tag{2.32}$$

where we have used the following definitions for the Mandelstam variables:

$$\begin{aligned}
s &= p_1{\cdot}p_2\,; &\quad t &= p_1{\cdot}p_3\\
p &= p_2{\cdot}D{\cdot}p_2\,; &\quad q &= p_3{\cdot}D{\cdot}p_3
\end{aligned}\tag{2.33}$$

The radial integrals in $\mathcal{I}_2$, $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_5$ have the following structure:

$$I = \int_0^1 dx \int_0^x dy\, x^a y^b (1-x)^c (1-y)^d \tag{2.34}$$

which has the solution (see the appendix in [28])

$$I = \frac{B(1+c, 2+a+b)}{1+b} {}_3F_2\left[\begin{matrix} 2+a+b,\ 1+b,\ -d \\ 3+a+b+c,\ 2+b \end{matrix};\ 1\right] \tag{2.35}$$

Using this formula, one finds

$$\begin{aligned}
\mathcal{I}_2 &= \pi \frac{B(s+t, 1+q)}{s} {}_3F_2\left[\begin{matrix} s,\ s+t,\ -p \\ 1+s,\ 1+s+t+q \end{matrix};\ 1\right] \\
\mathcal{J}_1 &= \frac{\pi}{t}\left(B(s+t,p)_3F_2\left[\begin{matrix} t,\ s+t,\ -q \\ 1+t,\ s+t+p \end{matrix};\ 1\right]\right. \\
&\quad \left. +B(1+s+t,p)_3F_2\left[\begin{matrix} t,\ 1+s+t,\ -q \\ 1+t,\ 1+s+t+p \end{matrix};\ 1\right]\right) \\
\mathcal{J}_2 &= \pi\left(\frac{B(s+t,1+q)}{s} {}_3F_2\left[\begin{matrix} s,\ s+t,\ 1-p \\ 1+s,\ 1+s+t+q \end{matrix};\ 1\right]\right. \\
&\quad \left. +\frac{B(1+s+t,1+q)}{1+s} {}_3F_2\left[\begin{matrix} 1+s,\ 1+s+t,\ 1-p \\ 2+s,\ 2+s+t+q \end{matrix};\ 1\right]\right) \\
\mathcal{J}_5 &= \pi\left(\frac{B(s+t,1+p)}{t} {}_3F_2\left[\begin{matrix} t,\ s+t,\ 1-q \\ 1+t,\ 1+s+t+p \end{matrix};\ 1\right]\right. \\
&\quad \left. +\frac{B(1+s+t,1+p)}{1+t} {}_3F_2\left[\begin{matrix} 1+t,\ 1+s+t,\ 1-q \\ 2+t,\ 2+s+t+p \end{matrix};\ 1\right]\right) - (2 \leftrightarrow 3)
\end{aligned}$$

The evaluation of the integrals $\mathcal{J}_{13}$ and $\mathcal{J}_{14}$ is presented in the appendix A.

Having found the explicit form of the integrals for the constraint kinematic setup (2.28), we now verify the relations between the integrals in (2.24). Since none of the integrals have simple pole at $p_2\cdot p_3$ or $p_2\cdot D\cdot p_3$, except $\mathcal{J}_{13}$ which has only simple massless pole at $p_2\cdot p_3$ (see appendix A), the relations (2.24) simplify to

$$\begin{aligned}
-2p_1\cdot N\cdot p_2 \mathcal{I}_1 + 2p_2\cdot V\cdot p_2 \mathcal{I}_7 &= 0 \\
-2\mathcal{I}_2 p_1\cdot N\cdot p_2 + \mathcal{J}_{13} p_2\cdot p_3 + 2\mathcal{J}_2 p_2\cdot V\cdot p_2 &= 0 \\
2\mathcal{I}_3 p_1\cdot N\cdot p_2 - 2\mathcal{J}_1 p_2\cdot V\cdot p_2 + \mathcal{J}_{13} p_2\cdot p_3 &= 0 \\
-2\mathcal{I}_4 p_1\cdot N\cdot p_2 + 2\mathcal{J}_3 p_2\cdot V\cdot p_2 &= 0
\end{aligned} \tag{2.36}$$

Note that $\mathcal{J}_{13}(p_2\cdot p_3)^2$ is zero whereas $\mathcal{J}_{13}p_2\cdot p_3$ is nonzero. Using the equation (2.32), one can easily verify the first and last relations. The subtraction of the second and the third relations give

$$-2p_1\cdot N\cdot p_2(\mathcal{I}_2+\mathcal{I}_3) + 2p_2\cdot V\cdot p_2(\mathcal{J}_1+\mathcal{J}_2) = 0$$

Using the integral representations in (2.29), one observes that $\mathcal{I}_2+\mathcal{I}_3 = -2\mathcal{I}_1$ and $\mathcal{J}_1+\mathcal{J}_2 = -2\mathcal{I}_7$. Hence the above relation reduces to the first relation in (2.36).

JHEP05(2011)066

To study the low energy limit of the amplitude (2.27), we expand $\mathcal{I}_1$, $\mathcal{I}_2, \mathcal{I}_4, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ and $\mathcal{J}_5$ at the low energy. The expansion of beta function is standard and for expanding the hypergeometric function we use the package [33]. The result is

$$\begin{aligned}
\mathcal{I}_1 &= -\frac{\pi}{2}\left(\frac{1}{s(s+t)}+\frac{1}{t(s+t)}-\frac{\pi^2}{6}\left(\frac{q}{s}+\frac{p}{t}\right)+\cdots\right)\\
\mathcal{I}_2 &= \pi\left(\frac{1}{s(s+t)}-\frac{\pi^2}{6}\frac{q}{s}+\cdots\right)\\
\mathcal{I}_4 &= -\frac{\pi}{2}\left(\frac{(2t+q)}{q}\right)\left(\frac{1}{st}-\frac{\pi^2}{6}\left[\frac{p}{t}+\frac{q}{s}\right]+\cdots\right)\\
\mathcal{J}_1 &= \pi\left(\frac{1}{t(s+t)}+\frac{2}{pt}-\frac{\pi^2}{6}\left[2+\frac{(2s+p)}{t}+\frac{2q}{p}\right]+\cdots\right)\\
\mathcal{J}_2 &= \pi\left(\frac{1}{s(s+t)}+\frac{\pi^2}{6}\left[2-\frac{q}{s}\right]+\cdots\right)\\
\mathcal{J}_3 &= -\frac{\pi}{2}\left(\frac{(2s+p)}{p}\frac{(2t+q)}{q}\right)\left(\frac{1}{st}-\frac{\pi^2}{6}\left[\frac{p}{t}+\frac{q}{s}\right]+\cdots\right)\\
\mathcal{J}_5 &= \pi\left(\frac{1}{t(s+t)}-\frac{1}{s(s+t)}+\frac{\pi^2}{6}\left[\frac{q}{s}-\frac{p}{t}\right]+\cdots\right)
\end{aligned}\tag{2.37}$$

From these expansions and the expansion (2.20) for $\mathcal{J}$, one finds the following contact terms at order $O(\alpha'^2)$:

$$\begin{aligned}
\mathcal{A}^{\rm contact} \sim \frac{\pi^3}{6}\bigg[&\frac{1}{2}(p_2)_a(p_2)_b H_2^{aa_0a_1}H_3^{ba_2a_3}+\frac{1}{2}(p_2)_a(p_2)_i H_2^{aa_0a_1}H_3^{ia_2a_3}\\
&+\frac{1}{3}(p_2)_a H_2^{a_0a_1a_2}(p_3)_b H_3^{aba_3}+\frac{1}{3}(p_2)_i H_2^{a_0a_1a_2}(p_3)_a H_3^{iaa_3}\\
&-\frac{1}{4}(p_2)_i H_2^{aa_0a_1}(p_3)_i H_3^{aa_2a_3}+\frac{1}{4}(p_2)_a H_2^{ia_0a_1}(p_3)_a H_3^{ia_2a_3}\\
&+\frac{1}{3}(p_2)_i(p_2)_a H_2^{a_0a_1a_2}H_3^{iaa_3}\bigg]\epsilon_{a_0a_1a_2a_3}+[2\leftrightarrow 3]
\end{aligned}\tag{2.38}$$

Even though we have found the expansion (2.37) for the constraint (2.28), the constants of the integrals which produce the above contact terms, are independent of $p_2{\cdot}p_3$ or $\phi_2{\cdot}D{\cdot}p_3$. Hence, the above result is valid for the general case.

The amplitude has also the following massless open string poles at order $O(\alpha'^2)$:

$$\begin{aligned}
\mathcal{A}^{\rm pole} \sim \frac{\pi^3}{12}\bigg[&2(p_2)_a H_2^{aa_0a_1}(p_3)_b H_3^{ba_2a_3}\left(\frac{q+2t}{p}\right)-(p_1)_i(p_2)_a H_2^{aa_0a_1}H_3^{ia_2a_3}\frac{q}{p}\\
&+\frac{2}{3}(p_2)_b H_2^{aba_3}(p_3)_a H_3^{a_0a_1a_2}\frac{q}{p}+(p_2)_a(p_2)_b H_2^{aa_0a_1}H_3^{ba_2a_3}\frac{q}{p}\bigg]\epsilon_{a_0a_1a_2a_3}+[2\leftrightarrow 3]
\end{aligned}\tag{2.39}$$

Since the expansion (2.37) for the integrals are valid for the constraint (2.28), there might be some other massless open string poles which are proportional to $p_2{\cdot}V{\cdot}p_3$ or $p_2{\cdot}N{\cdot}p_3$.

The amplitude (2.27) has also some massless closed string poles at order $O(\alpha'^2)$ which are in terms of H. However, they are not invariant under the RR gauge transformations.

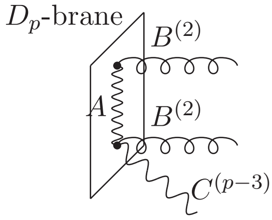

Figure 1. Feynman diagram for massless open string poles.

Recall that the RR gauge transformation of $(n-2)$-form potential in supergravity is canceled with the gauge transformation of n-form. So one does not expect to have Ward identity corresponding to a RR potential in the massless closed string poles. We have seen that the string amplitude (2.27) can be written in terms of RR field strength, i.e., (2.26). So this amplitude at order $O(\alpha'^2)$ which is equal to the sum of the above massless closed string amplitude, massless open string amplitude (2.39) and the contact terms (2.38), is invariant under the RR gauge transformation. We are not interested in the massless closed string poles, as they do not produce any new D-brane couplings.

2.2 Field theory amplitude

Having found the contact terms and the massless open string poles of string amplitude at order $O(\alpha'^2)$, we now reproduce them by appropriate couplings in field theory. The field theory couplings are those that we have already presented in section 1. To simplify the calculation we consider the RR scalar. In the next subsection we show that the field theory produces the the massless open string poles (2.39).

2.2.1 Open string pole

In field theory, the open string channel of the scattering amplitude of one RR potential $(p-3)$-form and two B-fields is given by the Feynman diagram in figure 1. The corresponding Feynman amplitude is given by:

$$\mathcal{A}_1^f = V_a(\varepsilon_3, A)G_{ab}(A)V_b(A, \varepsilon_2, \varepsilon_1^{(p-3)}) + (2 \leftrightarrow 3) \tag{2.40}$$

where A^a is the gauge field on the D$_p$-brane. The gauge field propagator and the vertex $V_a(\varepsilon_3, A)$ can be read from the DBI action (1.1), i.e.,

$$\begin{aligned} V_a(\varepsilon_3, A) &= (2\pi\alpha')T_3(p_3\!\cdot\! V\!\cdot\!\varepsilon_3)_a \\ G_{ab}(A) &= \left(\frac{-i}{T_3(2\pi\alpha')^2}\right)\frac{\eta_{ab}}{p_3\!\cdot\! V\!\cdot\! p_3} \end{aligned} \tag{2.41}$$

The couplings in section 1 produce the vertex $V_b(A, \varepsilon_2, \varepsilon_1^{(p-3)})$. We begin by considering the couplings in the second line of (1.7) for RR scalar. The vertex corresponding to the first term is given by

$$V_b(A, \varepsilon_2) = -2(\pi\alpha')^3 T_3 \epsilon_{a_0a_1a_2b}(p_1+p_2)^{a_2}(p_2)_a\left[H_2^{a_0a_1a}p_1\!\cdot\! N\!\cdot\! p_2\right]$$

The amplitude (2.40) then becomes

$$\mathcal{A}_1^f = i(\pi\alpha')^2 T_3 \frac{(p_3\cdot V\cdot\varepsilon_3)^{a_3} p_3^{a_2}}{p_3\cdot V\cdot p_3}\epsilon_{a_0\cdots a_3}(p_2)_a\left[H_2^{a_0a_1a}p_1\cdot N\cdot p_2\right] + (2\leftrightarrow 3)$$

This amplitude is of order $O(\alpha'^2)$ which has six momentum in the numerator and two momentum in the denominator. The couplings in the second line of (1.7) have also the following contact term:

$$\mathcal{A}_2^f = i\left(\frac{(\pi\alpha')^2 T_3}{2}\right)\varepsilon_3^{a_2a_3}\epsilon_{a_0\cdots a_3}(p_2)_a\left[H_2^{a_0a_1a}p_1\cdot N\cdot p_2\right] + (2\leftrightarrow 3) \tag{2.42}$$

Using the following identity:

$$\epsilon_{a_0\cdots a_3}(p_3\cdot V\cdot H_3)^{a_2a_3} = \epsilon_{a_0\cdots a_3}\left(2p_3\cdot V\cdot\varepsilon_3^{a_2}p_3^{a_3} + p_3\cdot V\cdot p_3\varepsilon_3^{a_2a_3}\right)$$

one can rewrite the sum of $\mathcal{A}_1^f$ and $\mathcal{A}_2^f$ as

$$\mathcal{A}^f = i\left(\frac{(\pi\alpha')^2 T_3}{2}\right)\frac{1}{p_3\cdot V\cdot p_3}\epsilon_{a_0\cdots a_3}(p_2)_a(p_3)_b\left[H_2^{a_0a_1a}H_3^{ba_2a_3}p_1\cdot N\cdot p_2\right] + (2\leftrightarrow 3) \tag{2.43}$$

Using the constraint (2.28), one writes $p_1\!\cdot\! N\!\cdot\! p_2 = s+p/2$. Hence, the above result is exactly the first term in the open string amplitude (2.39) provided that one fix the normalization of the string amplitude (2.27) to be

$$N = \frac{3i\alpha'^2 T_p}{\pi} \tag{2.44}$$

Now consider the second term in the second line of (1.7). It produces the following vertex and contact term:

$$\begin{aligned}V_b(A,\varepsilon_2) &= (\pi\alpha')^3 T_3\epsilon_{a_0a_1a_2b}(p_1+p_2)^{a_2}(p_1)_i\left[H_2^{a_0a_1i}p_2\cdot V\cdot p_2\right]\\ \mathcal{A}_2^f &= -i\left(\frac{(\pi\alpha')^2 T_3}{4}\right)\varepsilon_3^{a_2a_3}\epsilon_{a_0\cdots a_3}(p_1)_i\left[H_2^{a_0a_1i}p_2\cdot V\cdot p_2\right] + (2\leftrightarrow 3)\end{aligned}$$

Doing the same steps as before, one finds

$$\mathcal{A}^f = -i\left(\frac{(\pi\alpha')^2 T_3}{4}\right)\frac{1}{p_3\cdot V\cdot p_3}\epsilon_{a_0\cdots a_3}(p_1)_i(p_3)_a\left[H_2^{a_0a_1i}H_3^{aa_2a_3}p_2\cdot V\cdot p_2\right] + (2\leftrightarrow 3) \tag{2.45}$$

This is exactly the second term in the open string amplitude (2.39).

Next consider the first two terms in the last line of (1.8). They produce the following vertex and contact term:

$$\begin{aligned}V_b(A,\varepsilon_2) &= -\frac{(\pi\alpha')^3 T_3}{3}p_2\cdot V\cdot p_2 H_2^{a_0a_1a_2}\left[\epsilon_{a_0a_1a_2b}(p_1+p_2)\cdot V\cdot p_2 - \epsilon_{a_0a_1a_2a_3}(p_1+p_2)^{a_3}(p_2)_b\right]\\ \mathcal{A}_2^f &= i\left(\frac{(\pi\alpha')^2 T_3}{6}\right)p_2\cdot V\cdot p_2 H_2^{a_0a_1a_2}\epsilon_{a_0\cdots a_3}p_2\cdot V\cdot\varepsilon_3^{a_3} + (2\leftrightarrow 3)\end{aligned}$$

In this case, one finds

$$\mathcal{A}^f = -i\left(\frac{(\pi\alpha')^2 T_3}{6}\right)\frac{1}{p_3\cdot V\cdot p_3}\epsilon_{a_0\cdots a_3}(p_3)_a(p_2)_b\left[H_2^{a_0a_1a_2}H_3^{aba_3}p_2\cdot V\cdot p_2\right] + (2\leftrightarrow 3) \tag{2.46}$$

which is exactly the first term in the second line of (2.39).

Finally consider the last two terms in the last line of (1.8). They produce the following vertex and contact term:

$$
\begin{aligned}
V_b(A,\varepsilon_2) &= -(\pi\alpha')^3 T_3 p_2\!\cdot\! V\!\cdot\! p_2 H_2^{a_0a_1a}(p_1+p_2)_a(p_1+p_2)^{a_2}\epsilon_{a_0a_1a_2b} \\
\mathcal{A}_2^f &= i\left(\frac{(\pi\alpha')^2T_3}{4}\right) p_2\!\cdot\! V\!\cdot\! p_2 H_2^{a_0a_1a}\varepsilon_3^{a_2a_3}(p_3)_a\epsilon_{a_0\cdots a_3} + (2\leftrightarrow 3)
\end{aligned}
$$

In this case, one finds

$$
\mathcal{A}^f = i\left(\frac{(\pi\alpha')^2T_3}{4}\right)\frac{1}{p_3\!\cdot\! V\!\cdot\! p_3}\epsilon_{a_0\cdots a_3}(p_3)_a(p_3)_b\big[H_2^{a_0a_1a}H_3^{ba_2a_3}p_2\!\cdot\! V\!\cdot\! p_2\big] + (2\leftrightarrow 3) \tag{2.47}
$$

which is exactly the last term in the second line of (2.39).

The above calculation indicates that the couplings which include $(B+2\pi\alpha' f)$ appears as massless open string pole. Since the massless open string ampliude (2.39) does not includes terms which are proportional to $p_2\!\cdot\! V\!\cdot\! p_3$ or $p_2\!\cdot\! N\!\cdot\! p_3$, the above calculation can not fix the coefficient of all higher derivative couplings which contain $(B+2\pi\alpha' f)$. For example the higher derivative coupling $C\wedge\partial_a\partial_b f\wedge\partial^a\partial^b f$ can be read from the S-matrix element of one RR and two gauge field vertex operators [5, 35]. One may extend it to $C\wedge\partial_a\partial_b(B+2\pi\alpha' f)\wedge\partial^a\partial^b(B+2\pi\alpha' f)$. This coupling produces massless open string pole which is proportional to $p_2\!\cdot\! V\!\cdot\! p_3$. Hence, our calculation can not fix the presence of such couplings.

The couplings corresponding to the massless open string poles can also be extracted from the low energy limit of the S-matrix element of one RR, one B-field and one open string gauge field vertex operators. In that case, after fixing the SL$(2,R)$ symmetry, one would find a double integral which can be evaluated explicitly for the general kinematic setup [36].

2.2.2 Contact terms

Using the normalization (2.44), one finds the Lagrangian corresponding to the contact terms in (2.38) to be

$$
\begin{aligned}
\mathcal{L} = \frac{(\pi\alpha')^2T_3}{2}C^{(0)}\Big[&\frac{1}{2}H^{aa_0a_1,ab}H^{ba_2a_3} + \frac{1}{2}H^{aa_0a_1,ai}H^{ia_2a_3} + \frac{1}{3}H^{a_0a_1a_2,ia}H^{iaa_3} \\
&+\frac{1}{3}H^{a_0a_1a_2,a}H^{aba_3,b} + \frac{1}{3}H^{a_0a_1a_2,i}H^{iaa_3,a} \\
&-\frac{1}{4}H^{aa_0a_1,i}H^{aa_2a_3,i} + \frac{1}{4}H^{ia_0a_1,a}H^{ia_2a_3,a}\Big]\epsilon_{a_0a_1a_2a_3}
\end{aligned} \tag{2.48}
$$

The terms in the last line are exactly the couplings (1.4). All other terms are those appear in the first two lines of (1.8). Unlike the open string poles, there are no other couplings at order $O(\alpha^2)$.

The couplings in the last line above have been found in [6] by requiring the the Chern-Simons couplings (1.3) to be invariant under linear T-duality, and by requiring the new couplings to be invariant under B-field gauge transformation. These couplings, however, are not invariant under linear T-duality if one of the indices of B-field which is contracted with the volume form is the Killing coordinate. In that case one should add new couplings involving higher RR potential to make a complete T-dual multiplet [6]. The new couplings,

JHEP05(2011)066

however, are neither covariant nor invariant under the B-field gauge transformation. So one needs to add some other T-dual multiplets [6]. Having found the new couplings in the first two lines above, one should do the same steps for these couplings as well, to find all nonzero couplings of D_p-branes at order $O(\alpha'^2)$.

Extending the calculation of the S-matrix element in this paper to the case that $n = p-1$, $n = p+1$, $n = p+3$ and $n = p+5$, one would be able to find all nonzero couplings. We have performed the calculation for $n = p+5$ case and found that the S-matrix element is zero for two gravitons, and has only closed string poles for two B-fields. We present this result in the appendix B.

Acknowledgments

This work is supported by Ferdowsi University of Mashhad under grant 2/16340-1389/10/14.

A Evaluating $\mathcal{J}_{13}$ and $\mathcal{J}_{14}$

In this appendix we calculate the integrals $\mathcal{J}_{13}$ and $\mathcal{J}_{14}$. These integrals do not appear in the amplitude (2.27), however, they are needed to verifies the relations between the integrals that have been found in (2.24) and (2.25).

The integral $\mathcal{J}_{14}$ has no pole in $p_2{\cdot}p_3$. To take the θ-integral in $\mathcal{J}_{14}$, we first write it as

$$\mathcal{J}_{14} = \mathcal{J}'_{14} + \mathcal{J}''_{14}$$

where

$$\begin{aligned}
\mathcal{J}'_{14} &= \int_0^1 dr_2 \int_0^1 dr_3 \frac{(r_2^2 - r_3^2)^2}{r_2 r_3 (1-r_2^2)(1-r_3^2)} \int_0^{2\pi} d\theta \frac{\widetilde{K}}{(r_2^2 + r_3^2 - 2r_2 r_3 \cos(\theta))} \\
\mathcal{J}''_{14} &= \int_0^1 dr_2 \int_0^1 dr_3 \frac{(1 + r_2^2 r_3^2 - 2r_2^2)(1 + r_2^2 r_3^2 - 2r_3^2)}{r_2 r_3 (1-r_2^2)(1-r_3^2)} \int_0^{2\pi} d\theta \frac{\widetilde{K}}{(1 + r_2^2 r_3^2 - 2r_2 r_3 \cos(\theta))}
\end{aligned}$$

The θ-integrals then become

$$\begin{aligned}
\mathcal{J}'_{14} &= 2\pi \int_0^1 dr_2 \int_0^{r_2} dr_3 \frac{(r_2^2 - r_3^2)\widetilde{K}'}{r_2 r_3 (1-r_2^2)(1-r_3^2)} + (2 \leftrightarrow 3) \\
\mathcal{J}''_{14} &= -\pi \int_0^1 dr_2 \int_0^1 dr_3 \frac{\widetilde{K}'}{r_2 r_3} \left(\frac{(r_2^2 r_3^2 - 1)}{(1-r_2^2)(1-r_3^2)} + \frac{4}{(1-r_3^2)} + \frac{4}{(r_2^2 r_3^2 - 1)} \right) + (2 \leftrightarrow 3)
\end{aligned}$$

Writing $r_2^2 - r_3^2 = (r_2^2 - 1) + (1 - r_3^2)$ and using the formula (2.35) the radial integral in $\mathcal{J}'_{14}$ becomes

$$\begin{aligned}
\mathcal{J}'_{14} = \frac{\pi}{2t} \Bigg(& B(s+t, p)\, {}_3F_2\!\left[\begin{matrix} t,\ s+t,\ -q \\ 1+t,\ s+t+p \end{matrix} ;\ 1 \right] \\
& - B(s+t, 1+p)\, {}_3F_2\!\left[\begin{matrix} t,\ s+t,\ 1-q \\ 1+t,\ 1+s+t+p \end{matrix} ;\ 1 \right] \Bigg) + (2 \leftrightarrow 3)
\end{aligned}$$

The radial integral in the first two terms in $\mathcal{J}''_{14}$ gives a multiple of two beta functions. To take the radial integral in the last term we use the following identity:

$$\frac{1}{1-r_2^2r_3^2} = {}_2F_1\left[\begin{matrix}1,\ 1\\ 1\end{matrix}\ ;\ r_2^2r_3^2\right] \tag{A.1}$$

Then using the integral representation of the generalized hypergeometric function ${}_pF_q$ [34]:

$$\int_0^1 dx\, x^a(1-x)^b{}_pF_q\left[\begin{matrix}a_1,\ \cdots,a_p\\ b_1,\ \cdots,\ b_q\end{matrix}\ ;\ \lambda x\right] =$$
$${}_{p+1}F_{q+1}\left[\begin{matrix}1+a,\ a_1,\ \cdots,a_p\\ 2+a+b,\ b_1,\ \cdots,\ b_q\end{matrix}\ ;\ \lambda\right]B(1+a,1+b) \tag{A.2}$$

one can write the radial integral in terms of the the hypergeometric function ${}_4F_3$. The result is

$$\mathcal{J}''_{14} = \frac{\pi}{4}\Big(B(s,p)B(t,q) - B(1+s,p)B(1+t,q) - 4B(s,1+p)B(t,q)$$
$$+4B(t,1+q)B(s,1+p){}_3F_2\left[\begin{matrix}t,\ s,\ 1\\ 1+t+q,\ 1+s+p\end{matrix}\ ;\ 1\right]\Big) + (2\leftrightarrow 3) \tag{A.3}$$

where we have also used the identity:

$${}_4F_3\left[\begin{matrix}a,\ b,\ c,\ 1\\ d,\ e,\ 1\end{matrix}\ ;\ 1\right] = {}_3F_2\left[\begin{matrix}a,\ b,\ c\\ d,\ e\end{matrix}\ ;\ 1\right] \tag{A.4}$$

The integral $\mathcal{J}_{13}$, however, has simple pole $1/(p_2\cdot p_3)$. To see this we write it as

$$\mathcal{J}_{13} = \mathcal{J}'_{13} + \mathcal{J}''_{13}$$

where

$$\mathcal{J}'_{13} = -\int_0^1 dr_2\int_0^1 dr_3\frac{(1-r_2^2r_3^2)^2}{r_2r_3(1-r_2^2)(1-r_3^2)}\int_0^{2\pi}d\theta\frac{\widetilde{K}}{(1+r_2^2r_3^2-2r_2r_3\cos(\theta))}$$
$$\mathcal{J}''_{13} = \int_0^1 dr_2\int_0^1 dr_3\frac{(r_2^2+r_3^2-2)(2r_2^2r_3^2-r_2^2-r_3^2)}{r_2r_3(1-r_2^2)(1-r_3^2)}\int_0^{2\pi}d\theta\frac{\widetilde{K}}{(r_2^2+r_3^2-2r_2r_3\cos(\theta))}$$

The θ-integral in $\mathcal{J}'_{13}$ gives

$$\mathcal{J}'_{13} = -\pi\int_0^1 dr_2\int_0^1 dr_3\frac{(1-r_2^2r_3^2)\widetilde{K}'}{r_2r_3(1-r_2^2)(1-r_3^2)} + (2\leftrightarrow 3)$$

and the radial integrals give

$$\mathcal{J}'_{13} = -\frac{\pi}{4}\Big(B(s,p)B(t,q) - B(1+s,p)B(1+t,q)\Big) + (2\leftrightarrow 3) \tag{A.5}$$

The θ-integral in $\mathcal{J}''_{13}$ gives

$$\mathcal{J}''_{13} = 2\pi\int_0^1 dr_2\int_0^{r_2} dr_3\frac{[(r_2^2-r_3^2)+2(r_3^2-1)][2r_2^2(r_3^2-1)+(r_2^2-r_3^2)]\widetilde{K}'}{r_2r_3(1-r_2^2)(1-r_3^2)(r_2^2-r_3^2)} + (2\leftrightarrow 3)$$
$$= 2\pi\int_0^1 dr_2\int_0^{r_2} dr_3\Big(\frac{2r_2}{r_3(1-r_2^2)} - \frac{1}{r_2r_3(1-r_2^2)} - \frac{1}{r_2r_3(1-r_3^2)}$$
$$+\frac{4r_2}{r_3(r_2^2-r_3^2)}\Big) + (2\leftrightarrow 3) \tag{A.6}$$

The terms in the second line have no $p_2{\cdot}p_3$ pole. Their radial integrals can be evaluated using the formula (2.35), i.e.,

$$\mathcal{J}''_{13} = \left(\frac{B(p,1+s+t)}{t} {}_3F_2\left[\begin{matrix} 1+s+t,\ t,\ -q \\ 1+s+t+p,\ 1+t \end{matrix};1\right] - \frac{B(p,s+t)}{2t} {}_3F_2\left[\begin{matrix} s+t,\ t,\ -q \\ s+t+p,\ 1+t \end{matrix};1\right] \right.$$
$$\left. - \frac{B(1+p,s+t)}{2t} {}_3F_2\left[\begin{matrix} s+t,\ t,\ 1-q \\ 1+s+t+p,\ 1+t \end{matrix};1\right] \right) \pi + (2 \leftrightarrow 3) + J \tag{A.7}$$

The term in the last line of (A.6) which we have called it J, however, is infinite when $r_2 = r_3$ which means it has massless pole at $p_2{\cdot}p_3$. So this part of $\mathcal{J}''_{13}$ must be calculated for $p_2{\cdot}p_3 \neq 0$.

The θ-integral in this part has to be evaluated for $p_2{\cdot}p_3 \neq 0$. So we have to calculate the following integral:

$$J = \int_0^1 dx \int_0^1 dy\, x^s y^{t-1}(1-x)^p(1-y)^q \int_0^{2\pi} d\theta (x+y-2\sqrt{xy}\cos(\theta))^{-1+p_2\cdot p_3}$$

where we have chosen $x = r_2^2$, $y = r_3^2$. To take θ-integral we use the following formula [34]:

$$\int_0^{2\pi} d\theta \frac{\cos(n\theta)}{(1+a^2-2a\cos(\theta))^b} = 2\pi a^n \frac{\Gamma(b+n)}{n!\Gamma(b)} {}_2F_1\left[\begin{matrix} b,\ n+b \\ n+1 \end{matrix};a^2\right] \tag{A.8}$$

where $|a| < 1$. One finds

$$J = 2\pi \int_0^1 dx \int_0^x dy\, x^{-1+s+p_2\cdot p_3} y^{t-1}(1-x)^p(1-y)^q {}_2F_1\left[\begin{matrix} 1-p_2\cdot p_3,\ 1-p_2\cdot p_3 \\ 1 \end{matrix};y/x\right]$$
$$+2\pi \int_0^1 dy \int_0^y dx\, x^s y^{-2+t+p_2\cdot p_3}(1-x)^p(1-y)^q {}_2F_1\left[\begin{matrix} 1-p_2\cdot p_3,\ 1-p_2\cdot p_3 \\ 1 \end{matrix};x/y\right]$$

changing the variable in the first line as $y = xu$ and in the second line as $x = yu$, one finds

$$J = 2\pi \int_0^1 dx \int_0^1 du\, x^{-1+p_2\cdot p_3+s+t} u^{t-1}(1-x)^p(1-xu)^q {}_2F_1\left[\begin{matrix} 1-p_2\cdot p_3,\ 1-p_2\cdot p_3 \\ 1 \end{matrix};u\right]$$
$$+2\pi \int_0^1 dy \int_0^1 du\, u^s y^{-1+s+t+p_2\cdot p_3}(1-yu)^p(1-y)^q {}_2F_1\left[\begin{matrix} 1-p_2\cdot p_3,\ 1-p_2\cdot p_3 \\ 1 \end{matrix};u\right]$$

These integrals have no massless pole in the open string p- or q-channel. So for ease of calculation we set $p = q = 0$. Then using the integral representation of the generalized hypergeometric function (A.2) one finds

$$J = 2\pi \left(\frac{{}_3F_2\left[\begin{matrix} t,\ 1-p_2p_3,\ 1-p_2p_3 \\ 1+t,\ 1 \end{matrix};1\right]}{t(p_2\cdot p_3+s+t)} + \frac{{}_3F_2\left[\begin{matrix} 1+s,\ 1-p_2p_3,\ 1-p_2p_3 \\ 2+s,\ 1 \end{matrix};1\right]}{(1+s)(p_2\cdot p_3+s+t)} \right) \tag{A.9}$$

Having found the explicit form of $\mathcal{J}_{13}$ and $\mathcal{J}_{14}$, one can use the package [33] to expand them. The result is The low energy expansion of $\mathcal{J}_{13}$ and $\mathcal{J}_{14}$ are

$$\mathcal{J}_{14} = \frac{\pi}{2}\left(\frac{1}{st} - \frac{\pi^2}{6}\left[\frac{q}{s} + \frac{p}{t}\right] + \cdots\right)$$
$$\mathcal{J}_{13} = \frac{\pi}{2}\left(-\frac{3}{st} + \frac{\pi^2}{6}\left[-4 + \frac{3p}{t} + \frac{3q}{s}\right] + \cdots\right) + J$$

Note that these expansion for $\mathcal{J}_{14}$ and $\mathcal{J}_{13} - J$ are valid for the constraint (2.28). The expansion for J is:

$$J = 2\pi\left(\frac{1}{p_2\cdot p_3(s+t+p_2\cdot p_3)} + \frac{1}{t(s+t+p_2\cdot p_3)} - \frac{\pi^2}{6} + \cdots\right) \tag{A.10}$$

which is valid for $p = q = 0$ but $p_2\cdot p_3 \neq 0$.

We now check the relation (2.36). Using (2.37), one finds the following expansion:

$$2\mathcal{I}_3 p_1\cdot N\cdot p_2 - 2\mathcal{J}_1 p_2\cdot V\cdot p_2 = -2\pi\left(\frac{1}{(s+t)} + \cdots\right)$$

for the case that $p_2\cdot p_3 = p = q = 0$. Then the third relation in (2.36) gives

$$\mathcal{J}_{13} p_2\cdot p_3 = 2\pi\left(\frac{1}{(s+t)} + \cdots\right)$$

which is consistent with the expansion (A.10).

B S-matrix element for $n = p + 5$ case

In this appendix we consider the scattering amplitude (2.5) for the case $n = p + 5$. Since the RR potential is totally antisymmetric, it must have at least four transverse indices. We consider the case that the RR potential carries four transverse indices and $p + 1$ world volume indices. This is similar to the case $n = p - 3$ and the RR potential with only world volumes indices which we studied in section 2.1. The cases that the RR potential carries more transverse indices is similar to the case $n = p - 3$ and the RR potential with transverse and world volumes indices which we studied in [28].

Using the same steps as in section 2.1, one finds the scattering amplitude is zero for two graviton vertex operators. We find this result by explicit calculation in $(-3/2, -1/2)$-picture. One can find this result in $(-1/2, -1/2)$-picture without explicit calculation, and by taking into account the fact that the RR field strength in the vertex operators is totally antisymmetric.

JHEP05(2011)066

The explicit calculation in $(-3/2,-1/2)$-picture, as we have done in section 2.1, gives the following result for two B-fields:

$$\begin{aligned}
\mathcal{A}\sim{}&\frac{1}{4(p+1)!}\epsilon^{a_0\cdots a_p}\varepsilon_{1a_0\cdots a_pijkl}\varepsilon_3{}^{kl}\Big(p_2^ip_3^jp_3\!\cdot\! V\!\cdot\!\varepsilon_2\!\cdot\! N\!\cdot\! p_1\mathcal{I}_2-p_2^ip_3^jp_3\!\cdot\! N\!\cdot\!\varepsilon_2\!\cdot\! N\!\cdot\! p_1\mathcal{I}_3\\
&-p_2^ip_2\!\cdot\! V\!\cdot\! p_3p_3\!\cdot\! V\!\cdot\!\varepsilon_2{}^j\mathcal{J}_6-p_2^ip_3\!\cdot\! V\!\cdot\! p_3p_3\!\cdot\! V\!\cdot\!\varepsilon_2{}^j\mathcal{J}_4+p_3^ip_2\!\cdot\! V\!\cdot\! p_2p_3\!\cdot\! V\!\cdot\!\varepsilon_2{}^j\mathcal{J}_2\\
&-2p_3^ip_2\!\cdot\! V\!\cdot\! p_3p_3\!\cdot\! V\!\cdot\!\varepsilon_2{}^j\mathcal{J}_8-2p_2^ip_2\!\cdot\! N\!\cdot\! p_3p_3\!\cdot\! V\!\cdot\!\varepsilon_2{}^j\mathcal{J}_9-p_3^ip_2\!\cdot\! N\!\cdot\! p_3p_3\!\cdot\! V\!\cdot\!\varepsilon_2{}^j\mathcal{J}_6\\
&-2p_2^ip_3\!\cdot\! V\!\cdot\! p_3p_1\!\cdot\! N\!\cdot\!\varepsilon_2{}^j\mathcal{I}_4-p_3^ip_2\!\cdot\! V\!\cdot\! p_3p_1\!\cdot\! N\!\cdot\!\varepsilon_2{}^j\mathcal{I}_2-2p_2^ip_2\!\cdot\! V\!\cdot\! p_3p_3\!\cdot\! N\!\cdot\!\varepsilon_2{}^j\mathcal{J}_7\\
&+p_2^ip_3\!\cdot\! V\!\cdot\! p_3p_3\!\cdot\! N\!\cdot\!\varepsilon_2{}^j\mathcal{J}_{12}-p_3^ip_2\!\cdot\! V\!\cdot\! p_2p_3\!\cdot\! N\!\cdot\!\varepsilon_2{}^j\mathcal{J}_1-p_3^ip_2\!\cdot\! V\!\cdot\! p_3p_3\!\cdot\! N\!\cdot\!\varepsilon_2{}^j\mathcal{J}_6\\
&+p_3^ip_2\!\cdot\! N\!\cdot\! p_3p_1\!\cdot\! N\!\cdot\!\varepsilon_2{}^j\mathcal{I}_3-p_2^ip_2\!\cdot\! N\!\cdot\! p_3p_3\!\cdot\! N\!\cdot\!\varepsilon_2{}^j\mathcal{J}_6-2p_3^ip_2\!\cdot\! N\!\cdot\! p_3p_3\!\cdot\! N\!\cdot\!\varepsilon_2{}^jJ_{10}\\
&+\frac{1}{2}p_2\!\cdot\! V\!\cdot\! p_2p_3\!\cdot\! V\!\cdot\! p_3\varepsilon_2{}^{ij}\mathcal{J}_3+\frac{1}{4}(p_2\!\cdot\! V\!\cdot\! p_3)^2\varepsilon_2{}^{ij}\mathcal{J}_6+\frac{1}{2}p_2\!\cdot\! V\!\cdot\! p_3p_2\!\cdot\! N\!\cdot\! p_3\varepsilon_2{}^{ij}\mathcal{J}_{11}\\
&+\frac{1}{4}(p_2\!\cdot\! N\!\cdot\! p_3)^2\varepsilon_2{}^{ij}\mathcal{J}_6\Big)\\
&+\frac{1}{2(p+1)!}\epsilon^{a_0\cdots a_p}\varepsilon_{1a_0\cdots a_pijkl}p_2^ip_3^j\big(-p_1\!\cdot\! N\!\cdot\!\varepsilon_2{}^kp_1\!\cdot\! N\!\cdot\!\varepsilon_3{}^l\mathcal{I}_1+p_1\!\cdot\! N\!\cdot\!\varepsilon_2{}^kp_2\!\cdot\! N\!\cdot\!\varepsilon_3{}^l\mathcal{I}_2\\
&-p_1\!\cdot\! N\!\cdot\!\varepsilon_2{}^kp_2\!\cdot\! V\!\cdot\!\varepsilon_3{}^l\mathcal{I}_3\big)+(2\leftrightarrow 3)
\end{aligned}\tag{B.1}$$

The integrals are those appear in (2.21).

We have argued before that the sum of amplitudes (2.19) and (2.21) is invariant under linear T-duality when the world volume Killing coordinate is an index of the RR potential. However, it is not invariant under linear T-duality if one of the indices of B-field/graviton polarization tensor which is contracted with the volume form, is the Killing coordinate. In that case one should add new amplitude involving higher RR potential to make a complete T-dual amplitude [6]. In this way one would find new amplitude for RR potentials $C^{(p-1)}$, $C^{(p+1)}$, $C^{(p+3)}$ and C^{p+5}. The C^{p+5} part is exactly the terms in the seventh and eighth line above. All other terms involving p_2^i or p_3^i are not remnant of the amplitude (2.21) under T-duality. The string theory produce them for other consistencies.

The amplitude in terms of H is

$$\begin{aligned}
\mathcal{A}\sim{}&\frac{1}{8(p+1)!}\epsilon^{a_0\cdots a_p}\varepsilon_{1a_0\cdots a_pijkl}\Big(\frac{2}{3}H_3^{klj}H_2^{ami}(p_3)_a(p_1)_m\mathcal{I}_2-\frac{2}{3}H_3^{klj}H_2^{nmi}(p_3)_n(p_1)_m\mathcal{I}_3\\
&-H_3^{nlj}H_2^{mki}(p_1)_n(p_1)_m\mathcal{I}_1+H_3^{nlj}H_2^{mki}(p_2)_n(p_1)_m\mathcal{I}_2-H_3^{alj}H_2^{mki}(p_2)_a(p_1)_m\mathcal{I}_3\Big)+(2\leftrightarrow 3)
\end{aligned}$$

Since the integrals $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ have no constant and no massless open string poles (2.37), the above amplitude does not produce any coupling for $C^{(p+5)}$.

References

[1] R.G. Leigh, *Dirac-Born-Infeld Action from Dirichlet σ-model*, *Mod. Phys. Lett.* **A 4** (1989) 2767 [SPIRES].

[2] C. Bachas, *D-brane dynamics*, *Phys. Lett.* **B 374** (1996) 37 [hep-th/9511043] [SPIRES].

[3] C.P. Bachas, P. Bain and M.B. Green, *Curvature terms in D-brane actions and their M-theory origin*, *JHEP* **05** (1999) 011 [hep-th/9903210] [SPIRES].

[4] M.R. Garousi and R.C. Myers, *Superstring Scattering from D-branes*, Nucl. Phys. **B 475** (1996) 193 [hep-th/9603194] [SPIRES].

[5] A. Hashimoto and I.R. Klebanov, *Decay of Excited D-branes*, Phys. Lett. **B 381** (1996) 437 [hep-th/9604065] [SPIRES].

[6] M.R. Garousi, *T-duality of Curvature terms in D-brane actions*, JHEP **02** (2010) 002 [arXiv:0911.0255] [SPIRES].

[7] N. Wyllard, *Derivative corrections to D-brane actions with constant background fields*, Nucl. Phys. **B 598** (2001) 247 [hep-th/0008125] [SPIRES].

[8] S.R. Das, S. Mukhi and N.V. Suryanarayana, *Derivative corrections from noncommutativity*, JHEP **08** (2001) 039 [hep-th/0106024] [SPIRES].

[9] S. Mukhi and N.V. Suryanarayana, *Open-string actions and noncommutativity beyond the large-B limit*, JHEP **11** (2002) 002 [hep-th/0208203] [SPIRES].

[10] J. Polchinski, *Dirichlet-Branes and Ramond-Ramond Charges*, Phys. Rev. Lett. **75** (1995) 4724 [hep-th/9510017] [SPIRES].

[11] M.R. Douglas, *Branes within branes*, hep-th/9512077 [SPIRES].

[12] M.B. Green, J.A. Harvey and G.W. Moore, *I-brane inflow and anomalous couplings on D-branes*, Class. Quant. Grav. **14** (1997) 47 [hep-th/9605033] [SPIRES].

[13] Y.-K.E. Cheung and Z. Yin, *Anomalies, branes and currents*, Nucl. Phys. **B 517** (1998) 69 [hep-th/9710206] [SPIRES].

[14] R. Minasian and G.W. Moore, *K-theory and Ramond-Ramond charge*, JHEP **11** (1997) 002 [hep-th/9710230] [SPIRES].

[15] B. Craps and F. Roose, *Anomalous D-brane and orientifold couplings from the boundary state*, Phys. Lett. **B 445** (1998) 150 [hep-th/9808074] [SPIRES].

[16] J.F. Morales, C.A. Scrucca and M. Serone, *Anomalous couplings for D-branes and O-planes*, Nucl. Phys. **B 552** (1999) 291 [hep-th/9812071] [SPIRES].

[17] B. Stefanski, Jr., *Gravitational couplings of D-branes and O-planes*, Nucl. Phys. **B 548** (1999) 275 [hep-th/9812088] [SPIRES].

[18] K. Becker, G. Guo and D. Robbins, *Higher Derivative Brane Couplings from T-duality*, JHEP **09** (2010) 029 [arXiv:1007.0441] [SPIRES].

[19] M.R. Garousi, *T-duality of anomalous Chern-Simons couplings*, arXiv:1007.2118 [SPIRES].

[20] M.R. Garousi, *Ramond-Ramond field strength couplings on D-branes*, JHEP **03** (2010) 126 [arXiv:1002.0903] [SPIRES].

[21] T.H. Buscher, *Quantum corrections and extended supersymmetry in new σ-models*, Phys. Lett. **B 159** (1985) 127 [SPIRES].

[22] T.H. Buscher, *A Symmetry of the String Background Field Equations*, Phys. Lett. **B 194** (1987) 59 [SPIRES].

[23] T.H. Buscher, *Path Integral Derivation of Quantum Duality in Nonlinear σ-models*, Phys. Lett. **B 201** (1988) 466 [SPIRES].

[24] P. Meessen and T. Ortín, *An* Sl$(2, Z)$ *multiplet of nine-dimensional type-II supergravity theories*, Nucl. Phys. **B 541** (1999) 195 [hep-th/9806120] [SPIRES].

[25] E. Bergshoeff, C.M. Hull and T. Ortín, *Duality in the type-II superstring effective action*, Nucl. Phys. **B 451** (1995) 547 [hep-th/9504081] [SPIRES].

[26] E. Bergshoeff, M. de Roo, M.B. Green, G. Papadopoulos and P.K. Townsend, *Duality of Type II 7-branes and 8-branes*, Nucl. Phys. **B 470** (1996) 113 [hep-th/9601150] [SPIRES].

[27] S.F. Hassan, *T-duality, space-time spinors and RR fields in curved backgrounds*, Nucl. Phys. **B 568** (2000) 145 [hep-th/9907152] [SPIRES].

[28] M.R. Garousi and M. Mir, *On RR couplings on D-branes at order $O(\alpha'^2)$*, JHEP **02** (2011) 008 [arXiv:1012.2747] [SPIRES].

[29] M. Billó et al., *Microscopic string analysis of the D0-D8 brane system and dual RR states*, Nucl. Phys. **B 526** (1998) 199 [hep-th/9802088] [SPIRES].

[30] H. Liu and J. Michelson, **-trek III: The search for Ramond-Ramond couplings*, Nucl. Phys. **B 614** (2001) 330 [hep-th/0107172] [SPIRES].

[31] M.B. Green and N. Seiberg, *Contact Interactions In Superstring Theory*, Nucl. Phys. **B 299** (1988) 559 [SPIRES].

[32] M.R. Garousi, *Disk level S-matrix elements at eikonal Regge limit*, Phys. Lett. **B 696** (2011) 291 [arXiv:1010.4950] [SPIRES].

[33] T. Huber and D. Maître, *HypExp, a Mathematica package for expanding hypergeometric functions around integer-valued parameters*, Comput. Phys. Commun. **175** (2006) 122 [hep-ph/0507094] [SPIRES].

[34] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, San Diego U.S.A. (1994).

[35] M.R. Garousi and R.C. Myers, *World-volume interactions on D-branes*, Nucl. Phys. **B 542** (1999) 73 [hep-th/9809100] [SPIRES].

[36] A. Fotopoulos, *On $(\alpha')^2$ corrections to the D-brane action for non-geodesic world-volume embeddings*, JHEP **09** (2001) 005 [hep-th/0104146] [SPIRES].