

Universality and exactness of Schrödinger geometries in string and M-theory

Per Kraus and Eric Perlmutter

*Department of Physics and Astronomy, University of California,
Los Angeles, CA 90095, U.S.A.*

E-mail: pkraus@ucla.edu, perl@physics.ucla.edu

ABSTRACT: We propose an organizing principle for classifying and constructing Schrödinger-invariant solutions within string theory and M-theory, based on the idea that such solutions represent nonlinear completions of linearized vector and graviton Kaluza-Klein excitations of AdS compactifications. A crucial simplification, derived from the symmetry of AdS, is that the nonlinearities appear only quadratically. Accordingly, every AdS vacuum admits infinite families of Schrödinger deformations parameterized by the dynamical exponent z . We exhibit the ease of finding these solutions by presenting three new constructions: two from M5 branes, both wrapped and extended, and one from the D1-D5 (and S-dual F1-NS5) system. From the boundary perspective, perturbing a CFT by a null vector operator can lead to nonzero β -functions for spin-2 operators; however, symmetry restricts them to be at most quadratic in couplings.

This point of view also allows us to easily prove nonrenormalization theorems: to every Kaluza-Klein vector and graviton that lies in a short multiplet of an AdS supergroup, there exists at least one Sch(z) solution in which z is uncorrected to all orders in higher derivative corrections. Furthermore, we find infinite classes of 1/4 BPS solutions with 4-, 5- and 7-dimensional Schrödinger symmetry that are exact.

KEYWORDS: Gauge-gravity correspondence, AdS-CFT Correspondence, Holography and condensed matter physics (AdS/CMT)

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1 Introduction

Since the early days of bottom-up Galilean holography [1, 2], various techniques have been developed to embed Schrödinger solutions into string theory and M-theory. The TsT transformation, or Null Melvin Twist, has been used to transform the asymptotics of various black brane solutions [3–9], and zero temperature solutions have been embedded into consistent truncations of type IIB and eleven-dimensional supergravities that retain massive vector modes, thereby making contact with the original bottom-up approach [10–17]. More general solutions, formed directly in $D = 10, 11$, engineer non-relativistic conformal symmetry geometrically, drawing a connection between harmonic modes on internal manifolds and zero temperature Schrödinger solutions with infinite spectra of z , e.g. [18–25].

For extremal solutions, the construction of [20] is the most general to date. Building on their own work [19] and followed by [21], the authors’ framework subsumes all previously found zero temperature embeddings based on null deformations of the near-horizon D3 and M2-brane solutions, realizing them as members of infinite families of Schrödinger solutions. The spectrum of z is determined by the spectrum of harmonics on Sasaki-Einstein manifolds. The geometric nature of these solutions suggests an extension to other deformations of AdS vacua with some generality.

Given these developments, the outstanding question is no longer how top-down Schrödinger solutions can be found, but instead, what the fundamental principle is that allows us to explain their existence and classify them. In this work, largely inspired by [18, 20, 22], we attempt to answer the following question: given an $\text{AdS} \times \mathcal{M}$ solution, can it always be deformed to have Schrödinger symmetry? The answer, we argue, is yes.

We begin by elucidating and extending the constructions of [18, 20, 22], which stand in direct correspondence to linearized Kaluza-Klein vectors and gravitons of AdS compactification spectra. In fact, we show that this is more than a parallel: to every such excitation, there exists a nonlinear solution with Schrödinger symmetry that is obtained by “turning on” the KK mode. The most general solutions involve both vector and spin-2 excitations, which couple at quadratic order in the full solution.

The obvious puzzle is why linearized solutions can be extended to nonlinear order in such a simple way. We use perturbation theory around a generic $\text{AdS} \times \mathcal{M}$ vacuum to show that the nonlinearities are constrained by symmetry to arise only at second order. Scale invariance consistent with the first-order solutions determines the geometry to be of Schrödinger form. Putting this all together, for every $\text{AdS} \times \mathcal{M}$ vacuum in string and M-theory, there are Schrödinger solutions in direct correspondence with vector and graviton KK modes of the compactification spectra on \mathcal{M} , and the most general Schrödinger solutions superpose these modes. As we will show, for gravitons, and vectors which descend from a mixture of metric and flux components, the correspondence is one-to-one.

In this sense, the existence of Schrödinger solutions is universal, and their construction, mechanical. We support our arguments by providing three new categories of solutions explicitly, with three-, four- and seven-dimensional Schrödinger symmetry. This includes both supersymmetric and non-supersymmetric solutions.

We also provide an argument on the CFT side. The dual version of the quadratic

truncation of the AdS perturbation theory is that the β -function equations for all deforming operators truncate at second order in the couplings. When perturbing the theory by a source for a null vector operator, scale invariance is preserved provided that no marginal spin-2 operator appears in the OPE of two vector operators. Further parallels can be drawn by considering the Wilsonian β -functions equations for a tower of spin-2 operators. Their β -functions equations are again constrained by symmetry to be quadratic in the couplings, and the non-relativistic fixed point is easily solved in terms of operator dimensions and constants arising from OPEs between vector operators. This is particularly useful in light of recent work in setting up the precise holographic dictionary for Schrödinger gauge/gravity duality, which focused primarily on solutions to massive vector theories [26, 27]. But the most general embeddings of the type analyzed in this work do not fit into that framework: there are not only massive vectors but massive gravitons as well, dual to symmetric tensor operators with nontrivial Wilsonian β -functions.

This perspective has powerful implications for the existence of the Schrödinger backgrounds beyond the two-derivative approximation, first considered in [28]. In particular, we recall that conformal dimensions of operators in short multiplets of an AdS supergroup, which are determined via harmonics of \mathcal{M} , are unrenormalized by quantum and stringy corrections. This implies that any parameter of a Schrödinger solution that derives from the dimensionless AdS mass of a KK mode can be so protected. This is the case for the dynamical exponent, z . Furthermore, when we perturb the exact AdS vacua that arise in the near-horizon limit of M2, D3 and M5 branes, certain infinite families of the resulting Schrödinger solutions are also exact. These vacua preserve only eight Poincaré supersymmetries, and constitute a new contribution to the small set of exact solutions of string and M-theory. We back this claim with an explicit calculation showing that one such class of Sch_5 solutions of type IIB is unrenormalized to $O(\alpha'^3)$, inclusive of *all* correction terms, known [29] and unknown.

The rest of this paper is organized as follows. Section 2 reviews the Schrödinger solutions of type IIB obtained by a D3-brane deformation, and generalizes them with the perturbation theory arguments detailed above, both in the bulk and on the boundary. We also consider the implications for embedding Schrödinger solutions in consistent truncations with massive modes. Section 3 posits conditions for nonrenormalization. Section 4 presents the new Schrödinger constructions, details of which are in the appendices, and section 5 concludes with a discussion.

Note added. Appendix A now presents another infinite family of solutions with seven-dimensional Schrödinger symmetry obtained through deformation of the extended M5 brane geometry that does *not* lie in obvious correspondence with massive KK modes found in the compactification on $AdS_7 \times M_4$, as well as a single solution with six-dimensional Lifshitz symmetry with the same general structure as those Lifshitz solutions of [30, 31].

2 Infinite families of Schrödinger solutions from every AdS

In reviewing the type IIB solutions of [18, 20, 22], we emphasize how their structure parallels that seen in the harmonic analysis of the Kaluza Klein spectrum on $AdS_5 \times SE_5$.

We then show that infinite towers of Schrödinger solutions are guaranteed to exist, for any AdS vacuum, and are governed by the structure of the KK towers of massive vectors and gravitons. For convenience, we use the notation and supergravity conventions of [20], the latter of which are found in [32]. (As is probably clear by now, throughout this paper we use the term “Schrodinger” in reference to Galilean-symmetric spacetimes for any value of the dynamical exponent.)

2.1 Review and motivation

The type IIB solutions of [20] are¹

$$\begin{aligned}
 ds^2 &= \Phi^{-1/2} (2dx^+ dx^- + h(dx^+)^2 + 2Cdx^+ + d\mathbf{x}^2) + \Phi^{1/2} ds^2(CY_3) \\
 F_5 &= dx^+ \wedge dx^- \wedge dx_1 \wedge dx_2 \wedge d\Phi^{-1} + \star_{CY_3} d\Phi \\
 &\quad - dx^+ \wedge (d(\Phi^{-1}C) \wedge dx_1 \wedge dx_2 - \star_{CY_3} dC) \\
 G_3 &= dx^+ \wedge W
 \end{aligned}
 \tag{2.1}$$

This solution is a deformed version of D3-branes sitting at the tip of a CY_3 cone. Φ and h are functions; C is a one-form; and W is a complex two-form; all of which are defined on CY_3 . The axion-dilaton is set to zero, and $\mathbf{x} = (x_1, x_2)$. The field equations evaluated on this ansatz reduce to

$$\begin{aligned}
 \nabla_{CY_3}^2 \Phi &= 0 \\
 d \star_{CY_3} dC &= 0 \\
 dW &= d \star_{CY_3} W = 0 \\
 \nabla_{CY_3}^2 h &= -|W|_{CY_3}^2
 \end{aligned}
 \tag{2.2}$$

where $|W|_{CY_3}^2$ is the real square of W with indices contracted with the CY_3 metric. The first of these equations is the usual harmonic condition on Φ , which we use to zoom into the near-horizon region by making the substitution $\Phi = r^{-4}$, and writing the CY_3 as a cone over a Sasaki-Einstein space SE_5 as

$$ds^2(CY_3) = dr^2 + r^2 ds^2(SE_5).
 \tag{2.3}$$

Upon substitution, we can satisfy the remaining three equations by taking

$$\begin{aligned}
 C &= r^{z-2} \beta \\
 W &= d(r^z \sigma) \\
 h &= r^{2z-2} q
 \end{aligned}
 \tag{2.4}$$

where q , β and σ are a function, real one-form, and complex one-form, respectively, on SE_5 . The powers of r follow from insisting upon anisotropic scale invariance under the transformation

$$r \rightarrow r' = \lambda r, \quad x^+ \rightarrow x^{+'} = \frac{x^+}{\lambda^z}, \quad x^- \rightarrow x^{-'} = \frac{x^-}{\lambda^{2-z}}, \quad \mathbf{x} \rightarrow \mathbf{x}' = \frac{\mathbf{x}}{\lambda}
 \tag{2.5}$$

¹Note a minor sign error in [19, 20] of the $\star_{CY_3} dC$ term, where we, and they, use the convention $\epsilon_{+x_1 x_2 y_1 \dots y_6} = +\sqrt{-g}$, where the y_i are coordinates on CY_3 .

and the solution now reads

$$\begin{aligned}
 ds^2 &= r^{2z}q(dx^+)^2 + 2r^z dx^+ \beta + r^2 (2dx^+ dx^- + d\mathbf{x}^2) + \frac{dr^2}{r^2} + ds^2(SE_5) \\
 F_5 &= 4r^3 dx^+ \wedge dx^- \wedge dx_1 \wedge dx_2 \wedge dr + 4Vol_5 \\
 &\quad - d(r^{z+2} dx^+ \wedge dx_1 \wedge dx_2 \wedge \beta) + dx^+ \wedge (r^{z-1} dr \wedge \star_5 \beta + (z-2)r^z \star_5 d\beta) \\
 G_3 &= d(r^z dx^+ \wedge \sigma)
 \end{aligned}
 \tag{2.6}$$

where Vol_5 and \star_5 are the invariant volume form and Hodge star operator on SE_5 , respectively. The Schrödinger symmetry is now manifest, and the field equations reduce to

$$\begin{aligned}
 \Delta_5 \beta &= z(z-2)\beta, \quad \text{where } d\star_5 \beta = 0 \\
 \Delta_5 \sigma &= z(z+2)\sigma, \quad \text{where } d\star_5 \sigma = 0 \\
 \nabla_5^2 q + (2z-2)(2z+2)q &= -z^2|\sigma|_5^2 - |d\sigma|_5^2
 \end{aligned}
 \tag{2.7}$$

with $-\nabla_5^2$ and Δ_5 as the Laplace operator on functions and one-forms on SE_5 , respectively. The most general solution in which all of these fields are turned on is subject to the consistency of all equations.

Having reviewed these solutions, we now make some observations. The structure of this ansatz, and the form of the reduced field equations, is almost identical to that which arises in solving for linearized fluctuations on $AdS_5 \times SE_5$. The only difference, which we'll return to shortly, is the quadratic nonlinearity appearing in the final equation of (2.2). Take the $\sigma \neq 0$ solutions, for example. We have written the three-form flux as

$$G_3 = dA_2 = d(r^z dx^+ \wedge \sigma) \tag{2.8}$$

in order to emphasize the structure of the complex two-form gauge field A_2 : it has one leg along SE_5 , and one leg along the five noncompact directions that survive the compactification. The latter becomes the gauge field in $d = 5$, to wit, the Schrödinger gauge field of an effective $d = 5$ massive vector model,

$$A_1^{Sch} = r^z dx^+ \tag{2.9}$$

Furthermore, we found that σ must be a transverse vector harmonic on SE_5 , and $z(z+2)$ is identified with its eigenvalue. The spectrum of co-closed one-forms on SE_5 has eigenvalues $\Delta_5 \geq 8$, where this bound is saturated when σ is dual to a Killing vector on SE_5 . This gives a lower bound²

$$z \geq 2 \tag{2.10}$$

But this is exactly how the tower of KK vectors with a diagonal field equation³ arises in the harmonic analysis of linearized fluctuations: one expands the internal part of the

²Here and henceforth, we ignore negative z consistent with the field equations.

³Henceforth called “diagonal” vectors.

complex two-form gauge field in harmonics on SE_5 , plugs into the linearized field equations subject to a particular gauge choice, and the equation for the fluctuations A_1 becomes the eigenvalue equation of vector harmonics. The de Donder-Lorenz gauge choice forces these vector harmonics to be transverse, which is to say, co-closed on SE_5 : this is just the condition $d \star_5 \sigma = 0$ that we have imposed above. The resulting definition of z is simply that of a five-dimensional massive vector model,

$$(mL)^2 = z(z + 2) \tag{2.11}$$

where $(mL)^2$ is here the dimensionless AdS mass of the KK vector. Notice that the ansatz is linear in W , as it is in all of the deformation fields, and the axio-dilaton need not be turned on because it does not figure into the Kaluza-Klein vector spectrum.

There is another field equation to be obeyed, namely the Einstein equation given in (2.7). Notice that when $\sigma = 0$, the equation is simply that of a massive spin-2 excitation on $AdS_5 \times SE_5$, where q is a scalar harmonic with eigenvalue $4(z^2 - 1)$. Its harmonic level determines the mass of the spin-2 field. One should think of expanding h in SE_5 scalar harmonics as

$$h = \sum_k h_{\mu\nu}^k Y^k(SE_5) \tag{2.12}$$

thus identifying the spin-2 field as

$$h_{++} = r^{2z} \tag{2.13}$$

Notice that h_{++} is a transverse traceless mode because of the null Killing vector of the Schrödinger metric; this, too, parallels the gauge choice of the compactification.

Evidently, taking $\sigma \neq 0$ induces a quadratic correction to this equation; the solution is given as a linear superposition of scalar harmonics on SE_5 , as determined by the SE_5 dependence of the source term.⁴ Thus, the $\sigma \neq 0$ solutions require not only a vector perturbation to $AdS_5 \times SE_5$, but a tower of spin-2 perturbations whose amplitudes are proportional to the square of the vector harmonic.

A similar explanation holds for the $\beta \neq 0$ solutions: these make use of the vectors that descend from a mixture of the ten-dimensional metric and four-form RR gauge field.⁵ The five-form ansatz in (2.6) has various β terms which combine to make a self-dual F_5 . Focusing on the term of the form

$$F_5 = dA_4 \supset d(r^{z+2} dx^+ \wedge dx_1 \wedge dx_2 \wedge \beta) \tag{2.14}$$

we see that we have turned on a component of the four-form gauge field with *three* legs along the extended directions. Nevertheless, this is precisely the form of the ansatz needed to turn on the mixed KK vectors: in the linearized KK analysis, these components are algebraically

⁴There is an exception to this when $z = 2$, for which the σ -dependence of the spin-2 equation reduces to a constant. In this case, the most general solution is $g_{++} = qr^4$, where $q = a + bY(SE_5)_{12}$ for some constants a, b and $Y(SE_5)_{12}$ the scalar harmonic with eigenvalue 12 (if it exists). The choice $b = 0$ does not turn on a spin-2 field because there is no constant scalar harmonic on SE_5 other than zero. This is, in fact, the near-horizon limit of the TsT-transformed extremal D3-brane.

⁵Henceforth called “mixed” vectors.

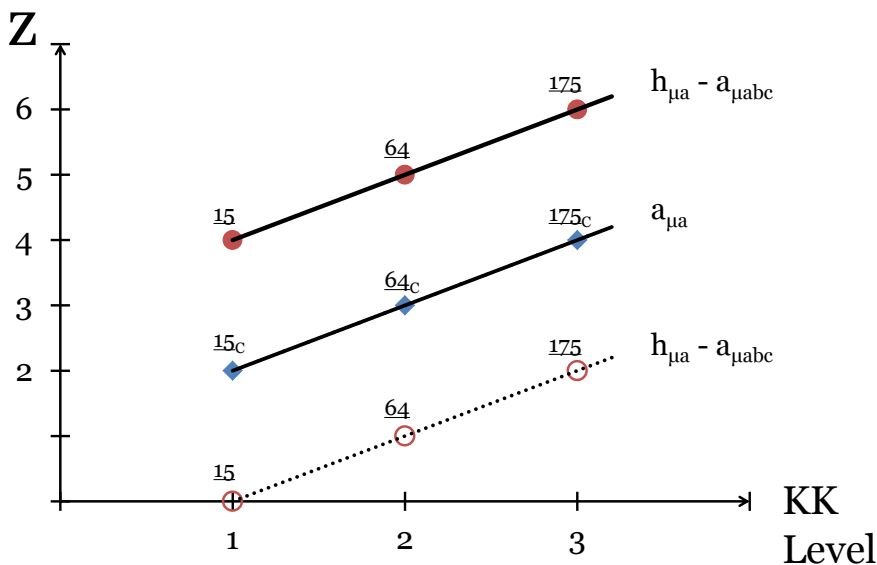


Figure 1. The lower states of the spectrum of $Sch_5(z)$ solutions formed from Kaluza-Klein vector deformations of $AdS_5 \times S^5$. This is an adaptation of figure 1 in [33] which depicts the vector mass spectrum, here converted via the massive vector relation $(mL)_{KK}^2 = z(z + 2)$. The $SO(6)$ representation of each vector, listed above each point, partly determines the degeneracy of solutions. At right are the $D = 10$ origin of fields, where μ lies along the noncompact directions and (a, b, c) lie on S^5 . The middle branch of solutions, formed from diagonal vectors descending from the complex two-form, also requires nonzero Kaluza-Klein gravitons not pictured. The lower branch of mixed vectors, while present in the spectrum, gives solutions with negative values of z ; hence, it appears dotted and unfilled. For details, see the main text.

eliminated in favor of a component with only one extended index. (See discussion above equation (2.17) of [33] when $SE_5 = S^5$.) These solutions will have $z \geq 4$, and the massive vector relation (2.11) still holds.⁶

Lastly, the solutions for which only $q \neq 0$ are, as articulated above, simply the elevation of a linearized massive graviton excitation to a full solution. Evidently, superpositions of all of these solutions may also be solutions, subject to the harmonic structure of SE_5 .

Upon taking $SE_5 = S^5$ [33], the above statements become rather transparent: the eigenvalue spectra of the Laplace operators are integer and stand in the right relation to give a spectrum of Schrödinger solutions with integer z . A graph of the vector spectrum about $AdS_5 \times S^5$ then supplies a visual representation of the associated Schrödinger deformations, provided in figure 1.

These comments apply equally to the M2-brane solutions constructed in [20]. In fact, there is nothing special about either of these two solutions in this regard, so one should expect this structure to extend to solutions built around any $AdS \times \mathcal{M}$ vacuum.

Summarizing, to each KK vector of an $AdS_{d+1} \times \mathcal{M}$ compactification, one can asso-

⁶In this and other constructions using mixed vectors, there are two such towers. Because the eigenvalue equation that determines z is quadratic, it is satisfied by two choices of z . The greater choice turns on a vector in the “upper” branch; the lesser choice, the “lower” branch. Typically, the values of z corresponding to the lower branch are negative, and we will ignore them for physical reasons.

ciate a solution with Sch_{d+1} symmetry where the Schrödinger vector is *identified* with the linearized KK fluctuation. The dynamical exponent z is given precisely by the relation for an effective d -dimensional massive vector model,

$$(mL)_{KK}^2 = z(z + d - 2) \tag{2.15}$$

where the mass is simply the AdS mass of the KK vector: by construction, the same differential operator of \mathcal{M} that gives the spectrum of vector masses is that which gives z . When the KK vector is diagonal, the metric deformation is induced by a tower of massive spin-2 fields that together solve an inhomogeneous Laplace equation with a quadratic source in the vector harmonic. One can also utilize the spin-2 fields alone to obtain a Schrödinger-symmetric solution, where z is determined by the effective mass of the spin-2 field as

$$(mL)_{KK}^2 = [d + 2(z - 1)] 2(z - 1) \tag{2.16}$$

(We will justify this relation for all d later in this section.) Finally, in all of these constructions, the degeneracy of Schrödinger solutions is given by the degeneracy of \mathcal{M} harmonics utilized in the solution.

Analysis of the KK spectrum is based on solving linearized field equations, while the Schrödinger solutions represent solutions of the fully nonlinear theory. In these terms, an obvious question is: how does a solution at linearized level become elevated to a full, nonlinear solution? We now explain how this occurs, first from the point of view of the bulk field equations, and subsequently from the point of view of the holographically dual boundary theory.

2.2 In the bulk: AdS perturbation theory

The message of the above remarks is that one should think of the nonlinear solution as a quadratic extension of a linearized solution around $\text{AdS}_5 \times SE_5$. Thus, our task is to explain why this perturbation theory truncates at second order, and to generalize this fact.

Still referring to the D3-brane case, our strategy will be to initiate a perturbation theory around the AdS solution by seeding it with a linearized vector fluctuation, where the three-form flux has a leg in the x^+ direction. The fluctuation has some definite weight under the symmetry transformations of the background. Going to higher order in the perturbation theory, the symmetries of this first order solution must be preserved at each order. Furthermore, the possible terms that can get generated must be covariant with respect to the symmetries of the background. In general, this perturbation series will not truncate at finite order; in particular, if one can combine positive powers of the first-order perturbations to form a singlet under all background symmetries, then one will expect this singlet to appear in equations at arbitrary order. Simplification occurs if, for any one symmetry of the background, there are no singlets.

In the case at hand, the Lorentz symmetry of AdS, acting in these coordinates as

$$x^+ \rightarrow x^{+'} = \frac{x^+}{\kappa}, \quad x^- \rightarrow x^{-'} = \kappa x^- \tag{2.17}$$

strongly constrains the possible terms that can be generated beyond linear order. Because we seed the perturbation theory with a field with a leg along the null x^+ direction, and with *no* legs along the x^- direction, this field has weight one under the Lorentz boost. That means that at n^{th} order, the only possible terms that can be generated are of weight n . But because the lone symmetric tensor field in our theory is the metric, which has two indices, this perturbation theory will necessarily truncate at quadratic order: there are no possible terms one can write down which turn on a field with Lorentz weight three.

Demonstrating explicitly with the $W \neq 0$ solution based on diagonal vectors, we seed the perturbation via the complex three-form flux, as

$$\begin{aligned}
 ds^2 &= r^2 (2dx^+ dx^- + d\mathbf{x}^2) + \frac{dr^2}{r^2} + ds^2(SE_5) \\
 F_5 &= 4r^3 dx^+ \wedge dx^- \wedge d\mathbf{x} \wedge dr + 4V ol_5 \\
 G_3 &= \delta G_3^{(1)}
 \end{aligned}
 \tag{2.18}$$

with

$$\delta G_3^{(1)} = dx^+ \wedge W
 \tag{2.19}$$

We assign W weight one under a Lorentz boost. The only term that can be generated at subsequent orders is the Schrödinger term in the metric,

$$\delta g_{AB}^{(2)} = g_{++} \delta A_+ \delta B_+
 \tag{2.20}$$

This fact ensures that the ∂_- Killing vector stays null at all orders. Note that we have merely used the Lorentz symmetry of AdS, so this same logic applies to null perturbations of any Lorentz-invariant background by a massive supergravity field. In particular, this argument makes no use of supersymmetry or scale invariance of the near-horizon limit. Note also that the resulting Schrödinger solutions will be at zero temperature, and of the same curvature scale as their AdS counterparts: $L_{Sch} = L_{AdS}$.

This procedure can also be carried out by seeding the perturbation with the mixed KK vectors (C), or with the massive spin-2 field (h). These first-order choices merely preserve different symmetries. In the former case, identical arguments to the above truncate the perturbation theory at second order; in the latter case, since our initial perturbation is of weight two, the linearized solution is in fact a full solution.

In fact, in this example the truncation of perturbation theory works even better than one might have expected. Notice that in the final equation in (2.2) it would be consistent with the Lorentz symmetry to have a $|dC|^2$ term appear alongside $|W|^2$, but the coefficient of this term is apparently zero. This may be a consequence of supersymmetry, and as we will see, has implications for the robustness of the $C \neq 0$ solutions against higher derivative corrections.

To actually obtain the Schrödinger metric, we turn now to the dilatation symmetry. The full nonlinear solution at hand is not Lorentz invariant, so it is consistent to allow for anisotropic scale invariance (2.5). This fixes the fields W and g_{++} to take their canonical Schrödinger scaling, and enforcement of the Bianchi identity on the three-form flux implies

its closure. Parameterizing the metric as $g_{++} = r^2 h$ gives

$$\begin{aligned} W &= d(r^z \sigma) \\ h &= r^{2z-2} q \end{aligned} \tag{2.21}$$

as in (2.4). Both h and W can have functional dependence only on CY_3 coordinates (r, Ω_5) : specifically, x^- is ruled out because of the x^\pm scaling argument, and \mathbf{x} is ruled out to preserve rotational symmetry along the brane.

Using the dilatation invariance, general covariance, and the fact that $z = 1$ must be an AdS solution for which we can set $W = 0$, one can essentially reproduce the reduced field equations (2.7), including the z -dependence. Focusing on the spin-2 equation, we know that a two-derivative operator, covariant with respect to the CY_3 on which the fields are defined, must act on h . Scale invariance determines the form of the equation as

$$\nabla_{CY_3}^2 (r^x h) = \alpha r^x |W|_{10}^2 \tag{2.22}$$

for some constants x, α , and where the square of W is taken with respect to the 10-dimensional metric. Each side has Lorentz weight two. Substituting from (2.21) gives

$$(\nabla_5^2 + (2z + x - 2)(2z + x + 2)) q = \alpha (z^2 |\sigma|_5^2 + |d\sigma|_5^2) \tag{2.23}$$

We impose one final constraint, which is that when $z = 1$, AdS is a solution with $\sigma = 0$ and q constant. This determines x as

$$x(x + 4) = 0 \tag{2.24}$$

The choice $x = 0, \alpha = -1$ is evidently made by plugging the ansatz into the Einstein equation itself. Up to this ambiguity, we have thus determined the relation between z and the spin-2 mass, which was unknown *a priori*, in contrast to the spectrum of z in the well-studied massive vector model.

Thus, it is clear why the field equations (2.7) for the $\sigma = 0$ solutions are simply those of the linearized level, and why the $\sigma \neq 0$ solutions have only a quadratic correction to those equations appearing in the Einstein equation. By using the *spacetime* symmetries of the AdS vacuum we have essentially reproduced all aspects of the nonlinear solution in this perturbative context; we will see in the next section that we can also use the *supersymmetry* of an AdS solution to make stronger statements about the existence of its Schrödinger deformations away from the supergravity approximation.

First, let us explain the existence of these Schrödinger fixed points from the CFT side.

2.3 On the boundary: conformal perturbation theory

Given the AdS/CFT correspondence, and its generalizations, it is instructive to give a version of our arguments that applies to the boundary theory. As we will see, this corresponds to a simple exercise in conformal perturbation theory. The response of a CFT to a null vector perturbation was studied (along with other aspects of holography for Schrödinger spacetimes) in [26, 27], and we briefly compare our conclusions at the end of this section.

Writing the CFT coordinates as (x^\pm, \mathbf{x}) , where x^\pm are lightcone coordinates, the generators corresponding to dilatations and Lorentz boosts in x^\pm act as

$$\begin{aligned} \mathcal{D}: \quad x^+ \rightarrow x^{+'} &= \frac{x^+}{\lambda}, \quad x^- \rightarrow x^{-'} = \frac{x^-}{\lambda}, \quad \mathbf{x} \rightarrow \mathbf{x}' = \frac{\mathbf{x}}{\lambda} \\ \mathcal{L}: \quad x^+ \rightarrow x^{+'} &= \frac{x^+}{\kappa}, \quad x^- \rightarrow x^{-'} = \kappa x^-, \quad \mathbf{x} \rightarrow \mathbf{x}' = \mathbf{x} \end{aligned} \tag{2.25}$$

Now consider an operator \mathcal{O}_- , whose dilatation weight is Δ , and whose behavior under Lorentz transformation is as indicated by its index structure. Under a combined dilatation and Lorentz transformation it transforms as

$$\mathcal{O}_-(x^+, x^-, \mathbf{x}) \rightarrow \mathcal{O}_-(x^{+'}, x^{-'}, \mathbf{x}') = \kappa^{-1} \lambda^\Delta \mathcal{O}_-(x^+, x^-, \mathbf{x}) \tag{2.26}$$

Adding this operator to the Lagrangian obviously breaks Lorentz invariance, and it also breaks scale invariance if $\Delta \neq d$. However, if we write Δ in the form

$$\Delta = d + z - 1 \tag{2.27}$$

we see that under the non-relativistic dilatation generator, \mathcal{D}_z , defined as

$$\mathcal{D}_z = \mathcal{D} + (z - 1)\mathcal{L} \tag{2.28}$$

and which acts as

$$\mathcal{D}_z: \quad x^+ \rightarrow x^{+'} = \frac{x^+}{\lambda^z}, \quad x^- \rightarrow x^{-'} = \frac{x^-}{\lambda^{2-z}}, \quad \mathbf{x} \rightarrow \mathbf{x}' = \frac{\mathbf{x}}{\lambda} \tag{2.29}$$

the operator \mathcal{O}_- acquires weight $\Delta_z = d$ under \mathcal{D}_z , and so $\int dx^+ dx^- d^{d-2}\mathbf{x} \mathcal{O}_-$ is scale invariant, as noted in [26, 27].

Similarly, a spin-2 operator \mathcal{O}_{--} will be marginal under \mathcal{D}_z provided that its relativistic conformal dimension is $\Delta = d + 2(z - 1)$.

Perturbing the action as

$$S \rightarrow S + g_+ \int dx^+ dx^- d^{d-2}\mathbf{x} \mathcal{O}_- + g_{++} \int dx^+ dx^- d^{d-2}\mathbf{x} \mathcal{O}_{--} \tag{2.30}$$

thus preserves, to first order in the couplings (g_+, g_{++}) , scale invariance generated by \mathcal{D}_z .

In parallel to our discussion on the gravity side, we can consider the couplings (g_+, g_{++}) as “seeds”, and study the renormalization group beyond first order to see whether scale invariance survives. We proceed using conformal perturbation theory (see [34], section 15.8). Inside the path integral we expand e^{-S} as a power series in the couplings. Ultraviolet divergences can occur when two or more operator insertions coincide in position space, and we regulate these by cutting out a small ball of radius Λ^{-1} around each operator, and adding counterterms to remove the divergences. A breakdown of scale invariance is then signalled by the appearance of $\log \Lambda$ terms, since their removal introduces a scale μ .

At second order in perturbation theory a log divergence can occur when two operators collide and produce a factor of $\int \frac{dx^+ dx^- d^{d-2}\mathbf{x}}{x^+ x^- |\mathbf{x}|^{d-2}}$. In particular, this will happen when the OPE of two vector operators behaves as

$$\mathcal{O}_-(x^+, x^-, \mathbf{x}) \mathcal{O}_-(0, 0, \mathbf{0}) \sim \frac{C}{x^+ x^- |\mathbf{x}|^{d-2}} \mathcal{O}_{--}(0, 0, \mathbf{0}) + \dots \tag{2.31}$$

where \mathcal{O}_{--} is a marginal (with respect to \mathcal{D}_z) spin-2 operator. In order to renormalize the theory we need to add to the action the operator \mathcal{O}_{--} with a coefficient that depends logarithmically on scale. The resulting theory thus breaks scale invariance at second order in g_+ unless $C = 0$, with a nonzero β -function $\beta_{++} \sim C(g_+)^2$.

This behavior corresponds to what we see on the gravity side. Consider an AdS vacuum whose field content includes a massive vector and a massive spin-2 field, whose masses are such that each admits a linearized solution preserving invariance under scale transformation with exponent z . In particular, consider the system of equations (2.2), where we assume that the appropriate CY_3 harmonics exist. Now seed a solution with the massive vector W . According to the last equation in (2.2) this will source h at quadratic order in W . By assumption, at linear order h admits a solution $h \sim r^{2z-2}$, in order to be compatible with the non-relativistic scale invariance. However, at quadratic order in W we will have to shift the power law of h in order to solve the field equations. Expanding in W , this gives rise to a term $h \sim |W|^2 r^{2z-2} \ln r$, whose presence is the bulk analog of the logarithmically running coupling in the CFT. Scale invariance survives only if the source for the marginal h mode vanishes, which is the bulk analog of the CFT condition $C = 0$.

In the CFT, checking scale invariance at orders g_+g_{++} and g_{++}^2 means looking for marginal spin-3 and spin-4 operators appearing in the OPEs $\mathcal{O}_-\mathcal{O}_{--}$ and $\mathcal{O}_{--}\mathcal{O}_{--}$. Nonzero OPE coefficients lead to nonzero β -functions for the spin-3 and spin-4 couplings. Going beyond second order in perturbation theory, we will similarly need to check for the appearance of operators of ever higher spin in OPEs. How do we deal with this? Here the key point is that in the supergravity limit the bulk theory contains no fields of spin larger than two. Such modes would correspond to stringy excitations, which we can think of as having been integrated out, yielding α' corrections in the supergravity action. Similarly, in order for the CFT to have a dual supergravity description it is necessary that all higher spin operators should acquire scaling dimensions that are parametrically large in the large (λ, N) limit. So for the CFTs dual to bulk theories with a good supergravity description, there will not exist any marginal operators beyond spin-2. Therefore, the full set of renormalization group equations governing the flows seeded by (g_+, g_{++}) consists of the equations $\beta_+ = 0, \beta_{++} \sim C(g_+)^2$. Existence of the fixed point thus boils down to checking the single condition $C = 0$. As noted above, this matches the behavior seen in the bulk.

The vanishing of C will follow from symmetry in some theories. Namely, if the theory has a global symmetry group under which the marginal spin-1 and spin-2 operators transform, C will vanish unless the product of two spin-1 representations contains the spin-2 representation. For example, in the case of $\mathcal{N} = 4$ SYM, operators will transform in representations of the $SU(4)$ R-symmetry. The same considerations apply on the gravity side, where for example in (2.7) the source terms appearing on the right hand side are subject to the rules governing products of harmonics. It was proven in [20] that the source for the marginal mode vanishes for all values of z when $SE_5 = S^5$, and furthermore, for $z = 2$ for generic SE_5 .

In the bulk, solving the last equation in (2.2) will involve turning on a tower of massive spin-2 two modes with amplitudes proportional to the square of the vector field. We can make a parallel observation on the CFT side. Let us consider a tower of generically *non-*

marginal spin-2 operators $\mathcal{O}_{--}^{(n)}$. In order to talk about the flow of their couplings $g_{++}^{(n)}$, let us now work in terms of the Wilsonian renormalization group, where we keep track of all couplings, not just the marginal and relevant ones. Lorentz invariance now constrains their β -functions to be of the form

$$\beta_{++}^{(n)} = \left(\Delta_z[\mathcal{O}_{--}^{(n)}] - d \right) g_{++}^{(n)} + C^{(n)}(g_+)^2 \tag{2.32}$$

where $\Delta_z[\mathcal{O}_{--}^{(n)}]$ are the operator dimensions at the Lorentz invariant fixed point. We are including only the effect of couplings with purely +-type indices, since all other couplings can be consistently set to zero by Lorentz invariance. Along with the fact that the equation $\beta_+ = 0$ is uncorrected by the presence of $g_{++}^{(n)}$, which again follows from Lorentz invariance, we find the fixed point by taking

$$g_{++}^{(n)} = -\frac{C^{(n)}}{\Delta_z[\mathcal{O}_{--}^{(n)}] - d} (g_+)^2 \tag{2.33}$$

We can now see the parallel with the gravity side. According to (2.7), scale invariant solutions are found by solving an inhomogeneous Laplace equation for q , with the source given by the square of the vector field. The solution can be decomposed into the harmonics for the massive spin-2 fields. The solution will then contain a tower of massive spin-2 fields with amplitudes proportional to the square of the vector, which is what we have in (2.33). To make this connection precise we would like a better understanding of the bulk interpretation of the Wilsonian couplings; see [35, 36] for recent work in this direction.

Finally, let us remark on references [26, 27], which argued that couplings for vector operators \mathcal{O}_- are exactly marginal with respect to non-relativistic scale transformations. The arguments in [26, 27] were based on showing that the 2-point function for \mathcal{O}_- is uncorrected by the addition of \mathcal{O}_- to the Lagrangian. This establishes that $\beta_+ = 0$; however one also needs to verify that β -functions for other couplings vanish as well. As we discussed above, this corresponds to checking that no marginal spin-2 operators appear in the OPE of the vector operators, and this has a direct correspondence with the field equations in the bulk.

2.4 A corollary: consistent truncations with massive KK modes

Before moving on, we can apply our conclusion to the construction of consistent truncations of string/M-theory. By now, there is a large amount of technology for constructing consistent truncations with massive modes, e.g. [10–17, 37], and it was in this manner that the first string/M-theory embeddings of Schrödinger solutions were found in [10]. We have shown here that one can always elevate the linearized vector and spin-2 fluctuations around AdS to part of a full, nonlinear Schrödinger solution; but if one can elevate some of these fields to the nonlinear level on the level of the action itself, then it is clear that Schrödinger solutions to such theories always exist.

Therefore, a corollary to our argument is that *anytime there exists a consistent truncation of an AdS \times \mathcal{M} KK spectrum that includes massive vector and/or spin-2 fields, the truncated theory admits the associated Schrödinger solutions.*

When can this be done? Although some recent work [11, 14] has argued for the possibility of including massive spin-2 fields in truncations to maximally supersymmetric supergravities, old lore [38] and new evidence [39] point to the contrary. If it is indeed true that no spin-2 fields can be so included — supersymmetrically or otherwise — then the Schrödinger solutions built on diagonal vectors that also turn on a tower of spin-2 fields cannot be found as solutions of consistent truncations. In type IIB solutions, this excludes all but those that can be attained by a TsT transformation.

On the other hand, the solutions built from mixed vectors have no associated spin-2 fields. In principle, all of these solutions are ripe for embedding; in practice, all massive truncations to date have only included vector harmonics that sit at the base of their respective KK towers.

It was conjectured in [13] that a necessary condition for the consistency of the inclusion of a massive KK mode is that the field sits at the bottom of its tower, based on consideration of structure groups of compactification manifolds. If we take this to be true as well, then the possibilities for embedding Schrödinger solutions into string/M-theory by fitting them into consistent truncations are severely limited; for example, no more of the D3-deformed Schrödinger solutions can find such an embedding.⁷

The flipside of this would be the implication that *anytime* a massive vector is included in a consistent truncation, it can be used to construct a Schrödinger solution. This would provide a trivial prescription for identifying non-relativistic vacua, and at least gives a rule of thumb. We give an example later on of a supersymmetric truncation to a $d = 4$, $\mathcal{N} = 2$ gauged supergravity arising from wrapped M5 branes that includes a massive vector multiplet [40], and show that, indeed, two Schrödinger solutions exist in correspondence to the theory's two AdS₄ vacua.

3 Quantum/string corrections

We turn now to the study of corrections to Schrödinger solutions due to quantum and string effects, which hinges on analyzing corrected KK spectra of AdS vacua. In what follows, there is no distinction made between α' corrections and G_N corrections, as both enter on equal footing as higher derivative terms. For convenience we refer to these collectively as α' corrections.

To begin, we can quickly establish that *there will always exist Schrödinger solutions at every order in a higher derivative expansion*. Since the worst that can happen to a supergravity AdS solution upon inclusion of α' corrections is that its radius gets renormalized, if one assumes that the curvature does not become so large as to invalidate the gravity description, then there will always exist Schrödinger solutions made by utilizing the KK spectrum around the new, corrected vacuum in the usual way. This is a generalization of a statement made in [28]. Using the dual arguments on the CFT side made in the previous section, the lone effect of heavy operators is to rescale terms in the β -function equations, so the non-relativistic fixed point will always exist for finite parameters $\{\Delta_z[\mathcal{O}_{--}^{(n)}], C^{(n)}\}$.

⁷Later, we will present new solutions with Sch₇ symmetry based on M5 branes; according to these criteria, only one of them could be a solution to a consistent truncation on AdS₇ × S⁴.

Compared to the two-derivative supergravity Schrödinger solutions, the corrected vacua will generically have parametric renormalization of z (when the vector mass spectrum is corrected) and L_{Sch} (when L_{AdS} is rescaled), and a different metric structure (on account of new mixing between linearized fluctuations).

What is required of the AdS background for these corrections to vanish? As one might expect, the answer is supersymmetry. And as we show momentarily, certain families of Schrödinger solutions built from perturbations of the maximally supersymmetry AdS spacetimes are exact.

The outline for this section is as follows. In 3.1, we utilize supersymmetry in presenting the conditions for nonrenormalization, and the aforementioned exact solutions. In 3.2, we explicitly show how one class of exact solutions with Sch_5 symmetry, obtained by deformation of $AdS_5 \times S^5$, remains a solution to $O(\alpha'^3)$ despite its reduced isometry and supersymmetry. This is done by direct calculation with the conjectured metric and five-form flux correction terms at this order in the type IIB supergravity action.

3.1 Uncorrected Schrödinger solutions from short multiplets

In the presence of supersymmetry, there will typically be some degree of nonrenormalization of the AdS background and its KK spectrum. In cases for which the AdS background itself is robust against renormalization to some order in a derivative expansion, Schrödinger solutions built from perturbations of this background are similarly robust, as $L_{AdS} = L_{Sch}$.

The most useful fact for our purposes is basic: if a supergravity mode lies in a shortened multiplet of the corresponding AdS supergroup, then the exponent appearing in a power law profile for such a mode cannot get renormalized at any order in higher derivative corrections. This can be understood intuitively as follows. The action of the dilatation operator directly relates the exponent to the scaling dimension of the dual boundary CFT operator; but operators in short multiplets have protected scaling dimensions. Since z appears as such an exponent, it will hence not be renormalized. Shortened multiplets occur, for instance, for *all* supergravity modes in maximally symmetric spacetimes because a long multiplet necessarily has fields with spins greater than two. Such shortened multiplets are actually ubiquitous in KK spectra, as many supergroups without maximal symmetry possess some number of short multiplets.

This immediately implies that if an AdS supergroup admits shortened multiplets, the Schrödinger solutions built from the massive fields in those multiplets will receive some level of protection from renormalization to all orders. Specifically, any property of the solution that depends only on the mass of such a field will be unrenormalized. Of course, this is the case for the dynamical exponent z which is determined by either a vector or spin-2 equation; see (2.7). Therefore, we conclude that *if the KK field that determines z sits in a short multiplet, z will remain uncorrected to all orders in an α' expansion.*

The reason that all solutions are not *fully* protected is that the perturbation theory around AdS truncates at quadratic, not linear, order. Those solutions formed from a diagonal vector ($\sigma \neq 0$) obey a spin-2 field equation corrected by a quadratic term. Thus, even if the spin-2 field h is in a short multiplet, the solution will receive a correction. On the other hand, the solutions without a diagonal vector receive no renormalization at

all, because *all* properties of the solution are derived from field equations which remain uncorrected in the full solution.

We summarize these results as follows:

Suppose we have an $\text{AdS} \times \mathcal{M}$ solution which remains unrenormalized to n^{th} order in α' or l_P . This implies that L_{Sch} will follow suit. We now ask what other renormalization can occur. Suppose the KK spectrum contains both spin-1 and spin-2 fields in short multiplets, and we turn these fields on in our perturbation theory. The following nonrenormalization theorems will hold:

- **No diagonal vector:** The reduced field equations are simply those of the linearized fluctuations. So the full solution, modulo L_{Sch} , remains unrenormalized to all orders in higher derivative corrections.
- **Diagonal vector:** This requires a tower of nonzero spin-2 fields as well, which collectively solve a Laplace equation with a quadratic source. This solution will have z unrenormalized, but the \mathcal{M} -dependence of g_{++} will be corrected at higher orders. (Again, L_{Sch} is unchanged to n^{th} order only.)

All of these arguments can be straightforwardly made on the CFT side, in accordance with the discussion in subsection 2.3.

Note that the mixed vector and spin-2 solutions are supersymmetric, generically preserving one half of the Poincaré supersymmetries of the AdS background [18, 19]. The diagonal vector solutions can be made supersymmetric, but need not be [20–22].

3.1.1 Exact solutions

When the AdS spacetime around which we perturb is maximally supersymmetric, $L_{\text{AdS}} = L_{Sch}$ receives no rescaling to all orders in higher-derivative corrections [41], and all supergravity fields belong to short multiplets. Such spacetimes are the near-horizon geometries of the conformal branes in type IIB string theory and M-theory. Therefore, *the towers of Schrödinger solutions based only on mixed vector and spin-2 perturbations of the maximally supersymmetric $\text{AdS}_{4/5/7} \times S^{7/5/4}$ vacua are exact.* These solutions preserve eight Poincaré supercharges [20], and are of course not maximally isometric. As such, we add them to the short list of exact solutions with less than maximal supersymmetry and isometry, including the non-supersymmetric plane wave spacetimes studied in [42] and the 1/4 BPS $\text{AdS}_2 \times S^2$ vacuum of $\mathcal{N} = 2, d = 4$ supergravity without matter multiplets [41].

On the CFT side, this says that there exist Galilean-invariant vacua for arbitrary 't Hooft coupling and number of colors. In $\mathcal{N} = 4$ SYM, the $(2, 0)$ theory on M5 branes and the CFT on M2 branes, the conformal phase can be broken to a Galilean-symmetric phase with z fixed as a function of λ or N .

When the spheres are replaced by manifolds with less isometry, the resulting AdS vacua will have partial or no supersymmetry. Still, it is known in some cases that L_{AdS} is norenormalized to a certain order in the derivative expansion. For example, using the conjectured $O(\alpha^3)$ correction terms of type IIB involving the metric and five-form only [29, 43], it has been shown that the $\text{AdS}_5 \times SE_5$ vacuum is unrenormalized to this order. Thus,

the solutions with Sch_5 symmetry built by perturbing this background with massive fields in short multiplets, while keeping $\sigma = 0$, are unrenormalized to this order too. The same goes for the $AdS_3 \times S^3 \times M_4$ geometry of the strongly coupled D1-D5 system, and its associated Sch_3 solutions which we present in section 4.

Indeed, the $SU(2,2|1)$ supergroup of the $AdS_5 \times SE_5$ vacuum is one of many that contains short multiplets. The shortening conditions for $SU(2,2|\mathcal{N})$, with $\mathcal{N} = 1, 2, 4$, are summarized in [44]. The most well-studied example of a geometry dual to an $\mathcal{N} = 1$ SCFT, $AdS_5 \times T^{1,1}$, has both spin-2 and spin-1 fields in semi-long multiplets [45], dual to boundary operators of protected, rational conformal dimension. So solutions based on these fields are unrenormalized to at least $O(\alpha'^3)$.

Other AdS vacua whose KK spectra contain states in short multiplets include: $AdS_3 \times S^3$ solutions of various $d = 6$ supergravities, the multiplet structure of which was given in [46]; $AdS_4 \times SE_7$ solutions, the supergravity states of which fall into unitary irreducible representations of $OSp(2|4)$ (e.g. [47, 48]); and orbifolded AdS vacua.

3.2 At $O(\alpha'^3)$, explicit nonrenormalization of Sch_5

We focus on type IIB Sch_5 solutions because, relatively speaking, we know a lot about type IIB correction terms. We possess an explicit form for the terms involving only the metric and five-form up to $O(\alpha'^3)$ relative to the supergravity action; we know that the supersymmetry of $AdS_5 \times SE_5$ vacua renders their length scales fixed to this order; and we know that when $SE_5 = S^5$, the solution is maximally symmetric, maximally supersymmetric, and hence unrenormalized to all orders, inclusive of corrections from *all* type IIB fields.

The full set of metric and five-form corrections at $O(\alpha'^3)$ was presented in [29], building on the conjecture of [43]. There exists a scheme, obtained by appropriate definition of the metric, in which the metric dependence can be written in terms of only the Weyl tensor, C_{ABCD} . All appearance of the five-form flux is via the following two-derivative tensor [49],

$$\mathcal{T}_{ABCDEF} = i\nabla_A F_{BCDEF} + \frac{1}{16}(F_{ABCMN}F_{DEF}{}^{MN} - 3F_{ABFMN}F_{DEC}{}^{MN}) \quad (3.1)$$

where $[A, B, C]$ and $[D, E, F]$ should be independently antisymmetrized, and the two sets symmetrized under interchange. Schematically, the corrections to the Lagrangian take the form

$$C^4, \quad C^3\mathcal{T}, \quad C^2\mathcal{T}^2, \quad C\mathcal{T}^3, \quad \mathcal{T}^4 \quad (3.2)$$

along with their complex conjugates, all entering at $O(\alpha'^3)$; the full set of contractions is determined in [29]. The terms enter the Einstein frame action as [29, 43]

$$S^{(3)} \sim \alpha'^3 \int d^{10}x \sqrt{-g} f^{(0,0)}(\tau, \bar{\tau})(C^4 + \dots) \quad (3.3)$$

where

$$f^{(0,0)}(\tau, \bar{\tau}) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^{3/2}}{|m + n\tau|^{3/2}} \quad (3.4)$$

is a modular form [50] written in terms of the axio-dilaton $\tau = \tau_1 + i\tau_2 = \chi + ie^{-\phi}$, and the well-known C^4 term [51] is

$$C^4 = C^{ABCD}C_{EBCF}C_A{}^{GHE}C^F{}_{GHD} + \frac{1}{2}C^{ABCD}C_{EFCD}C_A{}^{GHE}C^F{}_{GHB} \quad (3.5)$$

Beyond this order in α' , the structure of the corrections is not known.⁸

For the $\text{AdS}_5 \times S^5$ background, in which $L_{\text{AdS}_5} = L_{S^5}$, superspace arguments have been used to prove that the solution is exact [41]. This is clearly satisfied at $O(\alpha^3)$, as one can show that $C = \mathcal{T} = 0$ on this solution. The uncorrected KK spectrum of energies is also evidently unrenormalized at this order: the corrections are quartic in C and \mathcal{T} , so variation will never give a term linear in only one of these tensors.

Using this apparatus, we now consider corrections to the Sch_5 solutions obtained by deformations of $\text{AdS}_5 \times S^5$. We will explicitly demonstrate what we argued for earlier: that the metric and five-form solutions ($\sigma = 0$) are unrenormalized, but that those with three-form flux ($\sigma \neq 0$) can receive a renormalization of the S^5 part of g_{++} . We consider these in turn, keeping $\beta = 0$ for simplicity.

Before we begin, let us give away the punchline. It may seem like something of a mystery that these Sch_5 geometries, despite their greatly reduced supersymmetry and isometry, are as robust against these same corrections as the maximally supersymmetric and isometric $\text{AdS}_5 \times S^5$. As with many other aspects of Schrödinger holography, the null Killing vector is the key.

3.2.1 Sch_5 solutions from $\text{AdS}_5 \times S^5$: $\sigma = 0$

This solution has metric and five-form flux only, so the only terms that can contribute to renormalization are the terms given above, plus terms linear in the axio-dilaton and complex three-form flux which can turn on a decoupled field. We delay treatment of these latter terms temporarily, as we turn to the $O(\alpha^3)$ metric and five-form terms.

Our strategy will be to look for terms which contribute to C and \mathcal{T} — each of which vanishes in AdS_5 — and show that there are not enough terms to give nonzero contractions when plugged into the correction terms from (3.2).

The Sch_5 metric

$$ds^2 = r^{2z}q(dx^+)^2 + ds^2(\text{AdS}_5 \times S^5) \quad (3.6)$$

corrects the vanishing $\text{AdS}_5 \times S^5$ Weyl tensor only by terms which have two lower + indices and no - indices. Specifically, nonvanishing components are

$$C_{+r+\theta_i}; \quad C_{+\theta_i+\theta_j}; \quad C_{+r+r}; \quad C_{+\mathbf{x}+\mathbf{x}} \quad (3.7)$$

where $\{\theta_i\}$ parameterize S^5 . There are no lower - indices. Already, this implies that variation of the C^4 term will vanish on-shell, because the + indices have nothing to contract with: $g^{++} = 0$.⁹

⁸It has not been proven that this is the full set of correction terms, though there is a great amount of evidence in favor.

⁹Note that this same phenomenon for the $\text{Sch}_3 \times S^3$ solution from the D1-D5 system, to be presented in the next section, implies that the C^4 term does not renormalize that solution either [52].

The correction terms are quadratic in C and \mathcal{T} , so variation of these terms with respect to either F_5 or the metric will leave behind terms at least cubic in C and \mathcal{T} . Because $g^{++} = 0$, this implies that if \mathcal{T} has no components with lower - indices, all variation of the correction terms will vanish: the lower + indices of either tensor will not have any upper + indices to contract with, and the quartic order guarantees enough index contractions to force such a contraction. So, we examine the structure of \mathcal{T} , given in (3.1).

Recall that $\mathcal{T} = 0$ in $\text{AdS}_5 \times S^5$. We can uncover the structure of this term in the Schrödinger vacuum without actually plugging in for the flux — which is the same in both Schrödinger and AdS backgrounds, namely the sum of volume forms — by asking what new components of the Christoffel connection and contravariant metric are introduced by the deformation.

The new components of the connection are

$$\begin{aligned} \Gamma_{++}^r &= -qzr^{2z+1}, & \Gamma_{+r}^- &= qr^{2z-3}(z-1), \\ \Gamma_{+\theta_i}^- &= \frac{1}{2}r^{2z-2}\frac{\partial q}{\partial\theta_i}, & \Gamma_{++}^{\theta_i} &= -\frac{1}{2}r^{2z}\frac{\partial q}{\partial\theta_i} \end{aligned} \tag{3.8}$$

This implies that the flux remains covariantly constant in the Schrödinger background,

$$\nabla_A F_{BCDEF} = 0 \tag{3.9}$$

This relies on the fact that the flux is merely the sum of the two volume forms, just as in $\text{AdS}_5 \times S^5$, and the fact that none of the new Christoffel components has an identical upper and lower index.

Now we examine the terms quadratic in F_5 . Because the components of F_5 are the same as in AdS_5 , the only changes will come from new contravariant components of the metric, of which there is one: $g^{--} \neq 0$. This implies the existence of a single new nonzero term. Consider the contraction

$$F_{ABC-G}F_{DEF}^{-G} = F_{ABC-G}F_{DEF+}^G g^{+-} + F_{ABC-G}F_{DEF-}^G g^{--} \tag{3.10}$$

The second term is the new term; the first term merely contributes to the condition $\mathcal{T} = 0$ in the unperturbed AdS background. Looking at the new term, we see that $G \neq +$, otherwise it vanishes. So one index within each triplet $[A, B, C]$ and $[D, E, F]$ must be the + index. Hence, the new term takes the schematic form

$$\mathcal{T}_{+BC+EF} \sim F_{+BC-G}F_{+EF-}^G g^{--} \tag{3.11}$$

where $B, C, E, F \neq -$. More specifically, the only nonvanishing terms are

$$\mathcal{T}_{+xy+xy}, \quad \mathcal{T}_{+rx+rx}, \quad \mathcal{T}_{+ry+ry} \tag{3.12}$$

because $F_{+-rxy} \neq 0$ is the only nonvanishing flux component with legs along +.

To summarize, the only nonvanishing components of \mathcal{T} and C have no lower - indices, and hence any variation of the correction terms will vanish on-shell.

This is a perhaps surprising result. To further drive the point home, we point out an argument in [29] that any solution with all fields trivial except for the metric and five-form

that is more than 1/4 BPS is uncorrected at $O(\alpha^3)$ by the entire set of terms above. Here, we have shown the same for these 1/4 BPS solutions.

We now turn to possible terms linear in the axio-dilaton and three-form flux, which would act as sources for these fields. Symmetry under $G_3 \rightarrow -G_3$ rules out terms linear in G_3 . Terms linear in the axio-dilaton would, at this order, be multiplied against a scalar, eight-derivative object constructed from the Weyl tensor and five-form field strength. Further, any such scalars must vanish when evaluated on the AdS background, since this is a solution by assumption. Schematically, this would allow terms such as

$$(\phi, \chi) C_{ABCD} ((F_5)^6)^{ABCD}, \quad (\phi, \chi) \nabla_A C_{BCDE} ((F_5)^5)^{ABCDE} \dots \quad (3.13)$$

However, by examining the index structure and using the null Killing vector, one can check that all such scalar terms vanish. We conclude that the axio-dilaton is not sourced at this order.

3.2.2 Sch₅ solutions from AdS₅ × S⁵: $\sigma \neq 0$

Now we turn on three-form flux. This does not affect the metric and five-form correction analysis above. So if this solution is to be renormalized, it must come from the three-form terms. In the absence of knowledge of the actual form of the terms, the null constraint is not strong enough to rule this out.

Let us at least present the challenge. First, F_5 is not constrained to appear via the \mathcal{T} tensor in these terms. This introduces a tensor component with a lower - index, F_{+-rxy} , as well as one without any + or - indices at all, $F_{\theta_1 \dots \theta_5}$. The metric still must appear via the Weyl tensor, still as in (3.7), so the full set of nonzero components is

$$C_{+A+B}, \quad F_{+-rxy}, \quad F_{\theta_1 \dots \theta_5}, \quad G_{+r\theta_i}, \quad G_{+\theta_i\theta_j} \quad (3.14)$$

Additionally, G_3 is not covariantly constant, so we can have ∇G_3 appear in the correction terms.

Suppose we wish to ask what terms can contribute to the stress tensor. The following term is explicitly nonvanishing on this background, and could arise at $O(\alpha^3)$:

$$\begin{aligned} (G^2 F^6)_{++} &\sim G_{+BC} G_{+DE} F^{BCFGH} F^{DE}{}_{FGH} F^4 \\ &\sim G_{+\theta_i\theta_j} G_{+\theta_k\theta_l} F^{\theta_i\theta_j\theta_m\theta_n\theta_p} F^{\theta_k\theta_l}{}_{\theta_m\theta_n\theta_p} F^4 \end{aligned} \quad (3.15)$$

where F^4 includes all possible contractions. Another term is

$$\begin{aligned} (G^2 F^6)_{++} &\sim F_{+ABCD} F_{+E}{}^{BCD} G^{AGH} G^E{}_{GH} F^4 \\ &\sim F_{+-rxy} F_{+-}{}^{rxy} G^{-GH} G^-{}_{GH} F^4 \end{aligned} \quad (3.16)$$

The correct properties under symmetry are manifest for both of these terms.

Therefore, the null Killing vector does not imply an absence of renormalization of the g_{++} component of these Sch₅ solutions. It is quite possible, if not likely, that the true three-form correction terms involve a tensor structure, analogous to \mathcal{T} for the five-form and metric terms, that renders the corrections to the Sch₅ backgrounds zero.

4 New Schrödinger solutions

We put our theory into practice by presenting new Schrödinger solutions. Two of these are based on the compactification spectra of well-studied brane setups: the maximally supersymmetric M5 brane in flat space, and the half-supersymmetric D1-D5 system of type IIB with the D5 brane wrapped on T^4 or $K3$. The third construction presents evidence for our earlier suggestion that all consistent truncations of string/M-theory that retain massive modes admit Schrödinger solutions. In this instance, we build a pair of Sch_4 solutions dual to nonrelativistic phases of a 2+1 CFT living on M5 branes wrapped on special-Lagrangian 3-cycles of a Calabi-Yau.

The M5 construction includes two infinite families of solutions exact to all orders in l_P . The D1-D5 construction includes three families that are exact up to the order at which L_{AdS_3} is rescaled.¹⁰

4.1 Sch_7 from M5 branes

The KK spectrum around $AdS_7 \times M_4$ [53, 54] contains two towers of mixed vectors, no massive diagonal vectors, and massive gravitons as always. This spacetime has $L_{AdS_7} = 2L_{M_4} \equiv 2$. For $M_4 = S^4$, the upper branch of vectors has masses $m^2 = (l+3)(l+5)$ (where integer $l \geq 0$ henceforth), implying a (positive) z spectrum of $z = 6 + 2l$, in accordance with (2.15). The massive gravitons have spectrum $m^2 = (l+1)(l+4)$, which implies a (positive) z spectrum of $z = 2 + l$, in accordance with (2.16).

If one replaces S^4 with an M_4 which has nonzero second Betti number b_2 , there will be an additional set of Yang-Mills gauge fields in $d = 7$. These Betti vectors carry topological charge, descending directly from the three-form gauge field and appearing as a single harmonic level of diagonal vectors. Their masslessness gives an isolated Sch_7 solution with $z = -4$ and degeneracy b_2 ; the possibility $z = 0$ is ruled out, as the Schrödinger vector in this case would be pure gauge.

We will realize each one of these solutions in $D = 11$, again starting from the full M5-brane metric and then moving into the near-horizon. The most general solution¹¹ is

$$\begin{aligned}
 ds^2 &= \Phi^{-1/3}(2dx^+ dx^- + h(dx^+)^2 + 2dx^+ C + d\mathbf{x}^2) + \Phi^{2/3} ds^2(X_5) \\
 G &= \star_{11}(dx^+ \wedge dx^- \wedge d\mathbf{x} \wedge d\Phi^{-1}) + dx^+ \wedge V - \star_{11}(dx^+ \wedge d\mathbf{x} \wedge d(\Phi^{-1}C))
 \end{aligned}
 \tag{4.1}$$

This ansatz preserves rotational invariance. C is a one-form, V is a three-form, and h is a function, all defined on the space X_5 . The Maxwell and Einstein equations, and the

¹⁰It is known that at $O(\alpha'^3)$, L_{AdS_3} is unrenormalized.

¹¹Conventions for $D = 11$ supergravity are as in [55].

Bianchi identity, imply

$$\begin{aligned}
 d \star_{X_5} \Phi &= 0 \\
 d \star_{X_5} dC &= 0 \\
 d \star_{X_5} \left(\frac{V}{\Phi} \right) &= 0 \\
 dV &= 0 \\
 \nabla_{X_5}^2 h &= -\frac{1}{\Phi} |V|_{X_5}^2
 \end{aligned}
 \tag{4.2}$$

where $|V|_{X_5}^2 = \frac{1}{3!} V_{abc} V^{abc}$ with indices raised by the metric on X_5 . Notice the extra factors of Φ in the V terms relative to the D3 brane case (cf. equations (2.2)).

Set $V = 0$ for now. Zooming into the near-horizon region and writing X_5 as a cone over M_4 ,

$$ds^2(X_5) = dr^2 + r^2 ds^2(M_4)
 \tag{4.3}$$

we make the scale-invariant assignments

$$\begin{aligned}
 \Phi &= r^{-3} \\
 C &= r^{\frac{z-2}{2}} \beta \\
 h &= r^{z-1} q
 \end{aligned}
 \tag{4.4}$$

where q and β are a function and one-form, respectively, on M_4 . When $V = 0$, this is a solution so long as the reduced field equations

$$\begin{aligned}
 \Delta_4 \beta &= \frac{z(z-2)}{4} \beta \quad \text{where} \quad d \star_4 \beta = 0 \\
 \nabla_4^2 q &+ (z-1)(z+2)q = 0
 \end{aligned}
 \tag{4.5}$$

are satisfied, with $-\nabla_4^2$ and Δ_4 as the Laplace operator of M_4 acting on functions and transverse one-forms, respectively. For a spherical base space $M_4 = S^4$,

$$\Delta_4 \beta = (l+2)(l+3)\beta, \quad \nabla_4^2 q + l(l+3)q = 0
 \tag{4.6}$$

and we obtain the anticipated spectra of z outlined earlier. Notice that these two solutions can be consistently superposed, because there exist simultaneous eigenfunctions of Δ_4 and $-\nabla_4^2$ for all z .

Because all KK fields sit in short multiplets of the superalgebra $\text{OSp}(8|4)$ [54], these solutions are exact. By inspection of the structure of this solution, we expect it to preserve eight Poincaré supersymmetries, in analogy with the M2 and D3 brane solutions of [20].

Turning to $V \neq 0$, the discussion at the start of this subsection implies the existence of a single solution with $z = -4$ corresponding to deformation by the b_2 topological vector fields. Indeed, one can show that the following configuration solves the field equations,

$$V = d(r^{-2}\tau)
 \tag{4.7}$$

where τ is a harmonic two-form on M_4 , $d\tau = d \star_{M_4} \tau = 0$. So M_4 must have $b_2 \neq 0$, and scale invariance of G implies $z = -4$ as predicted, since V must scale as $r^{z/2}$. The associated Einstein equation is

$$\nabla_4^2 q + 10q = -4|\tau|_4^2 \tag{4.8}$$

Perhaps unexpectedly, this is but one of an infinite family of diagonal solutions, despite the absence of diagonal vectors in the spectrum of KK fields on $\text{AdS}_7 \times M_4$. We present the general construction in appendix A, along with a $\text{Lif}_6(z = 2)$ solution obtained in a similar manner.

4.2 Sch₃ from D1-D5 and F1-NS5 branes

These solutions come from deformations of $\text{AdS}_3 \times S^3 \times M_4$, with $M_4 = T^4, K3$. The KK spectra [56] around these two solutions are identical in everything but multiplicities of fields, on account of the different number of tensor multiplets in the chiral and non-chiral $d = 6$ supergravities. They are somewhat more involved than the examples presented so far because of the prior reduction on M_4 , combined with the self-duality properties of the tensor fields. As a result, we relegate a full treatment to an appendix and merely present the solutions here.

Once again, we expect these solutions to have the same supersymmetric structure as their D3 and M2-brane counterparts: the mixed solutions should preserve half of the Poincaré supersymmetry of the D1-D5 background, and the diagonal solutions can, but need not, be supersymmetric. We have not confirmed this explicitly, however.

The KK reduction on S^3 is done at the level of the $d = 6$ supergravity, which contains five self-dual and n_T anti-self-dual tensor fields that descend from the complex three-form and five-form fluxes, where n_T is the number of antisymmetric tensor multiplets.¹²

4.2.1 Diagonal vectors

There are two towers of diagonal vectors, each with mass $(mL)^2 = (l + 2)^2$, implying a z spectrum of $z^2 = (l + 2)^2$, where $l \in \mathbb{Z}$. One tower descends from the four self-dual tensor fields that do not source the D1-D5 background, and the other from the n_T anti-self-dual tensor fields.

The ansatz is

$$\begin{aligned} ds^2 &= r^{2z} q(dx^+)^2 + 2r^2 dx^+ dx^- + \frac{dr^2}{r^2} + ds^2(S^3) + ds^2(M_4) \\ iG_3 &= 2(1 + \star_6)d\Omega_3 + \delta F_3 - i\delta H_3 \\ F_5 &= \delta F_5 \end{aligned} \tag{4.9}$$

where q is a function defined on S^3 , $d\Omega_3$ is the invariant volume form on S^3 , and \star_6 is the Hodge star operator on the spacetime transverse to the M_4 . We have set $L_{\text{AdS}_3} = L_{S^3} = 1$. When $q = \delta H_3 = \delta F_3 = \delta F_5 = 0$, this is the D1-D5 solution.

¹²When $M_4 = T^4, K3$, one has $n_T = 5, 21$, respectively.

Because the $d = 6$ tensor fields descend from both G_3 and F_5 , we simply need to turn on the components of G_3 and F_5 that give rise to tensor fields with the right $d = 6$ self-duality property. This means that we can turn on anti-self dual parts of F_3, H_3 and F_5 , but only self-dual parts of H_3 and F_5 . This also implies that without doing any extra work, the F1-NS5 system obtained from the D1-D5 system by S-duality can also be engineered to give Schrödinger solutions in exactly the same way: from the six-dimensional perspective, S-duality merely shuffles the background flux to a different one of the five self-dual tensor fields.

Without further ado, the solutions are:

- Anti-self-dual RR two-form charge:

$$\delta F_3 = (1 - \star_6) dx^+ \wedge d(r^z \sigma) \tag{4.10}$$

- NS-NS two-form charge:

$$\delta H_3 = (1 \pm \star_6) dx^+ \wedge d(r^z \sigma) \tag{4.11}$$

- RR four-form charge:

$$\delta F_5 = \frac{1 + \star_{10}}{\sqrt{2}} [(1 \pm \star_6) dx^+ \wedge d(r^z \sigma) \wedge \alpha] \tag{4.12}$$

σ is a one-form on S^3 , α is a harmonic two-form on M_4 of norm $|\alpha|^2 = 1$ and definite M_4 self-duality property, and the reduced field equations are

$$\begin{aligned} \Delta_3 \sigma &= z^2 \sigma \quad \text{where} \quad d \star_3 \sigma = 0 \\ \nabla_3^2 q + 4z(z-1)q &= -2(z^2 |\sigma|_3^2 + |d\sigma|_3^2) \end{aligned} \tag{4.13}$$

with $-\nabla_3^2$ and Δ_3 as the Laplace operator of S^3 acting on functions and transverse one-forms, respectively. Note that the source terms in the Einstein equation are of the same form as those in the D3-brane case, given in (2.7). For future reference, let us denote this quantity

$$\Lambda(\sigma) \equiv z^2 |\sigma|_{\mathcal{M}}^2 + |d\sigma|_{\mathcal{M}}^2 \tag{4.14}$$

On S^3 , $\Delta_3 \sigma = (l+2)^2 \sigma$, where $l \in \mathbb{Z}$, so taking the positive branch gives

$$z = l + 2 \tag{4.15}$$

as expected.

Note that a $z = 2$ solution was constructed in [5] as a TsT-transformed D1-D5 system, by utilizing the Reeb Killing vector of S^3 , and studied more recently in [9]. (For the construction of another $z = 2$ solution in a somewhat different setting, see also [57].) That solution has nonzero H_3 charge, both self-dual and anti-self-dual. This is consistent with our result, as the lowest eigenvalue of Δ_3 is obtained for σ Killing; in fact, we see that it is actually redundant, because it turns on two different KK vectors with the same mass. The way this TsT solution fits into the ladder of solutions is qualitatively identical to the TsT D3 brane solution: in particular, $z = 2$ is the lone value which can give a direct product metric $Sch_3 \times S^3 \times M_4$ because $\Lambda(\sigma)$ is not constant otherwise. (See the appendix for more discussion on this point.)

4.2.2 Spin-2 fields

It is clear from the diagonal vector solutions that if we turn the vector off, we will get a Sch_3 solution from a spin-2 field alone, where z is now determined by the scalar Laplacian on S^3 as

$$4z(z - 1) = l(l + 2) \tag{4.16}$$

Note the agreement with (2.16). Starting at the first nonzero harmonic $l = 1$, we have solutions with

$$z = \frac{3}{2}, 2, \frac{5}{2}, \dots \tag{4.17}$$

4.2.3 Mixed vectors

There are three towers of mixed vectors, each with different mass spectra. Here, we construct Sch_3 solutions from two of them. One has $m^2 = l^2$, leading to a z spectrum of $z^2 = l^2$. The other has $m^2 = (l + 4)^2$, leading to a z spectrum of $z^2 = (l + 4)^2$.

The solution is

$$\begin{aligned} ds^2 &= 2(r^2 dx^+ dx^- + r^z dx^+ \beta) + \frac{dr^2}{r^2} + ds^2(S^3) + ds^2(M_4) \\ iG_3 &= 2(1 + \star_6)d\Omega_3 + (1 + \star_6)dx^+ \wedge d(r^z \beta) - \frac{r^2}{2}(1 + \star_6)dx^+ \wedge d(r^{z-2}\beta) \end{aligned} \tag{4.18}$$

where β is a one-form on S^3 and obeys the eigenvalue equation

$$\Delta_3 \beta = (z - 2)^2 \beta \quad \text{where} \quad d \star_3 \beta = 0 \tag{4.19}$$

This gives $z = -l, l + 4$, consistent with our expectation. As with other mixed vector constructions, each branch of the eigenvalue equation accesses one branch of vectors. The final term in the flux appears anomalous compared to other mixed vector solutions; as detailed in the appendix, this is a peculiarity of the dimensionality of these solutions.

4.3 Sch_4 from wrapped M5 branes

Finally, we construct two $Sch_4 \times H^3/\Gamma \times S^4$ solutions of $D = 11$ supergravity. This is based on a consistent (bosonic) truncation to an $\mathcal{N} = 2, d = 4$ gauged supergravity with one vector multiplet and two hypermultiplets, as performed in [40]. The M5 branes are wrapped on special-Lagrangian 3-cycles of CY_3 , giving a $d = 3, \mathcal{N} = 2$ CFT at low energies; the supersymmetry is preserved by virtue of the choice of cycle. With the consistent truncation in hand, two AdS_4 duals can be found directly within the $d = 4$ supergravity. Here, we show that these $d = 3$ CFTs also have non-relativistic phases by finding the dual Schrödinger geometries.

Most details of the truncation are unnecessary for our purposes; we refer the reader to [40], and use their notation in what follows.

The reduction is done first from $D = 11$ supergravity on S^4 to maximal seven-dimensional gauged supergravity, and then further on a three-manifold of constant curvature, $\Sigma_3 = S^3, \mathbb{R}^3, H^3$ or their quotients. One can parameterize the choice of Σ_3 by the sign of the curvature, $l = \pm 1, 0$, where S^3 is taken to have $l = 1$. The theory contains the

following field content: the metric; two three-forms h_3^α , where $\alpha = 1, 2$; one two-form B_2 ; two one-forms A_1, C_1 ; and nine scalars comprised of $\phi, \lambda, \beta, \theta_\alpha, \chi_\alpha$ and a symmetric matrix $\mathcal{T}_{\alpha\beta}$, which parameterizes a coset $\text{SL}(2, \mathbb{R})/\text{SO}(2)$. In addition to the parameter l which appears in the $d = 4$ field equations, there is a gauge parameter g with mass dimension one that comes from the $d = 7$ gauged supergravity potential.

We will not need most of these fields to construct the $\text{Sch}_4 \times H^3/\Gamma \times S^4$ solutions.

There are two known AdS_4 vacua of this theory, and the masses of the fields in each vacuum were calculated in [40]. Only the first is supersymmetric, and is given as

$$e^\phi = 2^{-1/20}, \quad e^\lambda = 2^{1/10}, \quad l = -1, \quad (gL)^2 = \sqrt{2} \tag{4.20}$$

with all other fields turned off. The vector fluctuations around this background can be diagonalized to give vectors with masses $(mL)^2 = 0, 4$, with C_1 as the massive vector. One can show that there is a Schrödinger solution, with

$$z(z+1) = 4 \quad \Rightarrow \quad z = -\frac{1}{2} + \frac{\sqrt{17}}{2} \approx 1.56 \tag{4.21}$$

in which the AdS fields and parameters above are unchanged, and we turn on the form-fields in the following manner:

$$\begin{aligned} C_1 &= r^z dx^+ \\ B_2 &= -\frac{1}{\sqrt{2}g} \star_4 dC_1 \\ F_2 &= dA_1 = -\frac{3}{g} dC_1 \end{aligned} \tag{4.22}$$

with x^+ a lightcone coordinate of AdS_4 , and z as in (4.21). Hence, the full $\text{Sch}_4 \times H^3/\Gamma \times S^4$ metric reads

$$ds^2 = -r^{2z}(dx^+)^2 + r^2(2dx^+dx^- + dx^2) + \frac{\sqrt{2}}{g^2} \frac{dr^2}{r^2} + ds^2(H^3/\Gamma \times S^4) \tag{4.23}$$

The massive vector sits in a long vector multiplet of $\text{OSp}(2|4)$, and so this solution will be subject to quantum corrections.

The second, non-supersymmetric AdS_4 solution is given as

$$e^\phi = 6^{-1/4}10^{1/5}, \quad e^\lambda = 10^{1/10}, \quad l = -1, \quad (gL)^2 = \frac{5\sqrt{2}}{3\sqrt{3}} \tag{4.24}$$

with all other fields turned off. The vector fluctuations around this background can be diagonalized to give vectors with masses $(mL)^2 = 0, \frac{28}{5}$; again, C_1 is the massive vector. One can show that there is a solution, with

$$z(z+1) = \frac{28}{5} \quad \Rightarrow \quad z = -\frac{1}{2} + \frac{3}{10}\sqrt{65} \approx 1.92 \tag{4.25}$$

in which the AdS fields and parameters above are unchanged, and we turn on the form-fields in the following manner:

$$\begin{aligned}
 C_1 &= r^z dx^+ \\
 B_2 &= -\frac{\sqrt{6}}{14g} \star_4 dC_1 \\
 F_2 = dA_1 &= -\frac{27}{7g} dC_1
 \end{aligned}
 \tag{4.26}$$

where dx^+ is again lightcone coordinate of AdS_4 , and now z is as in (4.25).

A useful way to see that these solutions exist is to consider the $d = 4$ field equations, and show that they can be truncated to those of a massive vector theory with the appropriate mass and cosmological constant, *provided* that the vector is null. The definitions of the parameters (L, g, l, m^2) used in these solutions fall out of this procedure. For a brief exposition of this, see appendix C.

The considerations in earlier sections tell us that these are but two of an infinite tower of Schrödinger deformations of these AdS_4 solutions. The others are implicit in the compactification spectrum. We were able to embed this one in a consistent truncation because, presumably, the massive vector C_1 sits at the bottom of a tower of vectors on H^3/Γ . This is supported by the fact that our solutions contain direct product metrics.

5 Discussion

Understanding Schrödinger solutions as Kaluza-Klein deformations of AdS has revealed some larger truths. We saw that they are universal, existing in infinite number for any AdS vacuum, by virtue of the maximal isometry of AdS; they can be robust against quantum and string corrections, by virtue of possible supersymmetry of AdS; and in some cases, they are exact. In essence, we have shown that the AdS/CFT correspondence is not only suggestive of a generalization to non-relativistic, Galilean gauge-gravity duality, but rather, the latter is truly contained within the former.

We expect these ideas to lend themselves to applications to condensed matter systems. Just as the universality of AdS leads to general thermodynamic results for holographically dual gauge theories (e.g. the KSS bound), one might expect that the universality of Schrödinger spacetimes implies similar statements about non-relativistic fixed points. Some work has already been done to show that the KSS bound is saturated for certain values of z (e.g. [3, 4] studied $z = 2$); the spirit of the present work suggests a universal extension of some kind.

Our analysis relies on the Lorentz and scaling symmetries of AdS, which precludes an extension of the perturbation theory method to finite temperature. On the other hand, any asymptotically AdS spacetime will, at infinity, possess the symmetries and KK spectrum required for the existence of Schrödinger deformations. It would be interesting to further investigate, in this KK framework, the extent to which asymptotically AdS black hole solutions can be universally deformed.

An aspect of this work that may lend itself to the study of RG flows with Schrödinger endpoints is the discovery of the role of relevant operators in generating Schrödinger backgrounds. We showed in section 2 that the generic $Sch(z)$ solutions in string and M-theory involve a tower of spin-2 fields which are non-marginal with respect to non-relativistic dilatations \mathcal{D}_z in the AdS vacuum. Some of these fields are relevant: any spin-2 field which has relativistic scaling weight $\Delta = d + 2(z' - 1)$ under \mathcal{D} is relevant with respect to \mathcal{D}_z when $z' < z$. In fact, in circumstances where the field theory possesses non-Abelian internal symmetries they may all be relevant.¹³ Supposing that one wants to construct an interpolation between a UV CFT and its IR Schrödinger phase of the same effective spacetime dimension, one may be able to utilize one of these relevant operators to stabilize an IR Schrödinger geometry.

Continuing with the topic of RG flows, it would be interesting to look for one that connects the $Sch_4 \times H^3/\Gamma \times S^4$ solution at low energies to an $AdS_7 \times S^4$ solution at high energies, in the spirit of [58]. The picture in the bulk is of an M5 brane, extended at asymptotic infinity, wrapping itself around H^3/Γ and turning on flux at small radii, dual to a nontrivial RG flow across dimension in which the scale invariance of the theory becomes anisotropic in the infrared. An example of this sort of behavior was recently found in [59] (though with Sch_3 symmetry and hence without Galilean boosts). It may also be that the Sch_4 solution built from a deformation of the supersymmetric AdS_4 solution preserves some of the eight supersymmetries, in which case one would be motivated to look for an analytic RG flow, generalizing the work of [58] to the non-relativistic regime.

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A More non-relativistic solutions from M5 branes

A.1 Diagonal Sch_7 solutions

Here, we present a family of Sch_7 solutions which do not appear to lie in correspondence with any KK fields on $AdS_7 \times M_4$.

Starting from the near-horizon ansatz

$$\begin{aligned}
 ds^2 &= r(2dx^+dx^- + h(dx^+)^2 + d\mathbf{x}^2) + \frac{dr^2}{r^2} + ds^2(M_4) \\
 G &= \star_{11}(3r^2 dx^+ \wedge dx^- \wedge d\mathbf{x} \wedge dr) + dx^+ \wedge V
 \end{aligned}
 \tag{A.1}$$

¹³One example is the set of solutions obtained by diagonal vector deformations of $AdS_5 \times S^5$, in which the global $SU(4)_R$ symmetry leads to a restricted set of lower harmonics appearing in the solution [20, 22].

the field equations are

$$\begin{aligned} d \star_{X_5} (V r^3) &= 0 \\ dV &= 0 \\ \nabla_{X_5}^2 h &= -r^3 |V|_{X_5}^2 \end{aligned} \tag{A.2}$$

where as usual, X_5 is the cone over M_4 . Scale invariance implies that V scales as $r^{z/2}$.

Following the strategy of the cases in which we *do* expect such diagonal solutions, we make the ansatz

$$V = d(r^{z/2} \tau) \tag{A.3}$$

where τ is a two-form on M_4 . The Maxwell equation can then be solved as

$$\Delta_4 \tau = \left(\frac{z}{2}\right) \left(\frac{z}{2} + 2\right) \tau, \quad \text{where } d \star_4 \tau = 0 \tag{A.4}$$

where Δ_4 is the Laplace operator of M_4 acting here on transverse two-forms. The flux now includes the term

$$G \supset d(r^{z/2} dx^+ \wedge \tau) \equiv d(A^{Sch} \wedge \tau) \tag{A.5}$$

which takes the usual form. The Einstein equation, upon writing $h = r^{z-1} q(M_4)$ in accordance with scale-invariance, is

$$\nabla_4^2 q + (z-1)(z+2)q = - \left[\left(\frac{z}{2}\right)^2 |\tau|_4^2 + |d\tau|_4^2 \right] \tag{A.6}$$

Note that the case $z = -4$ corresponds to the case of harmonic τ , and so we recover our earlier result as one of many.

The existence of a solution is contingent on solution of the two equations (A.4) and (A.6). For any M_4 , the Maxwell equation can always be solved to give a spectrum of z unbounded from above. Let us take this to define z . So the only way these solutions do not exist is if, for some such z , the scalar Laplacian of the Einstein equation admits a homogeneous solution and the quadratic source term includes this mode; this follows our earlier discussion in subsection 2.3.

But it is clear that there is at least some M_4 for which there are solutions. Indeed, this is true of the most straightforward choice $M_4 = S^4$, because its scalar and two-form spectra are not aligned so as to allow homogeneous spin-2 solutions for any z . On S^4 , the spectra of functions and transverse two-forms are (e.g. [60])

$$\nabla_{S^4}^2 q = -l(l+3)q, \quad \Delta_{S^4} \tau = (l+2)(l+3)\tau \tag{A.7}$$

where l is a non-negative integer. Immediately we see that z will not be rational; indeed, the solution is

$$z = -2 + 4\sqrt{(l+2)(l+3)+1} \tag{A.8}$$

where we took the positive branch. This gives irrational scalar eigenvalues, as

$$(z-1)(z+2) = 4\sqrt{(l+2)(l+3)+1} \left(4\sqrt{(l+2)(l+3)+1} - 3 \right) \tag{A.9}$$

So these are solutions on S^4 , and the solution of the Einstein equation for q will be some linear combination of scalar harmonics.

These solutions are evidently quadratic completions of linearized vector solutions, and are presumably “hiding” somewhere in the spectrum of KK fields on $\text{AdS}_7 \times M_4$. It is perhaps instructive to plug z into the effective massive vector relation $(mL)^2 = z(z+4)$, which gives integer masses

$$m^2 = 3 + 4(l+2)(l+3) \tag{A.10}$$

A.2 A $\text{Lif}_6(z=2)$ solution

We present this solution as a dimensional reduction of a $\text{Sch}_7(z=0)$ solution in the manner first studied by [30, 31], analogous to the Lif_4 and Lif_3 solutions from D3 and M2 branes. All of these, including the present solution, have dynamical exponent $z=2$.

Working once more from the ansatz (A.1), we substitute $h = r^{-1}q$, i.e. $z=0$, to give

$$\begin{aligned} ds^2 &= q(dx^+)^2 + r(2dx^+dx^- + d\mathbf{x}^2) + \frac{dr^2}{r^2} + ds^2(M_4) \\ G &= \star_{11}(3r^2dx^+ \wedge dx^- \wedge d\mathbf{x} \wedge dr) + dx^+ \wedge V \end{aligned} \tag{A.11}$$

The field equations are

$$\begin{aligned} dx^+ \wedge d\star_{X_5}(Vr^3) &= 0 \\ dx^+ \wedge dV &= 0 \\ r^{-6}(\nabla_4^2 q - 2q) &= -|V|_{X_5}^2 \end{aligned} \tag{A.12}$$

where we have retained the differentials dx^+ because x^+ no longer scales, and so the various fields can have x^+ dependence consistent with the field equations and scale invariance.

Consider the gauge field equations first. If we restrict V to live on M_4 and allow functional dependence on x^+ (as in the D3 and M2 brane cases), this will not lead to a solution: V is required by the field equations to be proportional to a harmonic three-form on M_4 , which does not exist assuming that M_4 has at least one continuous isometry.

Instead, we write a more general ansatz for V consistent with symmetry:

$$V = A(x^+)\omega + B(x^+)\frac{dr}{r} \wedge \tau \tag{A.13}$$

where ω and τ are a three-form and two-form on M_4 , respectively. Plugging this into the gauge field equations gives

$$\begin{aligned} dx^+ \wedge dV &= dx^+ \wedge (Ad\omega - B\frac{dr}{r} \wedge d\tau) = 0 \\ dx^+ \wedge d\star_{X_5}(r^3V) &= dx^+ \wedge (Ardr \wedge d\star_4\omega + 2Brdr \wedge \star_4\tau + Br^2d\star_4\tau) = 0 \end{aligned} \tag{A.14}$$

Note that we require $A, B \neq 0$. The solution to these equations is

$$\begin{aligned} \Delta_4\tau &= 0 \\ d\omega &= 0 \\ dx^+ \wedge d\star_4\omega &= dx^+ \wedge \left(-2\frac{B(x^+)}{A(x^+)} \star_4\tau \right) \end{aligned} \tag{A.15}$$

So M_4 must have $b_2 \neq 0$, reminiscent of the fact that analogous solutions for the M2 brane [31] require SE_7 to support harmonic 3-forms.

The Einstein equation for this solution then reads

$$\nabla_4^2 q - 2q = - [A^2 |\omega|_4^2 + B^2 |\tau|_4^2] \tag{A.16}$$

Expanding $|\omega|_4^2$ and $|\tau|_4^2$ in M_4 harmonics, the solution for q is a linear combination of such harmonics with x^+ -dependent coefficients,

$$q = \sum_n C_{i_1 i_2 \dots i_n}(x^+) Y^{i_1 i_2 \dots i_n}(M_4) \tag{A.17}$$

To summarize, the solution is

$$\begin{aligned} ds^2 &= q(dx^+)^2 + r(2dx^+ dx^- + d\mathbf{x}^2) + \frac{dr^2}{r^2} + ds^2(M_4) \\ G &= \star_{11}(3r^2 dx^+ \wedge dx^- \wedge d\mathbf{x} \wedge dr) + dx^+ \wedge V \end{aligned} \tag{A.18}$$

with fields q and V obeying

$$\begin{aligned} V &= A(x^+) \omega + B(x^+) \frac{dr}{r} \wedge \tau, \quad \text{where} \\ \Delta_4 \tau &= 0, \quad d\omega = 0, \quad dx^+ \wedge d \star_4 \omega = dx^+ \wedge \left(-\frac{2B}{A} \star_4 \tau \right) \end{aligned} \tag{A.19}$$

and

$$\nabla_4^2 q - 2q = - [A^2(x^+) |\omega|_4^2 + B^2(x^+) |\tau|_4^2] \tag{A.20}$$

Following [30, 31], we can write the metric as a circle fibration over a $\text{Lif}_6(z=2)$ solution as follows:

$$ds^2 = q \left(dx^+ + \frac{r}{q} dx^- \right)^2 - \frac{r^2}{q} (dx^-)^2 + r(2dx^+ dx^- + d\mathbf{x}^2) + \frac{dr^2}{r^2} + ds^2(M_4) \tag{A.21}$$

Identifying x^- with the time coordinate and x^+ with a compact angle, and restricting to $q \geq 0$, this solution has $\text{Lif}_6(z=2)$ symmetry, with the electric $U(1)$ gauge field given by $A = \frac{r}{q} dx^- = \frac{r}{q} dt$. The fact that $z=2$ is clear upon noting that time scales with twice the power of space under dilatations.

B Details of Sch_3 solutions from D1-D5

B.1 $\text{AdS}_3 \times S^3 \times M_4$ compactification spectrum

The compactification of the $\text{AdS}_3 \times S^3 \times K3$ solution down to three dimensions was done in [56]. The authors begin from the chiral $d=6, \mathcal{N}=(2,0)$ theory and reduce on the sphere. The matter content of this theory is comprised of the graviton supermultiplet — which contains five self-dual two-form tensor fields — coupled to 21 tensor multiplets, each of which contains a single anti-self-dual tensor field. These tensor fields, upon reduction on S^3 , produce the $d=3$ vectors we will use, and descend from the type IIB complex

three-form and five-form fluxes. The nonchiral $\mathcal{N} = (2, 2)$ theory which one gets from replacing K3 by T^4 has an equal number (five) of self-dual and anti-self-dual tensor fields; but since the reduction of bosonic fields on S^3 actually is independent of the number of matter multiplets n_T , we need not specify which $d = 6$ supergravity we work with.

The reduction on S^3 gives rise to five KK towers of vectors: three of these contain mixing between the two-forms and the metric, but two are diagonal. We will refer the reader to [56] for details, and just present the results which we need.

Of the five self-dual two-forms in the $d = 6$ theory, one of these descends from the RR two-form C_2 . In the D1-D5 background, C_2 has components turned on that gives a self-dual field strength in six dimensions, as

$$F_3 = dC_2 = 2(1 + \star_6)d\Omega_3 \tag{B.1}$$

so we can isolate one of the five self-dual two-forms as sourcing the $\text{AdS}_3 \times S^3$ geometry.¹⁴ We borrow the following notation from [56]:

- $B_{\mu\nu}^5$ is the two-form that sources the D1-D5 background, that is, the self-dual descendant of C_2 .
- $B_{\mu\nu}^{\underline{i}}$, where $\underline{i} = 1 \dots 4$, denotes the remainder of the self-dual two-forms.
- $B_{\mu\nu}^r$, where $r = 1 \dots n_T$, denotes the anti-self-dual two-forms

Upon perturbing around this background, there are vector components denoted $b_{\mu a}^5, b_{\mu a}^{\underline{i}}, b_{\mu a}^r$, respectively, where $a = 1, 2, 3$ denotes an S^3 coordinate. There are also vectors from components of the metric perturbations with one leg along S^3 , which are denoted K_μ , and of course spin-2 components from the metric with both legs along AdS_3 .

With this in hand, we delineate the KK tower structure. The three mixed vector towers involve mixing between K_μ and $b_{\mu a}^5$, that is, between the metric and the vector perturbations of the source field. One diagonal vector tower, $b_{\mu a}^{\underline{i}}$, comes from the remaining four self-dual tensor fields; the other, $b_{\mu a}^r$, comes from all of the anti-self-dual tensor fields. There is, of course, also a spin-2 tower of fields.

The only information about these towers which we need to determine the masses is their conformal weights. States of the $D = 1 + 1$ CFT dual to AdS_3 are classified by left- and right-moving conformal weights, denoted (h, \bar{h}) , dual to charges under the $\text{SL}(2, \mathbb{R})_L \times \text{SL}(2, \mathbb{R})_R$ global symmetry of AdS_3 . The relation between these weights and the bulk mass of a p -form field is

$$(mL)^2 = (h + \bar{h} - p)(h + \bar{h} + p - 2) \tag{B.2}$$

The three mixed vector towers have conformal weights

$$(h, \bar{h}) = \left(\frac{l + x \pm 1}{2}, \frac{l + x \mp 1}{2} \right) \tag{B.3}$$

¹⁴S-duality rotations in $D = 10$ rotate the source terms between the RR and NS-NS sectors.

where $x = 1, 3, 5$, and integer $l \geq 0$. This gives rise to masses,¹⁵ and hence values of z ,

$$(mL)^2 = z^2 = (l + x - 1)^2 \tag{B.4}$$

Both diagonal vector towers have

$$(h, \bar{h}) = \left(\frac{l + 3 \pm 1}{2}, \frac{l + 3 \mp 1}{2} \right) \tag{B.5}$$

again with integer $l \geq 0$, and hence

$$(mL)^2 = z^2 = (l + 2)^2 \tag{B.6}$$

Lastly, the spin-2 fields have

$$(h, \bar{h}) = \left(\frac{l + 2 \pm 1}{2}, \frac{l + 2 \mp 1}{2} \right) \tag{B.7}$$

again with integer $l \geq 0$.¹⁶ The spin-2 mass is defined as

$$(mL)^2 = (h + \bar{h})(h + \bar{h} - 2) \tag{B.8}$$

so we expect a spectrum

$$(mL)^2 = (2z - 2)2z = l(l + 2) \tag{B.9}$$

All of these KK states are organized into representations of the global symmetries of the $d = 6$ theory, so n_T determines, in part, the multiplicity of states at a given level. This enters into our Schrödinger constructions as determining the degeneracy of solutions.

We now have a guide to the construction of the $D = 10$ solutions. In particular, we cannot turn on self-dual vector components of C_2 without turning on components of the metric with one leg along the sphere. But we can turn on an anti-self-dual vector component of C_2 by itself, as well as both self-dual and anti-self-dual vector components of B_2 and A_4 , as these comprise all diagonal vectors $b_{\mu a}^i, b_{\mu a}^r$. Furthermore, when we take the one-form on S^3 to be Killing, we expect to reproduce the TsT-transformed D1-D5 system [5], in analogy with the D3-brane construction. We already have evidence for this, because the lowest rung of the diagonal solutions has $z = 2$.

Let us now provide complementary material to the solutions in the main text.

B.2 Diagonal vectors

B.2.1 Anti-self-dual RR two-form charge

In this case, the Maxwell and Bianchi equations reduce to¹⁷

$$dG = G \wedge G = G \wedge \star_{10} G = d \star_{10} G = 0 \tag{B.10}$$

¹⁵This is actually a bit subtle. The $x = 1$ vector branch resides in the spin-2 supermultiplet. The choice $l = 0$ vectors are part of a non-propagating supergravity multiplet. They can be gauged away in the three-dimensional theory, but in formulating our solutions in $D = 10$, we will not see this difference manifest. We treat this solution with caution.

¹⁶We have included the spin-2 field at $l = 0$ which is part of the aforementioned nonpropagating multiplet, so we should beware to discount the resulting Sch₃ solution.

¹⁷In the remainder of the appendix, we refer to the complex three-form $G_3 \equiv G$.

It is easy to show that

$$\star_{10}(iG) = [2(1 + \star_6)d\Omega_3 - (1 - \star_6)dx^+ \wedge d(r^z\beta)] \wedge dVol_4 \quad (\text{B.11})$$

where $dVol_4$ is the invariant volume form of M_4 , so a \star_{10} is essentially equivalent to a \star_6 . Then $G \wedge G = G \wedge \star_{10}G = 0$ by inspection: since the new component and its Hodge dual both have one leg along dx^+ and at least one legs along S^3 , their wedge product with the background components and with each other will vanish.

The equations $dG = d\star_{10}G = 0$ will give us an eigenvalue equation for the one-form β . The anti-self-duality of δF_3 is required for the Einstein equations in the $(+a)$ directions to be trivially satisfied.

This class of solutions includes the TsT-transformed D1-D5 system, in which q is constant, $z = 2$, and $\sigma = \eta$, the Reeb Killing vector on S^3 . From the Einstein equation (4.13), this requires

$$\Lambda(\eta)|_{z=2} = 4|\eta|^2 + |d\eta|^2 = \text{Constant} \quad (\text{B.12})$$

This was explicitly shown to be true for the TsT-transformed D3-brane in [22], where η lived on S^5 instead and was hence $\text{SO}(6)$ -valued. In fact, this statement is true for any Killing vector of $\text{SO}(N)$.

To summarize, the TsT-transformed brane solutions occupy a tidy niche in these constructions: they correspond to turning on the NS-NS two-form with the leg along the compact space given by the Reeb vector of that space. The resulting metrics are direct products.

B.2.2 NS-NS two-form charge

This solution is nearly the same as the previous one.

B.2.3 RR four-form charge

We note that the self-duality ($\star_4\alpha = \alpha$) or anti-self-duality ($\star_4\alpha = -\alpha$) of α with respect to the Euclidean metric on M_4 translates into a concordant statement about the $d = 6$ tensor fields' self-duality property. We calculate

$$\star_{10}((1 \pm \star_6)dx^+ \wedge d(r^z\beta) \wedge \alpha) = (\star_6 \pm 1)(dx^+ \wedge d(r^z\beta) \wedge \star_4\alpha) \quad (\text{B.13})$$

so that

$$F_5 = \frac{(1 \pm \star_6)}{\sqrt{2}}dx^+ \wedge d(r^z\beta) \wedge \alpha + \frac{(\star_6 \pm 1)}{\sqrt{2}}dx^+ \wedge d(r^z\beta) \wedge \star_4\alpha \quad (\text{B.14})$$

If $\star_4\alpha = \alpha$, then we must choose the upper sign — corresponding to self-duality in $d = 6$ — otherwise $F_5 = 0$. If $\star_4\alpha = -\alpha$, then we must choose the lower sign — corresponding to anti-self-duality in $d = 6$ — otherwise $F_5 = 0$. The expression for F_5 becomes

$$F_5 = \sqrt{2}(1 \pm \star_6)dx^+ \wedge d(r^z\beta) \wedge \alpha \quad (\text{B.15})$$

and the origin of the $d = 6$ tensor fields is clear, as is their multiplicity: the number of cycles which α can wrap gives the degeneracy of such fields.

The Maxwell equations and Bianchi identities are

$$\begin{aligned}
 G \wedge \star_{10} G &= dG = 0 \\
 dF_5 &= \frac{i}{2} G \wedge G^* \\
 d \star_{10} G &= -\frac{i}{6} G \wedge F_5
 \end{aligned}
 \tag{B.16}$$

Because we haven't added any G -flux, we have, from the D1-D5 solution,

$$G \wedge G^* = G \wedge \star_{10} G = d \star_{10} G = dG = 0 \tag{B.17}$$

so we must show that

$$G \wedge F_5 = dF_5 = 0 \tag{B.18}$$

The first of these is trivial, based on a previous argument. Imposing $dF_5 = 0$ gives us the eigenvalue equation.

B.3 Mixed vectors

For this subsection, we will ‘zoom out’ from the near-horizon limit of the D1-D5 system and consider the metric of the branes sitting in flat space: that is, we perturb a metric of the form

$$ds^2 = \Phi^{-1}(2dx^+ dx^-) + \Phi ds^2(\mathbb{R}^4) + ds^2(M_4) \tag{B.19}$$

where the D5 brane wraps M_4 with line element $ds^2(M_4)$, and the D1 and D5 intersection along the x^\pm directions sits transverse to \mathbb{R}^4 . This is as we did for the M5 brane. It should be clear that all of the results of the previous subsection for the diagonal vectors could have been found by starting from this metric, and taking $\Phi = r^{-2}$ to be harmonic on \mathbb{R}^4 as dictated by the Maxwell equations. We find it instructive to work in this context here, as it will give us a better guide to what goes wrong when we make our initial ansatz.

Informed by the D3 and M5 examples of mixed vector solutions, one might make an ansatz of the form

$$\begin{aligned}
 ds^2 &= \Phi^{-1}(2dx^+(dx^- + C)) + \Phi ds^2(\mathbb{R}^4) + ds^2(M_4) \\
 iG &= -(1 + \star_6) dx^+ \wedge dx^- \wedge d\Phi^{-1} + (1 + \star_6)(dx^+ \wedge d(\Phi^{-1}C))
 \end{aligned}
 \tag{B.20}$$

We work in the orthonormal frame

$$\begin{aligned}
 e^1 &= dx^+ \\
 e^2 &= 2\Phi^{-1}(dx^- + C) \\
 e^a &= \sqrt{\Phi} dx^a, \quad a = 3 \dots 6 \\
 e^i \cdot e^i &= ds^2(M_4)
 \end{aligned}
 \tag{B.21}$$

so the metric is given as

$$ds^2 = e^1 e^2 + e^a \cdot e^a + e^i \cdot e^i \tag{B.22}$$

The flux ansatz is then

$$iG = (1 + \star_6) \left(\frac{1}{2\Phi} e^1 \wedge e^2 \wedge d\Phi + \frac{1}{\Phi} e^1 \wedge dC \right) \tag{B.23}$$

Both the Bianchi and Maxwell equations give

$$d \star_{\mathbb{R}^4} d\Phi + d \left(\frac{e^1 \wedge \star_{\mathbb{R}^4} dC}{\Phi} \right) = 0 \quad (\text{B.24})$$

If we wish to assign the usual $\Phi = r^{-2}$ so as to see the AdS_3 region, the second term must vanish. But this is not the equation we want; specifically, the factor of Φ precludes the usual progression toward an eigenvalue of co-closed one-forms on S^3 . More to the point, the Einstein equations demand $dC = 0$.

If, however, we add a factor of $1/2$ to the flux ansatz in the orthonormal frame as

$$iG = (1 + \star_6) \left(\frac{1}{2\Phi} e^1 \wedge e^2 \wedge d\Phi + \frac{1}{2\Phi} e^1 \wedge dC \right) \quad (\text{B.25})$$

then this problem is remedied: the new reduced Maxwell and Bianchi equation is

$$d \star_{\mathbb{R}^4} d\Phi + \frac{1}{2} \left[d \left(\frac{e^1 \wedge \star_{\mathbb{R}^4} dC}{\Phi} \right) + d \left(\frac{e^1 \wedge dC}{\Phi} \right) \right] = 0 \quad (\text{B.26})$$

If we demand that the last two terms vanish as a pair, we are led to the condition

$$\star_{\mathbb{R}^4} dC = -dC \quad \Rightarrow \quad d \star_{\mathbb{R}^4} dC = 0 \quad (\text{B.27})$$

That is, dC must be anti-self-dual on \mathbb{R}^4 . The Einstein equations demand nothing more.

Making the usual substitutions

$$\begin{aligned} \Phi &= r^{-2} \\ C &= r^{z-2} \beta \end{aligned} \quad (\text{B.28})$$

we are led to the eigenvalue equation

$$\Delta_3 \beta = (z-2)^2 \beta, \quad \text{where} \quad d \star_3 \beta = 0 \quad (\text{B.29})$$

This gives us Schrödinger solutions for two mixed vector branches, with $z = -l, l+4$.

C Details of Sch_4 solutions from wrapped M5

The field equations of the $d = 4$ theory are rather long; we refer the reader to [40], specifically equations (A.11)–(A.24). In the field variables given earlier, one begins by making the following assignments:

$$\begin{aligned} \beta &= \theta_\alpha = \chi_\alpha = h_3^\alpha = 0 \\ \mathcal{T}_{\alpha\beta} &= \delta_{\alpha\beta} \\ \lambda, \phi &= \text{constant} \\ B_2 &= \gamma \star dC_1 \\ F_2 &\neq 0 \end{aligned} \quad (\text{C.1})$$

where γ is constant. We also impose the null conditions $C_1 \wedge \star C_1 = B_2 \wedge \star B_2 = F_2 \wedge \star F_2 = 0$, as we are looking for a solution to the field equations in which F_2 is also a function of the field strength $B_2 \sim dC_1$.

One finds that $l = -1$ is required for consistency, and that there are exactly two real choices for the values of the pair (ϕ, λ) , namely those given as part of the AdS solutions above. Plugging through the equations, one finds that F_2 is proportional to B_2 and all other Lagrangian parameters (L, g, m^2) are defined as in the above solutions, subject to the extra conditions

$$d \star C_1 = dC_1 \wedge dC_1 = \star dC_1 \wedge \star dC_1 = 0 \tag{C.2}$$

These amount to a gauge choice and a vanishing of instanton terms, respectively; the Schrödinger gauge field satisfies all constraints. So, for the non-supersymmetric choice $e^\phi = 6^{-1/4} 10^{1/5}$, $e^\lambda = 10^{1/10}$ for instance, one ends up with an action

$$\mathcal{L} = \int d^4x \sqrt{-g} \left(R + \frac{9\sqrt{6}g^2}{5} \right) - \frac{1}{2} \int dC_1 \wedge \star dC_1 - \frac{21\sqrt{6}}{25} g^2 \int C_1 \wedge \star C_1 \tag{C.3}$$

and the Schrödinger solution is implicit.

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