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Refined test of AdS_4/CFT_3 correspondence for $\mathcal{N} = 2, 3$ theories

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Abstract: We investigate the superconformal indices for the Chern-Simons-matter theories proposed for M2-branes probing the cones over $N^{010}/\mathbb{Z}_k$, Q^{111}, M^{32} with $\mathcal{N} = 2, 3$ supersymmetries and compare them with the corresponding dual gravity indices. For N^{010}, we find perfect agreements. In addition, for $N^{010}/\mathbb{Z}_k$, we also find an agreement with the gravity index including the contributions from two types of D6-branes wrapping $\mathbb{RP}^3$. For Q^{111}, we find that the model obtained by adding fundamental flavors to the $\mathcal{N}=6$ theory has the right structure to be the correct model. For M^{32}, we find the matching with the gravity index modulo contributions from peculiar saddle points.

Keywords: AdS-CFT Correspondence, Chern-Simons Theories, M-Theory

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1 Introduction

Recent years have seen the tremendous development in our understanding of AdS_4/CFT_3 correspondence. Important inflection point is proposal by J. Schwarz [1] that underlying CFT_3 can be written as Chern-Simons matter (CSM) theories without the usual kinetic term for the gauge fields, especially for higher supersymmetric theories with $N \geq 4$. Once that proposal is realized in specific examples, the understanding of AdS_4/CFT_3 correspondence has grown by leaps and bounds. After the realization that the Bagger-Lambert-Gustavsson theory [2–6] can be written as the usual SU(2) $\times$ SU(2) Chern-Simons matter theory [7], there appeared a paper by Gaiotto and Witten [8] where the attempt is made to write down $N = 4$ Chern-Simons matter theory with matter hypermultiplets. The attempt was generalized in [9] which includes twisted hypermultiplets as well, thereby writing down the general classes of $N = 4$ Chern-Simons matter theories. The special case of such construction is the famous $N = 6$ theory, known as ABJM theory [10, 11], describing coincident M2 branes on C^4/Z_k where $(k, -k)$ is the Chern-Simons level for two gauge groups of ABJM theory.

Since then, various tests and checks are made for $AdS4/CFT3$ correspondence with the most emphasis on ABJM theory. The first check is the matching of the moduli space [10] or chiral operators. Also the partition function on S^3 of ABJM theory is worked out to confirm the famous $N^{\frac{3}{2}}$ behavior of the membrane degrees of freedom [12]. Similar behvior is also observed for other $N = 4$ theories in [13]. There is a huge development on the integrability of the special sectors of AdS4/CFT3 correpondence. See [14] and subsequent reviews.

One of the most sophisticated test so far comes from the computation and comparison of the field theory/gravity index. For ABJM theory, such computation was first worked out for large k limit in [15] and was generalized for arbitrary k in [16] and especially for $k = 1$. The computation was generalized to $N = 4$ theories in [17]. The problem of the states arising in the gravity index from the twisted sector was resolved by [18] and again the field theory index perfectly matches with that of the gravity.

The index computation with the lower supersymmetric theories poses several problems. First of all, given the gravity background of the type $AdS^4 \times SE_7$ where SE_7 denotes a suitable Sasaki-Einstein 7-manifold, it's not clear in general what kind of field theory we have to compare with. There are plethora of examples which lead to the same moduli space in field theory. One might think that such theories are related to each other via dualities a la Seiberg. This could be true of some related theories but there are also counterexamples. In [19], it was shown that two theories, ABJM theory and $N = 3$ variant of the dual ABJM model lead to the differnt indices even though they have the identical moduli space C^4 with Chern-Simons level $k = 1$. Going down to $N = 2$ theories we have far more theories having identical chiral rings. On the other hand in [20], it was shown that the partition function of the theories, which are thought be related by Seiberg-like dualities, is the same. Thus the situation is far more subtle.

Secondly, if we consider general $N = 2$ theories, new subtleties arise since the conformal dimension and the corresponding R-charge of the fields, related by superconformal symmetry, can take nonconventional values in IR. Until recently, it's not clear how to tackle this problem. In [21], Jafferis proposes that the partition function on S^3 for a suitable superconformal field theory as a function of trial R charges is extremized on the actual value of the R-charges of the underlying fields. Along with this proposal, he writes down field theory action on S^3 with arbitrary R charge of the underlying fields with the symmetry $Osp(2|2)$. See also [22]. Similar calculation was applied to an index computation in [23]. Certainly it's worthwhile to apply these proposals in a specific example and to see if this gives rise to the desired check for a given AdS4/CFT3 dual pair.

With the subtleties mentioned above, the goal of the paper is to work out index for the gravity/field theory pair with $N = 2$ or $N = 3$ SUSY. The first case study is done for the proposed theory for M2 branes on N^{010}/Z_k, which has $N = 3$ supersymmetry. The proposed CSM theory is given by ABJM theory with flavors [24–26]. See also [27] for generalizations. Since ABJM theory has the gauge group $\mathrm{U}(N) \times \mathrm{U}(N)$ one can have m_1 flavors for the first factor and m_2 flavors for the 2nd factor. In order to have moduli space N^{010}/Z_k we have the relation $m_1 + m_2 = k$ where $(k, -k)$ are the Chern-Simons level for the gauge groups. In the Type IIA picture this theory is dual to type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ with D6-branes. At $k = 1$, the 'single D6-brane' is geometrized to N^{010}. Happily, we find the perfect matching of field theory/gravity index. Furthemore, the index captures nicely the picture of $AdS_4 \times CP^3$ with D6 branes at general k so that additional contributions to the index can be ascribed to the string modes between different D6 branes. In terms of the index computation, $N = 3$ field theory has less subtleties since we have firm control of F-terms and D-terms. This case study can be contrasted with the another case study of $N = 2$ theory.

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The next case studies concern on specific $N = 2$ theories. As we mentioned, given an M-theory background on $AdS_4 \times SE_7$, there's no known algorithm to tell you which CSM theories we have to consider. At this stage, one should rely on trial and errors so that we will scan through the various theories which give rise to the desired moduli space.

For the case of $C(Q^{111})$, there are many models suggested to describe M2-branes probing this background. We find that the best behaved model to be dual to the gravity on $AdS_4 \times Q^{111}$ is the one obtained by adding four fundamental flavors to the $\mathcal{N} = 6$ Chern-Simons theory [28, 29].[1] During the comparison of the gauge theory and gravity indices, we realized that the gravity spectrum on $AdS_4 \times Q^{111}$ obtained in [31] has been cross-checked in a very limited subsector in the literature. Among all the short OSp(2|4) multiplets listed in [31], we find that the large N field theory index agrees with the gravity index only after getting rid of a few towers of multiplets (which have not been cross-checked in the literature, as far as we are aware of). To conclusively check the validity of this model, one might have to carefully re-examine the results of [31]. One may also recall the model constructed in [32, 33] for Q^{111}. We only made a preliminary study of the index of this model, which has as many complications as our next example M^{32} has. A thorough study has not been made for the models of [32, 33], but it could be that a study similar to our section 4 may reveal a similar structure. At least, we have checked that the index for the latter model disagrees with [31] if we keep all multiplets claimed there.

We also find that in the former model of [28, 29], the trial R-charge does not appear in the large N low energy spectrum (as far as the index can see).

For the case of M^{32} models proposed in [34, 35], we find two clearly distinguished contributions to the index. The first part, coming from a set of saddle points of the localization calculation, does not depend on the trial R-charge and completely reproduces the gravity index on $AdS_4 \times M^{32}$. The second part, coming from the remaining saddle points, does depend on the trial R-charge and does not seem to correspond to any states on the gravity side. This may suggest that this model could not be correctly describing the desired gravity dual. However, see section 4 for the possible subtleties of the calculation and also for the possibility to construct a variant model to cure this discrepancy.

The content of the paper is as follows; After the introduction, in section 2, we work out the index for field theory/gravity pair for N^{010}/Z_k and find perfect matching. In section 3 and 4, we carry out similar computation for Q^{111} and M^{32}, respectively. In conclusion we enumerate the various future directions.

As this work is completed, we received the paper by Imamura et al [36], where similar topic is covered. However the comparison with gravity side is lacking in their paper.

2 The index for M2-branes probing $C(N^{010}/\mathbb{Z}_k)$

2.1 Field theory

The field theory dual proposed for M-theory on $AdS_4 \times N^{010}/\mathbb{Z}_k$ is given by the $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$ Chern-Simons theory coupled to two bifundamental hypermultiplets and m_1, $m_2 =$

[1] S.K. thanks D. Jafferis, I. Klebanov, S. Pufu and B. Safdi for emphasizing the importance of this model, especially related to their studies on the partition function on S^3 [30].

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fields	$\mathrm{U}(N)\times\mathrm{U}(N)$	$I_{\mathrm{SU}(2)_F}$	$\mathrm{U}(1)_B$	$\mathrm{U}(m_1)$	$\mathrm{U}(m_2)$	j_3	ϵ
$A_{1,2}$	$(N,\bar{N})$	$\pm\frac{1}{2}$	1	1	1	0	$\frac{1}{2}$
$B_{1,2}$	$(\bar{N},N)$	$\pm\frac{1}{2}$	-1	1	1	0	$\frac{1}{2}$
q_i	$(N,1)$	0	$\frac{1}{2}$	m_1	1	0	$\frac{1}{2}$
$\tilde{q}_i$	$(\bar{N},1)$	0	$-\frac{1}{2}$	$\bar{m}_1$	1	0	$\frac{1}{2}$
Q_I	$(1,N)$	0	$-\frac{1}{2}$	1	m_2	0	$\frac{1}{2}$
$\tilde{Q}_I$	$(1,\bar{N})$	0	$\frac{1}{2}$	1	$\bar{m}_2$	0	$\frac{1}{2}$

Table 1. Charges of bosonic fields in the CFT dual of N^{010}.

k–m_1 fundamental hypermultiplets in the first and second gauge group, respectively. As we shall review shortly below, the division of k fundamental hypermultiplets into m_1 and m_2 has to do with the presence of two types of D6-branes wrapping $AdS_4\times\mathbb{RP}^3$ in the gravity dual [26], with different $\mathbb{Z}_2$ valued Wilson lines on the worldvolume. The matter fields and their gauge and global charges are listed in table 1.[2] h_0 denotes the Cartan of $\mathrm{SU}(2)_R$ symmetry. In the table, $\mathrm{U}(1)_B$ is the so-called 'bayron-like' U(1) charge, named after similar symmetry of the $\mathcal{N}=6$ theory. This is expected to combine with $\mathrm{SU}(2)_F$ flavor symmetry to provide the enhanced $\mathrm{SU}(3)_F$ at $k=1$. Thus, it should not be confused with the real baryon symmetry, under which M5-branes wrapping topological 5-cycles are charged.

The R-charge of this theory is $\mathrm{SU}(2)_R$, so that its Cartan h_0 to be used to define the BPS sector and index takes the canonical value $\pm\frac{1}{2}$ for all fields.

The superconformal index of this theory can be computed by a procedure similar to [16]. The index is defined by

$$\mathrm{Tr}\left[(-1)^F x^{\epsilon+j_3} y_1^{I_F} y_2^B\right], \tag{2.1}$$

where the trace is taken over the space of local gauge invariant operators, and B denotes the $\mathrm{U}(1)_B$ charge. One may also insert chemical potentials for the $\mathrm{U}(m_1)\times\mathrm{U}(m_2)$ flavor symmetry, after which one would obtain factors of fundamental/anti-fundamental characters in the letter index explained below.

The index is given in terms of the so-called letter indices, which are given by

$$\begin{aligned} f^{\pm} &= \frac{x^{1/2}}{1+x}(y_1^{1/2}+y_1^{-1/2})y_2^{\pm 1} \quad (\text{from } B_a^\dagger, \psi_{A_a+} \text{ or } A_a^\dagger, \psi_{B_a+}) \\ f_1^{\pm} &= \frac{x^{1/2}}{1+x}y_2^{\pm 1/2}, \quad f_2^{\pm} = \frac{x^{1/2}}{1+x}y_2^{\mp 1/2}. \end{aligned} \tag{2.2}$$

The index is given by

$$I(x,y_1,y_2) = \sum_{\{n_i\},\{\tilde{n}_i\}} I_{\{n_i\},\{\tilde{n}_i\}}(x,y_1,y_2), \tag{2.3}$$

[2] In [24, 36], the definition of $\mathrm{U}(1)_B$ is different from ours by suitably mixing with diagonal $\mathrm{U}(1)\subset\mathrm{U}(N)\times\mathrm{U}(N)$ and $\mathrm{U}(1)^2\subset\mathrm{U}(m_1)\times\mathrm{U}(m_2)$ flavor symmetries. Our $\mathrm{U}(1)_B$ is the field theory dual of the KK momentum along the M-theory circle, which is an integer multiple of k.

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where

$$
\begin{aligned}
&I_{\{n_i\},\{\tilde{n}_i\}}(x,y_1,y_2)\\
&= x^{\epsilon_0}\int \frac{1}{(\text{symmetry})}\left[\frac{d\alpha_i d\tilde{\alpha}_i}{(2\pi)^2}\right] e^{ik\sum_{i=1}^N (n_i\alpha_i-\tilde{n}_i\tilde{\alpha}_i)}\\
&\quad\times\exp\left[-\sum_{i\neq j}\sum_{p=1}^{\infty}\frac{1}{p}(x^{p|n_i-n_j|}e^{-ip(\alpha_i-\alpha_j)}+x^{p|\tilde{n}_i-\tilde{n}_j|}e^{-ip(\tilde{\alpha}_i-\tilde{\alpha}_j)})\right]\\
&\quad\times\exp\left[\sum_{i,j=1}^{N}\sum_{p=1}^{\infty}\frac{1}{p}(f^+(x^p,y_1^p,y_2^p)x^{p|n_i-\tilde{n}_j|}e^{ip(\tilde{\alpha}_j-\alpha_i)}+f^-(x^p,y_1^p,y_2^p)x^{p|n_i-\tilde{n}_j|}e^{-ip(\tilde{\alpha}_j-\alpha_i)})\right]\\
&\quad\times\exp\left[\sum_{i=1}^{N}\sum_{p=1}^{\infty}\frac{1}{p}(m_1 f_1^+(x^p,y_1^p,y_2^p)x^{p|n_i|}e^{-ip\alpha_i}+m_1 f_1^-(x^p,y_1^p,y_2^p)x^{p|n_i|}e^{ip\alpha_i})\right.\\
&\qquad\left.+(m_1,\alpha,n,f_1^{\pm}\to m_2,\tilde{\alpha},\tilde{n},f_2^{\pm})\right].
\end{aligned}
\tag{2.4}
$$

Here the summation is over all integral magnetic monopole charges, $\{n_i,\tilde{n}_i\}$. The zero point energy ϵ_0 is given by

$$
\epsilon_0=\sum_{i,j=1}^{N}|n_i-\tilde{n}_j|-\sum_{i<j}|n_i-n_j|-\sum_{i<j}|\tilde{n}_i-\tilde{n}_j|+\frac{m_1}{2}\sum_{i=1}^{N}|n_i|+\frac{m_2}{2}\sum_{i=1}^{N}|\tilde{n}_i|. \tag{2.5}
$$

Symmetry factor in the expression is the dimension of the Weyl group for the gauge group unbroken by the magnetic flux [16].

In the large N limit, the integral over the holonomies $\alpha,\tilde{\alpha}$ which do not host magnetic flux can be done by Gaussian approximation, introducing the distribution functions $\rho(\alpha)=\sum_{n=-\infty}^{\infty}\rho_n e^{-in\alpha}$ and $\chi(\alpha)$. After performing the integral, similar to [16], the index takes the form of

$$
I_{N=\infty}(x,y_1,y_2)=I^{(0)}(x,y_1)I'(x,y_1,y_2),
$$

where

$$
\begin{aligned}
&I^{(0)}(x,y_1)\\
&=\prod_{n=1}^{\infty}\frac{1}{1-f^+(\cdot^n)f^-(\cdot^n)}\times\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\frac{\left(m_1^2+m_2^2+2m_1m_2\frac{x^{n/2}}{1+x^n}(y_1^{n/2}+y_1^{-n/2})\right)}{1-f^+f^-(\cdot^n)}\left(\frac{x^{n/2}}{1+x^n}\right)^2\right]\\
&=\prod_{n=1}^{\infty}\frac{(1-x^{2n})^2}{(1-x^n y_1^n)(1-x^n y_1^{-n})(1-x^n)^2}\\
&\quad\times\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\frac{\left(m_1^2+m_2^2+2m_1m_2\frac{x^{n/2}}{1+x^n}(y_1^{n/2}+y_1^{-n/2})\right)}{1-f^+f^-(\cdot^n)}\left(\frac{x^{n/2}}{1+x^n}\right)^2\right]\\
&\equiv\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}I^{(0)}_{\text{single}}(x^n,y_1^n)\right],
\end{aligned}
\tag{2.6}
$$

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with

$$I^{(0)}_{\text{single}}(x,y_1) = \frac{x}{y_1-x}+\frac{xy_1}{1-xy_1}+\frac{2x}{1-x}-\frac{2x^2}{1-x^2} + (m_1^2+m_2^2)\frac{x}{(1-xy_1)\left(1-xy_1^{-1}\right)}+2m_1m_2\frac{x^{3/2}(y_1^{1/2}+y_1^{-1/2})}{(1+x)(1-xy_1)\left(1-xy_1^{-1}\right)}\,. \tag{2.7}$$

This part does not refer to magnetic monopole flux, and thus only contains states which are $\mathrm{U}(1)_B$ neutral. The remaining part I' is given by

$$I'(x,y_1,y_2) = x^{\epsilon_0}\int \frac{1}{(\text{symmetry})}\left[\frac{d\alpha}{2\pi}\right]\left[\frac{d\tilde{\alpha}}{2\pi}\right]e^{ik\sum(n_i\alpha_i-\tilde{n}_i\tilde{\alpha}_i)} \times \exp\left[\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\frac{1}{n}f_{ij}^{\text{bif}}(\cdot^n)+\sum_{i,j=1}^{M_1}\frac{1}{n}f_{ij}^{\text{adj}}(\cdot^n)+\sum_{i,j=1}^{M_2}\frac{1}{n}\tilde{f}_{ij}^{\text{adj}}(\cdot^n)\right] \tag{2.8}$$

is an integral over the holonomies associated with nonzero flux, with

$$\begin{aligned} f_{ij}^{\text{bif}} &= (x^{|n_i-\tilde{n}_j|}-x^{|n_i|+|\tilde{n}_j|})(f^+e^{i(\tilde{\alpha}_j-\alpha_i)}+f^-e^{i(\alpha_i-\tilde{\alpha}_j)}),\\ f_{ij}^{\text{adj}} &= -\left[(1-\delta_{ij})x^{|n_i-n_j|}-x^{|n_i|+|n_j|}\right]e^{-i(\alpha_i-\alpha_j)},\\ \tilde{f}_{ij}^{\text{adj}} &= -\left[(1-\delta_{ij})x^{|\tilde{n}_i-\tilde{n}_j|}-x^{|\tilde{n}_i|+|\tilde{n}_j|}\right]e^{-i(\tilde{\alpha}_i-\tilde{\alpha}_j)}. \end{aligned} \tag{2.9}$$

M_1 and M_2 are number of nonzero fluxes. Like [16], $I'(x,y_1,y_2)$ can be factorized and yields

$$I_{N=\infty}(x,y_1,y_2) = I^{(0)}(x,y_1)I^{(+)}(x,y_1,y_2)I^{(-)}(x,y_1,y_2). \tag{2.10}$$

$I^{\pm}$ can be computed using the formula for I' in (2.8) with all monopole charges are positive/negative. $I^{\pm}$ can be expanded in positive/negative powers of y_2, respectively.

Let us leave some remarks on the structure of the large N index. Firstly, the 'single particle index' $I^{(0)}_{\text{single}}$ in (2.7) consists of two parts. As will be explained later, the first line is identical to the single graviton index on $AdS_4\times\mathbb{CP}^3$. The second line will be shown to be the single particle index of the open string degrees of freedom living on m_1 and m_2 D6-branes with different $\mathbb{Z}_2$ Wilson lines, wrapping $AdS_4\times\mathbb{RP}^3$ in $AdS_4\times\mathbb{CP}^3$. See the next subsection for more explanations.

Secondly, in the index $I^{\pm}$ with monopoles, the fundamental degrees of freedom encoded by the indices $f_1^{\pm}$ and $f_2^{\pm}$ totally disappears, and the only trace of their existence is in the zero point energy ϵ_0. The gravity dual interpretation of this phenomena would be the absence of open string modes between D0-D6 branes of type IIA theory on $AdS_4\times\mathbb{CP}^3$. Also, apart from the last zero point energy factor $x^{\frac{1}{2}\sum(m_1|n_i|+m_2|\tilde{n}_i|)}$, the integral is exactly the same as the I^+ of $\mathcal{N}=6$ Chern-Simons index at large N. The comparison of I^+ for the $\mathcal{N}=6$ Chern-Simons theory and gravity on $AdS_4\times S^7$ is studied in detail in [16]. So all analytic and numerical results there could be used in our context to show the agreement of the index of this section. In the next subsection, we shall prove that the the agreement of the gauge-gravity large N indices for the $\mathcal{N}=6$ theory directly implies the agreement of the large N index of our system for $k=1$.

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	j_3	ϵ	h_0	$\epsilon+j_3$	(M_1, M_2)
$p \geq 0$	$\frac{3}{2}$	$p+\frac{5}{2}$	$p+1$	$p+4$	(p,p)
$p \geq 0$	1	$p+2$	$p+1$	$p+3$	(p,p)
$p \geq 0$	1	$p+3$	$p+2$	$p+4$	$(p,p+3),(p+3,p)$
$p \geq 0$	$\frac{1}{2}$	$p+\frac{5}{2}$	$p+2$	$p+3$	$(p,p+3),(p+3,p)$
$p \geq 1$	$\frac{1}{2}$	$p+\frac{1}{2}$	p	$p+1$	(p,p)
$p \geq 1$	0	p	p	p	(p,p)
$\cdot$	$\frac{1}{2}$	$\frac{3}{2}$	1	2	$(0,0)$
$\cdot$	0	1	1	1	$(0,0)$

Table 2. Fields saturating the BPS bound for N^{010}.

From the absence of these fundamental degrees of freedom, the integrand in $I^{\pm}$ is invariant under the common shift of the holonomies $\alpha, \tilde{\alpha}$, implying that it acquires contribution only from monopoles satisfying

$$\sum n_i = \sum \tilde{n}_i \ . \tag{2.11}$$

This decoupling of the diagonal U(1) in $\mathrm{U}(N) \times \mathrm{U}(N)$ was an exact property in the $\mathcal{N}=6$ Chern-Simons-matter theory, while here it is true only in the large N limit.

2.2 Gravity

The complete Kaluza-Klein spectrum on $AdS_4 \times N^{010}$ was obtained in [37]. To calculate the index, it suffices to consider the fields in the short multiplets.[3] There exists a tower of short or massless graviton multiplets of $\mathrm{OSp}(3|4) \times \mathrm{SU}(3)$ with $M_1 = M_2 = p$, $J_0 = p$ and $p \geq 0$, where the lowest multiplet with $p=0$ is the massless multiplet. M_1, M_2 denote the two parameters of the SU(3) representation as used in [37], and J_0 denotes the $\mathrm{SU}(2)_R$ Casimir of the primary. There also exists a tower of short gravitino multiplets, with $M_1 = p$, $M_2 = p+3$, $J_0 = p+1$ and $p \geq 0$, all being massive. As these multiplets are in complex representations of SU(3) with $M_1 \neq M_2$, the corresponding 4 dimensional field is complex. Therefore, when calculating the index below, we should include the contribution from the conjugate modes with $M_1 = p+3$, $M_2 = p$. Also, a tower of short or massless vector multiplets comes with $M_1 = M_2 = p$, $J_0 = p$ and $p \geq 1$, where the multiplet with $p = 1$ is massless and corresponds to the gauge fields for the SU(3) isometry. Finally, there appears another massless vector multiplet with $M_1 = M_2 = 0$, $J_0 = 1$ which corresponds to the baryonic U(1) symmetry, under which the M5-brane wrapped on a 5-cycle in N^{010} is charged. From these multiplets, the fields (or representations of the conformal group) which satisfy the BPS relation $\epsilon = h_0 + j_3$ are listed in table 2.

[3] By 'fields' we mean fields in AdS_4. This is obtained by decomposing OSp(3|4) representations into the representations of the conformal group, where each conformal representation comes from a 4 dimensional field.

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Identifying SU(3) Cartans with the $SU(2)_F$ Cartan and $U(1)_B$ from theory theory as [25]

$$h_1 = \text{diag}\left(\frac{1}{2}, -\frac{1}{2}, 0\right) , \quad h_2 = \text{diag}\left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right) , \tag{2.12}$$

the single particle index from gravity is given by

$$\begin{aligned} I_{\text{sp}}(x, y_1, y_2) = & \frac{1}{1-x^2}\Bigg[\sum_{l=0}^{\infty}(x^{l+3} - x^{l+4})\chi^{(l,l)}_{\text{SU(3)}}(y_1, y_2) \\ & + \sum_{l=0}^{\infty}(x^{l+4} - x^{l+3})\big(\chi^{(l,l+3)}_{\text{SU(3)}}(y_1, y_2) + \chi^{(l+3,l)}_{\text{SU(3)}}(y_1, y_2)\big) \\ & + \sum_{l=1}^{\infty}(x^{l} - x^{l+1})\chi^{(l,l)}_{\text{SU(3)}}(y_1, y_2) + (x - x^2)\Bigg], \\ = & \frac{1}{1+x}\Bigg[x + \sum_{l=0}^{\infty} x^{l+3}\big(\chi^{(l,l)}_{\text{SU(3)}}(y_1, y_2) - \chi^{(l,l+3)}_{\text{SU(3)}}(y_1, y_2) - \chi^{(l+3,l)}_{\text{SU(3)}}(y_1, y_2)\big) \\ & + \sum_{l=1}^{\infty} x^{l}\chi^{(l,l)}_{\text{SU(3)}}(y_1, y_2)\Bigg]. \end{aligned} \tag{2.13}$$

The SU(3) character $\chi^{M_1,M_2}_{\text{SU(3)}}(y_1, y_2)$ for an irreducible representation is given by

$$\begin{aligned} \chi^{(M_1,M_2)}_{\text{SU(3)}}(y_1, y_2) \equiv & \,\text{tr}_{(M_1,M_2)} y_1^{h_1} y_2^{h_2} = \frac{(\text{numerator})}{(\text{denominator})}, \\ (\text{numerator}) = & \, y_1^{-\frac{1}{2}(M_1+M_2)} y_2^{-\frac{1}{6}(2M_1+M_2)}\big[y_1^{\frac{1}{2}(M_1+1)} - y_1^{\frac{1}{2}(M_1+3)+M_2} - y_2^{\frac{1}{2}(M_1+1)} \\ & + y_1^{(M_1+M_2+2)} y_2^{\frac{1}{2}(M_1+1)} + y_1^{\frac{1}{2}(M_2+1)} y_2^{\frac{1}{2}(M_1+M_2+2)} \\ & - y_1^{\frac{1}{2}(2M_1+M_2+3)} y_2^{\frac{1}{2}(M_1+M_2+2)}\big], \\ (\text{denominator}) = & \, (y_1 - 1)\big[y_2^{\frac{1}{2}} + y_1 y_2^{\frac{1}{2}} - y_1^{\frac{1}{2}}(y_2 + 1)\big]. \end{aligned} \tag{2.14}$$

From this single particle index, the full index over the gravity states is given by

$$I(x, y_1, y_2) = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n} I_{\text{sp}}(x^n, y_1^n, y_2^n)\right] . \tag{2.15}$$

Below, we compare the field theory and gravity indices when $k=1$. The case with $k\neq 1$ is studied in the next subsection, with D6-brane contributions taken into account.

For the field theory side at $k=1$, one either sets $(m_1, m_2) = (1, 0)$ or $(m_1, m_2) = (0, 1)$ to study the field theory dual of N^{010} (rather than other tri-Sasakian spaces). In the 'D6-brane' picture (although one needs the full M-theory at $k=1$), these two cases correspond to having different $\mathbb{Z}_2$ valued Wilson lines on the worldvolume of D6-brane. Geometrically uplifting D6-branes to N^{010}, this should lift to some kind of $\mathbb{Z}_2$ valued holonomy of the 3-form potential of M-theory. As such holonomy does not affect the spectrum of gravity fields, the two field theories would give identical large N spectrum that we consider in this

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section. This was what we encountered for the field theory in the previous subsection. We thus consider the case $(m_1, m_2) = (1, 0)$ for definiteness.

As the field theory index is factorized as $I^{(0)} I^+ I^-$ where the three factors are neutral or positively/negatively charged in h_2 $\mathrm{U}(1)_B$ charge, we can make the same decomposition of $I_{\rm sp}(x, y_1, y_2)$ on the gravity side and compare the three factors separately. The single particle index in the neutral sector is given by

$$\begin{aligned} I^{(0)}_{\rm sp}(x, y_1) &= \frac{1}{2\pi i} \oint \frac{dy_2}{y_2} I_{\rm sp}(x, y_1, y_2^2) \\ &= \frac{x}{y_1 - x} + \frac{x y_1}{1 - x y_1} + \frac{2x}{1 - x} - \frac{2x^2}{1 - x^2} + \frac{x y_1}{(y_1 - x)(1 - x y_1)}. \end{aligned} \tag{2.16}$$

The sum of first four terms is the same as the single particle index on $AdS_4 \times \mathbb{CP}^3$. The last term may be interpreted as a contribution from single 'D6-brane' wrapping $\mathbb{RP}^3 \subset \mathbb{CP}^3$, as explained in more detail in the next subsection. The neutral sector 'single particle index' $I^{(0)}_{\rm single}$ of (2.7) from field theory completely agrees with $I^{(0)}_{\rm sp}$ of (2.16) as one inserts $m_1 = 1$, $m_2 = 0$. This shows the agreement in the neutral sector.

To study the sector with positive h_2, one generally has to rely on numerical analysis like [16], as we do not know how to calculate holonomy integral (2.8) analytically. As we mentioned in the previous subsection, the field theory I^+ is almost identical for our $\mathcal{N} = 3$ theory and the $\mathcal{N} = 6$ theory so that all results (analytic or numerical) known for the latter case can be borrowed to study the former.

For instance, in the simplest sector with minimal positive value $h_2 = \frac{1}{2}$ which was treated analytically in [16], the corresponding gravity index in this sector comes from one particle states with $h_2 = \frac{1}{2}$,

$$\begin{aligned} I^{(1/2)}_{\rm sp}(x, y_1) &= \oint \frac{d\sqrt{y_2}}{2\pi i \sqrt{y_2}} y_2^{-1/2} I_{\rm sp}(x, y_1, y_2) \\ &= -\frac{x(1 + y_1)\left(-x^2 + y_1 - x^2 y_1 + 2x^3 y_1 - x^2 y_1^2\right)}{(-1 + x)(1 + x)(x - y_1)\sqrt{y_1}(-1 + x y_1)}, \end{aligned} \tag{2.17}$$

which is the $\mathcal{O}(y_2^{1/2})$ contribution to $I^+(x, y_1, y_2)$. The corresponding field theory index comes with nonzero fluxes $n_1 = \tilde{n}_1 = 1$ and all other fluxes being zero. Here one obtains from (2.8)

$$\begin{aligned} I^+_{(1)(1)} &= x^{k/2} \int_0^{2\pi} \frac{d\alpha d\tilde{\alpha}}{(2\pi)^2} e^{ik(\alpha - \tilde{\alpha})} \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \left((1 - x^{2n}) \left(f^+(\cdot^n) e^{in(\tilde{\alpha} - \alpha)} + f^-(\cdot^n) e^{in(\alpha - \tilde{\alpha})} \right) + 2x^{2n} \right) \right] \\ &= x^{k/2} \oint \frac{dz}{2\pi i z} z^{-k} \frac{\left(1 - \frac{x^{3/2}}{z\sqrt{y_1}}\right)\left(1 - \frac{x^{3/2} z}{\sqrt{y_1}}\right)\left(1 - \frac{x^{3/2}\sqrt{y_1}}{z}\right)\left(1 - x^{3/2} z \sqrt{y_1}\right)}{(1 - x^2)^2 \left(1 - \frac{\sqrt{x}}{z\sqrt{y_1}}\right)\left(1 - \frac{\sqrt{x} z}{\sqrt{y_1}}\right)\left(1 - \frac{\sqrt{x}\sqrt{y_1}}{z}\right)\left(1 - \sqrt{x} z \sqrt{y_1}\right)}. \end{aligned} \tag{2.18}$$

As emphasized, this is the same as the large N $\mathcal{N} = 6$ index apart from the extra factor $x^{k/2}$. At $k = 1$, the contour integral can be performed to yield

$$I^+_{(1)(1)} \Big|_{k=1} = -\frac{x(1 + y_1)\left(-x^2 + y_1 - x^2 y_1 + 2x^3 y_1 - x^2 y_1^2\right)}{(-1 + x)(1 + x)(x - y_1)\sqrt{y_1}(-1 + x y_1)}, \tag{2.19}$$

in perfect agreement with (2.17).

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Pushing this sort of analysis further, we assume the agreement between the gauge/gravity $\mathcal{N}=6$ indices at large N (shown in [16] up to three monopoles in various sectors) and show that this directly implies the agreement of the large N indices here. Firstly, in the positive flux sector, the $\mathcal{N}=6$ index $I^+_{ABJM}(x,y_1,y_2,y_3)$ is related to our I^+ by

$$I^+(x,y_1,y_2) = I^+_{ABJM}(x,y_1,y_2=1,y_3=y_2x)\ . \tag{2.20}$$

y_1, y_2 are chemical potentials of $\mathrm{SU}(2)\times\mathrm{SU}(2)\subset\mathrm{SO}(6)$ global symmetry of the latter system in the notation of [16]. As only the former SU(2) is the symmetry in our case, we set $y_2=1$. Also, as the charge conjugate to y_3 in the latter case is just given by $\frac{1}{2}\sum_i n_i$ from positive fluxes, the replacement $y_3\to y_2x$ yields the desired extra factor $x^{\frac{1}{2}\sum n_i}$ of zero point energy. Now the agreement between $\mathcal{N}=6$ large N gauge/gravity indices (which we accept) means $I^+_{ABJM}=\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}I^+_{\rm sp}(\cdot^n)_{S^7}\right]$, while we would like to show

$$I^+ = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}I^+_{\rm sp}(\cdot^n)_{N^{010}}\right]\ . \tag{2.21}$$

Here, $(I^+_{\rm sp})_{S^7}$ and $(I^+_{\rm sp})_{N^{010}}$ denote single particle gravity indices in appropriate backgrounds. Thus, from (2.20), it suffices for us to show

$$I^+_{\rm sp}(x,y_1,y_2)_{N^{010}} = I^+_{\rm sp}(x,y_1,y_2\to 1,y_3\to y_2x)_{S^7}. \tag{2.22}$$

Using computer, one can explicitly show to all order that

$$\begin{aligned} I^+_{\rm sp}(x,y_1,y_2)_{N^{010}} &= \sum_{m=1}^{\infty} I^{(m/2)}_{\rm sp}(x,y_1)_{N^{010}}y_2^{m/2} \\ &= \sum_{m=1}^{\infty}\left([Res_{y_2\to x\sqrt{y_1}}+Res_{y_2\to x/\sqrt{y_1}}]\frac{I_{\rm sp}(x,y_1,y_2^2)_{N^{010}}y_2^m}{y_2}\right)y_2^{m/2}, \\ &= \sum_{m=1}^{\infty}\left(\frac{(x^2y_1y_2)^{m/2}(x-y_1-x^3y_1+x^2y_1^2)}{(1-y_1)(1-x^2)(1-xy_1)}+(y_1\to y_1^{-1})\right). \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} I^{(+)}_{\rm sp}(x,y_1,1,y_3)_{S^7} &= \sum_{m=1}^{\infty} I^{(m/2)}_{\rm sp}(x,y_1,1)_{S^7}y_3^{m/2} \\ &= \sum_{m=1}^{\infty}\left([Res_{y_3\to\sqrt{xy_1}}+Res_{y_2\to\sqrt{x/y_1}}]\frac{I_{\rm sp}(x,y_1,1,y_3^2)_{S^7}y_3^m}{y_3}\right)y_3^{m/2}, \\ &= \sum_{m=1}^{\infty}\left(\frac{(xy_1y_3)^{m/2}(x-y_1-x^3y_1+x^2y_1^2)}{(1-y_1)(1-x^2)(1-xy_1)}+(y_1\to y_1^{-1})\right). \end{aligned} \tag{2.24}$$

The apparent $y_1\to 1$ singularity in each term cancels with a pair term with $y_1\to y_1^{-1}$ replacement. From these, (2.22) is obvious, which proves the agreement between the gauge/gravity indices for N^{010}.

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2.3 D6-branes

When $k \neq 1$, $N^{010}/\mathbb{Z}_k$ is singular so that there appear contributions beyond the supergravity approximation, coming from light degrees of freedom localized on the fixed points of $\mathbb{Z}_k$. Similar analysis was done for $\mathcal{N}=4$ M2-brane models in [17, 18].

In our case, one can use the type IIA D6-brane picture of the field theory model when $k \gg 1$.[4] In this picture, one starts from the $\mathcal{N}=6$ Chern-Simions-matter theory for M2-branes and add fundamental flavors by adding D6-branes wrapping $AdS_4 \times \mathbb{RP}^3 \subset AdS_4 \times \mathbb{CP}^3$ [24–26]. The type IIA index from gravity can then be computed by adding the single particle index from the bulk modes in $AdS_4 \times \mathbb{CP}^3$ and the modes on the D6-brane worldvolumes. As $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$, there appears two types of D6-branes with different values of discrete Wilson line on the worldvolume [26]. As mentioned in the previous subsection, the single particle index on $AdS_4 \times \mathbb{CP}^3$ is given by [15]

$$I_{\rm sp}^{\mathbb{CP}^3}(x, y_1) = \frac{x}{y_1 - x} + \frac{xy_1}{1 - xy_1} + \frac{2x}{1 - x} - \frac{2x^2}{1 - x^2} \tag{2.25}$$

where y_1 is the chemical potential for the Cartan of $\mathrm{SU}(2)_F \subset \mathrm{SU}(4)$ unbroken by the D6-branes.

The open string degrees of freedom on D6-branes are described by a 7 dimensional supersymmetric Yang-Mills theory on $AdS_4 \times \mathbb{RP}^3$. The quadratic part of this action on $AdS_4 \times S^3$ was obtained in [18], which in our case can be derived from the DBI action with Wess-Zumino term, as studied in [26]. Let m_1 and m_2 be the number of D6-branes which support two possible $\mathbb{Z}_2$ valued Wilson lines. These numbers are identified with the number of two fundamental hypermultiplets in the field theory. The fields in a vector multiplet are: 7 dimensional gauge field A_μ, three real scalars ϕ_i $(i=1,2,3)$, gauginos λ. The symmetry of this worldvolume theory is: $\mathrm{U}(m_1) \times \mathrm{U}(m_2)$ gauge symmetry, $\mathrm{SO}(4) \equiv \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$ isometry on S^3 or $\mathbb{RP}^3$, $\mathrm{SO}(3) \sim \mathrm{SU}(2)_3$ symmetry transverse to the D6-branes.[5] Viewing the four bi-fundamental scalars $A_1, B_1^\dagger, A_2, B_2^\dagger$ as spanning the $\mathbb{C}^4$ space (at least for $N=1$), the embedding condition for $\mathbb{RP}^3$ in $\mathbb{CP}^3$ should be specifying $\mathbb{C}^2 = \mathbb{R}^4 \subset \mathbb{C}^4$ embedding by taking one combination of the two hypermultiplets to zero, as the D6-brane embedding should be compatible with $\mathrm{SU}(2)_R$ R-symmetry. Therefore, one of the two hypermultiplets span transverse directions, acted by $\mathrm{SU}(2)_3$, while another one is acted by $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$. $\mathrm{SU}(2)_{1,2,3}$ are symmetries of the low energy theory on D6-branes, and some of them are broken in the whole theory. In particular, the diagonal combination of $\mathrm{SU}(1)_1$ and $\mathrm{SU}(2)_3$ is to be identified with our $\mathrm{SU}(2)_R$ symmetry, while $\mathrm{SU}(2)_2$ is our $\mathrm{SU}(2)_F$ symmetry.

The Kaluza-Klein spectrum of the 7 dimensional modes on $\mathbb{RP}^3$ or S^3 are summarized in table 3, following [18]. For S^3, the number s is half an integer. For $\mathbb{RP}^3$, s is an integer by the $\mathbb{Z}_2$ projection when the open strings connect D6-branes with same Wilson line. There are $m_1^2 + m_2^2$ such vector multiplets. On the other hand, s is half an odd integer when the open strings connect D6-branes of different Wilson lines, as there are half-integral shifts of the spectrum due to their coupling to nonzero Wilson lines [26]. The modes saturating the

[4] This $D6$-brane picture turns out to be appropriate for studying the index even for small k (including $k=1$), presumably due to simplifications coming from supersymmetry.

[5] In [18], the SU(2) symmetries are named $\mathrm{SU}(2)_{1,2,3} = \mathrm{SU}(2)'_R, \mathrm{SU}(2)'_F, \mathrm{SU}(2)_R$, respectively.

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fields	$SU(2)_j$	$SU(2)_3$	$SU(2)_1$	$SU(2)_2$	ϵ
ϕ_i	0	1	s	s	$s+2$
A_μ	0	0	$s+1$	s	$s+1$
	1	0	s	s	$s+2$
	0	0	$s-1$	s	$s+3$
λ	$\frac{1}{2}$	$\frac{1}{2}$	$s+\frac{1}{2}$	s	$s+\frac{3}{2}$
	$\frac{1}{2}$	$\frac{1}{2}$	$s-\frac{1}{2}$	s	$s+\frac{3}{2}$

Table 3. Spectrum of 7 dimensional vector supermultiplets on $AdS_4 \times S^3$ or $\mathbb{RP}^3$.

BPS bound come from the second and fifth lines of table 3. The single particle index (or to be more precise, the mode index) from the open strings is

$$\begin{aligned} I_{\rm sp}^{D6} &= \frac{m_1^2+m_2^2}{1-x^2}\sum_{s=0}^{\infty}\chi_s^{SU(2)_F}(y_1)\left[x^{s+1}-x^{s+2}\right]+\frac{2m_1m_2}{1-x^2}\sum_{s=\frac{1}{2}}^{\infty}\chi_s^{SU(2)_F}(y_1)\left[x^{s+1}-x^{s+2}\right] \\ &= \frac{(m_1^2+m_2^2)x}{(1-xy_1)(1-xy_1^{-1})}+\frac{2m_1m_2x^{3/2}(y_1^{1/2}+y_1^{-1/2})}{(1-xy_1)(1-xy_1^{-1})}\,, \end{aligned} \tag{2.26}$$

where $\chi_s^{SU(2)_F}(y_1)=\frac{y_1^{(s+1)/2}-y_1^{-(s+1)/2}}{y_1^{1/2}-y_1^{-1/2}}$. From this, one finds that $I_{\rm sp}^{\mathbb{CP}^3}+I_{\rm sp}^{D6}$ perfectly agrees with $I_{\rm single}^{(0)}$ in (2.7). In particular, extending the above result to $k=1$, we confirm that the $U(1)_B$ neutral part (2.16) of the index on N^{010} is the gravity index on $AdS_4\times\mathbb{CP}^3$ with one D6-brane.

3 The index for M2-branes probing $C(Q^{111})$

3.1 Gravity

The gravity spectrum on $AdS_4\times Q^{111}$ has been analyzed in [31]. The index will acquire contribution only from the short multiplets. In [31], 15 infinite towers of short multiplets (1 tower of short graviton, 9 towers of short gravitino, 4 towers of short vector multiplets and 1 tower of hypermultiplet) plus 6 massless multiplets (1 massless graviton, 5 massless vectors) belong to this category.

As far as we are aware of, only the massless multiplets as well as some low-lying entries in the hypermultiplet tower (containing states dual to chiral ring operators) have been studied or cross-checked in the literature. From the anlaysis of the field theory dual of [28] proposed for the Q^{111}, it still turns out that the large N field theory index agrees with the gravity multiplets after discarding 6 of the 9 proposed towers of gravitino multiplets in [31], and 1 of the 4 proposed towers of short vector multiplets with $SU(2)^3$ charge $(k/2,k/2,k/2)$. It could be that one might have to carefully re-examine the gravity spectrum of [31], which is beyond the scope of this work.

In table 4, we present the set of gravity multiplets which will be compared to the field theory results in the next subsection.

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	$\epsilon+j_3$	j_3	$\mathrm{SU}(2)_1$	$\mathrm{SU}(2)_2$	$\mathrm{SU}(2)_3$
$k\geq 1$	$k+4$	$3/2$	$k/2$	$k/2$	$k/2$
$k\geq 0$	$k+4$	1	$k/2$	$k/2+1$	$k/2+1$
	$k+4$	1	$k/2+1$	$k/2$	$k/2+1$
	$k+4$	1	$k/2+1$	$k/2+1$	$k/2$
$k\geq 1$	$k+2$	$1/2$	$k/2+1$	$k/2$	$k/2$
	$k+2$	$1/2$	$k/2$	$k/2+1$	$k/2$
	$k+2$	$1/2$	$k/2$	$k/2$	$k/2+1$
$k\geq 1$	k	0	$k/2$	$k/2$	$k/2$
$\cdot$	4	$3/2$	0	0	0
$\cdot$	2	$1/2$	1	0	0
$\cdot$	2	$1/2$	0	1	0
$\cdot$	2	$1/2$	0	0	1

Table 4. Gravity spectrum, ignoring 6 towers of short gravitino and 1 tower of short vector multiplet of [31]. The first line from short graviton, second-fourth lines from short gravitino, fifth-seventh lines from the short vector, eighth line from hypermultiplets. The last four lines are from massless graviton and vector multiplets.

The single particle index over the above fields is

$$
\begin{aligned}
(1-x^2)I_{\rm sp}(x,y,y_2) &= (1-x^2)\mathrm{tr}\left[(-1)^F x^{\epsilon+j_3} y^{2J_1} y_2^{J_2+J_3}\right] \qquad (3.1)\\
&= -\sum_{k=1}^{\infty}\chi_{k/2}(y)\chi_{k/2}(y_2^{1/2})^2 x^{k+4} - x^4\\
&\quad +\sum_{k=0}^{\infty}\Big(\chi_{k/2+1}(y_2^{1/2})^2\chi_{k/2}(y)+2\chi_{k/2}(y_2^{1/2})\chi_{k/2+1}(y_2^{1/2})\chi_{k/2+1}(y)\Big)x^{k+4}\\
&\quad -\sum_{k=1}^{\infty}\left(\chi_{k/2}(y_2)^2\chi_{k/2+1}(y)+2\chi_{k/2}(y_2^{1/2})\chi_{k/2+1}(y_2^{1/2})\chi_{k/2}(y)\right)x^{k+2}\\
&\quad -(\chi_1(y)+2\chi_1(y_2^{1/2})+2)x^2\\
I_{\rm sp} &= \frac{x\left(y_2^{1/2}+y_2^{-1/2}\right)^2\left((y+y^{-1})-x(y_2^{1/2}+y_2^{-1/2})^2+x^2(y+y^{-1})\right)}{(1-xy_1y_2)\,(1-xy/y_2)\,(1-xy_2/y)\,(1-x/(yy_2))}
\end{aligned}
$$

with $\chi_j(y)=\frac{y^{2j+1}-y^{-2j-1}}{y-y^{-1}}$, where $J_{1,2,3}$ is the Cartan of $\mathrm{SU}(2)_{1,2,3}$, respectively, and the combination J_2+J_3 is to be identified with the $\mathrm{U}(1)_B$ charge carried by the monopoles $\sum_i n_i=\sum_i \tilde{n}_i$ in the large N limit. y would later be identified with y_1 in field theory as $y=y_1^{1/2}$.

Expanding this result in powers of y_2, one obtains the single paraticle indices with

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definite $\mathrm{U}(1)_B$ (or J_2+J_3) charges as follows:

$$
\begin{aligned}
I^{(0)}_{\rm sp} &= 2\left(\frac{xy_1^{1/2}}{1-xy_1^{1/2}}+\frac{xy_1^{-1/2}}{1-xy_1^{-1/2}}-\frac{x^2}{1-x^2}\right) \\
I^{(1)}_{\rm sp} &= \frac{x\left(y+y^{-1}+4x^3+x(-y^{-1}+y)^2-3x^2(y^{-1}+y)\right)}{(1-x^2)(1-xy^{-1})(1-xy)} \\
I^{(2)}_{\rm sp} &= \frac{x^2(x^4y^3-3x^3(y^2+y^4)-y(1+y^2+y^4)+3x^2(y+y^5)-x(1-2y^2-2y^4+y^6))}{(-1+x^2)(x-y)y^2(-1+xy)},
\end{aligned}
\tag{3.2}
$$

and so on ($y=y_1^{1/2}$).

3.2 Field theory

The field theory model proposed for M2-branes on $C(Q^{111})$ adds four fundamental chiral supermultiplets to the $\mathcal{N}=6$ theory: two of them q_1, q_2 are anti-fundamental in the first gauge group, while the other two $\tilde{q}_1$, $\tilde{q}_2$ are fundamental in the second gauge group. This model is proposed to described M2-branes on $C(Q^{111})$ when the bare Chern-Simons level k is 0. One should also add the following superpotential

$$
\Delta W = q_1 A_1 \tilde{q}_1 + q_2 A_2 \tilde{q}_2 \tag{3.3}
$$

to the $\mathcal{N}=6$ theory.

In the $\mathrm{U}(1)\times\mathrm{U}(1)$ theory, the OPE of the monopole operators $T, \tilde{T}$ with charges $(1)(1)$ and $(-1)(-1)$ is given by $T\tilde{T}=A_1A_2$, which can be solved as

$$
A_1 = a_1b_2\,, \quad A_2 = b_1a_2\,, \quad T = a_1b_1\,, \quad \tilde{T} = a_2b_2\ . \tag{3.4}
$$

This solution is invariant under $(a_1,a_2,b_1,b_2)\to(\lambda a_1,\lambda a_2,\lambda^{-1}b_1,\lambda^{-1}b_2)$. On the space of $(B_1,B_2,a_1,a_2,b_1,b_2)$, $\mathbb{C}^6//\mathrm{U}(1)^2$ is given by $(\mu^{-2}B_1,\mu^{-2}B_2,\mu\lambda a_1,\mu\lambda a_2,\mu\lambda^{-1}b_1,\mu\lambda^{-1}b_2)$, or equivalently $(\lambda B_1,\lambda B_2,\mu a_1,\mu a_2,\nu b_1,\nu b_2)$ with $\lambda\mu\nu=1$, giving the cone over Q^{111}. The three SU(2) symmetries are given by rotating B_1,B_2 or a_1,a_2 or b_1,b_2.

In the non-Abelian field theory, only the SU(2) rotating B_1,B_2 is manifest. For the remaining SU(2)'s only the Cartans could be manifest. Under the Cartans of second and third SU(2)'s rotating a_1,a_2 and b_1,b_2, $A_{1,2}$ carry charge $\pm\frac{1}{2}$ and $\mp\frac{1}{2}$. Also, the monopole charge $\frac{1}{2}\sum n_i (=\frac{1}{2}\sum\tilde{n}_i$ in the large N limit) would contribute to this charge. Phase rotations of $A_{1,2}$ explained above for are not symmetries unless accompanied by appropriate rotations of fundamental fields, to make the superpotential invariant. For simplicity, we only weight the states with the sum of two SU(2) Cartans in this paper, carried by the monopole charges.

The bosonic fields and charges are listed in table 5. Note that the R-charges of fields are constrained to have the superpotential marginal, after which only one parameter b is left. This parameter is expected to be $b=\frac{1}{3}$, for instance from the study of baryonic states from the gravity dual. This can be determined by studying the extremization of the partition function [21], say in the large N limit [13], or by comparing the $\mathcal{O}(N)$ energy spectrum of the index with gravity. In this paper, we only study the low energy spectrum

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fields	$\mathrm{SU}(2)_1$	$\mathrm{SU}(2)_2+\mathrm{SU}(2)_3$	R	$\mathrm{U}(N)\times\mathrm{U}(N)$
B_1	$+1/2$	0	$b=1/3$	$(\bar{N},N)$
B_2	$-1/2$	0	$b=1/3$	
A_1	0	0	$a=1-b$	$(N,\bar{N})$
A_2	0	0	$a=1-b$	
q_1	0	0	$\frac{1+b}{2}$	$(\bar{N},1)$
q_2	0	0	$\frac{1+b}{2}$	
$\tilde{q}_1$	0	0	$\frac{1+b}{2}$	$(1,N)$
$\tilde{q}_2$	0	0	$\frac{1+b}{2}$	
$T=(1)(1)$	0	1	$1-b$	
$\tilde{T}=(-1)(-1)$	0	-1	$1-b$	

Table 5. Fields and charges: $\mathrm{SU}(2)_{1,2,3}$ denote the Cartan charges.

in the large N limit and work with the parameter b unfixed. It will turn out that the low energy large N index does not depend on b, after imposing the gauge invariance constraint (holonomy integrals, to be explained shortly).

From the general expressions in [23], one finds the following index with monopoles charges $\{n_i\}$, $\{\tilde{n}_i\}$:

$$
\begin{aligned}
I=&\frac{x^{\epsilon_0}}{(\text{symmetry})}\int\left[\frac{d\alpha d\tilde{\alpha}}{(2\pi)^2}\right]e^{ib_0}\exp\left[\sum_{i,j=1}^{N}\sum_{n=1}^{\infty}\frac{1}{n}x^{n|n_i-\tilde{n}_j|}\left(f^+(\cdot^n)e^{-in(\alpha_i-\tilde{\alpha}_j)}+f^-(\cdot^n)e^{in(\alpha_i-\tilde{\alpha}_j)}\right)\right]\\
&\exp\left[\sum_{i=1}^{N}\sum_{n=1}^{\infty}\frac{1}{n}x^{n|n_i|}\left(f_B(\cdot^n)e^{-in\alpha_i}+f_F(\cdot^n)e^{in\alpha_i}\right)+(n_i,\alpha_i\to\tilde{n}_i,-\tilde{\alpha}_i)\right]\\
&\exp\left[-\sum_{i\neq j}^{N}\sum_{n=1}^{\infty}\frac{1}{n}\left(x^{n|n_i-n_j|}e^{-in(\alpha_i-\alpha_j)}+(n_i,\alpha_i\to\tilde{n}_i,\tilde{\alpha}_i)\right)\right]
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
\epsilon_0=&\sum_{i,j=1}^{N}|n_i-\tilde{n}_j|-\sum_{i<j}(|n_i-n_j|+|\tilde{n}_i-\tilde{n}_j|)+\frac{1-b}{2}\sum_{i=1}^{N}(|n_i|+|\tilde{n}_i|)\\
b_0=&\sum_{i=1}^{N}(|n_i|\alpha_i-|\tilde{n}_i|\tilde{\alpha}_i)\ .
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
f^+=&\frac{x^b}{1-x^2}\left(y_1^{1/2}+y_1^{-1/2}\right)-\frac{2x^{1+b}}{1-x^2}\quad(\text{from } B_a^\dagger,\psi_{A_a})\\
f^-=&\frac{2x^{1-b}}{1-x^2}-\frac{x^{2-b}}{1-x^2}\left(y_1^{1/2}+y_1^{-1/2}\right)\quad(\text{from } A_a^\dagger,\psi_{B_a})\\
f_B=&\frac{2x^{\frac{1+b}{2}}}{1-x^2}\,,\quad f_F=-\frac{2x^{\frac{3-b}{2}}}{1-x^2}\ .
\end{aligned}
\tag{3.7}
$$

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Again after going to the large N limit with energy and monopoles fixed to $\mathcal{O}(1)$, the index factorizes as

$$I_{N\to\infty} = I_0 I^+ I^- \tag{3.8}$$

where

$$\begin{aligned} I_0 &= \prod_{n=1}^{\infty} \frac{1}{1-f^+(.^n)f^-(.^n)} \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\frac{2f_Bf_F+f^-f_B^2+f^+f_F^2}{1-f^+f^-}(.^n)\right] \\ &= \prod_{n=1}^{\infty}\frac{(1-x^{2n})^2}{\left(1-x^n y_1^{n/2}\right)^2\left(1-x^n y_1^{-n/2}\right)^2}\,. \end{aligned} \tag{3.9}$$

Note that the second factor involving fundamental letter indices is simply 1. The part I^+ depending on monopoles with positive flux is given by

$$\begin{aligned} I^+ = \frac{x^{\epsilon_0}}{(\text{symmetry})}\int\left[\frac{d\alpha d\tilde{\alpha}}{(2\pi)^2}\right] e^{i\sum_i(n_i\alpha-\tilde{n}_i\tilde{\alpha}_i)} \\ \exp\left[\sum_{i,j=1}\sum_{n=1}^{\infty}\frac{1}{n}\left(x^{n|n_i-\tilde{n}_j|}-x^{n|n_i|+n|\tilde{n}_j|}\right)\left(f^+(.^n)e^{-in(\alpha_i-\tilde{\alpha}_j)}+f^-(.^n)e^{in(\alpha_i-\tilde{\alpha}_j)}\right)\right] \\ \exp\left[-\sum_{i,j}\sum_{n=1}^{\infty}\frac{1}{n}\left((1-\delta_{ij})x^{n|n_i-n_j|}-x^{n|n_i|+n|n_j|}\right)e^{-in(\alpha_i-\alpha_j)}\right. \\ \left.-\sum_{i,j}\sum_{n=1}^{\infty}\frac{1}{n}\left((1-\delta_{ij})x^{n|\tilde{n}_i-\tilde{n}_j|}-x^{n|\tilde{n}_i|+n|\tilde{n}_j|}\right)e^{-in(\tilde{\alpha}_i-\tilde{\alpha}_j)}\right] \end{aligned} \tag{3.10}$$

Again the fundamental degrees disappear in I^+, apart from their contribution to ϵ_0 and b_0. On the gravity side, the flux $\sum n_i = \sum \tilde{n}_i$ in the large N corresponds to the sum of the Cartans of the two SU(2)'s. The $\mathrm{U}(1)_B$ neutral graviton contribution in (3.2) becomes

$$\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}I_{\rm sp}^{(0)}(x^n,y_1^n)\right],\quad I_{\rm sp}^{(0)}(x,y_1)=\frac{2xy_1^{1/2}}{1-xy_1^{1/2}}+\frac{2xy_1^{-1/2}}{1-xy_1^{-1/2}}-\frac{2x^2}{1-x^2}\,, \tag{3.11}$$

in perfect agreement with I_0 in (3.9) from field theory.

Let us now turn to the study of $\mathrm{U}(1)_B$ charged states. From the structure of the 'quantum' Chern-Simons phase factor $e^{i(|n_i|\alpha_i-|\tilde{n}_i|\tilde{\alpha}_i)}$, one can easily confirm that $I^+(x,y_1) = I^-(x,y_1)$. We thus study I^+ only. We again start from the flux (1)(1), which should agree with the single graviton particles having unit $\mathrm{U}(1)_B$ charge.

$$I^+_{(1)(1)}(x,y_1) = x^{1-b}\int\frac{d\alpha}{2\pi}e^{i\alpha}\frac{\left(1-x^{1+b}e^{-i\alpha}\right)^2\left(1-x^{2-b}y_1^{1/2}e^{i\alpha}\right)\left(1-x^{2-b}y_1^{-1/2}e^{i\alpha}\right)}{\left(1-x^by_1^{1/2}e^{-i\alpha}\right)\left(1-x^by_1^{-1/2}e^{-i\alpha}\right)(1-x^{1-b}e^{i\alpha})^2(1-x^2)^2}\,. \tag{3.12}$$

After performing the contour integral, one obtains

$$I^+_{(1)(1)} = -\frac{xy_1^{-1/2}}{1-x^2}-\frac{xy_1^{1/2}}{1-x^2}+\frac{2xy_1^{-1/2}}{1-xy_1^{-1/2}}+\frac{2xy_1^{1/2}}{1-xy_1^{1/2}}\,. \tag{3.13}$$

This completely agrees with the gravity index $I_{\rm sp}^{(1)}$.

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One can continue checking the agreement by studying the sector with two fluxes. One should consider the fluxes $(2)(2)$, $(1,1)(1,1)$, $(2)(1,1)$, $(1,1)(2)$. One finds

$$\begin{aligned}
I_{(2)(2)} &= x^2(1+1/y^2+y^2)+x^3(2/y^3+2y^3)+x^4(2/y^4-2/y^2-2y^2+2y^4)\\
&\quad +x^5(2/y^5+2/y+2y+2y^5)+x^6(-4+2/y^6+1/y^4-2/y^2-2y^2+y^4+2y^6)\\
&\quad +x^7(2/y^7-4/y^3+4/y+4y-4y^3+2y^7)\\
&\quad +x^8(-3+2/y^8+1/y^4+2/y^2+2y^2+y^4+2y^8)+\cdots\\
I_{(11)(11)} &= x^2(1+1/y^2+y^2)+x^3(2/y^3+2/y+2y+2y^3)+x^4(2+5/y^4+2/y^2+2y^2+5y^4)\\
&\quad +x^5(6/y^5+6y^5)+x^6(5+9/y^6-1/y^4+2/y^2+2y^2-y^4+9y^6)\\
&\quad +x^7(10/y^7+4/y^3+2/y+2y+4y^3+10y^7)\\
&\quad +x^8(-3+13/y^8-1/y^4-3/y^2-3y^2-y^4+13y^8)+\cdots\\
I_{(2)(11)} &= I_{(11)(2)} = x^6-x^7(2/y+2y)+x^8(5+1/y^2+y^2)+\cdots
\end{aligned} \tag{3.14}$$

where $y=y_1^{1/2}$. On the gravity side, the single particle index with $\mathrm{U}(1)_B$ charge 2 and two particle contributions from particles with $\mathrm{U}(1)_B$ charge 1 are

$$\begin{aligned}
I^{(2)}_{\rm sp}(x,y) &= x^2(1+1/y^2+y^2)+x^3(2/y^3+2y^3)+x^4(2/y^4-1/y^2-y^2+2y^4)\\
&\quad +x^5(2/y^5+2y^5)+x^6(2/y^6-1/y^2-y^2+2y^6)+x^7(2/y^7+2y^7)\\
&\quad +x^8(2/y^8-1/y^2-y^2+2y^8)+\cdots\\
\frac{I^{(1)}_{\rm sp}(x^2,y^2)+I^{(1)}(x,y)^2_{\rm sp}}{2} &= x^2(1+1/y^2+y^2)+x^3(2/y^3+2/y+2y+2y^3)\\
&\quad +x^4(2+5/y^4+1/y^2+y^2+5y^4)\\
&\quad +x^5(6/y^5+2/y+2y+6y^5)+x^6(3+9/y^6+1/y^2+y^2+9y^6)\\
&\quad +x^7(10/y^7+2/y+2y+10y^7)\\
&\quad +x^8(4+13/y^8+2/y^2+2y^2+13y^8)+\cdots\ .
\end{aligned} \tag{3.15}$$

One finds that

$$I_{(2)(2)}+I_{(11)(11)}+I_{(2)(11)}+I_{(11)(2)} = I^{(2)}_{\rm sp}(x,y)+\frac{I^{(1)}_{\rm sp}(x^2,y^2)+I^{(1)}(x,y)^2_{\rm sp}}{2}+\mathcal{O}(x^9)\,, \tag{3.16}$$

a perfect agreement up to the order we checked.

4 The index for M2-branes probing $C(M^{32})$

4.1 Gravity

The gravity spectrum of M-theory on $AdS_4\times M^{32}$ was obtained in [38]. Among them, in this paper we are only interested in the BPS states saturating the bound $\epsilon\geq h_0+j_3$, where h_0 is the U(1) R-charge. Exactly one field in AdS_4 appears to be BPS in each short OSp(2|4) supermultiplet. Those fields are listed in table 6 with their quantum numbers for energy ϵ, angular momentum j_3, SU(3) label (M_1,M_2) which follows the same convention as [37] and our presentation for N^{010} in section 2, SU(2) global symmetry. The Cartan of SU(2) global

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	ϵ	j_3	$\epsilon + j_3$	SU(3)	SU(2)
$p \geq 1$	$2p + 5/2$	$3/2$	$2p + 4$	$(0, 2p)$	p
$p \geq 0$	$2p + 3$	1	$2p + 4$	$(1, 3p + 1)$	$p + 1$
$p \geq 1$	$2p + 1$	1	$2p + 2$	$(0, 3p)$	$p - 1$
$p \geq 1$	$2p + 3/2$	$1/2$	$2p + 2$	$(1, 3p + 1)$	p
$p \geq 1$	$2p + 3/2$	$1/2$	$2p + 2$	$(0, 3p)$	$p + 1$
$p \geq 1$	$2p$	0	$2p$	$(0, 3p)$	p
$\cdot$	$5/2$	$3/2$	4	$(0, 0)$	0
$\cdot$	$3/2$	$1/2$	2	$(1, 1)$	0
$\cdot$	$3/2$	$1/2$	2	$(0, 0)$	1
$\cdot$	$3/2$	$1/2$	2	$(0, 0)$	0

Table 6. Supersymmetric fields on $AdS_4 \times M^{32}$.

symmetry is identified as a 'baryon-like' U(1) symmetry carried by the sum of monopole charge $\sum p_i$ in the first gauge group $\mathrm{U}(N)_{-2}$. This U(1) which we often call 'baryon-like' should not be confused with the real baryon U(1) symmetry, as M^{32} has topological 5-cycles on which M5-brane baryons can wrap. In the table, the first line comes from short graviton multiplets, the second and third lines from the short gravitino multiplets, the fourth and fifth lines from the short vector multiplets, sixth from the hypermultiplets. The seventh line is from the massless graviton multiplets. The eighth and ninth line is from the massless vector multiplet for the SU(3) and SU(2) isometry, respectively. The last line is from the U(1) baryon symmetry for M5-branes wrapped on 5-cycles in M^{32}.

Each field on the table contributes to the index. To calculate the single particle index, one also has to take the wavefunction factors in AdS_4 into account. This amounts to putting the factor $\frac{1}{1-x^2}$ to the index over the above fields. Denoting by h_1, h_2 the Cartans of SU(3) given by the diagonal matrices $\frac{1}{2}\mathrm{diag}(1, -1, 0)$, $\frac{1}{2}\mathrm{diag}(0, 1, -1)$ and by h_3 the SU(2) Cartan, one obtains

$$\begin{aligned} I_{\mathrm{sp}}(x, y_1, y_2, y_3) &= \mathrm{tr}\left[(-1)^F x^{\epsilon + j_3} y_1^{h_1} y_2^{h_2} y_3^{h_3}\right] \\ &= \frac{1}{1 - x^2}\left[-\sum_{p=1}^{\infty} x^{2p+4} \chi^{(0,2p)}_{\mathrm{SU}(3)}(y_1, y_2) \chi^{p}_{\mathrm{SU}(2)}(y_3) + \right] \end{aligned} \tag{4.1}$$

where

$$\chi^{M_1, M_2}_{\mathrm{SU}(3)}(y_1, y_2) = \frac{\begin{vmatrix} y_1^{M_1+M_2+2} & 1/y_2^{M_1+M_2+2} & (y_2/y_1)^{M_1+M_2+2} \\ y_1^{M_2+1} & 1/y_2^{M_2+1} & (y_2/y_1)^{M_2+1} \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} y_1^2 & 1/y_2^2 & y_2^2/y_1^2 \\ y_1 & 1/y_2 & y_2/y_1 \\ 1 & 1 & 1 \end{vmatrix}} \tag{4.2}$$

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and

$$\chi^j_{\mathrm{SU}(2)}(y_3) = \frac{y_3^{j+1/2} - y_3^{-(j+1/2)}}{y_3^{1/2} - y_3^{-1/2}} \tag{4.3}$$

are the SU(3) and SU(2) characters.

The general expression for $I_{\mathrm{sp}}(x, y_1, y_2, y_3)$ after the infinite sums is very messy, although an explicit closed form expression is available. The expressions we need below to compare with field theory would be those with definite SU(2) Cartan charge, labeled by y_3. This charge would correspond to the $\mathrm{U}(1)_B$ baryon-like charge carried by the monopoles. The single particle indices with this charge being 0, 1, 2 are

$$\begin{aligned}
I^{(0)}_{\mathrm{sp}} &= \frac{x^2/y_1^3}{1-x^2/y_1^3} + \frac{x^2 y_2^3}{1-x^2 y_2^3} + \frac{x^2 y_1^3/y_2^3}{1-x^2 y_1^3/y_2^3} - \frac{3x^2}{1-x^2} \\
I^{(1)}_{\mathrm{sp}} &= \frac{x^2/y_1^3}{1-x^2/y_1^3} + \frac{x^2 y_2^3}{1-x^2 y_2^3} + \frac{x^2 y_1^3/y_2^3}{1-x^2 y_1^3/y_2^3} - \frac{x^4}{1-x^2}\left(1/y_1^3 + y_2^3 + y_1^3/y_2^3\right) \\
&\quad + x^2\left(y_1^2/y_2 + y_2^2/y_1 + y_1/y_2^2 + y_2/y_1^2 + y_1 y_2 + 1/(y_1 y_2)\right) \\
I^{(2)}_{\mathrm{sp}} &= \frac{(x^8 + x^4 y_1^3 - 2x^6 y_1^3 + x^2 y_1^6 - x^4 y_1^6 - y_1^9 + x^2 y_1^9 - x^6 y_1^9 + x^8 y_1^9 + x^4 y_1^{12} - x^6 y_1^{12})}{(-1+x^2) y_1^6 (x^2 - y_1^3)} \\
&\quad + \frac{x^6 y_1^6}{(-1+x^2) y_2^6} + \frac{x^4 y_1^4}{y_2^5} + \frac{x^4 y_1^2 + x^4 y_1^5}{y_2^4} + \frac{x^4 - x^2 y_1^3}{y_2^3} + \frac{x^4 + x^4 y_1^6}{y_1^2 y_2^2} + \frac{x^4}{y_1^4 y_2} \\
&\quad + \frac{x^4 y_2}{y_1^5} + \frac{x^4(1+y_1^6) y_2^2}{y_1^4} + \frac{x^2(x^2 - y_1^3) y_2^3}{y_1^3} \\
&\quad + \frac{x^4(1+y_1^3) y_2^4}{y_1^2} + \frac{x^4 y_2^5}{y_1} + \frac{x^6 y_2^6}{-1+x^2} + \frac{x^2 y_1^3}{(-x^2 y 1^3 + y 2^3)} - \frac{1}{(-1+x^2 y_2^3)}
\end{aligned} \tag{4.4}$$

and so on, where $I^{(p)}_{\mathrm{sp}} = \oint_{|y_3|=1} \frac{dy_3}{y_3^{p+1}} I_{\mathrm{sp}}(x, y_1, y_2, y_3)$ is the single particle index with $\mathrm{U}(1)_B$ charge p. Note that, as one takes the SU(3) chemical potentials y_1, y_2 to 1, a dramatic simplification (or cancelation) appears:

$$I_{\mathrm{sp}}(x, y_1 = 1, y_2 = 1, y_3) = \frac{9x^2 y_3}{(1-x^2 y_3)^2} + \frac{9x^2 y_3^{-1}}{(1-x^2 y_3^{-1})^2}\,, \tag{4.5}$$

or

$$I^{(0)}_{\mathrm{sp}} = 0\,, \quad I^{(1)}_{\mathrm{sp}} = 9x^2\,, \quad I^{(2)}_{\mathrm{sp}} = 18x^4\,, \quad I^{(3)}_{\mathrm{sp}} = 27x^6\,, \ \cdots \tag{4.6}$$

and so on.

4.2 Field theory

The field theory proposed for M2-branes on $C(M^{32})$ is given by the quiver diagram in figure 1. The Chern-Simons levels for the three U(N) gauge fields are $(-2k, k, k)$ for $M^{32}/\mathbb{Z}_k$ [34, 35]. For the brevity of notation, we shall often denote the fields by $X^i_{23} = A^i$, $X^i_{31} = B^i$, $X^i_{12} = C^i$. $i = 1, 2, 3$ is the triplet index for the SU(3) global symmetry.

This $\mathcal{N} = 2$ model for M^{32} is chiral in a 4 dimensional sense. As the number of 'flavors' 3 for each bi-fundamental is odd, one has to worry about the issue of parity anomaly. The fermion determinant could pick up a minus sign under certain large gauge transformations,

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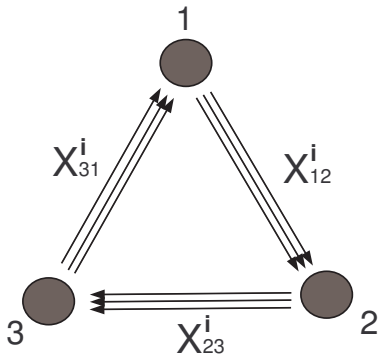

Figure 1. The quiver diagram of M2-branes on $C(M^{32})$.

which will make the path integral gauge non-invariant. More concretely, on $S^2 \times S^1$ that we are going to consider, the relevant large gauge transformations are 2π periodic shifts of the holonomy variables in our convention. The possible -1 sign will come from the phase e^{ib_0} defined below.

The canonical way of making such a theory consistent would be adding Chern-Simons term with appropriate half-integral levels, providing an extra minus sign to the path integral measure under the large gauge transformation which cancels that from the fermion determinant. As we shall see in the context of index shortly, the required Chern-Simons term is a mixed Chern-Simons term between the three gauge fields. As we shall see in the context of the index below, the required mixed Chern-Simons term is needed only for the overall $\mathrm{U}(1)^3$ part of the $\mathrm{U}(N)^3$ gauge fields. It should be very interesting to study the index for a suitably modified theory of this sort.[6]

In this section, we try to do best to make the proposed model of [34, 35] sensible in the context of the superconformal index. To this end, let us consider at this point another possible way of making the path integral consistent on $S^2 \times S^1$, which later will turn out to be a prescription yielding an interesting index structure containing all the gravity spectrum. At least when one considers a theory on $S^2 \times \mathbb{R}$ (or $S^2 \times S^1$) rather than generic case, there appear nontrivial sectors labeled by the magnetic monopole charges. Depending on these magnetic charges that we allow in this QFT as we shall explain below, it is possible to make the path integral measure to be invariant under the large gauge transformations of holonomies. We postpone the explanation until we have a concrete form of this measure.

We take the dimensions of three chiral fields A^i, B^i, C^i to be a, b, c, respectively. They should satisfy $a+b+c=2$ for the superpotential to be marginal. In principle, the parameters a, b, c, including the superpotential constaint, could be derivable from an extremization principle like that of [21]. We simply start with them unfixed, but at appropriate stage will constrain them by hand if necessary, to have a sensible model with the M^{32} gravity

[6] We thank I. Klebanov for discussions, and especially F. Benini for explaining to us his studies on this issue.

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dual. The only other constaint we impose will be $b = c$, which is anyway reasonable from the quiver diagram.

From the general expression in [23], one can immediately write down the integral expression for the index for this model:

$$
\begin{aligned}
I_{(p,q,r)} = \frac{x^{\epsilon_0}}{(\text{symmetry})} \int \left[\frac{d\alpha d\beta d\gamma}{(2\pi)^3}\right] e^{ik\sum_i(-2p_i\alpha_i+q_i\beta_i+r_i\gamma_i)} e^{ib_0(p,q,r,\alpha,\beta,\gamma)} \\
\exp\left[\sum_{i,j=1}^{N}\sum_{n=1}^{\infty}\frac{1}{n}x^{n|q_i-r_j|}\left(f_A^+(\cdot^n)e^{-in(\beta_i-\gamma_j)}+f_A^-e^{in(\beta_i-\gamma_j)}\right)\right. \\
+(A,\beta,\gamma,q,r\to B,\gamma,\alpha,r,p)+(A,\beta,\gamma,q,r\to C,\alpha,\beta,p,q) \\
\left.-\sum_{i\neq j}\sum_{n=1}^{\infty}\frac{1}{n}x^{n|p_i-p_j|}e^{-in(\alpha_i-\alpha_j)}+(p,\alpha\to q,\beta)+(p,\alpha\to r,\gamma)\right]
\end{aligned}
\tag{4.7}
$$

with

$$
\begin{aligned}
f_A^+ &= \frac{x^a}{1-x^2}\left(1/y_1+y_2+y_1/y_2\right), \qquad f_A^- = -\frac{x^{2-a}}{1-x^2}\left(y_1+1/y_2+y_2/y_1\right); \\
f_{B,C}^{\pm} &= (\text{replace } a\to b,c)
\end{aligned}
\tag{4.8}
$$

and

$$
\begin{aligned}
b_0 &= \frac{3}{2}\left[\sum|p_i-q_j|(\alpha_i-\beta_j)+\sum|q_i-r_j|(\beta_i-\gamma_j)+\sum|r_i-p_j|(\gamma_i-\alpha_j)\right] \\
\epsilon_0 &= \frac{3}{2}\left((1-c)\sum_{i,j}|p_i-q_j|+(1-a)\sum_{i,j}|q_i-r_j|+(1-b)\sum_{i,j}|r_i-p_j|\right) \\
&\quad -\sum_{i<j}\left(|p_i-p_j|+|q_i-q_j|+|r_i-r_j|\right).
\end{aligned}
\tag{4.9}
$$

The set of N integers $\{p_i\}$, $\{q_i\}$, $\{r_i\}$ are monopole charges in $\mathrm{U}(N)_{-2k}$, $\mathrm{U}(N)_k$, $\mathrm{U}(N)_k$ gauge groups. We shall be mostly interested in the case with $k = 1$ below.

Let us make comment on the 1-loop shift of the phase b_0, which is the index version of the 1-loop fermion determinant contributing to the Chern-Simons term. This could pick up dangerous minus signs by the large gauge transformations. The large gauge transformations in this case are periodic shifts of the holonomy variables α_i, β_i, γ_i by 2π, making them live on a $3N$ dimensional torus. With generic integers $\{p_i\}$, $\{q_i\}$, $\{r_i\}$, the phase e^{ib_0} may not be well-defined on this torus due to the $\frac{3}{2}$ factor. However, by suitably restricting these integers, constraining the monopole sectors of the theory, we can make this phase well defined. Let us see how this can happen.

Let us first consider the appearance of a variable α_i in the phase e^{ib_0}:

$$
\exp\left[\frac{3}{2}\alpha_i\sum_{j=1}^{N}\left(|p_i-q_j|-|p_i-r_j|\right)\right]=\exp\left[\frac{3}{2}\alpha_i\sum_{j=1}^{N}\left(|p_i|+|q_j|-|p_i|-|r_j|\right)\right]e^{\frac{3}{2}\alpha_i(\text{even ingeter})},
\tag{4.10}
$$

where we used the fact that $|m|+|n|-|m-n|$ is an even integer for integer m,n. The second phase in the last expression is thus well defined. To have the first part well defined, we demand the restriction

$$\sum_{i=1}^{N}|q_i| = \sum_{i=1}^{N}|r_i| \qquad \text{mod } 2\ . \tag{4.11}$$

Considering the large gauge transformation of β_i, γ_i, one finds that the restriction is $\sum|p| = \sum|q| = \sum|r|$ mod 2. This can again be written as

$$\sum_{i=1}^{N}p_i = \sum_{i=1}^{N}q_i = \sum_{i=1}^{N}r_i \qquad \text{mod } 2\ . \tag{4.12}$$

These are actually constraining the overall $\mathrm{U}(1)^3$ fluxes in $\mathrm{U}(N)^3$,

$$\int_{S^2}\mathrm{tr}F_1 = \int_{S^2}\mathrm{tr}F_2 = \int_{S^2}\mathrm{tr}F_3 \qquad \text{mod } 2\,, \tag{4.13}$$

which are $\mathrm{U}(N)^3$ gauge invariant constraints. One interpretation is that this condition comes from different quantization for Chern-Simons level for U(1) of $\mathrm{U}(N)$. Assigning different Chern-Simons level for U(1) part and $\mathrm{SU}(N)$ part out of $\mathrm{U}(N)$ is possible. This suggests that we have to choose the different Chern-Simons levels for U(1) and $\mathrm{SU}(N)$ to make theory free of the potential parity anomalies due to fermion one-loop determinants. It would be interesting to see if a similar consideration works for other manifolds, say on S^3. Below, we simply consider the index (4.7) with this restriction understood.

Again our main concern of this paper is the large N index with energies kept at $\mathcal{O}(1)$. In particular, we again require only $\mathcal{O}(1)$ number of magnetic charges to be nonzero among the above $3N$ of them. The details of the large N integration over holonomies with zero flux is analogous to the previous section. After this calculation, the large N index is given by

$$I_{N=\infty}(x,y_1,y_2,y_3) = I_0(x,y_1,y_2)I'(x,y_1,y_2,y_3)\,, \tag{4.14}$$

where

$$I_0 = \prod_{n=1}^{\infty}\frac{(1-x^{2n})^3}{(1-x^{2n}/y_1^{3n})(1-x^{2n}y_2^{3n})(1-x^{2n}y_1^{3n}/y_2^{3n})} \equiv \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f(x^n,y_1^n,y_2^n)\right] \tag{4.15}$$

with

$$f(x,y_1,y_2) = \frac{x^2/y_1^3}{1-x^2/y_1^3} + \frac{x^2y_2^3}{1-x^2y_2^3} + \frac{x^2y_1^3/y_2^3}{1-x^2y_1^3/y_2^3} - \frac{3x^2}{1-x^2}\ . \tag{4.16}$$

The monopole part I' is given by

$$I'_{(p,q,r)} = \frac{x^{\epsilon_0}}{(\text{symmetry})} \int \left[\frac{d\alpha d\beta d\gamma}{(2\pi)^3}\right] e^{ik\sum_i(-2p_i\alpha_i+q_i\beta_i+r_i\gamma_i)} e^{ib_0(p,q,r,\alpha,\beta,\gamma)} \tag{4.17}$$
$$\exp\left[\sum_{i,j=1}\sum_{n=1}^{\infty}\frac{1}{n}(x^{n|q_i-r_j|}-x^{n|q_i|+n|r_j|})\left(f_A^+(\cdot^n)e^{-in(\beta_i-\gamma_j)}+f_A^-e^{in(\beta_i-\gamma_j)}\right)\right.$$
$$+(A,\beta,\gamma,q,r\to B,\gamma,\alpha,r,p)+(A,\beta,\gamma,q,r\to C,\alpha,\beta,p,q)$$
$$-\sum_{i,j}\sum_{n=1}^{\infty}\frac{1}{n}\left((1-\delta_{ij})x^{n|p_i-p_j|}-x^{n|p_i|+n|p_i|}\right)e^{-in(\alpha_i-\alpha_j)}$$
$$\left.+(p,\alpha\to q,\beta)+(p,\alpha\to r,\gamma)\right]$$

where the holonomies $\alpha_i, \beta_i, \gamma_i$ range over those supporting nonzero fluxes. For a given set of fluxes, this index does not have any factorization structure.

As the diagonal combination of $\mathrm{U}(1) \subset \mathrm{U}(N)_{-2k} \times \mathrm{U}(N) \times \mathrm{U}(N)_k$ decouples with matter fields, the fluxes should satisfy

$$2\sum p_i = \sum q_i + \sum r_i \ . \tag{4.18}$$

Also, the magnetic charge

$$\sum_i p_i \tag{4.19}$$

for the first gauge group $\mathrm{U}(N)_{-2k}$ is to be identified with the Cartan of the SU(2) flavor symmetry at $k=1$, which is explicit in the gravity side. We call this the $\mathrm{U}(1)_B$ 'baryon-like' symmetry, again not to be confused withe the real baryon symmetry for wrapped M5-branes.

As our first goal is to study gravitons with $\mathcal{O}(1)$ numbers of magnetic flux, we consider the condition for the energy to remain $\mathcal{O}(1)$. Firstly, from the phase b_0, it happens that there can appear $O(N)$ number of phase variables in the exponent even with $\mathcal{O}(1)$ number of fluxes. This would cause gauge invariant operators to carry $\mathcal{O}(N)$ energy after integration. For this not to happen, one finds that

$$\sum |p| = \sum |q| = \sum |r| \tag{4.20}$$

has to be imposed, where the summations are over all nonzero fluxes in each gauge group. Of course this is compatible with the flux restriction we imposed above. Let us next investigate the possible $\mathcal{O}(N)$ contribution to ϵ_0. Denoting by N_1, N_2, N_3 the number of zero magnetic fluxes in each gauge group, one finds

$$N_1\left(\frac{3}{2}(1-c)\sum|q|+\frac{3}{2}(1-b)\sum|r|-\sum|p|\right)+cyclic. \tag{4.21}$$

When the condition (4.20) is met, the $\mathcal{O}(N)$ contribution always vanishes from $a+b+c=2$.

Firstly, the expression (4.15) for I_0 from field theory completely agrees with the gravity index $I^{(0)}$ coming from $\mathrm{U}(1)_B$ neutral particles, similar to the $\mathcal{N}=6$ models and also the

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models studied in the previous sections. Actually, when the large N index is factorized into $I_0 I^+ I^-$ acquiring contribution from $\mathrm{U}(1)_B$ neutal sector, positive monopoles and negative monopoles, respectively, this agreement $I_0 = I^{(0)}$ has to be the case. But as we shall see more concretely with examples, the index I' does not generally factorize to $I^+ I^-$ for separate saddle points (i.e. monopole fluxes). Below, we shall explain how such a factorization can appear *after* summing over the saddle points.

Let us first phrase our claim, and then illustrate by many examples. For a given saddle point, we decompose nonzero magnetic charges $\{p_i\}$ into positive and negative ones, and call them p_{i+}, p_{i-}, respectively. We will set $b = c$ for the trial R-charge, which we shall motivate shortly with examples, so that the only independent parameter left is b. Define

$$P_+ = \sum p_{i+} \,, \quad P_- = \sum |p_{i-}| \tag{4.22}$$

to be the sums of all positive/negative fluxes. Defining $I_{(P_+,P_-)}$ to be the sum over all indices whose saddle points have given values of P_+, P_-, this quantity can be written as

$$I_{(P_+,P_-)} = \hat{I}_{(P_+,P_-)} + I^b_{(P_+,P_-)} \,, \tag{4.23}$$

where $\hat{I}$ refers to the index summed over the sectors whose energies do not depend on b, and I^b that does depend on b.[7] We find that

$$\hat{I}_{(P_+,P_-)} = \hat{I}_{(P_+,0)} \hat{I}_{(0,P_-)} = I^{\mathrm{grav}}_{P_+} I^{\mathrm{grav}}_{P_-} \,, \tag{4.24}$$

i.e. $\hat{I}$ factorizes into two contribution which are identical to the multi-graviton index $I^{\mathrm{grav}}_{P_+}$ with $\mathrm{U}(1)_B$ charge P_+ coming from positively charged particles, and the index $I^{\mathrm{grac}}_{P_-}$ with charge $-P_-$ coming from negatively charge particles. If, for some reason, one has to discard all the seemingly existing saddle points contributing to I^b, then one finds from (4.24) that the following factorization is allowed. Labeling the index with its $\mathrm{U}(1)_B$ charge $P_+ - P_-$ with the chemical potential y_3, the above factorization implies

$$\hat{I}'(x, y_1, y_2, y_3) = \sum_{P_+,P_-=0}^{\infty} y_3^{P_+-P_-} \hat{I}_{(P_+,P_-)}(x, y_1, y_2) \tag{4.25}$$

$$= \sum_{P_+=0}^{\infty} y_3^{P_+} \hat{I}_{(P_+,0)}(x, y_1, y_2) \sum_{P_-=0}^{\infty} y_3^{-P_-} \hat{I}_{(0,P_-)}(x, y_1, y_2) \equiv I^+ I^- \,. \tag{4.26}$$

I^+ and I^- separately agrees with the gravity index coming from particles with positive/negative charges.

Before proceeding, let us explain what could be the meaning of discarding/keeping I^b. We have two possibilities in mind. Firstly, quite naturally, if we seriously keep this contribution, this could simply be reflecting the fact that the proposed field theory is not the correct one for M-theory on $AdS_4 \times M^{32}$, as the index captures extra branches of states other than those seen in the gravity. This looks a bit similar to the phenomenon observed

[7] Coefficients of the expansion with chemical potentials are always independent of b in both parts, being integers from the Plethystic exponential structure.

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in the so-called 'dual ABJM' model [39] that extra branches of moduli spaces develop [40], although appearing here in a much more nontrivial way of showing two sectors in the index. As we shall explain in more detail below, the fluxes contributing to $\hat{I}$ and I^b have clear distinctions in their properties. So even though this model itself does not seem to be showing the correct gravity spectrum, it still sounds quite amazing that the known gravity contribution is so well organized as above. So we could hope that a small modification of the existing model could provide a much better index structure.

One the other hand, let us also emphasize a potential subtlety of the localization calculation in quantum field theories which may justify discarding I^b as unphysical. Localization refers to a property of an integral with fermionic, or nilpotent, symmetry Q which makes the integral to acquire contribution only around the fixed points of the symmetry. In particular, a practical way of carrying out such integrals is to add to the integrand Q-exact measures with free parameters, using which one can perform a Gaussian evaluation of the exact result. The location of the fixed points of the symmetry in the integration domain could change during this deformation. This change of the fixed point profiles does not change the result *as long as the saddle points do not disappear or appear* during the deformation. This can happen if the integration domain is noncompact. A classic example of this sort appears in the calculation of the partition function of 2 dimensional topological Yang-Mills theory [41].

The localization calculation of the index, or the path integral for the partition function of Chern-Simons-matter theory on $S^2\times S^1$ in principle has the same issue when we introduce the Yang-Mills like deformation of [16]. It is not easy to determine a priori which saddle points are to be kept and which to be discarded. In the case of $\mathcal{N}=6$ theory, originally some saddle points which appeared in the deformed theory of [16] were not clearly understood in the QFT without deformation. Many of these saddle points were constructed later from the undeformed theory [42], justifying in the honest sense the prescription of [16].

In the present case of M^{32}, the observation summarized above suggest that all saddle points which contain the undetermined trial R-charge parameter b might have to be discarded as being unphysical ones. Also, the possibility of discarding I^b is not simply discarding all unwanted terms in the index, as the ambiguity mentioned above only allows one to keep or discard the whole contribution from a saddle point, not allowing to keep some term while discarding others. It should be interesting to see if this is the case or not for this model, using the QFT on $S^2\times\mathbb{R}$ written down in [23] with and without deformations.

Now let us present examples which shows that our claims (4.23), (4.24) are true in various sectors. The sector with $p=q=r=1$ is for smallest positive $\mathrm{U}(1)_B$ charge. Inserting $\epsilon_0=0$ and $b_0=0$, the field theory index is

$$\int_{\alpha,\beta,\gamma} e^{i(-2\alpha+\beta+\gamma)}\frac{(1-x^{2-c}e^{i(\alpha-\beta)})^3(1-x^{2-a}e^{i(\beta-\gamma)})^3(1-x^{2-b}e^{i(\gamma-\alpha)})^3}{(1-x^{c}e^{-i(\alpha-\beta)})^3(1-x^{a}e^{-i(\beta-\gamma)})^3(1-x^{b}e^{-i(\gamma-\alpha)})^3(1-x^2)^3}\,. \tag{4.27}$$

The result turns out to be extremely simple: $9x^{a+2b}$. Note that in this simple result, even the constant infinite series factor $\frac{1}{(1-x^2)^3}$ is canceled out, implying that there should be a large amount of cancelation between bosonic and fermionic operators.

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Let us explain the first few terms to illustrate how to understand this number, and how the higher order cancelation is happening. We should suitably excite the fields to obtain the phase $e^{i(2\alpha-\beta-\gamma)} \equiv \frac{u^2}{vw}$. There are two possible ways of obtaining it at the lowest energy x^{a+2b}. This is either by exciting scalars $A_i B_{(j} B_{k)}$ carrying $\frac{w}{v}\left(\frac{u}{w}\right)^2$ phase which yields $(3x^a)(6x^{2b})$, or by exciting $B_i \bar{\Psi}_{Cj}$ as $\frac{u}{v}\frac{u}{w}$ with $(3x^b)(-3x^{2-c})$, yielding $(28-9)x^{a+2b} = 9x^{a+2b}$ in total and explains the above result.[8] One may further wonder what happens at higher orders. One can easily find that the phase can also be provided as $\left(\frac{u}{v}\right)^2 \frac{v}{w}$ by $\psi_{C[i}\psi_{Cj]}\psi_{Ak}$ with the index $(3x^{2a+2b})(-3x^{b+c}) = -9x^{a+2b+2}$, and by $(AB^2 - B\psi_C)(ABC - B\psi_B)$ with $(6\cdot 10\cdot 3 - 3^3\cdot 6 - 10\cdot 3^2 + 6\cdot 3^2)x^{a+2b+2} = -18x^{a+2b+2}$. So at this order, one obtains $9x^{a+2b}(1-x^2)^3$, where the second factor cancels the overall factor $\frac{1}{(1-x^2)^3}$, consistent with our general computation.

We should identify $9x^{a+2b}$ with suitable gravity states, which will give us a hint about the values of the R-charge a, b. Obviously, there is a cancelation $18-9=9$ between bosonic and fermionic modes. As for the bosonic modes AB^2, it is easy to guess what are the true BPS operators, as they are chiral rings. One simply symmetrizes all three SU(3) indices, obtaining $10x^{a+2b}$. So this should be part of the hypermultiplet with $p=1$ and SU(3) representation $(0,3)$. Thus, we find $a+2b=2$, i.e. $b=c$. This in turn implies that only one out of 9 operators $B\psi_C$ survive to be BPS. It is easy to find what they are in the gravity dual. They are the fermions in the massless vector multiplet for SU(2) symmetry, as the operator carries nonzero $\mathrm{U}(1)_B$ to be the SU(2) Cartan. Therefore, only the SU(3) singlet survives to be BPS.

In fact, summing over an infinite tower of all single gravity particles with $\mathrm{U}(1)_B$ charge 1, one obtains $I^{(1)}_{\rm sp}(x) = 9x^2$ as shown in the previous subsection, agreeing with $I_{(1)(1)(1)}$ calculated above. More generally, with all chemical potentials y_1, y_2 kept, we have also checked that the integral (4.27) again completely reproduces the gravity index $I(1)_{\rm sp}$ of (4.4).

To get a more nontrivial test of our claim, we investigate the sector with charge 2 with $P_+ = 2$ and $P_- = 0$. We shall compare the index with the charge 2 gravity index made of positively charge particles only. Up to $\mathcal{O}(x^{12})$ and for any trial R-charge b, we find

$$\begin{aligned}
I_{(2)(2)(2)} &= 18x^4 + 9x^8 - 27x^{10} - 27x^{12}\cdots \\
I_{(1,1)(1,1)(1,1)} &= 45x^4 - 54x^8 + 378x^{10} - 1053x^{12}\cdots \\
I_{(1,1)(1,1)(2)} &= 0 + \cdots \\
I_{(1,1)(2)(1,1)} &= 45x^8 - 360x^{10} + 1251x^{12}\cdots \\
I_{(2)(1,1)(1,1)} &= 0 + \cdots \\
I_{(2)(2)(1,1)} &= 9x^8 - 18x^{10} - 171x^{12}\cdots \\
I_{(2)(1,1)(2)} &= -9x^8 + 27x^{10} + 27x^{12}\cdots \\
I_{(1,1)(2)(2)} &= -27x^{12}\cdots \ .
\end{aligned} \tag{4.28}$$

[8] Just by requiring gauge invariance, one might think that the operator of the form $B_i C_j^\dagger$ could screen the monopole charge (1)(1)(1) with scale dimension $b+c$. The gauge invariance certainly can be achieved. However, since all the fields are chiral, this operator is not BPS (i.e. does not belong to chiral ring) and would acquire large anomalous dimension at strong coupling. The operators of the form $B_i C_j$ are chiral, but cannot screen the monopole charge.

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At $\mathcal{O}(x^4)$, the first two contributions are the expected ones. The first is the single particle contribution with $\mathrm{U}(1)_B$ charge 2, from hypermultiplets scalars in $(0,6)$ and vector multiplet fermion in $(0,3)$: $28-10-18$. The second is simply the 2-particle states of the 9 charge 1 particles: $9+36=45$. At higher orders, the sum of the above indices is $63x^4-\mathcal{O}(x^{14})$.

From the results summarized in the previous subsection, the single particle gravity index at charge 2 and 2 particle index of charge 1 particles are given by

$$I^{(2)}_{\rm sp}(x)=18x^4\,,\quad \frac{I^{(1)}_{\rm sp}(x^2)+I^{(1)}_{\rm sp}(x)^2}{2}=45x^4\,, \tag{4.29}$$

again after a vast cancelation of the infinite tower of graviton states, similar to the charge 1 single particle index. Although the expression is too messy to be recorded here, we report that we have also checked the agreement after keeping the SU(3) chemical potentials y_1, y_2. Therefore, the positive flux part of the index agrees with the charge 2 field theory index restricted to the positive fluxes up to $\mathcal{O}(x^{14})$, and we claim this happens to all orders.

To study more nontrivial sectors, we study the sector with $\mathrm{U}(1)_B$ charge 1 with $P_+=2$, $P_-=1$. To classify the fluxes in this sector, we denote by $P_\pm\equiv\sum|p_\pm|$, $O_\pm\equiv\sum|q_\pm|$, $R_\pm\equiv\sum|r_\pm|$ the sum over positive/negative fluxes in a gauge group. The conditions for decoupling overall U(1) and $\mathcal{O}(1)$ energy are

$$P_++P_-=Q_++Q_-=R_++R_-\,,\quad Q_+-Q_-+R_+-R_-=2(P_+-P_-)=2\,. \tag{4.30}$$

These fluxes are again classified by a non-negative integer n, with $P_+=n+1$, $P_-=n$. As the sector labeled by n starts at the order x^{4n+2}, we study the saddle points with $P_+=2, P_-=1$ which corrects the $n=0$ sector (positive flux only) that we considered above. Possible values of flux sums are

$$(Q_+,Q_-,R_+,R_-)=(2,1,2,1),\ (1,2,3,0),\ (3,0,1,2)\,. \tag{4.31}$$

The fluxes in the first case and the correspondinc indices are as follows:

$$\begin{aligned}
I_{(2,-1)(2,-1)(2,-1)} &= I_{(2)(2)(2)}I_{(-1)(-1)(-1)}=9x^2(18x^4+9x^8-27x^{10}-27x^{12}+\cdots)\\
I_{(1,1,-1)(1,1,-1)(1,1,-1)} &= I_{(1,1)(1,1)(1,1)}I_{(-1)(-1)(-1)}=9x^2(45x^4-54x^8+378x^{10}-1053x^{12}\cdots)\\
I_{(2,-1)(2,-1)(1,1,-1)} &= I_{(2)(2)(1,1)}I_{(-1)(-1)(-1)}=9x^2(9x^8-18x^{10}-171x^{12}\cdots)\\
I_{(2,-1)(1,1,-1)(2,-1)} &= I_{(2)(1,1)(2)}I_{(-1)(-1)(-1)}=9x^2(-9x^8+27x^{10}+27x^{12}\cdots)\\
I_{(1,1,-1)(2,-1)(2,-1)} &= I_{(1,1)(2)(2)}I_{(-1)(-1)(-1)}=9x^2(-27x^{12}+\cdots)\\
I_{(2,-1)(1,1,-1)(1,1,-1)} &= 162(27x^{16}-125x^{18}+169x^{20}+\cdots)\\
I_{(1,1,-1)(2,-1)(1,1,-1)} &: 405x^{10}-3240x^{12}+11259x^{14}+\cdots\\
I_{(1,1,-1)(1,1,-1)(2,-1)} &: 0x^{14}+\cdots\,.
\end{aligned} \tag{4.32}$$

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The 12 fluxes in the second case are

$$\begin{aligned}
I_{(2,-1)(1,-2)(3)} &= x^{4b}\left(9x^6 - 9x^8 - 171x^{10} - 135x^{12} + 747x^{14} + 1323x^{16} + \cdots\right)\\
I_{(2,-1)(1,-2)(2,1)} &= x^{4b}\left(-63x^8 + 144x^{10} + 270x^{12} - 351x^{14} - 1440x^{16} + \cdots\right)\\
I_{(2,-1)(1,-2)(1,1,1)} &= x^{4b}\left(180x^{14} - 261x^{16} + \cdots\right)\\
I_{(2,-1)(1,-1,-1)(3)} &= x^{4b}\left(-x^{10} + x^{12} + 17x^{14} + 17x^{16} + \cdots\right)\\
I_{(2,-1)(1,-1,-1)(2,1)} &= x^{4b}\left(126x^{12} - 288x^{14} - 288x^{16} + \cdots\right)\\
I_{(2,-1)(1,-1,-1)(1,1,1)} &= x^{4b}\left(0x^{16} + \cdots\right)\\
I_{(1,1,-1)(1,-2)(3)} &= x^{4b}\left(-18x^{10} - 36x^{12} + 198x^{14} + 549x^{16} + \cdots\right)\\
I_{(1,1,-1)(1,-2)(2,1)} &= x^{4b}\left(90x^8 + 45x^{10} - 36x^{12} + \cdots\right)\\
I_{(1,1,-1)(1,-2)(1,1,1)} &= x^{4b}\left(0x^{12} + \cdots\right)\\
I_{(1,1,-1)(1,-1,-1)(3)} &= x^{4b}\left(36x^{14} + \cdots\right)\\
I_{(1,1,-1)(1,-1,-1)(2,1)} &= x^{4b}\left(-180x^{12} + \cdots\right)\\
I_{(1,1,-1)(1,-1,-1)(1,1,1)} &= x^{4b}\left(0x^{12} + \cdots\right)
\end{aligned} \tag{4.33}$$

Note the explicit appearance of the trial R-charge b. At generic b, this does not cancel with any other term in the field theory index: for instance, the first term $9x^{4b+6}$ on the first line, which is larger than x^{10}, cannot combine with other terms listed above and below. In particular, for the field theory index to match with the gravity index including these sectors, b has to be half an integer as the gravity spectrum of $\epsilon + j_3$ is all even. The fluxes in the last case are obtained by flipping the second and third fluxes. We find that, up to the order that we wish to check, these sectors all give zero contributions (which might mean that they are identically zero):

$$\begin{aligned}
I_{(2,-1)(3)(1,-2)} &= 0x^{16b}x^{20} + \cdots\,,\ I_{(2,-1)(2,1)(1,-2)} = 0x^{16b}x^{20} + \cdots\,,\\
I_{(2,-1)(1,1,1)(1,-2)} &= 0x^{16b}x^{12} + \cdots\,,\ I_{(2,-1)(3)(1,-1,-1)} = 0x^{16b}x^{12} + \cdots\\
I_{(2,-1)(2,1)(1,-1,-1)} &= 0x^{16b}x^{12} + \cdots\,,\ I_{(2,-1)(1,1,1)(1,-1,-1)} = 0x^{16b}x^{12} + \cdots\\
I_{(1,1,-1)(3)(1,-2)} &= 0x^{16b}x^{12} + \cdots\,,\ I_{(1,1,-1)(2,1)(1,-2)} = 0x^{8b}x^{10} + \cdots\\
I_{(1,1,-1)(1,1,1)(1,-2)} &= 0x^{8b}x^{10} + \cdots\,,\ I_{(1,1,-1)(3)(1,-1,-1)} = 0x^{8b}x^{10} +\\
I_{(1,1,-1)(2,1)(1,-1,-1)} &= 0x^{8b}x^{10} + \cdots\,,\ I_{(1,1,-1)(1,1,1)(1,-1,-1)} = 0x^{8b}x^{10} + \cdots\,.
\end{aligned} \tag{4.34}$$

We want to compute the full index up to $\mathcal{O}(x^{14})$ and show that this factorizes into the product of 'positive-flux part' $I_{(P_+,P_-)=(2,0)}$ and 'negative-flux part' $I_{(P_+,P_-)=(0,1)}$ after summing over all the saddle points with $P_+ = 2$, $P_- = 1$. Summing over 32 saddle points with $P_+ = 2, P_- = 1$, one obtains

$$\begin{aligned}
I_{(P_+,P_-)=(2,1)} &= \hat{I}_{(P_+,P_-)=(2,1)} + I^b_{(P_+,P_-)=(2,1)}\\
&\equiv 9x^2\left(63x^4 - 45x^8 + 360x^{10} - 1251x^{12} + \cdots\right)\\
&\quad + (405x^{10} - 3240x^{12} + 11259x^{14} + \cdots) + x^{4b}\left(9x^6 + 18x^8 - x^{10} + \cdots\right)\,,
\end{aligned} \tag{4.35}$$

where $\hat{I}$ is the expression on the second line while I^b is the expression on the last line containing the trial R-charge b. The first parenthesis of the second line is from the first six

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saddle points with factorized indices, the second term from the next three with unfactorized indices, and the last line is from the next 12 saddle points whose leading energy levels depend on the undetermined trial R-charge b. It is easy to find that the first two contributions show a nontrivial cancelation and yield $\hat{I} = (9x^2)(63x^4) = I_{(P_+,P_-)=(2,0)} I_{(P_+,P_-)=(0,1)}$, and this agrees with the graviton index consisting of positive charged particles with net charge 2 and negatively charged particles with charge -1,

$$\hat{I}(x) = I_{\rm sp}^{(-1)}(x) \left(I_{\rm sp}^{(+2)}(x) + \frac{I_{\rm sp}^{(+1)}(x^2) + I_{\rm sp}^{(+1)}(x)^2}{2} \right) = (9x^2)(63x^4) \ . \tag{4.36}$$

On the other hand, the I^b contribution does not appear to correspond to anything in the gravity dual. Especially, considering the allowed range $0 < b < 1$ from $a, b, c > 0$, the first term is or lower order than $\mathcal{O}(x^{10})$. As we have explained, if these saddle points are artifacts of our localization calculations introduced by unphysical saddle points 'flowing in from infinite' [41], the remaining index completely agrees with the gravity index. This however requires a careful study, similar to [42].

At this stage, let us comment that we empirically find from the the above saddle points a rule for the indices which do not contain b. Namely, in the first eight saddle points, if one decomposes the positive and negative fluxes into two, each of them separately satisfies the flux condition (4.18) for the fluxes. Note that this restriction turns out to pick up the factorizable saddle points only.

5 Discussions

We initiated the index computation of Chern-Simons matter theories with $N = 2, 3$ theories. We find the perfect matchings between the field theory index and the gravity index for $N = 3$ theory, which describes M2 brane probing N^{010}/Z_k theory. On the other hand we find the several subtleties for $N = 2$ models even with the impressive matching with the gravity index. For Q^{111} model, which is realized as flavored ABJM model, the field theory index gives the definite prediction for the gravity index, which begs the reexamination of the KK spectrum of Q^{111} in the gravity side available in the literature.

For M^{32} model, the presence of odd number ($i = 1, 2, 3$) fermions in each bifundamental sector seems to pose a potential issue on whether this theory is consistent or not. At least in the context of dealing with QFT on $S^2 \times S^1$ and partition function there, we find that appropriately restricting the magnetic charges of gauge fields in the overall U(1)'s of U(N)'s on S^2 makes the resulting partition function well-defined. This is basically due to shifting of Chern-Simons level for U(1) factors (among others) out of U(N) due to fermion determinant. It would be interesting to work out the structure of the potential mixed Chern-Simons terms. The comparison with the known gravity index is amazingly impressive but we face handling with the new saddle points. It could be important to understand this problem to understand better AdS4/CFT3 correspondence for vast classes of $N = 2$ theories. Even though we do not report our computation of the index for Q^{111} model suggested at [32, 33], they are suffering from the same problems as our M^{32} model has, i.e., potential anomaly issues due to chiral fermions and new saddle

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points. Now these problems are not appearing in the Q^{111} model [28, 29] we considered, we hope to construct new M^{32} model free from such subtleties, yet giving the impressive matching with the gravity index, which our current M^{32} model exhibits. We hope to report our progress in this direction in near future. However, we emphasize that final verdict on the current M^{32} model has not been made yet.

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