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Spectral form factor in the τ-scaling limit

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Abstract: We study the spectral form factor (SFF) of general topological gravity in the limit of large time and fixed temperature. It has been observed recently that in this limit, called the tau-scaling limit, the genus expansion of the SFF can be summed up and the late-time behavior of the SFF such as the ramp-plateau transition can be studied analytically. In this paper we develop a technique for the systematic computation of the higher order corrections to the SFF in the strict tau-scaling limit. We obtain the first five corrections in a closed form for the general background of topological gravity. As concrete examples, we present the results for the Airy case and Jackiw-Teitelboim gravity. We find that the above higher order corrections are the Fourier transforms of the corrections to the sine-kernel approximation of the Christoffel-Darboux kernel in the dual double-scaled matrix integral, which naturally explains their structure. Along the way we also develop a technique for the systematic computation of the corrections to the sine-kernel formula, which have not been fully explored in the literature before.

Keywords: 2D Gravity, Integrable Hierarchies, Matrix Models

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1 Introduction

Spectral form factor (SFF) is a useful measure of the level statistics of quantum chaotic systems [1] and it is widely studied in many areas of physics. In the context of quantum gravity and holography, the SFF of the Sachdev-Ye-Kitaev (SYK) model [2–4] was studied in [5, 6] as a useful diagnostics of the Maldacena's version of the information problem [7]. It is found that the SFF of the SYK model exhibits the behavior of the ramp and plateau as a function of time, which is consistent with the conjecture that the level statistics of quantum chaotic system is universally described by a random matrix model [8]. As shown in [9], Jackiw-Teitelboim (JT) gravity [10, 11], which is holographically dual to the low energy sector of the SYK model, is indeed described by a certain double-scaled random matrix model. In the bulk gravity picture, the ramp of the SFF of JT gravity comes from the contribution of a wormhole connecting two boundaries of spacetime [12]. In the matrix model picture, it is described by the connected part of the correlator of two macroscopic loop operators [13]. On the other hand, the plateau of the SFF is very mysterious from the viewpoint of bulk gravity and it was speculated that the appearance of the plateau is related to some non-perturbative effects in quantum gravity [6, 9]. In the matrix model picture, it is argued in [6, 9] that the plateau can be explained by the Andreev-Altshuler instantons [14]. See also [15] for an explanation of the plateau by eigenvalue instantons.

In this paper, we will consider the connected part of the SFF in 2d quantum gravity

$$\text{SFF} = \langle Z(\beta + \mathrm{i}t) Z(\beta - \mathrm{i}t) \rangle_{\mathrm{c}}, \tag{1.1}$$

where $Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$ and the expectation value is defined by averaging over a random matrix H. As shown in [16–18], 2d quantum gravity coupled to a conformal matter can be defined by a certain double-scaling limit of a random matrix model. More generally, one can introduce couplings $\{t_k\}$ $(k = 0, 1, 2, \ldots)$ to the matrix model potential. Then the free energy of the model is interpreted as a generating function of the intersection numbers on the moduli space of Riemann surfaces and the model is called topological gravity [19]. There is an underlying integrable structure in this model and the free energy serves as a tau-function of the KdV hierarchy [19, 20]. It turns out that the matrix model of JT gravity in [9] is a special case of topological gravity where infinitely many couplings $\{t_k\}$ are turned on in a specific way [21–23] and it corresponds to the computation of Weil-Petersson volumes [24, 25].

SFF exhibits the ramp-plateau transition around the time scale $t \sim \hbar^{-1}$, called the Heisenberg time, where $\hbar$ is the genus-counting parameter of the matrix model. Recently, it is observed in [26–28] that one can focus on the ramp-plateau transition regime by taking what is called the "τ-scaling limit"[1]

$$t \to \infty, \quad \hbar \to 0, \quad \tau = t\hbar : \text{fixed}, \tag{1.2}$$

with β fixed in (1.1). In this limit, SFF is expanded as

$$\text{SFF} = \sum_{g=0}^{\infty} \hbar^{2g-1} \text{SFF}_g(\tau, \beta). \tag{1.3}$$

Remarkably, it is found that the leading term SFF_0 can be computed in a closed form by just summing over the original genus expansion in the τ-scaling limit and the resulting SFF_0 approaches a constant as $\tau \to \infty$ [26]. This opens up an interesting avenue for a "perturbative plateau."

In this paper, we will develop a technique for the systematic computation of the higher order corrections SFF_g $(g \geq 1)$. By using our method, we obtain SFF_g up to $g = 5$ for arbitrary couplings $\{t_k\}$ of topological gravity. It turns out that SFF_g has a structure which is a natural generalization of SFF_0. In [26] it was shown that SFF_0 is essentially determined by the Fourier transform of the universal part of the two-body eigenvalue correlation, known as the sine kernel formula [30–32]. We find that the higher order correction SFF_g is closely related to the correction of the Christoffel-Darboux (CD) kernel to the naive sine kernel formula. Rather surprisingly, such corrections to the sine kernel formula have not been

[1]In [23, 29], another scaling limit where both β and t are of order $\hbar^{-1}$ was considered. This limit was called the "'t Hooft limit" in [23].

fully explored in the literature before, as far as we know.[2] We will also develop a technique for the systematic computation of the corrections of the CD kernel and confirm that SFF_g is correctly reproduced form the Fourier transform of the CD kernel by including the corrections to the sine kernel formula.

This paper is organized as follows. In section 2, we review the known results about the SFF in the τ-scaling limit. Along the way we consider the τ-scaling limit based on the genus expansion of the SFF and obtain the small τ expansion of SFF_g for small g. In section 3, we summarize our results of SFF_g and the higher order corrections of the CD kernel. We also explain their relations. In section 4, based on our general results we compute SFF_0 and SFF_1 for the Airy case and JT gravity, as concrete examples. In section 5, we formulate a systematic method of computing SFF_g. In section 6, we explain how to compute the higher order corrections of the CD kernel beyond the sine kernel approximation. Finally we conclude in section 7. In appendix A, we compute the corrections of the Airy kernel to the sine kernel formula. In appendix B, we present an explicit form of the coefficient that determines SFF_2.

2 τ-scaling limit of SFF

In this section we will briefly review the known results about the SFF in the τ-scaling limit. We will consider the τ-scaling limit for the general background of topological gravity based on the known genus expansion of the SFF, which enables us to compute the small τ expansion of SFF_g for small g.

For the general background $\{t_k\}$ of topological gravity, the connected two-boundary correlator is expanded as [29]

$$\begin{aligned}\langle Z(\beta_1)Z(\beta_2)\rangle_{\rm c} = \frac{\sqrt{\beta_1\beta_2}}{2\pi}e^{-(\beta_1+\beta_2)E_0}\Bigg[&\frac{1}{\beta_1+\beta_2}\\ &+\left(\frac{\beta_1^2+\beta_1\beta_2+\beta_2^2}{24s^2}+\frac{2(\beta_1+\beta_2)I_2+I_3}{24s^3}+\frac{I_2^2}{12s^4}\right)g_{\rm s}^2+\mathcal{O}(g_{\rm s}^4)\Bigg],\end{aligned} \tag{2.1}$$

where $g_{\rm s}$ is the genus-counting parameter related to $\hbar$ as[3]

$$g_{\rm s} = \sqrt{2}\hbar. \tag{2.2}$$

I_k denote the Itzykson-Zuber variables defined by [33]

$$I_k = \sum_{n=0}^{\infty}\frac{t_{k+n}}{n!}(-E_0)^n, \quad k\in\mathbb{Z}_{\geq 0} \tag{2.3}$$

and s in (2.1) is given by

$$s = 1-I_1. \tag{2.4}$$

The threshold energy E_0 is determined by the (genus-zero) string equation

$$I_0+E_0=0. \tag{2.5}$$

[2]In [32], general form of the large N limit of the CD kernel was studied from the asymptotic behavior of the orthogonal polynomials associated to an arbitrary matrix model potential, before taking the double-scaling limit.

[3]See [23] for more about our definition of $\hbar$ and its relation to the parameters S_0,γ in JT gravity.

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In the τ-scaling limit (1.2), after setting $\beta_{1,2} = \beta \pm \mathrm{i}\tau/\hbar$ in the two-boundary correlator (2.1) and expanding it in $\hbar$, we find the small τ expansion of SFF_0 and SFF_1

$$\begin{aligned} \mathrm{SFF}_0 &= \frac{e^{-2\beta E_0}}{2\pi}\left[\frac{\tau}{2\beta} - \frac{\tau^3}{12s^2} + \cdots\right], \\ \mathrm{SFF}_1 &= \frac{e^{-2\beta E_0}}{2\pi}\left[\frac{\beta}{4\tau} + \left(\frac{5\beta^2}{24s^2} + \frac{4\beta I_2 + I_3}{12s^3} + \frac{I_2^2}{6s^4}\right)\tau + \cdots\right]. \end{aligned} \tag{2.6}$$

In general, the two-boundary correlator is written as

$$\langle Z(\beta_1)Z(\beta_2)\rangle_{\mathrm{c}} = \langle Z(\beta_1+\beta_2)\rangle - \int dE_1 dE_2 e^{-\beta_1 E_1 - \beta_2 E_2} K(E_1,E_2)^2, \tag{2.7}$$

where $K(E_1, E_2)$ is the Christoffel-Darboux (CD) kernel. Thus, the SFF is written as

$$\mathrm{SFF} = \int \frac{dE}{2\pi} e^{-2\beta E}\rho(E) - \int dE_1 dE_2 e^{-\beta(E_1+E_2) - \mathrm{i}\tau\frac{E_1-E_2}{\hbar}} K(E_1,E_2)^2. \tag{2.8}$$

In the τ-scaling limit (1.2), the above integral is dominated by the region

$$E_1 - E_2 \sim \mathcal{O}(\hbar), \tag{2.9}$$

with finite $E_1 + E_2$. In this regime, the CD kernel is approximated by the universal two-body correlation of eigenvalues, known as the sine kernel [30–32]

$$K_{\sin}(E_1,E_2) = \frac{\sin\left[\frac{1}{2}\rho_0(E)\omega\right]}{\pi\hbar\omega}, \tag{2.10}$$

where we defined

$$E_1 = E + \frac{1}{2}\hbar\omega, \qquad E_2 = E - \frac{1}{2}\hbar\omega. \tag{2.11}$$

$\rho_0(E)$ is the genus-zero part of the eigenvalue density

$$\rho(E) = \sum_{g=0}^{\infty} \hbar^{2g-1}\rho_g(E) \tag{2.12}$$

which is related to the one-point function by

$$\langle Z(\beta)\rangle = \int_{E_0}^{\infty} \frac{dE}{2\pi} e^{-\beta E}\rho(E) = \sum_{g=0}^{\infty} \hbar^{2g-1}\langle Z(\beta)\rangle_g. \tag{2.13}$$

For the general background $\{t_k\}$, $\rho_0(E)$ is given by

$$\rho_0(E) = \sum_{k=1}^{\infty} \frac{(-1)^k (I_k - \delta_{k,1})\Gamma(1/2)}{\Gamma(k+1/2)}(E-E_0)^{k-\frac{1}{2}}. \tag{2.14}$$

In [26–28], it is argued that the above small τ expansion of SFF_0 (2.6) can be resummed and SFF_0 is obtained by replacing the CD kernel by the sine kernel in (2.8). Then, by the change of integration variables in (2.11), the leading term of the SFF becomes

$$\mathrm{SFF}_0 = \int_{E_0}^{\infty} \frac{dE}{2\pi} e^{-2\beta E}\rho_0(E) - \int_{E_0}^{\infty} dE e^{-2\beta E} \int_{-\infty}^{\infty} d\omega e^{-\mathrm{i}\omega\tau} \frac{\sin^2\left(\frac{1}{2}\rho_0(E)\omega\right)}{\pi^2\omega^2}. \tag{2.15}$$

The ω-integral is evaluated as [34]

$$\int_{-\infty}^{\infty} d\omega e^{-\mathrm{i}\omega\tau} \frac{\sin^2[\frac{1}{2}\rho_0(E)\omega]}{\pi^2\omega^2} = \frac{1}{2\pi}(\rho_0(E)-\tau)\theta(\rho_0(E)-\tau), \tag{2.16}$$

where $\theta(x)$ is the step function

$$\theta(x) = \begin{cases} 1, & (x>0), \\ 0, & (x<0). \end{cases} \tag{2.17}$$

Assuming that $\rho_0(E)$ is a monotonically increasing function of E, there is a unique solution E_τ to the equation

$$\rho_0(E_\tau) = \tau. \tag{2.18}$$

In terms of E_τ, SFF_0 in (2.15) is written as

$$\begin{aligned} \mathrm{SFF}_0 &= \int_{E_0}^{\infty} \frac{dE}{2\pi} e^{-2\beta E}\rho_0(E) - \int_{E_\tau}^{\infty} \frac{dE}{2\pi} e^{-2\beta E}(\rho_0(E)-\tau) \\ &= \frac{\tau}{4\pi\beta} e^{-2\beta E_\tau} + \int_{E_0}^{E_\tau} \frac{dE}{2\pi} e^{-2\beta E}\rho_0(E). \end{aligned} \tag{2.19}$$

The first term corresponds to the ramp and the second term approaches a constant $\langle Z(2\beta)\rangle_0$ in the $\tau\to\infty$ limit, corresponding to the plateau.

One can check that (2.19) reproduces the small τ expansion of SFF_0 in (2.6). To see this, we first notice that E_τ has the following small τ expansion[4]

$$E_\tau - E_0 = \frac{1}{4s^2}\tau^2 - \frac{I_2}{12s^5}\tau^4 + \left(\frac{7I_2^2}{144s^8} + \frac{I_3}{120s^7}\right)\tau^6 + \mathcal{O}(\tau^8). \tag{2.20}$$

This can be obtained by inverting the relation (2.18) using the expression of $\rho_0(E)$ in (2.14). Plugging this expansion of E_τ into (2.19), one can show that the small τ expansion of SFF_0 in (2.6) is indeed reproduced from (2.19).

In the rest of this paper, we will consider the higher order corrections SFF_g $(g\geq 1)$ to the spectral form factor. In the next section we will first summarize the result of SFF_g. The details of the computation will be postponed to section 5.

3 Summary of results

In this section we will summarize our results of SFF_g and discuss their relation to the corrections of the CD kernel.

3.1 SFF_g

It turns out that the expression of SFF_0 in (2.19) has a natural generalization to the higher order correction SFF_g. We find that SFF_g has the structure

$$\mathrm{SFF}_g = \frac{1}{2\pi} f_g(\tau,\beta) e^{-2\beta E_\tau} + \int_{E_0}^{E_\tau} \frac{dE}{2\pi} e^{-2\beta E}\rho_g(E). \tag{3.1}$$

[4] A fully explicit expression of this expansion is available (E_τ is given by $E(\lambda)$ in (5.12) with $\lambda = \mathrm{i}\tau$).

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By the relation (2.13) we can easily find the first few terms of $\rho_g(E)$ from the result of $\langle Z(\beta)\rangle_g$ in [23]. For instance,

$$\begin{aligned}\rho_1(E) &= \frac{1}{16sz^5} - \frac{I_2}{24s^2z^3},\\ \rho_2(E) &= -\frac{105}{1024s^3z^{11}} + \frac{203I_2}{1536s^4z^9}\\ &\quad + \frac{1}{z^7}\left(-\frac{7I_2^2}{64s^5} - \frac{29I_3}{768s^4}\right) + \frac{1}{z^5}\left(\frac{7I_2^3}{96s^6} + \frac{29I_2I_3}{480s^5} + \frac{I_4}{128s^4}\right)\\ &\quad + \frac{1}{z^3}\left(-\frac{7I_2^4}{144s^7} - \frac{5I_2^2I_3}{72s^6} - \frac{11I_2I_4}{720s^5} - \frac{29I_3^2}{2880s^5} - \frac{I_5}{576s^4}\right),\end{aligned} \tag{3.2}$$

where

$$z = \sqrt{E-E_0}. \tag{3.3}$$

Note that the second term of (3.1) is a formal expression since the integral has a divergence coming from $E=E_0$. This integral should be understood by a certain analytic continuation. For instance, the integral involving z^{-a} term is defined as

$$\begin{aligned}\int_{E_0}^{E_\tau} dEe^{-2\beta E}z^{-a} &= \int_{E_0}^{\infty} dEe^{-2\beta E}z^{-a} + \int_{\infty}^{E_\tau} dEe^{-2\beta E}z^{-a}\\ &= e^{-2\beta E_0}\Gamma(1-a/2)(2\beta)^{a/2-1} + \int_{\infty}^{E_\tau} dEe^{-2\beta E}z^{-a}.\end{aligned} \tag{3.4}$$

Using this prescription, one can show that the second term of (3.1) is written as

$$\int_{E_0}^{E_\tau}\frac{dE}{2\pi}e^{-2\beta E}\rho_g(E) = \langle Z(2\beta)\rangle_g \mathrm{Erf}(z_\tau\sqrt{2\beta}) + \frac{1}{2\pi}d_g(z_\tau,\beta)e^{-2\beta E_\tau}, \tag{3.5}$$

where $\mathrm{Erf}(z)$ denotes the error function and z_τ is defined by

$$z_\tau = \sqrt{E_\tau - E_0}. \tag{3.6}$$

It turns out that $d_g(z_\tau,\beta)$ in (3.5) is a polynomial in $1/z_\tau$. For instance, $d_1(z_\tau,\beta)$ is given by

$$d_1(z_\tau,\beta) = \frac{\beta}{6sz_\tau} - \frac{1}{24sz_\tau^3} + \frac{I_2}{12s^2z_\tau}. \tag{3.7}$$

Next, let us consider the first term of (3.1). We find that $f_g(\tau,\beta)e^{-2\beta E_\tau}$ can be rewritten as a sum of τ-derivatives

$$f_g(\tau,\beta)e^{-2\beta E_\tau} = \sum_{n=0}^{3g-2}\partial_\tau^n\left[h_{g,n}(\tau)e(1)e^{-2\beta E_\tau}\right]. \tag{3.8}$$

Here we introduced the notation

$$e(j) = \partial_\tau^j E_\tau. \tag{3.9}$$

The important point is that $h_{g,n}(\tau)$ is independent of β; the β-dependence of $f_g(\tau,\beta)$ arises solely from the τ-derivative in (3.8). We have computed $h_{g,n}$ up to $g=5$. For $g=1$ we find

$$h_{1,1} = -s(2), \qquad h_{1,0} = \frac{1}{16z_\tau^4}, \tag{3.10}$$

where $s(j)$ denotes

$$s(j) = \frac{\rho_0^{(j)}(E_\tau)}{(j+1)!2^j} \tag{3.11}$$

with $\rho_0^{(j)}(E) = \partial_E^j \rho_0(E)$. For $g = 2$ we find

$$\begin{aligned}
h_{2,4} &= -\frac{1}{2}s(2)^2, \\
h_{2,3} &= s(4) + \frac{1}{16z_\tau^4}s(2), \\
h_{2,2} &= -\frac{3}{256z_\tau^8} + \rho_1(E_\tau)s(2), \\
h_{2,1} &= \frac{7I_2}{768s^2z_\tau^7} - \frac{73}{1536sz_\tau^9}, \\
h_{2,0} &= -\frac{I_3}{96s^3z_\tau^6} - \frac{7}{128s^2z_\tau^{10}} - \frac{25}{2}\rho_1(E_\tau)^2.
\end{aligned} \tag{3.12}$$

From the definition of E_τ in (2.18), one can show that $\rho_0^{(j)}(E_\tau)$ can be written as a combination of $e(j)$. For instance,

$$\begin{aligned}
\rho_0^{(2)}(E_\tau) &= -\frac{e(2)}{e(1)^3}, \\
\rho_0^{(4)}(E_\tau) &= -\frac{e(4)}{e(1)^5} - \frac{15e(2)^3}{e(1)^7} + \frac{10e(3)e(2)}{e(1)^6}.
\end{aligned} \tag{3.13}$$

For general g, we find that $h_{g,n}$ with $n = 3g-2$ and $n = 3g-3$ are given by

$$h_{g,3g-2} = -\frac{s(2)^g}{g!}, \qquad h_{g,3g-3} = \frac{1}{16z_\tau^4}\frac{s(2)^{g-1}}{(g-1)!} + s(4)\frac{s(2)^{g-2}}{(g-2)!}. \tag{3.14}$$

In subsection 3.2, we will see that this structure can be naturally understood from the correction of the CD kernel to the sine kernel formula.

Let us take a closer look at the behavior of SFF_g in (3.1) as a function of τ. At late times, the first term of (3.1) vanishes exponentially and the second term approaches a constant

$$\lim_{\tau\to\infty} \text{SFF}_g = \langle Z(2\beta)\rangle_g. \tag{3.15}$$

This gives the higher genus correction to the value of plateau. On the other hand, at early times SFF_g diverges as

$$\lim_{\tau\to 0} \text{SFF}_g \sim e^{-2\beta E_0}\left(\frac{\beta}{\tau}\right)^{2g-1}. \tag{3.16}$$

This is just an artifact of the τ-scaling limit. This early time divergence can be traced back to the expansion of the original genus-zero term

$$\frac{\sqrt{\beta_1\beta_2}}{2\pi(\beta_1+\beta_2)} = \frac{\sqrt{\tau^2+\beta^2\hbar^2}}{4\pi\beta\hbar} = \frac{1}{4\pi}\sum_{g=0}^{\infty}\frac{(-1)^{g-1}(2g-3)!!}{g!2^g}\left(\frac{\beta\hbar}{\tau}\right)^{2g-1}. \tag{3.17}$$

Before expanding in $\hbar$, the original expression $\sqrt{\tau^2+\beta^2\hbar^2}$ is regular at $\tau = 0$, but after expanding it by $\hbar$ there appears an apparent divergence at $\tau = 0$. We have checked that the coefficient of $\mathcal{O}(\tau^{1-2g})$ term of SFF_g in the small τ expansion is indeed given by (3.17).

3.2 Relation to the corrections of CD kernel

From the general relation between the SFF and the CD kernel $K(E_1, E_2)$ in (2.8), the higher order correction SFF_g is closely related to the correction of the CD kernel to the naive sine kernel formula (2.10). The details of the computation of CD kernel will be presented in section 6. It turns out that the τ-derivative structure in (3.8) naturally appears from the Fourier transform of the CD kernel squared

$$\text{SFF} = \int \frac{dE}{2\pi} e^{-2\beta E}\rho(E) - \hbar \int dE e^{-2\beta E} \int d\omega e^{-\mathrm{i}\omega\tau} K\left(E + \frac{1}{2}\hbar\omega, E - \frac{1}{2}\hbar\omega\right)^2. \tag{3.18}$$

We find that the corrections of the CD kernel to the sine kernel formula is organized as

$$\begin{aligned} K\left(E + \frac{1}{2}\hbar\omega, E - \frac{1}{2}\hbar\omega\right) &= \left[\frac{2}{\hbar\omega} + \sum_{g=1}^{\infty} \hbar^{2g-1} k_{\text{s}}^{(g)}(E,\omega)\right] \frac{\sin\phi}{2\pi} \\ &\quad + \sum_{g=1}^{\infty} \hbar^{2g-1} \Big[\rho_g(E) + k_{\text{c}}^{(g)}(E,\omega)\Big] \frac{\cos\phi}{2\pi}, \end{aligned} \tag{3.19}$$

where ϕ is given by

$$\phi = \frac{1}{2\hbar} \int_{E-\frac{1}{2}\hbar\omega}^{E+\frac{1}{2}\hbar\omega} dE \rho_0(E) = \frac{1}{2} \sum_{j=0}^{\infty} \hbar^{2j} \omega^{2j+1} \frac{\rho_0^{(2j)}(E)}{(2j+1)!2^{2j}}. \tag{3.20}$$

Note that the diagonal part of the CD kernel is equal to the eigenvalue density

$$K(E,E) = \frac{1}{2\pi}\rho(E) = \frac{1}{2\pi} \sum_{g=0}^{\infty} \hbar^{2g-1} \rho_g(E). \tag{3.21}$$

We find that $k_{\text{s}}^{(g)}$ and $k_{\text{c}}^{(g)}$ vanishes as $\omega \to 0$ and hence our expansion (3.19) is consistent with the diagonal part (3.21). The explicit forms of $k_{\text{s}}^{(g)}$ and $k_{\text{c}}^{(g)}$ for $g = 1, 2$ are given by

$$\begin{aligned} k_{\text{s}}^{(1)} &= \frac{\omega}{16z^4}, \\ k_{\text{c}}^{(1)} &= 0, \\ k_{\text{s}}^{(2)} &= \frac{11\omega^3}{1024z^8} + \left(-\frac{7}{128s^2z^{10}} - \frac{I_3}{96s^3z^6} - \frac{49}{4}\rho_1(E)^2\right)\omega, \\ k_{\text{c}}^{(2)} &= \left(\frac{35}{768sz^9} - \frac{I_2}{128s^2z^7}\right)\omega^2. \end{aligned} \tag{3.22}$$

Expanding $\sin\phi$ and $\cos\phi$ further in $\hbar$, we find the $\hbar$ expansion of the CD kernel in (3.19)

$$K\left(E + \frac{1}{2}\hbar\omega, E - \frac{1}{2}\hbar\omega\right) = \sum_{g=0}^{\infty} \hbar^{2g-1} K_g(E,\omega). \tag{3.23}$$

One can see that the leading term is the sine kernel

$$K_0(E,\omega) = \frac{\sin\left[\frac{1}{2}\rho_0(E)\omega\right]}{\pi\omega}, \tag{3.24}$$

and the higher order correction $K_g(E,\omega)$ can be obtained from (3.19).

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From the general form of the CD kernel (3.19), one can see that the τ-derivative structure of (3.8) can be understood from the Fourier transform of (3.19). Let us consider a contribution of the term

$$K\left(E+\frac{1}{2}\hbar\omega, E-\frac{1}{2}\hbar\omega\right)^2 \supset \omega^n e^{\mathrm{i}\rho_0(E)\omega}, \quad (n \geq 0). \tag{3.25}$$

From the Fourier transformation of this term, we find

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-\mathrm{i}\omega\tau}\omega^n e^{\mathrm{i}\rho_0(E)\omega} &= (\mathrm{i}\partial_\tau)^n \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-\mathrm{i}\omega\tau+\mathrm{i}\rho_0(E)\omega} \\ &= (\mathrm{i}\partial_\tau)^n \delta(\rho_0(E)-\tau) \\ &= (\mathrm{i}\partial_\tau)^n \big[e(1)\delta(E-E_\tau)\big],\end{aligned} \tag{3.26}$$

where we used the relation

$$\rho^{(1)}(E_\tau) = \frac{1}{e(1)}. \tag{3.27}$$

After integrating over E in (3.18), we find the τ-derivative structure of $f_g(\tau,\beta)e^{-2\beta E_\tau}$ in (3.8). We have checked that $h_{1,n}$ in (3.10) and $h_{2,n}$ in (3.12) are correctly reproduced from the Fourier transform of the square of the CD kernel in (3.19). In particular, the appearance of $s(2j)$ can be naturally understood from the expansion of ϕ in (3.20).

The second term of (3.1) arises from the cross-term of $\sin\phi$ and $\cos\phi$ in the CD kernel squared

$$K\left(E+\frac{1}{2}\hbar\omega, E-\frac{1}{2}\hbar\omega\right)^2 \supset \frac{\sin[\rho_0(E)\omega]}{2\pi^2\hbar\omega}\sum_{g=1}^{\infty}\hbar^{2g-1}\rho_g(E). \tag{3.28}$$

Using the relation

$$\int_{-\infty}^{\infty} d\omega e^{-\mathrm{i}\omega\tau}\frac{\sin[\rho_0(E)\omega]}{2\pi^2\omega} = \frac{1}{2\pi}\theta(\rho_0(E)-\tau), \tag{3.29}$$

we find that the second term of (3.1) is reproduced from (3.18)

$$\begin{aligned}\mathrm{SFF}_g \supset &\int_{E_0}^{\infty}\frac{dE}{2\pi}e^{-2\beta E}\rho_g(E) - \int_{E_0}^{\infty}\frac{dE}{2\pi}e^{-2\beta E}\rho_g(E)\theta(\rho_0(E)-\tau) \\ = &\int_{E_0}^{\infty}\frac{dE}{2\pi}e^{-2\beta E}\rho_g(E) - \int_{E_\tau}^{\infty}\frac{dE}{2\pi}e^{-2\beta E}\rho_g(E) \\ = &\int_{E_0}^{E_\tau}\frac{dE}{2\pi}e^{-2\beta E}\rho_g(E).\end{aligned} \tag{3.30}$$

To summarize, the structure of SFF_g in (3.1) and (3.8) can be naturally understood from the Fourier transform of the CD kernel in (3.19). To our knowledge, corrections to the sine kernel formula are not known in the literature before and our (3.19) with (3.22) is a new result.[5]

[5]See also a comment in footnote 2.

4 Examples of Airy case and JT gravity

In this section, as concrete examples we compute SFF_0 and SFF_1 for the Airy case and JT gravity using our general result in the previous section.

4.1 Airy case

Let us first consider the Airy case. For the Airy case, all couplings t_k are set to zero. Then one can show that $E_0 = 0$ and $I_k = 0$ $(k \geq 1)$, and the genus-zero eigenvalue density (2.14) becomes

$$\rho_0(E) = 2\sqrt{E}. \tag{4.1}$$

From the definition of E_τ in (2.18), we find

$$E_\tau = \frac{\tau^2}{4}, \qquad z_\tau = \sqrt{E_\tau} = \frac{\tau}{2}. \tag{4.2}$$

From the general formula in the previous sections, we find that the SFF_0 and SFF_1 for the Airy case are given by

$$\begin{aligned} \mathrm{SFF}_0 &= \frac{1}{2\sqrt{\pi(2\beta)^3}}\mathrm{Erf}\left(\tau\sqrt{\frac{\beta}{2}}\right), \\ \mathrm{SFF}_1 &= \frac{\beta}{8\pi\tau}e^{-\frac{1}{2}\beta\tau^2} + \frac{\beta^{3/2}}{6\sqrt{2\pi}}\mathrm{Erf}\left(\tau\sqrt{\frac{\beta}{2}}\right). \end{aligned} \tag{4.3}$$

We can compare this with the exact result of two-boundary correlator for the Airy case [35]

$$\langle Z(\beta_1)Z(\beta_2)\rangle_{\mathrm{c}} = \frac{e^{\frac{\hbar^2}{12}(\beta_1+\beta_2)^3}}{2\hbar\sqrt{\pi(\beta_1+\beta_2)^3}}\mathrm{Erf}\left(\frac{\hbar}{2}\sqrt{\beta_1\beta_2(\beta_1+\beta_2)}\right). \tag{4.4}$$

One can check that the $\mathrm{SFF}_{0,1}$ in (4.3) are indeed reproduced from the τ-scaling limit of the exact result (4.4). See also appendix A for the corrections of the CD kernel in the Airy case.

4.2 JT gravity

Next consider the SFF of JT gravity. For JT gravity, the couplings t_k are given by [21–23]

$$t_0 = t_1 = 0, \qquad t_k = \frac{(-1)^k}{(k-1)!}, \quad (k \geq 2). \tag{4.5}$$

In this case, $E_0 = 0$ and the genus-zero eigenvalue density (2.14) becomes

$$\rho_0(E) = \sinh(2\sqrt{E}). \tag{4.6}$$

From (2.18), E_τ and z_τ are given by

$$E_\tau = \frac{1}{4}\mathrm{arcsinh}(\tau)^2, \quad z = \sqrt{E_\tau} = \frac{1}{2}\mathrm{arcsinh}(\tau). \tag{4.7}$$

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From the general formula (2.19), the leading term SFF_0 is evaluated as [26]

$$\mathrm{SFF}_0 = \frac{1}{2}\langle Z(2\beta)\rangle_{g=0}\left[\mathrm{Erf}\left(\frac{\beta\mathrm{arcsinh}(\tau)+1}{\sqrt{2\beta}}\right)+\mathrm{Erf}\left(\frac{\beta\mathrm{arcsinh}(\tau)-1}{\sqrt{2\beta}}\right)\right], \tag{4.8}$$

where the genus-zero one-point function is given by

$$\langle Z(\beta)\rangle_{g=0} = \frac{e^{\frac{1}{\beta}}}{2\sqrt{\pi\beta^3}}. \tag{4.9}$$

The next order correction SFF_1 for JT gravity can be found from the general result in the previous section. After some algebra, we find

$$\begin{aligned}\mathrm{SFF}_1 = \frac{1}{24\pi}e^{-\frac{1}{2}\beta\mathrm{arcsinh}(\tau)^2}\Bigg[&\frac{\tau}{1+\tau^2}\left(\beta+\frac{1}{\mathrm{arcsinh}(\tau)^2}\right)+\frac{4}{\mathrm{arcsinh}(\tau)^3}\left(\frac{1}{\sqrt{1+\tau^2}}-1\right)\\ &+\frac{1}{\mathrm{arcsinh}(\tau)}\left(2-\frac{1}{(1+\tau^2)^{3/2}}+\beta\left(4-\frac{1}{\sqrt{1+\tau^2}}\right)\right)\Bigg]\\ &+\langle Z(2\beta)\rangle_{g=1}\mathrm{Erf}\left(\sqrt{\frac{\beta}{2}}\mathrm{arcsinh}(\tau)\right),\end{aligned} \tag{4.10}$$

where the genus-one one-point function is given by

$$\langle Z(\beta)\rangle_{g=1} = \frac{(1+\beta)\sqrt{\beta}}{24\sqrt{\pi}}. \tag{4.11}$$

In figure 1, we show the plot of SFF_0 and SFF_1 of JT gravity. From figure 1a we can see that SFF_0 exhibits the behavior of the ramp and plateau. From figure 1b we see that SFF_1 approaches a constant at late times while it blows up as $\tau \to 0$. As we argued in the previous section, this diverging behavior of SFF_1 at $\tau = 0$ is an artifact of the τ-scaling limit. In fact, the small τ expansion of SFF_1 in (4.10) reads

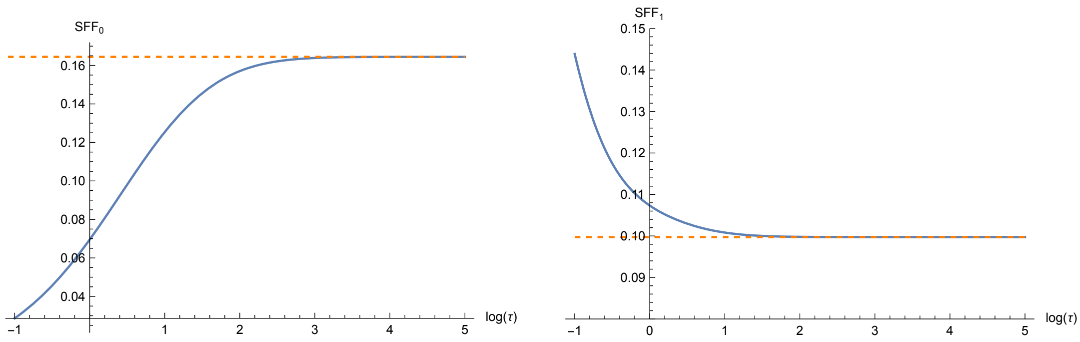
$$\mathrm{SFF}_1 = \frac{1}{2\pi}\left[\frac{\beta}{4\tau}+\frac{3+8\beta+5\beta^2}{24}\tau+\mathcal{O}(\tau^3)\right]. \tag{4.12}$$

One can see that the negative power term of τ agrees with the $\mathcal{O}(\hbar)$ term of (3.17), as expected. One can also check that this expansion (4.12) is consistent with the general case in (2.6).

5 Computation of SFF

In this section we formulate a systematic method of computing the small $\hbar$ expansion of the τ-scaling limit of the SFF in the general background $\{t_k\}$ of topological gravity. In section 5.1 we will first derive a key relation

$$\partial_\tau\partial_0\mathrm{SFF} = \frac{\mathrm{i}}{\hbar}\left[W_1\left(\beta+\frac{\mathrm{i}\tau}{\hbar}\right)\partial_0W_1\left(\beta-\frac{\mathrm{i}\tau}{\hbar}\right)-\partial_0W_1\left(\beta+\frac{\mathrm{i}\tau}{\hbar}\right)W_1\left(\beta-\frac{\mathrm{i}\tau}{\hbar}\right)\right], \tag{5.1}$$



(a) SFF_0. (b) SFF_1.

Figure 1. Plot of (a) SFF_0 and (b) SFF_1 for JT gravity. The horizontal axis is $\log \tau$. We set $\beta = 1$ in this figure. SFF_g approaches $\langle Z(2\beta)\rangle_g$ as $\tau \to \infty$ (shown by orange dashed lines).

which relates SFF with (the t_0-derivative of) the one-point function

$$W_1\left(\beta \pm \frac{\mathrm{i}\tau}{\hbar}\right) = \hbar\partial_0 \left\langle Z\left(\beta \pm \frac{\mathrm{i}\tau}{\hbar}\right)\right\rangle. \tag{5.2}$$

Here $\partial_k := \partial_{t_k}$. The relation (5.1) enables us to compute the small $\hbar$ expansion of the SFF from that of the one-point function $W_1(\beta \pm \frac{\mathrm{i}\tau}{\hbar})$. As we will describe in section 5.2, the small $\hbar$ expansion of $W_1(\beta \pm \frac{\mathrm{i}\tau}{\hbar})$ can be obtained by the method developed in [36] with a slight modification. This is based on the KdV equation. We will then explain in section 5.3 how to integrate $\partial_\tau\partial_0\mathrm{SFF}$ to obtain the small $\hbar$ expansion of the SFF.

5.1 Derivation of the key differential equation

In this subsection we will derive (5.1). As we have already seen in the previous sections, the one- and two-point functions can be expressed in terms of the CD kernel as [29]

$$\begin{aligned}\langle Z(\beta)\rangle &= \int dE e^{-\beta E} K(E,E),\\ \langle Z(\beta_1)Z(\beta_2)\rangle_{\mathrm{c}} &= \langle Z(\beta_1+\beta_2)\rangle - \int dE_1 dE_2 e^{-\beta_1 E_1 - \beta_2 E_2} K(E_1,E_2)^2.\end{aligned} \tag{5.3}$$

The CD kernel is written as

$$K(E_1,E_2) = \frac{\hbar\partial_0\psi(E_1)\psi(E_2) - \hbar\partial_0\psi(E_2)\psi(E_1)}{-E_1+E_2}, \tag{5.4}$$

where $\psi(E)$ is the Baker-Akhiezer function. It satisfies the Schrödinger equation

$$-(\hbar^2\partial_0^2 + u)\psi(E) = E\psi(E). \tag{5.5}$$

By using this equation it immediately follows from (5.4) that

$$\hbar\partial_0 K(E_1,E_2) = \psi(E_1)\psi(E_2). \tag{5.6}$$

Then we have

$$\begin{aligned}\hbar\partial_0\langle Z(\beta)\rangle = W_1(\beta) &= \int dE e^{-\beta E}\psi(E)^2,\\ \hbar\partial_0\langle Z(\beta_1)Z(\beta_2)\rangle_{\mathrm{c}} = W_1(\beta_1+\beta_2) &- \int dE_1 dE_2 e^{-\beta_1 E_1 - \beta_2 E_2} 2\psi(E_1)\psi(E_2)K(E_1,E_2).\end{aligned} \tag{5.7}$$

When $\beta_{1,2} = \beta \pm \mathrm{i}\tau\hbar^{-1}$, we find

$$\begin{aligned}
&\partial_\tau\partial_0\langle Z(\beta_1)Z(\beta_2)\rangle_{\mathrm{c}} \\
&= -\frac{\mathrm{i}}{\hbar^2}\int dE_1 dE_2 e^{-\beta_1 E_1-\beta_2 E_2} 2\psi(E_1)\psi(E_2)(-E_1+E_2)K(E_1,E_2) \\
&= -\frac{\mathrm{i}}{\hbar^2}\int dE_1 dE_2 e^{-\beta_1 E_1-\beta_2 E_2} 2\psi(E_1)\psi(E_2)\left[\hbar\partial_0\psi(E_1)\psi(E_2)-\hbar\partial_0\psi(E_2)\psi(E_1)\right] \\
&= -\frac{\mathrm{i}}{\hbar}\int dE_1 dE_2 e^{-\beta_1 E_1-\beta_2 E_2}\left[\partial_0\psi(E_1)^2\psi(E_2)^2-\psi(E_1)^2\partial_0\psi(E_2)^2\right] \\
&= -\frac{\mathrm{i}}{\hbar}\left[\partial_0 W_1(\beta_1)W_1(\beta_2)-W_1(\beta_1)\partial_0 W_1(\beta_2)\right].
\end{aligned} \tag{5.8}$$

Thus we have derived (5.1).

5.2 Small $\hbar$ expansion of one-point function

Let us next consider the small $\hbar$ expansion of $W_1\left(\beta \pm \frac{\mathrm{i}\tau}{\hbar}\right)$. We can restrict ourselves to the case of $W_1\left(\beta + \frac{\mathrm{i}\tau}{\hbar}\right)$ without loss of generality, as $W_1\left(\beta - \frac{\mathrm{i}\tau}{\hbar}\right)$ is immediately obtained by complex conjugation. The expansion can be done in two ways. One way, which is easier to understand, is to use the results of the 't Hooft expansion computed in [36]:

$$W_1(\beta) = \exp\left[\sum_{n=0}^{\infty}\hbar^{n-1}\widetilde{G}_n(\lambda)\right], \qquad \lambda = \hbar\beta \text{ fixed.} \tag{5.9}$$

Given this expansion, the small $\hbar$ expansion of $W_1\left(\beta + \frac{\mathrm{i}\tau}{\hbar}\right)$ is obtained by merely substituting $\lambda = \mathrm{i}\tau + \beta\hbar$ into the expansion and then re-expanding it in $\hbar$ as

$$\begin{aligned}
W_1\left(\beta+\frac{\mathrm{i}\tau}{\hbar}\right) = \exp\Big[&\hbar^{-1}\widetilde{G}_0(\mathrm{i}\tau) + \left(\widetilde{G}_1(\mathrm{i}\tau)+\widetilde{G}_0'(\mathrm{i}\tau)\beta\right) \\
&+\hbar\left(\widetilde{G}_2(\mathrm{i}\tau)+\widetilde{G}_1'(\mathrm{i}\tau)\beta+\frac{1}{2}\widetilde{G}_0''(\mathrm{i}\tau)\beta^2\right)+\mathcal{O}(\hbar^2)\Big].
\end{aligned} \tag{5.10}$$

More specifically, the first few $\widetilde{G}_n$ are given by [36][6]

$$\begin{aligned}
\widetilde{G}_0 &= -E_0\lambda + \sum_{\substack{j_a\ge 0\\ \sum_a j_a = k\\ \sum_a a j_a = n}} \frac{(2n+k+1)!}{(2n+3)!}\frac{\lambda^{2n+3}}{2^{n+1}s^{2n+k+2}}\prod_{a=1}^{\infty}\frac{I_{a+1}^{j_a}}{j_a!(2a+1)!!^{j_a}} \\
&= -E_0\lambda + \frac{1}{12s^2}\lambda^3 + \frac{I_2}{60s^5}\lambda^5 + \left(\frac{I_2^2}{144s^8}+\frac{I_3}{840s^7}\right)\lambda^7 + \mathcal{O}(\lambda^9),
\end{aligned} \tag{5.11}$$

$$\widetilde{G}_1 = \frac{1}{2}\log\left[\frac{\hbar}{8\pi}\frac{\partial_\lambda E(\lambda)}{E(\lambda)-E_0}\right],$$

$$\widetilde{G}_2 = \frac{\mathrm{i}I_2}{12s^2 z(\lambda)} - \frac{5\mathrm{i}}{24sz(\lambda)^3} - \frac{3E^{(1)}}{8z(\lambda)^4} + \frac{E^{(2)}}{4z(\lambda)^2E^{(1)}} - \frac{E^{(3)}}{8\left(E^{(1)}\right)^2} + \frac{\left(E^{(2)}\right)^2}{6\left(E^{(1)}\right)^3},$$

[6] The elements here and there are identified as $\widetilde{G}_n = 2^{(n-1)/2}G_n$, $E_0 = -u_0$, $s^{\text{here}} = t^{\text{there}} = 1 - I_1$, $E(\lambda) = -\xi_*$, $E^{(j)} = -2^{-n/2}\xi_*^{(n)}$, $z(\lambda) = -2^{-1/2}\mathrm{i}z_*$ and $\lambda = 2^{1/2}s^{\text{there}}$. The normalization of W_1 here (i.e. the constant part of $\widetilde{G}_1$) is also adjusted accordingly.

where

$$E(\lambda) = -\partial_\lambda \widetilde{G}_0, \qquad z(\lambda) = \sqrt{E(\lambda) - E_0}, \qquad E^{(j)} = \partial_\lambda^j E(\lambda). \tag{5.12}$$

I_k and $s = 1 - I_1$ were defined in section 2. From this one obtains

$$W_1\left(\beta + \frac{\mathrm{i}\tau}{\hbar}\right) = e^{G\left(\beta + \frac{\mathrm{i}\tau}{\hbar}\right)} = \exp\left[\sum_{n=0}^{\infty} \hbar^{n-1} G_n(\beta, \tau, \{I_k\})\right] \tag{5.13}$$

with

$$\begin{aligned}
G_0 &= \widetilde{G}_0(\mathrm{i}\tau), \\
G_1 &= \frac{1}{2}\log\left[\frac{\hbar}{8\pi\mathrm{i}}\frac{\partial_\tau E_\tau}{E_\tau - E_0}\right] - \beta E_\tau, \\
G_2 &= \mathrm{i}\left[\frac{I_2}{12 s^2 z_\tau} - \frac{5}{24 s z_\tau^3} + \frac{3e(1)}{8 z_\tau^4} - \frac{e(2)}{4 z_\tau^2 e(1)} + \frac{e(3)}{8 e(1)^2} - \frac{e(2)^2}{6 e(1)^3}\right. \\
&\qquad \left. + \left(\frac{e(1)}{2 z_\tau^2} - \frac{e(2)}{2 e(1)}\right)\beta + \frac{e(1)}{2}\beta^2\right].
\end{aligned} \tag{5.14}$$

Here $E_\tau = E(\mathrm{i}\tau)$, $z_\tau = \sqrt{E_\tau - E_0} = z(\mathrm{i}\tau)$ and $e(j) = \partial_\tau^j E_\tau = \mathrm{i}^j E^{(j)}(\mathrm{i}\tau)$.

The other way to compute the small $\hbar$ expansion of $W_1\left(\beta + \frac{\mathrm{i}\tau}{\hbar}\right)$, which is technically more efficient, is to solve the KdV equation directly with the expansion (5.13). This can be done by the method of [36] with only a slight modification. In what follows let us describe the method in our present notation. First, recall that $W_1 = W_1\left(\beta + \frac{\mathrm{i}\tau}{\hbar}\right)$ satisfies the KdV equation [23]

$$\partial_1 W_1 = u \partial_0 W_1 + \frac{\hbar^2}{6}\partial_0^3 W_1. \tag{5.15}$$

In terms of $G = \log W_1$, this is written as

$$\partial_1 G = u \partial_0 G + \frac{\hbar^2}{6}\left(\partial_0^3 G + 3\partial_0 G \partial_0^2 G + (\partial_0 G)^3\right). \tag{5.16}$$

Here, u is the specific heat of the general topological gravity. It obeys the KdV equation and its genus expansion

$$u = \sum_{g=0}^{\infty} 2^g u_g \hbar^{2g} \tag{5.17}$$

was computed by means of a recurrence relation.[7] Note, in particular, that the leading order part is equal to the threshold energy

$$u_0 = -E_0. \tag{5.18}$$

By plugging the expansions (5.17) and

$$G = \sum_{n=0}^{\infty} \hbar^{n-1} G_n \tag{5.19}$$

[7]See, e.g. [23] with the identification $t^{\text{there}} = s^{\text{here}}$, $B_n = (-1)^{n+1} I_{n+1}$ $(n \geq 1)$.

into (5.16), one obtains the recurrence relation

$$-\partial_s G_0 = \frac{1}{6}(\partial_0 G_0)^3, \tag{5.20}$$

$$DG_1 = \frac{1}{2}\partial_0 G_0 \partial_0^2 G_0 \tag{5.21}$$

for $n = 0, 1$ and

$$DG_n = \frac{1}{6}\sum_{\substack{0\le i,j,k<n\\ i+j+k=n}} \partial_0 G_i \partial_0 G_j \partial_0 G_k + \frac{1}{2}\sum_{k=0}^{n-1}\partial_0 G_{n-k-1}\partial_0^2 G_k + \frac{1}{6}\partial_0^3 G_{n-2} + \sum_{k=1}^{\lfloor \frac{n}{2}\rfloor} 2^k u_k \partial_0 G_{n-2k} \tag{5.22}$$

for $n \geq 2$.[8] Here we have introduced the differential operator

$$D := -\partial_s - \frac{1}{2}(\partial_0 G_0)^2 \partial_0. \tag{5.23}$$

Note also that instead of t_0, t_1 we regard s and E_0 as independent variables and treat $t_{k\geq 2}$ as parameters [23]. From this viewpoint $\partial_{0,1}$ are understood as

$$\partial_0 = -\frac{1}{s}\left(\partial_{E_0} + I_2 \partial_s\right), \qquad \partial_1 = -E_0\partial_0 - \partial_s. \tag{5.24}$$

To solve the recurrence relation (5.20)–(5.22), it is helpful to recall some relevant formulas (see [36]):

$$\begin{aligned}
&\partial_0 s = -\frac{I_2}{s}, \qquad \partial_0 I_n = \frac{I_{n+1}}{s} \quad (n\geq 1),\\
&\partial_0 G_0 = 2\mathrm{i} z_\tau, \qquad \partial_0 z_\tau = \frac{1}{2 s z_\tau} - \frac{\partial_\tau z_\tau}{z_\tau}, \qquad \partial_0 e(j) = -2\partial_\tau^{j+1} z_\tau,\\
&\partial_s z_\tau = -e(1).
\end{aligned} \tag{5.25}$$

The differential operator (5.23) is then written as

$$D = -\partial_s + 2 z_\tau^2 \partial_0. \tag{5.26}$$

It has the properties

$$Dz_\tau = \frac{z_\tau}{s}, \qquad DE_\tau = 0, \tag{5.27}$$

$$De(n) = 8\sum_{k=0}^{n-1}\sum_{\ell=0}^{k}\sum_{m=0}^{\ell}\frac{k!}{(k-\ell)!(\ell-m)!m!}\partial_\tau^{(n-\ell)} z_\tau\, \partial_\tau^{(\ell-m)} z_\tau\, \partial_\tau^{(m+1)} z_\tau. \tag{5.28}$$

As explained in [37], $\partial_\tau^n z_\tau$ can be expressed in terms of $e(j)$ and z_τ:

$$\begin{aligned}
\partial_\tau z_\tau &= \frac{e(1)}{2z_\tau},\\
\partial_\tau^2 z_\tau &= \frac{e(2)}{2z_\tau} - \frac{e(1)^2}{4z_\tau^3},\\
\partial_\tau^3 z_\tau &= \frac{e(3)}{2z_\tau} - \frac{3e(1)e(2)}{4z_\tau^3} + \frac{3e(1)^3}{8z_\tau^5}.
\end{aligned} \tag{5.29}$$

[8] G_n here is related with G_n in [36] as $G_n^{\text{here}}(\beta, \tau = 0, \{I_k\}) = 2^{(n-1)/2} G_n^{\text{there}}(s = 2^{-1/2}\hbar\beta, \{I_k\})$, up to the constant part of G_1.

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We are now in a position to solve the recurrence relation. In [36] the recurrence relation (5.20)–(5.22) was solved with the initial data

$$G_0^{\text{before}} = \widetilde{G}_0(\mathrm{i}\tau), \qquad G_1^{\text{before}} = \frac{1}{2}\log\left[\frac{\hbar}{8\pi\mathrm{i}}\frac{\partial_\tau E_\tau}{E_\tau - E_0}\right]. \tag{5.30}$$

Here, we would like to solve the same recurrence relation with the initial data given in (5.14). The only difference from [36] is that here G_1 has an extra β-dependent term $-\beta E_\tau$. It is clear from (5.27) that (5.21) is satisfied by G_1 with this term, given that (5.21) is satisfied by G_1^{before}.

To solve the recurrence relation, we can use almost the same algorithm we developed in [36]. The only difference is the step (v), which is modified accordingly so that it works for the β-dependent parts as well. The modified algorithm is as follows:

(i) Compute the r.h.s. of (5.22) and express it as a polynomial in the variables s^{-1}, $I_{k\geq 2}$, z_τ^{-1}, $e(1)^{-1}$ and $e(n),\ n \geq 1$.

(ii) Let $s^{-m} f(I_k, z_\tau, e(n))$ denote the highest-order part in s^{-1} of the obtained expression. This part can arise only from

$$D\left(\frac{f(I_k, z_\tau, e(n))}{2(m-2)s^{m-2} z_\tau^2 I_2}\right). \tag{5.31}$$

Therefore subtract this from the obtained expression.

(iii) Repeat procedure (ii) down to $m = 3$. Then all the terms of order s^{-2} automatically disappear and the remaining terms are of order s^{-1} or s^0. Note also that the expression does not contain any I_k.

(iv) In the result of (iii), collect all the terms of order s^{-1} and let $s^{-1} z_\tau \partial_{z_\tau} g(z_\tau, e(n))$ denote the sum of them. This part arises from

$$Dg(z_\tau, e(n)). \tag{5.32}$$

Therefore subtract this from the result of (iii). The remainder turns out to be independent of s.

(v) In the obtained expression, let

$$\frac{e(1)^m h(e(n \geq 2))}{z_\tau} \tag{5.33}$$

denote the part which is of the order z_τ^{-1} as well as of the highest order in $e(1)$. This part arises from

$$D\left[-\frac{1}{2e(1)}\int^{\tilde{a}} da \left(a^{-1}e(1)\right)^m h(e(n) \to a^{-n-1}e(n))\right]_{\tilde{a}=1}. \tag{5.34}$$

(Here, a and $\tilde{a}$ are intermediate formal variables.) Therefore subtract this from the obtained expression.

(vi) Repeat procedure (v) until the resulting expression vanishes.

(vii) By summing up all the above-obtained primitive functions, we obtain G_n.

Using the above algorithm we have computed G_n for $n \le 12$. For instance, G_3 is computed as

$$\begin{aligned}
G_3 = &-\frac{5}{32s^2z_\tau^6}+\frac{I_2}{8s^3z_\tau^4}-\frac{I_3}{24s^3z_\tau^2}-\frac{I_2^2}{12s^4z_\tau^2}-\frac{5I_2e(1)}{96s^2z_\tau^5}-\frac{3e(1)^2}{4z_\tau^8}+\frac{35e(1)}{64sz_\tau^7}+\frac{11e(2)}{16z_\tau^6}\\
&-\frac{3e(3)}{16z_\tau^4e(1)}-\frac{5e(2)}{32sz_\tau^5e(1)}+\frac{I_2e(2)}{48s^2z_\tau^3e(1)}+\frac{e(4)}{16z_\tau^2e(1)^2}+\frac{e(2)^2}{8z_\tau^4e(1)^2}-\frac{e(5)}{48e(1)^3}\\
&-\frac{7e(2)e(3)}{24z_\tau^2e(1)^3}+\frac{e(3)^2}{8e(1)^4}+\frac{e(2)^3}{4z_\tau^2e(1)^4}+\frac{e(2)e(4)}{6e(1)^4}-\frac{3e(2)^2e(3)}{4e(1)^5}+\frac{e(2)^4}{2e(1)^6}\\
&+\left(\frac{5e(1)}{16sz_\tau^5}-\frac{3e(1)^2}{4z_\tau^6}+\frac{5e(2)}{8z_\tau^4}-\frac{I_2e(1)}{24s^2z_\tau^3}-\frac{e(3)}{4z_\tau^2e(1)}+\frac{e(4)}{8e(1)^2}+\frac{e(2)^2}{4z_\tau^2e(1)^2}\right.\\
&\left.-\frac{7e(2)e(3)}{12e(1)^3}+\frac{e(2)^3}{2e(1)^4}\right)\beta+\left(-\frac{e(1)^2}{4z_\tau^4}+\frac{e(2)}{4z_\tau^2}+\frac{e(2)^2}{4e(1)^2}-\frac{e(3)}{4e(1)}\right)\beta^2+\frac{e(2)}{6}\beta^3.
\end{aligned} \tag{5.35}$$

Let us finally comment on the small $\hbar$ expansion of $W_1\left(\beta-\frac{\mathrm{i}\tau}{\hbar}\right)$. As we mentioned already, it is obtained from (5.13) by complex conjugation:

$$W_1\left(\beta-\frac{\mathrm{i}\tau}{\hbar}\right)=\exp\left[\sum_{n=0}^{\infty}\hbar^{n-1}\overline{G_n}\right]. \tag{5.36}$$

Furthermore, $\overline{G_n}$ is related to G_n by

$$\overline{G_n}=(-1)^{n+1}G_n+\frac{\pi\mathrm{i}}{2}\delta_{n,1}. \tag{5.37}$$

This can be shown directly for $n=0,1$ and inductively for $n\ge 2$ by using the recurrence relation (5.22).

5.3 Small $\hbar$ expansion of SFF

Using the results obtained above, one can compute the small $\hbar$ expansion of the SFF. Substituting (5.13), (5.36) and the explicit form of G_1 in (5.14) into (5.1), one obtains

$$\begin{aligned}
\partial_\tau\partial_0\mathrm{SFF}&=\frac{\mathrm{i}}{\hbar}(\partial_0\overline{G}-\partial_0 G)e^{G+\overline{G}}\\
&=-\frac{2\mathrm{i}}{\hbar}\sum_{k=0}^{\infty}\hbar^{2k-1}\partial_0G_{2k}\exp\left(G_1+\overline{G_1}+2\sum_{k=1}^{\infty}\hbar^{2k}G_{2k+1}\right)\\
&=\frac{e^{-2\beta E_\tau}}{4\pi\mathrm{i}}\frac{\partial_\tau E_\tau}{E_\tau-E_0}\sum_{k=0}^{\infty}\hbar^{2k-1}\partial_0G_{2k}\exp\left(\sum_{k=1}^{\infty}2G_{2k+1}\hbar^{2k}\right)\\
&=:\frac{e^{-2\beta E_\tau}}{2\pi\hbar}\sum_{n=0}^{\infty}c_n\hbar^{2n},
\end{aligned} \tag{5.38}$$

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where the first two of c_n are computed as

$$
\begin{aligned}
c_0 &= \frac{e(1)}{z_\tau}, \\
c_1 &= \frac{e(1)e(2)}{3z_\tau}\beta^3 + \left(-\frac{3e(1)^3}{8z_\tau^5} + \frac{e(1)e(2)}{4z_\tau^3} - \frac{e(3)}{2z_\tau} + \frac{e(2)^2}{2z_\tau e(1)}\right)\beta^2 \\
&\quad + \left(-\frac{15e(1)^3}{16z_\tau^7} + \frac{3e(1)e(2)}{4z_\tau^5} - \frac{e(3)}{4z_\tau^3} + \frac{3e(1)^2}{8sz_\tau^6} - \frac{I_2 e(1)^2}{12s^2 z_\tau^4}\right. \\
&\qquad \left. + \frac{e(4)}{4z_\tau e(1)} + \frac{e(2)^2}{4z_\tau^3 e(1)} - \frac{7e(2)e(3)}{6z_\tau e(1)^2} + \frac{e(2)^3}{z_\tau e(1)^3}\right)\beta \\
&\quad - \frac{5e(1)}{32s^2 z_\tau^7} + \frac{25e(1)e(2)}{32z_\tau^7} - \frac{3e(3)}{16z_\tau^5} + \frac{9e(1)^2}{16sz_\tau^8} + \frac{I_2 e(1)}{8s^3 z_\tau^5} - \frac{I_2^2 e(1)}{12s^4 z_\tau^3} - \frac{I_2 e(1)^2}{12s^2 z_\tau^6} - \frac{3e(2)}{16sz_\tau^6} \\
&\quad - \frac{I_3 e(1)}{24s^3 z_\tau^3} + \frac{I_2 e(2)}{24s^2 z_\tau^4} - \frac{105e(1)^3}{128z_\tau^9} + \frac{3e(2)^2}{32z_\tau^5 e(1)} + \frac{e(4)}{16z_\tau^3 e(1)} - \frac{e(5)}{24z_\tau e(1)^2} - \frac{7e(2)e(3)}{24z_\tau^3 e(1)^2} \\
&\quad + \frac{e(2)^3}{4z_\tau^3 e(1)^3} + \frac{e(3)^2}{4z_\tau e(1)^3} + \frac{e(2)e(4)}{3z_\tau e(1)^3} - \frac{3e(2)^2 e(3)}{2z_\tau e(1)^4} + \frac{e(2)^4}{z_\tau e(1)^5}.
\end{aligned}
\tag{5.39}
$$

Let us next consider the expansion

$$\partial_\tau \mathrm{SFF} = \frac{e^{-2\beta E_\tau}}{2\pi\hbar}\sum_{n=0}^{\infty} b_n \hbar^{2n}. \tag{5.40}$$

Comparing (5.40) with (5.38), we see that b_n and c_n are related as

$$
\begin{aligned}
c_n &= e^{2\beta E_\tau}\partial_0 e^{-2\beta E_\tau} b_n \\
&= (\partial_0 - 2\beta\partial_0 E_\tau)\, b_n \\
&= \left(\partial_0 + 2\beta\frac{e(1)}{z_\tau}\right) b_n.
\end{aligned}
\tag{5.41}
$$

For $n = 0$, we immediately see that $b_0 = 1/(2\beta)$ gives c_0 in (5.39). For $n \geq 1$, let us assume that b_n is a polynomial in β. Then, the highest-order term of this polynomial is determined as that of c_n divided by $2\beta e(1)/z_\tau$. Subtracting this contribution from both sides of (5.41), one can determine the next to the highest-order term in the same manner. In this way, one can determine b_n as a polynomial of β without integrating in t_0. For instance, the first two of them are

$$
\begin{aligned}
b_0 &= \frac{1}{2\beta}, \\
b_1 &= \frac{e(2)}{6}\beta^2 + \left(-\frac{e(1)^2}{8z_\tau^4} - \frac{e(3)}{6e(1)} + \frac{e(2)^2}{4e(1)^2}\right)\beta \\
&\quad - \frac{I_2 e(1)}{24s^2 z_\tau^3} + \frac{e(2)}{16z_\tau^4} + \frac{e(1)}{16sz_\tau^5} - \frac{e(1)^2}{8z_\tau^6} + \frac{e(4)}{24e(1)^2} - \frac{e(2)e(3)}{4e(1)^3} + \frac{e(2)^3}{4e(1)^4}.
\end{aligned}
\tag{5.42}
$$

Let us finally consider the small $\hbar$ expansion (1.3) of the SFF:

$$
\begin{aligned}
\mathrm{SFF} &= \mathrm{SFF}_{g=0} + \mathrm{SFF}_{g\geq 1} \\
&= \hbar^{-1}\mathrm{SFF}_0 + \sum_{g=1}^{\infty}\hbar^{2g-1}\mathrm{SFF}_g.
\end{aligned}
\tag{5.43}
$$

The leading term SFF_0 is given in (2.19). It is easy to verify that the τ-derivative of (2.19) reproduces the leading order part of (5.40) with $b_0 = 1/(2\beta)$. Here, we are interested in the higher order part $\mathrm{SFF}_{g\geq 1}$. As explained in section 3 (see, in particular (3.5)), a key feature of $\mathrm{SFF}_{g\geq 1}$ is that it takes the form

$$\mathrm{SFF}_{g\geq 1} = \mathrm{Erf}\left(\sqrt{2\beta} z_\tau\right) \langle Z(2\beta)\rangle_{g\geq 1} + \frac{e^{-2\beta E_\tau}}{2\pi\hbar} \sum_{g=1}^{\infty} a_g \hbar^{2g} \tag{5.44}$$

with a_g being polynomial in β. The small $\hbar$ expansion of the one-point function $\langle Z(2\beta)\rangle$ was computed in [23]:

$$\langle Z(2\beta)\rangle_{g\geq 1} = \frac{e^{-2\beta E_0}}{\sqrt{2\pi(2\beta)^3}\sqrt{2\hbar}} \sum_{g=1}^{\infty} \hbar^{2g} 2^g Z_g(2\beta), \tag{5.45}$$

where

$$\begin{aligned}
Z_1(\beta) &= \frac{1}{24s}\beta^3 + \frac{I_2}{24s^2}\beta^2, \\
Z_2(\beta) &= \frac{1}{1152s^3}\beta^6 + \frac{29I_2}{5760s^4}\beta^5 + \left(\frac{7I_2^2}{480s^5} + \frac{29I_3}{5760s^4}\right)\beta^4 + \left(\frac{7I_2^3}{288s^6} + \frac{29I_2I_3}{1440s^5} + \frac{I_4}{384s^4}\right)\beta^3 \\
&\quad + \left(\frac{7I_2^4}{288s^7} + \frac{5I_2^2I_3}{144s^6} + \frac{29I_3^2}{5760s^5} + \frac{11I_2I_4}{1440s^5} + \frac{I_5}{1152s^4}\right)\beta^2.
\end{aligned} \tag{5.46}$$

Comparing (5.44) with (5.40), we see that a_g and b_g are related as

$$b_g = \frac{e(1)}{2\beta z_\tau} 2^g Z_g(2\beta) + (\partial_\tau - 2\beta e(1)) a_g. \tag{5.47}$$

Given that a_g are polynomials in β, we can calculate them from b_g without integrating in τ, in essentially the same way as we obtained b_g from c_g. For instance, a_1 is obtained as

$$a_1 = \left(\frac{1}{6sz_\tau} - \frac{e(2)}{12e(1)}\right)\beta + \frac{e(1)}{16z_\tau^4} + \frac{I_2}{12s^2 z_\tau} - \frac{1}{24sz_\tau^3} + \frac{e(3)}{24e(1)^2} - \frac{e(2)^2}{12e(1)^3}. \tag{5.48}$$

One sees that the terms which do not contain $e(j)$ precisely give d_1 in (3.7); the other terms comprise f_1 given by (3.8) with (3.10). The expression of a_2 is already rather long and we relegate it to appendix B. We have computed a_g for $g \leq 5$.

6 Higher order corrections to CD kernel

In this section we consider the higher order correction of the CD kernel beyond the sine kernel approximation. That is, we consider a small $\hbar$ expansion of the CD kernel. In principle, this is not a difficult task since the CD kernel is expressed as a bilinear form (5.4) in terms of the BA function $\psi(E)$ whose $\hbar$-expansion was calculated [29]. The purpose of this section is to present a concrete, technically efficient method.

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6.1 Small $\hbar$ expansion of BA function

To compute the expansion of the CD kernel, one first needs to compute the small $\hbar$ expansion of $\psi(E+\frac{1}{2}\hbar\omega)$ rather than $\psi(E)$. Of course, this is obtained by re-expanding the results of [29] with the replacement $E\to E+\frac{1}{2}\hbar\omega$. For the sake of technical efficiency, however, here we compute the expansion

$$\log\psi(E+\tfrac{1}{2}\hbar\omega)=A(E+\tfrac{1}{2}\hbar\omega)=\sum_{n=0}^{\infty}\hbar^{n-1}A_n(E,\omega) \tag{6.1}$$

directly by means of a recurrence relation.

It was derived in [29] that

$$v(E):=\hbar\partial_0 A(E) \tag{6.2}$$

satisfies the equation

$$v^2+\hbar\partial_0 v=-E-u. \tag{6.3}$$

Here, u is the specific heat of the general topological gravity and we again assume that its genus expansion (5.17) is given. Since we want to compute the $\hbar$ expansion of ψ with a shifted argument, we replace E in (6.3) by $E+\frac{1}{2}\hbar\omega$ and plug the expansion of the form

$$v=v(E+\tfrac{1}{2}\hbar\omega)=\sum_{n=0}^{\infty}\hbar^n v_n(E,\omega) \tag{6.4}$$

into (6.3). Expanding both sides of the equation in $\hbar$, we obtain the recurrence relation

$$\begin{aligned}
&v_0^2=-E+E_0,\qquad 2v_0v_1+\partial_0 v_0=-\frac{\omega}{2},\\
&v_n=-\frac{1}{2v_0}\left(\partial_0 v_{n-1}+\sum_{k=1}^{n-1}v_k v_{n-k}+\begin{cases}2^{\frac{n}{2}}u_{\frac{n}{2}} & (n\text{ even})\\ 0 & (n\text{ odd})\end{cases}\right),\qquad n\geq 2.
\end{aligned} \tag{6.5}$$

The first two equations are solved as

$$\begin{aligned}
v_0&=\mathrm{i}z=\mathrm{i}\sqrt{E-E_0},\\
v_1&=-\frac{1}{4sz^2}+\frac{\mathrm{i}\omega}{4z}.
\end{aligned} \tag{6.6}$$

v_n $(n\geq 2)$ are also immediately obtained by the recurrence relation given the form of u_g. For instance, using

$$u_1=\frac{I_2^2}{12s^4}+\frac{I_3}{24s^3} \tag{6.7}$$

one obtains

$$\begin{aligned}
v_2&=\mathrm{i}\left(\frac{I_2^2}{12zs^4}+\frac{I_3}{24zs^3}-\frac{I_2}{8z^3s^3}+\frac{5}{32z^5s^2}\right)+\frac{1}{8z^4s}\omega-\mathrm{i}\frac{1}{32z^3}\omega^2,\\
v_3&=-\frac{I_2^3}{6s^6z^2}-\frac{7I_2I_3}{48s^5z^2}+\frac{11I_2^2}{48s^5z^4}-\frac{I_4}{48s^4z^2}+\frac{I_3}{12s^4z^4}-\frac{9I_2}{32s^4z^6}+\frac{15}{64s^3z^8}\\
&\quad+\mathrm{i}\left(-\frac{I_2^2}{48s^4z^3}-\frac{I_3}{96s^3z^3}+\frac{3I_2}{32s^3z^5}-\frac{25}{128s^2z^7}\right)\omega-\frac{1}{16sz^6}\omega^2+\mathrm{i}\frac{1}{128z^5}\omega^3.
\end{aligned} \tag{6.8}$$

Note that the form of the recurrence relation (6.5) for $v_{n\geq 2}$ is not affected by the shift $E \to E + \frac{1}{2}\hbar\omega$. The ω-dependence enters entirely through the initial data, i.e. the form of v_1.

One can easily obtain A_n from v_n through the relation $\partial_0 A_n = v_n$. Due to the structure (5.24) of ∂_0, the form of the s-dependent part of A_n can be determined without integrating in t_0. (We use a similar logic as we obtained b_n from c_n in the last section.) The s-independent part can also be determined with the help of the formulas (the proofs of which are straightforward)

$$\begin{aligned} \int^{t_0} dt_0 \frac{1}{sz^2} &= 2\log z, \\ \int^{t_0} dt_0 \frac{1}{sz^{2k}} &= -\frac{1}{(k-1)z^{2(k-1)}} \quad (k \geq 2), \\ \int^{t_0} dt_0 \frac{1}{z^{2k-1}} &= \frac{\Gamma(\frac{3}{2}-k)}{\Gamma(\frac{1}{2})}\rho_0^{(k-1)} = \frac{(-2)^{k-1}}{(2k-3)!!}\rho_0^{(k-1)} \quad (k \geq 0). \end{aligned} \tag{6.9}$$

Here

$$\rho_0^{(n)} = \partial_E^n \rho_0(E) \quad (n \geq 0), \qquad \rho_0^{(-1)} = \int_{E_0}^{E} dE \rho_0(E). \tag{6.10}$$

We thus obtain

$$\begin{aligned} A_0 &= \frac{\mathrm{i}}{2}\rho_0^{(-1)} = \frac{\mathrm{i}}{2}\int_{E_0}^{E} dE \rho_0(E), \\ A_1 &= -\frac{1}{2}\log z + \mathrm{i}\frac{\rho_0}{4}\omega - \frac{1}{2}\log(4\pi), \\ A_2 &= \mathrm{i}\left(\frac{I_2}{24s^2 z} - \frac{5}{48sz^3}\right) - \frac{1}{8z^2}\omega + \mathrm{i}\frac{\rho_0^{(1)}}{16}\omega^2, \\ A_3 &= -\frac{I_2^2}{24s^4z^2} - \frac{I_3}{48s^3z^2} + \frac{I_2}{16s^3z^4} - \frac{5}{64s^2z^6} + \mathrm{i}\left(-\frac{I_2}{96s^2z^3} + \frac{5}{64sz^5}\right)\omega \\ &\quad + \frac{1}{32z^4}\omega^2 + \mathrm{i}\frac{\rho_0^{(2)}}{96}\omega^3. \end{aligned} \tag{6.11}$$

The constant part of A_1 is determined in accordance with the $\omega = 0$ case [29].

6.2 Small $\hbar$ expansion of CD kernel

Let us now compute the small $\hbar$ expansion of the CD kernel. We should first note that this is a WKB-type asymptotic expansion. As is known [9], there appear some qualitative differences between the cases of $E > E_0$ and $E < E_0$. In this paper we consider the case of $E > E_0$. In this case the two independent BA functions are given by

$$\psi(E) \quad \text{and} \quad \overline{\psi(E)} = \psi(E)\Big|_{z\to -z}, \tag{6.12}$$

i.e. they are complex conjugate to each other. Due to this complex structure, for $E > E_0$ there are two saddles which equally contribute to the CD kernel [9]. Therefore, as long as the perturbative expansion in $\hbar$ concerns, the CD kernel is approximated by

$$K(E_1, E_2) = \widetilde{K}(E_1, E_2) + \widetilde{K}(E_2, E_1) \tag{6.13}$$

with

$$\widetilde{K}(E_1,E_2) = \frac{\hbar\partial_0\psi(E_1)\overline{\psi(E_2)} - \psi(E_1)\hbar\partial_0\overline{\psi(E_2)}}{-E_1+E_2}$$
$$= -\frac{v(E_1)-\overline{v(E_2)}}{\hbar\omega} e^{A(E_1)+\overline{A(E_2)}}. \tag{6.14}$$

By using the recurrence relation (6.5) it is easy to prove that

$$\overline{v_n(E,-\omega)} = (-1)^{n+1} v_n(E,\omega), \qquad \overline{A_n(E,-\omega)} = (-1)^{n+1} A_n(E,\omega). \tag{6.15}$$

Therefore (6.14) is written as

$$\widetilde{K}(E_1,E_2) = -\frac{1}{\hbar\omega}\left(\sum_{k=0}^{\infty} 2v_{2k}(E,\omega)\hbar^{2k}\right)\exp\left(\sum_{k=0}^{\infty} 2A_{2k+1}(E,\omega)\hbar^{2k}\right). \tag{6.16}$$

Similarly, we have

$$\widetilde{K}(E_2,E_1) = -\frac{1}{\hbar\omega}\sum_{k=0}^{\infty} 2\overline{v_{2k}(E,\omega)}\hbar^{2k}\exp\left[\sum_{k=0}^{\infty} 2\overline{A_{2k+1}(E,\omega)}\hbar^{2k}\right]. \tag{6.17}$$

Thus, the CD kernel is given by

$$K(E_1,E_2) = \mathrm{Re}\left\{2\widetilde{K}(E_1,E_2)\right\}. \tag{6.18}$$

Substituting the explicit form of $A_1(E,\omega)$ in (6.11) we obtain

$$\begin{aligned} 2\pi K(E_1,E_2) &= \mathrm{Re}\left\{-\frac{2}{\hbar\omega z}\left(\sum_{k=0}^{\infty} v_{2k}(E,\omega)\hbar^{2k}\right)\exp\left(2\sum_{k=1}^{\infty} A_{2k+1}(E,\omega)\hbar^{2k}\right)e^{\mathrm{i}\omega\rho_0/2}\right\} \\ &= \mathrm{Re}\left\{Xe^{\mathrm{i}\omega\rho_0/2}\right\} \\ &= \mathrm{Re}\{X\}\cos\frac{\omega\rho_0}{2} - \mathrm{Im}\{X\}\sin\frac{\omega\rho_0}{2}, \end{aligned} \tag{6.19}$$

where

$$X := -\frac{2}{\hbar\omega z}\left(\sum_{k=0}^{\infty} v_{2k}(E,\omega)\hbar^{2k}\right)\exp\left(2\sum_{k=1}^{\infty} A_{2k+1}(E,\omega)\hbar^{2k}\right). \tag{6.20}$$

With the forms of v_n and A_n obtained in (6.6), (6.8) and (6.11), this gives the small $\hbar$ expansion of the CD kernel.

A few comments are in order. First, substituting $v_0 = \mathrm{i}z$ one sees that the expression (6.19) correctly reproduces the sine kernel (2.10) at the order of $\hbar^{-1}$. Second, recall that the diagonal part of the CD kernel, i.e. (6.19) in the limit of $\omega \to 0$, is equal to the eigenvalue density (3.21). This is evident for the genus zero part at the order of $\hbar^{-1}$, but also implies that

$$\sum_{g=1}^{\infty} \hbar^{2g-1}\rho_g(E) = \mathrm{Re}\{X\}\Big|_{\omega=0}. \tag{6.21}$$

We checked that this indeed reproduces ρ_g in (3.2) which were obtained from the result of $\langle Z(\beta)\rangle_g$ in [23].

By taking (6.21) into account, the expansion (6.19) is written explicitly as

$$\begin{aligned}
&2\pi K(E_1, E_2) \\
&= \frac{2}{\hbar\omega}\left\{1+\frac{\omega^2}{32z^4}\hbar^2+\left[+\left(-\frac{49I_2^2}{4608s^4z^6}-\frac{I_3}{192s^3z^6}+\frac{49I_2}{1536s^3z^8}-\frac{105}{2048s^2z^{10}}\right)\omega^2\right.\right. \\
&\qquad\left.\left.+\left(\frac{I_2\rho_0^{(2)}}{2304s^2z^3}-\frac{\rho_0^{(2)}}{1536sz^5}+\frac{11}{2048z^8}\right)\omega^4-\frac{(\rho_0^{(2)})^2}{4608}\omega^6\right]\hbar^4+\mathcal{O}(\hbar^6)\right\}\sin\frac{\omega\rho_0}{2} \\
&\quad+\left\{\left[\rho_1+\frac{\rho_0^{(2)}}{24}\omega^2\right]\hbar\right. \\
&\qquad\left.+\left[\rho_2+\left(-\frac{I_2}{128s^2z^7}+\frac{35}{768sz^9}\right)\omega^2+\left(\frac{\rho_0^{(4)}}{1920}+\frac{\rho_0^{(2)}}{768z^4}\right)\omega^4\right]\hbar^3+\mathcal{O}(\hbar^5)\right\}\cos\frac{\omega\rho_0}{2}.
\end{aligned} \tag{6.22}$$

Observe that there appear several derivatives of ρ_0 at higher orders. These derivatives in fact have their origin in the integral (3.20) and are neatly absorbed if one recasts the above result into the form (3.19).

7 Conclusions and outlook

In this paper we have computed the higher order corrections of the SFF in the τ-scaling limit for an arbitrary background $\{t_k\}$ of topological gravity. We have also shown that these corrections of the SFF can be obtained by the Fourier transform of the CD kernel by including the corrections to the sine kernel formula. As we can see from figure 1, $\text{SFF}_{g\geq 1}$ approaches a constant at late times which gives a correction to the value of the plateau. On the other hand, $\text{SFF}_{g\geq 1}$ apparently diverges at $\tau = 0$, but this is just an artifact of the τ-scaling limit, as we argued at the end of subsection 3.1.

There are many interesting open questions. As far as we know, the corrections of the CD kernel to the sine kernel formula have not been fully explored in the literature before. In this paper we have computed these corrections of the CD kernel in the double-scaled matrix model of general topological gravity. It would be possible to compute the corrections of the CD kernel in a random matrix model before taking the double scaling limit. It would be interesting to carry out this computation along the lines of [32]. As emphasized in [26, 27], the τ-scaling limit enables us to reproduce the plateau of the SFF by just summing over the perturbative genus expansion. In general, the perturbative genus expansion is an asymptotic series, but it becomes a convergent series with a finite radius of convergence after we take the τ-scaling limit. What is happening is that, in the τ-scaling limit, the $(2g)!$ growing part of the Weil-Petersson volume is suppressed and only a non-growing part of the Weil-Petersson volume survives, and hence we get a convergent series for the SFF. In this sense, we throw away most of the contributions of the moduli space integral by taking the τ-scaling limit. It is fair to say that we still do not understand the non-perturbative effects which might contribute to the appearance of the plateau. In fact, on general grounds we expect that the sum over SFF_g in (1.3) is an asymptotic series. It would be interesting to study its resurgence structure and understand the non-perturbative effects associated with this asymptotic series (1.3).

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A Airy kernel

In this appendix, we consider the CD kernel for the Airy case, known as the Airy kernel. As the name suggests, the BA function for the Airy case is given by the Airy function

$$\psi(E) = \hbar^{-\frac{1}{6}} \mathrm{Ai}\big[-\hbar^{-\frac{2}{3}}(E+t_0)\big]. \tag{A.1}$$

Plugging this into (5.4), the Airy kernel is given by

$$K(E_1,E_2) = \frac{\mathrm{Ai}'(-\hbar^{-\frac{2}{3}}E_1)\mathrm{Ai}(-\hbar^{-\frac{2}{3}}E_2) - \mathrm{Ai}'(-\hbar^{-\frac{2}{3}}E_2)\mathrm{Ai}(-\hbar^{-\frac{2}{3}}E_1)}{E_1-E_2}, \tag{A.2}$$

where we have set $t_0 = 0$ after taking the t_0-derivative.

To see the corrections to the sine kernel formula, we first recall that $\mathrm{Ai}(-x)$ and $\mathrm{Ai}'(-x)$ have the following asymptotic expansion at large $x > 0$,[9]

$$\begin{aligned} \mathrm{Ai}(-x) &= \frac{1}{\sqrt{\pi}x^{\frac{1}{4}}}\left[\cos\left(A-\frac{\pi}{4}\right)\sum_{k=0}^{\infty}(-1)^k\frac{u_{2k}}{A^{2k}} + \sin\left(A-\frac{\pi}{4}\right)\sum_{k=0}^{\infty}(-1)^k\frac{u_{2k+1}}{A^{2k+1}}\right], \\ \mathrm{Ai}'(-x) &= \frac{x^{\frac{1}{4}}}{\sqrt{\pi}}\left[\sin\left(A-\frac{\pi}{4}\right)\sum_{k=0}^{\infty}(-1)^k\frac{v_{2k}}{A^{2k}} - \cos\left(A-\frac{\pi}{4}\right)\sum_{k=0}^{\infty}(-1)^k\frac{v_{2k+1}}{A^{2k+1}}\right], \end{aligned} \tag{A.3}$$

where $A = \frac{2}{3}x^{\frac{3}{2}}$ and

$$\begin{aligned} u_k &= \frac{\Gamma(k+5/6)\Gamma(k+1/6)}{2^{k+1}k!\pi}, \\ v_k &= \frac{1+6k}{1-6k}u_k = -\frac{\Gamma(k+7/6)\Gamma(k-1/6)}{2^{k+1}k!\pi}. \end{aligned} \tag{A.4}$$

The first few terms of the expansion (A.3) read

$$\begin{aligned} \mathrm{Ai}(-x) = &\frac{1}{\sqrt{\pi}x^{\frac{1}{4}}}\cos\left(A-\frac{\pi}{4}\right)\left[1-\frac{385}{10368A^2}+\cdots\right] \\ &+\frac{1}{\sqrt{\pi}x^{\frac{1}{4}}}\sin\left(A-\frac{\pi}{4}\right)\left[\frac{5}{72A}-\frac{85085}{2239488A^3}+\cdots\right], \end{aligned}$$

[9] See e.g. https://dlmf.nist.gov/9.7.

$$\mathrm{Ai}'(-x) = \frac{x^{\frac{1}{4}}}{\sqrt{\pi}} \sin\left(A - \frac{\pi}{4}\right)\left[1 + \frac{455}{10368A^2} + \cdots\right] - \frac{x^{\frac{1}{4}}}{\sqrt{\pi}} \cos\left(A - \frac{\pi}{4}\right)\left[-\frac{7}{72A} + \frac{95095}{2239488A^3} + \cdots\right]. \tag{A.5}$$

The corrections to the sine kernel formula can be obtained by plugging the above expansion (A.5) into (A.2) with $E_{1,2} = E \pm \frac{1}{2}\hbar\omega$. By keeping the terms proportional to $\sin(A_1 - A_2)$ and $\cos(A_1 - A_2)$, we find

$$\begin{aligned} K\left(E + \frac{\hbar\omega}{2}, E - \frac{\hbar\omega}{2}\right) &= \left[\frac{2}{\hbar\omega} + \frac{\omega}{16z^4}\hbar + \left(\frac{11\omega^3}{1024z^8} - \frac{105\omega}{1024z^{10}}\right)\hbar^3 + \mathcal{O}(\hbar^5)\right]\frac{\sin(A_1 - A_2)}{2\pi} \\ &+ \left[\frac{\hbar}{16z^5} + \left(\frac{35\omega^2}{768z^9} - \frac{105}{1024z^{11}}\right)\hbar^3 + \mathcal{O}(\hbar^5)\right]\frac{\cos(A_1 - A_2)}{2\pi}, \end{aligned} \tag{A.6}$$

where $z = \sqrt{E}$ and

$$A_{1,2} = \frac{2}{3\hbar}E_{1,2}^{\frac{3}{2}} = \frac{2}{3\hbar}\left(E \pm \frac{1}{2}\hbar\omega\right)^{\frac{3}{2}}. \tag{A.7}$$

One can check that (A.6) is consistent with our general formula of the CD kernel in (3.19). Note that if we plug (A.5) into (A.2), there appear terms proportional to $\sin(A_1 + A_2)$ and $\cos(A_1 + A_2)$ as well. However, they are highly oscillating in the limit $\hbar \to 0$ with fixed E, ω, and hence these terms can be ignored in the computation of the Fourier transform of the CD kernel.

B Coefficient a_2 of SFF_2 in (5.44)

The coefficient a_2 of SFF_2 in (5.44) is given by (for brevity we abbreviate z_τ to z)

$$\begin{aligned} a_2 = &\left(-\frac{e(2)^2}{72e(1)} + \frac{1}{18s^3z}\right)\beta^4 \\ &+ \left(\frac{e(1)e(2)}{48z^4} + \frac{e(4)}{240e(1)} - \frac{5e(2)^3}{144e(1)^3} + \frac{29I_2}{180s^4z} + \frac{e(2)e(3)}{72e(1)^2} - \frac{1}{72s^3z^3}\right)\beta^3 \\ &+ \left(-\frac{e(2)}{96sz^5} - \frac{e(3)}{32z^4} + \frac{e(1)e(2)}{16z^6} - \frac{e(5)}{160e(1)^2} + \frac{e(3)^2}{48e(1)^3} + \frac{e(2)^4}{24e(1)^5} + \frac{e(2)^2}{32z^4e(1)} \right.\\ &\quad - \frac{3e(1)^3}{64z^8} + \frac{7I_2^2}{30s^5z} + \frac{29I_3}{360s^4z} - \frac{29I_2}{720s^4z^3} + \frac{I_2e(2)}{144s^2z^3} + \frac{19e(2)e(4)}{480e(1)^3} - \frac{31e(2)^2e(3)}{288e(1)^4} \\ &\quad \left. + \frac{1}{96z^5s^3}\right)\beta^2 \\ &+ \left(\frac{29I_2I_3}{180s^5z} + \frac{e(3)}{96e(1)sz^5} - \frac{7I_2e(1)^2}{384s^2z^7} + \frac{I_2e(2)}{96s^2z^5} - \frac{e(2)^2}{64sz^5e(1)^2} - \frac{I_2e(3)}{144s^2z^3e(1)} \right.\\ &\quad + \frac{I_2e(2)^2}{96s^2z^3e(1)^2} + \frac{7I_2^3}{36s^6z} + \frac{I_4}{48s^4z} - \frac{7I_2^2}{120s^5z^3} - \frac{29I_3}{1440s^4z^3} + \frac{29I_2}{960s^4z^5} + \frac{73e(1)^2}{768sz^9} \\ &\quad - \frac{5e(2)}{192sz^7} + \frac{e(2)^2}{16z^6e(1)} + \frac{e(4)}{64z^4e(1)} - \frac{3e(1)^3}{16z^{10}} + \frac{13e(2)^5}{24e(1)^7} - \frac{e(3)}{16z^6} + \frac{e(6)}{320e(1)^3} - \end{aligned}$$

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$$
\begin{aligned}
&\quad -\frac{5}{384s^3z^7}-\frac{7e(2)e(3)}{96z^4e(1)^2}+\frac{21e(1)e(2)}{128z^8}+\frac{e(2)^3}{16z^4e(1)^3}+\frac{47e(2)e(3)^2}{144e(1)^5}-\frac{73e(2)^3e(3)}{72e(1)^6}\\
&\quad +\frac{181e(2)^2e(4)}{720e(1)^5}-\frac{61e(3)e(4)}{960e(1)^4}-\frac{113e(2)e(5)}{2880e(1)^4}\Bigg)\beta\\
&+\frac{25I_2e(1)}{384s^3z^8}-\frac{49I_2e(1)^2}{1536s^2z^9}-\frac{I_3e(1)}{96s^3z^6}-\frac{29I_2I_3}{720s^5z^3}-\frac{e(4)}{384sz^5e(1)^2}+\frac{5e(3)}{384sz^7e(1)}\\
&-\frac{e(2)^3}{64sz^5e(1)^4}-\frac{5e(2)^2}{256sz^7e(1)^2}-\frac{25I_2^2e(1)}{1152s^4z^6}+\frac{I_2e(2)}{64s^2z^7}+\frac{5I_2^2I_3}{36s^6z}+\frac{11I_2I_4}{360s^5z}\\
&-\frac{I_2e(3)}{192s^2z^5e(1)}+\frac{e(2)e(3)}{64sz^5e(1)^3}+\frac{I_2e(2)^2}{128s^2z^5e(1)^2}+\frac{I_2e(2)^3}{96s^2z^3e(1)^4}+\frac{I_2e(4)}{576s^2z^3e(1)^2}\\
&+\frac{13e(1)e(2)}{64z^{10}}+\frac{e(2)^3}{16z^6e(1)^3}+\frac{3e(2)^2}{64z^8e(1)}+\frac{e(2)^4}{16z^4e(1)^5}+\frac{e(3)^2}{64z^4e(1)^3}+\frac{e(4)}{64z^6e(1)}\\
&-\frac{e(5)}{384z^4e(1)^2}+\frac{e(2)e(4)}{48z^4e(1)^3}-\frac{7e(2)e(3)}{96z^6e(1)^2}-\frac{3e(2)^2e(3)}{32z^4e(1)^4}-\frac{119e(2)^4e(3)}{48e(1)^8}+\frac{7e(2)e(6)}{720e(1)^5}\\
&+\frac{29e(3)e(5)}{1440e(1)^5}+\frac{e(4)^2}{80e(1)^5}-\frac{e(7)}{1920e(1)^4}+\frac{7e(2)^6}{6e(1)^9}-\frac{7e(3)^3}{96e(1)^6}-\frac{9e(2)}{128sz^9}-\frac{53e(1)}{512s^2z^{10}}\\
&+\frac{219e(1)^2}{1024sz^{11}}-\frac{29I_2}{768s^4z^7}+\frac{29I_3}{1920s^4z^5}-\frac{I_4}{192s^4z^3}+\frac{7I_2^2}{160s^5z^5}-\frac{7I_2^3}{144s^6z^3}+\frac{I_5}{288s^4z}\\
&+\frac{29I_3^2}{1440s^5z}+\frac{7I_2^4}{72s^7z}-\frac{23e(2)e(3)e(4)}{72e(1)^6}+\frac{7e(2)^3e(4)}{12e(1)^7}+\frac{19e(2)^2e(3)^2}{16e(1)^7}-\frac{3e(2)^2e(5)}{32e(1)^6}\\
&-\frac{15e(1)^3}{64z^{12}}-\frac{15e(3)}{256z^8}+\frac{35}{1536s^3z^9}-\frac{I_2e(2)e(3)}{96s^2z^3e(1)^3}.
\end{aligned}
\tag{B.1}
$$

One can check that the $e(j)$-dependent part of (B.1) reproduces $h_{2,n}$ in (3.12); the $e(j)$-independent part of (B.1) becomes $d_2(z_\tau, \beta)$ defined in (3.5).

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