

Asymptotics in an asymptotic CFT

Lucas Schepers^a and Daniel C. Thompson^{a,b}

^a*Department of Physics, Swansea University,
Swansea SA2 8PP, U.K.*

^b*Theoretische Natuurkunde, Vrije Universiteit Brussel
and The International Solvay Institutes,
Pleinlaan 2, B-1050 Brussels, Belgium*

E-mail: lucas.schepers@swansea.ac.uk, daniel.c.thompson@swansea.ac.uk

ABSTRACT: In this work we illustrate the resurgent structure of the λ -deformation; a two-dimensional integrable quantum field theory that has an RG flow with an $SU(N)_k$ Wess-Zumino-Witten conformal fixed point in the UV. To do so we use modern matched asymptotic techniques applied to the thermodynamic Bethe ansatz formulation to compute the free energy to 38 perturbative orders in an expansion of large applied chemical potential. We find numerical evidence for factorial asymptotic behaviour with both alternating and non-alternating character which we match to an analytic expression. A curiosity of the system is that the leading non-alternating factorial growth vanishes when k divides N . The ambiguities associated to Borel resummation of this series are suggestive of non-perturbative contributions. This is verified with an analytic study of the TBA system demonstrating a cancellation between perturbative and non-perturbative ambiguities.

KEYWORDS: Field Theories in Lower Dimensions, Integrable Field Theories, Large-Order Behaviour of Perturbation Theory, Renormalons

ARXIV EPRINT: [2301.11803](https://arxiv.org/abs/2301.11803)

Contents

1	Introduction	1
2	The λ-model	3
2.1	Lagrangian construction	3
2.2	Renormalisation	5
2.3	Quantum integrability	5
3	TBA techniques	7
3.1	Free energy	8
3.2	Resolvent approach	8
3.2.1	Edge ansatz	9
3.2.2	Bulk ansatz	9
3.3	Matching	10
3.4	Perturbative result	10
4	Asymptotic analysis	12
5	Transseries and ambiguity cancellation	15
5.1	Borel resummation and large order perturbative ambiguity	16
5.2	Transseries and leading non-perturbative ambiguity	17
6	Discussion	23

1 Introduction

A complete understanding of the strong coupling dynamics of four dimensional asymptotically free (AF) non-supersymmetric gauge theory, i.e. QCD, remains elusive. To gain a foothold we may turn to simplified toy models. One strategy is to reduce the dimensionality of the problem considering instead two dimensional quantum field theories with similar RG behaviour. In the special case of integrable QFTs, an infinite set of symmetries completely determine the exact S-matrix [1] providing a powerful toolkit that can be used to tackle non-perturbative questions. An early success of this approach was the calculation of the exact ratio of the mass gap to cut-off of AF integrable QFTs theories [2–5].

More recently, techniques in integrable models have been used to elucidate even deeper questions of the nature of perturbation theory. Typically, perturbation theory is asymptotic in nature with perturbative coefficients growing factorially. The programme of resurgence asserts that this breakdown of convergence signals the need to include non-perturbative physics. Even more strongly, ambiguities inherent in resummations of asymptotic perturbative expansions should be cancelled by compensating ambiguities in a non-perturbative

sector. For a modern overview of resurgence from a physics view point see e.g. [6]. Again, integrable two-dimensional models provide an ideal test bed for resurgence.

In semi-classical approaches [7–12], an adiabatic compactification of two-dimension models is used to obtain a quantum mechanics which can be probed to large perturbative orders. In these cases, two-dimensional finite Euclidean action configurations, known as unitons¹ are shown to precisely resolve the semi-classical ambiguities of the perturbative sector. Whilst intriguing, such approaches intrinsically disregard degrees of freedom in compactification restricting to the lowest KK sector. Alongside this features of renormalisation group are disregarded in the truncation to Quantum Mechanics. Given these limitations, one thus prompted to ask if the resurgence paradigm can be established in a fully two-dimensional setting.

A breakthrough was the work of Volin [13, 14], recently refined in [15, 16] (see also the recent papers [17–25]), in addressing the Thermodynamic Bethe Ansatz (TBA) system that determines free energy in a large chemical potential. By comparing two scaling limits, it is possible to reduce the complicated integral TBA equation to a (complicated) set of algebraic equations that fix unknown coefficients in an ansatz for a perturbative expansion (in the chemical potential or other more refined coupling). This allows access to sufficient order in perturbation theory to reveal factorial divergence of perturbative coefficients.

Although we cannot identify an instanton or other semi-classical non-perturbative saddle in the TBA approach, we can find a matching ambiguity using a different method. Ref. [26] showed that it is possible to solve the TBA equations using a transseries. A critical step in their solution is an arbitrary choice of branch cut which introduces an ambiguity of the transseries. Although this approach, due to its computational difficulty, cannot be executed to large orders, it does exhibit a non-perturbative ambiguity that matches the ambiguity of the large-order behaviour found in the perturbative sector.

In this note we will adopt this toolbox to study the resurgent structure of a theory that exhibits a different renormalisation group dynamic. We consider a theory in which the UV is not Gaussian but instead is described by a non-trivial interacting conformal fixed point. The theory we will consider, known as the λ -model [27, 28], is realised as a flow away from an $SU(N)_k$ Wess-Zumino-Witten (WZW) model driven at leading order by a certain current-current bilinear. The IR of the theory is the principal chiral model, expressed in non-Abelian T-dual coordinates, and accordingly is gapped. Whilst this marginally relevant deformation breaks conformality and the full affine symmetry of the WZW current algebra, it does preserve an infinite symmetry associated to integrability. At the quantum level the exact S-matrix is known (based on symmetry grounds pre-dating the Lagrangian description) [29, 30] and has been shown to match the λ -model Lagrangian using a light cone lattice discretisation and Quantum Inverse Scattering [31].

The goal of this note is to match the perturbative ambiguity to that of the transseries the new context of a λ -model. The outline is as follows: section 2 provides a more in-depth discussion of the λ -model as we consider its RG flow in more detail and present its exact S-matrix. In section 3, we review the recent techniques to perturbatively solve TBA

¹Unlike instantons, unitons are not topologically protected.

equation [13–16] that determine free energy. Introducing a special coupling γ in section 3.4 results in a clean (i.e. log-free) series for the λ -model. We analyse its asymptotic behaviour in section 4 and compute the leading ambiguity. This ambiguity is matched by a transseries calculation in section 5. A particularly eye-catching result is that the leading UV ambiguity, and in fact all leading factorial behaviour, disappears when N divides k . We wrap up with ideas for future research in section 6.

2 The λ -model

In this section we outline the salient properties of the two-dimensional integrable QFT that we are considering: the λ -deformed model. Classically, the λ -model provides a Lagrangian interpolation between the conformal Wess-Zumino-Witten (WZW) model for a Lie-group G [32] and the principal chiral model (PCM) (written in non-Abelian T-dual variables). Remarkably this theory is integrable for all values of the eponymous interpolating parameter λ related to the level, k , of WZW and the radius, r , of the PCM by

$$\lambda = \frac{k}{k + r^2}. \tag{2.1}$$

Whilst the λ -model for the restricted case of $G = \text{SU}(2)$ was first proposed long ago [29, 33, 34], the pioneering work of Sfetsos [27] in constructing the general theory has prompted extensive recent development (for reviews see [35, 36]). The λ -model has been extended to \mathbb{Z}_2 graded symmetric spaces [27, 37] where it constitutes an interpolation between a G/H gauged WZW (representing the coset CFT) and the (non-abelian T-dual of) the principal chiral model on G/H and even to \mathbb{Z}_4 graded super-cosets relevant to the construction of the $AdS_5 \times S^5$ superstring [38] underpinned by an elegant quantum group at root-of-unity symmetry structure. In this string theory context, λ -deformation is in fact marginal, and the world sheet theory can be viewed as a σ -model in some target space super-gravity background [39–41]. Here however we will be considering the simpler bosonic case for which the λ -deformation does not define a CFT but rather a relevant RG flow from a WZW fixed point in the UV to the dualised PCM the IR [28, 42, 43]. A series of papers [44–48] have shown how the λ -model is actually part of a wide tapestry of integrable deformed models linked by (analytically continued) Poisson-Lie T-duality transformations.

2.1 Lagrangian construction

First we sketch the construction of the non-abelian T-dual of the PCM using the Buscher procedure [49] as it informs the construction of the λ -model. We start with the action of the PCM for a group valued field \tilde{g} ²

$$S_{\text{PCM}}[\tilde{g}] = -\frac{r^2}{4\pi} \int d^2\sigma \text{Tr} \left(\tilde{g}^{-1} \partial_+ \tilde{g} \tilde{g}^{-1} \partial_- \tilde{g} \right), \tag{2.2}$$

and downgrade the left symmetry $\tilde{g} \rightarrow h^{-1} \tilde{g}$ to a gauge symmetry by introducing a gauge connection transforming as $A \rightarrow h^{-1} dh + h^{-1} A h$ and replacing derivatives to covariant

²We use light cone coordinates $\sigma_{\pm} = \frac{1}{2}(t \pm x)$. Derivatives with respect to light cone coordinates are denoted by ∂_{\pm} .

derivatives $d \rightarrow D = d + A$. This yields the gauged PCM action which we denote as $S_{\text{gPCM}}[\tilde{g}, A]$. To ensure that the gauged theory is actually equivalent to the ungauged theory (at least in trivial topology which we assume throughout) we enforce that the connection is flat (i.e. the gauge field is pure gauge). This is implemented by introducing a Lagrange-multiplier term, $-\text{Tr}(\nu F_{+-})$, to the Lagrangian. Integrating out the field ν enforces that the field strength F_{+-} vanishes and we recover the original PCM after gauge-fixing $\tilde{g} = 1$. However, if instead we integrate out the gauge fields A , after gauge fixing $\tilde{g} = 1$, we obtain the non-abelian T-dual model in which the field ν becomes the fundamental field.

The construction of the λ -model by Sfetsos [27] is achieved through a modification of this Buscher procedure. Instead of adding a Lagrange multiplier term, we add a gauged WZW term. Recall that the WZW model is given by

$$S_{\text{WZW},k}[g] = -\frac{k}{2\pi} \int_{\Sigma} d^2\sigma \text{Tr} \left(g^{-1} \partial_{\mu} g g^{-1} \partial^{\mu} g \right) - \frac{ik}{6\pi} \int_{M_3} \text{Tr} \left(g^{-1} dg \right)^3, \quad (2.3)$$

in which g is extended to a 3-manifold M_3 with boundary $\partial(M_3) = \Sigma$. Standard arguments [32] ensure that the path integral is well-defined (independent of choice of extension) provided that k is appropriately quantised, and in particular for $G = \text{SU}(N)$ which we assume henceforth, $k \in \mathbb{Z}$. In this sector we gauge the diagonal symmetry $g \rightarrow h^{-1}gh$ leading to a gauged WZW model action $S_{\text{gWZW},k}[g, A]$ [50, 51].

To construct the λ -model we combine a gauged PCM and a gauged WZW model:

$$S_{\lambda,k}[g, \tilde{g}, A] = S_{\text{gPCM}}[\tilde{g}, A] + S_{\text{gWZW}}[g, A]. \quad (2.4)$$

Notice that the two models are coupled through the fact that they are gauged by the *same gauge field*. The Sfetsos procedure is concluded by gauge fixing $\tilde{g} = 1$ and integrating out the gauge field A using its on-shell value

$$A_+ = \lambda(1 - \lambda \text{Ad}_g)^{-1} R_+, \quad A_- = -\lambda(1 - \lambda \text{Ad}_{g^{-1}})^{-1} L_-, \quad (2.5)$$

where we defined $R_{\pm} = \partial_{\pm} g g^{-1}$ and $L_{\pm} = g^{-1} \partial_{\pm} g$ and the adjoint action $\text{Ad}_g X = gXg^{-1}$. Integrating out the gauge field then yields the action

$$S_{\lambda,k}[g] = S_{\text{WZW},k}[g] + \frac{k\lambda}{\pi} \int_{\Sigma} d^2\sigma \text{Tr} \left(R_+(1 - \lambda \text{Ad}_g)^{-1} L_- \right). \quad (2.6)$$

Though not vital for what follows we note that the equation of motion can be understood as a zero-curvature condition on the Lax connection [27, 52]

$$\mathcal{L}_{\pm}(z) = -\frac{2}{1 + \lambda} \frac{A_{\pm}}{1 \mp z}, \quad (2.7)$$

in which A_{\pm} are evaluated with the on-shell values eq. (2.5) and $z \in \mathbb{C}$ is a spectral parameter. This is the starting point of establishing the classical integrability of the theory. Further to this one requires *strong integrability* i.e. that the conserved charges built from the monodromy of this Lax are in involution. This is ensured provided that the Poisson algebra of the spatial component of the Lax has a particular r-s Maillet form as was demonstrated for the λ -model, and its generalisations, in [47, 53, 54].

2.2 Renormalisation

The parameter λ given by eq. (2.1) varies from 0 to 1 and we shall now discuss what happens in each of those limits. At a quantum level the parameter λ undergoes an RG flow [28, 42, 43] given by (to all orders in λ and leading in $\frac{1}{k}$)

$$\mu \frac{d\lambda}{d\mu} = \beta(\lambda) = -\frac{2N}{k} \left(\frac{\lambda}{1+\lambda} \right)^2 = -\beta_1 \lambda^2 - \beta_2 \lambda^3 + \mathcal{O}(\lambda^4). \quad (2.8)$$

The leading order behaviour, which shall be relevant later, is given by

$$\beta_1 = \frac{2N}{k}, \quad \beta_2 = -\frac{4N}{k}. \quad (2.9)$$

There is an evident UV fixed point at $\lambda = 0$, corresponding to the undeformed WZW model. In the vicinity of this $\lambda \approx 0$, or $k \ll r^2$, we obtain a current-current deformation of the WZW model:

$$S_{\lambda,k}[g] = S_{\text{WZW},k}[g] + \lambda \int_{\Sigma} d^2\sigma \text{Tr}(R_+ L_-) + \mathcal{O}(\lambda^2), \quad (2.10)$$

This, however, is *not* an *exactly* marginal deformation (cf. those of [55]), but relevant as it moves away from the WZW theory located in the UV.

To understand the IR regime as $\lambda \rightarrow 1$, we can force $k \rightarrow \infty$. If the group element is expanded as $g = 1 + \frac{i}{k} \nu^a t_a$, the action $S_{g\text{WZW},k}$ reduces to the Lagrange multiplier term $-\text{Tr}(\nu F_{+-})$. Thus in this limit the Sfetsos procedure reduces to the non-Abelian T-dualisation Buscher procedure described above. Hence, in this IR limit, we recover the non-abelian T-dual of the PCM. Further into the deep IR, one thus anticipates (as with the PCM) that the dimensionless parameter λ is transmuted into a mass gap mediated through a cut-off Λ .

2.3 Quantum integrability

Not only is the theory classical integrable, it remains so at the quantum level. The existence of higher spin conserved currents ensure that the scattering matrix of the theory factorises, and can be fully determined with the 2-to-2 particle scattering matrix the fundamental building block. The S-matrix for the $SU(N)$ λ -model was constructed many years ago [56] from an algebraic perspective, and was related directly to the Lagrangian description for the $SU(2)$ case in [29] by matching the free energy obtained by Lagrangian perturbation theory and by S-matrix TBA techniques. Following the introduction of the Lagrangian description λ -model by Sfetsos [27] for $SU(N)$ the exact S-matrix was conjectured [52] for general ranks. This conjecture was substantiated by Appadu et al. [31] in which the form of the S-matrix was ‘derived’ by the Quantum Inverse Scattering Method (i.e. a latticed version of the theory that takes the form of a spin chain such from the QFT particle states are obtained as excitations over the ground state in a continuum limit).³ Rather

³This QISM is in fact rather non-trivial as the $\delta'(\sigma)$ non-ultra-local terms in the fundamental Poisson bracket preclude a simple application of QISM. Instead what is proposed is a modification of the λ -model, that lies in the same universality class, to which QISM can be applied. This provides a description as a spin- k XXX spin chain with alternating inhomogeneities. This idea was expanded to a two-parameter integrable λ -type model [57] realised as a spin- k XXZ spin chain with alternating inhomogeneities.

than present the full details of the S-matrix (for which see [52]) we can give a schematic understanding somewhat mirroring the Sftsos procedure.

We start with the $SU(N)$ principal chiral model which has in particular an $SU(N)_L \times SU(N)_R$ global symmetry. The fundamental particles are massive and transform in fundamental antisymmetric tensor representations of the global symmetry. The scattering depends kinematically only on the rapidity difference θ of the particles.⁴ Reflecting this global symmetry, the S-matrix of these fundamental excitations has a schematic tensor form (suppressing explicit representation labels)

$$\mathbb{S}_{\text{PCM}}(\theta) = X(\theta)\mathcal{S}(\theta) \otimes \mathcal{S}(\theta), \tag{2.11}$$

where $X(\theta)$ is an overall scalar dressing factor to ensure all S-matrix axioms are obeyed, and the $\mathcal{S}(\theta)$ factors are separately $SU(N)$ invariant (in fact invariant under a larger Yangian symmetry). Recalling that in the Sftsos procedure the left acting $SU(N)_L$ symmetry was gauged, it is natural that the left hand block of the tensor product of eq. (2.11) is modified in the λ -theory and indeed this is the case with

$$\mathbb{S}_\lambda(\theta) = X_k(\theta)\mathcal{S}_k(\theta) \otimes \mathcal{S}(\theta). \tag{2.12}$$

Here $\mathcal{S}_k(\theta)$ is a block [56] that furnishes a quantum group symmetry at the $q^{2(k+N)} = 1$ root of unity taken in Restricted-Solid-On-Solid (RSOS) picture representing the scattering of kink degrees of freedom.

The particles of the λ -model then carry two types of quantum labels; first an $SU(N)_R$ representation label, specifying that the particles transform in the a^{th} fundamental (i.e. rank a anti-symmetric tensor). We let $[\omega]$ denote a representation obtained from the highest weight ω . With an over-complete basis $e_i \cdot e_j = \delta_{ij} = \frac{1}{2}$ roots are $\pm e_i \pm e_j$ and the a^{th} fundamental (antisymmetric) representations has highest weight vector $\omega_a = \sum_{i=1}^a e_i$. Particles transforming in this $[\omega_a]$ representation have a mass

$$M_a = m \sin \frac{\pi a}{N}.$$

In addition particles carry a second label corresponding to a kink structure defined by two weight vectors λ, λ' corresponding to highest weights of representations of level k (i.e. $(e_1 - e_N) \cdot \lambda \leq k$). A kink is allowed to connect two “vacua” provided that $[\lambda']$ is found in the tensor product of $[\lambda] \times [\omega_a]$.

Given knowledge of the exact S-matrix, the Thermodynamic Bethe Ansatz yields a set of rather complicated coupled-integral equations can be used to determine the free-energy of the theory. Solving these is quite formidable especially as the S-matrix is non-diagonal. A powerful simplification is achieved by exposing the system to a chemical potential h for a $U(1)$ charge such that only certain particles condense and contribute to the ground state. More precisely we define the charge corresponding to an $SU(N)$ weight ω to be

$$Q_\omega = \omega \cdot \vec{H}$$

⁴The mass shell is related to rapidity by $p_0 = m \cosh \theta$ and $p_1 = m \sinh \theta$.

for \vec{H} lying in the Cartan acting on the $SU(N)_R$ quantum label. Above threshold, only states of the largest mass to charge ratio contribute to the ground state. Following [31] we choose the charge $Q_{\omega_{\frac{N}{2}}}$ corresponding to the rank $N/2$ antisymmetric representation defined by highest weight $\omega_{\frac{N}{2}} = \sum_{i=1}^{\frac{N}{2}} e_n$. This ensures that only a single the state corresponding to the highest weight $\omega_{\frac{N}{2}}$ condenses in the ground state, and the e TBA system simplifies to a single integral equation determined by the identical scattering of this particle (as was demonstrated first in the case of $SU(2)$ in [29]).

In this case, the scattering “matrix” reduces to a simple phase factor $S(\theta)$ that governs transmission and reflection. It shall prove useful in this case to define the scattering kernel of this reduced S-matrix by

$$K(\theta) = \frac{1}{2\pi i} \frac{d}{d\theta} \log S(\theta), \tag{2.13}$$

and its Fourier transform

$$K(\omega) = \int_{-\infty}^{\infty} d\theta e^{i\omega\theta} K(\theta). \tag{2.14}$$

As a consequence of Hermitian analyticity on the reduced S-matrix, both $K(\theta)$ and its Fourier transform are symmetric functions. Explicitly we have that the relevant kernel is given by [57]

$$1 - K(\omega) = \frac{\sinh^2(\pi\omega/2)}{\sinh(\pi\omega) \sinh(\pi\kappa\omega)} \exp(\pi\kappa\omega), \tag{2.15}$$

where $\kappa = \frac{k}{N}$. In what follows, it shall prove useful to write the Fourier transform of the scattering kernel as a Wiener-Hopf (WH) decomposition

$$1 - K(\omega) = \frac{1}{G_+(\omega)G_-(\omega)}, \tag{2.16}$$

where $G_-(\omega) = G_+(-\omega)$, and $G_+(\omega)$ is analytic in the Upper Half Plane (UHP) and normalised such that $G_+(2is) = 1 + \mathcal{O}\left(\frac{1}{s}\right)$. Explicitly $G_+(\omega)$ is given by

$$G_+(\omega) = \sqrt{4\kappa} \frac{\Gamma(1 - i\omega/2)^2}{\Gamma(1 - i\omega)\Gamma(1 - i\kappa\omega)} \exp(ib\omega - i\kappa\omega \log(-i\omega)), \tag{2.17}$$

with

$$b = \kappa(1 - \log(\kappa)) - \log(2). \tag{2.18}$$

3 TBA techniques

Polyakov and Wiegmann [58–60] showed in the 80s that it is possible to compute the free energy of an integrable system with a chemical potential h turned on using a thermodynamic Bethe ansatz (TBA) technique. Using these techniques, Hasenfratz, Niedermayer and Maggiore [2, 3] showed in 1990⁵ that it is possible to calculate the mass gap in integrable

⁵This computation was initially performed for the $O(N)$ model, but was later also completed for Gross-Neveu models [4, 5] and PCM models [61, 62] and further examples as surveyed in the review article [30].

models by comparing the result from TBA with conventional Lagrangian perturbation theory. Building from this we will apply, in section 3.4, the techniques pioneered by [13–16] to extract an expansion for the free energy of λ -model in $\frac{1}{h}$ the large order behaviour of which we will study extensively in section 4.

3.1 Free energy

To present the TBA equations we will specialise to the case described above in which we introduce a chemical potential h such that only a single particle dominates the ensemble at large h .⁶ With $K(\theta)$ the appropriate scattering kernel, the TBA equations determine the density distribution of states, $\chi(\theta)$, via

$$m \cosh(\theta) = \chi(\theta) - \int_{-B}^B K(\theta - \theta') \chi(\theta') d\theta', \quad \theta^2 < B^2, \quad (3.1)$$

from which the charge and energy density follow

$$e = m \int_B^{-B} \chi(\theta) \cosh(\theta) \frac{d\theta}{2\pi}, \quad \rho = \int_B^{-B} \chi(\theta) \frac{d\theta}{2\pi}. \quad (3.2)$$

A critical complexity of this system is that the occupied states lie within a Fermi surface specified by B , which is however a function of h (with large B corresponding to large h). Supposing that we have calculated the energy density, thought of as a function of the charge density $e = e(\rho)$, then we can reconstruct a free energy density, $\mathcal{F}(h)$, from a Legendre transform:

$$\rho = -\mathcal{F}'(h), \quad \mathcal{F}(h) - \mathcal{F}(0) = e(\rho) - \rho h. \quad (3.3)$$

3.2 Resolvent approach

It will prove useful to recast the integral equation that determines $\chi(\theta)$ in terms of a resolvent function defined by

$$R(\theta) = \int_{-B}^B \frac{\chi(\theta')}{\theta - \theta'} d\theta'. \quad (3.4)$$

The resolvent is analytical everywhere except around the interval $[-B, B]$ where it has an ambiguity given by

$$\chi(\theta) = -\frac{1}{2\pi i} \left(R^+(\theta) - R^-(\theta) \right), \quad (3.5)$$

where we use the short hand notation $R^\pm(\theta) = R(\theta \pm i\epsilon)$. Suppose that the kernel can be cast in terms of some operator \mathcal{O} as $K(\theta) = \frac{1}{2\pi i} \mathcal{O} \frac{1}{\theta}$, then the eq. (3.1) is equivalent to a Riemann-Hilbert problem

$$R^+(\theta) - R^-(\theta) + \mathcal{O}R(\theta) = -2\pi i m \cosh \theta. \quad (3.6)$$

⁶That we can reduce the TBA system to involve just one species of particle from the fundamental representation singled out by the applied chemical potential is of course an assumption that makes the problem readily tractable. One anticipates that states of higher mass and higher charge are energetically disfavoured, but properly speaking this assumption ought to be proven starting from a complete nested TBA system (which we do not attempt here).

A determination of $R(\theta)$ is then equivalent to solving the TBA system and once known the charge density is immediately extracted as

$$\rho = -\frac{1}{2\pi} \text{Res}_{\theta=\infty} R(\theta). \tag{3.7}$$

We briefly now review the approach of [13–15] which does so by considering ansatz solutions for the resolvent in two limits (the *edge* and *bulk*) and matching them to fix all undetermined coefficients.

3.2.1 Edge ansatz

We begin first with the edge limit in which the weak coupling limit $B \rightarrow \infty$ is taken whilst keeping an edge coordinate $z = 2(\theta - B)$ fixed and small. This evidently scales to large θ and hence probes the properties of $\chi(\theta)$ around the vicinity of the Fermi energy, B . This limit is best studied by considering the Laplace transform of the resolvent (3.4) given by

$$R(z) = \int_0^\infty \widehat{R}(s)e^{-sz} ds, \quad \widehat{R}(s) = \frac{1}{2\pi i} \int_{-i\infty+\delta}^{i\infty+\delta} e^{sz} R(z) dz. \tag{3.8}$$

Note at large B the energy density is related to this Laplace transformation by

$$e = \frac{me^B}{4\pi} \widehat{R}(1/2). \tag{3.9}$$

The key result of [15, 16] is that in the edge limit the Laplace transformed resolvent has the following form

$$\widehat{R}(s) = me^B \Phi(s) \frac{\Phi\left(\frac{1}{2}\right)}{2} \left(\frac{1}{s+1/2} + Q(s) \right), \quad \Phi(s) = G_+(2is), \tag{3.10}$$

where $G_+(s)$ is the WH decomposition (2.16) of the (Fourier transformed) scattering kernel and $Q(s)$ is a series in large s and a perturbative expansion in $\frac{1}{B}$ of the form

$$Q(s) = \frac{1}{Bs} \sum_{m,n=0}^\infty \frac{Q_{n,m}}{B^{m+n} s^n}. \tag{3.11}$$

It should be noted that the coefficients $Q_{n,m}$ may still depend on $\log B$.

3.2.2 Bulk ansatz

In the bulk limit we let $B \rightarrow \infty$ and $\theta \rightarrow \infty$ but we keep $u = \theta/B$ fixed, we are hence studying the regime where θ is in the bulk, between 0 and B . The precise form of the Bulk ansatz depends on the model. For the λ -model, we shall take the same bulk ansatz for the Gross-Neveu model [15], which is given by

$$R(u) = \sum_{n=1}^\infty \sum_{m=0}^\infty \sum_{k=0}^{n+m} c_{n,m,k} \frac{u^{e(k+1)}}{B^{m+n}(u^2-1)^n} \left[\log \frac{u-1}{1+u} \right]^k, \tag{3.12}$$

where $e(k)$ is 0 if k is even and 1 if k is odd. The bulk ansatz can be motivated by constructing it using functions that are analytic outside the interval $[-B, B]$, where they have

a logarithmic branch cut.⁷ This is precisely the analytic structure demanded by eqs. (3.4) and (3.5).

3.3 Matching

If we re-expand the bulk ansatz (3.12) in an edge regime where $z = 2(\theta - B)$ is fixed, we should recover the expansion in the edge regime given by (3.10). Here a miraculous feature occurs: upon comparing expansions order by order in large B , then order by order in large z (which is small s) and then in $\log(z)$, we can solve for all the coefficients $c_{n,m,k}$ and $Q_{n,m}$. One peculiarity of the procedure is that we perform this matching only for the regular terms of the expansion z^{-n} ($n \geq 0$), while we disregard all divergent terms z^n ($n > 0$). Using a desktop PC, over the course of a week, we solved the system up to 38 orders. Once this calculation is completed, we compute e and ρ . Using equations (3.10) and (3.7) we can express ρ and e in terms of the coefficients by

$$\begin{aligned}
 e &= \frac{m^2 e^{2B} \Phi(1/2)^2}{8\pi} \left[1 + \sum_{m=1}^{\infty} \frac{1}{B^m} \sum_{n=0}^{m-1} 2^{n+1} Q_{n,m-1-n} \right], \\
 \rho &= 2\pi \sum_{m=0}^{\infty} \frac{c_{1,m,0}}{B^m}.
 \end{aligned}
 \tag{3.13}$$

Explicitly the first few coefficients required to determine up to order B^{-2} are given by

$$\begin{aligned}
 c_{1,0,0} &= 4\sqrt{\kappa}, & c_{1,1,0} &= -2\kappa^{3/2}, \\
 c_{1,2,0} &= \frac{1}{2}\kappa^{3/2}(2 - \kappa - 4\log 2 + 4\kappa \log(2B/\kappa)), \\
 Q_{0,0} &= 0, & Q_{1,0} &= 0, & Q_{0,1} &= \frac{\kappa}{4}.
 \end{aligned}
 \tag{3.14}$$

The last step is to calculate the quantity $\frac{e}{\rho^2}$ as an expansion in B the first terms of which are

$$\frac{8\kappa}{\pi} \frac{e}{\rho^2} = 1 + \frac{\kappa}{B} + \frac{\kappa}{B^2} \left(1 - \log(2) + \frac{\kappa}{2} + \kappa \log(2B/\kappa) \right) + \mathcal{O}(B^{-3}).
 \tag{3.15}$$

As this result depends on $\log(B)$, it is convenient to define a new effective coupling γ in terms of which the perturbative expansion is free from logarithms as we shall do in the next section.

3.4 Perturbative result

Before introducing the log-free coupling, we show our results are consistent with those of [31], which determines the mass gap of this theory. Using standard TBA techniques, they find an expansion for the free energy given by

$$\begin{aligned}
 \mathcal{F}(h) - \mathcal{F}(0) &= -\frac{2h^2\kappa}{\pi} \left\{ 1 - 2\kappa\alpha + 2\kappa\alpha^2 [2 + \kappa + \log 4 + 2\kappa \log \kappa + 2\kappa \log \alpha] \right. \\
 &\quad \left. - 8\kappa^2\alpha^3 \log(\alpha) [(-2 + 2\kappa + \log 4 + 2\kappa \log(\kappa) + \kappa \log(\alpha))] + \mathcal{O}(\alpha^3) \right\}.
 \end{aligned}
 \tag{3.16}$$

⁷This is different from the PCM bulk ansatz which also has a square root branch cut along the interval $[-B, B]$.

The coupling α is here defined by

$$\frac{1}{\alpha} = 2 \log \left(\frac{2h}{m} \sqrt{\frac{8\kappa}{\pi}} \right). \quad (3.17)$$

By using the Legendre transformation (3.3) we can compute the total energy e from eq. (3.16). Doing so, we obtain the expression

$$\begin{aligned} \frac{8\kappa}{\pi} \frac{e}{\rho^2} &= 1 + 2\alpha\kappa - 2\kappa\alpha^2(2\kappa \log(\alpha\kappa) - \kappa - 2 + \log(4)) \\ &+ 8\kappa^2\alpha^3[\kappa \log^2(\alpha) + (\log(\alpha) - 1)(-2\log(4) + 2\kappa \log(\kappa))] + \mathcal{O}(\alpha^4). \end{aligned} \quad (3.18)$$

From eq. (3.15), it follows that $\frac{e}{\rho^2} = \chi_0 + \mathcal{O}(\alpha)$ where $\chi_0 = \frac{\pi}{8\kappa}$. Therefore to leading order we have $h = \frac{\partial e}{\partial \rho} = 2\chi_0\rho$, which leads to $\rho = \frac{4h\kappa}{\pi}$. Looking at eq. (3.17), we should thus define a coupling by

$$\frac{1}{\alpha} = 2 \log \left(\frac{\rho}{m} \sqrt{\frac{2\pi}{\kappa}} \right). \quad (3.19)$$

This defines α in terms of B . Inverting the relation and substituting into the series (3.15) recovers precisely the expansion (3.18), providing an important consistency check for our programme.

We now take inspiration from the Gross-Neveu treatment of [15] to create a series expansions for $\frac{e}{\rho^2}$ that is log-free. This is appropriate because we have that to leading order $\Delta F \sim -h^2 + \mathcal{O}(\alpha)$, which leads to a coupling defined by⁸

$$\frac{1}{\gamma} + \xi \log \gamma = \log \frac{2\pi\rho}{m/c}, \quad \xi = \frac{\beta_2}{\beta_1^2} = -\frac{k}{N} = -\kappa. \quad (3.20)$$

One could demand that the right hand side be $\log \frac{2\pi\rho}{\Lambda_{MS}}$, where Λ_{MS} is the cut-off in the minimal subtraction scheme. To achieve this one has to tune the constant $c = c_{MS}$ such that $c_{MS}\Lambda_{MS} = m$. A key outcome of [31] determines that $c_{MS} = e^{3/2}N^{-1/2}$. However, we shall exercise the freedom to pick a c of our own choosing,

$$c = \frac{2^{-\kappa}\Gamma(\kappa)}{\pi}, \quad (3.21)$$

such that resulting expressions appear considerably simplified. This leads to an expansion that is log-free in the coupling, given by

$$\begin{aligned} \frac{8\kappa}{\pi} \frac{e}{\rho^2} &= \sum_{n=0}^{\infty} a_n \gamma^n = 1 + \kappa\gamma + \frac{\kappa}{2}[2 - \kappa]\gamma^2 \\ &+ \frac{\kappa}{2} [3 - 5\kappa + 2\kappa^2] \gamma^3 + \frac{\kappa}{8} [3(8 - \zeta(3)) - 61\kappa + 52\kappa^2 - 15\kappa^3] \gamma^4 \\ &+ \frac{\kappa}{12} [90 - 18\zeta(3) + \kappa(33\zeta(3) - 288) + 355\kappa^2 - 203\kappa^3 + 46\kappa^4] \gamma^5 \\ &+ \frac{\kappa}{32} \left[45(16 - 4\zeta(3) - \zeta(5)) + 2\kappa(259\zeta(3) - 1338) + \frac{1}{3}\kappa^2(12274 - 1329\zeta(3)) \right. \\ &\quad \left. - 3285\kappa^3 + 1412\kappa^4 - \frac{787\kappa^5}{3} \right] \gamma^6 + \mathcal{O}(\gamma^7). \end{aligned} \quad (3.22)$$

⁸This is in contrast to the PCM calculation where the free energy has a structure $\Delta F \sim -\frac{h^2}{\alpha} + \mathcal{O}(\alpha^0)$, which leads to a coupling $\frac{1}{\gamma} + (\xi - 1) \log \gamma \propto \log \rho$.

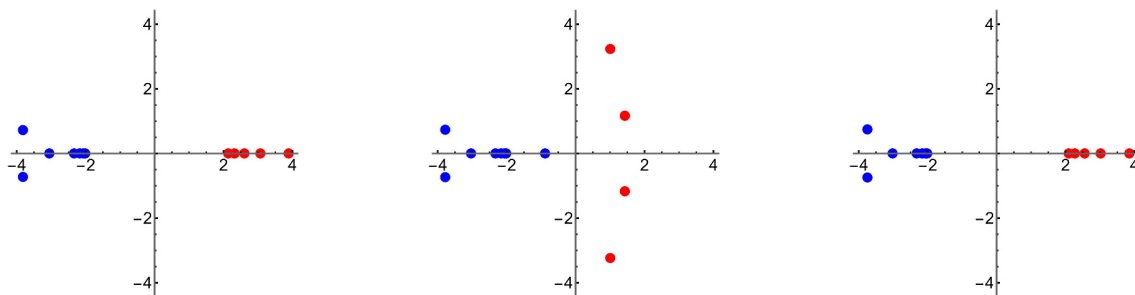


Figure 1. Left to right, for $\kappa = 0.98, 1$ and 1.02 , the Borel-Padé-poles in the ζ -plane. Evident are singularities at $\zeta = \pm 2$, with the positive pole removed for $\kappa = 1$.

In the next section we shall explore this perturbative expansion further.

4 Asymptotic analysis

In this section, we will quantitatively analyse the 38 orders of the perturbative series obtained in the previous section. The goal shall be to compute an asymptotic formula for the growth of the coefficients as a function of κ . After obtaining such a formula, we can compute its Borel ambiguity, which can later be compared against an ambiguity of a transseries.

As the perturbative series can readily be seen to exhibit factorial growth, as a first step to resummation we introduce the Borel transform

$$\mathcal{B} \left[\frac{8\kappa e^2}{\pi \rho} \right] \equiv \sum_{n=0}^{\infty} \frac{a_n}{n!} \zeta^n. \tag{4.1}$$

This series has a finite radius of convergence but typically has either, or both, poles and branch cuts. The pole/branch point closest to the origin in the ζ plane is governed by the leading asymptotic behaviour. Of course, numerically one does not have all orders with which to establish this Borel transformation, rather only a finite number of coefficients a_n for $n < N$ say. Here the Borel-Padé method can be employed: we compute $\mathcal{B}_N \left[\frac{8\kappa e^2}{\pi \rho} \right] = \sum_{n=0}^N \frac{a_n}{n!} \zeta^n = \frac{P(\zeta)}{Q(\zeta)} + \mathcal{O}(\zeta)^{N+1}$ in which P and Q are polynomials in ζ of degree $N/2$. This results in a picture in which an accumulation of poles (i.e. zeros of Q) is indicative of a branch point. We perform this numerically for various values of κ and generically we find evidence of branch points at $\zeta = \pm 2$ whose location is independent of κ except that for $\kappa \in \mathbb{Z}_{>0}$ the pole in the positive axis is removed - see figure 1. Pole/branch points in the negative real axis of the Borel plane indicate contributions to a_n of alternating sign whereas the contributions to a_n that result in poles on the positive axis would have the same sign. Here the analysis indicates that we have both. With 38 perturbative coefficients this analysis should only be regarded as indicative but is sufficient to inform an educated guess as to the asymptotic behaviour of the a_n which we will robustly verify below.

Motivated by the Borel-Padé analysis we assume the coefficients grow, to leading approximation, as

$$a_n \approx A_+ \Gamma(n+1)/S^n + A_- \Gamma(n+1)/(-S)^n + \mathcal{O}(n^{-1}). \tag{4.2}$$

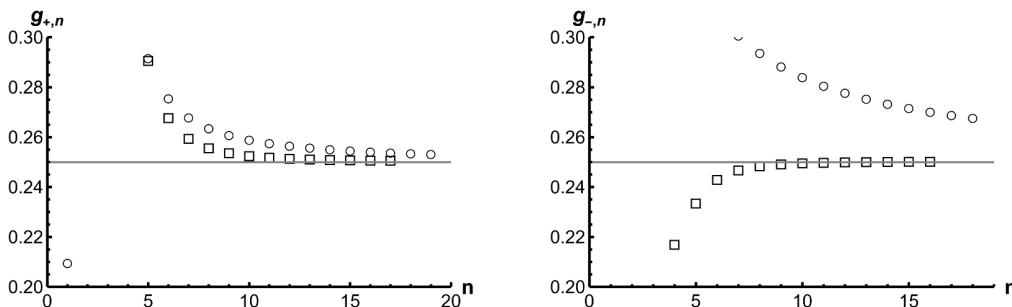


Figure 2. The series $g_{+,n}$ (left) and $g_{-,n}$ (right) given by eq. (4.3) displayed for $\kappa = 0.6$. Circle markers indicate the raw data, square markers the second Richardson transformation with accelerated convergence. The final values of the second Richardson transform differ by 0.11% and 0.05% respectively from the expected value $\frac{1}{4}$.

A first verification is to establish the factor S which can be done noting that

$$g_{+,n} := \frac{a_{2n}}{2n(2n-1)a_{2n-2}} \approx \frac{1}{S^2}, \quad g_{-,n} := \frac{a_{2n+1}}{2n(2n-1)a_{2n-1}} \approx \frac{1}{S^2}. \quad (4.3)$$

We find, see figure 2, that the series $g_{\pm,n}$ converge to $\frac{1}{4}$, independent of κ thus establishing $S = 2$ in accordance with the expectation from the Borel-Padé analysis.

Having established the factorially growing character of the perturbative series, we now propose a more refined ansatz for the a_n . Our central claim can be summarised by stating that the perturbative series has coefficients that have a leading large order behaviour as

$$a_n \approx \frac{A_+}{2^n} \sum_{l=0}^{\infty} \beta_l^+ \Gamma(n + a_+ - l) + \frac{A_-}{(-2)^n} \sum_{l=0}^{\infty} \beta_l^- \Gamma(n + a_- - l), \quad (4.4)$$

where we normalise $\beta_0^\pm = 1$ and the first few coefficients are

$$\begin{aligned} a_\pm &= \mp 2\kappa, \\ A_\pm &= \frac{8^{\pm 1}}{\pi} \sin(\mp \kappa) \frac{\Gamma(\pm \kappa)}{\Gamma(\mp \kappa)} = -\frac{8^{\pm 1}}{\Gamma(\mp \kappa)\Gamma(1 \mp \kappa)}, \\ \beta_1^- &= -\beta_2^- = -4\kappa. \end{aligned} \quad (4.5)$$

To support these claims, we shall define the auxiliary series

$$c_n = \frac{2^n}{\Gamma(n+1)} a_n, \quad (4.6)$$

to take care of the leading factorial and geometric growth. We project to the alternating and non-alternating parts of the series by considering

$$f_k^\pm = c_{2k} \pm c_{2k-1}, \quad (4.7)$$

which have asymptotics

$$f_n^\pm = 2A_\pm (2n)^{a_\pm - 1} \left(1 + \mathcal{O}\left(\frac{1}{n}\right) \right), \quad (4.8)$$

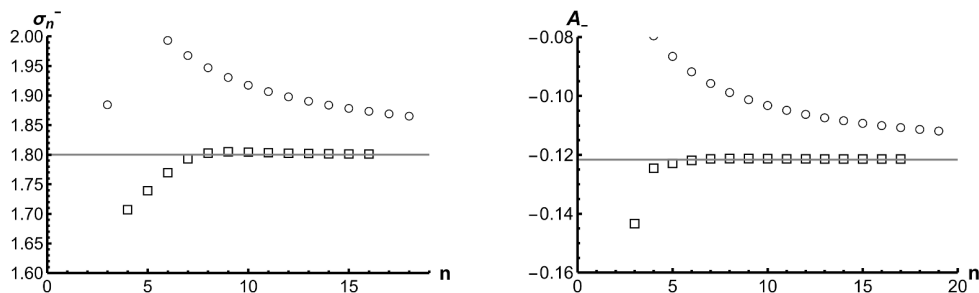


Figure 3. The series σ_n^- (left) converges to a_- using (4.9). Using eq. (4.8) we display (right) the sequence that converges to A_- . Circle markers indicate the raw data, square markers the second Richardson transformation. For both, we display results for $\kappa = 0.9$. The second Richardson transform converge to the expected results given by eq (4.5) up to errors of 0.011% and 0.00068% respectively.

such that the sequences

$$\sigma_n^\pm = 1 + n \log \frac{f_{n+1}^\pm}{f_n^\pm}, \tag{4.9}$$

converge to a_\pm . With a_\pm determined one can then directly consider the asymptotics of f^\pm to establish A_\pm . Figure 3 illustrates the convergence of this procedure for a fixed value of κ , and figure 4 establishes the functional form of these coefficients for various values of κ .

A methodological subtlety is that, from empirical observation, the contributions from the alternating sector, i.e. A_- (and associated subleading terms), are dominant for $\kappa > 0$ over those of the non-alternating A_+ sector. Thus to extract the non-alternating contributions we first establish the leading alternating contribution as described above and then repeat the process working instead with a new series in which the leading alternating contribution has been subtracted. However, when κ becomes sufficiently large, the sub-leading alternating contribution becomes comparable to that of the leading non-alternating contribution. This limits the reliability of determination numerically of the A_+, a_+ coefficients to small values of κ . However, these coefficients can be more readily verified by continuing to the $\kappa < 0$ regime where they are more dominant.

Having determined in this fashion the leading contributions to a_n , these can then be subtracted from the data, the analysis repeated *mutatis mutandis*, to determine the sub-leading β_k coefficients (and again for similar reasons to the above the β_k^- coefficients are more readily extracted). Figure 5 gives the numerical form of β_1^- and β_2^- as a function of κ indicating a linear relationship.

It becomes somewhat challenging to extract further subleading contributions from the data available. However, one can consider defining a new series, \tilde{a}_n , comprised by taking the data set and subtracting the already established asymptotic form of eq. (4.5). Using the Borel-Padé again to this subtracted series produces some evidence, see figure 6, of a compelling feature. Instead of poles accumulating at $\zeta = \pm 2$, as would be anticipated should the ansatz (4.4), one finds that leading positive poles appears to accumulate at $\zeta = +4$. The interpretation here is that the subtraction has removed the entire non-alternating terms with behaviour 2^{-n} , suggesting that all fluctuations $\beta_{n>0}^+ = 0$ That the

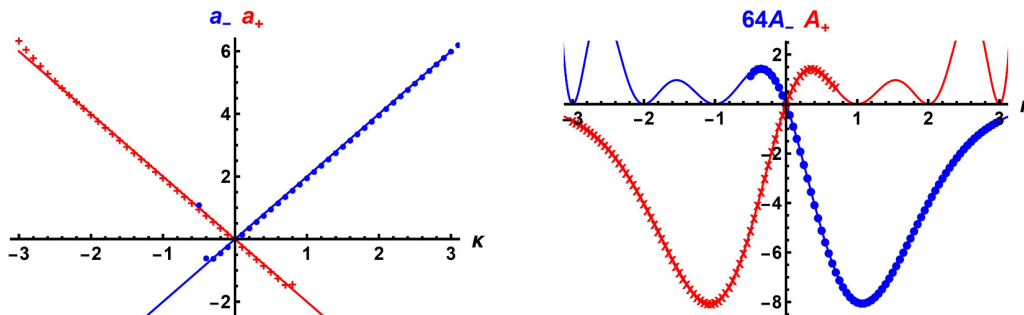


Figure 4. The second Richardson transformation of the sequences (4.9) (left) and (4.8) (right) to determine a_{\pm} and A_{\pm} as functions of κ . a_+, A_+ are indicated by red crosses and a_-, A_+ by blue points with solid lines showing the analytic formula of eq. (4.5).

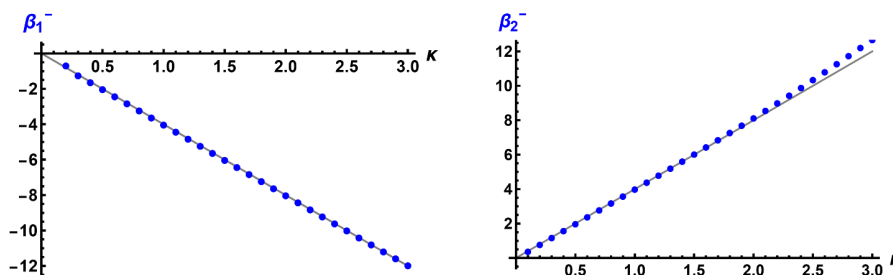


Figure 5. The sub-leading coefficient β_1^- (left) and β_2^- (right) for various values of κ . Shown is the terminal value of the third Richardson Transformation of the sequence that gives β_n^- constructed from f_n after subtraction of leading alternating and non-alternating asymptotics. Grey lines correspond to $\beta_1^- = -4\kappa$ and $\beta_2^- = +4\kappa$. A noticeable drift in β_2^- for larger values of κ suggests pollution from further sub-dominant terms contributing at this order of perturbation theory.

coefficients $\beta_{n>0}^+ = 0$ is a numerical prediction that we shall verify analytically in the sequel. That the next contribution comes with twice the “action” 4^{-n} is in accordance with the Parisi-’t Hooft conjecture [63–65]; the leading poles in the Borel plane at $\zeta = \pm 2$ lie at integer values and the values of $a_{\pm} = \mp 2\kappa = \pm 2\xi$ are as expected (see [15]).⁹ The accumulation of poles at $\zeta = +2$ is accordingly interpreted as an IR renormalon. A similar procedure of subtraction (removing the IR renormalon) used above (in figure 6) was performed in [26] to expose new Borel renormalon singularities that were *not* in accordance with Parisi-’t Hooft in cases including e.g. the Gross-Neveu model. Here however, figure 6 indicates that the next most proximate IR renormalon branch-point singularity is found in a location that *are* consistent with Parisi-’t Hooft.

5 Transseries and ambiguity cancellation

In this section we compute the leading ambiguity of $\frac{\epsilon}{\rho^2}$ in two different ways. First, we calculate the Borel ambiguity of the large order behaviour of the perturbative sector established in 4. This is compared against an approach which solves the TBA system in terms of a transseries.

⁹We thank M Mariño and T Reis for illuminating us on this point.

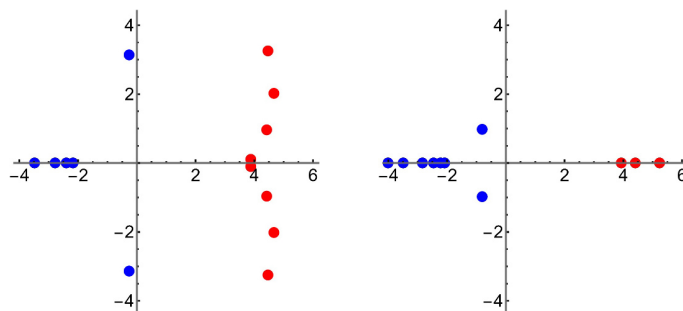


Figure 6. After subtracting the leading alternating and non-alternating contributions, we again perform a Borel-Padé computation for $\kappa = -0.75$ (left) and $\kappa = 0.4$ (right). This seems to suggest that there is no longer a Borel singularity at $\zeta = 2$, but instead finding one at $\zeta = 4$.

5.1 Borel resummation and large order perturbative ambiguity

Naively, one could try to resum the original asymptotic series by performing a Laplace transform on the Borel transform (4.1)

$$\frac{1}{\gamma} \int_0^\infty \mathcal{B} \left[\frac{8\kappa}{\pi} \frac{e}{\rho^2} \right] e^{-\zeta/\gamma} d\zeta = \frac{1}{\gamma} \int_0^\infty \sum_{n=0}^\infty \frac{a_n}{n!} \zeta^n e^{-\zeta/\gamma} \simeq \sum_{n=0}^\infty a_n \gamma^n = \frac{8\kappa}{\pi} \frac{e}{\rho^2}. \quad (5.1)$$

However, as we have seen, the Borel transform $\mathcal{B} \left[\frac{8\kappa}{\pi} \frac{e}{\rho^2} \right]$ generically has singularities along the positive real axis obstructing the contour of this integral. Therefore, we shall introduce a directional Borel resummation given by

$$S_\theta \left[\frac{8\kappa}{\pi} \frac{e}{\rho^2} \right] = \frac{1}{\gamma} \int_0^{e^{i\theta}\infty} \mathcal{B} \left[\frac{8\kappa}{\pi} \frac{e}{\rho^2} \right] e^{-\zeta/\gamma} d\zeta. \quad (5.2)$$

This procedure results, when integrating along a line without singularities, in a finite answer, which however, depends on the sign of θ , i.e. there is an ambiguity in the resummation of the perturbative series. This ambiguity, which is a Stokes phenomenon, is studied by considering $S_{+\epsilon} - S_{-\epsilon}$. This can be done analytically by using, instead of the numerically obtained results, a series whose coefficients are exactly the asymptotic form a_n given by eq. (4.4) for all values of n :

$$\begin{aligned} (S_{+\epsilon} - S_{-\epsilon}) \left[\frac{8\kappa}{\pi} \frac{e}{\rho^2} \right] (\gamma) &= 2\pi i A_+ \left(\frac{2}{\gamma} \right)^{a_+} e^{-2/\gamma} \sum_{k=0}^\infty \beta_k^+ \left(\frac{\gamma}{2} \right)^k \\ &= -\frac{16\pi i}{\Gamma(-\kappa)\Gamma(1-\kappa)} \left(\frac{\gamma}{2} \right)^{2\kappa} e^{-2/\gamma} [1 + \mathcal{O}(\gamma)]. \end{aligned} \quad (5.3)$$

Similarly, across the negative real axis we find a leading ambiguity given by

$$\begin{aligned} (S_{\pi+\epsilon} - S_{\pi-\epsilon}) \left[\frac{8\kappa}{\pi} \frac{e}{\rho^2} \right] (\gamma) &= 2\pi i A_- \left(-\frac{2}{\gamma} \right)^{a_-} e^{2/\gamma} \sum_{k=0}^\infty \beta_k^- \left(-\frac{z}{2} \right)^k \\ &= -\frac{\pi i}{4\Gamma(\kappa)\Gamma(1+\kappa)} \left(-\frac{\gamma}{2} \right)^{-2\kappa} e^{2/\gamma} [1 + \mathcal{O}(\gamma)]. \end{aligned} \quad (5.4)$$

In these expressions we note the presence of an exponentially small parameter, $\sqrt{q_\gamma} = \left(\frac{2}{\gamma}\right)^{2\kappa} e^{-2/\kappa}$ (the square root is for convenience later) characteristic of non-perturbative physics. The main thrust of the modern resurgence paradigm is that physical quantities, here e/ρ^2 , should be understood as a transseries, i.e. an expansion in $\sqrt{q_\gamma}$ whose terms are each formal (asymptotic) series in γ . It is critical that whilst resummation may be ambiguous when applied to any individual term in this (here the perturbative $\sqrt{q_\gamma}^0$ sector), taken altogether the final result is non-ambiguous. In particular, and this goes back to the pioneering work of Bogomol’nyi and Zinn-Justin [66–69], the ambiguity of this perturbative sector should be compensated by a leading order ambiguity in an appropriate non-perturbative sector. In the next section we shall verify that such an ambiguity cancellation does take place.

5.2 Transseries and leading non-perturbative ambiguity

In this series we shall apply a different type of analysis to the TBA equations which results in a transseries solution. The starting point shall be a reformulation of the TBA system as an integral equation for an auxiliary function $u(\omega)$,

$$u(\omega) = \frac{i}{\omega} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{e^{2iB\omega'} \varrho(\omega') u(\omega')}{\omega' + \omega + i\delta}, \tag{5.5}$$

where

$$\varrho(\omega) = \frac{1 - i\omega G_-(\omega)}{1 + i\omega G_+(\omega)}, \tag{5.6}$$

together with the boundary condition

$$u(i) = \frac{m}{2h} e^B \frac{G_+(i)}{G_+(0)}. \tag{5.7}$$

Having established the function u , the free energy is given by

$$\Delta F(h) = -\frac{1}{2\pi} h^2 u(i) G_-(0)^2 \left(1 - \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{2i\omega B} u(\omega) \varrho(\omega)}{\omega - i} \right). \tag{5.8}$$

We will now apply to the λ -model the techniques pioneered by [26] to solve this recursively order by order in a perturbative parameter and a non-perturbative parameter. The idea is to move the integration contour of the integral equation (5.5) into the UHP so that it envelops all the branch cuts and poles in the UHP. The Sine-Gordon model is special as it only has poles but no branch cut. This was studied in [70] and gives rise to a convergent rather than asymptotic expansion. However, in the case of the λ -deformed model, we are dealing with both poles and a branch cut along the imagine axis of $\rho(\omega)$.

To separate it from the poles, we slightly move the cut away from the imaginary axis to the ray $C_\pm = \{\xi e^{i\theta} | \theta = \frac{\pi}{2} \pm \delta\}$. By deforming the integration contour we isolate the contributions coming from the discontinuity over the cut and the residues at the poles (see figure 7). As explained in [26] the choice of moving the branch cut to C_+ or C_-

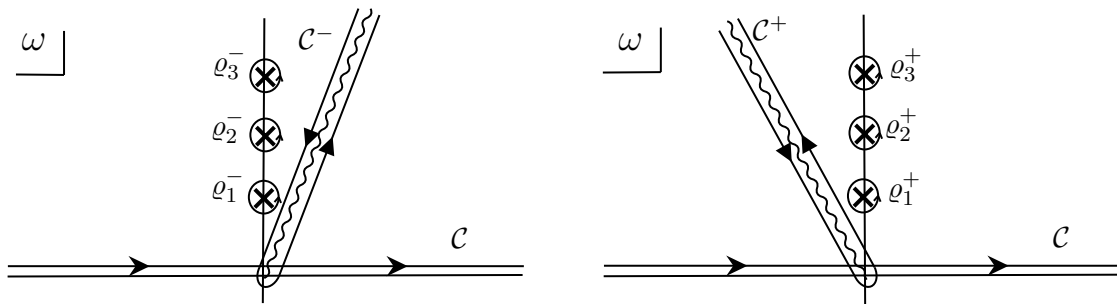


Figure 7. The contour $C = (-\infty, \infty)$ is deformed into either of two ways. The branch cut, represented by the curvy line is moved to either the ray C_+ or C_- . In those cases respectively, the contour is deformed into C^+ or C^- . In both cases we pick up residues ϱ_n^\pm , but their values differ by the branch cut discontinuity.

is arbitrary and gives rise to a leading non-perturbative ambiguity. Letting $\varrho_{n,\pm}$ be the residues at $x = x_n$ with the cut moved to C_\pm , after this contour pulling eq. (5.5) becomes

$$u(ix) = \frac{1}{x} + \frac{1}{2\pi i} \int_0^{\infty e^{\pm i\epsilon}} dx' \frac{e^{-2Bx'} u(ix') \delta \varrho(ix')}{x' + x} + \sum_n \frac{e^{-2Bx_n} u_n \varrho_{n,\pm}}{x_n + x}, \quad (5.9)$$

where $u_n \equiv u(ix_n)$ and $\delta \varrho$ is the discontinuity over the cut.¹⁰

From the WH-decomposition (2.17), we evaluate $\varrho(\omega)$ using (5.6) as

$$\varrho(\omega) = -\frac{\omega + i}{\omega - i} \frac{\Gamma\left(\frac{i\omega}{2} + 1\right)^2 \Gamma(1 - i\omega) \Gamma(1 - i\kappa\omega)}{\Gamma\left(1 - \frac{i\omega}{2}\right)^2 \Gamma(i\omega + 1) \Gamma(i\kappa\omega + 1)} e^{-2ib\omega} e^{i\kappa\omega(\log(i\omega) + \log(-i\omega))}. \quad (5.10)$$

For generic values of κ , this has poles on the positive real axis at $\omega = ix_n = i\mu n$ with $\mu = 2$ with residues given by

$$\varrho_{n,\pm} = \text{Res}_{x=x_n \pm i\epsilon} \varrho(ix) = -2ie^{2n(2b \pm i\pi\kappa - 2\kappa \log(2n))} n \frac{2n + 1}{2n - 1} \frac{((2n)!)^2 \Gamma(1 + 2n\kappa)}{(n!)^4 \Gamma(1 - 2n\kappa)}. \quad (5.11)$$

However, when κ is rational some of these poles are removed. Suppose we express $\kappa \equiv \frac{k}{N} = p/q$ as a reduced fraction with p, q coprime integers (i.e. $q = N/\text{gcd}(N, k)$), then the set of poles are located at $x \in 2\mathbb{N} \setminus q\mathbb{N}$ (rather than $x \in 2\mathbb{N}$). Hence, the residue $\varrho_{n,\pm}$ evaluates to zero if $2n \in 2\mathbb{N} \cap q\mathbb{N}$, i.e. $2n$ is a multiple of q . In particular, when k is an integer multiple of a half, i.e. $q = 1$ or $q = 2$, all poles are removed entirely. If $\varrho_1 = 0$, then $\varrho_n = 0$ for all n ; in what follows we shall consider only the case where $\varrho_1 \neq 0$ which is most relevant to our discussion.

The discontinuity function is given by

$$\delta \varrho(ix) = 2i \frac{x + 1}{x - 1} e^{2bx} e^{-2\kappa x \log x} \sin(\kappa\pi x) \frac{\Gamma(1 - x/2)^2 \Gamma(1 + x) \Gamma(1 + \kappa x)}{\Gamma(1 + x/2)^2 \Gamma(1 - x) \Gamma(1 - \kappa x)}. \quad (5.12)$$

¹⁰For the discontinuity function, we use the convention $\delta \rho(\omega) = \rho(\omega(1 - i\epsilon)) - \rho(\omega(1 + i\epsilon))$.

Notice this has simple poles at $x = 2n$, which have residues that vanish for κ half-integer. Lastly we shall need¹¹

$$\varrho(i \pm 0) = 8e^{2b \mp i\pi\kappa} \frac{\Gamma(1 + \kappa)}{\Gamma(1 - \kappa)} = \frac{8}{\pi\kappa} e^{2b \mp i\pi\kappa} \Gamma(1 + \kappa)^2 \sin(\pi\kappa). \quad (5.13)$$

Following again [26], the integral equation (5.9) is simplified by the introduction of $P(\eta, v)$ given by

$$e^{-2Bx} \delta\varrho(ix) = -2ive^{-\eta} P(\eta, v), \quad (5.14)$$

with a change of variables $(x, B) \rightarrow (\eta, v)$:

$$\frac{1}{v} + a \log v = 2B, \quad x = v\eta. \quad (5.15)$$

Here, a is a constant determined by demanding that $P(\eta, v)$ is regular in v with, in particular, no $\log(v)$ terms. From eq. (5.12), we have that $\delta\varrho(ix) \propto e^{\tilde{a}x \log x} \sum d_n x^n$, where $\tilde{a} = -2\kappa$, therefore this determines $\tilde{a} = a$. This yields an expansion of $P(\eta, v)$ given by

$$P(\eta, v) = d_{1,0}\eta + v\eta^2(d_{2,0} + d_{2,1} \log(\eta)) + \mathcal{O}(v^2), \quad (5.16)$$

$$d_{1,0} = \pi\kappa, \quad d_{2,0} = 2\pi\kappa(1 + (1 - \gamma_E - \log(\kappa))\kappa - \log(2)), \quad d_{2,1} = -2\pi\kappa^2.$$

With the introduction of an integral operator

$$\mathcal{I}[f](\eta) = -\frac{v}{\pi} \int_0^\infty d\eta' \frac{e^{-\eta'} P(\eta', v) f(\eta')}{\eta + \eta'}, \quad (5.17)$$

after this change of variables, eq. (5.9) can be written as

$$u(\eta) = \mathbf{u}(\eta) + \mathcal{I}[u](\eta), \quad (5.18)$$

in which the ‘seed’ solution is given as

$$\mathbf{u}(\eta) = \frac{1}{v\eta} + \frac{1}{v} \sum_n \frac{e^{-2Bv\eta_n} u_n \varrho_{n,\pm}}{\eta_n + \eta}. \quad (5.19)$$

The formal solution obtained by iteration is thus presented as

$$u(\eta) = \sum_{l=0}^\infty \mathcal{I}^l[\mathbf{u}](\eta) \equiv \mathcal{J}[\mathbf{u}](\eta). \quad (5.20)$$

To determine the unknown coefficients $u_n = u(\eta_n)$ we evaluate eq. (5.18) at $\eta = \eta_n = \mu n/v$ and define $\mathcal{I}_n[f] \equiv \mathcal{I}[f](\eta = \eta_n)$ to obtain

$$u_n = \frac{1}{\mu n} + \mathcal{I}_n[u] + \frac{1}{\mu} \sum_m \frac{e^{-2Bv\eta_m} u_m \varrho_{m,\pm}}{m + n}. \quad (5.21)$$

Here we have made a slight adaptation compared to [26] to suit the locations of the poles at $x_n = \mu n$ (with $\mu = 2$) (cf. the Gross-Neveu model for which $x_n = \frac{2n+1}{\Upsilon}$ for some

¹¹Because we are assuming that κ is *not* integer, $\varrho(i \pm 0)$ is non-zero. If $\kappa < 0$, then $\varrho(i \pm 0)$ generically has a finite ambiguity.

constant Υ). To treat the exponentially small contributions coming from the residue term we introduce the small parameter

$$q = e^{-2B\mu} = e^{-\mu/v} v^{-\mu a}. \quad (5.22)$$

Both the seed and formal solution, and the unknown values u_n , admit expansion in q

$$\mathbf{u}(\eta) = \sum_{s=1} \mathbf{u}^{(s)}(\eta) q^s, \quad u(\eta) = \sum u^{(s)}(\eta) q^s, \quad u_n = \sum_{s=0} u_n^{(s)} q^s. \quad (5.23)$$

As the operator \mathcal{J} does not introduce factors of q we can construct the full solution order by order in q noting $u^{(s)}(\eta) = \mathcal{J}[\mathbf{u}^{(s)}](\eta)$. Using (5.19) one finds that the first few terms¹² of the seed solution are given by

$$\mathbf{u}^{(0)} = \frac{1}{v\eta}, \quad \mathbf{u}^{(1)} = \frac{\varrho_{1,\pm} u_1^{(0)}}{v\eta + \mu}, \quad \mathbf{u}^{(2)} = \frac{\varrho_{1,\pm} u_1^{(1)}}{v\eta + \mu} + \frac{\varrho_{2,\pm} u_2^{(0)}}{v\eta + 2\mu}. \quad (5.25)$$

Applying the q -expansion to eq. (5.21) we have that

$$u_n^{(0)} = \mathcal{J}[\mathbf{u}^{(0)}](\eta_n) = \frac{1}{\mu n} + \mathcal{I}_n \left[\mathcal{J} \left[\frac{1}{v\eta} \right] \right], \quad u_n^{(1)} = \mathcal{I}_n[\mathcal{J}[\mathbf{u}^{(1)}]] + \frac{1}{\mu} \frac{\varrho_{1,\pm} u_1^{(0)}}{1+n}. \quad (5.26)$$

Let us assume that $\varrho_{1,\pm} \neq 0$ (i.e. κ is not half-integer), such that these two expressions are governing the leading behaviour. Suppose now we work formally¹³ to leading order in v and leading order in q . Because each application of \mathcal{I} carries a factor v , to leading order it is sufficient to consider only the identity operator $\mathcal{J} = 1 + \dots$ which results in¹⁴

$$u_n^{(0)} = \frac{1}{\mu n} - \frac{v}{n\pi\mu} d_{1,0} + \mathcal{O}(v^2), \quad u_n^{(1)} = \frac{\varrho_{1,\pm}}{\mu^2(n+1)} - \frac{d_{1,0}\varrho_{1,\pm}}{\mu^2\pi(n+1)} v + \mathcal{O}(v^2). \quad (5.27)$$

The leading orders of $u(\eta)$ are obtained by

$$u(\eta) = \left[\mathbf{u}^{(0)} + \mathcal{I}[\mathbf{u}^{(0)}] + \mathcal{O}(v) \right] + q \left[\mathbf{u}^{(1)} + \mathcal{I}[\mathbf{u}^{(1)}] + \mathcal{O}(v^2) \right] + \mathcal{O}(q^2). \quad (5.28)$$

To implement the boundary condition that relates the chemical potential h to q, v , we will need

$$u(i) = u\left(\eta = \frac{1}{v}\right) = \left[1 - \frac{d_{1,0}}{\pi} v + \mathcal{O}(v^2) \right] + \frac{q\varrho_{1,\pm}}{\mu(1+\mu)} \left[1 - \frac{d_{1,0}v}{\pi} + \mathcal{O}(v^2) \right] + \mathcal{O}(q^2). \quad (5.29)$$

¹²For $n \geq 1$, we have in general

$$\mathbf{u}^{(n)}(\eta) = \sum_{m=1}^n \frac{\varrho_{m,\pm} u_m^{(n-m)}}{v\eta + \mu m}. \quad (5.24)$$

¹³I.e. ignoring that q is exponentially smaller than higher order polynomial terms in v .

¹⁴The small v limit can be taken also in the integral:

$$\mathcal{I} \left[\frac{1}{v\eta} \right] (\eta_n) = -\frac{v}{\pi} \int_0^\infty \frac{e^{\eta'} d_{1,0} \eta}{v\eta' + n\mu} = -\frac{v}{\pi} \int_0^\infty \left(\frac{e^{\eta'} d_{1,0}}{n\mu} + \mathcal{O}(v) \right) = -\frac{v d_{1,0}}{n\pi\mu} + \mathcal{O}(v^2).$$

The next step is to do the Legendre transform and calculate $\frac{e}{\rho^2}$ from ΔF . This can then be used to compare against the perturbative calculation. The same procedure of resolving the cut away from the poles of ρ and deforming the contour appropriately yields

$$\Delta F(h) = -\frac{h^2}{2\pi} u(i) G_+(0)^2 \left(1 + \frac{v^2}{\pi} \int \frac{e^{-\eta} P(\eta, v) u(\eta)}{\eta v - 1} d\eta - \sum_{n \geq 1} \frac{q^n \varrho_{n, \pm} u_n}{\mu n - 1} \right) + \frac{1}{8\pi} m^2 G_+(i)^2 \rho(i \pm \epsilon), \tag{5.30}$$

in which we have made use of the boundary condition of eq. (5.7) to simplify the (h -independent) contribution proportional to $\rho(i \pm \epsilon)$. The leading orders of eq. (5.30) are given by¹⁵

$$\Delta F(h) = -\frac{G_+(0)^2 h}{2\pi} \left(1 - \frac{2d_{10}}{\pi} v + \mathcal{O}(v^2) \right) \times \left(1 + \frac{2\rho_{1, \pm}}{\mu(1 - \mu^2)} q + \mathcal{O}(q^2) \right) + \frac{1}{8\pi} m^2 G_+(i)^2 \rho(i \pm \epsilon) \tag{5.31}$$

The first step of the Legendre transform is to relate h to the parameters q and v . This is done by substituting the expansion (5.29) for $u(i)$ into the boundary condition (5.7) which, for $\mu = 2$, gives

$$h = \frac{m G_+(i)}{12\pi G_+(0)} q^{-1/4} (\pi + d_{1,0} v + \mathcal{O}(v^2)) (6 - \rho_{1, \pm} q + \mathcal{O}(q^2)). \tag{5.32}$$

As a consequence $\rho = -\frac{d\Delta F}{dh}$ is given by

$$\rho = \frac{G_+(i) G_+(0) m}{12\pi^2} (\pi - d_{1,0} v + \mathcal{O}(v^2)) \left(6q^{-1/4} + \rho_{1, \pm} q^{3/4} + \mathcal{O}(q^{7/4}) \right), \tag{5.33}$$

from which we obtain $\frac{e}{\rho^2}$ as a series in v and q :

$$\frac{e}{\rho^2} = \frac{1}{6G_+(0)^2} (\pi + 2d_{1,0} v + \mathcal{O}(v^2)) \left(3 + \rho_{1, \pm} q + \mathcal{O}(q^{3/2}) \right) + \frac{1}{8\pi} \frac{m^2}{\rho^2} G_+(i)^2 \rho(i \pm \epsilon). \tag{5.34}$$

We will now write this expansion in terms of the coupling (3.20) used in the previous sections. Let us introduce a parameter exponentially small in γ , analogous to q being exponentially small in v , given by $q_\gamma = e^{-4/\gamma} (\gamma/2)^{4\kappa}$. Notice that in particular that $m = 2\rho\Gamma(\kappa)q_\gamma^{1/4}$; this will allow us to exactly evaluate the h -independent contribution proportional to $\rho(i \pm \epsilon)$ in eq. (5.34). For the remaining terms in eq. (5.34) we use (3.20) to write v as a series in γ and q_γ . Upon substitution of this series for v (and $q = q(v)$) into (5.34), we arrive at

$$\frac{8\kappa}{\pi} \frac{e}{\rho^2} = \left(1 + \kappa\gamma + \mathcal{O}(\gamma^2) \right) - 8e^{\mp i\pi\kappa} \frac{\Gamma(\kappa)}{\Gamma(-\kappa)} q_\gamma^{1/2} + 2^{3-4\kappa} e^{\mp 2i\pi\kappa} \frac{\Gamma(2\kappa)}{\Gamma(-2\kappa)} q_\gamma (1 + \mathcal{O}(\gamma)). \tag{5.35}$$

¹⁵For aesthetic purposes we have displayed this equation, and those following, as a factored product of perturbative and non-perturbative parts. This factorisation however is coincidental to the order of expansion displayed here and not a general feature.

We see that the first two coefficient of the perturbative series match precisely with eq. (3.22). The presence of transseries parameters $q_\gamma = e^{-4/\gamma}(\gamma/2)^{4\kappa}$ provides concrete predictions of the resurgent structure of the perturbative series. In particular, we compute the ambiguity of the transseries (5.35) due to the difference in result if the branch cut is left or right of the poles. To leading order in q_γ and γ , it is given by

$$\frac{8\kappa}{\pi} \left[\left(\frac{e}{\rho^2} \right)_- - \left(\frac{e}{\rho^2} \right)_+ \right] = \frac{16\pi i}{\Gamma(-\kappa)\Gamma(1-\kappa)} \left(\frac{\gamma}{2} \right)^{2\kappa} e^{-2/\gamma}. \quad (5.36)$$

This is exactly the same ambiguity as obtained through the asymptotic analysis of our perturbative calculation - see eq. (5.3). We thus observe that the Borel-ambiguity of the perturbative series can be cancelled precisely by an ambiguity of a transmonomial. Therefore, the large order non-perturbative behaviour is unambiguous up to the order considered. This mirrors the fabled BZJJ ambiguity cancellation [66–69] in a field theory context. Further we emphasise that the $q_\gamma^{1/2}$ contribution above receives no polynomial corrections in γ ; this is exactly in accordance with the numerical non-alternating large order behaviour which pointed towards the vanishing of subleading corrections $\beta_{n>0}^+$ (recall that for BZJJ cancellation to hold subleading large order behaviour must match against perturbative fluctuations about the corresponding non-perturbative saddle and in the case at hand both are absent).

Beyond this leading ambiguity on the positive real axis of the Borel plane we can see that there is a non-perturbative ambiguity in the coefficient of q_γ which when translated back corresponds to the branch point singularity at $\zeta = +4$ observed in figure 6. Moreover, since the TBA kernel $\varrho(\omega)$ has poles in the positive imaginary axis at regular (twice integer) spacing $\omega = 2in$, all non perturbative-contributions will come with integer powers of q_γ . Since $q_\gamma^{1/4} \propto m/\rho$ this implies that all non perturbative-corrections are proportional to integer powers of the dynamically generated scale, and this is in accordance with the Parisi-'t Hooft conjecture [63–65].

We can do a similar analysis to recover the Borel branch singularity on the negative real axis. The critical modification of the programme, as realised by [15], is to deform the contour of the integral equation (5.5) into the lower half plane, instead of the upper half plan. The critical analytic data is then given by the branch cut and residues at the negative imaginary axis. In the lower half plane, $\varrho(-ix)$ has residues at $x_n = 2n + 1$ and at $\tilde{x}_n := \frac{n}{\kappa}$. However, as the latter set of residues is unambiguous with respect to the branch cut, they do not contribute.¹⁶ One subtlety when using this approach arises when computing $u(i)$. Deforming the contour of eq. (5.5) to an envelopment of the negative imaginary axis picks up a residue at $\omega = -i$, which introduces a contribution of $u(-i)\rho(-i \pm \epsilon)q$ not present in the analysis above. We will not present a detailed derivation as it is similar to the one above. Rather, we can report that the final result is a transseries with a leading ambiguity

¹⁶They would be part of a transseries solution, but as they are unambiguous, they are not of interest to us currently. As a further side remark, when choosing $\kappa < 0$, along the positive imaginary axis $\varrho(ix)$ also has such unambiguous residues at $x = \frac{n}{\kappa}$.

given by

$$\frac{8\kappa}{\pi} \left[\left(\frac{e}{\rho^2} \right)_- - \left(\frac{e}{\rho^2} \right)_+ \right] = -\frac{\pi i}{4\Gamma(\kappa)\Gamma(1+\kappa)} \left(-\frac{\gamma}{2} \right)^{-2\kappa} e^{2/\gamma}. \quad (5.37)$$

This precisely matches the ambiguity of the perturbative sector around the negative real axis found in eq. (5.4).

6 Discussion

In this note, we have studied the λ -model and brought it into the fold of resurgent analysis of [13–15, 26]. The model is particularly interesting, because, distinct from previously considered models, it has a interacting CFT fixed point in the UV.

We have found a perturbative series for the energy density at finite chemical potential of the λ -model in section 3.4 and identified with numerical techniques its asymptotic form in section 4. A key feature is that the Borel resummation of the large order behaviour is ambiguous when taken along either the positive or negative real axis. These ambiguities are exactly compensated/cancelled by a further ambiguity in a non-perturbative sector of a transseries solution in section 5. These cancellations provide the λ -model with a robustly defined foundation which may serve as a paradigmatic example for other theories with asymptotic CFT behaviour.

Of particular note is that the leading ambiguity on the positive axis (and associated features in the Borel plane) vanishes for $\kappa \in \mathbb{Z}_{>0}$, i.e. when the WZW level k divides the rank N of the gauge group $SU(N)$. This is reminiscent of Cheshire-cat resurgence [71–74] in which the full glory of resurgence only becomes apparent as you deform away from certain special points at which it truncates, here however the feature is more acute in that at the special values of κ where this happens both factorial large order growth and the non-perturbative contribution disappear entirely.

Let us finish with some broader questions to ponder following the analysis of the λ -model that we hope might stimulate further investigations on the topic:

- An interesting feature of the WZW CFT that defines the UV of the λ model is that it exhibits level-rank duality [75]. It would be valuable to understand the extent to which this property constrains, or is encapsulated, in the form of the transseries that defines the λ -model.
- In a QFT it is sometimes possible to directly link poles/branch points in the Borel plane to finite action non-perturbative saddle configurations. Remarkably, this can be done even in theories without instantons. In a series of paper [9, 11, 12] finite action ‘uniton’ configurations of 1+1d integrable QFTs were matched to Borel poles of a quantum mechanics that followed by dimensional reduction with twisted boundary conditions (akin to a chemical potential as deployed here). This poses a natural question: can the features of the Borel plane we have found here via TBA methods be related to some finite action saddles? Conversely, given the knowledge of such uniton configurations, what do they imply for the TBA method? Achieving this

would serve to put the semi-classical approaches of [9, 11, 12] on a surer-footing in quantum field theory.

- On the other hand there are a class of ambiguities which don't (yet at least) have an interpretation as semi-classical saddles. Instead they are renormalon ambiguities associated to certain classes of Feynman diagrams. In [18, 76] it was shown how to construct such a series of diagrams which source the renormalon ambiguities in $1/N$ expansion of the $O(N)$ vector model, the Gross-Neveu and the $SU(N)$ PCM. It would be interesting to investigate if there are diagrams that are responsible for the ambiguities in the λ -models.
- The landscape of integrable models in two dimensions has been vastly expanded in recent years through variants of this λ -model, and the related Yang-Baxter σ -models. It could be rewarding to deploy similar technique across this landscape included e.g. to models with multiple deformation parameters or theories based on cosets rather than group manifolds.
- In [57] the Quantum Inverse Scattering Method was applied to give a direct quantisation of the λ -models as a continuum limit of a spin k Heisenberg spin-chain with inhomogeneities. The parameter that governs the in-homogeneity becomes a mass. Although the ground state of the system is quite a complicated Fermi sea, one can identify holes as certain particle excitations. After taking the continuum limit, one can obtain a TBA system for these excitations matching that of the QFT. An exciting question is if the above resurgent structure can be given a similar *ab initio* derivation within the QISM framework.

Acknowledgments

DCT is supported by The Royal Society through a University Research Fellowship Generalised Dualities in String Theory and Holography URF 150185 and in part by STFC grant ST/P00055X/1 as well as by the FWO-Vlaanderen through the project G006119N and Vrije Universiteit Brussel through the Strategic Research Program “High-Energy Physics”. LS is supported by a PhD studentship from The Royal Society and the grant RF\ERE\210269. For the purpose of open access, the authors have applied a Creative Commons Attribution (CC BY) licence to any Author Accepted Manuscript version arising. We thank M Mariño and T Reis for helpful comments on a draft and I Aniceto for comments relating to this project.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited. SCOAP³ supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

- [1] A.B. Zamolodchikov, *Exact two particle S matrix of quantum sine-Gordon solitons*, *Pisma Zh. Eksp. Teor. Fiz.* **25** (1977) 499 [INSPIRE].
- [2] P. Hasenfratz, M. Maggiore and F. Niedermayer, *The exact mass gap of the $O(3)$ and $O(4)$ nonlinear sigma models in $d = 2$* , *Phys. Lett. B* **245** (1990) 522 [INSPIRE].
- [3] P. Hasenfratz and F. Niedermayer, *The exact mass gap of the $O(N)$ sigma model for arbitrary $N \geq 3$ in $d = 2$* , *Phys. Lett. B* **245** (1990) 529 [INSPIRE].
- [4] P. Forgacs, F. Niedermayer and P. Weisz, *The exact mass gap of the Gross-Neveu model. 1. The thermodynamic Bethe ansatz*, *Nucl. Phys. B* **367** (1991) 123 [INSPIRE].
- [5] P. Forgacs, F. Niedermayer and P. Weisz, *The exact mass gap of the Gross-Neveu model. 2. The $1/N$ expansion*, *Nucl. Phys. B* **367** (1991) 144 [INSPIRE].
- [6] I. Aniceto, G. Basar and R. Schiappa, *A primer on resurgent transseries and their asymptotics*, *Phys. Rept.* **809** (2019) 1 [arXiv:1802.10441] [INSPIRE].
- [7] G.V. Dunne and M. Ünsal, *Resurgence and trans-series in quantum field theory: the CP^{N-1} model*, *JHEP* **11** (2012) 170 [arXiv:1210.2423] [INSPIRE].
- [8] A. Cherman, D. Dorigoni, G.V. Dunne and M. Ünsal, *Resurgence in quantum field theory: nonperturbative effects in the principal chiral model*, *Phys. Rev. Lett.* **112** (2014) 021601 [arXiv:1308.0127] [INSPIRE].
- [9] A. Cherman, D. Dorigoni and M. Unsal, *Decoding perturbation theory using resurgence: Stokes phenomena, new saddle points and Lefschetz thimbles*, *JHEP* **10** (2015) 056 [arXiv:1403.1277] [INSPIRE].
- [10] G.V. Dunne and M. Unsal, *Resurgence and dynamics of $O(N)$ and Grassmannian sigma models*, *JHEP* **09** (2015) 199 [arXiv:1505.07803] [INSPIRE].
- [11] S. Demulder, D. Dorigoni and D.C. Thompson, *Resurgence in η -deformed principal chiral models*, *JHEP* **07** (2016) 088 [arXiv:1604.07851] [INSPIRE].
- [12] L. Schepers and D.C. Thompson, *Resurgence in the bi-Yang-Baxter model*, *Nucl. Phys. B* **964** (2021) 115308 [arXiv:2007.03683] [INSPIRE].
- [13] D. Volin, *From the mass gap in $O(N)$ to the non-Borel-summability in $O(3)$ and $O(4)$ sigma-models*, *Phys. Rev. D* **81** (2010) 105008 [arXiv:0904.2744] [INSPIRE].
- [14] D. Volin, *Quantum integrability and functional equations: applications to the spectral problem of AdS/CFT and two-dimensional sigma models*, *J. Phys. A* **44** (2011) 124003 [arXiv:1003.4725] [INSPIRE].
- [15] M. Mariño and T. Reis, *Renormalons in integrable field theories*, *JHEP* **04** (2020) 160 [arXiv:1909.12134] [INSPIRE].
- [16] M. Mariño and T. Reis, *Resurgence for superconductors*, arXiv:1905.09569 [DOI:10.1088/1742-5468/ab4802] [INSPIRE].
- [17] M. Mariño and T. Reis, *A new renormalon in two dimensions*, *JHEP* **07** (2020) 216 [arXiv:1912.06228] [INSPIRE].
- [18] M. Marino, R. Miravitllas and T. Reis, *Testing the Bethe ansatz with large N renormalons*, *Eur. Phys. J. ST* **230** (2021) 2641 [arXiv:2102.03078] [INSPIRE].

- [19] Z. Bajnok, J. Balog, A. Hegedus and I. Vona, *Instanton effects vs resurgence in the $O(3)$ sigma model*, *Phys. Lett. B* **829** (2022) 137073 [[arXiv:2112.11741](#)] [[INSPIRE](#)].
- [20] Z. Bajnok and R.A. Janik, *OPE coefficients and the mass-gap from the integrable scattering description of 2D CFT's*, *JHEP* **11** (2022) 128 [[arXiv:2209.10393](#)] [[INSPIRE](#)].
- [21] Z. Bajnok, J. Balog, A. Hegedus and I. Vona, *Running coupling and non-perturbative corrections for $O(N)$ free energy and for disk capacitor*, *JHEP* **09** (2022) 001 [[arXiv:2204.13365](#)] [[INSPIRE](#)].
- [22] Z. Bajnok, J. Balog and I. Vona, *Analytic resurgence in the $O(4)$ model*, *JHEP* **04** (2022) 043 [[arXiv:2111.15390](#)] [[INSPIRE](#)].
- [23] M.C. Abbott et al., *Resurgence in the $O(4)$ sigma model*, *JHEP* **05** (2021) 253 [[arXiv:2011.12254](#)] [[INSPIRE](#)].
- [24] M.C. Abbott, Z. Bajnok, J. Balog and Á. Hegedús, *From perturbative to non-perturbative in the $O(4)$ sigma model*, *Phys. Lett. B* **818** (2021) 136369 [[arXiv:2011.09897](#)] [[INSPIRE](#)].
- [25] Z. Bajnok, J. Balog and I. Vona, *The full analytic trans-series in integrable field theories*, [arXiv:2212.09416](#) [[INSPIRE](#)].
- [26] M. Marino, R. Miravitllas and T. Reis, *New renormalons from analytic trans-series*, *JHEP* **08** (2022) 279 [[arXiv:2111.11951](#)] [[INSPIRE](#)].
- [27] K. Sfetsos, *Integrable interpolations: from exact CFTs to non-Abelian T-duals*, *Nucl. Phys. B* **880** (2014) 225 [[arXiv:1312.4560](#)] [[INSPIRE](#)].
- [28] K. Sfetsos and K. Siampos, *Gauged WZW-type theories and the all-loop anisotropic non-Abelian Thirring model*, *Nucl. Phys. B* **885** (2014) 583 [[arXiv:1405.7803](#)] [[INSPIRE](#)].
- [29] J.M. Evans and T.J. Hollowood, *Integrable theories that are asymptotically CFT*, *Nucl. Phys. B* **438** (1995) 469 [[hep-th/9407113](#)] [[INSPIRE](#)].
- [30] J.M. Evans and T.J. Hollowood, *Exact results for integrable asymptotically-free field theories*, *Nucl. Phys. B Proc. Suppl.* **45** (1996) 130 [[hep-th/9508141](#)] [[INSPIRE](#)].
- [31] C. Appadu, T.J. Hollowood and D. Price, *Quantum inverse scattering and the lambda deformed principal chiral model*, *J. Phys. A* **50** (2017) 305401 [[arXiv:1703.06699](#)] [[INSPIRE](#)].
- [32] E. Witten, *Non-abelian bosonization in two dimensions*, in *Bosonization*, World Scientific (1994), p. 201.
- [33] J. Balog, P. Forgacs, Z. Horvath and L. Palla, *A new family of $SU(2)$ symmetric integrable sigma models*, *Phys. Lett. B* **324** (1994) 403 [[hep-th/9307030](#)] [[INSPIRE](#)].
- [34] K. Sfetsos, *Gauged WZW models and non-abelian duality*, *Phys. Rev. D* **50** (1994) 2784 [[hep-th/9402031](#)] [[INSPIRE](#)].
- [35] D.C. Thompson, *An introduction to generalised dualities and their applications to holography and integrability*, *PoS CORFU2018* (2019) 099 [[arXiv:1904.11561](#)] [[INSPIRE](#)].
- [36] B. Hoare, *Integrable deformations of sigma models*, *J. Phys. A* **55** (2022) 093001 [[arXiv:2109.14284](#)] [[INSPIRE](#)].
- [37] T.J. Hollowood, J.L. Miramontes and D.M. Schmidtt, *Integrable deformations of strings on symmetric spaces*, *JHEP* **11** (2014) 009 [[arXiv:1407.2840](#)] [[INSPIRE](#)].
- [38] T.J. Hollowood, J.L. Miramontes and D.M. Schmidtt, *An integrable deformation of the $AdS_5 \times S^5$ superstring*, *J. Phys. A* **47** (2014) 495402 [[arXiv:1409.1538](#)] [[INSPIRE](#)].

- [39] K. Sfetsos and D.C. Thompson, *Spacetimes for λ -deformations*, *JHEP* **12** (2014) 164 [[arXiv:1410.1886](#)] [[INSPIRE](#)].
- [40] S. Demulder, K. Sfetsos and D.C. Thompson, *Integrable λ -deformations: squashing coset CFTs and $AdS_5 \times S^5$* , *JHEP* **07** (2015) 019 [[arXiv:1504.02781](#)] [[INSPIRE](#)].
- [41] R. Borsato and L. Wulff, *Target space supergeometry of η and λ -deformed strings*, *JHEP* **10** (2016) 045 [[arXiv:1608.03570](#)] [[INSPIRE](#)].
- [42] A.A. Tseytlin, *On a ‘universal’ class of WZW type conformal models*, *Nucl. Phys. B* **418** (1994) 173 [[hep-th/9311062](#)] [[INSPIRE](#)].
- [43] C. Appadu and T.J. Hollowood, *Beta function of k deformed $AdS_5 \times S^5$ string theory*, *JHEP* **11** (2015) 095 [[arXiv:1507.05420](#)] [[INSPIRE](#)].
- [44] C. Klimčík, *Poisson-Lie T-duals of the bi-Yang-Baxter models*, *Phys. Lett. B* **760** (2016) 345 [[arXiv:1606.03016](#)] [[INSPIRE](#)].
- [45] B. Hoare and A.A. Tseytlin, *On integrable deformations of superstring sigma models related to $AdS_n \times S^n$ supercosets*, *Nucl. Phys. B* **897** (2015) 448 [[arXiv:1504.07213](#)] [[INSPIRE](#)].
- [46] K. Sfetsos, K. Siampos and D.C. Thompson, *Generalised integrable λ - and η -deformations and their relation*, *Nucl. Phys. B* **899** (2015) 489 [[arXiv:1506.05784](#)] [[INSPIRE](#)].
- [47] B. Vicedo, *Deformed integrable σ -models, classical R-matrices and classical exchange algebra on Drinfel’d doubles*, *J. Phys. A* **48** (2015) 355203 [[arXiv:1504.06303](#)] [[INSPIRE](#)].
- [48] C. Klimčík, *η and λ deformations as E-models*, *Nucl. Phys. B* **900** (2015) 259 [[arXiv:1508.05832](#)] [[INSPIRE](#)].
- [49] T.H. Buscher, *Path integral derivation of quantum duality in nonlinear sigma models*, *Phys. Lett. B* **201** (1988) 466 [[INSPIRE](#)].
- [50] K. Gawedzki and A. Kupiainen, *G/H conformal field theory from gauged WZW model*, *Phys. Lett. B* **215** (1988) 119 [[INSPIRE](#)].
- [51] E. Witten, *On holomorphic factorization of WZW and coset models*, *Commun. Math. Phys.* **144** (1992) 189 [[INSPIRE](#)].
- [52] T.J. Hollowood, J.L. Miramontes and D.M. Schmidt, *S-matrices and quantum group symmetry of k -deformed sigma models*, *J. Phys. A* **49** (2016) 465201 [[arXiv:1506.06601](#)] [[INSPIRE](#)].
- [53] G. Itsios, K. Sfetsos, K. Siampos and A. Torrielli, *The classical Yang-Baxter equation and the associated Yangian symmetry of gauged WZW-type theories*, *Nucl. Phys. B* **889** (2014) 64 [[arXiv:1409.0554](#)] [[INSPIRE](#)].
- [54] G. Georgiou, K. Sfetsos and K. Siampos, *Strong integrability of λ -deformed models*, *Nucl. Phys. B* **952** (2020) 114923 [[arXiv:1911.07859](#)] [[INSPIRE](#)].
- [55] S.F. Hassan and A. Sen, *Marginal deformations of WZNW and coset models from $O(d, d)$ transformation*, *Nucl. Phys. B* **405** (1993) 143 [[hep-th/9210121](#)] [[INSPIRE](#)].
- [56] C. Ahn, D. Bernard and A. LeClair, *Fractional supersymmetries in perturbed coset CFTs and integrable soliton theory*, *Nucl. Phys. B* **346** (1990) 409 [[INSPIRE](#)].
- [57] C. Appadu, T.J. Hollowood, D. Price and D.C. Thompson, *Quantum anisotropic sigma and lambda models as spin chains*, *J. Phys. A* **51** (2018) 405401 [[arXiv:1802.06016](#)] [[INSPIRE](#)].

- [58] A.M. Polyakov and P.B. Wiegmann, *Theory of non-Abelian Goldstone bosons*, *Phys. Lett. B* **131** (1983) 121 [INSPIRE].
- [59] A.M. Polyakov and P.B. Wiegmann, *Goldstone fields in two-dimensions with multivalued actions*, *Phys. Lett. B* **141** (1984) 223 [INSPIRE].
- [60] P.B. Wiegmann, *Exact solution of the $O(3)$ nonlinear two-dimensional sigma-model*, *JETP Lett.* **41** (1985) 95.
- [61] J. Balog, S. Naik, F. Niedermayer and P. Weisz, *Exact mass gap of the chiral $SU(n) \times SU(n)$ model*, *Phys. Rev. Lett.* **69** (1992) 873 [INSPIRE].
- [62] T.J. Hollowood, *The exact mass gaps of the principal chiral models*, *Phys. Lett. B* **329** (1994) 450 [hep-th/9402084] [INSPIRE].
- [63] G. Parisi, *On infrared divergences*, *Nucl. Phys. B* **150** (1979) 163 [INSPIRE].
- [64] G. Parisi, *Singularities of the Borel transform in renormalizable theories*, *Phys. Lett. B* **76** (1978) 65 [INSPIRE].
- [65] G. 't Hooft, *Can we make sense out of quantum chromodynamics?*, *Subnucl. Ser.* **15** (1979) 943 [INSPIRE].
- [66] E.B. Bogomolny, *Calculation of instanton-anti-instanton contributions in quantum mechanics*, *Phys. Lett. B* **91** (1980) 431 [INSPIRE].
- [67] J. Zinn-Justin and U.D. Jentschura, *Multi-instantons and exact results I: conjectures, WKB expansions, and instanton interactions*, *Annals Phys.* **313** (2004) 197 [quant-ph/0501136] [INSPIRE].
- [68] J. Zinn-Justin and U.D. Jentschura, *Multi-instantons and exact results II: specific cases, higher-order effects, and numerical calculations*, *Annals Phys.* **313** (2004) 269 [quant-ph/0501137] [INSPIRE].
- [69] J. Zinn-Justin, *Perturbation series at large orders in quantum mechanics and field theories: application to the problem of resummation*, *Phys. Rept.* **70** (1981) 109 [INSPIRE].
- [70] A.B. Zamolodchikov, *Mass scale in the sine-Gordon model and its reductions*, *Int. J. Mod. Phys. A* **10** (1995) 1125 [INSPIRE].
- [71] D. Dorigoni and P. Glass, *The grin of Cheshire cat resurgence from supersymmetric localization*, *SciPost Phys.* **4** (2018) 012 [arXiv:1711.04802] [INSPIRE].
- [72] D. Dorigoni and P. Glass, *Picard-Lefschetz decomposition and Cheshire cat resurgence in $3D$ $N = 2$ field theories*, *JHEP* **12** (2019) 085 [arXiv:1909.05262] [INSPIRE].
- [73] C. Kozçaz, T. Sulejmanpasic, Y. Tanizaki and M. Ünsal, *Cheshire cat resurgence, self-resurgence and quasi-exact solvable systems*, *Commun. Math. Phys.* **364** (2018) 835 [arXiv:1609.06198] [INSPIRE].
- [74] S. Kamata, T. Misumi, N. Sueishi and M. Ünsal, *Exact WKB analysis for SUSY and quantum deformed potentials: quantum mechanics with Grassmann fields and Wess-Zumino terms*, *Phys. Rev. D* **107** (2023) 045019 [arXiv:2111.05922] [INSPIRE].
- [75] S.G. Naculich and H.J. Schnitzer, *Duality between $SU(N)_k$ and $SU(k)_N$ WZW models*, *Nucl. Phys. B* **347** (1990) 687 [INSPIRE].
- [76] L. Di Pietro, M. Mariño, G. Sberveglieri and M. Serone, *Resurgence and $1/N$ expansion in integrable field theories*, *JHEP* **10** (2021) 166 [arXiv:2108.02647] [INSPIRE].