Published for SISSA by Springer

Received: November 12, 2022
Revised: March 23, 2023
Accepted: April 1, 2023
Published: April 18, 2023

Complete NNLO operator bases in Higgs effective field theory

Hao Sun,[a,b] Ming-Lei Xiao[c,d] and Jiang-Hao Yu[a,b,e,f,g]

[a] *CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, P.R. China*

[b] *School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, P.R. China*

[c] *Department of Physics and Astronomy, Northwestern University, Evanston, IL 60208, U.S.A.*

[d] *High Energy Physics Division, Argonne National Laboratory, Lemont, IL 60439, U.S.A.*

[e] *Center for High Energy Physics, Peking University, Beijing 100871, P.R. China*

[f] *School of Fundamental Physics and Mathematical Sciences, Hangzhou Institute for Advanced Study, UCAS, Hangzhou 310024, P.R. China*

[g] *International Centre for Theoretical Physics Asia-Pacific, Beijing/Hangzhou, P.R. China*

E-mail: sunhao@itp.ac.cn, minglei.xiao@northwestern.edu, jhyu@itp.ac.cn

Abstract: For the first time, we list the complete and independent set of operators at the next-to-next-to-leading order (NNLO) in the Higgs effective field theory (HEFT). The Young tensor technique utilized in this work guarantees the completeness and independence of the on-shell amplitude basis while the Adler zero condition imposes non-linear symmetry on the Nambu-Goldstone bosons that play the central role in the chiral Lagrangian. The spurion fields are incorporated into the gauge structure of operators following the Littlewood-Richardson rule to accommodate custodial symmetry breaking. We construct 11506 (1927574) NNLO operators for one (three) flavor of fermions for the electroweak chiral Lagrangian with the light Higgs, and enumerate 8065(1179181) operators for one (three) flavor of fermions when the right-handed neutrino is absent by Hilbert series technique. Below the electroweak symmetry breaking scale, the dimension-8 standard model effective field theory (SMEFT) operators could be matched to these HEFT operators at both the NLO and NNLO orders.

Keywords: Chiral Lagrangian, Effective Field Theories, SMEFT

ArXiv ePrint: 2210.14939

https://doi.org/10.1007/JHEP04(2023)086

JHEP04(2023)086

Contents

1 Introduction

In the last several decades, the standard model (SM) of particle physics has shown its grand validity. However, the failure of discovering particles beyond the SM at the Large Hadron Collider (LHC) implies that there is a considerable energy gap between the SM particles and new physics. Due to the scale separation, various new physics effects below the energy threshold of new physics particles can be characterized by the effective field theory (EFT) framework. Pioneered by Weinberg [1], EFT has been developed to be a systematic framework to parametrize the underlying physics at a low energy scale. In such a bottom-up approach, the building blocks at low energy are used to construct all the operators satisfying the specific symmetries with proper power counting. Therefore, all the operators can be organized order by order, and in each order, the Lagrangian is the linear combination of all the independent operators, and the coefficient of each operator is called the Wilson coefficient, carrying information from the underlying dynamics at high energy.

Adopting the SM particles as the building blocks, and imposing the gauge symmetry in the SM, the higher dimensional operators can be constructed and organized order by order in terms of the canonical dimension powers. The constructed EFT is the standard model effective field theory (SMEFT). Since the dimension-5 operators were presented by

JHEP04(2023)086

Weinberg [2], many progresses on the operator bases have been made [3–12]. A general algorithm, implemented in a Mathematica package ABC4EFT [13], has been proposed to construct the independent and complete SMEFT operator bases up to any canonical dimension.

Nowadays, the dimension-8 operators of the SMEFT [7, 8] receive more and more attention theoretically and experimentally. Below the electroweak (EW) scale, the SM gauge symmetry is broken down to $\mathrm{SU}(3) \times \mathrm{U}(1)_{\mathrm{em}}$, along with the Higgs doublet broken down to singlet. Thus below the EW scale, the SMEFT is not suitable to describe new physics effects anymore. On the other hand, due to the approximated custodial symmetry in the Higgs sector, the EFT can be characterized by the Coleman-Callan-Wess-Zumino (CCWZ) formalism [14, 15], known as the electroweak chiral Lagrangian with the light Higgs boson (H-EWChL), or the Higgs effective field theory (HEFT), see refs. [16–21] for early developments and refs. [22–30] after the Higgs discovery. The HEFT provides a more general realization of the EW dynamics, which includes the SMEFT as a particular case [31–33]. The HEFT Lagrangian has been constructed up to next-to-leading order (NLO), including the fermion sector [22–30], without considering the flavor structures. Recently the complete and independent NLO operators are presented in ref. [34], and the flavor structures of the operators are considered there.

If one would like to match the dimension-8 SMEFT operators to the HEFT operators after the EW symmetry breaking, the HEFT Lagrangian should be constructed up to the next-to-next-to-leading order (NNLO), to capture various effects only appearing at the dimension-8 SMEFT. For example, to investigate the effective operators contributing to the genuine quartic gauge-boson couplings, the relevant bosonic chiral Lagrangian at $\mathcal{O}(p^6)$ on the quartic gauge couplings has been written in ref. [35] and the connection to the bosonic dimension-8 SMEFT operators is also discussed. In addition, the one-loop renormalization of the LO HEFT operators has been considered [36–39], which the NLO operators complement according to the power-counting rule. Furthermore, the two-loop renormalization of the LO operators, as well as the one-loop renormalization of the diagrams with exactly one NLO interaction vertex need the complete and independent NNLO operators. However, in literature, the NNLO operators counted as the order $d_\chi = 5$ and $d_\chi = 6$ have not yet been constructed systematically.

For the first time, we construct the complete and independent NNLO operators with the flavor structures using the Young tensor technique developed in refs. [7, 9, 11, 13] with certain improvements on the operators involving in the Nambu-Goldstone Boson (NGB) and the spurion field parametrizing the custodial symmetry breaking:

- For the operators involving the NGBs, the Adler zero condition implies that the on-shell amplitudes corresponding to the Lorentz structure of the operators should vanish in the soft limit of the NGB momentum [34, 40–48]. Thus we need to impose this constraint on the Lorentz basis obtained by the Young tensor method to obtain the reduced Lorentz structures, which are usually of a subspace of the original Lorentz space.

- Since the spurions are frozen degrees of freedom, unlike the dynamical fields, the spurion should not enter the Lorentz sector. Instead, it only plays a role in constructing

JHEP04(2023)086

the SU(2) invariant together with other dynamical fields. Furthermore, we should avoid the appearance of self-contracted spurion fields, such as $\delta_{IJ}T^IT^J$, because they are redundant in describing the symmetry-breaking pattern.

With these improvements on the Young tensor technique, we could obtain that there are 11506 (1927574) NNLO operators with one (three) generations of the SM fermions assuming right-handed neutrino exists. In addition, we work out the numbers of the independent operators without the right-handed neutrino via the Hilbert series technique developed in refs. [49, 50], where there are 8065(1179181) operators with one (three) flavor of fermions, without explicitly marking their explicit forms in grey color as we did for the NLO operators in ref. [34].

The paper is organized as follows. In section 2, we briefly review the building blocks and the leading-order Lagrangian of the HEFT and present the chiral power-counting scheme. In section 3, we review the Young tensor method to construct the complete and independent effective operators, focus on the Adler zero condition on the operators involving the NGBs, and present how to deal with the spurion in the Young tensor method in section 3.3. Based the these, we construct the complete NNLO operators of the HEFT, of which the overview and some comments are presented in section 4, while the full operator list is so long that we present them in the supplementary material of this paper. Finally, we draw the conclusion in section 5.

2 Electroweak chiral Lagrangian

The Higgs sector in the SM has a larger global symmetry $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ than the gauge symmetry of the SM Lagrangian, which is spontaneously broken down to the custodial $\mathrm{SU}(2)_C$ symmetry by the Higgs vacuum expectation value (VEV). The coset pattern $\mathcal{G} \to \mathcal{H}$ with identifying $\mathcal{G} = S\mathrm{U}(2)_L \times S\mathrm{U}(2)_R$ and $\mathcal{H} = S\mathrm{U}(2)_V$ can be described by the non-linearized Nambu-Goldstone Boson fields along with the Higgs singlet using the CCWZ formalism [14, 15], adding additional explicit breaking terms from the gauge and Yukawa type interactions. This section will briefly review the construction of the leading order (LO) Lagrangian, and discuss how the NLO and NNLO operators are counted based on the chiral power-counting scheme.

2.1 Building blocks and the LO Lagrangian

The Lagrangian of the Higgs sector in the gaugeless limit $(g, g' \to 0)$ reads

$$\mathcal{L}_{\text{Higgs}} = \partial_\mu H^\dagger \partial^\mu H - \lambda \left(H^\dagger H - \frac{v^2}{2} \right)^2 , \tag{2.1}$$

where $H \equiv (\phi^+, \phi^0)^T$ denotes the $\mathrm{SU}(2)_L$ doublet Higgs and the Lagrangian is invariant under the $\mathrm{SU}(2)_L$ symmetry. In fact, there is another global $\mathrm{SU}(2)_R$ symmetry hidden in this Lagrangian if one rewrites the same Lagrangian by introducing another field $\tilde{H} \equiv (\phi^0, \phi^-)^T$. This enlarged symmetry $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ can be made explicit by re-expressing the Higgs field in terms of a bi-fundamental scalar field Σ that transforms under the global symmetry

$$\Sigma \equiv \begin{pmatrix} \tilde{H} & H \end{pmatrix} = \begin{pmatrix} \phi^{0*} & \phi^+ \\ -\phi^- & \phi^0 \end{pmatrix} \quad \longrightarrow \quad \mathfrak{g}_L \, \Sigma \, \mathfrak{g}_R^\dagger, \qquad (\mathfrak{g}_L, \mathfrak{g}_R) \in \mathcal{G} \, . \tag{2.2}$$

Then the Lagrangian in the Higgs sector becomes

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2} \langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \rangle - \frac{\lambda}{4} \left(\langle \Sigma^\dagger \Sigma \rangle - v^2 \right)^2 , \tag{2.3}$$

where $\langle \ldots \rangle$ represents $\text{SU}(2)_L$ matrix trace. It is more convenient to parametrize the Goldstone fields in terms of the unitary matrix $\mathbf{U}(x) = \exp(\Pi(x)/f)$:

$$\Sigma(x) \equiv \frac{h(x) + v}{\sqrt{2}} \, \mathbf{U} = \frac{h(x) + v}{\sqrt{2}} \exp\left[\Pi(x)/f \right] , \tag{2.4}$$

which separates the NGBs from the Higgs mode. The above Goldstone matrix contains three NGBs,

$$\Pi(x) = \vec{\pi}(x) \cdot \frac{\vec{\sigma}}{2} , \tag{2.5}$$

originated from the global symmetry breaking pattern $\text{SU}(2)_L \times \text{SU}(2)_R / \text{SU}(2)_V$. Thus the Lagrangian in eq. (2.3) takes the form that

$$\mathcal{L}_H = \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{v^2}{2} \langle \partial_\mu \mathbf{U} \partial^\mu \mathbf{U}^\dagger \rangle \, \mathcal{F}(h) - \frac{\lambda}{4} \left(\frac{h^2}{2} + hv - \frac{v^2}{2} \right)^2 , \tag{2.6}$$

where the $\mathcal{F}$ is dimensionless polynomial

$$\mathcal{F}(h) = 1 + 2\frac{h}{v} + \frac{h^2}{v^2} \, . \tag{2.7}$$

Recovering the gauge symmetry would introduce explicit custodial symmetry breaking. In the above Lagrangian, promoting the group $\text{SU}(2)_L$ and the third component of the group $\text{SU}(2)_R$ to be local, the ordinary derivatives would be replaced by the covariant derivatives defined as

$$D_\mu \Sigma = \partial_\mu \Sigma - ig \hat{W}_\mu \Sigma + ig' Y \Sigma \hat{B}_\mu \, , \quad \hat{W}_{\mu\nu} = \vec{W}_{\mu\nu} \cdot \frac{\vec{\sigma}}{2}, \quad \hat{B}_{\mu\nu} = B_{\mu\nu} \frac{\sigma_3}{2} \, , \tag{2.8}$$

where $\vec{W}_{\mu\nu}, B_{\mu\nu}$ are the gauge fields in the SM. The gauge fields $\hat{W}_{\mu\nu}$ transforms as the triplet of $\text{SU}(2)_L$,

$$\hat{W}_{\mu\nu} \to \mathfrak{g}_L \hat{W}_{\mu\nu} \mathfrak{g}_L^\dagger, \quad \mathfrak{g}_L \in \text{SU}(2)_L \, , \tag{2.9}$$

while $\hat{B}_{\mu\nu}$ transforms as $\text{SU}(2)_L$ singlet, and thus the explicit custodial symmetry breaking is parametrized by the spurion field $\mathcal{T}_R = \sigma_3/2$ with $\hat{B}_{\mu\nu} = B_{\mu\nu} \mathcal{T}_R$.

Introducing the SM fermions Yukawa terms would also break the custodial symmetry explicitly. Let us rewrite the SM fermion fields $\psi_{L/R} = P_{L/R} \psi$ that transform covariant under the global symmetry

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad \to \quad \mathfrak{g}_L Q_L\,, \qquad\qquad Q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix} \quad \to \quad \mathfrak{g}_R Q_R\,, \tag{2.10}$$

$$L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \quad \to \quad \mathfrak{g}_L L_L\,, \qquad\qquad L_R = \begin{pmatrix} \nu_R \\ e_R \end{pmatrix} \quad \to \quad \mathfrak{g}_R L_R\,, \tag{2.11}$$

Note that the right-handed fermions are the $\mathrm{SU}(2)_R$ doublets and thus the $\mathrm{U}(1)_Y$ symmetry in the SM is promoted to the $\mathrm{U}(1)_X$ symmetry, where $X = (B-L)/2$ is half of the baryon number B minus the lepton number L. The Yukawa Lagrangian takes a more compact form

$$\mathcal{L}_{\text{Yukawa}} = -v\overline{\psi_L}\mathbf{U}(\Pi)\mathcal{Y}_R^{\psi}\psi_R + h.c. \quad \text{with} \quad \mathcal{Y}_R^{\psi} \quad \longrightarrow \quad \mathfrak{g}_R\mathcal{Y}_R^{\psi}\mathfrak{g}_R^{\dagger}\,, \tag{2.12}$$

where ψ takes Q and L and $\mathcal{Y}_R^{\psi}$ is a 2×2 matrix and takes the form

$$\mathcal{Y}_R^Q = \frac{1}{2}(y_u + y_d) + \frac{\sigma_3}{2}(y_u - y_d), \quad \mathcal{Y}_R^L = \frac{1}{2}(y_\nu + y_e) + \frac{\sigma_3}{2}(y_\nu - y_e)\,, \tag{2.13}$$

where $y_\nu = 0$ if no right-handed neutrinos. Note that the $\frac{\sigma_3}{2}$ term above parametrizes the custodial symmetry breaking in the Yukawa term. Therefore, the spurion in the fermion sector takes the same form as the one in the gauge sector $\mathcal{T}_R = \sigma_3/2$.

The above form of the Lagrangian can also be obtained in the CCWZ formalism [14, 15] of the symmetry breaking pattern $\mathcal{G} \to \mathcal{H}$ with identifying $\mathcal{G} = S\mathrm{U}(2)_L \times S\mathrm{U}(2)_R$ and $\mathcal{H} = S\mathrm{U}(2)_V$, which provides a systematic way to write effective Lagrangian that allows manifesting the symmetries of the theory. The Goldstone matrix $\mathbf{U}$ takes the form $\mathbf{U}(x) = \exp(i\Pi(x)/v)$, which transforms under $\mathcal{G}$ as bi-doublet,

$$\mathbf{U} \to \mathfrak{g}_L\mathbf{U}\mathfrak{g}_R^{\dagger}, \quad (\mathfrak{g}_L, \mathfrak{g}_R) \in \mathcal{G}\,. \tag{2.14}$$

Let us collect all the building blocks that would appear in the chiral Lagrangian

$$h, \quad \mathbf{U}, \quad \psi_L, \quad \psi_R, \quad \hat{W}_{\mu\nu}, \quad \hat{B}_{\mu\nu}, \quad \hat{G}_{\mu\nu}, \quad \mathcal{T}_R. \tag{2.15}$$

which transforms differently under the global chiral symmetry. For the convenience of constructing higher-dimension operators, we can redefine these building blocks with $\mathbf{U}$ to make them transform solely under $\mathrm{SU}(2)_L$ [22–25, 28–30, 51–53],

$$\mathbf{V}_\mu(x) = i\mathbf{U}(x)D_\mu\mathbf{U}(x)^{\dagger}, \quad \longrightarrow \quad \mathfrak{g}_L\mathbf{V}_\mu\mathfrak{g}_L^{\dagger} \tag{2.16}$$

$$\hat{W}_{\mu\nu} \quad \longrightarrow \quad \mathfrak{g}_L\hat{W}_{\mu\nu}\mathfrak{g}_L^{\dagger} \tag{2.17}$$

$$\hat{B}_{\mu\nu} \quad \longrightarrow \quad \hat{B}_{\mu\nu} \tag{2.18}$$

$$\hat{G}_{\mu\nu} \quad \longrightarrow \quad \hat{G}_{\mu\nu} \tag{2.19}$$

$$\mathbf{T} = \mathbf{U}\mathcal{T}_R\mathbf{U}^{\dagger} \quad \longrightarrow \quad \mathfrak{g}_L\mathbf{T}\mathfrak{g}_L^{\dagger} \tag{2.20}$$

$$\psi_L \quad \longrightarrow \quad \mathfrak{g}_L\psi_L \tag{2.21}$$

$$\mathbf{U}\psi_R \quad \longrightarrow \quad \mathfrak{g}_L\mathbf{U}\psi_R \tag{2.22}$$

$$h \quad \longrightarrow \quad h \tag{2.23}$$

We summarise these building blocks and their representations of Lorentz and gauge groups in table 2. There are other redefinitions of the building blocks such as the refs. [26, 27, 54], and different schemes actually give the same operators set.

In terms of the building blocks, the LO Lagrangian takes the form that

$$\begin{aligned}\mathcal{L}_2 = &-\frac{1}{4}\left(G^a_{\mu\nu}G^{a\mu\nu}\right) - \frac{1}{2}\langle W_{\mu\nu}W^{\mu\nu}\rangle - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{g_s^2}{16\pi^2}\theta_s\left(G^a_{\mu\nu}\tilde{G}^{a\mu\nu}\right)\\ &+\frac{1}{2}\partial_\mu h\partial^\mu h - V(h) + \frac{v^2}{4}\langle\mathbf{V}_\mu\mathbf{V}^\mu\rangle\mathcal{F}_C(h) + \frac{v^2}{4}\langle\mathbf{T}\mathbf{V}_\mu\rangle\langle\mathbf{T}\mathbf{V}^\mu\rangle\mathcal{F}_T(h)\\ &+ i\bar{Q}_L\not{D}Q_L + i\bar{Q}_R\not{D}Q_R + i\bar{L}_L\not{D}L_L + i\bar{L}_R\not{D}L_R\\ &-\frac{v}{\sqrt{2}}(\bar{Q}_L\mathbf{U}\mathcal{Y}_R^Q(h)Q_R) + h.c.) - \frac{v}{\sqrt{2}}(\bar{L}_L\mathbf{U}\mathcal{Y}_R^L(h)L_R + h.c.)\,,\end{aligned} \tag{2.24}$$

where subscript 2 indicates the chiral dimension of the leading Lagrangian is 2, which will be discussed in the next subsection. The terms in the first line are the dynamic terms of gauge bosons and the theta term, and the second line contains the dynamic terms of NGBs, physical Higgs h, and its potential. The third line and the fourth line describe the dynamic term and the mass terms of the fermions. The Yukawa coupling matrix $\mathcal{Y}_R^{Q/L}$ takes the form in the eq. (2.13). $\mathcal{F}_C(h)$ and $\mathcal{F}_T(h)$ appearing in the second lines are dimensionless polynomials of Higgs h, which are actually arbitrary and are taken as Taylor-expansion form

$$\mathcal{F}(h)_{C/T} = 1 + b_1\frac{h}{v} + b_2\frac{h^2}{v^2} + O\left(\frac{h^3}{v^3}\right)\,,$$

where b_i are arbitrary dimensionless constants.

2.2 Power counting and higher order Lagrangian

The power-counting of the HEFT is similar to the one in the chiral perturbation theory (ChPT) using the chiral dimension d_χ [1, 55, 56], with certain improvements [22, 26, 27, 30, 52, 57–59]. Setting the LO Lagrangian be of the chiral dimension 2, the chiral dimensions of all the building blocks should be determined as follows.

- The gauge bosons $X_{\mu\nu} = G_{\mu\nu}, W_{\mu\nu}, B_{\mu\nu}$ are of $d_\chi = 1$, and the derivatives D or ∂ are of $d_\chi = 1$.
- The chiral dimension of the gauge coupling constants is 1, thus the dimension of the gauge vector fields is actually zero.
- The chiral dimension of $\mathbf{V}_\mu$ is 1, while the NGBs matrix $\mathbf{U}$ carries no chiral dimension.
- Every fermion doublet is of the chiral dimension 1/2, and the Yukawa coupling constants carry the chiral dimension 1.
- In particular, we make the convention that the spurion $\mathbf{T}$ is of no chiral dimension since this would describe the possible non-decoupling effects at the LO, for example, the triplet Higgs could develop a not-so-small VEV and causes custodial symmetry breaking effects at the LO Lagrangian, while in some literature such as refs. [27, 30, 54], the spurion is taken to be dimensional, in which the custodial symmetry breaking effects are always taking to be small.

This power-counting scheme is also consistent with the loop expansion [22, 27, 30, 52, 58, 59]. Based on the discussion above, a general type of operator in the HEFT can be denoted by

$$\kappa^{k_i}\psi^{F_i}X_{\mu\nu}^{V_i}\mathbf{U}hD^{d_i}\mathbf{T}^{S_i}\,, \tag{2.25}$$

where the number k_i of the gauge or Yukawa couplings κ, F_i the fermion fields ψ, V_i the field-strength tensor $X_{\mu\nu}$, d_i the covariant derivatives D, S_i the spurions, and an arbitrary number of both the NGBs $\mathbf{U}$ and the Higgs boson h. The total chiral dimension of such type of operator reads

$$d_\chi + d_i + k_i + \frac{F_i}{2} + V_i\,(+S_i) = 2L_i + 2 \tag{2.26}$$

where the spurion $\mathbf{T}$ is taken to be dimensionless (dimensional).

The above power counting on the gauge and Yukawa couplings implies that

$$\frac{p^2}{16\pi^2 v^2} \sim \frac{g^2}{(4\pi)^2}, \frac{y^2}{(4\pi)^2}, \frac{\lambda}{(4\pi)^2} \ll 1\,. \tag{2.27}$$

A similar argument applies to the cases where the fermions are weakly coupled. Thus for higher-order operators, every fermion bilinear $(\overline{\psi}\psi)$ and gauge field strength tenser $X_{\mu\nu}\, X = G, W, B$ carries chiral dimension 2 because of the weak coupling constants, for example, both the class $\psi^6 Uh$ and $\psi^4 XUh$ are of chiral dimension 6. In the constructions of higher-dimension operators, we absorb the gauge and fermion coupling constants into the redefinition of the gauge field strength tensor X and the fermions ψ and thus redefine the chiral power of X and ψ as 2 and 1, respectively. Thus, the coupling constants in the higher-dimension operators are always implicit.

We summarise the chiral dimensions of the building blocks in table 2, and the operator classes up to $d_\chi = 6$ are listed in table 1. We identify the operators of $d_\chi = 3, 4$ as the NLO operators and those of $d_\chi = 5, 6$ as the NNLO operators. In particular, the triple-gauge-boson type X^3 is excluded from the NNLO classes in this paper and is considered as NLO [34]. The classes listed in this paper respect the convention that each of them contains a factor Uh since $\mathbf{U}(x)$ is used to redefine the building blocks and $h(x)$ can be used freely in the operator constructions, as explained in section 4. Because the spurion $\mathbf{T}$ is of no chiral dimension, thus we do not write them in the classes explicitly, while it should be understood that each class in table 1 contains all possible sub-classes with different numbers of spurions.

There have been many discussions about the Lagrangian of the HEFT since the last century [16–21, 51, 60]. Recently, the NLO operators have been constructed [22–30, 52–54], but none of them presents the complete and independent operator set, and the full flavor structures have never been considered. In ref. [34], the complete result of NLO operators is presented by the Young tensor method [7, 9, 13], which is also used in this paper. At the NLO, there are 237 (8595) operators for one (three) generation fermions without right-handed neutrinos and 295 (11307) operators for one (three) generation fermions with right-handed neutrinos. In this work, we construct the complete and independent NNLO operators for the first time. As shown in section 4, there are 12 classes in this order, ranging from chiral dimension 5 to 6, and the numbers of

d_χ	fermion sector	boson sector
3	$\psi^2 UhD$	
4	$\psi^2 XUh, \psi^4 Uh, \psi^2 UhD^2$	$X^2 Uh, XUhD^2, UhD^4$
5	$\psi^2 XUhD, \psi^4 UhD, \psi^2 UhD^3$	
6	$\psi^2 X^2 Uh, \psi^4 XUh, \psi^6 Uh, \psi^2 XUhD^2, \psi^4 UhD^2, \psi^2 UhD^4$	$X^3 Uh, X^2 UhD^2, XUhD^4, UhD^6$

Table 1. Operator types of HEFT up to $d_\chi = 6$. The classes listed in this paper respect the convention that each of them contains a factor Uh since $\mathbf{U}(x)$ is used to 'dress' building blocks and $h(x)$ can be used freely in the operator constructions, as explained in section 4. Because the spurion $\mathbf{T}$ is of no chiral dimension, thus we do not write them in the classes explicitly, while it should be understood that each class here contains all possible sub-classes with different numbers of spurions.

operators in each class are listed in the table 3, and the total number of operators is $\frac{1}{9}(3672 + 25547{n_f}^2 + 420{n_f}^3 + 56684{n_f}^4 + 102{n_f}^5 + 17129{n_f}^6)$, corresponding to 11506 (1927574) for one (three) generations of fermions. The complete operators are present in a separate ancillary file.

3 The strategy of the basis construction

The following difficulties are present in the task of enumerating the NNLO operator basis:

- The usual redundancy relations for operators, such as the Equation of Motion (EOM), Integration by Part (IBP), the Covariant Derivative Commutator (CDC), and various operator identities like the Fierz rearrangement and the Cayley-Hamilton relation.
- The non-linear symmetry for the NGB imposes constraints on the operators.
- The operators in the broken phase organized in terms of spurions need special care regarding the group structures.

To tackle the first one, we briefly summarize the Young Tensor technique in section 3.1, which was implemented by the Mathematica package and applied to various EFT's [13]. The non-linear symmetry of the NGB is taken care of by imposing the Adler zero conditions on the corresponding amplitudes, as explained in section 3.2. Finally, in section 3.3 we elaborate on the treatment for the spurions in order to systematically organize the operators in the symmetry-broken phase.

3.1 Review on Young tensor method

An EFT operator should be singlet under both the Lorentz and gauge groups. For a specific field content, the independent Lorentz and gauge structures are of finite dimension, thus span two independent linear spaces respectively, the Lorentz space and the gauge space, in which the independent structures are called the Lorentz basis and the gauge basis. The whole space spanned by the independent operators constructed from this field content is the tensor product of these two spaces.

building blocks	spinor formalism	Lorentz group	$\mathrm{SU}(2)_L$	$\mathrm{SU}(3)_C$	d_χ
L_L	$L_{L\alpha}$	$(\frac{1}{2},0)$	Fundamental	Singlet	1
L_R	${L_R}^{\dot\alpha}$	$(0,\frac{1}{2})$	Fundamental	Singlet	1
Q_L	$Q_{L\alpha}$	$(\frac{1}{2},0)$	Fundamental	Fundamental	1
Q_R	${Q_R}^{\dot\alpha}$	$(0,\frac{1}{2})$	Fundamental	Fundamental	1
W_L	$W_{L\alpha\beta}$	$(1,0)$	Adjoint	Singlet	2
W_R	${W_R}^{\dot\alpha\dot\beta}$	$(0,1)$	Adjoint	Singlet	2
G_L	$G_{L\alpha\beta}$	$(1,0)$	Singlet	Adjoint	2
G_R	${G_R}^{\dot\alpha\dot\beta}$	$(0,1)$	Singlet	Adjoint	2
B_L	$B_{L\alpha\beta}$	$(1,0)$	Singlet	Singlet	2
B_R	${B_R}^{\dot\alpha\dot\beta}$	$(0,1)$	Singlet	Singlet	2
$\mathbf{V}^\mu \sim D^\mu\phi$	$(D\phi)_{\dot\alpha\beta}$	$(\frac{1}{2},\frac{1}{2})$	Adjoint	Singlet	1
D^μ	$D_{\alpha\dot\beta}$	$(\frac{1}{2},\frac{1}{2})$	Singlet	Singlet	1
$\mathbf{T}$	$\mathbf{T}$	$(0,0)$	Adjoint	Singlet	0

Table 2. The building blocks of HEFT, their representation under the Lorentz and gauge groups, and the chiral dimension of them. To satisfy the Adler zero condition, we use $D^\mu\phi$ replacing $\mathbf{V}^\mu$.

In the Young tensor method, the Lorentz basis of operators is related to the corresponding basis of local on-shell amplitudes. The on-shell solutions of the fields ϕ_i are given by the spinor-helicity variables $(\lambda_{i\alpha}, \tilde\lambda_i^{\dot\alpha})$ according to their helicities h_i, and thus we obtain the correspondence

$$(D^{r_i-h_i})\phi_i \sim \lambda_i^{r_i-h_i}\tilde\lambda^{r_i+h_i} \tag{3.1}$$

with free spinor indices. Due to the Lorentz invariance, these indices are contracted by invariant tensors $\epsilon_{\alpha\beta}$ and $\tilde\epsilon_{\dot\alpha\dot\beta}$ under the $\mathrm{SU}(2)_l$ and $\mathrm{SU}(2)_r$ subgroups of the Lorentz symmetry, with the numbers

$$n = \frac{1}{2}\sum_i (r_i - h_i) \equiv \frac{r-h}{2}\,, \quad \tilde n = \frac{1}{2}\sum_i (r_i + h_i) \equiv \frac{r+h}{2}\ . \tag{3.2}$$

where $r = \sum_i r_i$ and $h = \sum_i h_i$ are defined. The contracted spinor variables are denoted as usual

$$\lambda_i^\alpha\lambda_{j\alpha} = \langle ij\rangle, \quad \tilde\lambda_{i\dot\alpha}\tilde\lambda_j^{\dot\alpha} = [ij]\,. \tag{3.3}$$

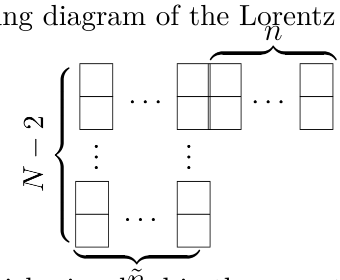
Therefore, any operator could be mapped to a unique (combination of) on-shell amplitude in terms of the spinor variables which can be written as

$$\mathcal{O} = \bigotimes_n \epsilon \bigotimes_{\tilde n} \tilde\epsilon \prod_i (D^{r_i-h_i})\phi_i \quad \sim \quad \mathcal{M} = \langle\cdot\rangle^{\otimes n}[\cdot]^{\otimes\tilde n} \tag{3.4}$$

With the amplitude-operator correspondence, we start to construct the Lorentz basis. For now, we consider the flavor-blind operators. At this stage, all the fields involved in

JHEP04(2023)086

the operators are taken to be different, and the basis obtained here is the so-called Lorentz y-basis, constructed by semi-standard Young tableaux (SSYT). The SSYTs are obtained by filling the primary Young diagram, which is completely determined by field contents. For example, the primary Young diagram of the Lorentz space takes the form [61, 62]


$$, \tag{3.5}$$

where N is the number of particles involved in the operator. The numbers of indices to fill in the primary Young diagram are determined by

$$\#i = \frac{1}{2}n_D + \sum_{h_i>0} |h_i| - 2h_i, \quad i = 1, 2, \ldots N\,, \tag{3.6}$$

where n_D is the number of derivatives. The primary Young diagram corresponds to the singlet representation of the Lorentz group, while the non-singlet representation of the SU(N) group. It has been proved that it is the primary Young diagram that eliminates the IBP redundancies [7, 9, 13], which form the independent y-basis of the amplitudes/operators. The explicit forms of the basis amplitudes $\mathcal{B}_i$ can be directly obtained by translating the Semi-Standard Young Tableau (SSYT) of the primary Young Diagram to the spinor brackets. Besides the simplicity of the construction, the y-basis is also convenient for the decomposition of any given local amplitudes/operators, which we call the reduction to the y-basis

$$\mathcal{M} = \sum_i c_i \mathcal{B}_i. \tag{3.7}$$

It turns out to be crucial in the various manipulation of amplitudes and operators in the related studies.

3.2 Adler zero condition on amplitude basis

Before the general effective Lagrangian of the nonlinearly realized symmetry developed by CCWZ [14, 15], Adler [40, 41] derived that the scattering amplitude with single pion emission vanishes in the soft limit of the pion momentum, which is called the Adler zero condition. With the CCWZ formalism, the pseudoscalar pions are considered as the NGBs after symmetry breaking, and the Adler zero condition is trivially fulfilled in the effective Lagrangian in the CCWZ formalism. But the Young tensor method starts from the general Lorentz structure, instead of the pion matrix field in the CCWZ formalism. Therefore, not all the Lorentz structures in the Young tensor satisfy Adler zero condition. According to the amplitude-operator correspondence, the Lorentz structures satisfying the Adler zero condition corresponds to the amplitudes satisfying soft limit [42–47].

In practice, we adopt the scalar field ϕ which is the adjoint representation under the $SU(2)_L$, and since it is the NGBs, the Adler zero condition implies that there is at least 1 derivative applied on ϕ. This is always possible by using the IBP relation (for amplitude,

JHEP04(2023)086

it is the momentum conservation relation $\sum_i \langle li\rangle[ik] = 0$). Thus in this method, we use $D^\mu\phi$ replacing $\mathbf{V}^\mu$, which can be regarded as the leading term of $\mathbf{V}^\mu$ defined in eq. (2.16). In the following, we will present the procedure of imposing the Adler zero condition which is also shown in refs. [34, 48].

Let us consider a type of operator with N particles, including at least one NGB π. Based on the Young tensor method above, the Lorentz basis can be expressed as the N-point on-shell amplitudes $\{\mathcal{B}_i^{(N)}, i = 1, 2, \dots d_N\}$, where d_N is the dimension of this Lorentz basis. In terms of such basis, any Lorentz structure of this type takes the form that

$$\mathcal{M}^{(N)} = \sum_{i=1}^{d_N} c_i \mathcal{B}_i^{(N)}, \tag{3.8}$$

where c_i are coefficients under this basis. If this amplitude satisfies the Adler zero condition, it vanishes when the external pion momentum p_π becomes soft:

$$\mathcal{M}^{(N)}(p_\pi \to 0) = 0 = \sum_{i=1}^{d_N} c_i \mathcal{B}_i^{(N)}(p_\pi \to 0). \tag{3.9}$$

Here $\mathcal{B}_i^{(N)}(p_\pi \to 0)$ becomes $(N-1)$-point on-shell amplitudes, which are generally not independent and can be expanded by the $(N-1)$-point basis with the soft particle π removed, $\{\mathcal{B}_i^{(N-1)}(\bar{\pi}), i = 1, 2, \dots d_{N-1}\}$, where d_{N-1} is the dimension of such $(N-1)$-point Lorentz basis, and $\bar{\pi}$ implies that the soft particle π is removed,

$$\mathcal{B}_i^{(N)}(p_\pi \to 0) = \sum_{j=1}^{d_{N-1}} f_{ij} \mathcal{B}_j^{(N-1)}(\bar{\pi})\,. \tag{3.10}$$

Furthermore, since the removed π is a scalar particles, all the $(N-1)$-point basis $\mathcal{B}_i^{(N-1)}(\bar{\pi})$ can be expanded by the original N-point basis $\mathcal{B}_i^{(N)}$

$$\mathcal{B}_i^{(N-1)}(\bar{\pi}) = \sum_{l=1}^{d_N} d_{il} \mathcal{B}_l^{(N)}\,. \tag{3.11}$$

Combining several equations above, we can obtain the expansion

$$0 = \sum_{l=1}^{d_N} \left(\sum_{i=1}^{d_N} c_i \mathcal{K}_{il} \right) \mathcal{B}_l^{(N)}\,, \tag{3.12}$$

where $\mathcal{K}_{il}$ is the expansion matrix

$$\mathcal{K}_{il} = \sum_{j=1}^{d_{N-1}} f_{ij} d_{jl}\,. \tag{3.13}$$

Since the basis $\{\mathcal{B}_i^{(N)}, i = 1, 2, \dots d_N\}$ are independent, this equation holds only if all the coefficients vanish,

$$0 = \sum_{i=1}^{d_N} c_i \mathcal{K}_{ij}, \quad (j = 1, 2, \dots d_N)\,. \tag{3.14}$$

JHEP04(2023)086

This is a system of linear equations about c_i, whose solutions span the subspace satisfying the Adler zero condition, which constitute the amplitude basis involving NGBs. If there are more than one NGBs, there is a system of linear equations for each of them, and the structures satisfying the Adler zero condition are their common solutions.

Let us present an explicit example of a five-particle class $F_L\phi^4D^4$, with the helicities that $\{-1,0,0,0,0\}$. We suppose that there are 4 derivatives in this class, and the second to the fourth spinless particles are NGBs. According to the Young tensor method, we can get the complete Lorentz basis of 14 dimensions,

$$\begin{aligned}
\mathcal{B}_1 &= -[45]^2\langle 45\rangle\langle 14\rangle\langle 15\rangle\,, & \mathcal{B}_2 &= -[34][35]\langle 13\rangle^2\langle 45\rangle\,,\\
\mathcal{B}_3 &= [35][45]\langle 45\rangle\langle 13\rangle\langle 14\rangle\,, & \mathcal{B}_4 &= -[34][45]\langle 45\rangle\langle 13\rangle\langle 14\rangle\,,\\
\mathcal{B}_5 &= [34][45]\langle 14\rangle^2\langle 35\rangle\,, & \mathcal{B}_6 &= [45][35]\langle 35\rangle\langle 14\rangle\langle 15\rangle\,,\\
\mathcal{B}_7 &= -[35]^2\langle 35\rangle\langle 13\rangle\langle 15\rangle\,, & \mathcal{B}_8 &= -[34]^2\langle 34\rangle\langle 13\rangle\langle 14\rangle\,,\\
\mathcal{B}_9 &= [34][35]\langle 35\rangle\langle 13\rangle\langle 14\rangle\,, & \mathcal{B}_{10} &= [35][25]\langle 25\rangle\langle 13\rangle\langle 15\rangle\,,\\
\mathcal{B}_{11} &= [34][24]\langle 24\rangle\langle 13\rangle\langle 14\rangle\,, & \mathcal{B}_{12} &= [24][35]\langle 13\rangle\langle 14\rangle\langle 25\rangle\,,\\
\mathcal{B}_{13} &= -[24][45]\langle 14\rangle^2\langle 25\rangle\,, & \mathcal{B}_{14} &= -[45][25]\langle 25\rangle\langle 14\rangle\langle 15\rangle\,,
\end{aligned} \tag{3.15}$$

but not all of them satisfy the Adler zero condition. Since there are 3 NGB particles in this class, their soft limits should be taken separately. For the second particle, we take the momentum $p_2 \to 0$, which is equivalent to the condition $|2\rangle, |2] \to 0$, the amplitude after the limitation

$$\mathcal{B}_1 \to \mathcal{B}_1(p_2 \to 0) = -[34]^2\langle 34\rangle\langle 13\rangle\langle 14\rangle \tag{3.16}$$

should belong to the type $\{-1,0,0,0\}$. Note that the 4-point Lorentz basis of the type $\{-1,0,0,0\}$ contains only 2 Lorentz structures

$$B_1(\bar{2}) = \langle 13\rangle\langle 14\rangle\langle 34\rangle[34]^2\,, \quad B_2(\bar{2}) = -\langle 13\rangle\langle 14\rangle\langle 24\rangle[24][34]\,. \tag{3.17}$$

Given that the two bases can be expanded in the original 5-point basis $d_{1i} = \delta_{i1}, d_{2i} = \delta_{i6}$, where $i = 1, \cdots, 14$ and the soft amplitude $\mathcal{B}_1(p_2 \to 0)$ can be expanded in the two 4-point basis $f_{11} = 1, f_{12} = 0$ the soft amplitude can be expanded on the original 5-point basis as

$$\begin{aligned}
\mathcal{B}_1 \to -[34]^2\langle 34\rangle\langle 13\rangle\langle 14\rangle &= \sum_{i=1}^{14}\sum_{j=1}^{2} f_{1j}d_{ji}\mathcal{B}_i\\
&= \sum_{i=1}^{14} f_{11}d_{1i}\mathcal{B}_i = \sum_{i=1}^{14} 1\times\delta_{i1}\mathcal{B}_i = \mathcal{B}_1\,,
\end{aligned} \tag{3.18}$$

A similar procedure can be applied on the other 5-point basis, and finally, we can get the full matrix $\mathcal{K}$ that

$$\mathcal{K} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.19}$$

and thus the non-trivial equations in eq. (3.14) are

$$\begin{cases} c_1 + c_2 + c_3 + c_4 - c_8 = 0 \\ c_5 + c_6 + c_7 + c_8 + c_9 = 0 \end{cases} \tag{3.20}$$

Similarly, we can get the equations for the third and forth soft particles,

$$\begin{cases} c_1 = 0 \\ c_{13} + c_{14} = 0 \end{cases}, \quad \begin{cases} c_7 = 0 \\ c_{10} = 0 \end{cases}. \tag{3.21}$$

There are total 6 constraints, and the solution space is of dimension 8. Thus there are 8 independent Lorentz basis satisfying the Adler zero condition, we present the transforming matrix that

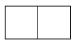
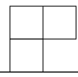
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.22}$$

Therefore, the resulting basis is usually polynomials of the original SSYT basis,

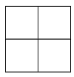
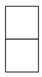
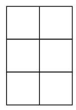
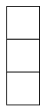
$$
\begin{aligned}
\mathcal{B}'_1 &= [24][45]\langle 14\rangle^2\langle 25\rangle - [45][25]\langle 25\rangle\langle 14\rangle\langle 15\rangle\,,\\
\mathcal{B}'_2 &= [24][35]\langle 13\rangle\langle 14\rangle\langle 25\rangle\,,\\
\mathcal{B}'_3 &= [34][24]\langle 24\rangle\langle 13\rangle\langle 14\rangle\,,\\
\mathcal{B}'_4 &= -[34][45]\langle 14\rangle^2\langle 35\rangle + [34][35]\langle 35\rangle\langle 13\rangle\langle 14\rangle\,,\\
\mathcal{B}'_5 &= -[34][45]\langle 14\rangle^2\langle 35\rangle + [34]^2\langle 34\rangle\langle 13\rangle\langle 14\rangle\,,\\
\mathcal{B}'_6 &= -[34][45]\langle 14\rangle^2\langle 35\rangle + [45][35]\langle 35\rangle\langle 14\rangle\langle 15\rangle\,,\\
\mathcal{B}'_7 &= [34][35]\langle 13\rangle^2\langle 45\rangle - [34][45]\langle 45\rangle\langle 13\rangle\langle 14\rangle\,,\\
\mathcal{B}'_8 &= [34][35]\langle 13\rangle^2\langle 45\rangle + [35][45]\langle 45\rangle\langle 13\rangle\langle 14\rangle\,.
\end{aligned} \tag{3.23}
$$

3.3 Spurion technique for the gauge structure

It is known in the group theory that every SU(N) irreducible representation (irrep) corresponds to a Young diagram. For example, the typical irreps of the SU(2) and SU(3) groups are

SU(2)	SU(3)
$t_i \in \mathbf{2} \sim$ 	$t_a \in \mathbf{3} \sim$ 
$\epsilon_{ij}t^j \in \overline{\mathbf{2}} \sim$ 	$\epsilon_{abc}t^c \in \overline{\mathbf{3}} \sim$ 
$t^I\tau_i^{Ik}\epsilon_{kj} \in \mathbf{3} \sim$ 	$t^A\lambda^{Ad}{}_a\epsilon_{dbc} \in \mathbf{8} \sim$ 

where τ, λ are the generators of the SU(2), SU(3) respectively. In the EFTs with unbroken symmetries, like the SMEFT, the effective operators belong to the singlets under the gauge groups SU(N), represented by Young diagrams with only N-row columns,

SU(2) $\sim$  ... , SU(3) $\sim$  ... . (3.24)

In the Young tensor method, we construct the independent basis of gauge structures by decomposing the tensor product of the Young diagrams of the dynamical fields following the Littlewood-Richardson rule (LR rule) [7, 9, 13], and select all the singlet terms therein.

However, in the HEFT, we are dealing with effective operators in the broken phase of custodial symmetry. The breaking pattern is characterized by a spurion field $\mathbf{T}$ under the adjoint representation of the SU(2)$_L$ group, which enters the gauge and Yukawa interactions. Typically the spurion fields are introduced in theories with broken symmetries to parametrize the symmetry-breaking effects. In the unbroken phase, the spurion is a dynamical degree of freedom, which makes sure the operators in the unbroken phase are singlet under the correspondent symmetry. In the broken phase, the spurion degree of freedom is frozen, and thus the operators in the broken phase become non-singlet under the same symmetry. Spurions are thus the auxiliary fields that contract with the operators to make a singlet, which is supposed to take VEV after the symmetry breaking, thus are

JHEP04(2023)086

frozen degrees of freedom, and do not participate in the Lorentz structures, which being said, we should not treat them as scalar building blocks in the Lorentz sector because the derivatives can not act on them. In practice, for a type of operator involving the spurions, they should be deleted from the helicity list, and their generators of symmetric group are taken to be the identity matrix.

Based on the above, the spurions are only used in constructing the group factor of the corresponding symmetry. Multiple copies of spurions may be used to construct operators under different non-singlet representations, and in turn, there may be different combinations of spurions that constitute the same representation. For example, suppose we have a spurion $\mathbf{T}^a$ under the adjoint representation $\mathbf{8}$ of SU(3), hence we have

$$\mathbf{T}^a \in \mathbf{8}\,, \quad d^{abc}\mathbf{T}^b\mathbf{T}^c \in \mathbf{8}\,, \tag{3.25}$$

where d^{abc} is the total symmetric tensor. Both the 2 combinations above are capable of contracting with an operator $\mathcal{O}^a$ under the adjoint representation of SU(3). It is obvious that the first would be the dominant one, and the second is sub-leading since it takes more spurions. Thus we do not count the second one for the fixing non-singlet operator $\mathcal{O}^a$ in the broken phase, which is equivalent to the eliminations of the gauge structures like $d^{abc}\mathbf{T}^b\mathbf{T}^c$ in the type with 2 spurions. Another restriction is the identity among the same kind of spurions, which demands totally symmetric combinations, thus $f^{abc}\mathbf{T}^b\mathbf{T}^c$, where f^{abc} is the totally anti-symmetric tensor, is zero. Furthermore, for a general type with j spurions, the only reserved structures are those of the traceless and totally symmetric combination,

$$\mathbf{T}^{\{I_1}\cdots\mathbf{T}^{I_j\}} \in \text{spin } j\,. \tag{3.26}$$

Thus we need to consider the representation of spurions and other fields separately, and both the Young diagrams of them are non-singlet, but their outer product can form singlet Young diagram. In particular, the SSYTs of the spurions representation should be symmetric under the permutations among the indices from different spurions. In general, this is difficult, but in the case of the SU(2) group, it is quite straightforward to deal with, as shown in below.

The spurion $\mathbf{T}^I$ in the HEFT belongs to the adjoint representation of the SU(2) group,

$$\mathbf{T}^I\tau^{Ik}{}_i\epsilon_{kj} \in \begin{array}{|c|c|}\hline i & j \\ \hline \end{array}\,, \tag{3.27}$$

where the two indices are symmetric. The tensor product of the 2 spurions can be expressed by the Young diagrams

$$\begin{array}{|c|c|}\hline & \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline & \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|}\hline & & & \\ \hline \end{array}\,, \tag{3.28}$$

and only the last one is traceless and totally symmetric,

$$\begin{array}{|c|c|c|c|}\hline i & j & m & n \\ \hline \end{array} = \mathbf{T}^{\{I}\mathbf{T}^{J\}}\tau^{Il}{}_i\tau^{Jk}{}_m\epsilon_{lj}\epsilon_{kn}\,, \tag{3.29}$$

since the first two diagrams contain columns of length 2, which correspond to anti-symmetrical tensor ϵ contracting with the indices of spurions to generate traces, for example, the tensor in the first diagram

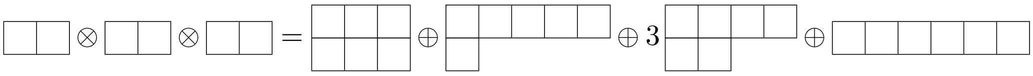
$$\begin{array}{|c|c|c|}\hline i & j & m \\ \hline n \\ \cline{1-1} \end{array} = \mathbf{T}^I\mathbf{T}^J\tau^{Ik}{}_i\tau^{Jl}{}_m\epsilon_{kj}\epsilon_{ln}\epsilon^{in} = \mathbf{T}^I\mathbf{T}^J\epsilon^{IJK}\tau^{Kk}{}_m\epsilon_{kj}\,, \tag{3.30}$$

JHEP04(2023)086

which is actually zero since there is only a single spurion field. The tensor of the second diagram takes the form

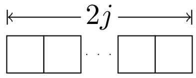
$$\begin{array}{|c|c|}\hline i & j \\ \hline m & n \\ \hline \end{array} = \mathbf{T}^I \mathbf{T}^J \tau^{I}{}^{k}_{i} \tau^{J}{}^{l}_{m} \epsilon_{kj} \epsilon_{ln} \epsilon^{im} \epsilon^{jn} = \mathbf{T}^I \mathbf{T}^I \,, \tag{3.31}$$

which is the self-contraction of spurions, and should be eliminated as well. Furthermore, the tensor product of the 3 spurions takes the form

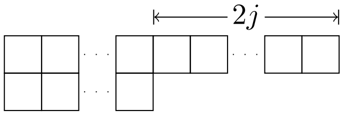


(3.32)

and only the last one is traceless and totally symmetric. Generally, the traceless and totally symmetric combinations of j spurions make a spin-j representation, which corresponds the irreducible representation with the highest weight in the direct-product decomposition of them, corresponding the diagram that



and the compensation formed by other dynamical fields of this representation to form the SU(2) singlet takes the form



which can be obtained by the application of the Littlewood-Richardson rule reversely. To eliminate the redundancies about the spurions mentioned above, we need to do the outer product of the j copies of the spurions and the other dynamical fields separately to form two SSYTs of the shape above, then combine them back to form the singlet Young diagrams by simply binding the two SSYTs together.

3.4 Flavor structure involving Goldstone and spurion

For both the Lorentz and gauge space, the y-basis is often polynomials and we can transform them to another basis in which all operators are monomials, the m-basis. The tensor product of the Lorentz and gauge m-basis constitutes the full space of all the independent flavor-blind operators, which is called the operator m-basis.

Considering the repeated fields in the operators, the operators in the m-basis obtained above are usually redundant. In this case, a field with flavor number n_f can be regarded as a n_f multiple of the flavor group SU(n_f). If an operator has n such fields, this operator behaves as an n-rank tensor under the group SU(n_f). This flavor tensor-product space is fully divided into several disjoint subspaces, each of which furnishes an irreducible representation of the symmetric group S_n. The operators in every irreducible representation of the S_n have specific permutation symmetries. Thus we introduce the p-basis composed of these operators. In the p-basis, not all the operators are physical, for example, if the repeated field has flavor number 1, all the operators in the p-basis are zero, except for those in the completely symmetric representation of S_n. Besides, if the p-basis contains operators with the mixed flavor symmetry such as $\yng(2,1)$, the irreducible subspace of $\mathrm{SU}(n_f)$ marked by this Young diagram has multiplicity equal to the dimension of the irreducible representation of the symmetry group S_n presented by the same Young diagram. It can be proved that these irreducible subspaces are isomorphic to each other [13], and only one of them needs to be reserved. After eliminating such redundancies, the remaining operators form the so-called f-basis or p'-basis, which serves as the final result. In practice, the p-basis can be obtained by applying the idempotent elements of group algebra $\tilde{S}_n$ on the m-basis.

$$\mathcal{O}^p_{\mathrm{rep}} = \mathcal{Y}_{\mathrm{rep}} \mathcal{O}^m \,, \tag{3.33}$$

where $\mathcal{Y}_{\mathrm{rep}}$ is the idempotent element of S_n's irreducible representation rep, which is symbolized by Young diagrams.

Let us illustrate the procedure above by the type $Q_L Q_R^\dagger B_R \phi^2 D^2$ at the NNLO. There are 5 fields in this type, and their helicities are $\{-1/2, -1/2, 1, 0, 0\}$, and the last two NGBs, marked by indices 4 and 5, are repeated fields. There are 2 derivatives, thus $n_D = 2$. According to table 2, we obtain that $h = 0, r = 4$, and the numbers of $\epsilon, \tilde{\epsilon}$ are $n = \tilde{n} = 2$. Thus the primary Young diagram takes the form that

$$\yng(4,4,2) \,. \tag{3.34}$$

To fill this diagram and obtain SSYTs, the numbers of all indices are needed, which can be obtained by eq. (3.6)

$$\#1 = 3, \quad \#2 = 3, \quad \#3 = 0, \quad \#4 = 2, \quad \#5 = 2 \,, \tag{3.35}$$

thus there are only 4 SSYTs,

$$\young(1114,2225,45)\,, \quad \young(1112,2255,44)\,, \quad \young(1112,2244,55)\,, \quad \young(1112,2245,45)\,, \tag{3.36}$$

and they correspond to the Lorentz y-basis

$$\begin{aligned}
\mathcal{B}^y_1 &= -\langle 12\rangle\langle 45\rangle[34][35]\,, \\
\mathcal{B}^y_2 &= \langle 15\rangle\langle 25\rangle[35]^2\,, \\
\mathcal{B}^y_3 &= \langle 14\rangle\langle 24\rangle[34]^2\,, \\
\mathcal{B}^y_4 &= -\langle 14\rangle\langle 25\rangle[34][3,5]\,.
\end{aligned} \tag{3.37}$$

The Adler zero condition constraints that the amplitudes should be zero whenever particle 4 or 5 becomes soft. If particle 4 becomes soft, all the four bases become zero except for the second one $\mathcal{B}^y_2$, thus the matrix $\mathcal{K}$ in eq. (3.13) takes the form that

$$\mathcal{K} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{3.38}$$

thus system of the linear equations in eq. (3.14) gives the solution that $c_2 = 0$. Similarly, if particle 5 becomes soft, only the third basis $\mathcal{B}_3^y$ is not zero, thus there is the solution $c_3 = 0$. Thus only the first and the last bases satisfy the Adler zero condition, thus the actual Lorentz y-basis is

$$\mathcal{B}_1^{ay} = \mathcal{B}_1^y, \quad \mathcal{B}_2^{ay} = \mathcal{B}_4^y \,, \tag{3.39}$$

with the projection matrix that

$$\mathcal{K}_y^{ay} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.40}$$

According to table 2, we translate these amplitudes to the form of operators

$$\begin{aligned} \mathcal{B}_1^{ay} &= {\psi_1}^{\alpha} \psi_{2\alpha} F_{R3\,\dot\beta\dot\gamma} (D^{\delta\dot\beta}\phi_4)(D_{\delta}^{\dot\gamma}\phi_5) \\ &= -4(\psi_1\psi_2) {F_{R3}}^{\mu\nu} (D_\mu\phi_4)(D_\nu\phi_5) \,, \end{aligned} \tag{3.41}$$

$$\begin{aligned} \mathcal{B}_2^{ay} &= {\psi_1}^{\alpha} {\psi_2}^{\beta} R_{R3\,\dot\gamma\dot\delta} (D_{\alpha}^{\dot\gamma}\phi_4)(D_{\beta}^{\dot\delta}\phi_5) \\ &= -2(\psi_1\psi_2) {F_{R3}}^{\mu\nu} (D_\mu\phi_4)(D_\nu\phi_5) + 2i(\psi_1\sigma_{\mu\nu}\psi_2) {F_{R3}}^{\nu\lambda} (D_\lambda\phi_4)(D^\mu\phi_5) \,, \end{aligned} \tag{3.42}$$

from which we choose the monomials

$$\mathcal{B}_1^m = (\psi_1\psi_2) {F_{R3}}^{\mu\nu} (D_\mu\phi_4)(D_\nu\phi_5) \,, \tag{3.43}$$

$$\mathcal{B}_2^m = (\psi_1\sigma_{\mu\nu}\psi_2) {F_{R3}}^{\nu\lambda} (D_\lambda\phi_4)(D^\mu\phi_5) \,, \tag{3.44}$$

as the Lorentz m-basis, with the transformation matrix

$$\mathcal{K}_{ay}^m = \begin{pmatrix} -\frac{1}{4} & 0 \\ \frac{i}{4} & -\frac{i}{2} \end{pmatrix} . \tag{3.45}$$

As for the gauge y-basis, the $SU(3)_C$ structure is simple, there is only one independent tensor δ_a^b, and the $SU(2)_L$ tensors can be obtained by similar SSYT technics. There are 2 gauge SSYTs for this type

$$\begin{array}{|c|c|c|} \hline i_1 & i_2 & i_4 \\ \hline j_4 & i_5 & j_5 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|} \hline i_1 & i_4 & j_4 \\ \hline i_2 & i_5 & j_5 \\ \hline \end{array}\,. \tag{3.46}$$

They correspond to the independent tensors

$$\mathcal{T}^{(y)}_{SU(2)_L,1} = \epsilon^{i_1 j_4} \epsilon^{i_2 i_5} \epsilon^{i_4 j_5} (\tau^{I_4})_{i_4 i_4} (\tau^{I_5})_{i_5 j_5} \epsilon_{i_2 j_2} \,, \tag{3.47}$$

$$\mathcal{T}^{(y)}_{SU(2)_L,2} = \epsilon^{i_1 i_2} \epsilon^{i_4 i_5} \epsilon^{j_4 j_5} (\tau^{I_4})_{i_4 j_4} (\tau^{I_5})_{i_5 j_5} \epsilon_{i_2 j_2} \,, \tag{3.48}$$

which can be further simplified to

$$\begin{aligned} \mathcal{T}^{(y)}_{SU(2)_L,1} &= \delta^{I_4 I_5} \delta^{i_1}_{j_2} - i\epsilon^{I_4 I_5 J} (\tau^J)^{i_1}_{j_2} \,, \\ \mathcal{T}^{(y)}_{SU(2)_L,2} &= 2\delta^{I_4 I_5} \delta^{i_1}_{j_2} \,. \end{aligned} \tag{3.49}$$

Similarly, we combine them to obtain the gauge m-basis

$$\mathcal{T}^{(m)}_{SU(2)_L,1} = \epsilon^{I_4 I_5 J} (\tau^J)^{i_1}_{j_2}, \quad \mathcal{T}^{(m)}_{SU(2)_L,2} = \delta^{I_4 I_5} \delta^{i_1}_{j_2} \,, \tag{3.50}$$

with the transformation matrix

$$\mathcal{K}^{my}_{\mathrm{SU}(2)_L} = \begin{pmatrix} i & -\frac{i}{2} \\ 0 & \frac{1}{2} \end{pmatrix} . \tag{3.51}$$

The tensor product of the gauge m-basis and the Lorentz m-basis gives $1 \times 2 \times 2 = 4$ operators, but they are not all physical when considering the repeated fields.

The NGB ϕ is the repeated field in this type, which carries no flavor number, thus only the operators symmetric under the permutations of them are physical. In the Lorentz space, the generator of S_2 in the y-basis[1] $\{\mathcal{B}^y_i, i = 1,2,3,4\}$ takes the form

$$\mathcal{D}^{(y)}_{\mathcal{B}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{3.52}$$

which is obtained directly from manipulations of the amplitudes in the Lorentz y-basis. When focusing on the subspace spanned $\{\mathcal{B}^m_i, i = 1,2\}$, the generator can be obtained by the linear transformation

$$\mathcal{D}^{(m)}_{\mathcal{B}} = \mathcal{K}^m_{ay} \mathcal{K}^{ay}_y \mathcal{D}^{(y)}_{\mathcal{B}} \mathcal{K}^{ay\,-1}_y \mathcal{K}^{m\,-1}_{ay} = \begin{pmatrix} -1 & 0 \\ i & -2i \end{pmatrix} , \tag{3.53}$$

where the matrix $\mathcal{K}^{ay}_y$ in the expression should be understood as

$$\mathcal{K}^{ay}_y \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{3.54}$$

and the 2×2 generator obtained above is the 4×4 one with the removal of all the null rows and columns. In the case of the $\mathrm{SU}(2)_L$ gauge basis, it is straightforward to obtain the generator of S_2 in the m-basis is

$$\mathcal{D}^{(m)}_{\mathrm{SU}(2)_L} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{3.55}$$

and $\mathcal{D}^{(m)}_{\mathrm{SU}(3)_C} = 1$. Thus the generator of the overall operator space is the tensor product of them,

$$\mathcal{D}^{(m)} = \mathcal{D}^{(m)}_{\mathrm{SU}(3)_C} \otimes \mathcal{D}^{(m)}_{\mathrm{SU}(2)_L} \otimes \mathcal{D}^{(m)}_{\mathcal{B}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2i & 0 \end{pmatrix} . \tag{3.56}$$

[1]The 2 generators of S_2 are identical, thus we treat them as one here.

The idempotent element of the subspace is symmetric under the permutations of the 2 ϕ's, which is symbolized by the young diagram $\square\square$

$$\mathcal{Y}[\boxed{4}\boxed{5}] = I_{4\times 4} + \mathcal{D}^{(m)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2i & 1 \end{pmatrix}, \tag{3.57}$$

whose rank is 2, thus there are only 2 independent operators in this type, and we write them as following

$$\begin{aligned} \mathcal{O}_1 &= \mathcal{Y}[\boxed{4}\boxed{5}]\epsilon^{IJK}\tau^{Ki}{}_j(Q_{Lai}\sigma_{\mu\nu}Q^{\dagger}_R{}^{aj})B_R{}^{\nu\lambda}(D_\lambda\phi^I)(D^\mu\phi^J)\,, \\ \mathcal{O}_2 &= \mathcal{Y}[\boxed{4}\boxed{5}](Q_{Lai}\sigma_{\mu\nu}Q^{\dagger}_R{}^{ai})B_R{}^{\nu\lambda}(D_\lambda\phi^I)(D^\mu\phi^I)\,. \end{aligned} \tag{3.58}$$

In the rest of this work, we will always write operators with the idempotent elements to indicate their flavor permutation symmetry, but for the operators with no repeated fields and/or the repeated fields carrying flavor number 1, the idempotent elements will be omitted.

Another non-trivial example is the type $B_L\phi^3hD^4$ in the bosonic sector, with the helicity structure $\{-1,0,0,0,0\}$, where the particles $2\sim 4$ are the NGBs, the repeated fields, and the last one is the physical Higgs h. The $\mathrm{SU}(3)_C$ gauge structure is trivial, and the $\mathrm{SU}(2)_L$ gauge space is of dimension 1, $\epsilon^{I_2I_3I_4}$. The Lorentz basis of this type is complicated since there are 4 derivatives. The filling of the SSYTs gives that the Lorentz space is of dimension 14, but the Adler zero condition constrains this space to the one of dimension 8, which has been discussed previously, and the 8 bases are presented in eq. (3.23). Combining the Lorentz and gauge structures together, we can obtain the f-basis is of dimension 1, and thus there is only 1 operator in this type

$$\epsilon^{IJK}B_L{}^{\lambda\nu}(D^\mu h)(D_\mu\phi^I)(D_\nu\phi^J)(D_\lambda\phi^K)\,. \tag{3.59}$$

To be consistent with the building blocks of the HEFT, we replace the fields such as $(D_\mu\phi^I)$ in the operators obtained by the Young tensor method by the $\mathbf{V}^I_\mu$. Thus the only operator in this type is written as

$$\epsilon^{IJK}B_L{}^{\lambda\nu}(D^\mu h)(\mathbf{V}^I_\mu)(\mathbf{V}^J_\nu)(\mathbf{V}^K_\lambda)\,. \tag{3.60}$$

This convention is respected in section 4.

Next, let us provide some examples involving the spurions. Let us consider the type $W_LL_LQ_LL^\dagger_LQ^\dagger_L\mathbf{T}^2$ from the class ψ^4XUh. There are 2 spurions in this type because they do not participate in the Lorentz structures, only the particles $\{W_L, L_L, Q_L, L^\dagger_L, Q^\dagger_L\}$, corresponding to helicities $\{-1,-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$ are considered in the Lorentz basis. The Lorentz basis of this type is simple, and there is only one y-basis

$$\mathcal{B}^{(y)}_1 = \langle 12\rangle\langle 13\rangle[45] \tag{3.61}$$

corresponding to the m-basis

$$W_{L\mu\nu}(L_L\sigma^\nu L^\dagger_L)(Q_L\sigma^\mu Q^\dagger_L) \tag{3.62}$$

with the trivial transformation matrix.

JHEP04(2023)086

For the gauge basis, the $SU(3)_C$ gauge basis is just δ^a_b, contracting with the quarks $Q_L\,,Q_L^\dagger$, but the $SU(2)_L$ structures are complicated. There are 13 independent tensors in the $SU(2)_L$ gauge m-basis, but most of them are redundant. To eliminate those, we consider the spurions and the other fields separately. The 2 spurions take the representation that

$$\begin{ytableau} i_6 & j_6 & i_7 & j_7 \end{ytableau} \tag{3.63}$$

according to the discussion around eq. (3.26), where all the indices are symmetric. Its compensation takes the form that

$$\ydiagram{5,1}\,, \tag{3.64}$$

thus we need to construct all the Young diagrams of such shape by the outer product from the dynamical fields except the spurions, whose representations are related to the SU(2) irreps

$$\begin{aligned} W_L^{I_1}\tau^{I_1 k}{}_{i_1}\epsilon_{kj_1} &\sim \begin{ytableau} i_1 & j_1 \end{ytableau} \\ L_{Li_2} &\sim \begin{ytableau} i_2 \end{ytableau} \\ Q_{Li_3} &\sim \begin{ytableau} i_3 \end{ytableau} \\ L_L^{\dagger\,l}\epsilon_{li_4} &\sim \begin{ytableau} i_4 \end{ytableau} \\ Q_L^{\dagger\,m}\epsilon_{mi_5} &\sim \begin{ytableau} i_5 \end{ytableau}\,, \end{aligned} \tag{3.65}$$

and the outer product of them can be determined by the LR rule,

$$\begin{aligned} &\begin{ytableau} i_1 & j_1 \end{ytableau} \otimes \begin{ytableau} i_2 \end{ytableau} \otimes \begin{ytableau} i_3 \end{ytableau} \otimes \begin{ytableau} i_4 \end{ytableau} \otimes \begin{ytableau} i_5 \end{ytableau} \\ &= \left(\begin{ytableau} i_1 & j_1 & i_2 \end{ytableau} \oplus \begin{ytableau} i_1 & j_1 \\ j_2 \end{ytableau} \right) \otimes \begin{ytableau} i_3 \end{ytableau} \otimes \begin{ytableau} i_4 \end{ytableau} \otimes \begin{ytableau} i_5 \end{ytableau} \\ &= \left(\begin{ytableau} i_1 & j_1 & i_2 & i_3 \end{ytableau} \oplus \begin{ytableau} i_1 & j_1 & i_2 \\ i_3 \end{ytableau} \oplus \begin{ytableau} i_1 & j_1 & i_3 \\ j_2 \end{ytableau} \right) \otimes \begin{ytableau} i_4 \end{ytableau} \otimes \begin{ytableau} i_5 \end{ytableau} \\ &= \left(\begin{ytableau} i_1 & j_1 & i_2 & i_3 & i_4 \end{ytableau} \oplus \begin{ytableau} i_1 & j_1 & i_2 & i_3 \\ i_4 \end{ytableau} \oplus \begin{ytableau} i_1 & j_1 & i_2 & i_4 \\ i_3 \end{ytableau} \oplus \begin{ytableau} i_1 & j_1 & i_3 & i_4 \\ j_2 \end{ytableau} \right) \otimes \begin{ytableau} i_5 \end{ytableau} \\ &= \begin{ytableau} i_1 & j_1 & i_2 & i_3 & i_4 \\ i_5 \end{ytableau} \oplus \begin{ytableau} i_1 & j_1 & i_2 & i_3 & i_5 \\ i_4 \end{ytableau} \oplus \begin{ytableau} i_1 & j_1 & i_2 & i_4 & i_5 \\ i_3 \end{ytableau} \oplus \begin{ytableau} i_1 & j_1 & i_3 & i_4 & i_5 \\ j_2 \end{ytableau}\,, \end{aligned} \tag{3.66}$$

where only the Young diagrams as the same as the one in eq. (3.64) is reserved. Combining the diagrams (3.63) and (3.66) together, we obtain the gauge singlet representations with all the redundancies of the spurions eliminated:

$$\begin{aligned} \mathcal{B}^{(y)}_{SU(2)_L,1} &= \begin{ytableau} i_1 & j_1 & i_2 & i_3 & i_4 \\ i_5 & i_6 & j_6 & i_7 & j_7 \end{ytableau} = \tau^{I_1 i_6}{}_{i_1}\epsilon^{i_1 i_5}(\mathbf{T}^{I_6}\tau^{I_6 i_2}{}_{i_6})(\mathbf{T}^{I_7}\tau^{I_7 i_4}{}_{i_7}\epsilon^{i_3 i_7})\,, \\ \mathcal{B}^{(y)}_{SU(2)_L,2} &= \begin{ytableau} i_1 & j_1 & i_2 & i_3 & i_5 \\ i_4 & i_6 & j_6 & i_7 & j_7 \end{ytableau} = \tau^{I_1 i_6}{}_{i_1}\epsilon^{i_1 i_4}(\mathbf{T}^{I_6}\tau^{I_6 i_2}{}_{i_6})(\mathbf{T}^{I_7}\tau^{I_7 i_5}{}_{i_7}\epsilon^{i_3 i_7})\,, \\ \mathcal{B}^{(y)}_{SU(2)_L,3} &= \begin{ytableau} i_1 & j_1 & i_2 & i_4 & i_5 \\ i_3 & i_6 & j_6 & i_7 & j_7 \end{ytableau} = \tau^{I_1 i_6}{}_{i_1}\epsilon^{i_1 i_3}(\mathbf{T}^{I_6}\tau^{I_6 i_2}{}_{i_6})(\mathbf{T}^{I_7}\tau^{I_7 i_5}{}_{i_7}\epsilon^{i_4 i_7})\,, \\ \mathcal{B}^{(y)}_{SU(2)_L,4} &= \begin{ytableau} i_1 & j_1 & i_3 & i_4 & i_5 \\ i_2 & i_6 & j_6 & i_7 & j_7 \end{ytableau} = \tau^{I_1 i_6}{}_{i_1}\epsilon^{i_1 i_2}(\mathbf{T}^{I_6}\tau^{I_6 i_3}{}_{i_6})(\mathbf{T}^{I_7}\tau^{I_7 i_5}{}_{i_7}\epsilon^{i_4 i_7})\,. \end{aligned} \tag{3.67}$$

JHEP04(2023)086

It should be emphasized that each tensor above can be further simplified, for example, the first tensor takes the form

$$
\begin{aligned}
\mathcal{B}^{(y)}_{\mathrm{SU}(2)_L,1} &= \mathbf{T}^{I_6}\mathbf{T}^{I_7}\tau^{I_1 i_6}{}_{i_1}\tau^{I_6 i_2}{}_{i_6}\tau^{I_7 i_4}{}_{i_7}\epsilon^{i_1 i_5}\epsilon^{i_3 i_7} \\
&= \mathbf{T}^{I_6}\mathbf{T}^{I_7}\tau^{I_1 i_6}{}_{i_1}\tau^{I_6 i_2}{}_{i_6}\tau^{I_7 i_4}{}_{i_7}(\delta^{i_1 i_3}\delta^{i_5 i_7} - \delta^{i_1 i_7}\delta^{i_5 i_3}) \\
&= \mathbf{T}^{I_6}\mathbf{T}^{I_7}\tau^{I_1 i_6}{}_{i_3}\tau^{I_6 i_2}{}_{i_6}\tau^{I_7 i_4}{}_{i_5} - \mathbf{T}^{I_6}\mathbf{T}^{I_7}\tau^{I_1 i_6}{}_{i_7}\tau^{I_6 i_2}{}_{i_6}\tau^{I_7 i_4}{}_{i_7}\delta^{i_3}_{i_5} \\
&= \mathbf{T}^{I_6}\mathbf{T}^{I_7}\epsilon^{I_6 I_1 K}\tau^{K i_2}{}_{i_3}\tau^{I_7 i_4}{}_{i_5} - \frac{1}{2}\mathbf{T}^{I_6}\mathbf{T}^{I_6}\tau^{I_1 i_2}{}_{i_4}\delta^{i_3}_{i_5} + \frac{1}{2}\mathbf{T}^{I_1}\mathbf{T}^{I_6}\tau^{I_6 i_2}{}_{i_4}\delta^{i_3}_{i_5}\,,
\end{aligned}
\tag{3.68}
$$

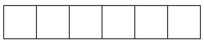
thus all the $\mathrm{SU}(2)_L$ structures are usually polynomials, consistent with the method in the ref. [34], which use the gauge j-basis techniques instead.

Thus there are 4 operators in this type, and we choose them as

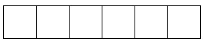
$$
\begin{aligned}
&i\tau^{K i}{}_{l}\tau^{M j}{}_{k}\epsilon^{IJM}\mathbf{T}^{J}\mathbf{T}^{K}W_L{}^{I}{}_{\mu\nu}(L_{Lp_i}\sigma^{\nu}L^{\dagger}_{Ls}{}^{k})(Q_{Lraj}\sigma^{\mu}Q^{\dagger}_{Lt}{}^{al})\,, \\
&i\tau^{J j}{}_{l}\mathbf{T}^{I}\mathbf{T}^{J}W_L{}^{I}{}_{\mu\nu}(L_{Lp_i}\sigma^{\nu}L^{\dagger}_{Ls}{}^{i})(Q_{Lraj}\sigma^{\mu}Q^{\dagger}_{Lt}{}^{al})\,, \\
&i\tau^{J i}{}_{k}\mathbf{T}^{I}\mathbf{T}^{J}W_L{}^{I}{}_{\mu\nu}(L_{Lp_i}\sigma^{\nu}L^{\dagger}_{Ls}{}^{k})(Q_{Lraj}\sigma^{\mu}Q^{\dagger}_{Lt}{}^{aj})\,, \\
&i\tau^{J i}{}_{l}\mathbf{T}^{I}\mathbf{T}^{J}W_L{}^{I}{}_{\mu\nu}(L_{Lp_i}\sigma^{\nu}L^{\dagger}_{Ls}{}^{j})(Q_{Lraj}\sigma^{\mu}Q^{\dagger}_{Lt}{}^{al})\,,
\end{aligned}
\tag{3.69}
$$

where the idempotent elements are omitted since the repeated fields $\mathbf{T}$ have the flavor number 1 and are symmetric under permutations.

Furthermore, if we consider the type with one more spurion, $W_L L_L Q_L L_L^{\dagger} Q_L^{\dagger}\mathbf{T}^3$, the Lorentz basis is unchanged while the $\mathrm{SU}(2)_L$ gauge space become dimension-30. The 3 spurions form the totally symmetric representation

 , (3.70)

and the compensation takes the same shape

 . (3.71)

According to the LR rule, there is only one way to form such a diagram from the outer product of the other fields, thus there is only one operator, though there are many spurions in this type,

$$
\tau^{J i}{}_{k}\tau^{K j}{}_{l}\mathbf{T}^{I}\mathbf{T}^{J}\mathbf{T}^{K}W_L{}^{I}{}_{\mu\nu}(L_{Lp_i}\sigma^{\nu}L^{\dagger}_{Ls}{}^{k})(Q_{Lraj}\sigma^{\mu}Q^{\dagger}_{Lt}{}^{al})\,. \tag{3.72}
$$

If the number of spurions in this type is more than 3, it is impossible to combine the totally symmetric combinations of spurions and other building blocks to form the gauge singlet, and thus there are no independent operators for this type.

Still, there are cases in that we have to consider many spurions in some types such as $\phi^6 D^6 \mathbf{T}^6$. Though the $\mathrm{SU}(2)_L$ gauge basis of this type is of high dimension, there is only one independent operator according to the similar argument with the previous example, where the Young diagrams of the spurions are of the same shape with its compensation.

Finally, it should be emphasized again that, the $\mathrm{SU}(2)_L$ is special and simple. If the spurions belong to the adjoint representation under the $\mathrm{SU}(N)\,, N > 2$ group, it is more difficult to deal with, since the corresponding Young diagrams are complicated.

Class	$\mathcal{N}_{\text{term}}$	$\mathcal{N}_{\text{operator}}$
UhD^6	114	114
X^2UhD^2	130	130
$XUhD^4$	164	164
ψ^2XUhD	184	$184{n_f}^2$
ψ^2UhX^2	192	$192{n_f}^2$
ψ^4UhD	1224	$\frac{4}{3}{n_f}^2(-2+3n_f+575{n_f}^2)$
ψ^4XUh	1988	$2{n_f}^2(-9+4n_f+519{n_f}^2)$
ψ^2UhD^3	272	$272{n_f}^2$
ψ^2XUhD^2	1044	$1044{n_f}^2$
ψ^4UhD^2	8260	$\frac{1}{3}{n_f}^2(155+78n_f+13525{n_f}^2)$
ψ^6Uh	7112	$\frac{1}{9}{n_f}^2(32+78n_f-133{n_f}^2+102nf^3+17129{n_f}^4)$
ψ^2UhD^4	1112	$1112{n_f}^2$
Total	21796	$\frac{1}{9}(3672+25547{n_f}^2+420{n_f}^3+56684{n_f}^4+102{n_f}^5+17129{n_f}^6)$ $n_f=1$: 11506, $n_f=3$: 1927574

Table 3. The numbers of the NNLO operators in each class.

4 Next-to-next-to-leading-order Lagrangian

There are 12 classes of operators in the NNLO Lagrangian, ranging from chiral dimension $d_\chi = 5$ and $d_\chi = 6$, and numbers of the NNLO operators are listed in the table 3. For cross-check, the Hilbert series result for the NNLO HEFT operators is presented in appendix A [49, 50]. The explicit operators are presented in an extra file, with several comments in order.

- We adopt the weyl spinors $\psi_{L/R}$ and $\psi^\dagger_{L/R}$ representing the SM fermions. The relations between the Weyl and the Dirac spinors $\Psi_{L/R}$ are

$$\Psi_L = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}, \quad \Psi_R = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}, \quad \bar{\Psi}_L = \begin{pmatrix} 0 \\ \psi^\dagger_L \end{pmatrix}, \quad \bar{\Psi}_R = \begin{pmatrix} \psi^\dagger_R \\ 0 \end{pmatrix}. \tag{4.1}$$

  The detailed transformation between these two different notations can be found in ref. [9]. Besides, the fermions contain the following SU(3), SU(2) and flavor indices. We adopt the indices set of them as

$$\begin{aligned} \mathrm{SU}(3)_{\text{color}} &\sim \{a,b,c,d,e,f\} \\ \mathrm{SU}(2)_{\text{L}} &\sim \{i,j,k,l,m,n\} \\ \mathrm{SU}(3)_{\text{flavor}} &\sim \{p,r,s,t,u,v\}\,, \end{aligned} \tag{4.2}$$

  Under this convention, the quark doublet field is denoted as

$$Q_{L p_{ai}}, \quad {Q^\dagger_{Lr}}^{ai}, \quad Q_{R p_{ai}}, \quad {Q^\dagger_{Rr}}^{ai}, \tag{4.3}$$

since the left- and right-handed spinors belong to fundamental representations while their hermitian conjugates belong to anti-fundamental representations, as shown in table 2. In particular, the spinor indices of the spinor fields are not written explicitly and are regarded as contracted with the neighbor spinor fields. For example, the bilinears should be understood as

$$\begin{aligned}(Q_L Q_R^\dagger) &= ({Q_L}^\alpha Q^\dagger_{R\alpha})\\ (Q_L \sigma^\mu Q_L^\dagger) &= ({Q_L}^\alpha \sigma^\mu_{\alpha\dot\beta} {Q_L^\dagger}^{\dot\beta})\,.\end{aligned} \tag{4.4}$$

- The operators listed assume there are right-handed neutrinos in the right-handed fermion doublet. If one removes the right-handed neutrinos in the fermion doublet, some operators would disappear, because the combination of the spurion $\mathbf{T}$ and the identity matrix would become the projector that picks up the right-handed neutrinos and the right-handed electrons. In this case, let us redefine the building blocks of the HEFT as the combination of the spurion $\mathbf{T}$ and the identity matrix

$$\begin{aligned}\mathbf{T}_+ &= \frac{I}{2} + \mathbf{T} = \mathbf{U}\left(\frac{I}{2} + \mathcal{T}_R\right)\mathbf{U}^\dagger = \mathbf{U}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathbf{U}^\dagger\,,\\ \mathbf{T}_- &= \frac{I}{2} - \mathbf{T} = \mathbf{U}\left(\frac{I}{2} - \mathcal{T}_R\right)\mathbf{U}^\dagger = \mathbf{U}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{U}^\dagger\,,\end{aligned} \tag{4.5}$$

which can be regarded as another choice of the building blocks to characterize the custodial symmetry breaking. The operators $\mathbf{T}_+$ and $\mathbf{T}_-$ apply to the right-handed leptons would project out the right-handed neutrino ν_R and right-handed electron e_R

$$\begin{aligned}\mathbf{T}_+ L_R &= \mathbf{U}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\nu_R\\ e_R\end{pmatrix} = \mathbf{U}\nu_R\,,\\ \mathbf{T}_- L_R &= \mathbf{U}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\nu_R\\ e_R\end{pmatrix} = \mathbf{U}e_R\,.\end{aligned} \tag{4.6}$$

Thus in the case where the right-handed neutrinos are absent, the operators with $\mathbf{T}_+$ acting on the right-handed lepton doublets should be eliminated. This becomes clear only when the operators presented below are re-expressed using the $\mathbf{T}_+$ and $\mathbf{T}_-$, which are usually combinations of the original operators. For example, there are 3 operators (not all operators)

$$\begin{aligned}\mathcal{O}_1 &= (L_{Lp_i} {L^\dagger_{Rr}}^i)\mathbf{V}^I{}_\mu \mathbf{V}^{I\mu}\mathbf{V}^K{}_\nu \mathbf{V}^{K\nu}\,,\\ \mathcal{O}_2 &= \tau^{O}{}^i_j \mathbf{T}^O (L_{Lp_i} {L^\dagger_{Rr}}^j)\mathbf{V}^I{}_\mu \mathbf{V}^{I\mu}\mathbf{V}^J{}_\nu \mathbf{V}^{J\nu}\,,\\ \mathcal{O}_3 &= \epsilon^{JKO}\mathbf{T}^O (L_{Lp_i}\sigma^{\lambda\nu} {L^\dagger_{Rr}}^i)\mathbf{V}^I{}_\lambda \mathbf{V}^I{}_\mu \mathbf{V}^{J\mu}\mathbf{V}^K{}_\nu\end{aligned} \tag{4.7}$$

in the NNLO class $\psi^2 UhD^4$. The combination of the first two operators $\mathcal{O}_1/2 + \mathcal{O}_2$ involves the right-handed neutrinos ν_R according to eq. (4.6) and should be eliminated if the ν_R's are absent. While the third operator $\mathcal{O}_3$ should be reserved since the spurion $\mathbf{T}$ is contracted with the NGBs rather than the right-handed leptons. Thus in these 3 operators, there are only 2 of them are independent if the right-handed neutrinos are absent. As the number of fermions increases, the fermions in a bilinear may not form SU(2) singlet, and the expansion structures may be complicated. In such cases, the projection may be not explicit. For an example of the type ${L_L}^2{L_R^\dagger}^2 L_L^\dagger L_R$ in the 6-fermion class, if right-handed neutrino exists, there are $2{n_f}^4(5{n_f}^2+1)$ independent operators, which can be justified by the Hilbert series result in appendix A. One of the operators in this type is

$$\mathcal{Y}[\young(pr),\young(st)](L_{Lu}^{\dagger}{}^{m}\bar{\sigma}_\mu L_{Lp_i})(L_{Rvm}\bar{\sigma}^\mu L_{Lrj})(L_{Rs}^{\dagger}{}^{i}L_{Rt}^{\dagger}{}^{j})\,, \tag{4.8}$$

where none of fermion bilinear is SU(2) singlet. We can express such operators in trace form

$$\mathrm{Tr}[(L_L^\dagger\bar{\sigma}_\mu L_L)(L_R^\dagger L_R^\dagger)(L_L\bar{\sigma}^\mu L_R)^T]\,, \tag{4.9}$$

where the flavor indices and the idempotent are dropped for convenience. In terms of trace form, if the right-handed neutrino does not exist, we can substitute the SU(2) doublet form of the fermions into it and expand it to a polynomial of the operators composed by SU(2) singlet fermions. After doing the same manipulation to all the operators in this type, it can be expected the condition $\nu_R=\nu_R^\dagger=0$ would eliminate some of them, and the complete set of independent operators without right-handed neutrinos in this type can be obtained. In this way, we can enumerate ${n_f}^4(3{n_f}^2+1)$ operators in the type ${L_L}^2{L_R^\dagger}^2 L_L^\dagger L_R$, which is consistent with the Hilbert series result in appendix B, and the original 40 operator terms in this type listed below are reduced to 12,

$$\begin{aligned}
&\mathcal{Y}[\young(pr),\young(st)](L_{Lu}^{\dagger}{}^{m}\bar{\sigma}_\mu L_{Lp_i})(L_{Rvm}\bar{\sigma}^\mu L_{Lrj})(L_{Rs}^{\dagger}{}^{i}L_{Rt}^{\dagger}{}^{j})\\
&\mathcal{Y}[\young(pr),\young(st)](L_{Lu}^{\dagger}{}^{m}\bar{\sigma}_\mu L_{Lp_i})(L_{Rvm}\bar{\sigma}_\nu L_{Lrj})(L_{Rs}^{\dagger}{}^{i}\sigma^{\nu\mu}L_{Rt}^{\dagger}{}^{j})\\
&\mathcal{Y}[\young(pr),\young(st)](L_{Lu}^{\dagger}{}^{j}\bar{\sigma}_\mu L_{Lp_i})(L_{Rvl}\bar{\sigma}^\mu L_{Lrj})(L_{Rs}^{\dagger}{}^{i}L_{Rt}^{\dagger}{}^{l})\\
&\mathcal{Y}[\young(pr),\young(s,t)](L_{Lu}^{\dagger}{}^{j}\bar{\sigma}_\mu L_{Lp_i})(L_{Rvl}\bar{\sigma}^\mu L_{Lrj})(L_{Rs}^{\dagger}{}^{i}L_{Rt}^{\dagger}{}^{l})\\
&\mathcal{Y}[\young(pr),\young(s,t)](L_{Lu}^{\dagger}{}^{j}\bar{\sigma}_\mu L_{Lp_i})(L_{Rvl}\bar{\sigma}_\nu L_{Lrj})(L_{Rs}^{\dagger}{}^{i}\sigma^{\nu\mu}L_{Rt}^{\dagger}{}^{l})\\
&\mathcal{Y}[\young(p,r),\young(st)](L_{Lu}^{\dagger}{}^{j}\bar{\sigma}_\mu L_{Lp_i})(L_{Rvl}\bar{\sigma}^\mu L_{Lrj})(L_{Rs}^{\dagger}{}^{i}L_{Rt}^{\dagger}{}^{l})\\
&\mathcal{Y}[\young(p,r),\young(st)](L_{Lu}^{\dagger}{}^{j}\bar{\sigma}_\mu L_{Lp_i})(L_{Rvl}\bar{\sigma}_\nu L_{Lrj})(L_{Rs}^{\dagger}{}^{i}\sigma^{\nu\mu}L_{Rt}^{\dagger}{}^{l})\\
&\mathcal{Y}[\young(p,r),\young(s,t)](L_{Lu}^{\dagger}{}^{m}\bar{\sigma}_\mu L_{Lp_i})(L_{Rvm}\bar{\sigma}^\mu L_{Lrj})(L_{Rs}^{\dagger}{}^{i}L_{Rt}^{\dagger}{}^{j})\\
&\mathcal{Y}[\young(p,r),\young(s,t)](L_{Lu}^{\dagger}{}^{m}\bar{\sigma}_\mu L_{Lp_i})(L_{Rvm}\bar{\sigma}_\nu L_{Lrj})(L_{Rs}^{\dagger}{}^{i}\sigma^{\nu\mu}L_{Rt}^{\dagger}{}^{j})\\
&\mathcal{Y}[\young(p,r),\young(s,t)](L_{Lu}^{\dagger}{}^{j}\bar{\sigma}_\mu L_{Lp_i})(L_{Rvl}\bar{\sigma}_\nu L_{Lrj})(L_{Rs}^{\dagger}{}^{i}\sigma^{\nu\mu}L_{Rt}^{\dagger}{}^{l})\\
&\mathcal{Y}[\young(pr),\young(st)]\tau^{Ii}{}_{l}\tau^{Jj}{}_{k}\tau^{Kn}{}_{m}\mathbf{T}^I\mathbf{T}^J\mathbf{T}^K(L_{Lu}^{\dagger}{}^{m}\bar{\sigma}_\mu L_{Lp_i})(L_{Rvn}\bar{\sigma}^\mu L_{Lrj})(L_{Rs}^{\dagger}{}^{k}L_{Rt}^{\dagger}{}^{l})\\
&\mathcal{Y}[\young(p,r),\young(s,t)]\tau^{Ii}{}_{l}\tau^{Jj}{}_{k}\tau^{Kn}{}_{m}\mathbf{T}^I\mathbf{T}^J\mathbf{T}^K(L_{Lu}^{\dagger}{}^{m}\bar{\sigma}_\mu L_{Lp_i})(L_{Rvn}\bar{\sigma}_\nu L_{Lrj})(L_{Rs}^{\dagger}{}^{k}\sigma^{\nu\mu}L_{Rt}^{\dagger}{}^{l})
\end{aligned} \tag{4.10}$$

Class	$\mathcal{N}_{\text{operator}}$
UhD^6	114
X^2UhD^2	130
$XUhD^4$	164
ψ^2XUhD	$156{n_f}^2$
ψ^2UhX^2	$156{n_f}^2$
ψ^4UhD	$2{n_f}^2(-1+2n_f+259{n_f}^2)$
ψ^4XUh	${n_f}^2(-16+9n_f+727{n_f}^2)$
ψ^2UhD^3	$224{n_f}^2$
ψ^2XUhD^2	$816{n_f}^2$
ψ^4UhD^2	$\frac{1}{2}{n_f}^2(91+56n_f+6059{n_f}^2)$
ψ^6Uh	$\frac{1}{9}{n_f}^2(28-24n_f+73{n_f}^2+15n{f}^3+10006{n_f}^4)$
ψ^2UhD^4	$834{n_f}^2$
Total	$\frac{1}{18}(7452+39899{n_f}^2+690{n_f}^3+77087{n_f}^4+30{n_f}^5+20012{n_f}^6)$ $n_f=1$: 8065, $n_f=3$: 1179181

Table 4. The numbers of the NNLO operators in each class without right-handed neutrino.

Because of the huge number of operators at NNLO, we can not make the explicit classification for all the types here, but all the operators without right-handed neutrinos can be obtained in a similar way above in principle. For cross-check, we also present the Hilbert series result for the HEFT NNLO operators in the appendix and make a brief overview in table 4.

- Every building block $\mathbf{V}_\mu$ in the class corresponds to a pairing of the ϕ and D in such type, while in the operators, the building block $\mathbf{V}_\mu$ instead of $D_\mu\phi$ is adopted.
- The flavor structures of the operators are indicated by the idempotent elements before that, and the indices in the Young diagram are the flavor indices of the repeated fields, just like

$$\mathcal{Y}[\tiny{\begin{array}{|c|}\hline p\\\hline r\\\hline\end{array}},\tiny{\begin{array}{|c|c|}\hline s&t\\\hline\end{array}}]\tau^{K i}{}_{l}\epsilon^{IJK}(L_{L p_i}\sigma_\lambda{}^\mu L_{L r j})(L^{\dagger}_{R s}{}^{j}\sigma^{\lambda\nu}L^{\dagger}_{R t}{}^{l})\mathbf{V}^I{}_\mu\mathbf{V}^J{}_\nu\,. \tag{4.11}$$

- The triple gauge bosons class X^3Uh, though carrying the chiral dimension 6, is attributed to the NLO operators, and is presented in the ref. [34], for the convenience of comparing with other literature.
- The spurion is of the chiral dimension $d_\chi = 0$ to capture the possible custodial symmetry breaking effects at the LO Lagrangian, while in some literature [27, 30, 54], the spurion is taken to be of chiral dimension 1, then the NNLO Lagrangian would be the combination of the NLO operators listed in ref. [34] and the NNLO operators listed below.

- In each class of operators, the physical Higgs singlet $h(x)$ could take an arbitrary number in every operator, and thus we neglect the dimensionless Higgs function in each operator
$$\mathcal{F}(h) = 1 + a\frac{h}{v} + b\frac{h^2}{v^2} + \cdots . \tag{4.12}$$

- In some cases, certain powers of the Higgs field in the operator needs to be kept to be consistent with the derivatives on the Higgs field. In this case, there is a minimum of the $h(x)$ fields for the independent operators. We utilize the convention that the number of Higgs h in such type of operators keeps to be minimal to construct the complete and independent operators in this type.

Let us take the first type of operator UhD^6 as an example to illustrate the above convention. In this type UhD^6, composed purely by Higgs h and derivatives, the operators are listed as

$$h^2(D_\lambda D_\mu D_\nu h)(D^\lambda D^\mu D^\nu h), \quad h^3(D_\mu D_\nu h)(D_\lambda D^\mu h)(D^\lambda D^\nu h). \tag{4.13}$$

where the first operator can be constructed by 4 and 5 Higgses and 6 derivatives, but the second one can not be constructed until the number of Higgses is 6. This should be understood as the following.

- Suppose there are only 4 Higgses in the operator, we can only construct one single operator
$$h^2(D_\lambda D_\mu D_\nu h)(D^\lambda D^\mu D^\nu h). \tag{4.14}$$

- Suppose there are 5 Higgses in the operator, similarly, we can construct one single operator
$$h^3(D_\lambda D_\mu D_\nu h)(D^\lambda D^\mu D^\nu h). \tag{4.15}$$

- Suppose the number of the Higgses is 6 and more, there are two kinds of operators with dimensionless Higgs function
$$h^2(D_\lambda D_\mu D_\nu h)(D^\lambda D^\mu D^\nu h)\,\mathcal{F}(h)\,, \quad h^3(D_\mu D_\nu h)(D_\lambda D^\mu h)(D^\lambda D^\nu h)\,\mathcal{F}(h)\,, \tag{4.16}$$

Since there are more than 6 Higgses, there is no more independent operator in this type, thus we write this type with the exact 6 Higgs and 6 derivatives, h^6D^6.

Because of the huge amounts of the NNLO operators, instead of listing them all here, we present them in the supplementary material of this paper. Note that the operators without sterile neutrino is also contained in the list of operators, although we do not pick those out from the list as we did for the NLO operators in ref. [34].

JHEP04(2023)086

5 Conclusion

In this work, we present the independent and complete NNLO operators of HEFT involving the right-handed neutrino for the first time, by means of the Young tensor technique on the Lorentz, gauge, and flavor structures, giving rise to the on-shell amplitude basis for each type of operators. For the operators involved in the Nambu-Goldstone bosons, the on-shell amplitude basis is further reduced to the subspace satisfying Adler zero condition in the soft momentum limit. The spurion field in the HEFT is carefully treated: they behave like ordinary building blocks with certain representation in the gauge sector, while we avoided the appearance of self-contractions among them. Thus the spurions can be implemented into the gauge structure by utilizing the Littlewood-Richardson rule on the symmetric Young diagram of the spurions and the ones of other dynamical fields. These new improvements on the NGBs and spurions in the Young tensor method are quite general and can thus be applied to other chiral effective field theories. In the case of no right-handed neutrino, we enumerate 8065(1179181) independent operators via the Hilbert series technique without picking them out from the listed operators in a separate ancillary file.

In the HEFT, we treat the power counting rules similar to the chiral Lagrangian, with the spurion field of no chiral dimension. We obtain that there are 11506 (1927574) NNLO operators, corresponding to the order $\mathcal{O}(p^5)$ and $\mathcal{O}(p^6)$, with one (three) generation fermion flavors. If the power counting rules treat the spurion field of chiral dimension one, then only a subset of the NNLO operators and partially the NLO operators with the spurion field consist of the complete and independent operators at the NNLO order.

Finally we expect the NNLO operators would be comparable with the two-loop corrections of the LO operators and the one-loop corrections with exactly one NLO vertex in the HEFT. It also benefits the phenomenological studies of the dimension-8 standard model effective field theory in the broken phase. Especially when one performs the matching between the dimension-8 SMEFT and the NNLO HEFT operators, it is necessary to list the complete sets of the SMEFT and HEFT operators.

Acknowledgments

J.-H.Y. and H. S. are supported by the National Science Foundation of China under Grants No. 12022514, No. 11875003, and No. 12047503, and National Key Research and Development Program of China Grant No. 2020YFC2201501, No. 2021YFA0718304, and CAS Project for Young Scientists in Basic Research YSBR-006, the Key Research Program of the CAS Grant No. XDPB15. M.-L.X. is supported in part by the U.S. Department of Energy under contract No. DE-AC02-06CH11357 at Argonne and No.DE-SC0010143 at Northwestern.

A Hilbert series result with right-handed neutrino

In this appendix we present the Hilbert series result for the NNLO HEFT operators with flavor number n_f, assuming the right-handed neutrino exists, the result of the NLO

JHEP04(2023)086

operators has been given in refs. [49, 50]. We follow the convention that, the polynomial of physical Higgs h is represented by $\mathcal{F}$, the derivative is represented by D, the 3 gauge bosons are represented by B, W, G, the leptons, and quarks are represented by $L_L, L_L^\dagger, L_R, L_R^\dagger, Q_L, Q_L^\dagger, Q_R, Q_R^\dagger$, and the NGB is represented by u.

$d_\chi = 5$.

$$
\begin{aligned}
hs ={}& 4n_f^2 BDL_L L_L^\dagger \mathcal{F} + 4n_f^2 D^3 L_L L_L^\dagger \mathcal{F} + 4n_f^2 BDL_R L_R^\dagger \mathcal{F} + 4n_f^2 D^3 L_R L_R^\dagger \mathcal{F} + n_f^3(9n_f-1) DL_L {L_L^\dagger}^2 L_R \mathcal{F} \\
&+ n_f^3(9n_f-1) D{L_L}^2 L_L^\dagger L_R^\dagger \mathcal{F} + n_f^3(9n_f-1) DL_L^\dagger {L_R}^2 L_R^\dagger \mathcal{F} + n_f^3(9n_f-1) DL_L L_R {L_R^\dagger}^2 \mathcal{F} \\
&+ n_f^3(3n_f+1) DL_R {Q_L}^3 \mathcal{F} + 4n_f^2 BDQ_L Q_L^\dagger \mathcal{F} + 4n_f^2 D^3 Q_L Q_L^\dagger \mathcal{F} + 4n_f^2 DGQ_L Q_L^\dagger \mathcal{F} + 18n_f^4 DL_L^\dagger L_R Q_L Q_L^\dagger \mathcal{F} \\
&+ 18n_f^4 DL_L L_R^\dagger Q_L Q_L^\dagger \mathcal{F} + n_f^3(3n_f+1) DL_R^\dagger {Q_L^\dagger}^3 \mathcal{F} + n_f^3(9n_f+1) DL_L {Q_L}^2 Q_R \mathcal{F} + 18n_f^4 DL_L L_L^\dagger Q_L^\dagger Q_R \mathcal{F} \\
&+ 18n_f^4 DL_R L_R^\dagger Q_L^\dagger Q_R \mathcal{F} + 18n_f^4 DQ_L {Q_L^\dagger}^2 Q_R \mathcal{F} + n_f^3(9n_f+1) DL_R Q_L {Q_R}^2 \mathcal{F} + n_f^3(3n_f+1) DL_L {Q_R}^3 \mathcal{F} \\
&+ 18n_f^4 DL_L L_L^\dagger Q_L Q_R^\dagger \mathcal{F} + 18n_f^4 DL_R L_R^\dagger Q_L Q_R^\dagger \mathcal{F} + 18n_f^4 D{Q_L}^2 Q_L^\dagger Q_R^\dagger \mathcal{F} + n_f^3(9n_f+1) DL_L^\dagger {Q_L^\dagger}^2 Q_R^\dagger \mathcal{F} \\
&+ 4n_f^2 BDQ_R Q_R^\dagger \mathcal{F} + 4n_f^2 D^3 Q_R Q_R^\dagger \mathcal{F} + 4n_f^2 DGQ_R Q_R^\dagger \mathcal{F} + 18n_f^4 DL_L^\dagger L_R Q_R Q_R^\dagger \mathcal{F} + 18n_f^4 DL_L L_R^\dagger Q_R Q_R^\dagger \mathcal{F} \\
&+ 18n_f^4 DQ_L^\dagger {Q_R}^2 Q_R^\dagger \mathcal{F} + n_f^3(9n_f+1) DL_R^\dagger Q_L^\dagger {Q_R^\dagger}^2 \mathcal{F} + 18n_f^4 DQ_L Q_R {Q_R^\dagger}^2 \mathcal{F} + n_f^3(3n_f+1) DL_L^\dagger {Q_R^\dagger}^3 \mathcal{F} \\
&+ 8n_f^2 BL_L L_L^\dagger u\mathcal{F} + 16n_f^2 D^2 L_L L_L^\dagger u\mathcal{F} + 8n_f^2 BL_R L_R^\dagger u\mathcal{F} + 16n_f^2 D^2 L_R L_R^\dagger u\mathcal{F} + 14n_f^4 L_L {L_L^\dagger}^2 L_R u\mathcal{F} \\
&+ 14n_f^4 {L_L}^2 L_L^\dagger L_R^\dagger u\mathcal{F} + 14n_f^4 L_L^\dagger {L_R}^2 L_R^\dagger u\mathcal{F} + 14n_f^4 L_L L_R {L_R^\dagger}^2 u\mathcal{F} + \frac{2}{3} n_f^2(7n_f^2-1) L_R {Q_L}^3 u\mathcal{F} \\
&+ 8n_f^2 BQ_L Q_L^\dagger u\mathcal{F} + 16n_f^2 D^2 Q_L Q_L^\dagger u\mathcal{F} + 8n_f^2 GQ_L Q_L^\dagger u\mathcal{F} + 28n_f^4 L_L^\dagger L_R Q_L Q_L^\dagger u\mathcal{F} + 28n_f^4 L_L L_R^\dagger Q_L Q_L^\dagger u\mathcal{F} \\
&+ \frac{2}{3} n_f^2(7n_f^2-1) L_R^\dagger {Q_L^\dagger}^3 u\mathcal{F} + 14n_f^4 L_L {Q_L}^2 Q_R u\mathcal{F} + 28n_f^4 L_L L_L^\dagger Q_L^\dagger Q_R u\mathcal{F} + 28n_f^4 L_R L_R^\dagger Q_L^\dagger Q_R u\mathcal{F} \\
&+ 28n_f^4 Q_L {Q_L^\dagger}^2 Q_R u\mathcal{F} + 14n_f^4 L_R Q_L {Q_R}^2 u\mathcal{F} + \frac{2}{3} n_f^2(7n_f^2-1) L_L {Q_R}^3 u\mathcal{F} + 28n_f^4 L_L L_L^\dagger Q_L Q_R^\dagger u\mathcal{F} \\
&+ 28n_f^4 L_R L_R^\dagger Q_L Q_R^\dagger u\mathcal{F} + 28n_f^4 {Q_L}^2 Q_L^\dagger Q_R^\dagger u\mathcal{F} + 14n_f^4 L_L^\dagger {Q_L^\dagger}^2 Q_R^\dagger u\mathcal{F} + 8n_f^2 BQ_R Q_R^\dagger u\mathcal{F} + 16n_f^2 D^2 Q_R Q_R^\dagger u\mathcal{F} \\
&+ 8n_f^2 GQ_R Q_R^\dagger u\mathcal{F} + 28n_f^4 L_L^\dagger L_R Q_R Q_R^\dagger u\mathcal{F} + 28n_f^4 L_L L_R^\dagger Q_R Q_R^\dagger u\mathcal{F} + 28n_f^4 Q_L^\dagger {Q_R}^2 Q_R^\dagger u\mathcal{F} \\
&+ 14n_f^4 L_R^\dagger Q_L^\dagger {Q_R^\dagger}^2 u\mathcal{F} + 28n_f^4 Q_L Q_R {Q_R^\dagger}^2 u\mathcal{F} + \frac{2}{3} n_f^2(7n_f^2-1) L_L^\dagger {Q_R^\dagger}^3 u\mathcal{F} + 30n_f^2 DL_L L_L^\dagger u^2\mathcal{F} \\
&+ 30n_f^2 DL_R L_R^\dagger u^2\mathcal{F} + 30n_f^2 DQ_L Q_L^\dagger u^2\mathcal{F} + 30n_f^2 DQ_R Q_R^\dagger u^2\mathcal{F} + 18n_f^2 L_L L_L^\dagger u^3\mathcal{F} + 18n_f^2 L_R L_R^\dagger u^3\mathcal{F} \\
&+ 18n_f^2 Q_L Q_L^\dagger u^3\mathcal{F} + 18n_f^2 Q_R Q_R^\dagger u^3\mathcal{F} + 8n_f^2 DL_L L_L^\dagger W\mathcal{F} + 8n_f^2 DL_R L_R^\dagger W\mathcal{F} + 8n_f^2 DQ_L Q_L^\dagger W\mathcal{F} \\
&+ 8n_f^2 DQ_R Q_R^\dagger W\mathcal{F} + 20n_f^2 L_L L_L^\dagger uW\mathcal{F} + 20n_f^2 L_R L_R^\dagger uW\mathcal{F} + 20n_f^2 Q_L Q_L^\dagger uW\mathcal{F} + 20n_f^2 Q_R Q_R^\dagger uW\mathcal{F}
\end{aligned}
\tag{A.1}
$$

$d_\chi = 6.$ The Hilbert series at this order is so long that it is divided into 3 parts $hs_{1,2,3}$.

$$
\begin{aligned}
hs_1 = {} & 3B^2D^2\mathcal{F} + 2D^6\mathcal{F} + 3D^2G^2\mathcal{F} + 2G^3\mathcal{F} + n_f^2(3n_f^2-1)B{L_L}^2{L_L^\dagger}^2\mathcal{F} \\
& + \frac{1}{2}n_f^2(27n_f^2-2n_f+5)D^2{L_L}^2{L_L^\dagger}^2\mathcal{F} + 4n_f^2B^2L_L^\dagger L_R\mathcal{F} + 10n_f^2BD^2L_L^\dagger L_R\mathcal{F} \\
& + 10n_f^2D^4L_L^\dagger L_R\mathcal{F} + 4n_f^2G^2L_L^\dagger L_R\mathcal{F} + \frac{1}{2}n_f^2(9n_f^2-2n_f-1)B{L_L^\dagger}^2{L_R}^2\mathcal{F} \\
& + 2(9n_f^4+n_f^2)D^2{L_L^\dagger}^2{L_R}^2\mathcal{F} + \frac{1}{9}n_f^2(25n_f^4-12n_f^3+7n_f^2-6n_f+4){L_L^\dagger}^3{L_R}^3\mathcal{F} + 4n_f^2B^2L_LL_R^\dagger\mathcal{F} \\
& + 10n_f^2BD^2L_LL_R^\dagger\mathcal{F} + 10n_f^2D^4L_LL_R^\dagger\mathcal{F} + 4n_f^2G^2L_LL_R^\dagger\mathcal{F} + \frac{1}{2}n_f^2(9n_f^2-2n_f-1)B{L_L}^2{L_R^\dagger}^2\mathcal{F} \\
& + 2(9n_f^4+n_f^2)D^2{L_L}^2{L_R^\dagger}^2\mathcal{F} + n_f^2(3n_f^2-1)B{L_R}^2{L_R^\dagger}^2\mathcal{F} + \frac{1}{2}n_f^2(27n_f^2-2n_f+5)D^2{L_R}^2{L_R^\dagger}^2\mathcal{F} \\
& + \frac{1}{9}n_f^2(25n_f^4-12n_f^3+7n_f^2-6n_f+4){L_L}^3{L_R^\dagger}^3\mathcal{F} + \frac{1}{3}n_f^3(10n_f^3+4n_f^2-n_f-1){L_L}^2{L_L^\dagger}^3L_R\mathcal{F} \\
& + \frac{1}{3}n_f^3(10n_f^3+4n_f^2-n_f-1){L_L}^3{L_L^\dagger}^2L_R^\dagger\mathcal{F} + \frac{1}{3}n_f^3(10n_f^3+4n_f^2-n_f-1)L_L^\dagger{L_R}^3{L_R^\dagger}^2\mathcal{F} \\
& + \frac{1}{3}n_f^3(10n_f^3+4n_f^2-n_f-1)L_L{L_R}^2{L_R^\dagger}^3\mathcal{F} + 12n_f^4BL_LL_L^\dagger L_RL_R^\dagger\mathcal{F} + 54n_f^4D^2L_LL_L^\dagger L_RL_R^\dagger\mathcal{F} \\
& + 2(5n_f^6+n_f^4)L_L{L_L^\dagger}^2{L_R}^2L_R^\dagger\mathcal{F} + 2(5n_f^6+n_f^4){L_L}^2L_L^\dagger L_R{L_R^\dagger}^2\mathcal{F} + n_f^3(3n_f+1)BL_L{Q_L}^3\mathcal{F} \\
& + \frac{1}{3}n_f^3(25n_f^3+4n_f^2-4n_f-1){L_L}^2L_R^\dagger{Q_L}^3\mathcal{F} + \frac{1}{3}n_f^3(10n_f^3+4n_f^2-n_f-1){L_R}^2L_R^\dagger{Q_L}^3\mathcal{F} \\
& + 12n_f^4D^2L_L{Q_L}^3\mathcal{F} + 6n_f^4GL_L{Q_L}^3\mathcal{F} + \frac{2}{3}n_f^4(10n_f^2-1)L_LL_L^\dagger L_R{Q_L}^3\mathcal{F} + 12n_f^4BL_LL_L^\dagger Q_LQ_L^\dagger\mathcal{F} \\
& + 54n_f^4D^2L_LL_L^\dagger Q_LQ_L^\dagger\mathcal{F} + 12n_f^4GL_LL_L^\dagger Q_LQ_L^\dagger\mathcal{F} + 12n_f^4BL_RL_R^\dagger Q_LQ_L^\dagger\mathcal{F} + 54n_f^4D^2L_RL_R^\dagger Q_LQ_L^\dagger\mathcal{F} \\
& + 12n_f^4GL_RL_R^\dagger Q_LQ_L^\dagger\mathcal{F} + 20n_f^6L_L{L_L^\dagger}^2L_RQ_LQ_L^\dagger\mathcal{F} + 20n_f^6{L_L}^2L_L^\dagger L_R^\dagger Q_LQ_L^\dagger\mathcal{F} \\
& + 20n_f^6L_L^\dagger{L_R}^2L_R^\dagger Q_LQ_L^\dagger\mathcal{F} + 20n_f^6L_LL_R{L_R^\dagger}^2Q_LQ_L^\dagger\mathcal{F} + n_f^4(5n_f^2-1)L_R{Q_L}^4Q_L^\dagger\mathcal{F} + 6n_f^4 \\
& - 2n_f^2B{Q_L}^2{Q_L^\dagger}^2\mathcal{F} + n_f^2(27n_f^2+5)D^2{Q_L}^2{Q_L^\dagger}^2\mathcal{F} + 12n_f^4G{Q_L}^2{Q_L^\dagger}^2\mathcal{F} + 20n_f^6L_L^\dagger L_R{Q_L}^2{Q_L^\dagger}^2\mathcal{F} \\
& + 20n_f^6L_LL_R^\dagger{Q_L}^2{Q_L^\dagger}^2\mathcal{F} + n_f^3(3n_f+1)BL_L^\dagger{Q_L^\dagger}^3\mathcal{F} + \frac{1}{3}n_f^3(25n_f^3+4n_f^2-4n_f-1){L_L^\dagger}^2L_R{Q_L^\dagger}^3\mathcal{F} \\
& + \frac{1}{3}n_f^3(10n_f^3+4n_f^2-n_f-1)L_R{L_R^\dagger}^2{Q_L^\dagger}^3\mathcal{F} + 12n_f^4D^2L_L^\dagger{Q_L^\dagger}^3\mathcal{F} + 6n_f^4GL_L^\dagger{Q_L^\dagger}^3\mathcal{F} \\
& + \frac{2}{3}n_f^4(10n_f^2-1)L_LL_L^\dagger L_R^\dagger{Q_L^\dagger}^3\mathcal{F} + n_f^4(5n_f^2-1)L_R^\dagger Q_L{Q_L^\dagger}^4\mathcal{F} + n_f^3(27n_f+1)D^2L_R{Q_L}^2Q_R\mathcal{F} \\
& + 2n_f^4(5n_f^2-1){L_L}^2L_L^\dagger{Q_L}^2Q_R\mathcal{F} + 6n_f^4BL_R{Q_L}^2Q_R\mathcal{F} + 12n_f^4GL_R{Q_L}^2Q_R\mathcal{F} \\
& + 2n_f^5(5n_f-2)L_L^\dagger{L_R}^2{Q_L}^2Q_R\mathcal{F} + 20n_f^6L_LL_RL_R^\dagger{Q_L}^2Q_R\mathcal{F} + 4n_f^2B^2Q_L^\dagger Q_R\mathcal{F} + 10n_f^2BD^2Q_L^\dagger Q_R\mathcal{F} \\
& + 10n_f^2D^4Q_L^\dagger Q_R\mathcal{F} + 6n_f^2BGQ_L^\dagger Q_R\mathcal{F} + 10n_f^2D^2GQ_L^\dagger Q_R\mathcal{F} + 10n_f^2G^2Q_L^\dagger Q_R\mathcal{F} + 18n_f^4BL_L^\dagger L_RQ_L^\dagger Q_R\mathcal{F} \\
& + 72n_f^4D^2L_L^\dagger L_RQ_L^\dagger Q_R\mathcal{F} + 18n_f^4GL_L^\dagger L_RQ_L^\dagger Q_R\mathcal{F} + n_f^4(25n_f^2-4n_f+1){L_L^\dagger}^2{L_R}^2Q_L^\dagger Q_R\mathcal{F} \\
& + 12n_f^4BL_LL_R^\dagger Q_L^\dagger Q_R\mathcal{F} + 54n_f^4D^2L_LL_R^\dagger Q_L^\dagger Q_R\mathcal{F} + 12n_f^4GL_LL_R^\dagger Q_L^\dagger Q_R\mathcal{F} + 2(5n_f^6+n_f^4){L_L}^2{L_R^\dagger}^2Q_L^\dagger Q_R\mathcal{F} \\
& + 2n_f^5(5n_f+2){L_L}^2{L_L^\dagger}^2Q_L^\dagger Q_R\mathcal{F} + 2n_f^5(5n_f+2){L_R}^2{L_R^\dagger}^2Q_L^\dagger Q_R\mathcal{F} + 40n_f^6L_LL_L^\dagger L_RL_R^\dagger Q_L^\dagger Q_R\mathcal{F} \\
& + 20n_f^6L_L{Q_L}^3Q_L^\dagger Q_R\mathcal{F} + 40n_f^6L_LL_L^\dagger Q_L{Q_L^\dagger}^2Q_R\mathcal{F} + 40n_f^6L_RL_R^\dagger Q_L{Q_L^\dagger}^2Q_R\mathcal{F} + 20n_f^6{Q_L}^2{Q_L^\dagger}^3Q_R\mathcal{F} \\
& + \frac{1}{2}n_f^4(25n_f^2+4n_f-1)L_L^\dagger{Q_L^\dagger}^4Q_R\mathcal{F} + n_f^3(27n_f+1)D^2L_LQ_L{Q_R}^2\mathcal{F} + 6n_f^4BL_LQ_L{Q_R}^2\mathcal{F} \\
& + 12n_f^4GL_LQ_L{Q_R}^2\mathcal{F} + 2n_f^4(5n_f^2-1){L_R}^2L_R^\dagger Q_L{Q_R}^2\mathcal{F} + 2n_f^5(5n_f-2){L_L}^2L_R^\dagger Q_L{Q_R}^2\mathcal{F} \\
& + 20n_f^6L_LL_L^\dagger L_RQ_L{Q_R}^2\mathcal{F} + 2n_f^5(15n_f-2)L_R{Q_L}^2Q_L^\dagger{Q_R}^2\mathcal{F} + n_f^2(9n_f^2-1)B{Q_L^\dagger}^2{Q_R}^2\mathcal{F} \\
& + 4(9n_f^4+n_f^2)D^2{Q_L^\dagger}^2{Q_R}^2\mathcal{F} + 18n_f^4G{Q_L^\dagger}^2{Q_R}^2\mathcal{F} + 2(25n_f^6+n_f^4)L_L^\dagger L_R{Q_L^\dagger}^2{Q_R}^2\mathcal{F} \\
& + 4(5n_f^6+n_f^4)L_LL_R^\dagger{Q_L^\dagger}^2{Q_R}^2\mathcal{F} + \frac{1}{3}n_f^3(10n_f^3+4n_f^2-n_f-1){L_L}^2L_L^\dagger{Q_R}^3\mathcal{F} + n_f^3(3n_f+1)BL_R{Q_R}^3\mathcal{F}
\end{aligned}
\tag{A.2}
$$

JHEP04(2023)086

$$
\begin{aligned}
hs_2 = {} & \frac{1}{3}n_f^3(25n_f^3+4n_f^2-4n_f-1)L_L^\dagger {L_R}^2 {Q_R}^3\mathcal{F} + 12n_f^4D^2L_R{Q_R}^3\mathcal{F} + 6n_f^4GL_R{Q_R}^3\mathcal{F} \\
& + \frac{2}{3}n_f^4(10n_f^2-1)L_LL_RL_R^\dagger {Q_R}^3\mathcal{F} + 20n_f^6L_LQ_LQ_L^\dagger {Q_R}^3\mathcal{F} + \frac{2}{3}n_f^2(25n_f^4+3n_f^2+2){Q_L^\dagger}^3{Q_R}^3\mathcal{F} \\
& + \frac{1}{2}n_f^4(25n_f^2+4n_f-1)L_RQ_L^\dagger {Q_R}^4\mathcal{F} + 4n_f^2B^2Q_LQ_R^\dagger\mathcal{F} + 10n_f^2BD^2Q_LQ_R^\dagger\mathcal{F} \\
& + 10n_f^2D^4Q_LQ_R^\dagger\mathcal{F} + 6n_f^2BGQ_LQ_R^\dagger\mathcal{F} + 10n_f^2D^2GQ_LQ_R^\dagger\mathcal{F} + 10n_f^2G^2Q_LQ_R^\dagger\mathcal{F} \\
& + 12n_f^4BL_L^\dagger L_RQ_LQ_R^\dagger\mathcal{F} + 54n_f^4D^2L_L^\dagger L_RQ_LQ_R^\dagger\mathcal{F} + 12n_f^4GL_L^\dagger L_RQ_LQ_R^\dagger\mathcal{F}\mathcal{F} \\
& + 2(5n_f^6+n_f^4){L_L^\dagger}^2{L_R}^2Q_LQ_R^\dagger\mathcal{F} + 18n_f^4BL_LL_R^\dagger Q_LQ_R^\dagger\mathcal{F} + 72n_f^4D^2L_LL_R^\dagger Q_LQ_R^\dagger\mathcal{F} \\
& + 18n_f^4GL_LL_R^\dagger Q_LQ_R^\dagger\mathcal{F} + n_f^4(25n_f^2-4n_f+1){L_L}^2{L_R^\dagger}^2Q_LQ_R^\dagger\mathcal{F} + 2n_f^5(5n_f+2){L_L}^2{L_L^\dagger}^2Q_LQ_R^\dagger\mathcal{F} \\
& + 2n_f^5(5n_f+2){L_R}^2{L_R^\dagger}^2Q_LQ_R^\dagger\mathcal{F} + 40n_f^6L_LL_L^\dagger L_RL_R^\dagger Q_LQ_R^\dagger\mathcal{F} + \frac{1}{2}n_f^4(25n_f^2+4n_f-1)L_L{Q_L}^4Q_R^\dagger\mathcal{F} \\
& + 40n_f^6L_LL_L^\dagger {Q_L}^2Q_L^\dagger Q_R^\dagger\mathcal{F} + 40n_f^6L_RL_R^\dagger {Q_L}^2Q_L^\dagger Q_R^\dagger\mathcal{F} + n_f^3(27n_f+1)D^2L_R^\dagger {Q_L^\dagger}^2Q_R^\dagger\mathcal{F} \\
& + 2n_f^4(5n_f^2-1)L_L{L_L^\dagger}^2{Q_L^\dagger}^2Q_R^\dagger\mathcal{F} + 6n_f^4BL_R^\dagger {Q_L^\dagger}^2Q_R^\dagger\mathcal{F} + 12n_f^4GL_R^\dagger {Q_L^\dagger}^2Q_R^\dagger\mathcal{F} \\
& + 2n_f^5(5n_f-2)L_L{L_R^\dagger}^2{Q_L^\dagger}^2Q_R^\dagger\mathcal{F} + 20n_f^6L_L^\dagger L_RL_R^\dagger {Q_L^\dagger}^2Q_R^\dagger\mathcal{F} + 20n_f^6{Q_L}^3{Q_L^\dagger}^2Q_R^\dagger + 20n_f^6L_L^\dagger Q_L{Q_L^\dagger}^3Q_R^\dagger\mathcal{F} \\
& + 12n_f^4BL_LL_L^\dagger Q_RQ_R^\dagger\mathcal{F} + 54n_f^4D^2L_LL_L^\dagger Q_RQ_R^\dagger\mathcal{F} + 12n_f^4GL_LL_L^\dagger Q_RQ_R^\dagger\mathcal{F} + 12n_f^4BL_RL_R^\dagger Q_RQ_R^\dagger\mathcal{F} \\
& + 54n_f^4D^2L_RL_R^\dagger Q_RQ_R^\dagger\mathcal{F} + 12n_f^4GL_RL_R^\dagger Q_RQ_R^\dagger\mathcal{F} + 20n_f^6L_L{L_L^\dagger}^2L_RQ_RQ_R^\dagger\mathcal{F} + 20n_f^6{L_L}^2L_L^\dagger L_R^\dagger Q_RQ_R^\dagger\mathcal{F} \\
& + 20n_f^6L_L^\dagger {L_R}^2L_R^\dagger Q_RQ_R^\dagger\mathcal{F} + 20n_f^6L_LL_R{L_R^\dagger}^2Q_RQ_R^\dagger\mathcal{F} + 20n_f^6L_R{Q_L}^3Q_RQ_R^\dagger\mathcal{F} + 24n_f^4BQ_LQ_L^\dagger Q_RQ_R^\dagger\mathcal{F} \\
& + 108n_f^4D^2Q_LQ_L^\dagger Q_RQ_R^\dagger\mathcal{F} + 48n_f^4GQ_LQ_L^\dagger Q_RQ_R^\dagger\mathcal{F} + 80n_f^6L_L^\dagger L_RQ_LQ_L^\dagger Q_RQ_R^\dagger\mathcal{F} \\
& + 80n_f^6L_LL_R^\dagger Q_LQ_L^\dagger Q_RQ_R^\dagger\mathcal{F} + 20n_f^6L_R^\dagger {Q_L^\dagger}^3Q_RQ_R^\dagger\mathcal{F} + 2n_f^5(15n_f-2)L_L{Q_L}^2{Q_R}^2Q_R^\dagger\mathcal{F} \\
& + 40n_f^6L_LL_L^\dagger Q_L^\dagger {Q_R}^2Q_R^\dagger\mathcal{F} + 40n_f^6L_RL_R^\dagger Q_L^\dagger {Q_R}^2Q_R^\dagger\mathcal{F} + 4(15n_f^6+n_f^4)Q_L{Q_L^\dagger}^2{Q_R}^2Q_R^\dagger\mathcal{F} \\
& + 20n_f^6L_RQ_L{Q_R}^3Q_R^\dagger\mathcal{F} + n_f^4(5n_f^2-1)L_L{Q_R}^4Q_R^\dagger\mathcal{F} + n_f^2(9n_f^2-1)B{Q_L}^2{Q_R^\dagger}^2\mathcal{F} \\
& + 4(9n_f^4+n_f^2)D^2{Q_L}^2{Q_R^\dagger}^2\mathcal{F} + 18n_f^4G{Q_L}^2{Q_R^\dagger}^2\mathcal{F} + 4(5n_f^6+n_f^4)L_L^\dagger L_R{Q_L}^2{Q_R^\dagger}^2\mathcal{F} \\
& + 2(25n_f^6+n_f^4)L_LL_R^\dagger {Q_L}^2{Q_R^\dagger}^2\mathcal{F} + n_f^3(27n_f+1)D^2L_L^\dagger Q_L^\dagger {Q_R^\dagger}^2\mathcal{F} + 6n_f^4BL_L^\dagger Q_L^\dagger {Q_R^\dagger}^2\mathcal{F} \\
& + 12n_f^4GL_L^\dagger Q_L^\dagger {Q_R^\dagger}^2\mathcal{F} + 2n_f^4(5n_f^2-1)L_R{L_R^\dagger}^2Q_L^\dagger {Q_R^\dagger}^2\mathcal{F} + 2n_f^5(5n_f-2){L_L^\dagger}^2L_RQ_L^\dagger {Q_R^\dagger}^2\mathcal{F} \\
& + 20n_f^6L_LL_L^\dagger L_R^\dagger Q_L^\dagger {Q_R^\dagger}^2\mathcal{F} + 2n_f^5(15n_f-2)L_R^\dagger Q_L{Q_L^\dagger}^2{Q_R^\dagger}^2\mathcal{F} + 40n_f^6L_LL_L^\dagger Q_LQ_R{Q_R^\dagger}^2\mathcal{F} \\
& + 40n_f^6L_RL_R^\dagger Q_LQ_R{Q_R^\dagger}^2\mathcal{F} + 4(15n_f^6+n_f^4){Q_L}^2Q_L^\dagger Q_R{Q_R^\dagger}^2\mathcal{F} + 2n_f^5(15n_f-2)L_L^\dagger {Q_L^\dagger}^2Q_R{Q_R^\dagger}^2\mathcal{F} \\
& + 6n_f^4-2n_f^2B{Q_R}^2{Q_R^\dagger}^2\mathcal{F} + n_f^2(27n_f^2+5)D^2{Q_R}^2{Q_R^\dagger}^2\mathcal{F} + 12n_f^4G{Q_R}^2{Q_R^\dagger}^2\mathcal{F} + 20n_f^6L_L^\dagger L_R{Q_R}^2{Q_R^\dagger}^2\mathcal{F} \\
& + 20n_f^6L_LL_R^\dagger {Q_R}^2{Q_R^\dagger}^2\mathcal{F} + 20n_f^6Q_L^\dagger {Q_R}^3{Q_R^\dagger}^2\mathcal{F} + \frac{1}{3}n_f^3(10n_f^3+4n_f^2-n_f-1)L_L{L_L^\dagger}^2{Q_R^\dagger}^3\mathcal{F} \\
& + n_f^3(3n_f+1)BL_R^\dagger {Q_R^\dagger}^3\mathcal{F} + \frac{1}{3}n_f^3(25n_f^3+4n_f^2-4n_f-1)L_L{L_R^\dagger}^2{Q_R^\dagger}^3\mathcal{F} \\
& + 12n_f^4D^2L_R^\dagger {Q_R^\dagger}^3\mathcal{F} + 6n_f^4GL_R^\dagger {Q_R^\dagger}^3\mathcal{F} + \frac{2}{3}n_f^4(10n_f^2-1)L_L^\dagger L_RL_R^\dagger {Q_R^\dagger}^3\mathcal{F} \\
& + \frac{2}{3}n_f^2(25n_f^4+3n_f^2+2){Q_L}^3{Q_R^\dagger}^3\mathcal{F} + 20n_f^6L_L^\dagger Q_LQ_L^\dagger {Q_R^\dagger}^3\mathcal{F} + 20n_f^6L_R^\dagger Q_L^\dagger Q_R{Q_R^\dagger}^3\mathcal{F} \\
& + 20n_f^6Q_L{Q_R}^2{Q_R^\dagger}^3\mathcal{F} + \frac{1}{2}n_f^4(25n_f^2+4n_f-1)L_R^\dagger Q_L{Q_R^\dagger}^4\mathcal{F} + n_f^4(5n_f^2-1)L_L^\dagger Q_R{Q_R^\dagger}^4\mathcal{F} \\
& + 3B^2Du\mathcal{F} + 4BD^3u\mathcal{F} + 3D^5u\mathcal{F} + 3DG^2u\mathcal{F} + \frac{1}{2}n_f^2(63n_f^2-6n_f+1)D{L_L}^2{L_L^\dagger}^2u\mathcal{F} \\
& + 32n_f^2BDL_L^\dagger L_Ru\mathcal{F} + 44n_f^2D^3L_L^\dagger L_Ru\mathcal{F} + 32n_f^2BDL_LL_R^\dagger u\mathcal{F} + 44n_f^2D^3L_LL_R^\dagger u\mathcal{F} \\
& + \frac{1}{2}n_f^2(63n_f^2-6n_f+1)D{L_R}^2{L_R^\dagger}^2u\mathcal{F} + 6n_f^3(7n_f-1)D{L_L^\dagger}^2{L_R}^2u\mathcal{F} + 6n_f^3(7n_f-1)D{L_L}^2{L_R^\dagger}^2u\mathcal{F} \\
& + 126n_f^4DL_LL_L^\dagger L_RL_R^\dagger u\mathcal{F} + 2n_f^3(14n_f+3)DL_L{Q_L}^3u\mathcal{F} + 126n_f^4DL_LL_L^\dagger Q_LQ_L^\dagger u\mathcal{F} \\
& + 126n_f^4DL_RL_R^\dagger Q_LQ_L^\dagger u\mathcal{F} + 63n_f^4+n_f^2D{Q_L}^2{Q_L^\dagger}^2u\mathcal{F} + 2n_f^3(14n_f+3)DL_L^\dagger {Q_L^\dagger}^3u\mathcal{F}
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
hs_3 = {} & 3n_f^3(21n_f+1)DL_RQ_L^2Q_Ru\mathcal{F} + 32n_f^2BDQ_L^\dagger Q_Ru\mathcal{F} + 44n_f^2D^3Q_L^\dagger Q_Ru\mathcal{F} + 32n_f^2DGQ_L^\dagger Q_Ru\mathcal{F} \\
& + 168n_f^4DL_L^\dagger L_RQ_L^\dagger Q_Ru\mathcal{F} + 126n_f^4DL_LL_R^\dagger Q_L^\dagger Q_Ru\mathcal{F} + 3n_f^3(21n_f+1)DL_LQ_LQ_R^2u\mathcal{F} \\
& + 84n_f^4DQ_L^{\dagger 2}Q_R^2u\mathcal{F} + 2n_f^3(14n_f+3)DL_RQ_R^3u\mathcal{F} + 32n_f^2BDQ_LQ_R^\dagger u\mathcal{F} + 44n_f^2D^3Q_LQ_R^\dagger u\mathcal{F} \\
& + 32n_f^2DGQ_LQ_R^\dagger u\mathcal{F} + 126n_f^4DL_L^\dagger L_RQ_LQ_R^\dagger u\mathcal{F} + 168n_f^4DL_LL_R^\dagger Q_LQ_R^\dagger u\mathcal{F} \\
& + 3n_f^3(21n_f+1)DL_R^\dagger Q_L^{\dagger 2}Q_R^\dagger u\mathcal{F} + 126n_f^4DL_LL_L^\dagger Q_RQ_R^\dagger u\mathcal{F} + 126n_f^4DL_RL_R^\dagger Q_RQ_R^\dagger u\mathcal{F} \\
& + 252n_f^4DQ_LQ_L^\dagger Q_RQ_R^\dagger u\mathcal{F} + 84n_f^4DQ_L^2Q_R^{\dagger 2}u\mathcal{F} + 3n_f^3(21n_f+1)DL_L^\dagger Q_L^\dagger Q_R^{\dagger 2}u\mathcal{F} + 63n_f^4 \\
& + n_f^2DQ_R^2Q_R^{\dagger 2}u\mathcal{F} + 2n_f^3(14n_f+3)DL_R^\dagger Q_R^{\dagger 3}u\mathcal{F} + 6B^2u^2\mathcal{F} + 14BD^2u^2\mathcal{F} + 14D^4u^2\mathcal{F} + 6G^2u^2\mathcal{F} \\
& + 2(9n_f^4+n_f^2)L_L^2L_L^{\dagger 2}u^2\mathcal{F} + 24n_f^2BL_L^\dagger L_Ru^2\mathcal{F} + 98n_f^2D^2L_L^\dagger L_Ru^2\mathcal{F} + \frac{1}{2}n_f^2(43n_f^2-6n_f+5)L_L^{\dagger 2}L_R^2u^2\mathcal{F} \\
& + 24n_f^2BL_LL_R^\dagger u^2\mathcal{F} + 98n_f^2D^2L_LL_R^\dagger u^2\mathcal{F} + \frac{1}{2}n_f^2(43n_f^2-6n_f+5)L_L^2L_R^{\dagger 2}u^2\mathcal{F} + 2(9n_f^4+n_f^2)L_R^2L_R^{\dagger 2}u^2\mathcal{F} \\
& + 72n_f^4L_LL_L^\dagger L_RL_R^\dagger u^2\mathcal{F} + \frac{1}{3}n_f^2(43n_f^2+9n_f-4)L_LQ_L^3u^2\mathcal{F} + 72n_f^4L_LL_L^\dagger Q_LQ_L^\dagger u^2\mathcal{F} \\
& + 72n_f^4L_RL_R^\dagger Q_LQ_L^\dagger u^2\mathcal{F} + 4(9n_f^4+n_f^2)Q_L^2Q_L^{\dagger 2}u^2\mathcal{F} + \frac{1}{3}n_f^2(43n_f^2+9n_f-4)L_L^\dagger Q_L^{\dagger 3}u^2\mathcal{F} \\
& + 36n_f^4L_RQ_L^2Q_Ru^2\mathcal{F} + 24n_f^2BQ_L^\dagger Q_Ru^2\mathcal{F} + 98n_f^2D^2Q_L^\dagger Q_Ru^2\mathcal{F} + 24n_f^2GQ_L^\dagger Q_Ru^2\mathcal{F} \\
& + 86n_f^4L_L^\dagger L_RQ_L^\dagger Q_Ru^2\mathcal{F} + 72n_f^4L_LL_R^\dagger Q_L^\dagger Q_Ru^2\mathcal{F} + 36n_f^4L_LQ_LQ_R^2u^2\mathcal{F} + n_f^2(43n_f^2+5)Q_L^{\dagger 2}Q_R^2u^2\mathcal{F} \\
& + \frac{1}{3}n_f^2(43n_f^2+9n_f-4)L_RQ_R^3u^2\mathcal{F} + 24n_f^2BQ_LQ_R^\dagger u^2\mathcal{F} + 98n_f^2D^2Q_LQ_R^\dagger u^2\mathcal{F} + 24n_f^2GQ_LQ_R^\dagger u^2\mathcal{F} \\
& + 72n_f^4L_L^\dagger L_RQ_LQ_R^\dagger u^2\mathcal{F} + 86n_f^4L_LL_R^\dagger Q_LQ_R^\dagger u^2\mathcal{F} + 36n_f^4L_R^\dagger Q_L^{\dagger 2}Q_R^\dagger u^2\mathcal{F} + 72n_f^4L_LL_L^\dagger Q_RQ_R^\dagger u^2\mathcal{F} \\
& + 72n_f^4L_RL_R^\dagger Q_RQ_R^\dagger u^2\mathcal{F} + 144n_f^4Q_LQ_L^\dagger Q_RQ_R^\dagger u^2\mathcal{F} + n_f^2(43n_f^2+5)Q_L^2Q_R^{\dagger 2}u^2\mathcal{F} + 36n_f^4L_L^\dagger Q_L^\dagger Q_R^{\dagger 2}u^2\mathcal{F} \\
& + 4(9n_f^4+n_f^2)Q_R^2Q_R^{\dagger 2}u^2\mathcal{F} + \frac{1}{3}n_f^2(43n_f^2+9n_f-4)L_R^\dagger Q_R^{\dagger 3}u^2\mathcal{F} + 18BDu^3\mathcal{F} + 25D^3u^3\mathcal{F} \\
& + 94n_f^2DL_L^\dagger L_Ru^3\mathcal{F} + 94n_f^2DL_LL_R^\dagger u^3\mathcal{F} + 94n_f^2DQ_L^\dagger Q_Ru^3\mathcal{F} + 94n_f^2DQ_LQ_R^\dagger u^3\mathcal{F} + 8Bu^4\mathcal{F} \\
& + 37D^2u^4\mathcal{F} + 32n_f^2L_L^\dagger L_Ru^4\mathcal{F} + 32n_f^2L_LL_R^\dagger u^4\mathcal{F} + 32n_f^2Q_L^\dagger Q_Ru^4\mathcal{F} + 32n_f^2Q_LQ_R^\dagger u^4\mathcal{F} \\
& + 23Du^5\mathcal{F} + 10u^6\mathcal{F} + 4BD^2W\mathcal{F} + n_f^2(7n_f^2-1)L_L^2L_L^{\dagger 2}W\mathcal{F} + 12n_f^2BL_L^\dagger L_RW\mathcal{F} \\
& + 20n_f^2D^2L_L^\dagger L_RW\mathcal{F} + \frac{1}{2}n_f^2(21n_f^2-6n_f-1)L_L^{\dagger 2}L_R^2W\mathcal{F} + 12n_f^2BL_LL_R^\dagger W\mathcal{F} + 20n_f^2D^2L_LL_R^\dagger W\mathcal{F} \\
& + \frac{1}{2}n_f^2(21n_f^2-6n_f-1)L_L^2L_R^{\dagger 2}W\mathcal{F} + n_f^2(7n_f^2-1)L_R^2L_R^{\dagger 2}W\mathcal{F} + 28n_f^4L_LL_L^\dagger L_RL_R^\dagger W\mathcal{F} \\
& + n_f^3(7n_f+3)L_LQ_L^3W\mathcal{F} + 28n_f^4L_LL_L^\dagger Q_LQ_L^\dagger W\mathcal{F} + 28n_f^4L_RL_R^\dagger Q_LQ_L^\dagger W\mathcal{F} + 2n_f^2(7n_f^2-1)Q_L^2Q_L^{\dagger 2}W\mathcal{F} \\
& + n_f^3(7n_f+3)L_L^\dagger Q_L^{\dagger 3}W\mathcal{F} + 14n_f^4L_RQ_L^2Q_RW\mathcal{F} + 12n_f^2BQ_L^\dagger Q_RW\mathcal{F} + 20n_f^2D^2Q_L^\dagger Q_RW\mathcal{F} \\
& + 12n_f^2GQ_L^\dagger Q_RW\mathcal{F} + 42n_f^4L_L^\dagger L_RQ_L^\dagger Q_RW\mathcal{F} + 28n_f^4L_LL_R^\dagger Q_L^\dagger Q_RW\mathcal{F} + 14n_f^4L_LQ_LQ_R^2W\mathcal{F} \\
& + n_f^2(21n_f^2-1)Q_L^{\dagger 2}Q_R^2W\mathcal{F} + n_f^3(7n_f+3)L_RQ_R^3W\mathcal{F} + 12n_f^2BQ_LQ_R^\dagger W\mathcal{F} + 20n_f^2D^2Q_LQ_R^\dagger W\mathcal{F} \\
& + 12n_f^2GQ_LQ_R^\dagger W\mathcal{F} + 28n_f^4L_L^\dagger L_RQ_LQ_R^\dagger W\mathcal{F} + 42n_f^4L_LL_R^\dagger Q_LQ_R^\dagger W\mathcal{F} + 14n_f^4L_R^\dagger Q_L^{\dagger 2}Q_R^\dagger W\mathcal{F} \\
& + 28n_f^4L_LL_L^\dagger Q_RQ_R^\dagger W\mathcal{F} + 28n_f^4L_RL_R^\dagger Q_RQ_R^\dagger W\mathcal{F} + 56n_f^4Q_LQ_L^\dagger Q_RQ_R^\dagger W\mathcal{F} + n_f^2(21n_f^2-1)Q_L^2Q_R^{\dagger 2}W\mathcal{F} \\
& + 14n_f^4L_L^\dagger Q_L^\dagger Q_R^{\dagger 2}W\mathcal{F} + 2n_f^2(7n_f^2-1)Q_R^2Q_R^{\dagger 2}W\mathcal{F} + n_f^3(7n_f+3)L_R^\dagger Q_R^{\dagger 3}W\mathcal{F} + 18BDuW\mathcal{F} \\
& + 12D^3uW\mathcal{F} + 80n_f^2DL_L^\dagger L_RuW\mathcal{F} + 80n_f^2DL_LL_R^\dagger uW\mathcal{F} + 80n_f^2DQ_L^\dagger Q_RuW\mathcal{F} + 80n_f^2DQ_LQ_R^\dagger uW\mathcal{F} \\
& + 22Bu^2W\mathcal{F} + 34D^2u^2W\mathcal{F} + 62n_f^2L_L^\dagger L_Ru^2W\mathcal{F} + 62n_f^2L_LL_R^\dagger u^2W\mathcal{F} + 62n_f^2Q_L^\dagger Q_Ru^2W\mathcal{F} \\
& + 62n_f^2Q_LQ_R^\dagger u^2W\mathcal{F} + 50Du^3W\mathcal{F} + 24u^4W\mathcal{F} + 2BW^2\mathcal{F} + 7D^2W^2\mathcal{F} + 16n_f^2L_L^\dagger L_RW^2\mathcal{F} \\
& + 16n_f^2L_LL_R^\dagger W^2\mathcal{F} + 16n_f^2Q_L^\dagger Q_RW^2\mathcal{F} + 16n_f^2Q_LQ_R^\dagger W^2\mathcal{F} + 21DuW^2\mathcal{F} + 34u^2W^2\mathcal{F} + 2W^3\mathcal{F}
\end{aligned}
\tag{A.4}
$$

JHEP04(2023)086

B Hilbert series result without right-handed neutrino

For the purpose of cross-checking with the result without the right-handed neutrino discussed in the section 4, we present the Hilbert series result for the NNLO HEFT operators with flavor number n_f, assuming the right-handed neutrino does not exist. The convention is consistent with that in appendix A.

$d_\chi = 5$.

$$
\begin{aligned}
hs = {} & 4n_f^2 BDL_L L_L^\dagger \mathcal{F} + 4n_f^2 D^3 L_L L_L^\dagger \mathcal{F} + 2n_f^2 BDL_R L_R^\dagger \mathcal{F} + 2n_f^2 D^3 L_R L_R^\dagger \mathcal{F} + n_f^3(9n_f - 1) DL_L{}^2 L_L^\dagger L_R^\dagger \mathcal{F} \\
& + n_f^3(3n_f - 1) DL_L L_R L_R^{\dagger\,2} \mathcal{F} + 4n_f^2 BDQ_L Q_L^\dagger \mathcal{F} + 4n_f^2 D^3 Q_L Q_L^\dagger \mathcal{F} + 4n_f^2 DGQ_L Q_L^\dagger \mathcal{F} \\
& + 18n_f^4 DL_L L_R^\dagger Q_L Q_L^\dagger \mathcal{F} + n_f^3(3n_f + 1) DL_R^\dagger Q_L^{\dagger\,3} \mathcal{F} + n_f^3(9n_f + 1) DL_L Q_L{}^2 Q_R \mathcal{F} + 18n_f^4 DL_L L_L^\dagger Q_L^\dagger Q_R \mathcal{F} \\
& + 6n_f^4 DL_R L_R^\dagger Q_L^\dagger Q_R \mathcal{F} + 18n_f^4 DQ_L Q_L^{\dagger\,2} Q_R \mathcal{F} + n_f^3(3n_f + 1) DL_L Q_R{}^3 \mathcal{F} + 18n_f^4 DL_L L_L^\dagger Q_L Q_R^\dagger \mathcal{F} \\
& + 6n_f^4 DL_R L_R^\dagger Q_L Q_R^\dagger \mathcal{F} + 18n_f^4 DQ_L{}^2 Q_L^\dagger Q_R^\dagger \mathcal{F} + n_f^3(9n_f + 1) DL_L^\dagger Q_L^{\dagger\,2} Q_R^\dagger \mathcal{F} + 4n_f^2 BDQ_R Q_R^\dagger \mathcal{F} \\
& + 4n_f^2 D^3 Q_R Q_R^\dagger \mathcal{F} + 4n_f^2 DGQ_R Q_R^\dagger \mathcal{F} + 18n_f^4 DL_L L_R^\dagger Q_R Q_R^\dagger \mathcal{F} + 18n_f^4 DQ_L^\dagger Q_R{}^2 Q_R^\dagger \mathcal{F} \\
& + n_f^3(9n_f + 1) DL_R^\dagger Q_L^\dagger Q_R^{\dagger\,2} \mathcal{F} + 18n_f^4 DQ_L Q_R Q_R^{\dagger\,2} \mathcal{F} + n_f^3(3n_f + 1) DL_L^\dagger Q_R^{\dagger\,3} \mathcal{F} + 8n_f^2 BL_L L_L^\dagger u \mathcal{F} \\
& + 16n_f^2 D^2 L_L L_L^\dagger u \mathcal{F} + 2n_f^2 BL_R L_R^\dagger u \mathcal{F} + 4n_f^2 D^2 L_R L_R^\dagger u \mathcal{F} + 14n_f^4 L_L{}^2 L_L^\dagger L_R^\dagger u \mathcal{F} + 4n_f^4 L_L L_R L_R^{\dagger\,2} u \mathcal{F} \\
& + 8n_f^2 BQ_L Q_L^\dagger u \mathcal{F} + 16n_f^2 D^2 Q_L Q_L^\dagger u \mathcal{F} + 8n_f^2 GQ_L Q_L^\dagger u \mathcal{F} + 28n_f^4 L_L L_R^\dagger Q_L Q_L^\dagger u \mathcal{F} \\
& + \frac{2}{3} n_f^2(7n_f^2 - 1) L_R^\dagger Q_L^{\dagger\,3} u \mathcal{F} + 14n_f^4 L_L Q_L{}^2 Q_R u \mathcal{F} + 28n_f^4 L_L L_L^\dagger Q_L^\dagger Q_R u \mathcal{F} + 8n_f^4 L_R L_R^\dagger Q_L^\dagger Q_R u \mathcal{F} \\
& + 28n_f^4 Q_L Q_L^{\dagger\,2} Q_R u \mathcal{F} + \frac{2}{3} n_f^2(7n_f^2 - 1) L_L Q_R{}^3 u \mathcal{F} + 28n_f^4 L_L L_L^\dagger Q_L Q_R^\dagger u \mathcal{F} + 8n_f^4 L_R L_R^\dagger Q_L Q_R^\dagger u \mathcal{F} \\
& + 28n_f^4 Q_L{}^2 Q_L^\dagger Q_R^\dagger u \mathcal{F} + 14n_f^4 L_L^\dagger Q_L^{\dagger\,2} Q_R^\dagger u \mathcal{F} + 8n_f^2 BQ_R Q_R^\dagger u \mathcal{F} + 16n_f^2 D^2 Q_R Q_R^\dagger u \mathcal{F} \\
& + 8n_f^2 GQ_R Q_R^\dagger u \mathcal{F} + 28n_f^4 L_L L_R^\dagger Q_R Q_R^\dagger u \mathcal{F} + 28n_f^4 Q_L^\dagger Q_R{}^2 Q_R^\dagger u \mathcal{F} + 14n_f^4 L_R^\dagger Q_L^\dagger Q_R^{\dagger\,2} u \mathcal{F} \\
& + 28n_f^4 Q_L Q_R Q_R^{\dagger\,2} u \mathcal{F} + \frac{2}{3} n_f^2(7n_f^2 - 1) L_L^\dagger Q_R^{\dagger\,3} u \mathcal{F} + 30n_f^2 DL_L L_L^\dagger u^2 \mathcal{F} + 9n_f^2 DL_R L_R^\dagger u^2 \mathcal{F} \\
& + 30n_f^2 DQ_L Q_L^\dagger u^2 \mathcal{F} + 30n_f^2 DQ_R Q_R^\dagger u^2 \mathcal{F} + 18n_f^2 L_L L_L^\dagger u^3 \mathcal{F} + 5n_f^2 L_R L_R^\dagger u^3 \mathcal{F} + 18n_f^2 Q_L Q_L^\dagger u^3 \mathcal{F} \\
& + 18n_f^2 Q_R Q_R^\dagger u^3 \mathcal{F} + 8n_f^2 DL_L L_L^\dagger W \mathcal{F} + 2n_f^2 DL_R L_R^\dagger W \mathcal{F} + 8n_f^2 DQ_L Q_L^\dagger W \mathcal{F} + 8n_f^2 DQ_R Q_R^\dagger W \mathcal{F} \\
& + 20n_f^2 L_L L_L^\dagger uW \mathcal{F} + 6n_f^2 L_R L_R^\dagger uW \mathcal{F} + 20n_f^2 Q_L Q_L^\dagger uW \mathcal{F} + 20n_f^2 Q_R Q_R^\dagger uW \mathcal{F}
\end{aligned}
\tag{B.1}
$$

$d_\chi = 6$. The Hilbert series at this order is so long that it is divided into 3 parts $hs_{1,2,3}$.

$$
\begin{aligned}
hs_1 = {} & 3B^2D^2\mathcal{F} + 2D^6\mathcal{F} + 3D^2G^2\mathcal{F} + 2G^3\mathcal{F} + n_f^2(3n_f^2-1)B{L_L}^2{L_L^\dagger}^2\mathcal{F} + \frac{1}{2}n_f^2(27n_f^2-2n_f+5)D^2{L_L}^2{L_L^\dagger}^2\mathcal{F} \\
& + 4n_f^2B^2L_LL_R^\dagger\mathcal{F} + 10n_f^2BD^2L_LL_R^\dagger\mathcal{F} + 10n_f^2D^4L_LL_R^\dagger\mathcal{F} + 4n_f^2G^2L_LL_R^\dagger\mathcal{F} \\
& + \frac{1}{2}n_f^2(6n_f^2-3n_f-1)B{L_L}^2{L_R^\dagger}^2\mathcal{F} + 2(6n_f^4+n_f^2)D^2{L_L}^2{L_R^\dagger}^2\mathcal{F} + \frac{1}{2}n_f^2(n_f^2-1)B{L_R}^2{L_R^\dagger}^2\mathcal{F} \\
& + \frac{1}{4}n_f^2(9n_f^2-2n_f+5)D^2{L_R}^2{L_R^\dagger}^2\mathcal{F} + \frac{1}{9}n_f^2(10n_f^4-9n_f^3+4n_f^2-9n_f+4){L_L}^3{L_R^\dagger}^3\mathcal{F} \\
& + \frac{1}{2}(n_f^3-3n_f^4)B{L_L^\dagger}^2{L_R}^2\mathcal{F} + -\frac{1}{6}n_f^3(5n_f^3-4n_f^2+n_f-2){L_L^\dagger}^3{L_R}^3\mathcal{F} \\
& + \frac{1}{3}n_f^3(10n_f^3+4n_f^2-n_f-1){L_L}^3{L_L^\dagger}^2L_R^\dagger\mathcal{F} + \frac{1}{3}(n_f-1)n_f^3(n_f+1)^2L_L{L_R}^2{L_R^\dagger}^3\mathcal{F} + -6n_f^4D^2{L_L^\dagger}^2{L_R}^2\mathcal{F} \\
& + 4n_f^4BL_LL_L^\dagger L_RL_R^\dagger\mathcal{F} + 18n_f^4D^2L_LL_L^\dagger L_RL_R^\dagger\mathcal{F} + 3n_f^6 + n_f^4{L_L}^2L_L^\dagger L_R{L_R^\dagger}^2\mathcal{F} + n_f^3(3n_f+1)BL_L{Q_L}^3\mathcal{F} \\
& + \frac{1}{3}n_f^3(25n_f^3+4n_f^2-4n_f-1){L_L}^2L_R^\dagger{Q_L}^3\mathcal{F} + 12n_f^4D^2L_L{Q_L}^3\mathcal{F} + 6n_f^4GL_L{Q_L}^3\mathcal{F} + 12n_f^4BL_LL_L^\dagger Q_LQ_L^\dagger\mathcal{F} \\
& + 54n_f^4D^2L_LL_L^\dagger Q_LQ_L^\dagger\mathcal{F} + 12n_f^4GL_LL_L^\dagger Q_LQ_L^\dagger\mathcal{F} + 4n_f^4BL_RL_R^\dagger Q_LQ_L^\dagger\mathcal{F} + 18n_f^4D^2L_RL_R^\dagger Q_LQ_L^\dagger\mathcal{F} \\
& + 4n_f^4GL_RL_R^\dagger Q_LQ_L^\dagger\mathcal{F} + 20n_f^6{L_L}^2L_L^\dagger L_R^\dagger Q_LQ_L^\dagger\mathcal{F} + 6n_f^6L_LL_R{L_R^\dagger}^2Q_LQ_L^\dagger\mathcal{F} + 6n_f^4 - 2n_f^2B{Q_L}^2{Q_L^\dagger}^2\mathcal{F} \\
& + n_f^2(27n_f^2+5)D^2{Q_L}^2{Q_L^\dagger}^2\mathcal{F} + 12n_f^4G{Q_L}^2{Q_L^\dagger}^2\mathcal{F} + 20n_f^6L_LL_R^\dagger{Q_L}^2{Q_L^\dagger}^2\mathcal{F} + n_f^3(3n_f+1)BL_L^\dagger{Q_L^\dagger}^3\mathcal{F} \\
& + 12n_f^4D^2L_L^\dagger{Q_L^\dagger}^3\mathcal{F} + 6n_f^4GL_L^\dagger{Q_L^\dagger}^3\mathcal{F} + \frac{2}{3}n_f^4(10n_f^2-1)L_LL_L^\dagger L_R^\dagger{Q_L^\dagger}^3\mathcal{F} + n_f^5(n_f+1)L_R{L_R^\dagger}^2{Q_L^\dagger}^3\mathcal{F} \\
& + n_f^4(5n_f^2-1)L_R^\dagger Q_L{Q_L^\dagger}^4\mathcal{F} + 2n_f^4(5n_f^2-1){L_L}^2L_L^\dagger{Q_L}^2Q_R\mathcal{F} + n_f^5 - 3n_f^6L_L^\dagger{L_R}^2{Q_L}^2Q_R\mathcal{F} \\
& + 6n_f^6L_LL_RL_R^\dagger{Q_L}^2Q_R\mathcal{F} + 4n_f^2B^2Q_L^\dagger Q_R\mathcal{F} + 10n_f^2BD^2Q_L^\dagger Q_R\mathcal{F} + 10n_f^2D^4Q_L^\dagger Q_R\mathcal{F} \\
& + 6n_f^2BGQ_L^\dagger Q_R\mathcal{F} + 10n_f^2D^2GQ_L^\dagger Q_R\mathcal{F} + 10n_f^2G^2Q_L^\dagger Q_R\mathcal{F} + -\frac{1}{2}n_f^4(15n_f^2-4n_f+1){L_L^\dagger}^2{L_R}^2Q_L^\dagger Q_R\mathcal{F} \\
& + 12n_f^4BL_LL_R^\dagger Q_L^\dagger Q_R\mathcal{F} + 54n_f^4D^2L_LL_R^\dagger Q_L^\dagger Q_R\mathcal{F} + 12n_f^4GL_LL_R^\dagger Q_L^\dagger Q_R\mathcal{F} + n_f^4(7n_f^2+3){L_L}^2{L_R^\dagger}^2Q_L^\dagger Q_R\mathcal{F} \\
& + 2n_f^5(5n_f+2){L_L}^2{L_L^\dagger}^2Q_L^\dagger Q_R\mathcal{F} + n_f^5(n_f+1){L_R}^2{L_R^\dagger}^2Q_L^\dagger Q_R\mathcal{F} + 12n_f^6L_LL_L^\dagger L_RL_R^\dagger Q_L^\dagger Q_R\mathcal{F} \\
& + 20n_f^6L_L{Q_L}^3Q_L^\dagger Q_R\mathcal{F} + 40n_f^6L_LL_L^\dagger Q_L{Q_L^\dagger}^2Q_R\mathcal{F} + 12n_f^6L_RL_R^\dagger Q_L{Q_L^\dagger}^2Q_R\mathcal{F} + 20n_f^6{Q_L}^2{Q_L^\dagger}^3Q_R\mathcal{F} \\
& + \frac{1}{2}n_f^4(25n_f^2+4n_f-1)L_L^\dagger{Q_L^\dagger}^4Q_R\mathcal{F} + n_f^3(27n_f+1)D^2L_LQ_L{Q_R}^2\mathcal{F} + 6n_f^4BL_LQ_L{Q_R}^2\mathcal{F} \\
& + 12n_f^4GL_LQ_L{Q_R}^2\mathcal{F} + 2n_f^5(5n_f-2){L_L}^2L_R^\dagger Q_L{Q_R}^2\mathcal{F} + n_f^2(9n_f^2-1)B{Q_L^\dagger}^2{Q_R}^2\mathcal{F} \\
& + 4(9n_f^4+n_f^2)D^2{Q_L^\dagger}^2{Q_R}^2\mathcal{F} + 18n_f^4G{Q_L^\dagger}^2{Q_R}^2\mathcal{F} + 4(5n_f^6+n_f^4)L_LL_R^\dagger{Q_L^\dagger}^2{Q_R}^2\mathcal{F} \\
& + \frac{1}{3}n_f^3(10n_f^3+4n_f^2-n_f-1){L_L}^2L_L^\dagger{Q_R}^3\mathcal{F} + \frac{1}{2}(n_f^4-5n_f^6)L_L^\dagger{L_R}^2{Q_R}^3\mathcal{F} + 2n_f^6L_LL_RL_R^\dagger{Q_R}^3\mathcal{F} \\
& + 20n_f^6L_LQ_LQ_L^\dagger{Q_R}^3\mathcal{F} + \frac{2}{3}n_f^2(25n_f^4+3n_f^2+2){Q_L^\dagger}^3{Q_R}^3\mathcal{F} + 4n_f^2B^2Q_LQ_R^\dagger\mathcal{F} \\
& + 10n_f^2BD^2Q_LQ_R^\dagger\mathcal{F} + 10n_f^2D^4Q_LQ_R^\dagger\mathcal{F} + 6n_f^2BGQ_LQ_R^\dagger\mathcal{F} + 10n_f^2D^2GQ_LQ_R^\dagger\mathcal{F} + 10n_f^2G^2Q_LQ_R^\dagger\mathcal{F} + \\
& -n_f^4(3n_f^2+1){L_L^\dagger}^2{L_R}^2Q_LQ_R^\dagger\mathcal{F} + 18n_f^4BL_LL_R^\dagger Q_LQ_R^\dagger\mathcal{F} + 72n_f^4D^2L_LL_R^\dagger Q_LQ_R^\dagger\mathcal{F} + 18n_f^4GL_LL_R^\dagger Q_LQ_R^\dagger\mathcal{F} \\
& + \frac{1}{2}n_f^4(35n_f^2-10n_f+3){L_L}^2{L_R^\dagger}^2Q_LQ_R^\dagger\mathcal{F} + 2n_f^5(5n_f+2){L_L}^2{L_L^\dagger}^2Q_LQ_R^\dagger\mathcal{F} + n_f^5(n_f+1){L_R}^2{L_R^\dagger}^2Q_LQ_R^\dagger\mathcal{F} \\
& + 12n_f^6L_LL_L^\dagger L_RL_R^\dagger Q_LQ_R^\dagger\mathcal{F} + \frac{1}{2}n_f^4(25n_f^2+4n_f-1)L_L{Q_L}^4Q_R^\dagger\mathcal{F} + 40n_f^6L_LL_L^\dagger{Q_L}^2Q_L^\dagger Q_R^\dagger\mathcal{F} \\
& + 12n_f^6L_RL_R^\dagger{Q_L}^2Q_L^\dagger Q_R^\dagger\mathcal{F} + n_f^3(27n_f+1)D^2L_R^\dagger{Q_L^\dagger}^2Q_R^\dagger\mathcal{F} + 2n_f^4(5n_f^2-1)L_L{L_L^\dagger}^2{Q_L^\dagger}^2Q_R^\dagger\mathcal{F}
\end{aligned}
\tag{B.2}
$$

JHEP04(2023)086

$$
\begin{aligned}
hs_2 = {} & 6n_f^4 B L_R^\dagger {Q_L^\dagger}^2 Q_R^\dagger \mathcal{F} + 12n_f^4 G L_R^\dagger {Q_L^\dagger}^2 Q_R^\dagger \mathcal{F} + n_f^5(7n_f-3) L_L {L_R^\dagger}^2 {Q_L^\dagger}^2 Q_R^\dagger \mathcal{F} + 6n_f^6 L_L^\dagger L_R L_R^\dagger {Q_L^\dagger}^2 Q_R^\dagger \mathcal{F} \\
& + 20n_f^6 {Q_L}^3 {Q_L^\dagger}^2 Q_R^\dagger \mathcal{F} + 20n_f^6 L_L^\dagger Q_L {Q_L^\dagger}^3 Q_R^\dagger \mathcal{F} + 12n_f^4 B L_L L_L^\dagger Q_R Q_R^\dagger \mathcal{F} + 54n_f^4 D^2 L_L L_L^\dagger Q_R Q_R^\dagger \mathcal{F} \\
& + 12n_f^4 G L_L L_L^\dagger Q_R Q_R^\dagger \mathcal{F} + 4n_f^4 B L_R L_R^\dagger Q_R Q_R^\dagger \mathcal{F} + 18n_f^4 D^2 L_R L_R^\dagger Q_R Q_R^\dagger \mathcal{F} + 4n_f^4 G L_R L_R^\dagger Q_R Q_R^\dagger \mathcal{F} \\
& + 20n_f^6 {L_L}^2 L_L^\dagger L_R^\dagger Q_R Q_R^\dagger \mathcal{F} + 6n_f^6 L_L L_R {L_R^\dagger}^2 Q_R Q_R^\dagger \mathcal{F} + 24n_f^4 B Q_L Q_L^\dagger Q_R Q_R^\dagger \mathcal{F} \\
& + 108n_f^4 D^2 Q_L Q_L^\dagger Q_R Q_R^\dagger \mathcal{F} + 48n_f^4 G Q_L Q_L^\dagger Q_R Q_R^\dagger \mathcal{F} + 80n_f^6 L_L L_R^\dagger Q_L Q_L^\dagger Q_R Q_R^\dagger \mathcal{F} \\
& + 20n_f^6 L_R^\dagger {Q_L^\dagger}^3 Q_R Q_R^\dagger \mathcal{F} + 2n_f^5(15n_f-2) L_L {Q_L}^2 {Q_R}^2 Q_R^\dagger \mathcal{F} + 40n_f^6 L_L L_L^\dagger Q_L^\dagger {Q_R}^2 Q_R^\dagger \mathcal{F} \\
& + 12n_f^6 L_R L_R^\dagger Q_L^\dagger {Q_R}^2 Q_R^\dagger \mathcal{F} + 4(15n_f^6+n_f^4) Q_L {Q_L^\dagger}^2 {Q_R}^2 Q_R^\dagger \mathcal{F} + n_f^4(5n_f^2-1) L_L {Q_R}^4 Q_R^\dagger \mathcal{F} \\
& + n_f^2(9n_f^2-1) B {Q_L}^2 {Q_R^\dagger}^2 \mathcal{F} + 4(9n_f^4+n_f^2) D^2 {Q_L}^2 {Q_R^\dagger}^2 \mathcal{F} + 18n_f^4 G {Q_L}^2 {Q_R^\dagger}^2 \mathcal{F} \\
& + 2(25n_f^6+n_f^4) L_L L_R^\dagger {Q_L}^2 {Q_R^\dagger}^2 \mathcal{F} + n_f^3(27n_f+1) D^2 L_L^\dagger Q_L^\dagger {Q_R^\dagger}^2 \mathcal{F} + 6n_f^4 B L_L^\dagger Q_L^\dagger {Q_R^\dagger}^2 \mathcal{F} \\
& + 12n_f^4 G L_L^\dagger Q_L^\dagger {Q_R^\dagger}^2 \mathcal{F} + n_f^4(3n_f^2-1) L_R {L_R^\dagger}^2 Q_L^\dagger {Q_R^\dagger}^2 \mathcal{F} + 20n_f^6 L_L L_L^\dagger L_R^\dagger Q_L^\dagger {Q_R^\dagger}^2 \mathcal{F} \\
& + 2n_f^5(15n_f-2) L_R^\dagger Q_L {Q_L^\dagger}^2 {Q_R^\dagger}^2 \mathcal{F} + 40n_f^6 L_L L_L^\dagger Q_L Q_R {Q_R^\dagger}^2 \mathcal{F} + 12n_f^6 L_R L_R^\dagger Q_L Q_R {Q_R^\dagger}^2 \mathcal{F} \\
& + 4(15n_f^6+n_f^4) {Q_L}^2 Q_L^\dagger Q_R {Q_R^\dagger}^2 \mathcal{F} + 2n_f^5(15n_f-2) L_L^\dagger {Q_L^\dagger}^2 Q_R {Q_R^\dagger}^2 \mathcal{F} + 6n_f^4 \\
& - 2n_f^2 B {Q_R}^2 {Q_R^\dagger}^2 \mathcal{F} + n_f^2(27n_f^2+5) D^2 {Q_R}^2 {Q_R^\dagger}^2 \mathcal{F} + 12n_f^4 G {Q_R}^2 {Q_R^\dagger}^2 \mathcal{F} + 20n_f^6 L_L L_R^\dagger {Q_R}^2 {Q_R^\dagger}^2 \mathcal{F} \\
& + 20n_f^6 Q_L^\dagger {Q_R}^3 {Q_R^\dagger}^2 \mathcal{F} + \frac{1}{3} n_f^3(10n_f^3+4n_f^2-n_f-1) L_L {L_L^\dagger}^2 {Q_R^\dagger}^3 \mathcal{F} + n_f^3(3n_f+1) B L_R^\dagger {Q_R^\dagger}^3 \mathcal{F} \\
& + \frac{1}{6} n_f^3(35n_f^3+2n_f^2-11n_f-2) L_L {L_R^\dagger}^2 {Q_R^\dagger}^3 \mathcal{F} + 12n_f^4 D^2 L_R^\dagger {Q_R^\dagger}^3 \mathcal{F} + 6n_f^4 G L_R^\dagger {Q_R^\dagger}^3 \mathcal{F} \\
& + 2n_f^6 L_L^\dagger L_R L_R^\dagger {Q_R^\dagger}^3 \mathcal{F} + \frac{2}{3} n_f^2(25n_f^4+3n_f^2+2) {Q_L}^3 {Q_R^\dagger}^3 \mathcal{F} + 20n_f^6 L_L^\dagger Q_L Q_L^\dagger {Q_R^\dagger}^3 \mathcal{F} + 20n_f^6 L_R^\dagger Q_L^\dagger Q_R {Q_R^\dagger}^3 \mathcal{F} \\
& + 20n_f^6 Q_L {Q_R}^2 {Q_R^\dagger}^3 \mathcal{F} + \frac{1}{2} n_f^4(25n_f^2+4n_f-1) L_R^\dagger Q_L {Q_R^\dagger}^4 \mathcal{F} + n_f^4(5n_f^2-1) L_L^\dagger Q_R {Q_R^\dagger}^4 \mathcal{F} + 3B^2 D u \mathcal{F} \\
& + 4BD^3 u \mathcal{F} + 3D^5 u \mathcal{F} + 3DG^2 u \mathcal{F} + \frac{1}{2} n_f^2(63n_f^2-6n_f+1) D {L_L}^2 {L_L^\dagger}^2 u \mathcal{F} + 32n_f^2 BD L_L L_R^\dagger u \mathcal{F} \\
& + 44n_f^2 D^3 L_L L_R^\dagger u \mathcal{F} + \frac{1}{4} n_f^2(9n_f^2-2n_f+1) D {L_R}^2 {L_R^\dagger}^2 u \mathcal{F} + 3(1-4n_f) n_f^3 D {L_L^\dagger}^2 {L_R}^2 u \mathcal{F} \\
& + n_f^3(30n_f-7) D {L_L}^2 {L_R^\dagger}^2 u \mathcal{F} + 36n_f^4 D L_L L_L^\dagger L_R L_R^\dagger u \mathcal{F} + 2n_f^3(14n_f+3) D L_L {Q_L}^3 u \mathcal{F} \\
& + 126n_f^4 D L_L L_L^\dagger Q_L Q_L^\dagger u \mathcal{F} + 36n_f^4 D L_R L_R^\dagger Q_L Q_L^\dagger u \mathcal{F} + 63n_f^4 + n_f^2 D {Q_L}^2 {Q_L^\dagger}^2 u \mathcal{F} \\
& + 2n_f^3(14n_f+3) D L_L^\dagger {Q_L^\dagger}^3 u \mathcal{F} + 32n_f^2 BD Q_L^\dagger Q_R u \mathcal{F} + 44n_f^2 D^3 Q_L^\dagger Q_R u \mathcal{F} + 32n_f^2 DG Q_L^\dagger Q_R u \mathcal{F} \\
& + 126n_f^4 D L_L L_R^\dagger Q_L^\dagger Q_R u \mathcal{F} + 3n_f^3(21n_f+1) D L_L Q_L {Q_R}^2 u \mathcal{F} + 84n_f^4 D {Q_L^\dagger}^2 {Q_R}^2 u \mathcal{F} + 32n_f^2 BD Q_L Q_R^\dagger u \mathcal{F} \\
& + 44n_f^2 D^3 Q_L Q_R^\dagger u \mathcal{F} + 32n_f^2 DG Q_L Q_R^\dagger u \mathcal{F} + 168n_f^4 D L_L L_R^\dagger Q_L Q_R^\dagger u \mathcal{F} + 3n_f^3(21n_f+1) D L_R^\dagger {Q_L^\dagger}^2 Q_R^\dagger u \mathcal{F} \\
& + 126n_f^4 D L_L L_L^\dagger Q_R Q_R^\dagger u \mathcal{F} + 36n_f^4 D L_R L_R^\dagger Q_R Q_R^\dagger u \mathcal{F} + 252n_f^4 D Q_L Q_L^\dagger Q_R Q_R^\dagger u \mathcal{F} + 84n_f^4 D {Q_L}^2 {Q_R^\dagger}^2 u \mathcal{F} \\
& + 3n_f^3(21n_f+1) D L_L^\dagger Q_L^\dagger {Q_R^\dagger}^2 u \mathcal{F} + 63n_f^4 + n_f^2 D {Q_R}^2 {Q_R^\dagger}^2 u \mathcal{F} + 2n_f^3(14n_f+3) D L_R^\dagger {Q_R^\dagger}^3 u \mathcal{F} + 6B^2 u^2 \mathcal{F} \\
& + 14BD^2 u^2 \mathcal{F} + 14D^4 u^2 \mathcal{F} + 6G^2 u^2 \mathcal{F} + 2(9n_f^4+n_f^2) {L_L}^2 {L_L^\dagger}^2 u^2 \mathcal{F} + -\frac{1}{2} n_f^2(12n_f^2-3n_f+1) {L_L^\dagger}^2 {L_R}^2 u^2 \mathcal{F} \\
& + 24n_f^2 B L_L L_R^\dagger u^2 \mathcal{F} + 98n_f^2 D^2 L_L L_R^\dagger u^2 \mathcal{F} + \frac{1}{2} n_f^2(31n_f^2-7n_f+6) {L_L}^2 {L_R^\dagger}^2 u^2 \mathcal{F} + \frac{1}{2}(3n_f^4+n_f^2) {L_R}^2 {L_R^\dagger}^2 u^2 \mathcal{F} \\
& + 20n_f^4 L_L L_L^\dagger L_R L_R^\dagger u^2 \mathcal{F} + \frac{1}{3} n_f^2(43n_f^2+9n_f-4) L_L {Q_L}^3 u^2 \mathcal{F} + 72n_f^4 L_L L_L^\dagger Q_L Q_L^\dagger u^2 \mathcal{F} \\
& + 20n_f^4 L_R L_R^\dagger Q_L Q_L^\dagger u^2 \mathcal{F} + 4(9n_f^4+n_f^2) {Q_L}^2 {Q_L^\dagger}^2 u^2 \mathcal{F} + \frac{1}{3} n_f^2(43n_f^2+9n_f-4) L_L^\dagger {Q_L^\dagger}^3 u^2 \mathcal{F} \\
& + 24n_f^2 B Q_L^\dagger Q_R u^2 \mathcal{F} + 98n_f^2 D^2 Q_L^\dagger Q_R u^2 \mathcal{F} + 24n_f^2 G Q_L^\dagger Q_R u^2 \mathcal{F} + 72n_f^4 L_L L_R^\dagger Q_L^\dagger Q_R u^2 \mathcal{F}
\end{aligned}
\tag{B.3}
$$

JHEP04(2023)086

$$
\begin{aligned}
hs_3 = {} & 36n_f^4 L_L Q_L {Q_R}^2 u^2 \mathcal{F} + n_f^2(43n_f^2+5){Q_L^\dagger}^2 {Q_R}^2 u^2 \mathcal{F} + 24n_f^2 B Q_L Q_R^\dagger u^2 \mathcal{F} + 98n_f^2 D^2 Q_L Q_R^\dagger u^2 \mathcal{F} \\
& + 24n_f^2 G Q_L Q_R^\dagger u^2 \mathcal{F} + 86n_f^4 L_L L_R^\dagger Q_L Q_R^\dagger u^2 \mathcal{F} + 36n_f^4 L_R^\dagger {Q_L^\dagger}^2 Q_R^\dagger u^2 \mathcal{F} + 72n_f^4 L_L L_L^\dagger Q_R Q_R^\dagger u^2 \mathcal{F} \\
& + 20n_f^4 L_R L_R^\dagger Q_R Q_R^\dagger u^2 \mathcal{F} + 144n_f^4 Q_L Q_L^\dagger Q_R Q_R^\dagger u^2 \mathcal{F} + n_f^2(43n_f^2+5){Q_L}^2 {Q_R^\dagger}^2 u^2 \mathcal{F} \\
& + 36n_f^4 L_L^\dagger Q_L^\dagger {Q_R^\dagger}^2 u^2 \mathcal{F} + 4(9n_f^4+n_f^2){Q_R}^2 {Q_R^\dagger}^2 u^2 \mathcal{F} + \frac{1}{3}n_f^2(43n_f^2+9n_f-4)L_R^\dagger {Q_R^\dagger}^3 u^2 \mathcal{F} \\
& + 18BDu^3\mathcal{F} + 25D^3u^3\mathcal{F} + 94n_f^2 D L_L L_R^\dagger u^3 \mathcal{F} + 94n_f^2 D Q_L^\dagger Q_R u^3 \mathcal{F} + 94n_f^2 D Q_L Q_R^\dagger u^3 \mathcal{F} + 8Bu^4\mathcal{F} \\
& + 37D^2u^4\mathcal{F} + 32n_f^2 L_L L_R^\dagger u^4 \mathcal{F} + 32n_f^2 Q_L^\dagger Q_R u^4 \mathcal{F} + 32n_f^2 Q_L Q_R^\dagger u^4 \mathcal{F} + 23Du^5\mathcal{F} + 10u^6\mathcal{F} \\
& + 4BD^2W\mathcal{F} + n_f^2(7n_f^2-1){L_L}^2 {L_L^\dagger}^2 W\mathcal{F} + \frac{1}{2}(-6n_f^4+3n_f^3+n_f^2){L_L^\dagger}^2 {L_R}^2 W\mathcal{F} + 12n_f^2 B L_L L_R^\dagger W\mathcal{F} \\
& + 20n_f^2 D^2 L_L L_R^\dagger W\mathcal{F} + \frac{1}{2}n_f^2(15n_f^2-7n_f-2){L_L}^2 {L_R^\dagger}^2 W\mathcal{F} + \frac{1}{2}n_f^2(n_f^2-1){L_R}^2 {L_R^\dagger}^2 W\mathcal{F} \\
& + 8n_f^4 L_L L_L^\dagger L_R L_R^\dagger W\mathcal{F} + n_f^3(7n_f+3)L_L {Q_L}^3 W\mathcal{F} + 28n_f^4 L_L L_L^\dagger Q_L Q_L^\dagger W\mathcal{F} + 8n_f^4 L_R L_R^\dagger Q_L Q_L^\dagger W\mathcal{F} \\
& + 2n_f^2(7n_f^2-1){Q_L}^2 {Q_L^\dagger}^2 W\mathcal{F} + n_f^3(7n_f+3)L_L^\dagger {Q_L^\dagger}^3 W\mathcal{F} + 12n_f^2 B Q_L^\dagger Q_R W\mathcal{F} + 20n_f^2 D^2 Q_L^\dagger Q_R W\mathcal{F} \\
& + 12n_f^2 G Q_L^\dagger Q_R W\mathcal{F} + 28n_f^4 L_L L_R^\dagger Q_L^\dagger Q_R W\mathcal{F} + 14n_f^4 L_L Q_L {Q_R}^2 W\mathcal{F} + n_f^2(21n_f^2-1){Q_L^\dagger}^2 {Q_R}^2 W\mathcal{F} \\
& + 12n_f^2 B Q_L Q_R^\dagger W\mathcal{F} + 20n_f^2 D^2 Q_L Q_R^\dagger W\mathcal{F} + 12n_f^2 G Q_L Q_R^\dagger W\mathcal{F} + 42n_f^4 L_L L_R^\dagger Q_L Q_R^\dagger W\mathcal{F} \\
& + 14n_f^4 L_R^\dagger {Q_L^\dagger}^2 Q_R^\dagger W\mathcal{F} + 28n_f^4 L_L L_L^\dagger Q_R Q_R^\dagger W\mathcal{F} + 8n_f^4 L_R L_R^\dagger Q_R Q_R^\dagger W\mathcal{F} + 56n_f^4 Q_L Q_L^\dagger Q_R Q_R^\dagger W\mathcal{F} \\
& + n_f^2(21n_f^2-1){Q_L}^2 {Q_R^\dagger}^2 W\mathcal{F} + 14n_f^4 L_L^\dagger Q_L^\dagger {Q_R^\dagger}^2 W\mathcal{F} + 2n_f^2(7n_f^2-1){Q_R}^2 {Q_R^\dagger}^2 W\mathcal{F} \\
& + n_f^3(7n_f+3)L_R^\dagger {Q_R^\dagger}^3 W\mathcal{F} + 18BDuW\mathcal{F} + 12D^3uW\mathcal{F} + 80n_f^2 D L_L L_R^\dagger uW\mathcal{F} \\
& + 80n_f^2 D Q_L^\dagger Q_R uW\mathcal{F} + 80n_f^2 D Q_L Q_R^\dagger uW\mathcal{F} + 22Bu^2W\mathcal{F} + 34D^2u^2W\mathcal{F} + 62n_f^2 L_L L_R^\dagger u^2 W\mathcal{F} \\
& + 62n_f^2 Q_L^\dagger Q_R u^2 W\mathcal{F} + 62n_f^2 Q_L Q_R^\dagger u^2 W\mathcal{F} + 50Du^3W\mathcal{F} + 24u^4W\mathcal{F} + 2BW^2\mathcal{F} + 7D^2W^2\mathcal{F} \\
& + 16n_f^2 L_L L_R^\dagger W^2\mathcal{F} + 16n_f^2 Q_L^\dagger Q_R W^2\mathcal{F} + 16n_f^2 Q_L Q_R^\dagger W^2\mathcal{F} + 21DuW^2\mathcal{F} + 34u^2W^2\mathcal{F} + 2W^3\mathcal{F}
\end{aligned}
\tag{B.4}
$$

References

[1] S. Weinberg, *Phenomenological Lagrangians*, *Physica A* **96** (1979) 327 [INSPIRE].

[2] S. Weinberg, *Baryon and lepton nonconserving processes*, *Phys. Rev. Lett.* **43** (1979) 1566 [INSPIRE].

[3] W. Buchmuller and D. Wyler, *Effective Lagrangian analysis of new interactions and flavor conservation*, *Nucl. Phys. B* **268** (1986) 621 [INSPIRE].

[4] B. Grzadkowski, M. Iskrzynski, M. Misiak and J. Rosiek, *Dimension-six terms in the standard model Lagrangian*, *JHEP* **10** (2010) 085 [arXiv:1008.4884] [INSPIRE].

[5] L. Lehman, *Extending the standard model effective field theory with the complete set of dimension-7 operators*, *Phys. Rev. D* **90** (2014) 125023 [arXiv:1410.4193] [INSPIRE].

[6] Y. Liao and X.-D. Ma, *Renormalization group evolution of dimension-seven baryon- and lepton-number-violating operators*, *JHEP* **11** (2016) 043 [arXiv:1607.07309] [INSPIRE].

[7] H.-L. Li et al., *Complete set of dimension-eight operators in the standard model effective field theory*, *Phys. Rev. D* **104** (2021) 015026 [arXiv:2005.00008] [INSPIRE].

[8] C.W. Murphy, *Dimension-8 operators in the standard model effective field theory*, *JHEP* **10** (2020) 174 [arXiv:2005.00059] [INSPIRE].

JHEP04(2023)086

[9] H.-L. Li et al., *Complete set of dimension-nine operators in the standard model effective field theory*, *Phys. Rev. D* **104** (2021) 015025 [arXiv:2007.07899] [INSPIRE].

[10] Y. Liao and X.-D. Ma, *An explicit construction of the dimension-9 operator basis in the standard model effective field theory*, *JHEP* **11** (2020) 152 [arXiv:2007.08125] [INSPIRE].

[11] H.-L. Li, J. Shu, M.-L. Xiao and J.-H. Yu, *Depicting the landscape of generic effective field theories*, arXiv:2012.11615 [INSPIRE].

[12] B. Henning, X. Lu, T. Melia and H. Murayama, 2, 84, 30, 993, 560, 15456, 11962, 261485, *. . .: higher dimension operators in the SM EFT*, *JHEP* **08** (2017) 016 [*Erratum ibid.* **09** (2019) 019] [arXiv:1512.03433] [INSPIRE].

[13] H.-L. Li et al., *Operators for generic effective field theory at any dimension: on-shell amplitude basis construction*, *JHEP* **04** (2022) 140 [arXiv:2201.04639] [INSPIRE].

[14] S.R. Coleman, J. Wess and B. Zumino, *Structure of phenomenological Lagrangians.* 1, *Phys. Rev.* **177** (1969) 2239 [INSPIRE].

[15] C.G. Callan Jr., S.R. Coleman, J. Wess and B. Zumino, *Structure of phenomenological Lagrangians.* 2, *Phys. Rev.* **177** (1969) 2247 [INSPIRE].

[16] T. Appelquist and C.W. Bernard, *Strongly interacting Higgs bosons*, *Phys. Rev. D* **22** (1980) 200 [INSPIRE].

[17] A.C. Longhitano, *Heavy Higgs bosons in the Weinberg-Salam model*, *Phys. Rev. D* **22** (1980) 1166 [INSPIRE].

[18] A.C. Longhitano, *Low-energy impact of a heavy Higgs boson sector*, *Nucl. Phys. B* **188** (1981) 118 [INSPIRE].

[19] F. Feruglio, *The chiral approach to the electroweak interactions*, *Int. J. Mod. Phys. A* **8** (1993) 4937 [hep-ph/9301281] [INSPIRE].

[20] M.J. Herrero and E. Ruiz Morales, *The electroweak chiral lagrangian for the standard model with a heavy Higgs*, *Nucl. Phys. B* **418** (1994) 431 [hep-ph/9308276] [INSPIRE].

[21] M.J. Herrero and E. Ruiz Morales, *Nondecoupling effects of the SM Higgs boson to one loop*, *Nucl. Phys. B* **437** (1995) 319 [hep-ph/9411207] [INSPIRE].

[22] G. Buchalla and O. Cata, *Effective theory of a dynamically broken electroweak standard model at NLO*, *JHEP* **07** (2012) 101 [arXiv:1203.6510] [INSPIRE].

[23] R. Alonso et al., *The effective chiral Lagrangian for a light dynamical "Higgs particle"*, *Phys. Lett. B* **722** (2013) 330 [*Erratum ibid.* **726** (2013) 926] [arXiv:1212.3305] [INSPIRE].

[24] G. Buchalla, O. Catà and C. Krause, *Complete electroweak chiral Lagrangian with a light Higgs at NLO*, *Nucl. Phys. B* **880** (2014) 552 [arXiv:1307.5017] [INSPIRE].

[25] I. Brivio et al., *Disentangling a dynamical Higgs*, *JHEP* **03** (2014) 024 [arXiv:1311.1823] [INSPIRE].

[26] A. Pich, I. Rosell, J. Santos and J.J. Sanz-Cillero, *Low-energy signals of strongly-coupled electroweak symmetry-breaking scenarios*, *Phys. Rev. D* **93** (2016) 055041 [arXiv:1510.03114] [INSPIRE].

[27] A. Pich, I. Rosell, J. Santos and J.J. Sanz-Cillero, *Fingerprints of heavy scales in electroweak effective Lagrangians*, *JHEP* **04** (2017) 012 [arXiv:1609.06659] [INSPIRE].

JHEP04(2023)086

[28] I. Brivio, J. Gonzalez-Fraile, M.C. Gonzalez-Garcia and L. Merlo, *The complete HEFT Lagrangian after the LHC run I*, *Eur. Phys. J. C* **76** (2016) 416 [arXiv:1604.06801] [INSPIRE].

[29] L. Merlo, S. Saa and M. Sacristán-Barbero, *Baryon non-invariant couplings in Higgs effective field theory*, *Eur. Phys. J. C* **77** (2017) 185 [arXiv:1612.04832] [INSPIRE].

[30] C. Krause et al., *Colorful imprints of heavy states in the electroweak effective theory*, *JHEP* **05** (2019) 092 [arXiv:1810.10544] [INSPIRE].

[31] A. Falkowski and R. Rattazzi, *Which EFT*, *JHEP* **10** (2019) 255 [arXiv:1902.05936] [INSPIRE].

[32] P. Agrawal et al., *Determining the shape of the Higgs potential at future colliders*, *Phys. Rev. D* **101** (2020) 075023 [arXiv:1907.02078] [INSPIRE].

[33] T. Cohen, N. Craig, X. Lu and D. Sutherland, *Is SMEFT enough?*, *JHEP* **03** (2021) 237 [arXiv:2008.08597] [INSPIRE].

[34] H. Sun, M.-L. Xiao and J.-H. Yu, *Complete NLO operators in the Higgs effective field theory*, arXiv:2206.07722 [INSPIRE].

[35] O.J.P. Éboli and M.C. Gonzalez-Garcia, *Classifying the bosonic quartic couplings*, *Phys. Rev. D* **93** (2016) 093013 [arXiv:1604.03555] [INSPIRE].

[36] F.-K. Guo, P. Ruiz-Femenía and J.J. Sanz-Cillero, *One loop renormalization of the electroweak chiral Lagrangian with a light Higgs boson*, *Phys. Rev. D* **92** (2015) 074005 [arXiv:1506.04204] [INSPIRE].

[37] R. Alonso, K. Kanshin and S. Saa, *Renormalization group evolution of Higgs effective field theory*, *Phys. Rev. D* **97** (2018) 035010 [arXiv:1710.06848] [INSPIRE].

[38] G. Buchalla et al., *Complete one-loop renormalization of the Higgs-electroweak chiral Lagrangian*, *Nucl. Phys. B* **928** (2018) 93 [arXiv:1710.06412] [INSPIRE].

[39] G. Buchalla et al., *Higgs-electroweak chiral Lagrangian: one-loop renormalization group equations*, *Phys. Rev. D* **104** (2021) 076005 [arXiv:2004.11348] [INSPIRE].

[40] S.L. Adler, *Consistency conditions on the strong interactions implied by a partially conserved axial vector current*, *Phys. Rev.* **137** (1965) B1022 [INSPIRE].

[41] S.L. Adler, *Consistency conditions on the strong interactions implied by a partially conserved axial-vector current. II*, *Phys. Rev.* **139** (1965) B1638 [INSPIRE].

[42] I. Low, *Adler's zero and effective Lagrangians for nonlinearly realized symmetry*, *Phys. Rev. D* **91** (2015) 105017 [arXiv:1412.2145] [INSPIRE].

[43] I. Low, *Minimally symmetric Higgs boson*, *Phys. Rev. D* **91** (2015) 116005 [arXiv:1412.2146] [INSPIRE].

[44] C. Cheung, K. Kampf, J. Novotny and J. Trnka, *Effective field theories from soft limits of scattering amplitudes*, *Phys. Rev. Lett.* **114** (2015) 221602 [arXiv:1412.4095] [INSPIRE].

[45] C. Cheung et al., *On-shell recursion relations for effective field theories*, *Phys. Rev. Lett.* **116** (2016) 041601 [arXiv:1509.03309] [INSPIRE].

[46] I. Low and Z. Yin, *Soft bootstrap and effective field theories*, *JHEP* **11** (2019) 078 [arXiv:1904.12859] [INSPIRE].

[47] L. Dai, I. Low, T. Mehen and A. Mohapatra, *Operator counting and soft blocks in chiral perturbation theory*, *Phys. Rev. D* **102** (2020) 116011 [arXiv:2009.01819] [INSPIRE].

[48] I. Low, J. Shu, M.-L. Xiao and Y.-H. Zheng, *Amplitude/operator basis in chiral perturbation theory*, *JHEP* **01** (2023) 031 [arXiv:2209.00198] [INSPIRE].

[49] L. Gráf et al., *Hilbert series, the Higgs mechanism, and HEFT*, *JHEP* **02** (2023) 064 [arXiv:2211.06275] [INSPIRE].

[50] H. Sun, Y.-N. Wang and J.-H. Yu, *Hilbert series and operator counting on the Higgs effective field theory*, arXiv:2211.11598 [INSPIRE].

[51] B. Grinstein and M. Trott, *A Higgs-Higgs bound state due to new physics at a TeV*, *Phys. Rev. D* **76** (2007) 073002 [arXiv:0704.1505] [INSPIRE].

[52] G. Buchalla, O. Catà and C. Krause, *On the power counting in effective field theories*, *Phys. Lett. B* **731** (2014) 80 [arXiv:1312.5624] [INSPIRE].

[53] M.B. Gavela et al., *CP violation with a dynamical Higgs*, *JHEP* **10** (2014) 044 [arXiv:1406.6367] [INSPIRE].

[54] A. Pich, *Effective field theory with Nambu-Goldstone modes*, arXiv:1804.05664 [IFIC/18-13] [DOI:10.1093/oso/9780198855743.003.0003] [INSPIRE].

[55] J. Gasser and H. Leutwyler, *Chiral perturbation theory to one loop*, *Annals Phys.* **158** (1984) 142 [INSPIRE].

[56] J. Gasser and H. Leutwyler, *Chiral perturbation theory: expansions in the mass of the strange quark*, *Nucl. Phys. B* **250** (1985) 465 [INSPIRE].

[57] G. Buchalla, O. Cata, A. Celis and C. Krause, *Comment on "Analysis of general power counting rules in effective field theory"*, arXiv:1603.03062 [LMU-ASC-10-16] [INSPIRE].

[58] B.M. Gavela, E.E. Jenkins, A.V. Manohar and L. Merlo, *Analysis of general power counting rules in effective field theory*, *Eur. Phys. J. C* **76** (2016) 485 [arXiv:1601.07551] [INSPIRE].

[59] J. Hirn and J. Stern, *Lepton-number violation and right-handed neutrinos in Higgs-less effective theories*, *Phys. Rev. D* **73** (2006) 056001 [hep-ph/0504277] [INSPIRE].

[60] A. Dobado and M.J. Herrero, *Phenomenological Lagrangian approach to the symmetry breaking sector of the standard model*, *Phys. Lett. B* **228** (1989) 495 [INSPIRE].

[61] B. Henning and T. Melia, *Conformal-helicity duality & the Hilbert space of free CFTs*, arXiv:1902.06747 [INSPIRE].

[62] B. Henning and T. Melia, *Constructing effective field theories via their harmonics*, *Phys. Rev. D* **100** (2019) 016015 [arXiv:1902.06754] [INSPIRE].