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Consistent truncations and dualities

Daniel Butter,[a] Falk Hassler,[a,b] Christopher N. Pope[a] and Haoyu Zhang[a]

[a] *George P. & Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy, Texas A&M University, College Station, TX 77843, U.S.A.*

[b] *University of Wrocław, Faculty of Physics and Astronomy, Maksa Borna 9, 50-204 Wrocław, Poland*

E-mail: dbutter@tamu.edu, falk.hassler@uwr.edu.pl, pope@physics.tamu.edu, zhanghaoyu@tamu.edu

Abstract: Recent progress in generalised geometry and extended field theories suggests a deep connection between consistent truncations and dualities, which is not immediately obvious. A prime example is generalised Scherk-Schwarz reductions in double field theory, which have been shown to be in one-to-one correspondence with Poisson-Lie T-duality. Here we demonstrate that this relation is only the tip of the iceberg. Currently, the most general known classes of T-dualities (excluding mirror symmetry) are based on dressing cosets. But as we discuss, they can be further extended to the even larger class of generalised cosets. We prove that the latter give rise to consistent truncations for which the ansatz can be constructed systematically. Hence, we pave the way for many new examples of T-dualities and consistent truncations. The arising structures result in covariant tensors with more than two derivatives and we argue how they might be key to understand generalised T-dualities and consistent truncations beyond the leading two derivative level.

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1 Introduction

The notions of consistent truncations and dualities are at first sight seemingly unrelated. The former singles out degrees of freedom in a physical theory which decouple from the rest. They are important because it is often easier to analyse a system, i.e. solve its equations of motion, if the number of degrees of freedom is reduced. Therefore, consistent truncations provide a crucial tool for finding solutions in (super)gravity, with a wide range of applications. More precisely, this idea can be summarised by the commuting diagram

$$\begin{array}{ccc} \text{full action } S & \longrightarrow & \text{truncated action } S_{\text{red}} \\ \big\downarrow & & \big\downarrow \\ \text{field equations } \delta S = 0 & \longrightarrow & \text{truncated field equations } \delta S_{\text{red}} = 0. \end{array}$$
$$\underset{\text{uplift}}{\overset{\text{truncation}}{\rightleftarrows}} \tag{1.1}$$

The truncation is said to be consistent if the two pathways to the truncated equations of motion yield the same result. Otherwise, the chosen truncation ansatz does not single

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out decoupled degrees of freedom. One should note that it is in general very difficult to find consistent truncations, because the standard Kaluza-Klein ansatz with massless gauge fields is in general inconsistent [1]. For a long time, only few exceptions have been known, including sphere reductions [2] and reductions on group manifolds [3].

The second central concept for this paper is dualities. They are ubiquitous in physics, and find applications from models in condensed matter to high energy physics. The basic idea is that two seemingly very different models ultimately still share the same (quantum/classical) dynamics. Here, we are particularly interested in target-space dualities (T-dualities) of two-dimensional σ-models. They can be studied from two major perspectives: the worldsheet and the target space. The former is the surface on which the σ-model is defined as a field theory. In the classical limit, T-duality acts as a canonical transformation relating at least two different σ-models on different target spaces. Alternatively, one can consider the low-energy, effective target-space action that captures the dynamics of the strings described by the σ-model. Here, T-duality maps existing solutions of the field equations to new solutions. In this context it plays a role as a solution generating technique. It is known that a small subset of all T-dualities, namely abelian T-dualities, are preserved under quantum corrections on the worldsheet and the corresponding higher-derivative corrections in the target space effective action. For the remaining, generalised, T-dualities, their fate under quantum corrections is not yet known. There are some preliminary results [4–7] that suggest that they might also cover higher derivative corrections. Here, we are mostly concerned with the leading two-derivative effective action, and such higher-derivative corrections will just touch the discussion tangentially.

Historically, one distinguishes between non-abelian T-duality [8], Poisson-Lie T-duality [9, 10], which might be supplemented by a Wess-Zumino-Witten term [11], and dressing cosets [12]. Our results will apply to all of these, and we shall call them *generalised T-dualities*. Note that we will not consider mirror symmetries, which relate different Calabi-Yau manifolds; they are related to abelian T-duality by the SYZ conjecture [13]. For completeness, let us quickly explain the two central mathematical objects mentioned here. A Poisson-Lie group is a Lie group G equipped with a Poisson bracket that satisfies

$$\{f_1, f_2\}(gg') = \{f_1 \circ L_g, f_2 \circ L_g\}(g') + \{f_1 \circ R_{g'}, f_2 \circ R_{g'}\}(g) \tag{1.2}$$

where $L_h(g) = hg$ denotes the left-multiplication on G and $R_h(g) = gh$ the right-multiplication, respectively. The classical Lie group → Lie algebra correspondence was extended by Drinfeld to Poisson-Lie group → Lie bialgebra, which contains in addition to the Lie algebra Lie(G) also a dual Lie algebra Lie($\widetilde{G}$) corresponding to dual Lie group $\widetilde{G}$. Together, these two Lie groups form a double Lie group $\mathbb{D}$ with the Lie algebra Lie($\mathbb{D}$)=Lie(G)⊕Lie($\widetilde{G}$). Poisson-Lie groups are called dual, if they share the same $\mathbb{D}$. For example, they can be related by exchanging G and $\widetilde{G}$. Poisson-Lie T-duality got its name because it identifies σ-models whose target spaces are dual Poisson-Lie groups. In addition to the right- and left-action of G on G, Poisson-Lie groups admit a so-called dressing action [14]. It can be most easily seen on the level of the doubled Lie group $\mathbb{D}$ where any element of the subgroup $F \subset \mathbb{D}$ generates the dressing transformation $g \to g'$ on G with

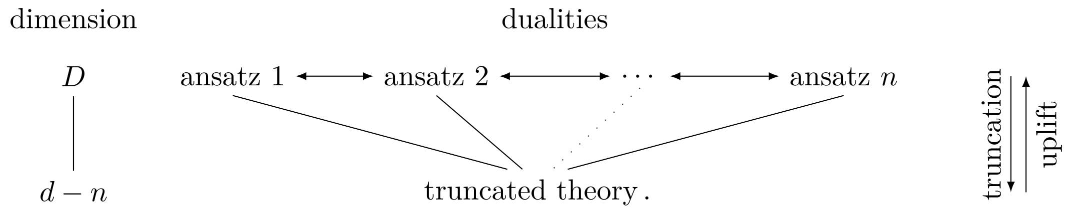
$$fg = g'\widetilde{h}\,, \qquad g, g' \in G\,, \quad \widetilde{h} \in \widetilde{G}\,, \quad \text{and} \quad f \in F\,. \tag{1.3}$$

Here the group multiplication is the multiplication on $\mathbb{D}$. The set of orbits of this action is called the dressing coset [12]. In this paper, we will even work with a more general notion than dressing cosets, which are called generalised cosets [15]. They drop the requirement that the doubled Lie group $\mathbb{D}$ has to originate from a Poisson-Lie group and can be understood as the lift of the concept of a coset in differential geometry to generalised geometry. At the end, the following dependencies arise:

$$\begin{array}{ccccccc} \text{abelian} & \subset & \text{non-abelian} & \subset & \text{Poisson-Lie} & \subset & \text{WZW-Poisson} \\ & & & & \cap & & \cap \\ & & & & \text{dressing coset} & \subset & \underline{\text{generalised coset}}\,. \end{array} \tag{1.4}$$

At first glance these two concepts, consistent truncations and generalised T-duality, seem unrelated. However, in recent years it has become evident that a deeper understanding of each of these concepts can be achieved using similar tools, most prominently (exceptional) generalised geometry [16, 17] and double/exceptional field theories. In particular the latter were initially developed [18–20] with abelian T-duality (and later, its extension to U-duality) in mind; before its utility for understanding consistent truncations was appreciated [21–23]. While double field theory is in principle able to describe also non-geometric setups, we shall use it in its most conservative form, with the standard solution $\widetilde{\partial}^i = 0$ of the section condition, thus rendering it equivalent to generalised geometry. Later, it was shown that generalised T-dualities can also be conveniently studied in double field theory [24–26]. Moreover, they provided the first systematic construction of the generalised frame fields that underlie generalised Scherk-Schwarz reductions [27–29], and which still dominate the landscape of consistent truncations. Motivated by this connection and examples like [30], the goal of this paper is to explore further the relation between generalised T-dualities and consistent truncations. It is already known that all generalised T-dualities except for dressing cosets give rise to consistent generalised Scherk-Schwarz reductions. Therefore, we shall take a closer look at generalised cosets, and show how they can be used to construct a very rich class of consistent truncations that go beyond generalised Scherk-Schwarz reductions.

Our starting point will be the most general known class of consistent truncations [23] that arise from generalised geometry. In section 2, we review the underlying construction. At the end of this section, we then show how the already mentioned generalised Scherk-Schwarz reductions are connected to Poisson-Lie T-duality. At this point, we already see that dualities can be understood naturally in the context of consistent truncations, by relating different truncation ansätze for a higher-dimensional theory that give rise to the same truncated theory:



(1.5)

Inspired by this finding, we are led to pose the question:

Is there a one-to-one correspondence between generalised T-dualities and consistent truncations in generalised geometry?

To answer it, section 3.1 introduces a new construction for the truncation ansätze depicted in (1.5). It employs a higher-dimensional, auxiliary space to geometrise the generalised structure group that underlies each consistent truncation. A similar technique was used by Poláček and Siegel [31] some time ago to find a natural construction of the generalised Riemann tensor in double field theory. It provides a structured construction of covariant torsion and curvature tensors which, as we shall show, are of central interest in consistent truncations also. With this tool at hand, we establish in section 3.3 that all spaces that admit generalised T-dualities give rise to consistent truncations. Establishing the converse, namely that any consistent truncation can be constructed from a dressing coset, is more involved. On the one hand, we know that consistent truncations can be constructed on Sasaki-Einstein spaces [23] that are clearly not double cosets. However, this is not, of itself, a problem, because as the illustration (1.5) shows, there can exist different ansätze that result in the same truncation. To check if at least one of them originates from a dressing coset we have to find solutions for the Jacobi identity of the underlying Lie algebra in which certain components of the structure coefficients are fixed while others remain free. We set up the computation in section 3.4, but it is hard to find solutions because the Jacobi identities lead to many coupled quadratic equations. This problem gets easier the more components of the structure coefficients are fixed, because they do not then appear as unknowns in the quadratic equations. Fortunately, this is exactly what happens when consistent truncations are considered in connection with higher-derivative corrections. We explore this idea in section 3.5, and interpret it as a hint that there might indeed be a one-to-one connection between consistent truncations and generalised T-dualities. We plan to return to the problem of constructing a complete proof in future work. Finally, section 4 is concerned with the explicit construction of truncation anätze for generalised cosets, the basis for any generalised T-duality. Here, we extend the results of previous work [15], by presenting a completely systematic construction. This culminates with the observation 2 that on any dressing coset $H\backslash\mathbb{D}/F$ one can construct a consistent truncation with generalised structure group F.

2 Generalised geometry and consistent truncations

2.1 (Super)gravity and generalised geometry

We are interested in consistent truncations of (super)gravity, such as arises as the low-energy effective action in string theory. For the sake of simplicity, we only consider the NS/NS sector, which is governed by the action

$$S = \int \mathrm{d}^D x \, \sqrt{g} e^{-2\phi} \left(R + 4\partial_i \phi \partial^i \phi - \frac{1}{12} H_{ijk} H^{ijk} \right) . \tag{2.1}$$

We use the convention here that g is the determinant of the metric g_{ij}, and R the corresponding curvature scalar. Besides the metric, there is also the dilaton ϕ and the B-field B_{ij} on the D-dimensional spacetime M_D (also referred to as the target space). The action does not incorporate B_{ij} directly, but instead its field strength $H_{ijk} = 3\partial_{[i} B_{jk]}$. Conformal invariance of the string at the quantum level fixes $D = 10$ for superstrings and $D = 26$ for

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their bosonic counterpart. (2.1) possesses two local symmetries, namely diffeomorphisms that account for coordinate changes and B-field gauge transformations $B_{ij} \to B_{ij} + 2\partial_{[i}\xi_{j]}$.

The infinitesimal versions of these two symmetries can be written in a unified form in terms of the generalised Lie derivative

$$\mathcal{L}_U V^I = U^J \partial_J V^I - \left(\partial_J U^I - \partial^I U_J\right) V^J . \tag{2.2}$$

$U^I = \begin{pmatrix} u^i & u_i \end{pmatrix}$, $U_I = \begin{pmatrix} u_i & u^i \end{pmatrix}$ and V^I are generalised vectors on the generalised tangent space $TM \oplus T^*M$. While the original tangent space TM is always extended in this setup, the manifold M_D might be extended (double/exceptional field theories) or not (generalised geometry). The two alternatives are related by the section condition, which singles out the non-constant, physical, directions on M_D. We always solve the section condition in the trivial way, namely $\partial_I = \begin{pmatrix} \partial_i & 0 \end{pmatrix}$. Thus we do not need to distinguish between the two different approaches, and we just use the doubling of the partial derivative index as a convenient book-keeping device. The form of (2.2) is fixed by the requirement that the natural pairing between a vector and a one-form,

$$U^I V^J \eta_{IJ} = u^i v_i + u_i v^i , \quad \text{with} \quad \eta_{IJ} = \begin{pmatrix} 0 & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix} , \tag{2.3}$$

is preserved. The invariance of this pairing introduces an O(D,D) structure. This structure allows the metric and B-field to be captured in terms of one unified object, the generalised metric

$$\mathcal{H}_{IJ} = \begin{pmatrix} g_{ij} - B_{ik} g^{kl} B_{lj} & -B_{ik} g^{kj} \\ g^{ik} B_{kj} & g^{ij} \end{pmatrix} . \tag{2.4}$$

It has two defining properties: 1) it is symmetric and 2) it is an element of O(D,D); $\mathcal{H}_{IK}\eta^{KL}\mathcal{H}_{LJ} = \eta_{IJ}$, where η^{IJ} is the inverse of η_{IJ}. Moreover, its parameterisation in terms of g_{ij} and B_{ij} is chosen such that the infinitesimal action of diffeomorphisms and B-field gauge transformation is mediated by the generalised Lie derivative.

The generalised metric introduces more structure than might be immediately obvious from the parameterisation (2.4). To reveal it, we re-express $\mathcal{H}_{IJ}$ in terms of the generalised frame field $E^A{}_I$, with the defining properties

$$\eta_{IJ} = E^A{}_I \eta_{AB} E^B{}_J , \qquad \eta_{AB} = \begin{pmatrix} 0 & \delta_a^b \\ \delta_b^a & 0 \end{pmatrix} \quad \text{and} \tag{2.5}$$

$$\mathcal{H}_{IJ} = E^A{}_I \mathcal{H}_{AB} E^B{}_J , \qquad \mathcal{H}_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta^{ab} \end{pmatrix} , \tag{2.6}$$

where η_{ab} is either the Lorentzian or Euclidean metric and η^{ab} its inverse. These two relations still do not fix the generalised frame completely. One can perform coordinate dependent double Lorentz transformations, $E_A{}^I \to \Lambda_A{}^B E_B{}^I$, without changing the resulting η_{IJ} and $\mathcal{H}_{IJ}$, assuming that $\Lambda_A{}^C \eta_{CD} \Lambda_B{}^D = \Lambda_A{}^C \mathcal{H}_{CD} \Lambda_B{}^D = 0$. They furnish the double Lorentz group H_D=O(D-1,1)×O(1,D-1) for Lorentzian or H_D=O(D)×O(D) for Euclidean spacetimes.

We can use these ingredients to rewrite the action (2.1). There are in fact different ways to do this. For our purpose the so-called flux formulation is most suitable [29, 32, 33]. To make the invariance under generalised diffeomorphisms (generated by the generalised Lie derivative) of the action manifest, the flux formulation introduces two generalised fluxes, namely

$$\mathcal{L}_{E_A} E_B = F_{AB}{}^C E_C \quad \text{and} \quad \mathcal{L}_{E_A} e^{-2d} = -F_A e^{-2d} . \tag{2.7}$$

Taking into account the definition of the generalised Lie derivative (2.2), the first equation leads to

$$F_{ABC} = 3 E_{[A}{}^I \partial_I E_B{}^J E_{C]J} , \tag{2.8}$$

while the second relation requires more explanation because we encounter a new quantity, d, the generalised dilaton. It is defined by

$$e^{-2d} = \sqrt{g} e^{-2\phi} \quad \text{or} \quad d = \phi - \frac{1}{4} \log g . \tag{2.9}$$

Importantly, e^{-2d} does not transform as a scalar under the generalised Lie derivative, but rather, as a scalar density, resulting in

$$\mathcal{L}_U e^{-2d} = \partial_I (U^I e^{-2d}) . \tag{2.10}$$

Therefore, the second generalised flux F_A is given by

$$F_A = 2 E_A{}^I \partial_I d - \partial_I E_A{}^I . \tag{2.11}$$

The two fluxes F_{ABC} and F_A repackage the information contained in the metric, dilaton and B-field. Hence, the action (2.1) can be alternatively written in terms of them as [29, 33]

$$S = \int \mathrm{d}^D x \, e^{-2d} \mathcal{R} \tag{2.12}$$

with the generalised Ricci scalar being given by

$$\mathcal{R} = P^{AB} P^{CD} \left(\overline{P}^{EF} + \frac{1}{3} P^{EF} \right) F_{ACE} F_{BDF} + 2 P^{AB} (2 D_A F_B - F_A F_B) . \tag{2.13}$$

We encounter two new objects here: first, the flat derivative

$$D_A = E_A{}^I \partial_I \tag{2.14}$$

and second, the projectors

$$P^{AB} = \frac{1}{2} (\eta^{AB} + \mathcal{H}^{AB}) \quad \text{and} \quad \overline{P}^{AB} = \frac{1}{2} (\eta^{AB} - \mathcal{H}^{AB}) . \tag{2.15}$$

They are called projectors because of their properties

$$P_A{}^C P_C{}^B = P_A{}^B , \qquad \overline{P}_A{}^C \overline{P}_C{}^B = \overline{P}_A{}^B , \quad \text{and} \quad P_A{}^C \overline{P}_C{}^B = \overline{P}_A{}^C P_C{}^B = 0 . \tag{2.16}$$

At this point, we just have a rewriting of (2.1) in terms of quantities that appear "naturally" in generalised geometry or double field theory.

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2.2 Systematics of consistent truncations

Next, we explain how this form of the action helps in identifying consistent truncations. The answer is given by the following theorem, which was established in [23]:

Theorem 1. *Let M_D be a D-dimensional manifold with a generalised F-structure defining a set of invariant tensors $f^{(j)}$ with $F \subset H_D$ and only constant, singlet intrinsic torsion. Then there is a consistent truncation of the action* (2.1) *on M_D defined by expanding all bosonic fields in terms of invariant tensors.*

A key observation to understand this theorem is that invariant tensors are covariantly constant with respect to an appropriate $\mathrm{O}(D,D)$ covariant derivative ∇_A, i.e. $\nabla_A f^{(j)} = 0$ for all $f^{(j)}$. This derivative acts as a selector for degrees of freedom that are retained in the truncation. An important feature of this derivative is that since it obeys the Leibniz rule, the product of any two invariant tensors is again covariantly constant, and thus will be part of the truncation. Usually, the derivative ∇_I is not the generalised Levi-Civita connection $\overline{\nabla}_I$ from which the generalised curvature scalar in the action (2.12) is derived. Fortunately though, the two are related. To see how, we have to look closer at covariant derivatives in generalised geometry/double field theory.

Different generalised covariant derivatives differ by their generalised torsion. Similarly to the situation in standard geometry, the torsion is defined by comparing the generalised Lie derivative written with partial or covariant derivatives as in [16]:

$$\left(\mathcal{L}_U^{\nabla} - \mathcal{L}_U^{\partial}\right) V^I = U^J V^K T_{JK}{}^I \,. \tag{2.17}$$

To compute this quantity directly, we need an explicit definition of the covariant derivative:

$$\nabla_I E_A{}^J = \partial_I E_A{}^J - \Omega_{IA}{}^B E_B{}^J + \Gamma_{IK}{}^J E_A{}^K \,, \tag{2.18}$$

involving the generalised spin connection $\Omega_{IA}{}^B E_B{}^J$ and the generalised affine connection $\Gamma_{IJ}{}^K$. The two are related by the vielbein postulate $\nabla_I E_A{}^J = 0$, leading to

$$\Gamma_{IJK} = \partial_I E^A{}_J E_{AK} + \Omega_{IJK} \,. \tag{2.19}$$

With these definitions, the generalised torsion evaluates to

$$T_{IJK} = 3\Gamma_{[IJK]} \,. \tag{2.20}$$

Additionally, the generalised covariant derivative should at least satisfy two more constraints [16]:

1. Compatibility with the η-metric, $\nabla_I \eta_{JK} = 0$, implying

$$\Gamma_{IJK} = -\Gamma_{IKJ} \,. \tag{2.21}$$

2. Compatibility with the generalised metric, $\nabla_I \mathcal{H}_{JK} = 0$, implying

$$P_{(J}{}^M \overline{P}_{K)}{}^N \Gamma_{IMN} = \frac{1}{2} P_{(J}{}^M \overline{P}_{K)}{}^N \partial_I \mathcal{H}_{MN} \,. \tag{2.22}$$

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3. (optional) Compatibility with integration by parts $\int \mathrm{d}^D x\, e^{-2d} \nabla_I V^I = 0$, implying

$$\overline{\Gamma}^J{}_{JI} = 2\partial_I d\,. \tag{2.23}$$

Note that for the generalised Levi-Civita connection, point 3 applies, but it does not have to hold for ∇_I. In the definition of the generalised torsion (2.17), the generalised Lie derivative acts on a vector. We obtain a similar result for higher rank generalised tensors, where $T_{IJ}{}^K$ acts on each index individually. But there are also densities, like the generalised dilaton d. Hence, we introduce an additional torsion for them, too,

$$\left(\mathcal{L}_U^{\nabla} - \mathcal{L}_U^{\overline{\nabla}}\right) e^{-2d} = U^I T_I e^{-2d}\,, \tag{2.24}$$

which gives rise to

$$T_I = 2\partial_I d - \Gamma^J{}_{JI}\,. \tag{2.25}$$

As in conventional geometry, the generalised Levi-Civita connection, $\overline{\nabla}_I$, is defined to have vanishing torsion. Thus, we can deduce the relation between the generalised Christoffel symbols $\overline{\Gamma}_{IJK}$ of $\overline{\nabla}_I$ and Γ_{IJK} of ∇_I,

$$\begin{aligned}\Sigma_{IJK} &= \overline{\Gamma}_{IJK} - \Gamma_{IJK}\\ &= -\left(\frac{1}{3}PPP + \overline{P}PP + P\overline{PP} + \frac{1}{3}\overline{PPP}\right)_{IJK}{}^{LMN}\left(T_{LMN} + \frac{6}{D-1}\eta_{L[M}T_{N]}\right)\,.\end{aligned} \tag{2.26}$$

Note that Σ_{IJK} is not fully fixed by all constraints above. There are some undetermined components [16], which we have set to zero here. However, it is known that these components do not contribute to the two-derivative effective action (2.1) or its field equations. Therefore, it is safe to neglect them.

After this digression, we recognise that the action of the generalised Levi-Civita connection on the invariant tensors has to be of the form

$$\overline{\nabla}_I f^{(j)} = \Sigma_I \cdot f^{(j)}\,. \tag{2.27}$$

According to theorem 1, the torsions T_{IJK} and T_I of ∇, called the intrinsic torsion, have to be covariantly constant. From this observation and (2.26), it is obvious that

$$\nabla_I \Sigma_{JKL} = 0 \tag{2.28}$$

holds. Therefore, any tensor expression which is a function u of $f^{(j)}$ and their $\overline{\nabla}$-derivatives satisfies

$$\nabla_I u(f^{(j)}, \overline{\nabla}) = 0 \tag{2.29}$$

automatically. Still, this does not guarantee that the resulting tensor is contained in the set $f^{(j)}$ that forms the truncation. To overcome this problem, we note that all elements of this set are invariant under the action of the group F. Any function $u(f^{(j)}, \overline{\nabla})$ will share this property as long as the intrinsic torsion is a singlet (i.e. it is invariant) under the F action. In this case Σ_{IJK} is invariant because, due to $F \subset H_D$, the projectors in its definition are automatically invariant, too.

Let us summarise the ingredients of the construction implied by theorem 1 and fix some notation for later. For a consistent truncation we need:

1. The set of all invariant tensors $\{f^{(j)}\}$ which satisfy

$$\nabla_I f^{(j)} = 0 \quad \text{and} \quad \nabla_\alpha f^{(j)} = 0\,. \tag{2.30}$$

   Here ∇_α denotes the infinitesimal action by generators $t_\alpha \in \mathrm{Lie}(F)$ of the structure group F. The reason why we use this particular notation, will become clear in the next section.

2. Constant, singlet intrinsic torsion that implies

$$\nabla_I T_{JKL} = 0\,, \quad \nabla_I T_J = 0 \qquad \text{and} \qquad \nabla_\alpha T_{IJK} = 0\,, \quad \nabla_\alpha T_I = 0\,. \tag{2.31}$$

2.3 The truncated theory

At this point, we can say more about the truncated theory. In general, the consistent truncation still has an infinite number of degrees of freedom. To accommodate them, the D-dimensional target space is split into two parts

$$M_D = M_{D-n} \times M_n\,, \tag{2.32}$$

comprising an external manifold M_{D-n} of dimension $D-n$ where no truncation is performed, and the internal manifold M_n that hosts the invariant tensors discussed above. Accordingly, we split the coordinates, namely x^μ on M_{D-n} and y^i on M_n. On the external space, we do not require generalised geometry because no truncation takes place here. Therefore, we do not use doubled indices on M_{D-n}. Only on the internal space M_n, we have the $\mathrm{O}(n,n)$-invariant pairing η_{IJ}, a generalised metric $\mathcal{H}_{IJ}$, and all the other objects introduced above. In this setup the metric, B-field and dilaton on the full space split into five contributions:

- external metric $g_{\mu\nu}(x)\,,$
- external B-field $B_{\mu\nu}(x)\,,$
- dilaton $\phi(x,y)\,,$
- gauge connection $A_\mu{}^I(x,y)\,,$ and
- scalar field $\mathcal{H}_{IJ}(x,y)\,.$

All shall be understood as fields in the truncated theory with their y-dependence totally fixed by the truncation ansatz.

The action (2.1) can be rewritten in terms of these fields by using a Kaluza-Klein ansatz. This rewriting is cumbersome, but can be found in different papers, e.g. [28, 29, 34]. Here, we start from the result of [34], because it is fully general and does not commit to any specific ansatz on the internal manifold yet:

$$\begin{aligned} S = \int \mathrm{d}^{(D-n)}x\, \mathrm{d}^n y \sqrt{g} e^{-2\phi} \Big(& \overline{R} + \mathcal{D}_\mu \phi \mathcal{D}^\mu \phi - \frac{1}{12} \mathcal{H}_{\mu\nu\rho} \mathcal{H}^{\mu\nu\rho} \\ & + \frac{1}{8} \mathcal{D}_\mu \mathcal{H}_{IJ} \mathcal{D}^\mu \mathcal{H}^{IJ} - \frac{1}{4} \mathcal{H}_{IJ} \mathcal{F}_{\mu\nu}{}^I \mathcal{F}^{\mu\nu J} - V \Big)\,. \end{aligned} \tag{2.33}$$

In the following, we will refine this action by using all the insights from the previous subsection. But before doing so, let us discuss the new objects that appear in (2.33). $\overline{R}$ is the curvature scalar for the external metric. Furthermore, the truncated theory is a gauge theory coupled to (super)gravity, normally referred to as a gauged (super)gravity. It comes with the gauge covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - \mathcal{L}_{A_\mu} \tag{2.34}$$

that incorporates the connection one-form ${A_\mu}^I$. Like in Yang-Mills theory, one has to add a kinetic term containing the corresponding 2-form field strength tensor

$${\mathcal{F}_{\mu\nu}}^I = 2\partial_{[\mu}{A_{\nu]}}^I - [A_\mu, A_\nu]^I_{\mathrm{C}} \tag{2.35}$$

to the action. Instead of a Lie bracket, it employs the Courant bracket

$$[U, V]_{\mathrm{C}} = \frac{1}{2}(\mathcal{L}_U V - \mathcal{L}_V U)\,. \tag{2.36}$$

This is because we have not yet performed the truncation. Later, we will see that this bracket turns into a Lie bracket after the truncation, as is required for a gauge theory. Moreover, the 3-form H_{ijk} in the original action has to be complemented by a Chern-Simons term giving rise to

$$\mathcal{H}_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} + 3{A_{[\mu}}^I\partial_\nu A_{\rho]I} - A_{[\mu|I|}[A_\nu, A_{\rho]}]^I_{\mathrm{C}}\,. \tag{2.37}$$

Finally, we have the scalar potential

$$V = -\mathcal{R}\,. \tag{2.38}$$

It is expressed in terms of the generalised Ricci scalar (2.13) *on the internal manifold* M_n. For completeness, we note that gauge transformations of the connection 1-form ${A_\mu}^I$ are mediated by

$$\delta_\Lambda {A_\mu}^I = \partial_\mu \Lambda^I + \mathcal{L}_\Lambda {A_\mu}^I\,. \tag{2.39}$$

For all the covariant fields, the generalised Lie derivative mediates gauge transformation, namely

$$\delta_\Lambda = \mathcal{L}_\Lambda\,. \tag{2.40}$$

At this stage, there are still an infinite number of scalars $\mathcal{H}_{IJ}$ and vectors ${A_\mu}^I$. But, as suggested above, we should expand them in terms of tensors that are invariant under the structure group in order to obtain a consistent truncation. Let us start with the generalised metric

$$\mathcal{H}_{IJ}(x,y) = {\widetilde{E}^A}_I(y) h_{AB}(x) {\widetilde{E}^B}_J(y)\,, \tag{2.41}$$

and adopt the notation that quantities which depend only on y are decorated with a tilde. In contrast to section 2.1, the generalised metric h_{AB} here is not restricted to the diagonal form given in (2.6). Rather, it is an element of the coset $\mathrm{O}(n,n)/H_n$ and it thereby captures the scalar moduli space of the truncated theory. However, this choice has to be refined

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because it is not automatically invariant under the action of the structure group F. Thus, we restrict h_{AB} to the coset

$$h_{AB}(x) \in \frac{C_{\mathrm{O}(n,n)}(F)}{C_{H_n}(F)} \tag{2.42}$$

where $C_G(H)$ denotes all element of the Lie group G that commute with all element of H. Similarly, the vectors of the truncated theory are formed from all n_{v} F-invariant vectors $\widetilde{K}_{\dot\alpha}{}^I(y)$, with $\dot\alpha = 1, \dots, n_{\mathrm{v}}$,

$$A_\mu{}^I(x,y) = A_\mu{}^{\dot\alpha}(x)\widetilde{K}_{\dot\alpha}{}^I(y)\,. \tag{2.43}$$

Thanks to the analysis in the last section, it is not hard to see how these vectors act on other invariant tensors through the generalised Lie derivative.[1] In particular, we find

$$\mathcal{L}_{\widetilde{K}_{\dot\alpha}}\widetilde{K}_{\dot\beta}{}^I = (\widetilde{T}_{\dot\alpha})_J{}^I\widetilde{K}_{\dot\beta}{}^J \tag{2.44}$$

$$\mathcal{L}_{\widetilde{K}_{\dot\beta}}\mathcal{H}_{IJ} = 2(\widetilde{T}_{\dot\alpha})_{(I}{}^K\mathcal{H}_{J)K} \tag{2.45}$$

with

$$(\widetilde{T}_{\dot\alpha})_I{}^J = -\widetilde{K}_{\dot\alpha}{}^K\widetilde{T}_{KI}{}^J\,. \tag{2.46}$$

Here, $\widetilde{T}_{IJK}$ is the generalised torsion from (2.20). We only decorated it with a tilde to emphasis that it depends only on the internal coordinates y^i. One can interpret $(\widetilde{T}_{\dot\alpha})_I{}^J$ as $2n \times 2n$-matrices. They are elements of the Lie algebra $\mathrm{Lie}[C_{\mathrm{O}(n,n)}(F)]$ and generate the gauge group $G \subset C_{\mathrm{O}(n,n)}(F)$ [23]. The corresponding Lie algebra has the structure *constants*

$$[\widetilde{T}_{\dot\alpha}, \widetilde{T}_{\dot\beta}] = f_{\dot\alpha\dot\beta}{}^{\dot\gamma}\widetilde{T}_{\dot\gamma}\,. \tag{2.47}$$

This result is quite remarkable, because it tells us that the torsion T_{IJK} controls how the gauge group is embedded in the global symmetry group of the truncated theory. It plays the role of the embedding tensor, which is known from truncations that preserve maximal or half-maximal supersymmetry (see [35, 36] for reviews).

Eventually, we want to get rid of the internal manifold's coordinate dependence. To this end, we note that $K_{\dot\alpha}{}^A = \widetilde{K}_{\dot\alpha}{}^I\widetilde{E}^A{}_I$ in flat indices is constant. The same of course also holds for the torsion T_{ABC} and T_A. Thus, we define the constant tensors

$$\eta_{\dot\alpha\dot\beta} = K_{\dot\alpha}{}^A K_{\dot\beta}{}^B\eta_{AB}\,, \tag{2.48}$$

$$h_{\dot\alpha\dot\beta} = K_{\dot\alpha}{}^A K_{\dot\beta}{}^B h_{AB}(x)\,, \quad \text{and} \tag{2.49}$$

$$(T_{\dot\alpha})_A{}^B = \widetilde{K}_{\dot\alpha}{}^I T_{IJ}{}^K\widetilde{E}_A{}^J\widetilde{E}^B{}_K\,. \tag{2.50}$$

Note that $\eta_{\dot\alpha\dot\beta}$ is invariant under the action of the gauge group, and we use it to raise and lower dotted Greek indices. Moreover, we have to deal with the dilaton. It decomposes into

$$\phi(x,y) = \overline{\phi}(x) + \widetilde{d}(y)\,. \tag{2.51}$$

[1]The trick here is to write the generalised Lie derivative in terms of the covariant derivative ∇_A and keep in mind that it annihilates all F-invariant tensors on the internal manifold. Thus, the only non-vanishing contribution comes from the torsion tensor.

With these definitions in place, we can eventually restrict the action (2.33) to M_{D-n},

$$S = V_{\rm int} \int d^{(D-n)}x \sqrt{g} e^{-2\overline{\phi}} \Big(\overline{R} + 4\mathcal{D}_\mu \overline{\phi} \mathcal{D}^\mu \overline{\phi} - \frac{1}{12} \mathcal{H}_{\mu\nu\rho} \mathcal{H}^{\mu\nu\rho} \\ \frac{1}{8} \mathcal{D}_\mu h_{AB} \mathcal{D}^\mu h^{AB} - \frac{1}{4} h_{\dot\alpha\dot\beta} \mathcal{F}_{\mu\nu}{}^{\dot\alpha} \mathcal{F}^{\mu\nu\dot\beta} - V \Big) . \tag{2.52}$$

This reduced action employs the covariant derivative and field strengths

$$\mathcal{D}_\mu \overline{\phi} = \partial_\mu \overline{\phi} - \frac{1}{2} A_\mu \,, \tag{2.53}$$

$$\mathcal{D}_\mu h_{AB} = \partial_\mu h_{AB} - 2 A_\mu{}^{\dot\alpha} (T_{\dot\alpha})_{(A}{}^C h_{B)C} \,, \tag{2.54}$$

$$\mathcal{F}_{\mu\nu}{}^{\dot\alpha} = 2\partial_{[\mu} A_{\nu]}{}^{\dot\alpha} - f_{\dot\beta\dot\gamma}{}^{\dot\alpha} A_\mu{}^{\dot\beta} A_\nu{}^{\dot\gamma} \,, \qquad \text{and} \tag{2.55}$$

$$\mathcal{H}_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]} + 3 A_{[\mu}{}^{\dot\alpha} \partial_\nu A_{\rho]\dot\alpha} - f_{\dot\alpha\dot\beta\dot\gamma} A_\mu{}^{\dot\alpha} A_\nu{}^{\dot\beta} A_\rho{}^{\dot\gamma} \,, \tag{2.56}$$

and the gauge transformations

$$\delta_\Lambda A_\mu{}^{\dot\alpha} = \partial_\mu A_\mu{}^{\dot\alpha} + \Lambda^{\dot\beta} A_\mu{}^{\dot\gamma} f_{\dot\beta\dot\gamma}{}^{\dot\alpha} \,, \tag{2.57}$$

$$\delta_\Lambda \overline{\phi} = \frac{1}{2} \Lambda^{\dot\alpha} T_{\dot\alpha} \,, \tag{2.58}$$

$$\delta_\Lambda h_{AB} = 2\Lambda^{\dot\alpha} (T_{\dot\alpha})_{(A}{}^C h_{B)C} \,. \tag{2.59}$$

Two new quantities,

$$A_\mu(x) = A_\mu{}^{\dot\alpha}(x) T_{\dot\alpha}, \quad \text{and} \quad T_{\dot\alpha} = K_{\dot\alpha}{}^A T_A \,, \tag{2.60}$$

have appeared here. As intended, neither depends on the internal coordinates y because the torsion T_A is constant by theorem 1. There is only one imprint of the manifold M_d left, namely its generalised volume

$$V_{\rm int} = \int \mathrm{d}^n y \, e^{-2\tilde{d}}, \tag{2.61}$$

which appears as an overall prefactor.

Before, we turn to an example, let us summarise the salient features of all truncated theories that arise from theorem 1 in generalised geometry:

- They are gauged (super)gravities in dimensions $D < 10$ for the superstring or $D < 26$ for the bosonic string.
- Their field content and gauge group are completely fixed by the constant, singlet intrinsic torsion T_{ABC} and T_A.
- The only part of the action that is not so easily fixed is the scalar potential V. It requires detailed knowledge about the geometry of the internal manifold and its dependence on the scalar moduli fields.

However, the scalar potential is central for most applications. That is the reason why a particular subclass of consistent truncations, called generalised Scherk-Schwarz reductions [28, 29], currently dominates most applications.

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2.4 Generalised Scherk-Schwarz reductions and Poisson-Lie T-duality

Remarkably, these reductions are directly related to the second central pillar of this paper, dualities. In this subsection, we explain how.

Consistent truncation perspective. Generalised Scherk-Schwarz reductions implement a special case of theorem 1 with trivial structure group F. Therefore, any tensor in flat indices which is annihilated by the covariant derivative ∇_I forms part of the truncation. In particular, we do not have to deal with ∇_α, introduced in (2.30) and (2.31). Moreover, the spin connection $\Omega_{IA}{}^B$ in (2.18) vanishes.

This situation is similar to a group manifold in standard geometry, where one can always introduce a flat derivative without curvature. The only difference is that in generalised geometry instead of the Riemann tensor, the generalised Riemann tensor [37–39]

$$\mathcal{R}_{IJKL} = 2\partial_{[I}\Gamma_{J]KL} + 2\Gamma_{[I|ML}\Gamma_{|J]K}{}^M + \frac{1}{2}\Gamma_{MIJ}\Gamma^M{}_{KL} + (IJ) \leftrightarrow (KL)\,, \tag{2.62}$$

or in flat indices after using (2.19)

$$\begin{aligned}\mathcal{R}_{ABCD} = 2E_{[A}{}^I\partial_I\Omega_{B]CD} + 2\Omega_{[A|C}{}^E\Omega_{B]DE} + \frac{1}{2}\Omega_{EAB}\Omega^E{}_{CD} \\ - F_{AB}{}^E\Omega_{ECD} + (AB) \leftrightarrow (CD)\,,\end{aligned} \tag{2.63}$$

has to vanish for trivial F. Taking into account the vanishing spin connection, we find the generalised affine connection

$$\Gamma_{IJK} = \partial_I E^A{}_J E_{AK} \tag{2.64}$$

by using (2.19). From it, we next compute the generalised torsions from (2.20),

$$T_{ABC} = -F_{ABC}\,, \tag{2.65}$$

and from (2.25),

$$T_A = F_A\,. \tag{2.66}$$

Hence, the conditions for consistent truncations become

$$F_{ABC} = \text{const.} \qquad \text{and} \qquad F_A = \text{const.}\,, \tag{2.67}$$

because the covariant derivative ∇_A acting on quantities with just flat indices reduces to $D_A = E_A{}^I\partial_I$.

At this point, it is instructive to look at the Bianchi identities for ∇_A. They reduce to [33]

$$D_{[A}F_{BCD]} - \frac{3}{4}F_{[AB}{}^E F_{CD]E} = 0\,, \quad \text{and} \tag{2.68}$$

$$D_{[A}F_{B]} + \frac{1}{2}D^C F_{CAB} - \frac{1}{2}F^C F_{CAB} = 0\,. \tag{2.69}$$

Because of (2.67), all terms with derivatives D_A drop out. Consequentially, (2.68) becomes the Jacobi identity of a Lie algebra, Lie($\mathbb{D}$). Assume that this Lie algebra has the generators t_A, satisfying

$$[t_A, t_B] = F_{AB}{}^C t_C\,. \tag{2.70}$$

Furthermore, (2.69) states that $t_{\rm F} = F^A t_A$ has to be in the center of Lie($\mathbb{D}$), saying that the generator $t_{\rm F}$ commutes with all other elements of the Lie algebra.

Another effect of a trivial generalised structure group is that we can identify the index $\dot{\alpha}$, enumerating invariant constant vectors, with the O(D,D) index A, resulting in $K_A{}^B = \delta_A^B$. Hence, the Lie group $\mathbb{D}$ has a natural interpretation as the gauge group of the truncated gauged (super)gravity. In the same vein, we identify the scalar manifold as $O(n,n)/H_n$. Finally, one can directly read off the scalar potential[2]

$$V = F_{ACE} F_{BDF} \left(\frac{1}{12} h^{AB} h^{CD} h^{EF} - \frac{1}{4} h^{AB} \eta^{CD} \eta^{EF} \right) + F_A F_B h^{AB} \tag{2.71}$$

from (2.13) by dropping all terms with a derivative and expanding the projectors $P^{AB} = \frac{1}{2}(\eta^{AB} + h^{AB})$ and $\overline{P}^{AB} = \frac{1}{2}(\eta^{AB} - h^{AB})$.

Summarising, the truncation ansatz for a consistent truncation with a trivial structure group is built from following data:

1. A doubled Lie group $\mathbb{D}$ (=gauge group), generated by t_A
   (a) with the structure coefficients $F_{AB}{}^C$ and
   (b) and an elements in the center $t_{\rm F} = F^A t_A$ ($t_{\rm F} = 0$ always works).
2. $\mathbb{D}$ has to be a subgroup of O(n,n). Otherwise, its adjoint action would not leave η_{AB} invariant and consequently, $F_{ABC} = F_{AB}{}^D \eta_{DC}$ would only be antisymmetric with respect to the first two indices A and B, but it has to be totally antisymmetric. Therefore, $F_{AB}{}^C$ actually describes how $\mathbb{D}$ is embedded into O(n,n), and is called the embedding tensor.
3. A constant generalised metric h_{AB} on the internal manifold, to construct the projectors P^{AB} and $\overline{P}^{AB}$.

Generalised T-duality perspective. Intriguingly, exactly the same data are needed to describe a Poisson-Lie symmetric target space in the $\mathcal{E}$-model formalism [10, 40]. More precisely, one needs the following ingredients to construct an $\mathcal{E}$-model:

1. A doubled Lie group $\mathbb{D}$, generated by t_A
   (a) with the structure coefficients $F_{AB}{}^C$.
   (b) there is no item (b).
2. A non-degenerate pairing $\langle t_A, t_B \rangle = \eta_{AB}$, that is invariant under the adjoint action of $\mathbb{D}$.
3. An $\mathcal{E}$-operator $\mathcal{E} : \text{Lie}(\mathbb{D}) \to \text{Lie}(\mathbb{D})$, which squares to the identity. In the language we use here, this is just the generalised metric $\mathcal{E}_A{}^B = \mathcal{H}_A{}^B$.

[2] We drop the contribution

$$F_A F^A - \frac{1}{6} F_{ABC} F^{ABC}$$

because it vanishes under the section condition, we always impose in this paper.

The underlying classical σ-model

$$S_\Sigma = \frac{1}{4\pi\alpha'} \int_\Sigma \left(g_{ij} \mathrm{d}x^i \wedge \star \mathrm{d}x^j + B_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j \right) \tag{2.72}$$

does not incorporate the dilaton, therefore item 1b is not contained in this list. To define the $\mathcal{E}$-model, one first transitions to the Hamiltonian formalism with the Hamiltonian [41]

$$H = \frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma \mathcal{J}^M \mathcal{H}_{MN} \mathcal{J}^N \,. \tag{2.73}$$

In addition to the generalised metric in (2.4), we find the generalised currents

$$\mathcal{J}_M = \begin{pmatrix} p_m & \partial_\sigma x^m \end{pmatrix} \tag{2.74}$$

defined by using the embedding coordinates x^m and their canonical momenta

$$p_m = g_{mn} \partial_\tau x^n + B_{mn} \partial_\sigma x^n \,. \tag{2.75}$$

Taking into account canonical, equal-time Poisson brackets $\{x_m(\sigma), p^n(\sigma')\} = \delta_m^n \delta(\sigma - \sigma')$, one obtains

$$\{\mathcal{J}^M(\sigma), \mathcal{J}^N(\sigma')\} = 2\pi\alpha' \delta'(\sigma - \sigma') \eta^{MN} \tag{2.76}$$

which introduces η^{MN} from the worldsheet perspective. The $\mathcal{E}$-models arises after dressing these currents with the generalised frame $E^A{}_I$ to obtain the Kac-Moody algebra

$$\{\mathcal{J}^A(\sigma), \mathcal{J}^B(\sigma')\} = \delta(\sigma - \sigma') F^{AB}{}_C \mathcal{J}^C(\sigma) + \delta'(\sigma - \sigma') \eta^{AB} \tag{2.77}$$

with

$$\mathcal{J}^A = \frac{1}{\sqrt{2\pi\alpha'}} E^A{}_M \mathcal{J}^M \,. \tag{2.78}$$

As in our previous discussion, it is crucial here that F_{ABC}, η_{AB} and $\mathcal{H}_{AB}$ are all constant, resulting in the $\mathcal{E}$-model Hamiltonian

$$H = \frac{1}{2} \int \mathrm{d}\sigma \mathcal{J}^A \mathcal{H}_{AB} \mathcal{J}^B \tag{2.79}$$

that is quadratic in the generalised currents and therefore results in the simple equations of motion

$$\mathrm{d}\mathcal{J} + \frac{1}{2}[\mathcal{J}, \mathcal{J}] = 0 \tag{2.80}$$

with the Lie($\mathbb{D}$)-valued, worldsheet one-forms

$$\mathcal{J} = t_A \left(\mathcal{E}^A{}_B \mathcal{J}^B \mathrm{d}\tau + \mathcal{J}^A \mathrm{d}\sigma \right) \,. \tag{2.81}$$

Poisson-Lie T-duality relates different choices for the metric g_{ij} and the B-field B_{ij} in (2.72) by a canonical transformation. How this exactly works is best seen in the Hamiltonian formalism where the canonical transformation leaves the Poisson brackets and the Hamiltonian invariant, but changes the composition of the currents $\mathcal{J}^M$ in terms of the fundamental field x^n. Consequently, there is in general not just one choice of the generalised frame $E^A{}_I$ that results in some fixed structure constants $F_{AB}{}^C$, but multiple ones.

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There is an important lesson to be learned from this new perspective: the truncation ansatz, which is fixed by the same generalised frame ${E_A}^I$ as the currents $\mathcal{J}^A$ in the $\mathcal{E}$-model, is in general not unique. Instead one can find for every maximally isotropic subgroup H_i of $\mathbb{D}$ a generalised frame on the coset $M^{(i)} = H_i \backslash \mathbb{D}$ that results in the same generalised fluxes F_{ABC} [24, 25, 42]. All of them are connected by Poisson-Lie T-duality. We will discuss the details of the construction of generalised frames in section 4.

It is known that gauged $\mathcal{E}$-models give rise to an even broader notion of T-duality [43], called dressing cosets [12]. There are hints that they are also closely related to consistent truncations [15]. In the rest of this paper, we follow these hints and eventually show that generalised cosets provide a very large class of new consistent truncations for which the scalar potential can be computed.

3 The Poláček-Siegel construction

In the last section, we have identified the generalised structure group F and the singlet, intrinsic torsion as the fundamental building blocks of consistent truncations. We will now present a construction of the associated generalised frames ${E_A}^I$ and spin connection ${\Omega_{AB}}^C$, which treats them as first class citizens. The basic idea for our approach first came up in the paper [31], and we therefore refer to it as the Poláček-Siegel construction. In its original form, it was restricted to the case $F = H_n$, which is not of much use for the application to consistent truncations. Fortunately, one of the authors extended the discussion to general F's in ref. [44]. We shall review the construction in the following, and adapt it to our conventions before applying it to truncations.

3.1 Generalised frame on the mega-space

First, we define generators $t_\alpha \in \mathrm{Lie}(F)$ that generate the generalised structure group and that are governed by the commutators

$$[t_\alpha, t_\beta] = {f_{\alpha\beta}}^\gamma t_\gamma \,. \tag{3.1}$$

Next, we introduce the auxiliary coordinates z^μ to parameterise group elements $f(z^\mu) \in F$. In combination with the coordinates y^i on the internal manifold M_n, they describe what we call mega-space. It is important to keep in mind that the mega-space is not physical. It is, rather, a useful book-keeping device, as will become clear by the end of this section.

In order to make contact with the discussion in the last section, we have to fix at least two quantities on the mega-space, namely the generalised frame and the η-metric. For the former, we will use the parameterisation

$$\widehat{E}_{\widehat{A}}{}^{\widehat{I}} = \widetilde{M}_{\widehat{A}}{}^{\widehat{B}} \begin{pmatrix} {\delta_\beta}^\gamma & 0 & 0 \\ -{\Omega^\gamma}_B & {E_B}^J & 0 \\ \rho^{\beta\gamma} - \frac{1}{2}{\Omega^\beta}_K \Omega^{\gamma K} & \Omega^{\beta J} & {\delta^\beta}_\gamma \end{pmatrix} \begin{pmatrix} {\widehat{\widetilde{v}}_\gamma}{}^\mu & 0 & 0 \\ 0 & \delta^I_J & 0 \\ 0 & 0 & {\widetilde{v}^\gamma}{}_\mu \end{pmatrix} . \tag{3.2}$$

Before motivating it, let us take a moment to explain the new conventions we encounter here: indices come in two kinds: flat $\widehat{A}$, $\widehat{B}$, ... and curved $\widehat{I}$, $\widehat{J}$, ... The latter split as

$x^{\widehat{I}} = \begin{pmatrix} z^\mu & y^i & \widetilde{y}_i & \widetilde{z}_\mu \end{pmatrix}$, $x_{\widehat{I}} = \begin{pmatrix} \widetilde{z}_\mu & \widetilde{y}_i & y^i & z^\mu \end{pmatrix}$. Note that the section condition is still trivially solved, because nothing depends on the coordinates $\widetilde{y}_i$ and $\widetilde{z}_\mu$ and we use the partial derivative $\partial_{\widehat{I}} = \begin{pmatrix} \partial_\mu & \partial_i & 0 & 0 \end{pmatrix}$. Moreover, the $\mathrm{O}(n+\dim F, n+\dim F)$ invariant metric $\eta_{\widehat{I}\widehat{J}}$ is given by

$$\eta_{\widehat{I}\widehat{J}} = \begin{pmatrix} 0 & 0 & \delta_\mu^\nu \\ 0 & \eta_{IJ} & 0 \\ \delta_\nu^\mu & 0 & 0 \end{pmatrix} \tag{3.3}$$

where η_{IJ} is already known from (2.3). Flat indices behave in the same way. In particular, we encounter the flat η-metric

$$\eta_{\widehat{A}\widehat{B}} = \begin{pmatrix} 0 & 0 & \delta_\alpha^\beta \\ 0 & \eta_{AB} & 0 \\ \delta_\beta^\alpha & 0 & 0 \end{pmatrix} . \tag{3.4}$$

With the index conventions established, we can say more about the form of (3.2). At first glance, it has three distinguishing features:

1. All physically relevant quantities are contained in the middle matrix.
2. This matrix is of lower triangular form.
3. It is dressed from left and right by two matrices which only depend on the auxiliary coordinates z^μ.

Note that we use here tilde quantities which depend on the auxiliary coordinates z^μ. $\widetilde{M}_{\widehat{A}}{}^{\widehat{B}}$ does not appear explicitly in either [31] or [44], but later it will be very helpful for understanding how the mega-generalised frame field relates to the covariant derivative ∇_A. This matrix mediates the adjoint action of F on the mega-space. Therefore, it has two defining properties:

$$\widetilde{M}_{\widehat{A}}{}^{\widehat{C}} \widetilde{M}_{\widehat{B}}{}^{\widehat{D}} \eta_{\widehat{C}\widehat{D}} = \eta_{\widehat{A}\widehat{B}} \tag{3.5}$$

and

$$\partial_\mu \widetilde{M}_{\widehat{A}}{}^{\widehat{B}} = \widetilde{v}^\alpha{}_\mu \widetilde{M}_{\widehat{A}}{}^{\widehat{C}} f_{\alpha\widehat{C}}{}^{\widehat{B}} . \tag{3.6}$$

Here $\widetilde{v}^\alpha{}_\mu$ denotes the components of the right-invariant one-forms $\mathrm{d}f f^{-1} = t_\alpha \widetilde{v}^\alpha{}_\mu \mathrm{d}z^\mu$ and $\widehat{\widetilde{v}}_\alpha{}^\mu$ are the dual vector fields, defined by $\widehat{\widetilde{v}}_\alpha{}^\mu \widetilde{v}^\beta{}_\mu = \delta_\alpha^\beta$. The infinitesimal action of F is specified by the constants $f_{\alpha\widehat{B}\widehat{C}}$. Due to (3.5), they have to satisfy $f_{\alpha\widehat{B}\widehat{C}} = -f_{\alpha\widehat{C}\widehat{B}}$. Furthermore, (3.6) should not spoil the lower triangular form of the middle matrix in (3.2) and therefore we find

$$f_{\alpha\beta}{}^C = 0 , \qquad \text{and} \qquad f_{\alpha\beta\gamma} = 0 . \tag{3.7}$$

Owing to these constraints, the full mega-generalised frame field is lower triangular. This observation motivates the parameterisation of the two right-most matrices in (3.2). Together, they implement the most general lower triangular matrix that leaves the η-metric invariant. Thereby, we encounter three fields on the physical space M_n:

- The generalised frame $E_A{}^I$ that we already discussed in the last section.

- An unconstrained field $\Omega^{\alpha}{}_{B}$ which, as we will see later, is directly related to the spin connection Ω_{ABC} of ∇_A.
- An antisymmetric tensor $\rho^{\alpha\beta} = -\rho^{\beta\alpha}$. It is the most interesting result of the Poláček-Siegel construction because it has no analog in standard geometry. Therefore, in honor of these two gentlemen, we call it the Poláček-Siegel field.

Finally, we come to the question: how is this setup related to the covariant derivative ∇_A? The relation becomes manifest when we split all tensors into a physical and an auxiliary part as, for example, in

$$\widehat{V}_{\widehat{A}} = \widetilde{M}_{\widehat{A}}{}^{\widehat{B}} V_{\widehat{B}} \,. \tag{3.8}$$

To indicate that $\widehat{V}_{\widehat{A}}$ depends on z^μ and y^i while $V_{\widehat{A}}$ depends only on y^i, we decorate the former with a hat. This is also the reason that the generalised frame field on the mega-space is called $\widehat{E}_{\widehat{A}}{}^{\widehat{I}}$ instead of just $E_{\widehat{A}}{}^{\widehat{I}}$. In the same vein, we introduce the flat derivative

$$\widehat{D}_{\widehat{A}} = \widehat{E}_{\widehat{A}}{}^{\widehat{I}} \partial_{\widehat{I}} = \widetilde{M}_{\widehat{A}}{}^{\widehat{B}} D_{\widehat{A}} \,, \qquad \text{with} \qquad D_{\widehat{A}} = E_{\widehat{A}}{}^{\widehat{I}} \partial_{\widehat{I}} \,. \tag{3.9}$$

A natural question at this point is: can get rid of the auxiliary coordinates z^μ completely? This would be the case if these coordinates always appeared only in the form of (3.8). There is only one place where this could go wrong, namely for flat derivatives. But fortunately, for them we find

$$\widehat{D}_{\widehat{A}} \widehat{V}_{\widehat{B}} = \widetilde{M}_{\widehat{A}}{}^{\widehat{C}} \widetilde{M}_{\widehat{B}}{}^{\widehat{D}} \left(D_{\widehat{C}} V_{\widehat{D}} + E_{\widehat{C}}{}^{\alpha} f_{\alpha\widehat{D}}{}^{\widehat{E}} V_{\widehat{E}} \right) \quad \text{with} \quad E_{\widehat{A}}{}^{\beta} = \begin{pmatrix} \delta_\alpha^\beta \\ -\Omega^{\beta}{}_{A} \\ \rho^{\alpha\beta} - \frac{1}{2} \Omega^{\alpha}{}_{I} \Omega^{\beta I} \end{pmatrix} . \tag{3.10}$$

Hence, we conclude that the z^μ-dependence of any quantity with flat indices arises only from the twist of each index with $\widetilde{M}_{\widehat{A}}{}^{\widehat{B}}$.

Looking more closely at (3.10), one can interpret the two terms in the brackets on the right hand side of the first equation as a covariant derivative. Thus, we write

$$\widehat{D}_{\widehat{A}} \widehat{V}_{\widehat{B}} = \widetilde{M}_{\widehat{A}}{}^{\widehat{C}} \widetilde{M}_{\widehat{B}}{}^{\widehat{D}} \nabla'_{\widehat{C}} V_{\widehat{D}} \tag{3.11}$$

for flat derivatives on the mega-space, and we see that it can alternatively be interpreted as a covariant derivative on M_n. Comparing (3.10) and (3.11), we find

$$\nabla'_A = E_A{}^I \partial_I - \Omega^{\beta}{}_{A} \nabla'_\beta \,, \tag{3.12}$$

$$\nabla'^{\alpha} = \Omega^{\alpha B} D_B + \left(\rho^{\alpha\beta} - \frac{1}{2} \Omega^{\alpha}{}_{I} \Omega^{\beta I} \right) \nabla'_\beta \,, \tag{3.13}$$

and furthermore

$$\nabla'_\alpha V_\beta = f_{\alpha\beta}{}^{\gamma} V_\gamma \,, \tag{3.14}$$

$$\nabla'_\alpha V_B = f_{\alpha B}{}^{C} V_C + f_{\alpha B}{}^{\gamma} V_\gamma \,, \tag{3.15}$$

$$\nabla'_\alpha V^\beta = -f_{\alpha\gamma}{}^{\beta} V^\gamma - f_{\alpha C}{}^{\beta} V^C + f_{\alpha}{}^{\beta\gamma} V_\gamma \,. \tag{3.16}$$

Most important for our purpose is to relate $\nabla'_{\widehat{A}}$ to the covariant derivative ∇_A and ∇_α that is required for constructing consistent truncations. More precisely, we want to identify

$$\nabla_A V_B = \nabla'_A V_B \,, \tag{3.17}$$

or equally

$$\Omega_{IBC} = \Omega^\delta{}_I f_{\delta BC} \,. \tag{3.18}$$

Hence, we impose

$$f_{\alpha B}{}^\gamma = 0 \,. \tag{3.19}$$

This condition can be relaxed in the context of generalised T-dualities.[3] However, for the consistent truncations we study here, it arises naturally. Thus, we shall keep the $\cancel{f_{\alpha B}{}^\gamma}$-terms in intermediate results and only remove them in the final expression on M_n.

Finally, we also need to know how the generalised structure group F acts on the physical generalised tangent space $TM_n \oplus T^*M_n$. Remember, the corresponding infinitesimal action is mediated by ∇_α, introduced in (2.30). Thus, it is natural to relate

$$\nabla_\alpha V_B := f_{\alpha B}{}^C V_C = \nabla'_\alpha V_B \,, \tag{3.20}$$

too. This also explains the initially arbitrary looking notation for this operation. On the mega-space it has exactly the same origin as the covariant derivative ∇_A.

3.2 Torsion and curvature

We have seen that the Poláček-Siegel construction transforms flat derivatives $\widehat{D}_{\widehat{A}}$ on the mega-space into covariant derivatives on the physical space M_n. From a conceptual point of view, one might say that it geometrises a generalised connection by adding auxiliary coordinates z^μ. Therefore, analysing the properties of the derivatives $\widehat{D}_{\widehat{A}}$ becomes the main objective for this subsection. Fortunately, we already learned in section 2.4 that they are exclusively controlled by the torsions (see (2.8) and (2.11))

$$\widehat{f}_{\widehat{A}\widehat{B}\widehat{C}} = \widehat{D}_{[\widehat{A}} \widehat{E}_{\widehat{B}}{}^{\widehat{I}} \widehat{E}_{\widehat{C}]\widehat{I}} \,, \quad \text{and} \tag{3.21}$$

$$\widehat{f}_{\widehat{A}} = 2\widehat{D}_{\widehat{A}} \widehat{d} - \partial_{\widehat{I}} \widehat{E}_{\widehat{A}}{}^{\widehat{I}} \,. \tag{3.22}$$

Here, we switched from capital F's to f's, because we want to reserve the F for the physical space M_n. At the moment this might seem arbitrary, but it will become obvious shortly. Next, we will evaluate (3.21). As already discussed, it is convenient to strip off $\widetilde{M}_{\widehat{A}}{}^{\widehat{B}}$ from generalised tensors, as we did in (3.8) while going from $\widehat{V}_{\widehat{A}}$ to $V_{\widehat{A}}$. The generalised frame field in (3.2) is no exception. Therefore, we split it according to

$$\widehat{E}_{\widehat{A}}{}^{\widehat{I}} = \widetilde{M}_{\widehat{A}}{}^{\widehat{B}} \overline{E}_{\widehat{B}}{}^{\widehat{C}} \mathcal{V}_{\widehat{C}}{}^{\widehat{I}} \qquad \text{with} \qquad \mathcal{V}_{\widehat{A}}{}^{\widehat{I}} = \begin{pmatrix} \widehat{\widetilde{v}}_\alpha{}^\mu & 0 & 0 \\ 0 & E_A{}^I & 0 \\ 0 & 0 & \widetilde{v}^\alpha{}_\mu \end{pmatrix} \,, \tag{3.23}$$

[3] We thank Yuho Sakatani for pointing out this possibility to us.

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and first compute

$$3\mathcal{V}_{[\widehat{A}}{}^{\widehat{I}}\partial_{\widehat{I}}\mathcal{V}_{\widehat{B}}{}^{\widehat{J}}\mathcal{V}_{\widehat{C}]\widehat{J}} = \begin{cases} f_{\alpha\beta}{}^{\gamma} \text{ and cyclic} \\ F_{ABC}\,. \end{cases} \tag{3.24}$$

The latter is then used to obtain

$$f_{\widehat{A}\widehat{B}\widehat{C}} = 3\overline{E}_{\widehat{A}}{}^{\widehat{D}}\overline{E}_{\widehat{B}}{}^{\widehat{E}}\overline{E}_{\widehat{C}}{}^{\widehat{F}}\mathcal{V}_{[\widehat{D}}{}^{\widehat{I}}\partial_{\widehat{I}}\mathcal{V}_{\widehat{E}}{}^{\widehat{J}}\mathcal{V}_{\widehat{F}]\widehat{J}} + 3\nabla'_{[\widehat{A}}\overline{E}_{\widehat{B}}{}^{\widehat{I}}\overline{E}_{\widehat{C}]\widehat{I}}\,. \tag{3.25}$$

Note that ∇'_A here just acts on the flat index $\widehat{B}$ of the generalised frame field $\overline{E}_{\widehat{B}}{}^{\widehat{I}}$. There is no affine connection fixed by a vielbein postulate. Before we can turn to $\widehat{f}_{\widehat{A}}$, we have to specify how the generalised dilaton on the mega-space depends on the auxiliary coordinates z^μ. It turns out that the right choice is

$$\widehat{d}(y,z) = d(y) - \frac{1}{2}\log\det\widetilde{e}(z) \tag{3.26}$$

with

$$t_\alpha \widetilde{e}^{\alpha}{}_{\mu}(z)\mathrm{d}z^\mu = f^{-1}\mathrm{d}f\,. \tag{3.27}$$

Only for this choice does the z^μ-dependence of $\widehat{f}_{\widehat{A}}$ completely factor into a $\widetilde{M}_{\widehat{A}}{}^{\widehat{B}}$ twist, as we required in (3.8). After removing this twist, we find

$$f_{\widehat{A}} = 2\overline{E}_{\widehat{A}}{}^{B}D_B d - \partial_I E_{\widehat{A}}{}^{I} - f_{\alpha\widehat{A}}{}^{\widehat{B}}\overline{E}_{\widehat{B}}{}^{\alpha}\,. \tag{3.28}$$

It is very instructive to compute the individual components of both $\widehat{f}_{\widehat{A}\widehat{B}\widehat{C}}$ and $\widehat{f}_{\widehat{A}}$. But before doing so, we need a way to keep track of all contributions. To this end, we introduce the ϵ-dimension, which is defined in the following way: assume we scale the generators of the generalised structure group F according to $t_\alpha \to \epsilon^{-1}t_\alpha$. In this case, the structure coefficients $f_{\alpha\beta}{}^{\gamma}$ introduced in (3.1) scale as $f_{\alpha\beta}{}^{\gamma} \to \epsilon^{-1}f_{\alpha\beta}{}^{\gamma}$. To find out how other tensors scale, assign -1 to each lowered Greek index, $+1$ to each raised Greek index and 0 to each Latin index. Summing over all indices of the tensor then gives its ϵ-dimension. The motivation for this particular scaling comes from an alternative approach in the literature to the construction of dressing cosets [45, 46]. It considers an $\mathcal{E}$-model on the mega-space where the generalised structure group F describes a global symmetry. After scaling the generators t_α as shown above, and sending ϵ to 0, the $\mathcal{E}$-model degenerates and the global symmetry becomes a gauge symmetry. This limit is subtle, but several examples suggest that the relevant quantities on the dressing coset are those which are invariant under the scaling or, equally, have vanishing ϵ-dimension.

To find all independent components of $\widehat{f}_{\widehat{A}\widehat{B}\widehat{C}}$, recall that it is by construction totally antisymmetric. For each of its components, we can therefore order the indices by their ϵ-dimension (of course still keeping track of the sign). The results for the ten independent classes of components are then given by

$$
\begin{array}{r|ll|l}
\epsilon\text{-dim.} & & & \\
-3 & f_{\alpha\beta\gamma}=0 & & \\
-2 & f_{\alpha\beta C}=0 & & \\
-1 & f_{\alpha\beta}{}^{\gamma} & f_{\alpha AB} & \\
0 & f_{\alpha B}{}^{\gamma}=0 & & f_{ABC} \\
+1 & f_{\alpha}{}^{\beta\gamma} & & f_{AB}{}^{\gamma} \\
+2 & & & f_{A}{}^{\beta\gamma} \\
+3 & & & f^{\alpha\beta\gamma}\,.
\end{array}
\tag{3.29}
$$

All the components in the first column are just the parts of $f_{\alpha\widehat{B}}{}^{\widehat{C}}$ that describe the F-action on the mega-space (see (3.6)). They are constant, and constrained by the Jacobi identity

$$2 f_{[\alpha|\widehat{C}}{}^{\widehat{E}} f_{|\beta]\widehat{E}}{}^{\widehat{D}} = -f_{\alpha\beta}{}^{\gamma} f_{\gamma\widehat{C}}{}^{\widehat{D}}\,, \tag{3.30}$$

which arises from $\mathrm{d}^2 M_{\widehat{A}}{}^{\widehat{B}}=0$ and (3.6), because infinitesimal F-transformations have to close into a group. Non-constant contributions only arise from the components in the second column. In the following, we compute them one by one.

First, we look at

$$f_{ABC} = F_{ABC} - 3\Omega^{\delta}{}_{[A} f_{\delta BC]} = F_{ABC} - 3\Omega_{[ABC]}\,. \tag{3.31}$$

Comparing this result with (2.19) and (2.20), we find the remarkable identification

$$f_{ABC} = -T_{ABC}\,. \tag{3.32}$$

On the other hand, (2.31) requires that T_{BCD} is annihilated by ∇_A and ∇_α for theorem 1 to apply. This condition can now be written as

$$\nabla_A f_{BCD} = \nabla'_A f_{BCD} = 0 \quad \text{and} \tag{3.33}$$

$$\nabla_\alpha f_{BCD} = \nabla'_\alpha f_{BCD} = 0 \tag{3.34}$$

by using (3.17) and (3.20). Due to (3.12), these two equations automatically imply

$$f_{ABC} = \text{const.} \tag{3.35}$$

for all consistent truncations resulting from theorem 1. Next in line is

$$f_{AB}{}^{\gamma} = -2 D_{[A}\Omega^{\gamma}{}_{B]} + f_{\alpha\beta}{}^{\gamma}\Omega^{\alpha}{}_{A}\Omega^{\beta}{}_{B} - \frac{1}{2} f_{\alpha AB}\Omega^{\alpha}{}_{C}\Omega^{\gamma C} + \cancel{2 f_{\alpha[A}{}^{\gamma}\Omega^{\alpha}{}_{B]}} - f_{\alpha AB}\rho^{\alpha\gamma} + F_{AB}{}^{C}\Omega^{\gamma}{}_{C}\,. \tag{3.36}$$

To find an interpretation for this quantity, we first get rid of the Greek index γ by contracting with $f_{\gamma CD}$, resulting in the new quantity

$$f_{ABCD} := f_{AB}{}^{\gamma} f_{\gamma CD}\,. \tag{3.37}$$

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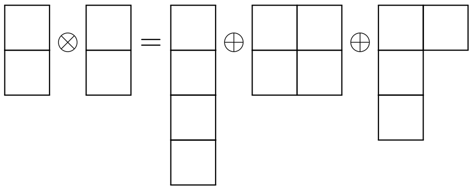
In the same vein, we introduce

$$r_{ABCD} := \rho^{\alpha\beta} f_{\alpha AB} f_{\beta CD}\,, \tag{3.38}$$

which is actually the original form of the Poláček-Siegel-field introduced in [31]. With these two new quantities, (3.36) can be rewritten exclusively in terms of Latin indices as

$$f_{ABCD} = -2D_{[A}\Omega_{B]CD} - 2\Omega_{[A|C}{}^{E}\Omega_{B]DE} - \frac{1}{2}\Omega_{EAB}\Omega^{E}{}_{CD} + F_{AB}{}^{E}\Omega_{ECD} - r_{ABCD}\,. \tag{3.39}$$

From (3.37), it follows that f_{ABCD} is antisymmetric with respect to its first two and last two indices. Thus, it can be decomposed into the three contributions

$f_{ABCD} =$  . (3.40)

The first two diagrams on the right hand side contain the parts of f_{ABCD} that are symmetric under the pairwise exchange of AB and CD. For them, r_{ABCD} drops out and we find

$$f_{ABCD} + f_{CDAB} = -\mathcal{R}_{ABCD}\,. \tag{3.41}$$

At this point, the power of the Poláček-Siegel construction becomes fully apparent: we recover the generalised curvature on the physical space M_n purely from a torsion component on the mega-space!

Additionally, the Bianchi identity components

$$\widehat{D}_{[A}\widehat{f}_{BCD]} - \frac{3}{4}\widehat{f}_{[AB}{}^{\widehat{E}}\widehat{f}_{CD]\widehat{E}} = 0 \tag{3.42}$$

on the mega-space imply the algebraic Bianchi identity of $\mathcal{R}_{ABCD}$, namely

$$\mathcal{R}_{[ABCD]} = f_{[AB}{}^{E} f_{CD]E} - \frac{4}{3}\nabla_{[A} f_{BCD]}\,. \tag{3.43}$$

Note that $\nabla_A f_{BCD} = 0$ for consistent truncations, rendering the identity completely algebraic. Finally, there is the hook in (3.40), which is antisymmetric under pairwise exchange of AB and CD and contains the Poláček-Siegel-field. As we will show in the next subsection, it will not contribute to the two-derivative action or its equations of motion. The same holds for the remaining components $f_A{}^{\beta\gamma}$ and $f^{\alpha\beta\gamma}$. Hence, we postpone deriving detailed expressions for them until section 3.5.

We still have to deal with the one-index torsion (3.28). In analogy with (3.29), we decompose it into

$$\begin{array}{r|c|l} \epsilon\text{-dim.} & & \\ -1 & f_\alpha = f_{\alpha\beta}{}^{\beta} & \\ 0 & & f_A \\ 1 & & f^\alpha\,. \end{array} \tag{3.44}$$

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Again, we need to compute the two contributions, f_A and f^α, in the right-hand column. For the first, we obtain

$$f_A = F_A - \Omega^B{}_{BA} - \cancel{f_{\beta A}{}^\beta} \tag{3.45}$$

by expanding (3.28). Similar to f_{ABC}, we recover the one-index torsion for the last section and identify

$$T_A = f_A \tag{3.46}$$

Therefore,

$$\nabla_A T_B = \nabla'_A f_B = 0 \qquad \text{and} \qquad \nabla_\alpha T_B = \nabla'_\alpha f_B = 0 \tag{3.47}$$

hold according to (2.31) for all consistent truncations that are governed by theorem 1. Just as in (3.35), we therefore deduce

$$f_A = \text{const.} \tag{3.48}$$

Finally, there is

$$f^\alpha = -D_B \Omega^{\alpha}{}_B - \cancel{\Omega^\beta{}_C f_\beta{}^{C\alpha}} - \rho^{\beta\gamma} f_{\beta\gamma}{}^\alpha + \Omega^{\alpha B} F_B + f_\beta{}^{\beta\alpha}\,, \tag{3.49}$$

which we rewrite as

$$\begin{aligned} f_{AB} &:= \left(f^\alpha - f_\gamma{}^{\gamma\alpha}\right) f_{\alpha AB} && (3.50)\\ &= -D^C \Omega_{CAB} + F^C \Omega_{CAB} + 2 r^C{}_{ABC}\,. && (3.51)\end{aligned}$$

However, it is not a new independent quantity. Instead it can be reconstructed from already encountered f's by using the Bianchi identity

$$\widehat{D}_{[A} \widehat{f}_{B]} + \frac{1}{2} \widehat{D}^{\widehat{C}} \widehat{f}_{\widehat{C}AB} - \frac{1}{2} \widehat{f}^{\widehat{C}} \widehat{f}_{\widehat{C}AB} = 0 \tag{3.52}$$

on the mega-space, resulting in

$$f_{AB} = 2\nabla_{[A} f_{B]} + \nabla^C f_{CAB} - f^C f_{CAB} - 2 f^C{}_{[AB]C} - f_\alpha f^\alpha{}_{AB}\,. \tag{3.53}$$

3.3 Truncation of the generalised Ricci scalar/tensor and dualities

We now come to the core objective of this paper, and reveal the relation between consistent truncations and dualities. To do so, we revisit the generalised Ricci scalar in (2.13). Currently, it is written in terms of the flat derivative D_A. But we rather want to write it in terms of the covariant derivatives ∇_A that are relevant for consistent truncations. While F_A and F_{ABC} are directly related to the torsion of D_A, they lack this kind of geometric interpretation for ∇_A. Therefore, we will ultimately replace them with f_A, f_{ABC}, and f_{ABCD}. In general, one might worry that after performing these substitutions, some naked connection terms would remain. By naked, we mean any generalised spin connection $\Omega_{AB}{}^C$ which is not part of a corresponding covariant derivative ∇_A. However, the generalised Ricci scalar transforms covariantly under the action of the generalised structure group F. Therefore, one should in the end be able to rewrite it just in terms of manifestly covariant quantities, and these are exactly all the f's introduced above (for more details see

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section 3.6) and their covariant derivatives. Indeed, all terms including a naked connection vanish and one finds

$$\mathcal{R} = P^{AB} P^{CD} \left[\left(\overline{P}^{EF} + \frac{1}{3} P^{EF} \right) f_{ACE} f_{BDF} + 2 f_{ACDB} \right] + 2 P^{AB} (2 \nabla_A f_B - f_A f_B) . \quad (3.54)$$

Note that the Poláček-Siegel-field gets projected out from f_{ABCD} and we can also write

$$2 P^{AB} P^{CD} f_{ACBD} = -\mathcal{R}_{ACBD} P^{AB} P^{CD} := \mathcal{R}^{\nabla} \quad (3.55)$$

by using (3.41). Hence, can rewrite $\mathcal{R}$ purely in terms of the torsions and scalar curvature of ∇_A as

$$\mathcal{R} = \mathcal{R}^{\nabla} + P^{AB} P^{CD} \left(\overline{P}^{EF} + \frac{1}{3} P^{EF} \right) T_{ACE} T_{BDF} + 2 P^{AB} (2 \nabla_A T_B - T_A T_B) . \quad (3.56)$$

As shown in diagram (1.1), it is crucial for any consistent truncation to preserve the relation between the action and its equations of motion. Therefore, we compute the latter by varying the action (2.12) and dropping all boundary terms, to obtain

$$\delta S = \int \mathrm{d}^D x \, e^{-2d} \left(-2 \mathcal{R} \delta d + \mathcal{G}_{AB} \delta E^{AB} \right) = 0 , \quad (3.57)$$

with [33]

$$\begin{aligned} \mathcal{G}_{AB} = 4 P_{[A}{}^{C} \overline{P}_{B]}{}^{D} \Big(& F_{CEG} F_{DFH} P^{EF} \overline{P}^{GH} + F_{CDE} F_F P^{EF} + \\ & D_D F_C - D_E F_{CDF} P^{EF} \Big) \end{aligned} \quad (3.58)$$

and

$$\delta E_{AB} := (\delta E_A{}^I) E_{BI} . \quad (3.59)$$

Note that $\mathcal{G}_{AB}$ is not yet the generalised Ricci tensor $\mathcal{R}_{IJ}$ that arises if we vary with respect to the generalised metric $\mathcal{H}_{IJ}$

$$\delta S = \int \mathrm{d}^D x \, e^{-2d} \left(-2 \mathcal{R} \delta d + \mathcal{R}_{IJ} \delta \mathcal{H}^{IJ} \right) = 0 . \quad (3.60)$$

Both $\mathcal{G}_{AB}$ and $\mathcal{R}_{IJ}$ are commonly used in the literature. They can be related by identifying

$$\delta \mathcal{H}^{IJ} = 4 \delta E_{KL} P^{K(I} \overline{P}^{J)L} \qquad \text{and} \qquad \delta E^{AB} = \delta \mathcal{H}_{CD} P^{C[A} \overline{P}^{B]D} \quad (3.61)$$

first, and afterwards comparing (3.57) with (3.60) to eventually obtain

$$\mathcal{R}_{IJ} = \mathcal{G}^{LK} P_{L(I} \overline{P}_{J)K} . \quad (3.62)$$

We follow the same strategy as we previously did for $\mathcal{R}$, and rewrite $\mathcal{G}_{AB}$ in terms of ∇_A, f_A, f_{ABC} and f_{ABCD}. Again, we find that all terms with naked connections cancel. The result

$$\begin{aligned} \mathcal{G}_{AB} = 4 P_{[A}{}^{C} \overline{P}_{B]}{}^{D} \Big(& f_{CEG} f_{DFH} P^{EF} \overline{P}^{GH} + f_{CDE} f_F P^{EF} \\ & + \nabla_D f_C - \nabla_E f_{CDF} P^{EF} - f_{EDCF} P^{EF} \Big) \end{aligned} \quad (3.63)$$

again looks similar to (3.58), with only one new contribution from f_{ABCD}. If we look at the definition (3.37), we note that the last two indices of f_{ABCD} originate from $f_{\alpha CD}$ which captures the infinitesimal action of the generalised structure group F on the generalised tangent space. Because F is a subgroup of the double Lorentz group H_D, we find

$$f_{\alpha CD} P^C{}_A \overline{P}^D{}_B = f_{\alpha CD} \overline{P}^C{}_A P^D{}_B = 0\,. \tag{3.64}$$

Exploiting this property, we rewrite

$$-4P_{[A}{}^C \overline{P}_{B]}{}^D f_{EDCF} P^{EF} = 2P_{[A}{}^C \overline{P}_{B]}{}^D \mathcal{R}_{ECDF} P^{EF} = \mathcal{G}^{\nabla}_{AB} \tag{3.65}$$

and see that only the box diagram in the decomposition (3.40) of f_{ABCD} contributes to the generalised Ricci tensor. Like for $\mathcal{R}$ in (3.56), we can write it exclusively in terms of ∇_A's torsions and curvature:

$$\begin{aligned} \mathcal{G}_{AB} = \mathcal{G}^{\nabla}_{AB} + 4P_{[A}{}^C \overline{P}_{B]}{}^D \Big(& T_{CEG} T_{DFH} P^{EF} \overline{P}^{GH} - T_{CDE} T_F P^{EF} + \\ & \nabla_D T_C + \nabla_E T_{CDF} P^{EF} \Big)\,. \end{aligned} \tag{3.66}$$

For any consistent truncation governed by theorem 1, $\mathcal{R}$, $\mathcal{R}_{AB}$ and $\mathcal{G}_{AB}$ have to be constant and singlets. Written in terms of equations this imposes

$$\nabla_{A/\alpha} \mathcal{R} = 0 \qquad \text{and} \qquad \nabla_{A/\alpha} \mathcal{G}_{BC} = 0\,. \tag{3.67}$$

Because the intrinsic torsions f_A and f_{ABC} share this property, the two new quantities $\mathcal{R}^{\nabla}$ and $\mathcal{G}^{\nabla}_{AB}$ defined in (3.65) and (3.55) have to satisfy

$$\nabla_{A/\alpha} \mathcal{R}^{\nabla} = 0 \qquad \text{and} \qquad \nabla_{A/\alpha} \mathcal{G}^{\nabla}_{AB} = 0\,, \tag{3.68}$$

too. This fixes some of the components of $F_{\widehat{A}\widehat{B}\widehat{C}}$, which we need for the Poláček-Siegel construction, to be constants, but clearly not all of them.

However, assuming that we have fixed all physically relevant data (at least at the leading two derivative level), we should rather ask: is there a way to choose the remaining components of $f_{\widehat{A}\widehat{B}\widehat{C}}$ such that they are compatible with the ones we already fixed? From a physical point of view this corresponds to obtaining a truncation ansatz which reproduces a target truncated theory. In general, there can be multiple solutions to this problem. Each of them provides a different metric and B-field on the internal space, but eventually they all reproduce the same truncated theory. We encountered this behaviour, which is depicted in (1.5), already, for the motivating example of generalised Scherk-Schwarz reductions and their relation to Poisson-Lie T-duality. But a main difference is that we now have two possible mechanisms for generating dualities instead of one. First, we will see that there is the possibility of finding different generalised frames on the mega-space that realise the same generalised fluxes $f_{\widehat{A}\widehat{B}\widehat{C}}$. We already encountered this mechanism in section 2.4; it provides the foundation of Poisson-Lie T-duality. Moreover, for a non-trivial generalised structure group, we have a second option since only some components of $f_{\widehat{A}\widehat{B}\widehat{C}}$ are fixed. The remaining ones are only constrained by the Bianchi identities on the mega-space.

3.4 Jacobi identities

Exploring the space of all these dual backgrounds is beyond the scope of the present paper. We rather want to restrict our discussion to a specific family of backgrounds with constant $f_{\widehat{A}\widehat{B}\widehat{C}}$, like for generalised Scherk-Schwarz reductions. In section 4, we show that all of them are realised in terms of generalised cosets. Thus, they 1) form the foundation of all generalised T-dualities currently known, and 2) provide explicit constructions of the full truncation ansätze. Hence, the question from above has to be refined: can we find for any given truncated theory at least one ansatz with constant $f_{\widehat{A}\widehat{B}\widehat{C}}$ that uplifts the theory to D dimensions?

To answer this question, one has to analyse the Bianchi identities

$$\widehat{f}_{[\widehat{A}\widehat{B}}{}^{\widehat{E}}\widehat{f}_{\widehat{C}\widehat{D}]\widehat{E}} = 0 \quad \text{and} \tag{3.69}$$

$$\widehat{f}^{\widehat{C}}\widehat{f}_{\widehat{C}\widehat{A}\widehat{B}} = 0\,, \tag{3.70}$$

on the mega-space, which arise after dropping the flat derivative $\widehat{D}_{\widehat{A}}$ because we are assuming that $f_{\widehat{A}\widehat{B}\widehat{C}}$ and $f_{\widehat{A}}$ are constant. It turns out that this problem is still hard, because it entails solving systems of coupled quadratic equations. Therefore, we postpone a thorough analysis to future work. But even without a full solution, already the structure of the various components of (3.69) and (3.70) is interesting.

We recognise (3.69) as the Jacobi identity for a Lie algebra. It has 15 independent contributions which can be organised into four categories:

1. Closure of the F-action on the mega-space (6 constraints)

   These components arise from the requirement that the action of the generalised structure group on the mega-space closes. They are governed by (3.30). Only three of them are non-trivial.

   (a) ϵ-dimension -2 implements the Jacobi identity of Lie(F),

   $$3 f_{[\alpha\beta}{}^{\epsilon} f_{\gamma]\epsilon}{}^{\delta} = 0\,. \tag{3.71}$$

   (b) ϵ-dimension -2 also requires that $f_{\alpha B}{}^{C}$ generates this action on the generalised tangent space $TM_n \oplus T^*M_n$,

   $$2 f_{[\alpha|C}{}^{E} f_{|\beta]E}{}^{D} = -f_{\alpha\beta}{}^{\epsilon} f_{\epsilon C}{}^{D}\,. \tag{3.72}$$

   (c) ϵ-dimension 0 imposes the condition

   $$f_{\alpha\beta}{}^{\epsilon} f_{\epsilon}{}^{\gamma\delta} - 4 f_{[\alpha|\epsilon}{}^{[\gamma|} f_{|\beta]}{}^{\epsilon|\delta]} = 0\,. \tag{3.73}$$

   A simple solution is $f_{\alpha}{}^{\beta\gamma} = -2\,\mathbf{r}^{\delta[\beta} f_{\delta\alpha}{}^{\gamma]}$ for antisymmetric $\mathbf{r}^{\alpha\beta}$. This condition (and the existence of possibly other non-trivial solutions) can be understood using the language of Lie algebra cohomology (see e.g. appendix A of [47], whose conventions we follow here). One introduces one-forms θ^{α} (essentially the left-invariant one-forms on F) as well as scalar 0-forms e_A valued in some representation — here we are concerned with $\Lambda^2\text{Lie}(F)$, so we denote $e_{\alpha\beta} = t_\alpha \wedge t_\beta$, where the wedge product merely emphasizes that the result is antisymmetrized in $\alpha\beta$. Then $f_\alpha{}^{\beta\gamma}$ is a one-form φ valued in $\Lambda^2\text{Lie}(F)$, i.e.

$$\varphi = \frac{1}{2}\theta^\gamma\, f_\gamma{}^{\alpha\beta}\, t_\alpha \wedge t_\beta\,. \tag{3.74}$$

The condition (3.73) amounts to closure of this one-form, where we take the exterior derivative to obey

$$\mathrm{d}\theta^\alpha = -\frac{1}{2}\theta^\beta \wedge \theta^\gamma f_{\beta\gamma}{}^\alpha \qquad \text{and} \qquad \mathrm{d}t_\alpha = \theta^\beta f_{\beta\alpha}{}^\gamma t_\gamma\,. \tag{3.75}$$

Nilpotence of the exterior derivative is guaranteed by (3.71). Then one indeed finds

$$\mathrm{d}\varphi = \frac{1}{4}\theta^\alpha \wedge \theta^\beta \left(-f_{\alpha\beta}{}^\epsilon f_\epsilon{}^{\gamma\delta} + 4 f_{\alpha\epsilon}{}^\gamma f_\beta{}^{\epsilon\delta}\right) t_\gamma \wedge t_\delta = 0 \tag{3.76}$$

so φ must be closed (i.e. a cocycle). An immediate solution is to take φ to be exact (i.e. a coboundary),

$$\varphi = \mathrm{d}\mathbf{r}\,, \qquad \mathbf{r} = \frac{1}{2}r^{\alpha\beta} t_\alpha \wedge t_\beta \qquad \Longrightarrow \qquad f_\alpha{}^{\beta\gamma} = -2\,\mathbf{r}^{\delta[\beta} f_{\delta\alpha}{}^{\gamma]}\,. \tag{3.77}$$

But in general, there may also be cocycles that are not coboundaries and therefore cannot be written as $\varphi = \mathrm{d}\mathbf{r}$. These are governed by the Lie algebra cohomology $H^1(\text{Lie}(F), \Lambda^2\text{Lie}(F))$.

2. Invariance under the F-action (4 constraints)

   Next, we find four constraints describing how the components in the last column of (3.29) transform under the infinitesimal action of the generalised structure group F.

   (a) ϵ-dimension -1 requires that the torsion f_{ABC} is invariant, namely

   $$3 f_{\alpha[B}{}^E f_{CD]E} = 0\,. \tag{3.78}$$

   An alternative way to write this equation is by using ∇_α:

   $$\nabla_\alpha f_{BCD} = 0\,. \tag{3.79}$$

   (b) ϵ-dimension 0 gives rise to

   $$\nabla_\alpha f_{BCDE} - f_\alpha{}^{\beta\gamma} f_{\beta BC} f_{\gamma DE} = 0\,. \tag{3.80}$$

   By using (3.77), we can further simplify this equation to

   $$\nabla_\alpha \left(f_{BCDE} + \mathbf{r}_{BCDE}\right) = 0 \tag{3.81}$$

   with

   $$\mathbf{r}_{ABCD} := \mathbf{r}^{\alpha\beta} f_{\alpha AB} f_{\beta CD}\,. \tag{3.82}$$

This is quite interesting, because it tells us that f_{ABCD} does not need to be fully invariant under F-action. Instead the invariant quantity is rather the sum of f_{ABCD} and the newly introduced $\mathbf{r}_{ABCD}$. Note that $\mathbf{r}_{ABCD}$ is antisymmetric under the exchange (AB)↔(CD). This implies that the generalised Riemann tensor $\mathcal{R}_{ABCD}$ defined by (3.41) is always invariant, as

$$\nabla_\alpha \mathcal{R}_{ABCD} = 0 \tag{3.83}$$

has to hold.

(c) ϵ-dimension 1 requires

$$\nabla_\alpha f_{ABCDE} = 0 \tag{3.84}$$

with

$$f_{ABCDE} := f_A{}^{\alpha\beta} f_{\alpha BC} f_{\beta DE} \,. \tag{3.85}$$

(d) ϵ-dimension 2 relates $f_\alpha{}^{\beta\gamma}$ with $f^{\alpha\beta\gamma}$,

$$3 f_\alpha{}^{\epsilon[\beta} f_\epsilon{}^{\gamma\delta]} + 3 f_{\alpha\epsilon}{}^{[\beta} f^{\gamma\delta]\epsilon} = 0 \,. \tag{3.86}$$

With (3.77), this expression simplifies to

$$\nabla_\alpha (f_{BCDEFG} + \mathbf{r}_{BCDEFG}) = 0 \tag{3.87}$$

with

$$\begin{aligned} \mathbf{r}_{ABCDEF} &=: \mathbf{r}^{\alpha\beta\gamma} f_{\alpha AB} f_{\beta CD} f_{\gamma EF} \,, \\ f_{ABCDEF} &=: f^{\alpha\beta\gamma} f_{\alpha AB} f_{\beta CD} f_{\gamma EF} \,, \end{aligned} \tag{3.88}$$

and the classical Yang-Baxter tensor

$$\mathbf{r}^{\alpha\beta\gamma} = \mathbf{r}^{\delta\alpha} \mathbf{r}^{\beta\epsilon} f_{\delta\epsilon}{}^{\gamma} + \mathbf{r}^{\delta\gamma} f_\delta{}^{\alpha\beta} \,. \tag{3.89}$$

We use this terminology because $\mathbf{r}^{\alpha\beta\gamma}$ describes the left hand side of the classical Yang-Baxter equation. It is manifestly antisymmetric with respect to its first two indices, α and β. After substituting $f_\alpha{}^{\beta\gamma} = -2\mathbf{r}^{\delta[\beta} f_{\delta\alpha}{}^{\gamma]}$ one can show that it is actually totally antisymmetric.

3. Bianchi identities (3 constraints)

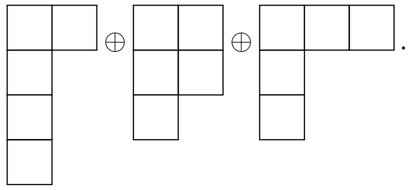
Next, we find Bianchi identities for the three curvatures f_{ABCD}, f_{ABCDE} and f_{ABCDEF}:

(a) ϵ-dimension 0 gives the Bianchi identity for f_{ABCD} in (3.43), which we already discussed in section 3.2.

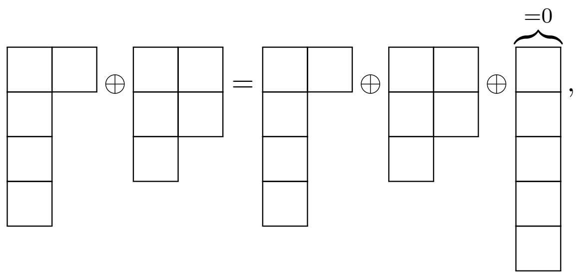
(b) ϵ-dimension 1 implies

$$f_{[ABC]DE} = f_{[AB|}{}^F f_{F|C]DE} \,. \tag{3.90}$$

To better understand the implications of this equation, we decompose

$$f_{ABCDE} =$$  (3.91)

The antisymmetrisation on the right hand side of (3.90) projects out the last contribution. If we also take into account the left hand side, we find

 (3.92)

showing that the totally antisymmetric contribution from the left hand side has to vanish. As result, we obtain the first quadratic constraint, namely

$$f_{[AB|}{}^{F} f_{F|CDE]} = 0\,. \tag{3.93}$$

(c) ϵ-dimension 2 results finally in

$$\begin{aligned} f_{ABCDEF} = -\frac{1}{2} f_{AB}{}^{G} f_{GCDEF} + f_{G[A|CD} f^{G}{}_{|B]EF} + f_{G[A|CD} f^{G}{}_{|B]EF} + \\ 2 f_{AB[C}{}^{G} f_{|G|D]EF} - (CD) \leftrightarrow (EF)\,. \end{aligned} \tag{3.94}$$

The right hand side is by definition totally antisymmetric under pairwise exchange of (AB), (CD) and (EF) but the left hand side is not. Therefore we find additional quadratic constraints.

4. Additional quadratic constraints (2 constraints)

There are two more constraints that do not contain any linear contribution. Together with the quadratic constraints we just encountered in point 3, they represent the real challenge in identifying admissible curvatures that result in a dressing coset. There is not much more we can do about them at the moment, and we therefore just list them here:

(a) ϵ-dimension 3 gives rise to

$$-\frac{1}{2} f_{ABIJ} f^{B}{}_{CDEF} - 2 f_{ACD[I}{}^{G} f_{|G|J]EF} + \mathrm{asym}(IJ, CD, EF) = 0\,, \tag{3.95}$$

where $\mathrm{asym}(IJ, CD, EF)$ denotes all permutations of the index pairs (IJ), (CD), (EF) in the given expression weighted by -1 for odd permutations.

(b) ϵ-dimension 4 finally results in

$$f_{IJCD[K}{}^G f_{|G|L]EF} + \frac{1}{8} f_{ACDKL} f^A{}_{IJEF} + \text{asym}(CD, IJ, EF, KL) = 0\,. \tag{3.96}$$

Next, we decompose the second constraint (3.70). It has six contributions which again can be ordered according to their ϵ-dimension:

- ϵ-dimension -2 requires
$$f_\gamma f^\gamma{}_{\alpha\beta} = 0\,, \tag{3.97}$$
which is automatically satisfied for $f_\gamma = f_{\gamma\delta}{}^\delta$.

- ϵ-dimension -1 imposes
$$\nabla_\alpha f_B = 0\,, \tag{3.98}$$
saying that the one-index torsion is a singlet under F-action. This is required by theorem 1.

- ϵ-dimension 0 gives rise to the two constraints
$$f_{AB} = -f_{AB}{}^C f_C - f_\gamma f^\gamma{}_{AB} + f_\gamma{}^{\gamma\delta} f_{\delta AB} \tag{3.99}$$
$$\text{and} \qquad f_{\gamma\alpha}{}^\beta f^\gamma = -f_\gamma f_\alpha{}^{\beta\gamma}\,. \tag{3.100}$$
The first equation is used to fix f_{AB}. (Recall that we did exactly the same in (3.53) for the more general case of non-constant f's.) In the second equation, we contract the free index β with $f_{\beta BC}$ to obtain
$$\nabla_\alpha(f_{BC} + \mathbf{r}_{BC}) = 0 \quad \text{where} \quad \mathbf{r}_{AB} := r^{\alpha\beta} f_{\beta\alpha}{}^\gamma f_{\gamma AB} = 2\,\mathbf{r}^C{}_{ABC}\,. \tag{3.101}$$
Satisfying this equation requires that the combination $f_{AB} + \mathbf{r}_{AB}$ to be a singlet. This result is not very surprising, because we observed the same phenomena already in (3.81) and (3.87).

There are two further constraints that do not admit such nice interpretations. In general, we note that the higher the ϵ-dimensions becomes, the more complicated it gets. The only advantage here is that both are linear in the components of $f_{\widehat{A}}$. Therefore, they can be always solved easily:

- ϵ-dimension 1 results in
$$f^A f_{ABCD} - f_\alpha f_B{}^{\alpha\beta} f_{\beta CD} = 0\,, \tag{3.102}$$
while

- ϵ-dimension 2 requires
$$f^C f_{CABDE} + f^\gamma f_\gamma{}^{\alpha\beta} f_{\alpha AB} f_{\beta DE} + f_\gamma f^{\gamma\alpha\beta} f_{\alpha AB} f_{\beta DE} = 0\,. \tag{3.103}$$

Concluding, the most problematic are the quadratic constraints that originate from (3.69). There is currently no obvious way to find general solutions. Instead one has to treat them on a case by case basis.

3.5 Higher derivative curvature

In subsection 3.3, one might have gained the impression that the components ${f_A}^{\beta\gamma}$ and $f^{\alpha\beta\gamma}$ are irrelevant for the action or its equations of motion. In fact, this is only true at the leading two-derivative level. If we want to consider higher-derivative corrections, there is no obvious reason why they should not contribute.

To make this point clear, we first compute

$$\begin{aligned} {f_A}^{\beta\gamma} = D_A\rho^{\beta\gamma} + D_A\Omega^{[\beta}{}_D\Omega^{\gamma]D} - 2D_D\Omega^{[\beta}{}_A\Omega^{\gamma]D} - \Omega^\delta{}_A f_\delta{}^{\beta\gamma} + F_{ADE}\Omega^{\beta D}\Omega^{\gamma E} - \\ \Omega^\delta{}_A f_{\delta\epsilon}{}^{[\beta}\Omega^{\gamma]F}\Omega^\epsilon{}_F - 2\Omega^\delta{}_A f_{\delta\epsilon}{}^{[\beta}\rho^{\gamma]\epsilon} + \cancel{2f_{\delta A}{}^{[\beta}\rho^{\gamma]\delta}} + \cancel{f_{\delta A}{}^{[\beta}\Omega^{\gamma]E}\Omega^\delta}{}_E\,. \end{aligned} \tag{3.104}$$

As before, we rewrite this quantity in terms of indices on $TM \oplus T^*M$ only, as

$$\begin{aligned} f_{ABCDE} = &\frac{1}{2}D_A r_{BCDE} + 2\Omega_A{}^F{}_{[B}r_{C]FDE} - 2\Omega_A{}^F{}_{[B}f_{C]FDE} - D_F\Omega_{ABC}\Omega^F{}_{DE} \\ &+ \frac{1}{2}D_A\Omega_{FED}\Omega^F{}_{BC} - \Omega_{AI[E}\Omega^{FI}{}_{D]}\Omega_{FBC} - \frac{1}{2}F_{FAG}\Omega^F{}_{BC}\Omega^G{}_{DE} \\ &- (BC) \leftrightarrow (DE) \end{aligned} \tag{3.105}$$

There is one more component left, $f^{\alpha\beta\gamma}$, which we have to compute. In analogy with f^α, it is not independent of the ones we computed already. Rather, it arises from the Bianchi identity

$$2\nabla^{[\alpha}f^{\beta]}{}_{CD} + 2\nabla_{[C}f_{D]}{}^{\alpha\beta} - f^{\alpha\beta\widehat{E}}f_{CD\widehat{E}} + 2f^{\widehat{E}[\alpha}{}_{[C}f^{\beta]}{}_{D]\widehat{E}} = 0 \tag{3.106}$$

which eventually gives rise to

$$\begin{aligned} f_{ABCDEF} = &r_A{}^G{}_{EF}r_{GBCD} + 2r_{G[D|EF|}f_{|AB|C]}{}^G - \frac{1}{2}D_G r_{EFCD}\Omega^G{}_{AB} - f_{G[D|EF|}\Omega_{|I|C]}{}^G\Omega^I{}_{AB} \\ &+ r_{G[D|EF|}\Omega^I{}_{C]}{}^G\Omega_{IAB} + \frac{1}{4}\Omega_{IG[D}\Omega^I{}_{|EF|}\Omega_{|K|C]}{}^G\Omega^K{}_{AB} + \frac{1}{6}F^{GHI}\Omega_{HEF}\Omega_{IAB}\Omega_{GCD} \\ &+ \frac{1}{2}D_I\Omega_{GEF}\Omega^G{}_{AB}\Omega^I{}_{CD} + \text{asym}(AB, CD, EF) \end{aligned} \tag{3.107}$$

To count the number of derivatives in each component of $f_{\widehat{A}\widehat{B}\widehat{C}}$, we define that either D_A or Ω_{ABC} count as one, because they both contribute equally to the covariant derivative ∇_A defined in (2.18). Hence, from the Poláček-Siegel construction, we find the following independent quantities:

$$\begin{array}{c|c|c|c} \text{derivatives} & 1 & 2 & 3 \\ \hline \text{quantity} & f_A\,,\, f_{ABC} & f_{ABCD} & f_{ABCDE} \end{array} \tag{3.108}$$

At the two-derivative level, we have seen that the action and its equations of motion incorporate only the one-derivative torsions $f_A = T_A$ and $f_{ABC} = -T_{ABC}$, and the projectors of the two derivative curvature f_{ABCD}. Starting with the leading α'-corrections at four derivatives, the situation becomes more complicated because the form of the action [48] is only fixed up to field redefinitions. If it is possible to find a field basis in which the generalised structure group F still acts by ∇_α, then the action and its field equations should also incorporate f_{ABCDE}. Moreover, the singlet condition that is required by theorem 1 has to be imposed at each order of α' separately. This will likely result in more constraints in addition to (3.68). Thus, on a qualitative level, we note that consistent truncations with higher derivative corrections should require that more and more components of the mega-space generalised fluxes $f_{\widehat{A}\widehat{B}\widehat{C}}$ are constant. Taking the Bianchi identities into account, this might even at some order prohibit any non-constant contributions. In this case, there would be a one-to-one correspondence between consistent truncations, à la theorem 1, and generalised cosets. We leave the exploration of this idea to future work.

3.6 Gauge transformations

We observed in section 3.3 that, by what appeared to be a miracle, all naked connections in the rewriting of the action and its equations of motion vanish and just the various f's remain. Of course this is not actually a miracle, but rather the consequence of a gauge symmetry. To see how this symmetry emerges, we define its infinitesimal action on the mega-space by

$$\delta_{\widehat{\lambda}} \widehat{E}_{\widehat{A}\widehat{B}} = \left(\mathcal{L}_{\widehat{\lambda}} \widehat{E}_{\widehat{A}}{}^{\widehat{I}} \right) \widehat{E}_{\widehat{B}\widehat{I}} . \tag{3.109}$$

A short calculation, using the definition of the generalised Lie derivative (2.2), gives rise to

$$\delta_{\widehat{\lambda}} \widehat{E}_{\widehat{A}\widehat{B}} = -2 \widehat{D}_{[\widehat{A}} \widehat{\lambda}_{\widehat{B}]} + \widehat{\lambda}^{\widehat{C}} \widehat{f}_{\widehat{C}\widehat{A}\widehat{B}} . \tag{3.110}$$

Essential for our purpose is that all transformations generated by $\widehat{\lambda}^{\widehat{A}}$ do not change the lower-triangular form of $\widehat{E}_{\widehat{A}}{}^{\widehat{I}}$ given in (3.2). To see why, we first note that

$$\lambda^{\widehat{A}} = \widetilde{M}_{\widehat{B}}{}^{\widehat{A}} \widehat{\lambda}^{\widehat{B}} \tag{3.111}$$

should only depend of the internal coordinates y but not on the auxiliary coordinates z. In this case, (3.110) implies

$$\delta_{\lambda} E_{\widehat{A}\widehat{B}} = -2 \nabla'_{[\widehat{A}} \lambda_{\widehat{B}]} + \lambda^{\widehat{C}} f_{\widehat{C}\widehat{A}\widehat{B}} . \tag{3.112}$$

On the other hand, we can also compute $\delta_{\lambda} E_{\widehat{A}\widehat{B}}$ directly from the variation of (3.2), giving rise to

$$\delta_{\lambda} E_{\widehat{A}\widehat{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta_{\lambda} E_{AB} & -\delta_{\lambda} \Omega^{\beta}{}_{A} - \Omega^{\beta}{}_{C} \delta_{\lambda} E^{C}{}_{B} \\ 0 & \delta_{\lambda} \Omega^{\alpha}{}_{B} + \Omega^{\alpha}{}_{C} \delta_{\lambda} E^{C}{}_{B} & \delta_{\lambda} \rho^{\alpha\beta} + 2 \Omega^{[\alpha}{}_{C} \delta_{\lambda} \Omega^{\beta]C} + \Omega^{\alpha}{}_{C} \delta_{\lambda} E^{CD} \Omega^{\beta}{}_{D} \end{pmatrix} . \tag{3.113}$$

We see two important things here: first, the transformation of $E_{A}{}^{I}$, $\Omega^{\alpha}{}_{B}$ and $\rho^{\alpha\beta}$ can easily be extracted from this result, and second, $\delta E_{\alpha\widehat{B}} = \delta E_{\widehat{B}\alpha} = 0$. The latter is not automatically guaranteed by (3.112). It requires the restriction

$$\lambda_{\widehat{A}} = \begin{pmatrix} \lambda^{\alpha} & \lambda_{A} & 0 \end{pmatrix} . \tag{3.114}$$

But the second parameter λ_A only mediates a generalised diffeomorphism on the physical space M_d. For the following discussion, we therefore concern ourselves only with λ^α, which mediates gauge transformations, and compute

$$\delta_\lambda E_{AB} = \lambda^\gamma f_{\gamma AB}\,, \tag{3.115}$$

$$\delta_\lambda E^\alpha{}_B = D_B\lambda^\alpha + \Omega^\gamma{}_B f_{\gamma\delta}{}^\alpha\lambda^\delta - \cancel{\lambda^\gamma f_{\gamma B}{}^\alpha}\,, \quad \text{and} \tag{3.116}$$

$$\delta_\lambda E^{\alpha\beta} = 2D_C\lambda^{[\alpha}\Omega^{\beta]C} - 2\rho^{\gamma[\alpha}f_{\gamma\delta}{}^{\beta]}\lambda^\delta - \Omega^{\gamma I}\Omega^{[\alpha}{}_I f_{\gamma\delta}{}^{\beta]}\lambda^\delta + \lambda^\gamma f_\gamma{}^{\alpha\beta}\,, \tag{3.117}$$

eventually extracting

$$\delta_\lambda\Omega^\alpha{}_B = D_B\lambda^\alpha - \Omega^\alpha{}_C\lambda^C{}_B + \Omega^\gamma{}_B f_{\gamma\delta}{}^\alpha\lambda^\delta\,, \tag{3.118}$$

$$\delta_\lambda\rho^{\alpha\beta} = 4D_C\lambda^{[\alpha}\Omega^{\beta]C} - 2\rho^{\gamma[\alpha}f_{\gamma\delta}{}^{\beta]}\lambda^\delta - 3\Omega^{[\alpha}{}_E f_{\gamma\delta}{}^{\beta]}\Omega^{\gamma E}\lambda^\delta + 3\Omega^{[\alpha}{}_C\Omega^{\beta]}_D\lambda^{DC} + \lambda^\gamma f_\gamma{}^{\alpha\beta}\,. \tag{3.119}$$

As in earlier calculations, it is useful to write the results just with indices for the generalised tangent space on the internal manifold $TM \oplus T^*M$. To do so, we introduce

$$\lambda_{AB} := \lambda^\gamma f_{\gamma AB} \tag{3.120}$$

and obtain

$$\delta_\lambda E_{AB} = \lambda_{AB}\,, \tag{3.121}$$

$$\delta_\lambda\Omega_{ABC} = D_A\lambda_{BC} - \lambda_A{}^D\Omega_{DBC} - 2\lambda_{D[C}\Omega_{|A|B]}{}^D\,, \tag{3.122}$$

$$\begin{aligned}\delta_\lambda r_{ABCD} = {} & 2D_E\lambda_{AB}\Omega^E{}_{CD} - 2D_E\lambda_{CD}\Omega^E{}_{AB} + 2\lambda^E{}_{[D}r_{C]EAB} - 2\lambda^E{}_{[B}r_{A]ECD} \\ & + 3\Omega_{FAB}\lambda^E{}_{[D}\Omega^F{}_{C]E} - 3\lambda_{E[B}\Omega^F{}_{A]E}\Omega_{FCD} + 3\Omega_{EAB}\Omega_{FCD}\lambda^{EF} \\ & + 2\lambda_{[A}{}^G f_{|G|B]CD} + 2\lambda_{[C}{}^G f_{|ABG|D]}\,.\end{aligned} \tag{3.123}$$

Here, we recognise the double Lorentz transformation rules for the frame and the spin connection Ω_{ABC}. Additionally, there is a new transformation rule for the Poláček-Siegel field r_{ABCD}. To understand how it arises, we look at the transformation of the generalised fluxes $\widehat{f}_{\widehat{A}\widehat{B}\widehat{C}}$ on the mega-space. Under generalised diffeomorphisms they transform as scalars,

$$\delta_{\widehat{\lambda}}\widehat{f}_{\widehat{A}\widehat{B}\widehat{C}} = \widehat{\lambda}^{\widehat{D}}\widehat{D}_{\widehat{D}}\widehat{f}_{\widehat{A}\widehat{B}\widehat{C}}\,, \tag{3.124}$$

or, equally,

$$\delta_\lambda f_{\widehat{A}\widehat{B}\widehat{C}} = \lambda^\delta\nabla'_\delta f_{\widehat{A}\widehat{B}\widehat{C}}\,. \tag{3.125}$$

A similar equation,

$$\delta_\lambda f_{\widehat{A}} = \lambda^\beta\nabla'_\beta f_{\widehat{A}}\,, \tag{3.126}$$

holds for the one-index generalised flux $\widehat{f}_{\widehat{A}}$, too. These two equations become even more useful if we exchange the derivative ∇'_α for ∇_α, which captures the covariant action of the generalised structure group on $T^M \oplus T^*M$. Various quantities we already encountered do not transform covariantly. Examples are the connections Ω_{ABC} and r_{ABCD}. To quantify the deviation from a covariant transformation, it is convenient to introduce the operator

$$\Delta_\lambda = \delta_\lambda - \lambda^\alpha\nabla_\alpha\,. \tag{3.127}$$

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Any quantity it annihilates is covariant. In general, there are two perspectives one can take on the Poláček-Siegel construction. We started by looking at the extended, mega-space with $\widehat{f}_{\widehat{A}\widehat{B}\widehat{C}}$ and $\widehat{E}_{\widehat{A}}{}^{\widehat{I}}$. On the other hand, it also has to be possible to extract the same information just from the internal space M. Indeed this is when we consider the following data on M:

1. A generalised structure group F which acts faithfully on the generalised tangent space $TM \oplus T^*M$. This will fix $f_{\alpha BC}$, $f_{\alpha\beta}{}^{\gamma}$ and set $f_{\alpha B}{}^{\gamma} = 0$.

2. All dynamic contributions to the generalised fluxes on the mega-space. They are listed in the last columns of (3.29) and (3.44) which are completely encoded in the torsions and curvatures

$$f_A\,, \quad f_{ABC}\,, \quad f_{ABCD} \quad \text{and} \quad f_{ABCDE}\,.$$

3. Everything else is fixed by the Bianchi identity on the mega-space.

Taking into account (3.125), we find that all torsions and curvatures mentioned transform covariantly, except for

$$\Delta_\lambda f_{ABCD} = \lambda^\alpha f_\alpha{}^{\beta\gamma} f_{\beta AB} f_{\gamma CD}\,. \tag{3.128}$$

This is the power of the Poláček-Siegel formalism. It provides a systematic method for finding covariant tensors. Moreover, we know that both the generalised Ricci scalar $\mathcal{R}$ and the generalised Ricci tensor $\mathcal{R}_{AB}$ transform covariantly under double Lorentz transformations. Hence, when we rewrite them in terms of f's in section 3.3, they must remain covariant. Therefore, all the non-covariant naked connections Ω_{ABC} must in the end be formed from covariant f's or covariant derivatives of them. We also see that already at the two derivative level the antisymmetric part (with respect to $(AB) \leftrightarrow (CD)$) of f_{ABCD} is not covariant. Thus, it cannot appear in the two derivative action or its field equations.

4 Construction of the mega-generalised frame

Currently all known generalised T-dualities can be captured by dressing cosets. In section 3.3, we concluded that these cosets result in consistent truncations where the Poláček-Siegel construction has constant generalised fluxes $\widehat{f}_{\widehat{A}\widehat{B}\widehat{C}}$ on the mega-space. The objective now is to show how we can construct the corresponding mega-generalised frames that make the truncation ansatz discussed in section 2.3 fully explicit. As we have seen in section 2.4, these frames are not only very valuable for consistent truncations, but they also are an important tool for studying generalised T-dualities on the worldsheet, and for constructing the canonical transformations of the underlying σ-models.

4.1 Generalised frames on group manifolds

Our starting point is the constant generalised fluxes on the mega-space, and we are looking for a generalised frame $\widehat{E}_{\widehat{A}}{}^{\widehat{I}}$ that satisfies

$$\widehat{f}_{\widehat{A}\widehat{B}\widehat{C}} = \widehat{D}_{[\widehat{A}} \widehat{E}_{\widehat{B}}{}^{\widehat{I}} \widehat{E}_{\widehat{C}]\widehat{I}} = \text{const.} \tag{4.1}$$

As outlined in section 3.4, the corresponding Bianchi identity becomes the Jacobi identity (3.69) of a Lie algebra. We denote it as Lie($\mathbb{D}$) and interpret the generalised fluxes $\widehat{f}_{\widehat{A}\widehat{B}\widehat{C}}$ as its structure coefficients. An important consequence is that the adjoint action of any element of $\mathbb{D}$ will leave $\widehat{f}_{ABC}$ invariant. Therefore, we can identify $\widehat{f}_{\widehat{A}\widehat{B}\widehat{C}} = f_{\widehat{A}\widehat{B}\widehat{C}}$. It is known that one can construct $\widehat{E}_{\widehat{A}}{}^{\widehat{I}}$ systematically [24, 25, 42, 49], based on the following data:

1. A doubled Lie group $\mathbb{D}$, which is generated by the generators $t_{\widehat{A}}$ with the structure coefficients
$$[t_{\widehat{A}}, t_{\widehat{B}}] = f_{\widehat{A}\widehat{B}}{}^{\widehat{C}} t_{\widehat{C}} \,. \tag{4.2}$$
In the following, we label them by $t_{\widehat{A}} = \begin{pmatrix} t_{\widehat{a}} & t^{\widehat{a}} \end{pmatrix}$, where $\widehat{A} = 1, \dots, 2\widehat{D}$ and $\widehat{a} = 1, \dots, \widehat{D}$.

2. A non-degenerate, pairing, $\langle t_{\widehat{A}}, t_{\widehat{B}} \rangle = \eta_{\widehat{A}\widehat{B}} = \begin{pmatrix} 0 & \delta_{\hat{a}}{}^{\hat{b}} \\ \delta^{\hat{a}}{}_{\hat{b}} & 0 \end{pmatrix}$, that is invariant under the adjoint action of $\mathbb{D}$.

3. A maximally isotropic subgroup $H \subset \mathbb{D}$, generated by $t^{\hat{a}}$, with $\langle t^{\widehat{a}}, t^{\widehat{b}} \rangle = 0$.

Explicitly, the generalised frame is defined on the coset $M = H \backslash \mathbb{D}$ in terms of

$$\widehat{E}_{\widehat{A}}{}^{\widehat{I}} = M_{\widehat{A}}{}^{\widehat{B}} \begin{pmatrix} \widetilde{v}_{\widehat{b}}{}^{\widehat{i}} & \widetilde{v}_{\widehat{b}}{}^{\widehat{j}} B_{\widehat{j}\widehat{i}} \\ 0 & v^{\widehat{b}}{}_{\widehat{i}} \end{pmatrix} . \tag{4.3}$$

In studying its properties, it is convenient to use the differential forms $v^{\widehat{a}} = v^{\widehat{a}}{}_{\widehat{i}} \mathrm{d}x^{\widehat{i}}$, $A_{\widehat{a}} = A_{\widehat{a}\widehat{i}} \mathrm{d}x^{\widehat{i}}$, $B = \frac{1}{2} B_{\widehat{i}\widehat{j}} \mathrm{d}x^{\widehat{i}} \wedge x^{\widehat{j}}$ which are defined by

$$\mathrm{d}m m^{-1} = t_{\widehat{a}} v^{\widehat{a}} + t^{\widehat{a}} A_{\widehat{a}} \,, \qquad m \in M \tag{4.4}$$
$$B = \frac{1}{2} v^{\widehat{a}} \wedge A_{\widehat{a}} + B_{\mathrm{WZW}} \,, \tag{4.5}$$
$$\mathrm{d}B_{\mathrm{WZW}} = H_{\mathrm{WZW}} = -\frac{1}{12} \langle \mathrm{d}m m^{-1}, [\mathrm{d}m m^{-1}, \mathrm{d}m m^{-1}] \rangle \,. \tag{4.6}$$

In general, H_{WZW} is closed but not exact. If this is the case, B_{WZW} can only be defined in a local patch and the patches have to be connected by appropriate gauge transformations. Moreover, we need the adjoint action

$$m t_{\widehat{A}} m^{-1} = M_{\widehat{A}}{}^{\widehat{B}} t_{\widehat{B}} \,, \tag{4.7}$$

and the dual vector field $\widetilde{v}_{\widehat{a}}{}^{\widehat{i}}$, defined by $\widetilde{v}_{\widehat{a}}{}^{\widehat{i}} v^{\widehat{b}}{}_{\widehat{i}} = \delta_{\widehat{a}}{}^{\widehat{b}}$, to complete the list of ingredients that enter (4.3).

At this point, we finally fully review how Poisson-Lie T-duality is realised by generalised Scherk-Schwarz reductions. Looking at the diagram in (1.5), we want to preserve the truncated theory, and, most importantly, with it the generalised torsions T_A and T_{ABC}. With them the generalised Ricci scalar $\mathcal{R}$ and tensor $\mathcal{R}_{AB}$ are also preserved, because there is no curvature $\mathcal{R}_{ABCD}$; all that counts for generalised Scherk-Schwarz reductions are the torsions. To preserve them, we are looking for different generalised frames $E_A^{(i)\,I}$ which still produce the same constant generalised fluxes F_{ABC}. We used the construction presented above (after removing all hats) to achieve this goal. The generalised fluxes depend only on the double Lie group $\mathbb{D}$, but not on the choice of the maximally isotropic subalgebra. On the other hand, the constructed generalised frame $E_A{}^I$ crucially depends on the subalgebra used in the construction. For any maximally isotropic subalgebra H_i, we obtain a new generalised frame field $E_A^{(i)\,I}$ that still gives rise to the same generalised fluxes F_{ABC}. On the string worldsheet, the same mechanism was used to define Poisson-Lie T-duality and to show that it is a canonical transformation between two σ-models with dual target spaces.

We are not completely done yet, because there is still the one-index generalised flux $\widehat{f}_{\widehat{A}}$. As it is constant, its Bianchi identity on the mega-space simplifies to (3.70). As a consequence the adjoint action of any element in $\mathbb{D}$ will leave $\widehat{f}_{\widehat{A}}$ invariant and we identify $\widehat{f}_{\widehat{A}} = f_{\widehat{A}}$. In analogy with (4.1), we have now to find a $\widehat{d}$ such that

$$\widehat{f}_{\widehat{A}} = 2\widehat{D}_{\widehat{A}}\widehat{d} - \partial_{\widehat{I}}\widehat{E}_{\widehat{A}}{}^{\widehat{I}} = \text{const.} \tag{4.8}$$

holds. The component

$$f^{\widehat{a}} = f_{\widehat{b}}{}^{\widehat{b}\widehat{a}} \tag{4.9}$$

is automatically constant and from $f_{\widehat{a}}$, we obtain the differential equation

$$\begin{aligned} \mathrm{d}\overline{d} &= \frac{1}{2}v^{\widehat{a}}\left(f_{\widehat{a}} - f_{\widehat{a}\widehat{b}}{}^{\widehat{b}} + \iota_{v_{\widehat{b}}}A_{\widehat{c}}f_{\widehat{a}}{}^{\widehat{b}\widehat{c}}\right) - \frac{1}{2}\iota_{v_{\widehat{a}}}\mathrm{d}v^{\widehat{a}} \\ &= \frac{1}{2}\left(v^{\widehat{a}}f_{\widehat{a}} + A_{\widehat{a}}f^{\widehat{a}}\right) = \frac{1}{2}\langle \mathrm{d}mm^{-1}, t_{\widehat{A}}\rangle f^{\widehat{A}} \end{aligned} \tag{4.10}$$

with

$$\widehat{d} = \overline{d} - \frac{1}{2}\log\det v \tag{4.11}$$

that fixes $\widehat{d}$ up to a constant. The integrability condition for $\mathrm{d}^2\overline{d} = 0$ for this equation follows immediately from the Bianchi identity because

$$\mathrm{d}^2\overline{d} = \frac{1}{4}\langle[\mathrm{d}mm^{-1}, \mathrm{d}mm^{-1}], t_{\widehat{A}}\rangle f^{\widehat{A}} = \frac{1}{4}v^{\widehat{A}} \wedge v^{\widehat{B}}\, f_{\widehat{A}\widehat{B}}{}^{\widehat{C}} f_{\widehat{C}} = 0\,, \tag{4.12}$$

where $v^{\widehat{A}}$ denotes $\mathrm{d}mm^{-1} = t_{\widehat{A}}v^{\widehat{A}}$. This observation has been already made in [42]. Also one should note that according to (4.9) the generalised fluxes $\widehat{f}_{\widehat{A}}$ and $\widehat{f}_{\widehat{A}\widehat{B}\widehat{C}}$ are not completely independent. One half of the former is completely fixed by the latter. It is not possible to break this connection in supergravity. But there is also the framework of generalised supergravity [50], where this additional constraint besides the Bianchi identities is not required [42].

***H*-shift of the coset representative and *B*-field gauge transformations.** The coset representative we used in the last section is defined only up to the action of H from the left. There, one might ask what happens to the generalised frame if we shift $m \to m' = hm$, with $h \in H$. The adjoint action in (4.3) transforms as

$$M'_{\widehat{A}}{}^{\widehat{B}} = M_{\widehat{A}}{}^{\widehat{C}}\Lambda_{\widehat{C}}{}^{\widehat{B}}\,, \qquad hT_{\widehat{A}}h^{-1} =: \Lambda_{\widehat{A}}{}^{\widehat{B}}T_{\widehat{B}}\,. \tag{4.13}$$

The fact that H is an isotropic subgroup guarantees that $\Lambda^{\widehat{a}\widehat{b}}$ vanishes. To evaluate the remaining quantities in (4.3), we compute first

$$\mathrm{d}m'm'^{-1} = h\,\mathrm{d}mm^{-1}\,h^{-1} + \mathrm{d}hh^{-1}\,. \tag{4.14}$$

Defining $\mathrm{d}hh^{-1} = \omega_{\widehat{a}}T^{\widehat{a}}$, one can show that the one forms $v^{\widehat{a}}$ and $A_{\widehat{a}}$ shift as

$$v'^{\widehat{a}} = v^{\widehat{b}}\Lambda_{\widehat{b}}{}^{\widehat{a}}\,, \qquad A'_{\widehat{a}} = v^{\widehat{b}}\Lambda_{\widehat{b}\widehat{a}} + A_{\widehat{b}}\Lambda^{\widehat{b}}{}_{\widehat{a}} + \omega_{\widehat{a}}\,. \tag{4.15}$$

For the WZW contribution to the B-field, we find

$$B'_{\mathrm{WZW}} \cong B_{\mathrm{WZW}} - \frac{1}{2}v^{\widehat{a}}\Lambda_{\widehat{a}}{}^{\widehat{b}}\omega_{\widehat{b}} \tag{4.16}$$

where $\cong$ denotes equality up to an undetermined exact term. This exact term cannot be determined explicitly: the ansatz (4.3) involves only a locally defined B_{WZW} via (4.6). This leads to

$$B' \cong B + \frac{1}{2}v^{\widehat{a}} \wedge v^{\widehat{b}}\Lambda_{\widehat{a}}{}^{\widehat{c}}\Lambda_{\widehat{b}\widehat{c}}. \tag{4.17}$$

The combination $\Lambda_{\widehat{a}}{}^{\widehat{c}}\Lambda_{\widehat{b}\widehat{c}}$ is antisymmetric in $\widehat{a}\widehat{b}$ from the orthogonality condition on Λ. The difference in these two B-fields is not closed, but this is not crucial, because the field denoted B here is not *precisely* the physical B-field; the latter is encoded in the generalized metric (2.4) and $M_A{}^B$ contributes non-trivially. When the shift in M to M' is accounted for, one finds indeed that it cancels the transformations not only of B but also of $\widehat{v}$, leading to

$$\widehat{E}'_{\widehat{A}}{}^{\widehat{I}} = M'_{\widehat{A}}{}^{\widehat{B}}\begin{pmatrix} \widehat{v}'_{\widehat{b}}{}^{\widehat{i}} & \widehat{v}'_{\widehat{b}}{}^{\widehat{j}}B'_{\widehat{j}\widehat{i}} \\ 0 & v'^{\widehat{b}}{}_{\widehat{i}}\end{pmatrix} \cong M_{\widehat{A}}{}^{\widehat{B}}\begin{pmatrix} \widehat{v}_{\widehat{b}}{}^{\widehat{i}} & \widehat{v}_{\widehat{b}}{}^{\widehat{j}}B_{\widehat{j}\widehat{i}} \\ 0 & v^{\widehat{b}}{}_{\widehat{i}}\end{pmatrix} = \widehat{E}_{\widehat{A}}{}^{\widehat{I}} \tag{4.18}$$

where $\cong$ denotes equality up to an exact shift in the B-field. Hence, changing the coset representative m just amounts to making a B-field gauge transformation. This is nice, because it implies that different possible choices for m are all related to each other, justifying the identification of the space as a coset.

As one might expect, the generalised dilaton is unchanged (up to a constant) since it is determined by integrating the expression for $f_{\widehat{a}}$. From (4.10), one finds $\mathrm{d}\overline{d}' = \mathrm{d}\overline{d} + \frac{1}{2}\omega_{\widehat{a}}f^{\widehat{a}}$ where the shift $\omega_{\widehat{a}}f^{\widehat{a}}$ is closed. In fact, it is exact since

$$\mathrm{d}\log\det\Lambda_{\widehat{a}}{}^{\widehat{b}} = f_{\widehat{b}}{}^{\widehat{b}\widehat{a}}\omega_{\widehat{a}} = \omega_{\widehat{a}}f^{\widehat{a}} \tag{4.19}$$

and this relation implies that

$$\widehat{d}' = \widehat{d} + \mathrm{const}\,. \tag{4.20}$$

4.2 Poláček-Siegel form of the mega frame

Next, we have to check if the generalised frame we just constructed can be brought into the Poláček-Siegel form (3.2) introduced in section 3.1. Because this form is tightly constrained by $\mathrm{O}(\widehat{D}, \widehat{D})$, we have only to check the last column, namely

$$\widehat{E}_{\widehat{A}\mu} = \widetilde{M}_{\widehat{A}}{}^{\widehat{B}} \begin{pmatrix} 0 \\ 0 \\ \widetilde{v}^{\beta}{}_{\mu} \end{pmatrix} . \tag{4.21}$$

Considering the insight gained in [15], we expect that this condition can only be satisfied if we consider a dressing coset. Therefore, we first decompose the coset representative m as

$$m = nf \,, \quad \text{with} \quad n \in H\backslash\mathbb{D}/F \quad \text{and} \quad f \in F \,. \tag{4.22}$$

Now, n is a representative of a double coset, which is called a dressing coset in the context of generalised T-duality [12]. In particular, F has to be an isotropic subgroup, but not necessarily maximally isotropic. The coordinates that we choose to parameterise n are called y^i, while for f, we use z^μ. As before, we adapt all constituents of the generalised frame in (4.3) to this new decomposition, starting with

$$M_{\widehat{A}}{}^{\widehat{B}} = \widetilde{M}_{\widehat{A}}{}^{\widehat{C}} \overline{M}_{\widehat{C}}{}^{\widehat{B}} \,, \qquad \text{with} \qquad \widetilde{M}_{\widehat{A}}{}^{\widehat{B}} t_{\widehat{B}} = f t_{\widehat{A}} f^{-1} \tag{4.23}$$

$$\overline{M}_{\widehat{A}}{}^{\widehat{B}} t_{\widehat{B}} = n t_{\widehat{A}} n^{-1} \,. \tag{4.24}$$

We also find

$$B = \frac{1}{2}\left(v^{\widehat{a}} \wedge A_{\widehat{a}} - \langle \mathrm{d}f f^{-1}, n^{-1}\mathrm{d}n \rangle \right) + \overline{B}_{\mathrm{WZW}} \quad \text{with} \tag{4.25}$$

$$\overline{H}_{WZW} = \mathrm{d}\overline{B}_{\mathrm{WZW}} = -\frac{1}{12} \langle \mathrm{d}n n^{-1}, [\mathrm{d}n n^{-1}, \mathrm{d}n n^{-1}] \rangle \,, \tag{4.26}$$

which allows us to compute

$$\iota_{\widetilde{\widetilde{v}}_\alpha} B = -\langle n t_\alpha n^{-1}, t_{\widehat{a}} \rangle v^{\widehat{a}} \,, \tag{4.27}$$

$$\iota_{\widetilde{\widetilde{v}}_\alpha} \iota_{\widetilde{\widetilde{v}}_\beta} B = \langle n t_\alpha n^{-1}, t_{\widehat{b}} \rangle \langle t^{\widehat{b}}, n t_\beta n^{-1} \rangle \,, \tag{4.28}$$

by taking into account that F is isotropic. (Recall that $\widetilde{\widetilde{v}}_\alpha$ is the dual vector field to the Maurer-Cartan form $t_\alpha \widetilde{v}^\alpha{}_\mu \mathrm{d}x^\mu = \mathrm{d}f f^{-1}$, which we introduced in section 3.1.)

Equation (4.21) can now be equivalently written as

$$\iota_{\widetilde{\widetilde{v}}_\beta} \langle n t_{\widehat{A}} n^{-1}, t^{\widehat{c}} \iota_{\widetilde{\widetilde{v}}_{\widehat{c}}} B + t_{\widehat{c}} v^{\widehat{c}} \rangle = \langle t_{\widehat{A}}, t_\beta \rangle \,, \tag{4.29}$$

which is better suited to be checked. Using, (4.27), $v^{\widehat{a}} = \langle t^{\widehat{a}}, \mathrm{d}n n^{-1} + n \mathrm{d}f f^{-1} n^{-1} \rangle$, $\iota_{\widetilde{\widetilde{v}}_\alpha} \mathrm{d}f f^{-1} = t_\alpha$, and $\iota_{\widetilde{\widetilde{v}}_{\widehat{a}}} v^{\widehat{b}} = \delta_{\widehat{a}}^{\widehat{b}}$, one can easily show that this equation indeed holds. Therefore, just using the appropriate B-field given in (4.25), the generalised frame field on the mega-space (4.3) takes the Poláček-Siegel form.

We also have to check if the generalised dilaton $\widehat{d}$, which we fixed in (4.11), is compatible with the ansatz (3.28) on the mega-space. To this end, we first write

$$v^{\widehat{a}}{}_{\widehat{i}} = \underbrace{\begin{pmatrix} \eta^{\widehat{a}}{}_{\beta} & \overline{M}^{\widehat{a}}{}_{b} \end{pmatrix}}_{:= \mathrm{m}} \begin{pmatrix} \widetilde{v}^{\beta}{}_{\mu} & 0 \\ 0 & \overline{v}^{b}{}_{i} \end{pmatrix} , \tag{4.30}$$

with the one-forms $\overline{v}^{a}$ which are defined by $t_a \overline{v}^a{}_i \mathrm{d}y^i = \mathrm{d}nn^{-1}$. After plugging this expression into (4.11), we obtain

$$\widehat{d} = \overline{d} - \frac{1}{2}\log\det \mathrm{m}(y) - \frac{1}{2}\log\det v(y) - \frac{1}{2}\log\det \widetilde{v}(z) \,. \tag{4.31}$$

Comparing this equation with (3.28), we have to identity

$$d(y) = \overline{d} - \frac{1}{2}\log\det \mathrm{m}(y) - \frac{1}{2}\log\det v(y) - \frac{1}{2}\log\det \widetilde{\mathrm{m}}(z) \tag{4.32}$$

with $\widetilde{\mathrm{m}}_{\alpha}{}^{\beta} = \widetilde{M}_{\alpha}{}^{\beta}$. However, this only works if the final result, $d(y)$, does not depend on the auxiliary coordinates z. To check if this is indeed the case, we compute

$$\iota_{\widehat{v}_\alpha} \mathrm{d}d = \iota_{\widehat{v}_\alpha} \mathrm{d}\overline{d} - f_{\alpha\beta}{}^{\beta} = f_{\alpha} - f_{\alpha\beta}{}^{\beta} = 0 \,. \tag{4.33}$$

Fortunately, this is exactly the constraint we already imposed in (3.44). Hence, we conclude that also the dilaton is of the right Poláček-Siegel form.

This completes the proof of the central result of this paper:

Theorem 2. *For every dressing coset $H\backslash\mathbb{D}/F$, where $\mathbb{D}$ and H satisfy the properties 1–3 in section 4.1, and F is an isotropic subgroup of $\mathbb{D}$, there exists a consistent truncation, governed by theorem 1, with generalised structure group F.*

4.3 F-shift and gauge transformations

We already figured out in section 4.1 what happens to the coset representative under H-shifts from the right. To extend this discussion to generalised cosets, we now study the infinitesimal F-action from the left on the representative $n \in H\backslash\mathbb{D}/F$. To this end, we shift $\delta n = n\delta h$ with $\delta h = \lambda^{\alpha} t_{\alpha} := \lambda$. Under this shift, we first find

$$\delta M_{\widehat{A}}{}^{\widehat{B}} = \lambda^{\alpha} \widetilde{M}_{\widehat{A}}{}^{\widehat{C}} \widehat{f}_{\alpha\widehat{C}}{}^{\widehat{D}} \overline{M}_{\widehat{D}}{}^{\widehat{B}} \tag{4.34}$$

and after a bit more of calculation

$$\delta(\mathrm{d}mm^{-1}) = t_{\widehat{a}}\delta v^{\widehat{a}} + t^{\widehat{a}}\delta A_{\widehat{a}} = n(\mathrm{d}\lambda + [\lambda, \mathrm{d}ff^{-1}])n^{-1} \,, \tag{4.35}$$

$$\delta B_{\mathrm{WZW}} = -\frac{1}{2}\langle \delta(\mathrm{d}mm^{-1}), \mathrm{d}nn^{-1}\rangle \,, \quad \text{and} \tag{4.36}$$

$$\delta B = v^{\widehat{a}} \wedge \delta A_{\widehat{a}} \,. \tag{4.37}$$

We also need the transformation of the one-form $v^{\widehat{a}}$ and its dual vector fields $\widehat{v}_{\widehat{a}}$. It is convenient not to treat them separately, but instead combine them into

$$V_{\widehat{A}}{}^{\widehat{I}} = \begin{pmatrix} \widehat{v}_{\widehat{a}}{}^{\widehat{i}} & 0 \\ 0 & v^{\widehat{a}}{}_{\widehat{i}} \end{pmatrix} \tag{4.38}$$

and compute

$$\delta V_{\widehat{A}\widehat{B}} = \overline{M}_{\widehat{A}}{}^{\widehat{C}}\delta V_{\widehat{C}}{}^{\widehat{I}} V^{\widehat{D}}{}_{\widehat{I}}\overline{M}_{\widehat{B}\widehat{D}} = -2\iota_{E_{[\widehat{A}}}\delta v^{\widehat{c}}\overline{M}_{\widehat{B}]\widehat{c}}. \tag{4.39}$$

In the same vein, we introduce

$$\delta B_{\widehat{A}\widehat{B}} = \overline{M}_{\widehat{A}}{}^{\widehat{c}}\overline{M}_{\widehat{B}}{}^{\widehat{d}}\iota_{\widehat{v}_d}\iota_{\widehat{v}_c}\delta B = -2\iota_{E_{[\widehat{A}}}\delta A_{\widehat{c}}\overline{M}_{\widehat{B}]}{}^{\widehat{c}}. \tag{4.40}$$

These two quantities allow us to write the shift of the mega generalised frame in the compact form

$$\delta\widehat{E}_{\widehat{A}\widehat{B}} := \delta\widehat{E}_{\widehat{A}}{}^{\widehat{I}}\widehat{E}_{\widehat{B}\widehat{I}} = \widetilde{M}_{\widehat{A}}{}^{\widehat{C}}\widetilde{M}_{\widehat{B}}{}^{\widehat{D}}\left(-2\nabla'_{[\widehat{C}}\lambda_{\widehat{D}]} + \lambda_{\widehat{C}\widehat{D}}\right) \tag{4.41}$$

with

$$\lambda_{\widehat{A}} = \begin{pmatrix} 0 & 0 & \lambda^{\alpha}\end{pmatrix} \qquad \text{and} \qquad \lambda_{\widehat{A}\widehat{B}} = \lambda^{\gamma} f_{\gamma\widehat{A}\widehat{B}}. \tag{4.42}$$

Comparing these equations with (3.112) and (3.114) in section 3.6, we notice that they are exactly same gauge transformations we discussed in section 3.6. Therefore, we know that they leave the constant generalised fluxes $f_{\widehat{A}\widehat{B}\widehat{C}}$ invariant.

5 Conclusion and outlook

The results in this paper reveal a new, deep connection between consistent truncations and dualities. In particular, we established in theorem 2 that generalised cosets, which underlie the most general formulation of T-duality currently known (excluding mirror symmetry), automatically give rise to a large family of consistent truncations. The relation between them is not immediately obvious, and we therefore developed a new geometrical approach that makes the generalised structure group of the consistent truncation manifest by introducing an auxiliary space.

We showed that the relation represented by the solid arrow in the diagram

$$\text{generalised T-dualities} \;\underset{\dashleftarrow}{\longrightarrow}\; \text{consistent truncations} \tag{5.1}$$

holds. This means that all currently known backgrounds which give rise to generalised T-dualities also produce consistent truncations. We did not yet manage to determine the fate of the other direction, represented by the dashed arrow. There are two interesting alternatives that our analysis currently hints at:

1. Our new approach to generalised cosets suggests that their description in generalised geometry automatically leads to curvatures with more than two derivatives. This brings into question what happens for consistent truncations in (super)gravity beyond the leading, two-derivative level. To the best of our knowledge there is currently not much known. The results from section 3.5 indicate that there might be new constraints that complement those already known from theorem 1. At the moment it seems that we know more consistent truncations than generalised T-dualities. However, these new constraints could level the field and even in the end result in a one-to-one correspondence.

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2. Alternatively, there may also be some new generalised T-dualities, waiting to be found. The relation between the former and consistent truncations, sketched in diagram (1.5), could then be a useful tool for searching for new dualities by studying existing examples of consistent truncations beyond generalised cosets.

Besides these conceptual questions, there are also important applications for our results. They originate from both sides of (5.1).

- Generalised T-dualities: one particularly active sub-branch of this field is concerned with the construction and analysis of integrable deformations. Although it is still not completely understood why, there is a very close relation between generalised T-dualities and integrable string theories. The latter are among the primary means of exploring new concepts in theoretical physics, because they provide a superior level of computational control in comparison to models which are not integrable. All results we derived here apply mostly to bosonic strings or the NS/NS sectors of superstrings. This is sufficient for answering conceptual questions, but for concrete applications, such as integrable Green-Schwarz strings required for probing the AdS/CFT correspondence, the full R/R sector is needed also, together with supersymmetry. We shall address this problem in a forthcoming paper [51], by extending the results of the current paper to supergroups, in a supersymmetric version of double field theory [44] proposed by one of the authors.

- Consistent truncations: the last years have seen significant progress in constructing and understanding consistent truncations. They are mostly centered around exceptional generalised geometry and exceptional field theory, and have applications reaching from the AdS/CFT correspondence to supporting or disproving swampland conjectures. Exceptional field theory goes beyond the string and incorporates membranes too. At the same time, T-duality enhances to U-duality, which takes into account S-duality as well. Another advantage is that R/R fluxes are automatically implemented. However, one should not think that they arise in the same ways as in the Green-Schwarz string discussed above. While, for the latter, supersymmetry and its fermionic degrees of freedom result in R/R fluxes, in exceptional geometry/field theory U-duality is the driving force. In particular, it relates strings and D-branes, which source R/R fluxes. One important consequence is that exceptional field theory still requires an explicit splitting of the spacetime, which makes it perfectly suited for studying consistent truncations in supergravity. Therefore, one should also try to extend the results we have presented here for O(n,n) generalised geometry to the corresponding exceptional groups $E_{n+1(n+1)}$. If possible, and there are no obvious conceptual problems, this would open up a route to many new consistent truncations, with a wide range of applications. On the other hand, it will also shed new light on extending generalised T-duality to U-duality, which has been already initiated [52, 53] based on the results in double field theory.

We hope that there will be more insights into all of these points in the future, strengthening the connection between consistent truncations and dualities even further.

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References

[1] M.J. Duff, B.E.W. Nilsson, C.N. Pope and N.P. Warner, *On the Consistency of the Kaluza-Klein Ansatz*, *Phys. Lett. B* **149** (1984) 90 [INSPIRE].

[2] M. Cvetic, H. Lu and C.N. Pope, *Consistent Kaluza-Klein sphere reductions*, *Phys. Rev. D* **62** (2000) 064028 [hep-th/0003286] [INSPIRE].

[3] J. Scherk and J.H. Schwarz, *How to Get Masses from Extra Dimensions*, *Nucl. Phys. B* **153** (1979) 61 [INSPIRE].

[4] F. Hassler and T. Rochais, *α'-Corrected Poisson-Lie T-Duality*, *Fortsch. Phys.* **68** (2020) 2000063 [arXiv:2007.07897] [INSPIRE].

[5] R. Borsato and L. Wulff, *Quantum Correction to Generalized T Dualities*, *Phys. Rev. Lett.* **125** (2020) 201603 [arXiv:2007.07902] [INSPIRE].

[6] T. Codina and D. Marques, *Generalized Dualities and Higher Derivatives*, *JHEP* **10** (2020) 002 [arXiv:2007.09494] [INSPIRE].

[7] F. Hassler and T.B. Rochais, *O(D,D)-covariant two-loop β-functions and Poisson-Lie T-duality*, *JHEP* **10** (2021) 210 [arXiv:2011.15130] [INSPIRE].

[8] X.C. de la Ossa and F. Quevedo, *Duality symmetries from nonAbelian isometries in string theory*, *Nucl. Phys. B* **403** (1993) 377 [hep-th/9210021] [INSPIRE].

[9] C. Klimčik and P. Severa, *Dual nonAbelian duality and the Drinfeld double*, *Phys. Lett. B* **351** (1995) 455 [hep-th/9502122] [INSPIRE].

[10] C. Klimčik and P. Severa, *Poisson-Lie T duality and loop groups of Drinfeld doubles*, *Phys. Lett. B* **372** (1996) 65 [hep-th/9512040] [INSPIRE].

[11] C. Klimčik and T. Strobl, *WZW — Poisson manifolds*, *J. Geom. Phys.* **43** (2002) 341 [math/0104189] [INSPIRE].

[12] C. Klimčik and P. Severa, *Dressing cosets*, *Phys. Lett. B* **381** (1996) 56 [hep-th/9602162] [INSPIRE].

[13] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is T duality*, *Nucl. Phys. B* **479** (1996) 243 [hep-th/9606040] [INSPIRE].

JHEP04(2023)007

[14] J.-H. Lu and A. Weinstein, *Poisson Lie groups, dressing transformations, and Bruhat decompositions*, J. Diff. Geom. **31** (1990) 501.

[15] S. Demulder, F. Hassler, G. Piccinini and D.C. Thompson, *Generalised Cosets*, JHEP **09** (2020) 044 [arXiv:1912.11036] [INSPIRE].

[16] A. Coimbra, C. Strickland-Constable and D. Waldram, *Supergravity as Generalised Geometry I: Type II Theories*, JHEP **11** (2011) 091 [arXiv:1107.1733] [INSPIRE].

[17] A. Coimbra, C. Strickland-Constable and D. Waldram, *Supergravity as Generalised Geometry II: $E_{d(d)} \times \mathbb{R}^+$ and M theory*, JHEP **03** (2014) 019 [arXiv:1212.1586] [INSPIRE].

[18] W. Siegel, *Superspace duality in low-energy superstrings*, Phys. Rev. D **48** (1993) 2826 [hep-th/9305073] [INSPIRE].

[19] C. Hull and B. Zwiebach, *Double Field Theory*, JHEP **09** (2009) 099 [arXiv:0904.4664] [INSPIRE].

[20] O. Hohm, C. Hull and B. Zwiebach, *Generalized metric formulation of double field theory*, JHEP **08** (2010) 008 [arXiv:1006.4823] [INSPIRE].

[21] K. Lee, C. Strickland-Constable and D. Waldram, *Spheres, generalised parallelisability and consistent truncations*, Fortsch. Phys. **65** (2017) 1700048 [arXiv:1401.3360] [INSPIRE].

[22] O. Hohm and H. Samtleben, *Consistent Kaluza-Klein Truncations via Exceptional Field Theory*, JHEP **01** (2015) 131 [arXiv:1410.8145] [INSPIRE].

[23] D. Cassani, G. Josse, M. Petrini and D. Waldram, *Systematics of consistent truncations from generalised geometry*, JHEP **11** (2019) 017 [arXiv:1907.06730] [INSPIRE].

[24] F. Hassler, *Poisson-Lie T-Duality in Double Field Theory*, Phys. Lett. B **807** (2020) 135455 [arXiv:1707.08624] [INSPIRE].

[25] S. Demulder, F. Hassler and D.C. Thompson, *Doubled aspects of generalised dualities and integrable deformations*, JHEP **02** (2019) 189 [arXiv:1810.11446] [INSPIRE].

[26] Y. Sakatani, *Type II DFT solutions from Poisson-Lie T-duality/plurality*, arXiv:1903.12175 [DOI:10.1093/ptep/ptz071] [INSPIRE].

[27] A. Dabholkar and C. Hull, *Duality twists, orbifolds, and fluxes*, JHEP **09** (2003) 054 [hep-th/0210209] [INSPIRE].

[28] G. Aldazabal, W. Baron, D. Marques and C. Nunez, *The effective action of Double Field Theory*, JHEP **11** (2011) 052 [*Erratum ibid.* **11** (2011) 109] [arXiv:1109.0290] [INSPIRE].

[29] D. Geissbuhler, *Double Field Theory and $N = 4$ Gauged Supergravity*, JHEP **11** (2011) 116 [arXiv:1109.4280] [INSPIRE].

[30] G. Itsios, Y. Lozano, E. O Colgáin and K. Sfetsos, *Non-Abelian T-duality and consistent truncations in type-II supergravity*, JHEP **08** (2012) 132 [arXiv:1205.2274] [INSPIRE].

[31] M. Poláček and W. Siegel, *Natural curvature for manifest T-duality*, JHEP **01** (2014) 026 [arXiv:1308.6350] [INSPIRE].

[32] O. Hohm and S.K. Kwak, *Frame-like Geometry of Double Field Theory*, J. Phys. A **44** (2011) 085404 [arXiv:1011.4101] [INSPIRE].

[33] D. Geissbuhler, D. Marques, C. Nunez and V. Penas, *Exploring Double Field Theory*, JHEP **06** (2013) 101 [arXiv:1304.1472] [INSPIRE].

JHEP04(2023)007

[34] O. Hohm and H. Samtleben, *Gauge theory of Kaluza-Klein and winding modes*, *Phys. Rev. D* **88** (2013) 085005 [arXiv:1307.0039] [INSPIRE].

[35] H. Samtleben, *Lectures on Gauged Supergravity and Flux Compactifications*, *Class. Quant. Grav.* **25** (2008) 214002 [arXiv:0808.4076] [INSPIRE].

[36] M. Trigiante, *Gauged Supergravities*, *Phys. Rept.* **680** (2017) 1 [arXiv:1609.09745] [INSPIRE].

[37] I. Jeon, K. Lee and J.-H. Park, *Differential geometry with a projection: Application to double field theory*, *JHEP* **04** (2011) 014 [arXiv:1011.1324] [INSPIRE].

[38] I. Jeon, K. Lee and J.-H. Park, *Stringy differential geometry, beyond Riemann*, *Phys. Rev. D* **84** (2011) 044022 [arXiv:1105.6294] [INSPIRE].

[39] O. Hohm and B. Zwiebach, *On the Riemann Tensor in Double Field Theory*, *JHEP* **05** (2012) 126 [arXiv:1112.5296] [INSPIRE].

[40] C. Klimčik, *η and λ deformations as E-models*, *Nucl. Phys. B* **900** (2015) 259 [arXiv:1508.05832] [INSPIRE].

[41] A.A. Tseytlin, *Duality Symmetric Formulation of String World Sheet Dynamics*, *Phys. Lett. B* **242** (1990) 163 [INSPIRE].

[42] R. Borsato, S. Driezen and F. Hassler, *An algebraic classification of solution generating techniques*, *Phys. Lett. B* **823** (2021) 136771 [arXiv:2109.06185] [INSPIRE].

[43] C. Klimčík, *Dressing cosets and multi-parametric integrable deformations*, *JHEP* **07** (2019) 176 [arXiv:1903.00439] [INSPIRE].

[44] D. Butter, *Type II double field theory in superspace*, *JHEP* **02** (2023) 187 [arXiv:2209.07296] [INSPIRE].

[45] K. Sfetsos, *Duality invariant class of two-dimensional field theories*, *Nucl. Phys. B* **561** (1999) 316 [hep-th/9904188] [INSPIRE].

[46] Y. Sakatani, *Poisson-Lie T-plurality for dressing cosets*, *PTEP* **2022** (2022) 063B01 [arXiv:2112.14766] [INSPIRE].

[47] F. Hassler, *The Topology of Double Field Theory*, *JHEP* **04** (2018) 128 [arXiv:1611.07978] [INSPIRE].

[48] D. Marques and C.A. Nunez, *T-duality and α'-corrections*, *JHEP* **10** (2015) 084 [arXiv:1507.00652] [INSPIRE].

[49] F. Hassler, D. Lüst and F.J. Rudolph, *Para-Hermitian geometries for Poisson-Lie symmetric σ-models*, *JHEP* **10** (2019) 160 [arXiv:1905.03791] [INSPIRE].

[50] G. Arutyunov et al., *Scale invariance of the η-deformed $AdS_5 \times S^5$ superstring, T-duality and modified type II equations*, *Nucl. Phys. B* **903** (2016) 262 [arXiv:1511.05795] [INSPIRE].

[51] D. Butter, F. Hassler, C.N. Pope and H. Zhang, *Generalised Dualities and Supergroups*, in preparation.

[52] Y. Sakatani, *U-duality extension of Drinfel'd double*, *PTEP* **2020** (2020) 023B08 [arXiv:1911.06320] [INSPIRE].

[53] E. Malek and D.C. Thompson, *Poisson-Lie U-duality in Exceptional Field Theory*, *JHEP* **04** (2020) 058 [arXiv:1911.07833] [INSPIRE].

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