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SU(5) GUTs with A_4 modular symmetry

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ABSTRACT: We combine SU(5) Grand Unified Theories (GUTs) with A_4 modular symmetry and present a comprehensive analysis of the resulting quark and lepton mass matrices for all the simplest cases. Classifying the models according to the representation assignments of the matter fields under A_4 , we find that there are seven types of SU(5) models with A_4 modular symmetry. We present 53 benchmark models with the fewest free parameters. The parameter space of each model is scanned to optimize the agreement between predictions and experimental data, and predictions for the masses and mixing parameters of quarks and leptons are given at the best fitting points. The best fit predictions for the leptonic CP violating Dirac phase, the lightest neutrino mass and the neutrinoless double beta decay parameter when displayed graphically are observed to cover a wide range of possible values, but are clustered around particular regions, allowing future neutrino experiments to discriminate between the different types of models.

KEYWORDS: Discrete Symmetries, GUT, Neutrino Physics, Quark Masses and SM Parameters

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1 Introduction

The standard model (SM) of electroweak interactions based on $SU(2)_L \times U(1)_Y$ is sixty years old this year [1], although the inclusion of the Higgs mechanism and QCD required a further decade of work as has been well documented elsewhere. It is by now well established and has successfully passed many precision tests at various energy levels. After the discovery of the Higgs boson at the LHC, all particles predicted by the three family SM have been observed. Although there is no concrete experimental evidence for new physics beyond the SM, apart from the neutrino masses and mixing, most physicists believe that the SM should be a low energy effective theory rather than a fundamental theory. One essential question of the SM is how to understand the hierarchical mass spectrum and flavor mixing patterns of quarks and leptons, including neutrinos whose masses lie beyond the SM. The masses of quarks and leptons span about 13 orders of magnitude. The Planck collaboration gives $\sum_i m_i < 0.120$ eV which implies the lightest neutrino mass is smaller than 0.04 eV [2], while the top quark mass is around 173 GeV. The three quark mixing angles are hierarchical and the largest one is the Cabibbo angle $\theta_{12}^q \simeq 13.09^\circ$, and the CP phase of the quark sector is $\delta_{CP}^q \simeq 68.53^\circ$ [3]. In contrast to the quark sector, the solar mixing angle $\theta_{12}^l \simeq 33.44^\circ$ and

the atmospheric mixing angle $\theta_{23}^l \simeq 49.2^\circ$ are large, and the reactor mixing angle is small with $\theta_{13}^l \simeq 8.57^\circ$, while the CP violation of the lepton sector has not been confirmed yet [4]. The quark and charged lepton masses are described by different interaction strengths with the Higgs doublets within the SM, but their values cannot be predicted. Understanding the flavor structure of the SM from the first principles is one of the greatest challenges in particle physics. Much effort has been devoted to addressing the flavor puzzle, and symmetry has been an important guiding principle. It is found that the non-Abelian discrete flavor symmetry is particularly suitable to explain the large lepton mixing angles θ_{12}^l and θ_{23}^l [5]. In conventional flavor symmetry models, a complicated vacuum alignment is frequently required, since flavons are generally necessary and their vacuum expectation values (VEVs) should be oriented along certain directions in flavor space.

Recently modular invariance has been suggested to play the role of flavor symmetry, especially in the neutrino sector [6], an approach which is inspired by superstring theory with compactified extra dimensions. The finite discrete flavor symmetry groups arise as the quotient group of modular group $SL(2, \mathbb{Z})$ over the principal congruence subgroups. The quark and lepton fields transform nontrivially under the finite modular groups and are assigned to various modular weights, thus modular invariance requires the Yukawa couplings are modular forms which are holomorphic functions of the complex modulus τ . The flavon fields other than the modulus might not be needed and the flavor symmetry can be entirely broken by the vacuum expectation value (VEV) of the modulus τ . Moreover, all higher dimensional operators in the superpotential are completely fixed by the modular invariance. The modular form of level N and integral weight k can be arranged into some modular multiplets of the homogeneous finite modular group $\Gamma'_N \equiv \Gamma/\Gamma(N)$ [7], and they can be organized into modular multiplets of the inhomogeneous finite modular group $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$ if k is an even number [6]. The inhomogeneous finite modular group Γ_N of lower levels $N = 2$ [8–11], $N = 3$ [6, 8, 9, 12–37], $N = 4$ [25, 38–45], $N = 5$ [43, 46, 47] and $N = 7$ [48] have been considered and a large number of models have been constructed. All the modular forms of integral weights can be generated from the tensor products of weight one modular forms and the odd weight modular forms are in the representations with $\rho_{\mathbf{r}}(S^2) = -1$. The homogeneous finite modular groups Γ'_N provide richer structure of modular forms for flavor model building, and the small groups $\Gamma'_3 \cong T'$ [7, 49], $\Gamma'_4 \cong S'_4$ [50, 51] and $\Gamma'_5 \cong A'_5$ [52] have been studied. If the modular weight k of the operator is not an integer, $(c\tau+d)^k$ is not the automorphy factor anymore and some multiplier is needed, consequently the modular group should be extended to its metaplectic covering [53]. The framework of modular invariance have been extended to include the modular forms of rational weights $k/2$ [53] and $k/5$ [54]. The generalized CP symmetry can be consistently imposed in the context of modular symmetry, the modulus is determined to transform as $\tau \rightarrow -\tau^*$ under the action of CP [55–57]. The CP transformation matrix is completely fixed by the consistency condition up to an overall phase, and it is exactly the canonical CP transformation in the symmetric basis [55]. The more fundamental theory such as string theory sometimes requires several compact space with more than one modulus parametrizing its shape. The modular invariance approach has been extended to incorporate several factorizable [40] and non-factorizable moduli [58].

Grand unified theories (GUTs) are amongst the most well motivated theories beyond the SM, realising the elegant aspiration to unify the three gauge interactions of the SM into a simple gauge group [59]. The fermion representations of the SM are unified into a smaller number of multiplets under the GUT gauge group, with quarks and leptons being assigned to the same GUT multiplet, thereby providing an explanation for charge quantisation. Imposing a family symmetry together with GUTs can help to address the problem of quark and lepton mass and mixing hierarchies [60]. Among many different GUT scenarios, $SU(5)$ [59] is the minimal choice, being the smallest simple group which can accommodate the gauge symmetry of the SM. Although neutrino masses are not predicted by $SU(5)$, they can easily be accommodated as singlet representations of the GUT group. Among the many choices of family symmetry, A_4 is the minimal choice which admits triplet representations [61]. Combining A_4 family symmetry with $SU(5)$ GUTs [62] also requires vacuum alignment of the flavons in order to break the A_4 , and so there is a strong motivation for introducing modular symmetry in such frameworks. Indeed, modular symmetry in the context of $SU(5)$ GUTs was first studied in an $(\Gamma_3 \simeq A_4) \times SU(5)$ model in [14]. Other modular GUT models were subsequently constructed based on $(\Gamma_2 \simeq S_3) \times SU(5)$ [10, 63], and $(\Gamma_4 \simeq S_4) \times SU(5)$ [64].

In this paper we shall perform a comprehensive study of the $\Gamma_3 \simeq A_4$ modular symmetry in the framework of supersymmetric (SUSY) $SU(5)$ GUTs. It is known that the fifteen matter fields in each generation are embedded into two chiral supermultiplets $\bar{\mathbf{5}}$ and $\mathbf{10}$. The $SU(5)$ gauge symmetry is spontaneously broken down to the SM gauge group $SU(3)_c \times SU(2) \times U(1)_Y$ by the VEV of the Higgs $H_{\mathbf{24}}$ in the adjoint representation of $SU(5)$. In the minimal $SU(5)$ model, two Higgs multiplets $H_{\mathbf{5}}$ and $H_{\bar{\mathbf{5}}}$ in the fundamental representation $\mathbf{5}$ and antifundamental representation $\bar{\mathbf{5}}$ of $SU(5)$ further break the SM gauge symmetry into $SU(3)_c \times U(1)_{EM}$. The charged leptons and down-type quarks are in the same GUT multiplets, the down type quark mass matrix is the transpose of the charged lepton mass matrix in the minimal $SU(5)$ such that the masses of the charged leptons and down quarks would be identical. In order to account the different masses of down quarks and charged leptons, additional Higgs multiplet $H_{\bar{\mathbf{45}}}$ is introduced in our models [65]. The neutrinos are massless in the minimal $SU(5)$ GUT because of the absence of right-handed neutrinos. In the present work, we extend the matter contents by including right-handed neutrinos which are $SU(5)$ singlets, and the neutrino masses are generated by the type-I seesaw mechanism [66–70]. The most minimal version of the seesaw mechanism involves two additional right-handed neutrinos [71, 72], consequently both versions of seesaw models with two and three right-handed neutrinos are considered. The A_4 group has three singlet representations $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$ and a triplet representation $\mathbf{3}$. We assume the three Higgs fields are modular invariants and their modular weights are vanishing, and the three generations of matter fields are assigned to transform as singlets or triplet of A_4 . The purpose of this work is to find phenomenologically viable $SU(5)$ GUT models based on A_4 modular symmetry with less parameters. We classify all possible A_4 modular $SU(5)$ GUT models into seven different types according to transformation properties of the matter superfields under A_4 . Intensive numerical analysis is performed for each model and we have optimized the free parameters of the models in order to match the experimental data.

This paper is organized as follows. In section 2, we briefly review modular symmetry and modular forms, and modular forms of level 3 are listed. In section 3, we present the most general form of the A_4 modular quark and lepton mass matrices in $SU(5)$, and all possible A_4 representation assignments of matter fields are considered. In section 4, according to the representation assignments of matter fields, we find that there are seven different types of $SU(5)$ GUT models with A_4 modular symmetry, and some benchmark models are presented. We don't consider the cases in which no matter fields are assigned to triplet of A_4 since generally more free parameters would be involved. In section 5, we show the results of a scan of the parameter space for each benchmark model, we search for the minimum of the χ^2 function to optimize the agreement between predictions and experimental data. Finally we draw the conclusions in section 6. Appendix A gives the best fit values of the input parameters and the predictions for the masses, mixing angles and CP violating phases of quarks and leptons.

2 Modular symmetry and modular forms of level $N = 3$

The modular group $SL(2, \mathbb{Z})$ consists of matrices with integer entries and determinant 1, it acts on the upper half plane as fractional linear(Mobius) transformations:

$$\tau \mapsto \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad \text{Im}(\tau) > 0. \quad (2.1)$$

It is easily verified that the kernel of this action is the center $C = \{I, -I\}$ where I denotes the two dimensional unit matrix. Therefore the group of fractional linear transformations is isomorphic to the projective special linear group $PSL(2, \mathbb{Z}) \cong SL(2, \mathbb{Z})/C = \bar{\Gamma}$ which is the quotient of $SL(2, \mathbb{Z})$ by its center C . Although the group $PSL(2, \mathbb{Z})$ is of infinite order, it can be generated by two transformations S and T ,

$$\begin{aligned} S &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \tau \rightarrow -\frac{1}{\tau}, \\ T &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \tau \rightarrow \tau + 1. \end{aligned} \quad (2.2)$$

Taking the quotient of $PSL(2, \mathbb{Z})$ over the principal congruence subgroup of level N , we can obtain the (inhomogeneous) finite modular group $\Gamma_N \cong \bar{\Gamma}/\bar{\Gamma}(N)$ with

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \pmod{N} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad (2.3)$$

and $\bar{\Gamma}(N) = \Gamma(N)$ for $N > 2$ and $\bar{\Gamma}(N) = \Gamma(N)/\{I, -I\}$ for $N = 1, 2$. Obviously $T^N \in \Gamma(N)$, consequently the finite modular group Γ_N can be generated by the generators S and T satisfying

$$S^2 = (ST)^3 = T^N = 1. \quad (2.4)$$

Additional relations are needed to render the group finite make for $N > 5$ [48]. In the present we are interested in the minimal finite modular group $\Gamma_3 = \bar{\Gamma}/\bar{\Gamma}(3) \cong A_4$ which

can be seen as the symmetry group of the tetrahedron. There are 4 inequivalent irreducible representations of A_4 : three singlets $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$ and a triplet $\mathbf{3}$. In the three singlet representations, the generators S and T are

$$\begin{aligned} \mathbf{1} : S &= 1, & T &= 1, \\ \mathbf{1}' : S &= 1, & T &= \omega, \\ \mathbf{1}'' : S &= 1, & T &= \omega^2, \end{aligned} \tag{2.5}$$

where $\omega = e^{2\pi i/3}$ refers to a cubic root of unity. We shall denote $\mathbf{1} \equiv \mathbf{1}^0$, $\mathbf{1}' \equiv \mathbf{1}^1$ and $\mathbf{1}'' \equiv \mathbf{1}^2$ when constructing models in the following. The triplet representation $\mathbf{3}$ in the basis where T is diagonal is given by

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \tag{2.6}$$

The decompositions of the direct product of A_4 representations are

$$\begin{aligned} \mathbf{1}' \otimes \mathbf{1}' &= \mathbf{1}'', & \mathbf{1}' \otimes \mathbf{1}'' &= \mathbf{1}, & \mathbf{1}'' \otimes \mathbf{1}'' &= \mathbf{1}', \\ \mathbf{3} \otimes \mathbf{3} &= \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3} \oplus \mathbf{3}. \end{aligned} \tag{2.7}$$

Given two triplets $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$, the irreducible representations obtained from their product are:

$$\begin{aligned} \mathbf{1} &= \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2, \\ \mathbf{1}' &= \alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1, \\ \mathbf{1}'' &= \alpha_2\beta_2 + \alpha_1\beta_3 + \alpha_3\beta_1, \\ \mathbf{3}_S &= (2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2, 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1, 2\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1), \\ \mathbf{3}_A &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_1\beta_2 - \alpha_2\beta_1, \alpha_3\beta_1 - \alpha_1\beta_3), \end{aligned} \tag{2.8}$$

where $\mathbf{3}_{S(A)}$ denotes the symmetric (antisymmetric) combination. The modular forms of level N and even weight k span a linear space of finite dimension, and they can be organized into some modular multiplets $Y_{\mathbf{r}}^{(k)}(\tau)$ transforming in the irreducible representation \mathbf{r} of Γ_N up to the automorphy factor,

$$Y_{\mathbf{r}}^{(k)}(\gamma\tau) = (c\tau + d)^k \rho_{\mathbf{r}}(\gamma) Y_{\mathbf{r}}^{(k)}(\tau), \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma}. \tag{2.9}$$

The modular forms of level 3 has been constructed [6], and it turns out that there are only three linearly independent modular forms $Y_{1,2,3}(\tau)$ of weight 2 and level 3,

$$\begin{aligned} Y_1(\tau) &= \frac{i}{2\pi} \left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right], \\ Y_2(\tau) &= \frac{-i}{\pi} \left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2 \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} \right], \\ Y_3(\tau) &= \frac{-i}{\pi} \left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2 \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} \right], \end{aligned} \tag{2.10}$$

where $\eta(\tau)$ is the famous Dedekind eta-function,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i \tau}. \quad (2.11)$$

Thus the q -expansion of $Y_{1,2,3}(\tau)$ reads

$$\begin{aligned} Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + 84q^4 + 72q^5 + \dots, \\ Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + 18q^3 + 14q^4 + \dots), \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + 4q^3 + 8q^4 + \dots). \end{aligned} \quad (2.12)$$

The above three modular forms can be arranged into a triplet $Y_{\mathbf{3}}^{(2)} = (Y_1, Y_2, Y_3)^T$ which transforms as a three-dimensional irreducible representation of A_4 up to the automorphy factor:

$$Y_{\mathbf{3}}^{(2)}(\tau) \xrightarrow{S} Y_{\mathbf{3}}^{(2)}(-1/\tau) = \tau^2 \rho_{\mathbf{3}}(S) Y_{\mathbf{3}}^{(2)}(\tau), \quad Y_{\mathbf{3}}^{(2)}(\tau) \xrightarrow{T} Y_{\mathbf{3}}^{(2)}(\tau + 1) = \rho_{\mathbf{3}}(T) Y_{\mathbf{3}}^{(2)}(\tau), \quad (2.13)$$

where $\rho_{\mathbf{3}}(S)$ and $\rho_{\mathbf{3}}(T)$ are the representation matrices of S and T respectively given in eq. (2.6). The weight $2k$ modular forms of level 3 are homogeneous polynomials $Y_{i_1} Y_{i_2} \dots Y_{i_k}$ of degree k in $Y_i(\tau)$. There are five linearly independent weight four modular forms and they decompose as $\mathbf{3} \oplus \mathbf{1} \oplus \mathbf{1}'$ under A_4 ,

$$\begin{aligned} Y_{\mathbf{3}}^{(4)} &= \frac{1}{2} (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{3}}^{(2)})_{\mathbf{3}} = \begin{pmatrix} Y_1^2 - Y_2 Y_3 \\ Y_3^2 - Y_1 Y_2 \\ Y_2^2 - Y_1 Y_3 \end{pmatrix}, \\ Y_{\mathbf{1}}^{(4)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{3}}^{(2)})_{\mathbf{1}} = Y_1^2 + 2Y_2 Y_3, \\ Y_{\mathbf{1}'}^{(4)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{3}}^{(2)})_{\mathbf{1}'} = Y_3^2 + 2Y_1 Y_2. \end{aligned} \quad (2.14)$$

The tensor products of weight 2 and weight 4 modular forms give rise to weight 6 modular forms,

$$\begin{aligned} Y_{\mathbf{1}}^{(6)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{3}}^{(4)})_{\mathbf{1}} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1 Y_2 Y_3, \\ Y_{\mathbf{3}I}^{(6)} &= Y_{\mathbf{3}}^{(2)} Y_{\mathbf{1}}^{(4)} = (Y_1^2 + 2Y_2 Y_3) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}, \\ Y_{\mathbf{3}II}^{(6)} &= Y_{\mathbf{3}}^{(2)} Y_{\mathbf{1}'}^{(4)} = (Y_3^2 + 2Y_1 Y_2) \begin{pmatrix} Y_3 \\ Y_1 \\ Y_2 \end{pmatrix}. \end{aligned} \quad (2.15)$$

weight k	representation \mathbf{r}
2	$\mathbf{3}$
4	$\mathbf{1}, \mathbf{1}', \mathbf{3}$
6	$\mathbf{1}, \mathbf{3}, \mathbf{3}$
8	$\mathbf{1}, \mathbf{1}', \mathbf{1}'', \mathbf{3}, \mathbf{3}$

Table 1. The transformation properties of the weight k and the level 3 modular forms $Y_{\mathbf{r}}^{(k)}$ under A_4 .

The weight 8 modular forms which can be arranged into two triplets and three singlets of A_4

$$\begin{aligned}
 Y_{\mathbf{1}}^{(8)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{3}I}^{(6)})_{\mathbf{1}} = (Y_1^2 + 2Y_2 Y_3)^2, \\
 Y_{\mathbf{1}'}^{(8)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{3}I}^{(6)})_{\mathbf{1}'} = (Y_1^2 + 2Y_2 Y_3)(Y_3^2 + 2Y_1 Y_2), \\
 Y_{\mathbf{1}''}^{(8)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{3}II}^{(6)})_{\mathbf{1}''} = (Y_3^2 + 2Y_1 Y_2)^2, \\
 Y_{\mathbf{3}I}^{(8)} &= Y_{\mathbf{3}}^{(2)} Y_{\mathbf{1}}^{(6)} = (Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1 Y_2 Y_3) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}, \\
 Y_{\mathbf{3}II}^{(8)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{3}II}^{(6)})_{\mathbf{3}_A} = (Y_3^2 + 2Y_1 Y_2) \begin{pmatrix} Y_2^2 - Y_1 Y_3 \\ Y_1^2 - Y_2 Y_3 \\ Y_3^2 - Y_1 Y_2 \end{pmatrix}.
 \end{aligned} \tag{2.16}$$

Notice that modular forms in all irreducible representations of A_4 appears at weight 8, both $Y_{\mathbf{3}I}^{(6)}(\tau)$ and $Y_{\mathbf{3}I}^{(8)}(\tau)$ are proportional to $Y_{\mathbf{3}}^{(2)}(\tau)$. We summarize the even weight modular forms of level 3 and their transformation under A_4 in table 1.

3 General form of fermion mass matrices

We formulate our models in the framework of supersymmetric SU(5) GUT. The neutrino masses are generated by the type-I seesaw mechanism [66–70], consequently three (or two) right-handed neutrinos N are introduced and they are SU(5) singlets. Thus all left-handed quark and lepton superfields within each family are embedded into three SU(5) multiplets $\mathbf{1}$, $\bar{\mathbf{5}}$ and $\mathbf{10}$ with

$$N = \nu^c, \quad \bar{F} = \begin{pmatrix} d_r^c \\ d_g^c \\ d_b^c \\ e \\ -\nu \end{pmatrix}, \quad T = \begin{pmatrix} 0 & u_b^c & -u_g^c & -u_r & -d_r \\ -u_b^c & 0 & u_r^c & -u_g & -d_g \\ u_g^c & -u_r^c & 0 & -u_b & -d_b \\ u_r & u_g & u_b & 0 & e^c \\ d_r & d_g & d_b & -e^c & 0 \end{pmatrix}, \tag{3.1}$$

where the superscript c denotes charge conjugation of the right-handed superfields. The weight and transformation of the above matter superfields under A_4 are denoted as k_{ψ} and ρ_{ψ} respectively with $\psi = N, \bar{F}, T$. Moreover, the SU(5) Higgs fields H_5 and $H_{\bar{5}}, H_{\bar{45}}$ is

considered to account for the mass differences of down quarks and charged leptons. All the three Higgs fields are assumed to be invariant under A_4 and their modular weights are zero. Each of these GUT Higgs representations contains an $SU(2)_L$ Higgs doublet, the low energy doublet H_u in the minimal supersymmetric standard model arises from H_5 , and H_d originates from a linear combination of $H_{\bar{5}}$ and $H_{\bar{45}}$. The three generations of F , T and N can transform as either a triplet $\mathbf{3}$ or three singlets under A_4 modular symmetry. The most general superpotential for quark and lepton masses is of the following form,

$$\mathcal{W} = NNf_M(Y) + N\bar{F}H_5f_N(Y) + \bar{F}TH_{\bar{5}}f_D(Y) + \bar{F}TH_{\bar{45}}f'_D(Y) + TTH_5f_U(Y), \quad (3.2)$$

where $f_M(Y)$, $f_N(Y)$, $f_D(Y)$, $f'_D(Y)$ and $f_U(Y)$ are functions of modular forms, and they can be determined by the requirement that each term of \mathcal{W} is modular invariant with total vanishing modular weight. Because both $H_{\bar{5}}$ and $H_{\bar{45}}$ are assumed to be A_4 invariant singlet and their modular weights are vanishing, the modular functions $f_D(Y)$ and $f'_D(Y)$ are of the same form. In the following, we shall present the most general form of the mass matrices of quarks and leptons in $SU(5)$ models with A_4 modular symmetry. We shall consider modular forms up to weight 8 in the following, and extension to higher weight modular forms is straightforward.

3.1 Majorana mass matrix of the right-handed neutrinos

The first term $NNf_M(Y)$ in eq. (3.2) gives rise to Majorana mass matrix of the right-handed neutrinos, and the modular function $f_M(Y)$ is completely fixed by the weight and representation of N .

- $\rho_N = \mathbf{3}$

Using the contraction rules in eq. (2.8), we can read off the mass matrix of the right-handed neutrinos for different values of N 's weight k_N ,

$$\begin{aligned} k_N = 0 & : M_N = \Lambda S_{\mathbf{1}}^{(0)}, \\ k_N = 1 & : M_N = \Lambda S_{\mathbf{3}}^{(2)}, \\ k_N = 2 & : M_N = \Lambda_1 S_{\mathbf{3}}^{(4)} + \Lambda_2 S_{\mathbf{1}}^{(4)} + \Lambda_3 S_{\mathbf{1}'}^{(4)}, \\ k_N = 3 & : M_N = \Lambda_1 S_{\mathbf{3}I}^{(6)} + \Lambda_2 S_{\mathbf{3}II}^{(6)} + \Lambda_3 S_{\mathbf{1}}^{(6)}, \\ k_N = 4 & : M_N = \Lambda_1 S_{\mathbf{3}I}^{(8)} + \Lambda_2 S_{\mathbf{3}II}^{(8)} + \Lambda_3 S_{\mathbf{1}}^{(8)} + \Lambda_4 S_{\mathbf{1}'}^{(8)} + \Lambda_5 S_{\mathbf{1}''}^{(8)}, \end{aligned} \quad (3.3)$$

where Λ_i are the characteristic scale of flavour dynamics. For simplicity we have defined

$$S_{\mathbf{1}}^{(k)} = Y_{\mathbf{1}}^{(k)}(\tau) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_{\mathbf{1}'}^{(k)} = Y_{\mathbf{1}'}^{(k)}(\tau) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_{\mathbf{1}''}^{(k)} = Y_{\mathbf{1}''}^{(k)}(\tau) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.4)$$

and

$$S_{\mathbf{3}}^{(k)}(\tau) = \begin{pmatrix} 2Y_{\mathbf{3}1}^{(k)}(\tau) & -Y_{\mathbf{3}3}^{(k)}(\tau) & -Y_{\mathbf{3}2}^{(k)}(\tau) \\ -Y_{\mathbf{3}3}^{(k)}(\tau) & 2Y_{\mathbf{3}2}^{(k)}(\tau) & -Y_{\mathbf{3}1}^{(k)}(\tau) \\ -Y_{\mathbf{3}2}^{(k)}(\tau) & -Y_{\mathbf{3}1}^{(k)}(\tau) & 2Y_{\mathbf{3}3}^{(k)}(\tau) \end{pmatrix}, \quad (3.5)$$

where we denote $Y_{\mathbf{1}}^{(0)}(\tau) = 1$ and $\mathbf{3}$ stands for $\mathbf{3}$, $\mathbf{3}I$ or $\mathbf{3}II$.

- $\rho_N = \mathbf{1}, \mathbf{1}', \mathbf{1}''$

In this case, $f_M(Y)$ has to be modular form transforming as singlet under A_4 . It is known that the mass matrix M_N of right-handed neutrinos is a symmetric matrix. For the modular weight $k_{N_i} + k_{N_j} \leq 8$, the possible nonzero elements depend on their weights and A_4 kronecker products

$$\begin{aligned}
 k_{N_i} + k_{N_j} = 0 & : (M_N)_{ij} = \alpha_{ij}^M \Lambda \text{ for } \rho_{N_i} \otimes \rho_{N_j} = \mathbf{1}, \\
 k_{N_i} + k_{N_j} = 4 & : (M_N)_{ij} = \begin{cases} \alpha_{ij}^M \Lambda Y_{\mathbf{1}}^{(4)}(\tau) & \text{for } \rho_{N_i} \otimes \rho_{N_j} = \mathbf{1}, \\ \alpha_{ij}^M \Lambda Y_{\mathbf{1}'}^{(4)}(\tau) & \text{for } \rho_{N_i} \otimes \rho_{N_j} = \mathbf{1}'', \end{cases} \\
 k_{N_i} + k_{N_j} = 6 & : (M_N)_{ij} = \alpha_{ij}^M \Lambda Y_{\mathbf{1}}^{(6)}(\tau) \text{ for } \rho_{N_i} \otimes \rho_{N_j} = \mathbf{1}, \\
 k_{N_i} + k_{N_j} = 8 & : (M_N)_{ij} = \begin{cases} \alpha_{ij}^M \Lambda Y_{\mathbf{1}}^{(8)}(\tau) & \text{for } \rho_{N_i} \otimes \rho_{N_j} = \mathbf{1}, \\ \alpha_{ij}^M \Lambda Y_{\mathbf{1}''}^{(8)}(\tau) & \text{for } \rho_{N_i} \otimes \rho_{N_j} = \mathbf{1}', \\ \alpha_{ij}^M \Lambda Y_{\mathbf{1}'}^{(8)}(\tau) & \text{for } \rho_{N_i} \otimes \rho_{N_j} = \mathbf{1}'', \end{cases} \quad (3.6)
 \end{aligned}$$

where α_{ij}^M are coupling constants.

3.2 Dirac neutrino Yukawa coupling

The Dirac neutrino Yukawa coupling arises from the term $N\bar{F}H_5 f_N(Y)$ in eq. (3.2). In the following, we report the Dirac neutrino Yukawa coupling for different weight and representation assignments of N and \bar{F} .

- $\rho_N = \rho_{\bar{F}} = \mathbf{3}$

For different values of k_N and $k_{\bar{F}}$, we can read out the neutrino Yukawa couplings \mathcal{Y}_5^ν as follows.

$$\begin{aligned}
 k_N + k_{\bar{F}} = 0 & : \mathcal{Y}_5^\nu = S_{\mathbf{1}}^{(0)}, \\
 k_N + k_{\bar{F}} = 2 & : \mathcal{Y}_5^\nu = S_{\mathbf{3}}^{(2)} + A_{\mathbf{3}}^{(2)}, \\
 k_N + k_{\bar{F}} = 4 & : \mathcal{Y}_5^\nu = S_{\mathbf{3}}^{(4)} + A_{\mathbf{3}}^{(4)} + S_{\mathbf{1}}^{(4)} + S_{\mathbf{1}'}^{(4)}, \\
 k_N + k_{\bar{F}} = 6 & : \mathcal{Y}_5^\nu = S_{\mathbf{3}I}^{(6)} + A_{\mathbf{3}I}^{(6)} + S_{\mathbf{3}II}^{(6)} + A_{\mathbf{3}II}^{(6)} + S_{\mathbf{1}}^{(6)}, \\
 k_N + k_{\bar{F}} = 8 & : \mathcal{Y}_5^\nu = S_{\mathbf{3}I}^{(8)} + A_{\mathbf{3}I}^{(8)} + S_{\mathbf{3}II}^{(8)} + A_{\mathbf{3}II}^{(8)} + S_{\mathbf{1}}^{(8)} + S_{\mathbf{1}'}^{(8)} + S_{\mathbf{1}''}^{(8)}, \quad (3.7)
 \end{aligned}$$

where we have omitted the coupling coefficients of each independent term and

$$A_{\mathbf{3}}^{(k)}(\tau) = \begin{pmatrix} 0 & Y_{\mathbf{3},\mathbf{3}}^{(k)}(\tau) & -Y_{\mathbf{3},\mathbf{2}}^{(k)}(\tau) \\ -Y_{\mathbf{3},\mathbf{3}}^{(k)}(\tau) & 0 & Y_{\mathbf{3},\mathbf{1}}^{(k)}(\tau) \\ Y_{\mathbf{3},\mathbf{2}}^{(k)}(\tau) & -Y_{\mathbf{3},\mathbf{1}}^{(k)}(\tau) & 0 \end{pmatrix}. \quad (3.8)$$

- $\rho_N = \mathbf{3}, \rho_{\bar{F}} = \mathbf{1}, \mathbf{1}', \mathbf{1}''$

In this case, the modular function $f^N(Y)$ should be a modular form in the representation $\mathbf{3}$ of A_4 . For notational simplicity, we denote $\mathbf{1}^{0,1,2} = \mathbf{1}, \mathbf{1}', \mathbf{1}''$ and $(\rho_{\bar{F}_1}, \rho_{\bar{F}_2}, \rho_{\bar{F}_3}) =$

$(\mathbf{1}^{a_{\bar{F}_1}}, \mathbf{1}^{a_{\bar{F}_2}}, \mathbf{1}^{a_{\bar{F}_3}})$ with $a_{\bar{F}_j} = 0, 1, 2$. Then the (ij) element of the Dirac neutrino Yukawa coupling matrix is given by,

$$\begin{aligned}
 k_N + k_{\bar{F}_j} = 2, 4 : \quad (\mathcal{Y}_5^\nu)_{ij} &= \alpha_{\nu_j} Y_{\mathbf{3}, \langle i+a_{\bar{F}_j} \rangle}^{(k_N+k_{\bar{F}_j})}, \\
 k_N + k_{\bar{F}_j} = 6, 8 : \quad (\mathcal{Y}_5^\nu)_{ij} &= \alpha_{\nu_j} Y_{\mathbf{3I}, \langle i+a_{\bar{F}_j} \rangle}^{(k_N+k_{\bar{F}_j})} + \alpha'_{\nu_j} Y_{\mathbf{3II}, \langle i+a_{\bar{F}_j} \rangle}^{(k_N+k_{\bar{F}_j})},
 \end{aligned} \tag{3.9}$$

with $\langle 1 \rangle = \langle 4 \rangle = 1$, $\langle 2 \rangle = \langle 5 \rangle = 3$ and $\langle 0 \rangle = \langle 3 \rangle = 2$. Here i and j are the generation indices of N and \bar{F} respectively. The neutrino Yukawa couplings can be compactly written as

$$\mathcal{Y}_5^\nu = \left(C_{a_{\bar{F}_1}}^{k_N+k_{\bar{F}_1}}, C_{a_{\bar{F}_2}}^{k_N+k_{\bar{F}_2}}, C_{a_{\bar{F}_3}}^{k_N+k_{\bar{F}_3}} \right), \tag{3.10}$$

with

$$C_{a_{\bar{F}_j}}^{k_N+k_{\bar{F}_j}} = \begin{pmatrix} Y_{\mathbf{3}, \langle 1+a_{\bar{F}_j} \rangle}^{(k_N+k_{\bar{F}_j})}(\tau) \\ Y_{\mathbf{3}, \langle 2+a_{\bar{F}_j} \rangle}^{(k_N+k_{\bar{F}_j})}(\tau) \\ Y_{\mathbf{3}, \langle 3+a_{\bar{F}_j} \rangle}^{(k_N+k_{\bar{F}_j})}(\tau) \end{pmatrix}. \tag{3.11}$$

To be more specific, we have

$$\begin{aligned}
 k_N + k_{\bar{F}_j} = 2, 4 : \quad C_{a_{\bar{F}_j}}^{k_N+k_{\bar{F}_j}} &= \begin{cases} \left(Y_{\mathbf{3},1}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3},3}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3},2}^{(k_N+k_{\bar{F}_j})} \right)^T & \text{for } a_{\bar{F}_j} = 0, \\ \left(Y_{\mathbf{3},3}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3},2}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3},1}^{(k_N+k_{\bar{F}_j})} \right)^T & \text{for } a_{\bar{F}_j} = 1, \\ \left(Y_{\mathbf{3},2}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3},1}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3},3}^{(k_N+k_{\bar{F}_j})} \right)^T & \text{for } a_{\bar{F}_j} = 2, \end{cases} \\
 k_N + k_{\bar{F}_j} = 6, 8 : \quad C_{a_{\bar{F}_j}}^{k_N+k_{\bar{F}_j}} &= \begin{cases} \left(Y_{\mathbf{3I},1}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3I},3}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3I},2}^{(k_N+k_{\bar{F}_j})} \right)^T \\ + \left(Y_{\mathbf{3II},1}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3II},3}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3II},2}^{(k_N+k_{\bar{F}_j})} \right)^T & \text{for } a_{\bar{F}_j} = 0, \\ \left(Y_{\mathbf{3I},3}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3I},2}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3I},1}^{(k_N+k_{\bar{F}_j})} \right)^T \\ + \left(Y_{\mathbf{3II},3}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3II},2}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3II},1}^{(k_N+k_{\bar{F}_j})} \right)^T & \text{for } a_{\bar{F}_j} = 1, \\ \left(Y_{\mathbf{3I},2}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3I},1}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3I},3}^{(k_N+k_{\bar{F}_j})} \right)^T \\ + \left(Y_{\mathbf{3II},2}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3II},1}^{(k_N+k_{\bar{F}_j})}, Y_{\mathbf{3II},3}^{(k_N+k_{\bar{F}_j})} \right)^T & \text{for } a_{\bar{F}_j} = 2, \end{cases}
 \end{aligned}$$

where we have omitted the overall coefficient associated with each independent contraction for every column.

- $\rho_N = \mathbf{1}, \mathbf{1}', \mathbf{1}'', \rho_{\bar{F}} = \mathbf{3}$

Similar to previous case, we denote $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}) = (\mathbf{1}^{a_{N_1}}, \mathbf{1}^{a_{N_2}}, \mathbf{1}^{a_{N_3}})$ with $a_{N_i} = 0, 1, 2$ and the neutrino Yukawa couplings read as

$$\begin{aligned} k_{N_i} + k_{\bar{F}} = 2, 4 & : (\mathcal{Y}_5^\nu)_{ij} = \alpha_{\nu_i} Y_{\mathbf{3}, \langle j+a_{N_i} \rangle}^{(k_{N_i}+k_{\bar{F}})}, \\ k_{N_i} + k_{\bar{F}} = 6, 8 & : (\mathcal{Y}_5^\nu)_{ij} = \alpha_{\nu_i} Y_{\mathbf{3}I, \langle j+a_{N_i} \rangle}^{(k_{N_i}+k_{\bar{F}})} + \alpha'_{\nu_i} Y_{\mathbf{3}II, \langle j+a_{N_i} \rangle}^{(k_{N_i}+k_{\bar{F}})}, \end{aligned} \quad (3.12)$$

where i and j are the flavor indices of N and \bar{F} respectively, α_{ν_i} and α'_{ν_i} are free coupling constants. We can also write

$$\mathcal{Y}_5^\nu = \left(C_{a_{N_1}}^{k_{N_1}+k_{\bar{F}}}, C_{a_{N_2}}^{k_{N_2}+k_{\bar{F}}}, C_{a_{N_3}}^{k_{N_3}+k_{\bar{F}}} \right)^T. \quad (3.13)$$

3.3 The charged lepton and down quark sectors

The masses of the charged lepton and down type quarks are described by the terms $\bar{F}TH_{\bar{5}}f_d(Y)$ and $\bar{F}TH_{\bar{45}}f'_d(Y)$ in eq. (3.2). Modular invariance strongly constrains the form of the modular functions $f_D(Y)$ and $f'_D(Y)$.

- $\rho_{\bar{F}} = \rho_T = \mathbf{3}$

The Yukawa coupling with the Higgs $H_{\bar{5}}$ is determined by the modular weight of matter fields.

$$\begin{aligned} k_{\bar{F}} + k_T = 0 & : \mathcal{Y}_{\bar{5}}^d = S_{\mathbf{1}}^{(0)}, \\ k_{\bar{F}} + k_T = 2 & : \mathcal{Y}_{\bar{5}}^d = S_{\mathbf{3}}^{(2)} + A_{\mathbf{3}}^{(2)}, \\ k_{\bar{F}} + k_T = 4 & : \mathcal{Y}_{\bar{5}}^d = S_{\mathbf{3}}^{(4)} + A_{\mathbf{3}}^{(4)} + S_{\mathbf{1}}^{(4)} + S_{\mathbf{1}'}^{(4)}, \\ k_{\bar{F}} + k_T = 6 & : \mathcal{Y}_{\bar{5}}^d = S_{\mathbf{3}I}^{(6)} + A_{\mathbf{3}I}^{(6)} + S_{\mathbf{3}II}^{(6)} + A_{\mathbf{3}II}^{(6)} + S_{\mathbf{1}}^{(6)}, \\ k_{\bar{F}} + k_T = 8 & : \mathcal{Y}_{\bar{5}}^d = S_{\mathbf{3}I}^{(8)} + A_{\mathbf{3}I}^{(8)} + S_{\mathbf{3}II}^{(8)} + A_{\mathbf{3}II}^{(8)} + S_{\mathbf{1}}^{(8)} + S_{\mathbf{1}'}^{(8)} + S_{\mathbf{1}''}^{(8)}, \end{aligned} \quad (3.14)$$

where the dimensionless coefficient in front of each term has been omitted. Analogously the Yukawa coupling $\mathcal{Y}_{\bar{45}}^d$ with $H_{\bar{45}}$ can be obtained, and it is found to be of the same form as $\mathcal{Y}_{\bar{5}}^d$ because both $H_{\bar{5}}$ and $H_{\bar{45}}$ are assumed to be A_4 singlets with vanishing modular weight.

- $\rho_{\bar{F}} = \mathbf{3}, \rho_T = \mathbf{1}, \mathbf{1}', \mathbf{1}''$

The charged lepton and down quark Yukawa couplings depend on the A_4 representation of the ten-dimensional matter fields T and their weights. We can read out

$$\begin{aligned} k_{\bar{F}} + k_{T_j} = 2, 4 & : (\mathcal{Y}_{\bar{5}}^d)_{ij} = \alpha_{d_j} Y_{\mathbf{3}, \langle i+a_{T_j} \rangle}^{(k_{\bar{F}}+k_{T_j})}, \\ k_{\bar{F}} + k_{T_j} = 6, 8 & : (\mathcal{Y}_{\bar{5}}^d)_{ij} = \alpha_{d_j} Y_{\mathbf{3}I, \langle i+a_{T_j} \rangle}^{(k_{\bar{F}}+k_{T_j})} + \alpha'_{d_j} Y_{\mathbf{3}II, \langle i+a_{T_j} \rangle}^{(k_{\bar{F}}+k_{T_j})}, \end{aligned} \quad (3.15)$$

where α_{d_j} and α'_{d_j} are coupling constants, and i and j stand for the flavor indices of \bar{F} and T respectively. Note $\mathcal{Y}_{\bar{45}}^d$ is of the same form as $\mathcal{Y}_{\bar{5}}^d$.

- $\rho_{\bar{F}} = \mathbf{1}, \mathbf{1}', \mathbf{1}'', \rho_T = \mathbf{3}$

For this assignment, the (ij) element of the Yukawa coupling matrix is found to be

$$\begin{aligned}
 k_{\bar{F}_i} + k_T = 2, 4 & : (\mathcal{Y}_{\bar{5}}^d)_{ij} = \alpha_{d_i} Y_{\mathbf{3}, \langle j+a_{\bar{F}_i} \rangle}^{(k_{\bar{F}_i}+k_T)}, \\
 k_{\bar{F}_i} + k_T = 6, 8 & : (\mathcal{Y}_{\bar{5}}^d)_{ij} = \alpha_{d_i} Y_{\mathbf{3I}, \langle j+a_{\bar{F}_i} \rangle}^{(k_{\bar{F}_i}+k_T)} + \alpha'_{d_i} Y_{\mathbf{3II}, \langle j+a_{\bar{F}_i} \rangle}^{(k_{\bar{F}_i}+k_T)}. \quad (3.16)
 \end{aligned}$$

The electroweak Higgs field H_d is a linear combination of the doublet components of the Higgs fields $H_{\bar{5}}$ and $H_{\bar{45}}$, and the mixing angle can be absorbed into the coupling constants. Hence the charged lepton and down quark mass matrices are of the following form:

$$M_e = (\mathcal{Y}_{\bar{5}}^d - 3\mathcal{Y}_{\bar{45}}^d)^T v_d, \quad M_d = (\mathcal{Y}_{\bar{5}}^d + \mathcal{Y}_{\bar{45}}^d) v_d. \quad (3.17)$$

The Georgi-Jarlskog factor “−3” can account for the mass differences of charged leptons and down quarks [65].

3.4 The up quark sector

The up quark mass matrix is symmetric in the SU(5) GUT theory and it is predicted to be of the same form as the right-handed neutrino mass matrix M_N in the current context of A_4 modular symmetry.

- $\rho_T = \mathbf{3}$

We can read out the up quark Yukawa coupling matrix for different values of modular weight k_T as follows,

$$\begin{aligned}
 k_T = 0 & : \mathcal{Y}_{\bar{5}}^u = S_{\mathbf{1}}^{(0)}, \\
 k_T = 1 & : \mathcal{Y}_{\bar{5}}^u = S_{\mathbf{3}}^{(2)}, \\
 k_T = 2 & : \mathcal{Y}_{\bar{5}}^u = S_{\mathbf{3}}^{(4)} + S_{\mathbf{1}}^{(4)} + S_{\mathbf{1}'}^{(4)}, \\
 k_T = 3 & : \mathcal{Y}_{\bar{5}}^u = S_{\mathbf{3I}}^{(6)} + S_{\mathbf{3II}}^{(6)} + S_{\mathbf{1}}^{(6)}, \\
 k_T = 4 & : \mathcal{Y}_{\bar{5}}^u = S_{\mathbf{3I}}^{(8)} + S_{\mathbf{3II}}^{(8)} + S_{\mathbf{1}}^{(8)} + S_{\mathbf{1}'}^{(8)} + S_{\mathbf{1}''}^{(8)}. \quad (3.18)
 \end{aligned}$$

- $\rho_T = \mathbf{1}, \mathbf{1}', \mathbf{1}''$

For the modular weight $k_{T_i} + k_{T_j} \leq 8$, the possible non-vanishing elements of $\mathcal{Y}_{\bar{5}}^u$ are

$$\begin{aligned}
 k_{T_i} + k_{T_j} = 0 & : (\mathcal{Y}_{\bar{5}}^u)_{ij} = \alpha_{ij}^u \text{ for } \rho_{T_i} \otimes \rho_{T_j} = \mathbf{1}, \\
 k_{T_i} + k_{T_j} = 4 & : (\mathcal{Y}_{\bar{5}}^u)_{ij} = \begin{cases} \alpha_{ij}^u Y_{\mathbf{1}}^{(4)}(\tau) & \text{for } \rho_{T_i} \otimes \rho_{T_j} = \mathbf{1}, \\ \alpha_{ij}^u Y_{\mathbf{1}'}^{(4)}(\tau) & \text{for } \rho_{T_i} \otimes \rho_{T_j} = \mathbf{1}'', \end{cases} \\
 k_{T_i} + k_{T_j} = 6 & : (\mathcal{Y}_{\bar{5}}^u)_{ij} = \alpha_{ij}^u Y_{\mathbf{1}}^{(6)}(\tau) \text{ for } \rho_{T_i} \otimes \rho_{T_j} = \mathbf{1}, \\
 k_{T_i} + k_{T_j} = 8 & : (\mathcal{Y}_{\bar{5}}^u)_{ij} = \begin{cases} \alpha_{ij}^u Y_{\mathbf{1}}^{(8)}(\tau) & \text{for } \rho_{T_i} \otimes \rho_{T_j} = \mathbf{1}, \\ \alpha_{ij}^u Y_{\mathbf{1}''}^{(8)}(\tau) & \text{for } \rho_{T_i} \otimes \rho_{T_j} = \mathbf{1}', \\ \alpha_{ij}^u Y_{\mathbf{1}'}^{(8)}(\tau) & \text{for } \rho_{T_i} \otimes \rho_{T_j} = \mathbf{1}''. \end{cases} \quad (3.19)
 \end{aligned}$$

4 Benchmark models

We can classify all the possible A_4 modular $SU(5)$ models according to the transformation properties of the matter fields N , \bar{F} and T under A_4 . We don't consider the cases in which all matter fields are assigned to A_4 singlets, since the Yukawa superpotentials would be less constrained by modular symmetry and generally more free parameters would be involved to produce the observed mass and mixing patterns of quarks and leptons. If three right-handed neutrinos are introduced in the seesaw mechanism, we have the following five types of models:

$$\text{Type-I : } \quad N \sim \mathbf{3}, \quad \bar{F} \sim \mathbf{3}, \quad T_{1,2,3} \sim \mathbf{1}(\mathbf{1}', \mathbf{1}''), \quad (4.1)$$

$$\text{Type-II : } \quad N \sim \mathbf{3}, \quad \bar{F}_{1,2,3} \sim \mathbf{1}(\mathbf{1}', \mathbf{1}''), \quad T \sim \mathbf{3}, \quad (4.2)$$

$$\text{Type-III : } \quad N_{1,2,3} \sim \mathbf{1}(\mathbf{1}', \mathbf{1}''), \quad \bar{F} \sim \mathbf{3}, \quad T \sim \mathbf{3}, \quad (4.3)$$

$$\text{Type-IV : } \quad N \sim \mathbf{3}, \quad \bar{F} \sim \mathbf{3}, \quad T \sim \mathbf{3}, \quad (4.4)$$

$$\text{Type-V : } \quad N_{1,2,3} \sim \mathbf{1}(\mathbf{1}', \mathbf{1}''), \quad \bar{F} \sim \mathbf{3}, \quad T_{1,2,3} \sim \mathbf{1}(\mathbf{1}', \mathbf{1}''). \quad (4.5)$$

As shown in ref. [20], there are ten possibilities for the singlet assignment of the three generations of matter fields,

$$\begin{aligned} C_1 : \{ \mathbf{1}, \mathbf{1}, \mathbf{1} \}, \quad C_2 : \{ \mathbf{1}', \mathbf{1}', \mathbf{1}' \}, \quad C_3 : \{ \mathbf{1}'', \mathbf{1}'', \mathbf{1}'' \}, \\ C_4 : \{ \mathbf{1}, \mathbf{1}, \mathbf{1}' \}, \quad C_5 : \{ \mathbf{1}, \mathbf{1}, \mathbf{1}'' \}, \quad C_6 : \{ \mathbf{1}', \mathbf{1}', \mathbf{1} \}, \quad (4.6) \\ C_7 : \{ \mathbf{1}', \mathbf{1}', \mathbf{1}'' \}, \quad C_8 : \{ \mathbf{1}'', \mathbf{1}'', \mathbf{1} \}, \quad C_9 : \{ \mathbf{1}'', \mathbf{1}'', \mathbf{1}' \}, \quad C_{10} : \{ \mathbf{1}, \mathbf{1}', \mathbf{1}'' \}. \end{aligned}$$

The three generations of matter fields are distinguished by their modular weights and transformation rules under A_4 . If any two generations of fermion fields are assigned to the same singlet representation of A_4 , their modular weight should be different. One can also permute the above singlet assignments together with modular weights, for instance, the three generations of matter fields can be assigned to transform as $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{1}''$ or $\mathbf{1}'$, $\mathbf{1}$, $\mathbf{1}''$, this amounts to a redefinition of the fields. As a consequence, the predictions for the masses and mixing matrices of the quarks and leptons would keep invariant.

Moreover, if neutrino masses are generated by the seesaw mechanism with two right-handed neutrinos which transform as singlets under A_4 , then we have another two kinds of models:

$$\text{Type-VI : } \quad N_{1,2} \sim \mathbf{1}(\mathbf{1}', \mathbf{1}''), \quad \bar{F} \sim \mathbf{3}, \quad T \sim \mathbf{3}, \quad (4.7)$$

$$\text{Type-VII : } \quad N_{1,2} \sim \mathbf{1}(\mathbf{1}', \mathbf{1}''), \quad \bar{F} \sim \mathbf{3}, \quad T_{1,2,3} \sim \mathbf{1}(\mathbf{1}', \mathbf{1}''). \quad (4.8)$$

As is generally true in two right-handed neutrino models, the lightest neutrino would be massless. The right-handed neutrinos N_1 and N_2 can transform in either the same or different manner under A_4 ,

$$\begin{aligned} D_1 : \{ \mathbf{1}, \mathbf{1} \}, \quad D_2 : \{ \mathbf{1}', \mathbf{1}' \}, \quad D_3 : \{ \mathbf{1}'', \mathbf{1}'' \}, \\ D_4 : \{ \mathbf{1}, \mathbf{1}' \}, \quad D_5 : \{ \mathbf{1}, \mathbf{1}'' \}, \quad D_6 : \{ \mathbf{1}', \mathbf{1}'' \}. \quad (4.9) \end{aligned}$$

Note that N_1 and N_2 are assigned to the same A_4 singlet for the cases of $D_{1,2,3}$ and they should carry different modular weights to be distinguished. For the Type-VII models, the three generations of the ten-plets T_1 , T_2 and T_3 have ten possible representation assignments as shown in eq. (4.6). For each type of models, we comprehensively scan over the weights and representations of N , \bar{F} and T . The requirement that each term of the superpotential \mathcal{W} has vanishing total weight entails the modular weights of the matter fields N , \bar{F} and T are integers. We neglect these cases that one column or one row of the fermion mass matrix is vanishing, since at least one fermion would be massless. A numerical analysis is performed for each model, and the strategy of numerical analysis is discussed in section 5. In the following, we present some benchmark models which can give acceptable masses and mixing parameters of both quark and leptons for certain values of the input parameters. The assignments of the matter fields for these viable models are summarized in table 2.

We see that either \bar{F} or T are A_4 singlets in Type-I, Type-II, Type-V and Type-VII models, and the mass hierarchies are achieved by introducing one free parameters for each generation. By comparison, the three generations of matter fields \bar{F} and T are both assigned to transform as triplet $\mathbf{3}$ under the modular symmetry A_4 in the Type-III, Type-IV and Type-VI models, thus the three generations of quark and lepton fields are treated on an equal footing. In order to reproduce the huge mass hierarchies among the quarks and leptons, fine-tuning of the coupling constants is necessary so that some comparable terms should cancel with each other to generate the small masses of the electron, up quark and down quark. Therefore the Type-III, Type-IV and Type-VI models are not favored by the current experiments, and more free parameters are needed to accommodate the mass hierarchies of quarks and leptons. For instance, we find a phenomenologically viable Type-III model involving 40 real parameters, which is significantly larger than the number of parameters required for the Type-I, Type-II, Type-V and Type-VII models. In summary, we shall not analyse the Type-III, Type-IV and Type-VI models further in this paper since they are more fine tuned and require more parameters.

4.1 Type-I

- Model \mathcal{I}_1 : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1})$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (1, 1, 1, 1, 3)$

The modular invariant superpotentials for quark and lepton masses are of the following form,

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\overline{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\overline{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\overline{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_2} (T_2 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(6)} H_5. \tag{4.10}
 \end{aligned}$$

Type-I	#P	$(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3})$	k_N	k_F	$(k_{T_1}, k_{T_2}, k_{T_3})$
\mathcal{I}_1	18	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1})$	1	1	(1,1,3)
\mathcal{I}_2	22	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}', \mathbf{1}')$	0	2	(2,0,4)
\mathcal{I}_3	22	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$	0	2	(0,2,4)
\mathcal{I}_4	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1})$	0	2	(0,0,4)
\mathcal{I}_5	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1})$	0	2	(0,2,4)
\mathcal{I}_6	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1})$	0	2	(2,2,4)
\mathcal{I}_7	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}', \mathbf{1}, \mathbf{1}'')$	0	2	(0,0,4)
\mathcal{I}_8	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$	2	0	(2,2,2)
\mathcal{I}_9	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$	0	2	(2,2,4)
\mathcal{I}_{10}	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}', \mathbf{1}', \mathbf{1}'')$	0	2	(0,2,4)
\mathcal{I}_{11}	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}', \mathbf{1}')$	0	2	(2,2,4)
\mathcal{I}_{12}	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$	0	2	(2,4,2)
\mathcal{I}_{13}	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$	2	0	(2,2,4)
\mathcal{I}_{14}	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$	2	0	(2,4,4)
\mathcal{I}_{15}	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}', \mathbf{1}')$	2	0	(2,2,4)
\mathcal{I}_{16}	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$	2	0	(2,4,2)
\mathcal{I}_{17}	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$	2	0	(2,4,4)
Type-II	#P	$(\rho_N, \rho_T, \rho_{F_1}, \rho_{F_2}, \rho_{F_3})$	k_N	$(k_{F_1}, k_{F_2}, k_{F_3})$	k_T
\mathcal{II}_1	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}')$	1	(1,3,1)	3
\mathcal{II}_2	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1})$	1	(1,3,1)	3
Type-V	#P	$(\rho_F, \rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_{T_1}, \rho_{T_2}, \rho_{T_3})$	$(k_{N_1}, k_{N_2}, k_{N_3})$	k_F	$(k_{T_1}, k_{T_2}, k_{T_3})$
\mathcal{V}_1	22	$(\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}, \mathbf{1})$	(1,3,3)	1	(1,1,3)
\mathcal{V}_2	22	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}, \mathbf{1})$	(3,1,3)	1	(1,1,3)
\mathcal{V}_3	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}, \mathbf{1}')$	(0,2,2)	2	(2,2,2)
\mathcal{V}_4	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}, \mathbf{1}')$	(4,2,2)	0	(2,2,2)
\mathcal{V}_5	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}', \mathbf{1}')$	(4,2,2)	0	(2,2,4)
\mathcal{V}_6	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$	(4,2,2)	0	(2,4,2)
\mathcal{V}_7	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$	(4,2,2)	0	(2,4,4)
\mathcal{V}_8	24	$(\mathbf{3}, \mathbf{1}', \mathbf{1}', \mathbf{1}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$	(2,4,2)	0	(2,2,2)
\mathcal{V}_9	24	$(\mathbf{3}, \mathbf{1}', \mathbf{1}', \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}')$	(2,4,2)	0	(2,2,4)
\mathcal{V}_{10}	24	$(\mathbf{3}, \mathbf{1}', \mathbf{1}', \mathbf{1}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$	(2,4,2)	0	(2,4,2)
\mathcal{V}_{11}	24	$(\mathbf{3}, \mathbf{1}', \mathbf{1}', \mathbf{1}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$	(2,4,2)	0	(2,4,4)
\mathcal{V}_{12}	24	$(\mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$	(2,4,2)	0	(2,2,2)
\mathcal{V}_{13}	24	$(\mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$	(2,4,2)	0	(2,4,4)
\mathcal{V}_{14}	24	$(\mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}, \mathbf{1}'', \mathbf{1}, \mathbf{1})$	(2,4,2)	0	(2,2,4)
\mathcal{V}_{15}	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}'')$	(0,0,0)	2	(0,0,4)
\mathcal{V}_{16}	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1})$	(0,0,0)	2	(0,0,4)
\mathcal{V}_{17}	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1})$	(0,0,0)	4	(0,0,4)
\mathcal{V}_{18}	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1})$	(2,0,0)	2	(0,0,4)
\mathcal{V}_{19}	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}'')$	(0,0,0)	4	(0,0,4)
\mathcal{V}_{20}	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}'')$	(2,0,0)	2	(0,0,4)
\mathcal{V}_{21}	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}'')$	(0,0,0)	2	(2,0,4)
\mathcal{V}_{22}	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}'')$	(2,0,0)	2	(2,0,4)
\mathcal{V}_{23}	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}'')$	(0,0,0)	2	(0,2,4)
\mathcal{V}_{24}	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}'')$	(2,0,0)	2	(0,2,4)
\mathcal{V}_{25}	24	$(\mathbf{3}, \mathbf{1}', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}'', \mathbf{1})$	(0,2,0)	2	(0,0,4)
\mathcal{V}_{26}	24	$(\mathbf{3}, \mathbf{1}', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}'', \mathbf{1}'')$	(0,2,0)	2	(0,0,4)
\mathcal{V}_{27}	24	$(\mathbf{3}, \mathbf{1}', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$	(0,2,0)	2	(2,0,4)
\mathcal{V}_{28}	24	$(\mathbf{3}, \mathbf{1}', \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$	(0,2,0)	2	(0,2,4)
Type-VII	#P	$(\rho_F, \rho_{N_1}, \rho_{N_2}, \rho_{T_1}, \rho_{T_2}, \rho_{T_3})$	(k_{N_1}, k_{N_2})	k_F	$(k_{T_1}, k_{T_2}, k_{T_3})$
\mathcal{VII}_1	22	$(\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}, \mathbf{1})$	(1,3)	3	(1,1,3)
\mathcal{VII}_2	22	$(\mathbf{3}, \mathbf{1}', \mathbf{1}', \mathbf{1}'', \mathbf{1}, \mathbf{1})$	(1,3)	3	(1,1,3)
\mathcal{VII}_3	22	$(\mathbf{3}, \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}, \mathbf{1})$	(1,3)	3	(1,1,3)
\mathcal{VII}_4	23	$(\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$	(0,2)	2	(0,2,4)
\mathcal{VII}_5	23	$(\mathbf{3}, \mathbf{1}, \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$	(0,2)	2	(0,2,4)
\mathcal{VII}_6	23	$(\mathbf{3}, \mathbf{1}, \mathbf{1}', \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$	(2,2)	2	(0,2,4)

Table 2. Summary of phenomenologically viable SU(5) GUT models based on A_4 modular symmetry with less free parameters. Here the real and imaginary part of τ are taken as free parameters.

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\begin{aligned}
 m_M^\nu &= \Lambda \begin{pmatrix} 2\alpha_{N_1} Y_{\mathbf{3},1}^{(2)} & -\alpha_{N_1} Y_{\mathbf{3},3}^{(2)} & -\alpha_{N_1} Y_{\mathbf{3},2}^{(2)} \\ -\alpha_{N_1} Y_{\mathbf{3},3}^{(2)} & 2\alpha_{N_1} Y_{\mathbf{3},2}^{(2)} & -\alpha_{N_1} Y_{\mathbf{3},1}^{(2)} \\ -\alpha_{N_1} Y_{\mathbf{3},2}^{(2)} & -\alpha_{N_1} Y_{\mathbf{3},1}^{(2)} & 2\alpha_{N_1} Y_{\mathbf{3},3}^{(2)} \end{pmatrix}, \\
 \mathcal{Y}_5^\nu &= \begin{pmatrix} 2\alpha_{\nu_1} Y_{\mathbf{3},1}^{(2)} & -(\alpha_{\nu_1} - \alpha_{\nu_2}) Y_{\mathbf{3},3}^{(2)} & -(\alpha_{\nu_1} + \alpha_{\nu_2}) Y_{\mathbf{3},2}^{(2)} \\ -(\alpha_{\nu_1} + \alpha_{\nu_2}) Y_{\mathbf{3},3}^{(2)} & 2\alpha_{\nu_1} Y_{\mathbf{3},2}^{(2)} & -(\alpha_{\nu_1} - \alpha_{\nu_2}) Y_{\mathbf{3},1}^{(2)} \\ -(\alpha_{\nu_1} - \alpha_{\nu_2}) Y_{\mathbf{3},2}^{(2)} & -(\alpha_{\nu_1} + \alpha_{\nu_2}) Y_{\mathbf{3},1}^{(2)} & 2\alpha_{\nu_1} Y_{\mathbf{3},3}^{(2)} \end{pmatrix}, \\
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},1}^{(4)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},3}^{(4)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},2}^{(4)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} 0 & 0 & \alpha_{u_1} Y_{\mathbf{1}'}^{(4)} \\ 0 & 0 & \alpha_{u_2} Y_{\mathbf{1}'}^{(4)} \\ \alpha_{u_1} Y_{\mathbf{1}'}^{(4)} & \alpha_{u_2} Y_{\mathbf{1}'}^{(4)} & \alpha_{u_3} Y_{\mathbf{1}'}^{(6)} \end{pmatrix}. \tag{4.11}
 \end{aligned}$$

Since both $H_{\bar{5}}$ and $H_{\overline{45}}$ are assumed to be invariant under A_4 and their modular weights are vanishing, \mathcal{Y}_5^d and $\mathcal{Y}_{\overline{45}}^d$ are of the same form except that the couplings α_{d_i} are replaced by α'_{d_i} . Hence we shall not give the explicit form of the Yukawa coupling $\mathcal{Y}_{\overline{45}}^d$ here and below. The parameters α_{N_1} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and α_{ν_2} , α'_{d_2} , α'_{d_3} , α_{u_1} and α_{u_2} are generically complex.

- Model \mathcal{I}_2 : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}', \mathbf{1}')$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 2, 0, 4)$

We can read out the superpotential for quark and leptons as follows,

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda (NN)_{\mathbf{1}} + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\overline{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\overline{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\overline{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\overline{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(6)} H_5 + \alpha_{u_2} (T_2 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(8)} H_5. \tag{4.12}
 \end{aligned}$$

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\begin{aligned}
 m_M^\nu &= \Lambda \begin{pmatrix} \alpha_{N_1} & 0 & 0 \\ 0 & 0 & \alpha_{N_1} \\ 0 & \alpha_{N_1} & 0 \end{pmatrix}, \\
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},I,3}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,3}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},I,2}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,2}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},I,1}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,1}^{(6)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} 0 & 0 & \alpha_{u_1} Y_{\mathbf{1}'}^{(6)} \\ 0 & 0 & \alpha_{u_2} Y_{\mathbf{1}'}^{(4)} \\ \alpha_{u_1} Y_{\mathbf{1}'}^{(6)} & \alpha_{u_2} Y_{\mathbf{1}'}^{(4)} & \alpha_{u_3} Y_{\mathbf{1}'}^{(8)} \end{pmatrix}. \tag{4.13}
 \end{aligned}$$

The \mathcal{Y}_5^{ν} matrix is in common with that of the model \mathcal{I}_1 . The parameters α_{N_1} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and α_{ν_2} , α_{d_4} , α'_{d_2} , α'_{d_3} , α'_{d_4} , α_{u_1} and α_{u_2} are generically complex.

- Model \mathcal{I}_3 : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 0, 2, 4)$

The modular invariant superpotential is of the following form:

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_1 + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_2} (T_2 T_3)_1 Y_{\mathbf{1}}^{(6)} H_5 + \alpha_{u_3} (T_3 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(8)} H_5,
 \end{aligned} \tag{4.14}$$

which leads to

$$\begin{aligned}
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,3}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,3}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,2}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,2}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,1}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,1}^{(6)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} 0 & 0 & \alpha_{u_1} Y_{\mathbf{1}}^{(4)} \\ 0 & 0 & \alpha_{u_2} Y_{\mathbf{1}}^{(6)} \\ \alpha_{u_1} Y_{\mathbf{1}}^{(4)} & \alpha_{u_2} Y_{\mathbf{1}}^{(6)} & \alpha_{u_3} Y_{\mathbf{1}'}^{(8)} \end{pmatrix}.
 \end{aligned} \tag{4.15}$$

The right-handed neutrino mass matrix M_N matrix is the same as that of the model \mathcal{I}_2 , and \mathcal{Y}_5^{ν} is the same as that of the model \mathcal{I}_1 . The phases of the parameters α_{N_1} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be removed by field redefinition, and the remaining parameters α_{ν_2} , α_{d_4} , α'_{d_2} , α'_{d_3} , α'_{d_4} , α_{u_1} and α_{u_2} are complex.

- Model \mathcal{I}_4 : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1})$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 0, 0, 4)$

The quark and lepton masses are described by the following superpotential:

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_1 + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_2} (T_2 T_2)_1 H_5 + \alpha_{u_3} (T_2 T_3)_1 Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_4} (T_3 T_3)_1 Y_{\mathbf{1}}^{(8)} H_5.
 \end{aligned} \tag{4.16}$$

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\begin{aligned}
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},I,1}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,1}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},I,3}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,3}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},I,2}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,2}^{(6)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} 0 & 0 & \alpha_{u_1} Y_{\mathbf{1}'}^{(4)} \\ 0 & \alpha_{u_2} & \alpha_{u_3} Y_{\mathbf{1}}^{(4)} \\ \alpha_{u_1} Y_{\mathbf{1}'}^{(4)} & \alpha_{u_3} Y_{\mathbf{1}}^{(4)} & \alpha_{u_4} Y_{\mathbf{1}}^{(8)} \end{pmatrix}.
 \end{aligned} \tag{4.17}$$

The matrices M_N and \mathcal{Y}_5^{ν} coincide with those of the model \mathcal{I}_3 as well. The phases of the parameters α_{N_1} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be removed by field redefinition, and the other parameters are complex.

- Model \mathcal{I}_5 : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1})$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 0, 2, 4)$

The modular invariant superpotentials for quark and lepton masses are of the following form,

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_1 + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_2} (T_2 T_2)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_3} (T_2 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(6)} H_5 + \alpha_{u_4} (T_3 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(8)} H_5.
 \end{aligned} \tag{4.18}$$

The right-handed neutrino mass matrix and the Yukawa coupling matrices are predicted to be

$$\begin{aligned}
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,1}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,1}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,3}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,3}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,2}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,2}^{(6)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} 0 & 0 & \alpha_{u_1} Y_{\mathbf{1}'}^{(4)} \\ 0 & \alpha_{u_2} Y_{\mathbf{1}}^{(4)} & \alpha_{u_3} Y_{\mathbf{1}}^{(6)} \\ \alpha_{u_1} Y_{\mathbf{1}'}^{(4)} & \alpha_{u_3} Y_{\mathbf{1}}^{(6)} & \alpha_{u_4} Y_{\mathbf{1}}^{(8)} \end{pmatrix}.
 \end{aligned} \tag{4.19}$$

The matrices M_N and \mathcal{Y}_5^{ν} are the same as the corresponding ones of the model \mathcal{I}_3 as well. The parameters α_{N_1} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and the other parameters are complex.

- Model \mathcal{I}_6 : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1})$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 2, 2, 4)$

The quark and lepton masses are described by the following superpotential:

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_1 + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_2} (T_2 T_2)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_3} (T_2 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(6)} H_5 + \alpha_{u_4} (T_3 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(8)} H_5,
 \end{aligned} \tag{4.20}$$

which gives rise to

$$\mathcal{Y}_5^d = \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_3} Y_{\mathbf{3}I,1}^{(6)} + \alpha_{d_4} Y_{\mathbf{3}II,1}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_3} Y_{\mathbf{3}I,3}^{(6)} + \alpha_{d_4} Y_{\mathbf{3}II,3}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_3} Y_{\mathbf{3}I,2}^{(6)} + \alpha_{d_4} Y_{\mathbf{3}II,2}^{(6)} \end{pmatrix},$$

$$\mathcal{Y}_5^u = \begin{pmatrix} 0 & \alpha_{u_1} Y_{\mathbf{1}'}^{(4)} & 0 \\ \alpha_{u_1} Y_{\mathbf{1}'}^{(4)} & \alpha_{u_2} Y_{\mathbf{1}}^{(4)} & \alpha_{u_3} Y_{\mathbf{1}}^{(6)} \\ 0 & \alpha_{u_3} Y_{\mathbf{1}}^{(6)} & \alpha_{u_4} Y_{\mathbf{1}}^{(8)} \end{pmatrix}. \quad (4.21)$$

The matrices M_N and \mathcal{Y}_5^{ν} are in common with those of the model \mathcal{I}_3 as well. The parameters α_{N_1} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and α_{ν_2} , α_{d_4} , α'_{d_2} , α'_{d_3} , α'_{d_4} , α_{u_1} , α_{u_2} and α_{u_4} are complex.

- Model \mathcal{I}_7 : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}', \mathbf{1}, \mathbf{1}'')$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 0, 0, 4)$

The modular invariant superpotential is of the following form:

$$\begin{aligned} \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_{\mathbf{1}} + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\ \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}II}^{(6)} H_{\bar{5}} \\ &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}II}^{(6)} H_{\bar{45}}, \\ \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_2} (T_2 T_2)_{\mathbf{1}} H_5 + \alpha_{u_3} (T_2 T_3)_{\mathbf{1}''} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_4} (T_3 T_3)_{\mathbf{1}'} Y_{\mathbf{1}}^{(8)} H_5. \end{aligned} \quad (4.22)$$

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\mathcal{Y}_5^d = \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_3} Y_{\mathbf{3}I,2}^{(6)} + \alpha_{d_4} Y_{\mathbf{3}II,2}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_3} Y_{\mathbf{3}I,1}^{(6)} + \alpha_{d_4} Y_{\mathbf{3}II,1}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_3} Y_{\mathbf{3}I,3}^{(6)} + \alpha_{d_4} Y_{\mathbf{3}II,3}^{(6)} \end{pmatrix},$$

$$\mathcal{Y}_5^u = \begin{pmatrix} 0 & 0 & \alpha_{u_1} Y_{\mathbf{1}}^{(4)} \\ 0 & \alpha_{u_2} & \alpha_{u_3} Y_{\mathbf{1}'}^{(4)} \\ \alpha_{u_1} Y_{\mathbf{1}}^{(4)} & \alpha_{u_3} Y_{\mathbf{1}'}^{(4)} & \alpha_{u_4} Y_{\mathbf{1}''}^{(8)} \end{pmatrix}. \quad (4.23)$$

The matrices M_N and \mathcal{Y}_5^{ν} coincide with those of the model \mathcal{I}_3 as well. The parameters α_{N_1} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and the other parameters are complex.

- Model \mathcal{I}_8 : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 0, 2, 2, 2)$

The modular invariant superpotential is of the following form:

$$\begin{aligned} \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} + \alpha_{N_2} \Lambda(NN)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} + \alpha_{N_3} \Lambda(NN)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(4)} + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 \\ &\quad + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\ \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} \\ &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}}, \\ \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_2} (T_1 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_3} (T_2 T_2)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_4} (T_3 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5. \end{aligned} \quad (4.24)$$

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\begin{aligned}
 m_M^\nu &= \Lambda \begin{pmatrix} \alpha_{N_1} Y_{\mathbf{1}}^{(4)} + 2\alpha_{N_3} Y_{\mathbf{3},1}^{(4)} & -\alpha_{N_3} Y_{\mathbf{3},3}^{(4)} & \alpha_{N_2} Y_{\mathbf{1}'}^{(4)} - \alpha_{N_3} Y_{\mathbf{3},2}^{(4)} \\ -\alpha_{N_3} Y_{\mathbf{3},3}^{(4)} & \alpha_{N_2} Y_{\mathbf{1}'}^{(4)} + 2\alpha_{N_3} Y_{\mathbf{3},2}^{(4)} & \alpha_{N_1} Y_{\mathbf{1}}^{(4)} - \alpha_{N_3} Y_{\mathbf{3},1}^{(4)} \\ \alpha_{N_2} Y_{\mathbf{1}'}^{(4)} - \alpha_{N_3} Y_{\mathbf{3},2}^{(4)} & \alpha_{N_1} Y_{\mathbf{1}}^{(4)} - \alpha_{N_3} Y_{\mathbf{3},1}^{(4)} & 2\alpha_{N_3} Y_{\mathbf{3},3}^{(4)} \end{pmatrix}, \\
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},3}^{(2)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},2}^{(2)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},1}^{(2)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} 0 & \alpha_{u_1} Y_{\mathbf{1}'}^{(4)} & \alpha_{u_2} Y_{\mathbf{1}}^{(4)} \\ \alpha_{u_1} Y_{\mathbf{1}'}^{(4)} & \alpha_{u_3} Y_{\mathbf{1}}^{(4)} & 0 \\ \alpha_{u_2} Y_{\mathbf{1}}^{(4)} & 0 & \alpha_{u_4} Y_{\mathbf{1}'}^{(4)} \end{pmatrix}. \tag{4.25}
 \end{aligned}$$

The \mathcal{Y}_5^ν matrix coincide with those of the model \mathcal{I}_2 as well. The phases of the parameters α_{N_1} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} are unphysical, and the remaining parameters are complex.

- Model \mathcal{I}_9 : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 2, 2, 4)$

We can read out the superpotentials relevant to quark and lepton masses

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda (NN)_{\mathbf{1}} + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_{\mathbf{1}'} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_2} (T_1 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(6)} H_5 + \alpha_{u_3} (T_2 T_2)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_4} (T_3 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(8)} H_5, \tag{4.26}
 \end{aligned}$$

which lead to

$$\begin{aligned}
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,3}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,3}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,2}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,2}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,1}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,1}^{(6)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} 0 & \alpha_{u_1} Y_{\mathbf{1}'}^{(4)} & \alpha_{u_2} Y_{\mathbf{1}}^{(6)} \\ \alpha_{u_1} Y_{\mathbf{1}'}^{(4)} & \alpha_{u_3} Y_{\mathbf{1}}^{(4)} & 0 \\ \alpha_{u_2} Y_{\mathbf{1}}^{(6)} & 0 & \alpha_{u_4} Y_{\mathbf{1}'}^{(8)} \end{pmatrix}. \tag{4.27}
 \end{aligned}$$

The matrices M_N and \mathcal{Y}_5^ν are identical with those of the model \mathcal{I}_3 as well. The phases of the parameters α_{N_1} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} are unphysical, and the other parameters are complex.

- Model \mathcal{I}_{10} : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}', \mathbf{1}', \mathbf{1}'')$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 0, 2, 4)$

We can read out the superpotential for quark and leptons as follows,

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_1 + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_1^{(4)} H_5 + \alpha_{u_2} (T_2 T_2)_{1''} Y_{1'}^{(4)} H_5 + \alpha_{u_3} (T_2 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_4} (T_3 T_3)_{1'} Y_{1''}^{(8)} H_5.
 \end{aligned} \tag{4.28}$$

The right-handed neutrino mass matrix and the Yukawa coupling matrices are predicted to be

$$\begin{aligned}
 \mathcal{Y}_{\bar{5}}^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,2}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,2}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,1}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,1}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,3}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,3}^{(6)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} 0 & 0 & \alpha_{u_1} Y_1^{(4)} \\ 0 & \alpha_{u_2} Y_{1'}^{(4)} & \alpha_{u_3} Y_1^{(6)} \\ \alpha_{u_1} Y_1^{(4)} & \alpha_{u_3} Y_1^{(6)} & \alpha_{u_4} Y_{1''}^{(8)} \end{pmatrix}.
 \end{aligned} \tag{4.29}$$

The matrices M_N and \mathcal{Y}_5^u are the same as the corresponding ones of the model \mathcal{I}_3 as well. The parameters α_{N_1} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} are real, and the remaining parameters are complex.

- Model \mathcal{I}_{11} : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}', \mathbf{1}')$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 2, 2, 4)$

The modular invariant superpotential for quark and lepton mass is given by

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_1 + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_1 Y_1^{(4)} H_5 + \alpha_{u_2} (T_1 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_3} (T_2 T_2)_{1''} Y_{1'}^{(4)} H_5 + \alpha_{u_4} (T_3 T_3)_{1''} Y_{1'}^{(8)} H_5.
 \end{aligned} \tag{4.30}$$

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\begin{aligned}
 \mathcal{Y}_{\bar{5}}^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,3}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,3}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,2}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,2}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,1}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,1}^{(6)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} 0 & \alpha_{u_1} Y_1^{(4)} & \alpha_{u_2} Y_1^{(6)} \\ \alpha_{u_1} Y_1^{(4)} & \alpha_{u_3} Y_{1'}^{(4)} & 0 \\ \alpha_{u_2} Y_1^{(6)} & 0 & \alpha_{u_4} Y_{1'}^{(8)} \end{pmatrix}.
 \end{aligned} \tag{4.31}$$

The matrices M_N and \mathcal{Y}_5^u are identical with those of the model \mathcal{I}_3 as well. The parameters α_{N_1} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and the remaining parameters are complex.

- Model \mathcal{I}_{12} : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 2, 4, 2)$

The quark and lepton masses are described by the following superpotential:

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_{\mathbf{1}} + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_2)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_2)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_2} (T_2 T_2)_{\mathbf{1}'} Y_{\mathbf{1}''}^{(8)} H_5 + \alpha_{u_3} (T_2 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(6)} H_5 + \alpha_{u_4} (T_3 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5,
 \end{aligned} \tag{4.32}$$

which gives rise to

$$\begin{aligned}
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},I,2}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,2}^{(6)} & \alpha_{d_3} Y_{\mathbf{3},3}^{(4)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},I,1}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,1}^{(6)} & \alpha_{d_3} Y_{\mathbf{3},2}^{(4)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},I,3}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,3}^{(6)} & \alpha_{d_3} Y_{\mathbf{3},1}^{(4)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} 0 & 0 & \alpha_{u_1} Y_{\mathbf{1}}^{(4)} \\ 0 & \alpha_{u_2} Y_{\mathbf{1}''}^{(8)} & \alpha_{u_3} Y_{\mathbf{1}}^{(6)} \\ \alpha_{u_1} Y_{\mathbf{1}}^{(4)} & \alpha_{u_3} Y_{\mathbf{1}}^{(6)} & \alpha_{u_4} Y_{\mathbf{1}'}^{(4)} \end{pmatrix}.
 \end{aligned} \tag{4.33}$$

The matrices M_N and \mathcal{Y}_5^u are identical with the corresponding ones of the model \mathcal{I}_3 as well. The phases of the parameters α_{N_1} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be removed by field redefinition, and the remaining parameters are complex.

- Model \mathcal{I}_{13} : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 0, 2, 2, 4)$

The modular invariant superpotential is of the following form:

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} + \alpha_{N_2} \Lambda(NN)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} + \alpha_{N_3} \Lambda(NN)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(4)} + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 \\
 &\quad + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_2} (T_1 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(6)} H_5 + \alpha_{u_3} (T_2 T_2)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_4} (T_3 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(8)} H_5.
 \end{aligned} \tag{4.34}$$

which gives rise to

$$\mathcal{Y}_5^d = \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},3}^{(4)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},2}^{(4)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},1}^{(4)} \end{pmatrix}. \tag{4.35}$$

The M_N matrix coincides with that of the model \mathcal{I}_8 . The \mathcal{Y}_5^u matrix coincides with that of the model \mathcal{I}_1 . The \mathcal{Y}_5^d matrix is identical with the corresponding one of the model \mathcal{I}_9 . The parameters α_{N_3} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} are real, and the remaining parameters α_{N_1} , α_{N_2} , α_{ν_2} , α'_{d_2} , α'_{d_3} , α_{u_1} , α_{u_2} and α_{u_4} are complex.

- Model \mathcal{I}_{14} : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 0, 2, 4, 4)$

We can read out the superpotential for the fermion masses as,

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_1 Y_1^{(4)} + \alpha_{N_2} \Lambda(NN)_{1''} Y_{1'}^{(4)} + \alpha_{N_3} \Lambda(NN)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(4)} + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 \\
 &\quad + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_2} (T_2 T_2)_1 Y_1^{(8)} H_5 + \alpha_{u_3} (T_2 T_3)_{1'} Y_{1''}^{(8)} H_5 + \alpha_{u_4} (T_3 T_3)_{1''} Y_{1'}^{(8)} H_5.
 \end{aligned} \tag{4.36}$$

which gives rise to

$$\begin{aligned}
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},3}^{(4)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},2}^{(4)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},1}^{(4)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} 0 & 0 & \alpha_{u_1} Y_1^{(6)} \\ 0 & \alpha_{u_2} Y_1^{(8)} & \alpha_{u_3} Y_{1''}^{(8)} \\ \alpha_{u_1} Y_1^{(6)} & \alpha_{u_3} Y_{1''}^{(8)} & \alpha_{u_4} Y_{1'}^{(8)} \end{pmatrix}.
 \end{aligned} \tag{4.37}$$

The matrices M_N and \mathcal{Y}_5^l are identical with those of the models \mathcal{I}_8 and \mathcal{I}_1 respectively. The parameters α_{N_3} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken to be real without loss of generality, while α_{N_1} , α_{N_2} , α_{ν_2} , α'_{d_2} , α'_{d_3} , α_{u_1} , α_{u_2} and α_{u_4} are generically complex.

- Model \mathcal{I}_{15} : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}', \mathbf{1}')$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 0, 2, 2, 4)$

The modular invariant superpotential for quark and lepton mass is given by

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_1 Y_1^{(4)} + \alpha_{N_2} \Lambda(NN)_{1''} Y_{1'}^{(4)} + \alpha_{N_3} \Lambda(NN)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(4)} + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 \\
 &\quad + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_1 Y_1^{(4)} H_5 + \alpha_{u_2} (T_1 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_3} (T_2 T_2)_{1''} Y_{1'}^{(4)} H_5 + \alpha_{u_4} (T_3 T_3)_{1''} Y_{1'}^{(8)} H_5,
 \end{aligned} \tag{4.38}$$

which leads to

$$\mathcal{Y}_5^d = \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},3}^{(4)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},2}^{(4)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_3} Y_{\mathbf{3},1}^{(4)} \end{pmatrix}. \tag{4.39}$$

The M_N matrix coincides with the corresponding one of the model \mathcal{I}_8 . The \mathcal{Y}_5^l matrix coincides with that of the model \mathcal{I}_1 . The \mathcal{Y}_5^u matrix is identical with that of the model \mathcal{I}_{11} . The parameters α_{N_3} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and the remaining parameters are complex.

- Model \mathcal{I}_{16} : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 0, 2, 4, 2)$

We can read out the superpotentials relevant to quark and lepton masses

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_1 Y_1^{(4)} + \alpha_{N_2} \Lambda(NN)_{1''} Y_{1''}^{(4)} + \alpha_{N_3} \Lambda(NN)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(4)} + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 \\
 &\quad + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_1^{(4)} H_5 + \alpha_{u_2} (T_2 T_2)_{1'} Y_{1''}^{(8)} H_5 + \alpha_{u_3} (T_2 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_4} (T_3 T_3)_{1''} Y_{1'}^{(4)} H_5.
 \end{aligned} \tag{4.40}$$

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\mathcal{Y}_5^d = \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},3}^{(2)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},2}^{(2)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},1}^{(2)} \end{pmatrix}. \tag{4.41}$$

The M_N matrix is the same as the corresponding one of the model \mathcal{I}_8 . The \mathcal{Y}_5^ν matrix is in common with the corresponding one of the model \mathcal{I}_1 . The \mathcal{Y}_5^u matrix coincides with the corresponding one of the model \mathcal{I}_{12} . The parameters α_{N_3} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken to be real without loss of generality, while α_{N_1} , α_{N_2} , α_{ν_2} , α'_{d_2} , α'_{d_3} , α_{u_1} , α_{u_2} and α_{u_4} are generically complex.

- Model \mathcal{I}_{17} : $(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{3}, \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$, $(k_N, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 0, 2, 4, 4)$

We can read out the superpotential for the fermion masses as,

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_1 Y_1^{(4)} + \alpha_{N_2} \Lambda(NN)_{1''} Y_{1''}^{(4)} + \alpha_{N_3} \Lambda(NN)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(4)} + \alpha_{\nu_1} (NF)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} H_5 \\
 &\quad + \alpha_{\nu_2} (NF)_{\mathbf{3}_A} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_2} (T_2 T_2)_{1'} Y_{1''}^{(8)} H_5 + \alpha_{u_3} (T_2 T_3)_1 Y_1^{(8)} H_5 + \alpha_{u_4} (T_3 T_3)_{1''} Y_{1'}^{(8)} H_5.
 \end{aligned} \tag{4.42}$$

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\begin{aligned}
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},3}^{(4)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},2}^{(4)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},1}^{(4)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} 0 & 0 & \alpha_{u_1} Y_1^{(6)} \\ 0 & \alpha_{u_2} Y_{1''}^{(8)} & \alpha_{u_3} Y_1^{(8)} \\ \alpha_{u_1} Y_1^{(6)} & \alpha_{u_3} Y_1^{(8)} & \alpha_{u_4} Y_{1'}^{(8)} \end{pmatrix}.
 \end{aligned} \tag{4.43}$$

The matrices M_N and \mathcal{Y}_5^ν coincide with the corresponding ones of the model \mathcal{I}_{14} as well. The parameters α_{N_3} , α_{ν_1} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and the other parameters are complex.

4.2 Type-II

- Model \mathcal{IL}_1 : $(\rho_N, \rho_{F_1}, \rho_{F_2}, \rho_{F_3}, \rho_T) = (\mathbf{3}, \mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3})$, $(k_N, k_{F_1}, k_{F_2}, k_{F_3}, k_T) = (1, 1, 3, 1, 3)$

The quark and lepton masses are described by the following superpotential:

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} + \alpha_{\nu_1} (NF_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (NF_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_3} (NF_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (F_1 T)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_2} (F_2 T)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (F_2 T)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} + \alpha_{d_3} (F_3 T)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (F_1 T)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_2} (F_2 T)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (F_2 T)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}} + \alpha'_{d_3} (F_3 T)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (TT)_1 Y_1^{(6)} H_5 + \alpha_{u_2} (TT)_{\mathbf{3}_S} Y_{\mathbf{3},I}^{(6)} H_5 + \alpha_{u_3} (TT)_{\mathbf{3}_S} Y_{\mathbf{3},II}^{(6)} H_5.
 \end{aligned} \tag{4.44}$$

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\begin{aligned}
 \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{\nu_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{\nu_3} Y_{\mathbf{3},3}^{(2)} \\ \alpha_{\nu_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{\nu_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{\nu_3} Y_{\mathbf{3},2}^{(2)} \\ \alpha_{\nu_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{\nu_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{\nu_3} Y_{\mathbf{3},1}^{(2)} \end{pmatrix}, \\
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_1} Y_{\mathbf{3},2}^{(4)} \\ \alpha_{d_2} Y_{\mathbf{3},I,2}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,2}^{(6)} & \alpha_{d_2} Y_{\mathbf{3},I,1}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,1}^{(6)} & \alpha_{d_2} Y_{\mathbf{3},I,3}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,3}^{(6)} \\ \alpha_{d_3} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},1}^{(4)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} \alpha_{u_1} Y_1^{(6)} + 2\alpha_{u_2} Y_{\mathbf{3},I,1}^{(6)} + 2\alpha_{u_3} Y_{\mathbf{3},II,1}^{(6)} & -\alpha_{u_2} Y_{\mathbf{3},I,3}^{(6)} - \alpha_{u_3} Y_{\mathbf{3},II,3}^{(6)} & -\alpha_{u_2} Y_{\mathbf{3},I,2}^{(6)} - \alpha_{u_3} Y_{\mathbf{3},II,2}^{(6)} \\ -\alpha_{u_2} Y_{\mathbf{3},I,3}^{(6)} - \alpha_{u_3} Y_{\mathbf{3},II,3}^{(6)} & 2(\alpha_{u_2} Y_{\mathbf{3},I,2}^{(6)} + \alpha_{u_3} Y_{\mathbf{3},II,2}^{(6)}) & \alpha_{u_1} Y_1^{(6)} - \alpha_{u_2} Y_{\mathbf{3},I,1}^{(6)} - \alpha_{u_3} Y_{\mathbf{3},II,1}^{(6)} \\ -\alpha_{u_2} Y_{\mathbf{3},I,2}^{(6)} - \alpha_{u_3} Y_{\mathbf{3},II,2}^{(6)} & \alpha_{u_1} Y_1^{(6)} - \alpha_{u_2} Y_{\mathbf{3},I,1}^{(6)} - \alpha_{u_3} Y_{\mathbf{3},II,1}^{(6)} & 2(\alpha_{u_2} Y_{\mathbf{3},I,3}^{(6)} + \alpha_{u_3} Y_{\mathbf{3},II,3}^{(6)}) \end{pmatrix}.
 \end{aligned} \tag{4.45}$$

The M_N matrix coincides with the corresponding one of the model \mathcal{I}_1 . The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and the remaining parameters are complex.

- Model \mathcal{IL}_2 : $(\rho_N, \rho_{F_1}, \rho_{F_2}, \rho_{F_3}, \rho_T) = (\mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}, \mathbf{3})$, $(k_N, k_{F_1}, k_{F_2}, k_{F_3}, k_T) = (1, 1, 3, 1, 3)$

The modular invariant superpotential for quark and lepton mass is given by

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(NN)_{\mathbf{3}_S} Y_{\mathbf{3}}^{(2)} + \alpha_{\nu_1} (NF_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (NF_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_3} (NF_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (F_1 T)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_2} (F_2 T)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (F_2 T)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} + \alpha_{d_3} (F_3 T)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (F_1 T)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_2} (F_2 T)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (F_2 T)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}} + \alpha'_{d_3} (F_3 T)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (TT)_1 Y_1^{(6)} H_5 + \alpha_{u_2} (TT)_{\mathbf{3}_S} Y_{\mathbf{3},I}^{(6)} H_5 + \alpha_{u_3} (TT)_{\mathbf{3}_S} Y_{\mathbf{3},II}^{(6)} H_5.
 \end{aligned} \tag{4.46}$$

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\begin{aligned}
 \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{\nu_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{\nu_3} Y_{\mathbf{3},1}^{(2)} \\ \alpha_{\nu_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{\nu_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{\nu_3} Y_{\mathbf{3},3}^{(2)} \\ \alpha_{\nu_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{\nu_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{\nu_3} Y_{\mathbf{3},2}^{(2)} \end{pmatrix}, \\
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_1} Y_{\mathbf{3},3}^{(4)} \\ \alpha_{d_2} Y_{\mathbf{3},I,2}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,2}^{(6)} & \alpha_{d_2} Y_{\mathbf{3},I,1}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,1}^{(6)} & \alpha_{d_2} Y_{\mathbf{3},I,3}^{(6)} + \alpha_{d_4} Y_{\mathbf{3},II,3}^{(6)} \\ \alpha_{d_3} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},2}^{(4)} \end{pmatrix}.
 \end{aligned} \tag{4.47}$$

The matrices M_N and \mathcal{Y}_5^u are identical with those of the models \mathcal{I}_1 and \mathcal{II}_1 respectively. The phases of the parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} are unphysical, and α_{d_4} , α'_{d_2} , α'_{d_4} , α'_{d_3} , α_{u_1} and α_{u_2} are generically complex.

4.3 Type-V

- Model \mathcal{V}_1 : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}, \mathbf{1}'', \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}),$
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (1, 3, 3, 1, 1, 1, 3)$

We can read out the superpotential for quark and leptons as follows,

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_2)_1 Y_1^{(4)} + \alpha_{N_2} \Lambda(N_1 N_3)_{1''} Y_{1'}^{(4)} + \alpha_{N_3} \Lambda(N_2 N_2)_1 Y_1^{(6)} + \alpha_{\nu_1} (N_1 F)_3 Y_3^{(2)} H_5 \\
 &\quad + \alpha_{\nu_2} (N_2 F)_3 Y_3^{(4)} H_5 + \alpha_{\nu_3} (N_3 F)_3 Y_3^{(4)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_3^{(4)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_3^{(4)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_{1''} Y_{1'}^{(4)} H_5 + \alpha_{u_2} (T_2 T_3)_1 Y_1^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_1 Y_1^{(6)} H_5.
 \end{aligned} \tag{4.48}$$

We can straightforwardly read out the right-handed neutrino mass matrix and the Yukawa matrices as follows,

$$\begin{aligned}
 m_M^\nu &= \Lambda \begin{pmatrix} 0 & \alpha_{N_1} Y_1^{(4)} & \alpha_{N_2} Y_{1'}^{(4)} \\ \alpha_{N_1} Y_1^{(4)} & Y_1^{(6)} \alpha_{N_3} & 0 \\ \alpha_{N_2} Y_{1'}^{(4)} & 0 & 0 \end{pmatrix}, \\
 \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{3,1}^{(2)} & \alpha_{\nu_1} Y_{3,3}^{(2)} & \alpha_{\nu_1} Y_{3,2}^{(2)} \\ \alpha_{\nu_2} Y_{3,1}^{(4)} & \alpha_{\nu_2} Y_{3,3}^{(4)} & \alpha_{\nu_2} Y_{3,2}^{(4)} \\ \alpha_{\nu_3} Y_{3,2}^{(4)} & \alpha_{\nu_3} Y_{3,1}^{(4)} & \alpha_{\nu_3} Y_{3,3}^{(4)} \end{pmatrix}.
 \end{aligned} \tag{4.49}$$

The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are the same as those of the model \mathcal{I}_1 as well. The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and α_{N_2} , α_{N_3} , α'_{d_2} , α'_{d_3} , α_{u_1} and α_{u_2} are complex.

- Model \mathcal{V}_2 : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}),$
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (3, 1, 3, 1, 1, 1, 3)$

The modular invariant superpotentials for quark and lepton masses are of the following form,

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_1 Y_1^{(6)} + \alpha_{N_2} \Lambda(N_1 N_2)_{1''} Y_{1'}^{(4)} + \alpha_{N_3} \Lambda(N_2 N_3)_1 Y_1^{(4)} + \alpha_{\nu_1} (N_1 F)_3 Y_3^{(4)} H_5 \\
 &\quad + \alpha_{\nu_2} (N_2 F)_3 Y_3^{(2)} H_5 + \alpha_{\nu_3} (N_3 F)_3 Y_3^{(4)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_3^{(4)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_3^{(4)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_{1''} Y_{1'}^{(4)} H_5 + \alpha_{u_2} (T_2 T_3)_1 Y_1^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_1 Y_1^{(6)} H_5.
 \end{aligned} \tag{4.50}$$

We can straightforwardly read out the right-handed neutrino mass matrix and the Yukawa matrices as follows,

$$\begin{aligned}
 m_M^\nu &= \Lambda \begin{pmatrix} Y_1^{(6)} \alpha_{N_1} & \alpha_{N_2} Y_{1'}^{(4)} & 0 \\ \alpha_{N_2} Y_{1'}^{(4)} & 0 & \alpha_{N_3} Y_1^{(4)} \\ 0 & \alpha_{N_3} Y_1^{(4)} & 0 \end{pmatrix}, \\
 \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{\nu_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{\nu_1} Y_{\mathbf{3},2}^{(4)} \\ \alpha_{\nu_2} Y_{\mathbf{3},2}^{(2)} & \alpha_{\nu_2} Y_{\mathbf{3},1}^{(2)} & \alpha_{\nu_2} Y_{\mathbf{3},3}^{(2)} \\ \alpha_{\nu_3} Y_{\mathbf{3},3}^{(4)} & \alpha_{\nu_3} Y_{\mathbf{3},2}^{(4)} & \alpha_{\nu_3} Y_{\mathbf{3},1}^{(4)} \end{pmatrix}. \tag{4.51}
 \end{aligned}$$

The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u coincide with the corresponding ones of the model \mathcal{V}_1 as well. The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and α_{N_2} , α_{N_3} , α'_{d_2} , α'_{d_3} , α_{u_1} and α_{u_2} are generically complex.

- Model \mathcal{V}_3 : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 2, 2, 2, 2, 2)$

The modular invariant superpotential for quark and lepton mass is given by

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_{\mathbf{1}} + \alpha_{N_2} \Lambda(N_2 N_3)_{\mathbf{1}} Y_1^{(4)} + \alpha_{N_3} \Lambda(N_3 N_3)_{\mathbf{1}''} Y_{1'}^{(4)} + \alpha_{\nu_1} (N_1 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 \\
 &\quad + \alpha_{\nu_2} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_3} (N_3 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_{\mathbf{1}''} Y_{1'}^{(4)} H_5 + \alpha_{u_2} (T_1 T_3)_{\mathbf{1}} Y_1^{(4)} H_5 + \alpha_{u_3} (T_2 T_2)_{\mathbf{1}} Y_1^{(4)} H_5 + \alpha_{u_4} (T_3 T_3)_{\mathbf{1}''} Y_{1'}^{(4)} H_5, \tag{4.52}
 \end{aligned}$$

which lead to

$$\begin{aligned}
 m_M^\nu &= \Lambda \begin{pmatrix} \alpha_{N_1} & 0 & 0 \\ 0 & 0 & \alpha_{N_2} Y_1^{(4)} \\ 0 & \alpha_{N_2} Y_{1'}^{(4)} & \alpha_{N_3} Y_1^{(4)} \end{pmatrix}, \\
 \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{\nu_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{\nu_1} Y_{\mathbf{3},2}^{(2)} \\ \alpha_{\nu_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{\nu_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{\nu_2} Y_{\mathbf{3},3}^{(4)} \\ \alpha_{\nu_3} Y_{\mathbf{3},3}^{(4)} & \alpha_{\nu_3} Y_{\mathbf{3},2}^{(4)} & \alpha_{\nu_3} Y_{\mathbf{3},1}^{(4)} \end{pmatrix}, \\
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},3}^{(4)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},2}^{(4)} \\ \alpha_{d_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},1}^{(4)} \end{pmatrix}. \tag{4.53}
 \end{aligned}$$

The \mathcal{Y}_5^u matrix coincides with the corresponding one of the model \mathcal{I}_8 . The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and α_{N_2} , α_{N_3} , α'_{d_2} , α'_{d_3} , α_{u_1} , α_{u_2} and α_{u_4} are complex.

- Model \mathcal{V}_4 : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (4, 2, 2, 0, 2, 2, 2)$

We can read out the superpotential for quark and leptons as follows,

$$\begin{aligned} \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1) \mathbf{1} Y_{\mathbf{1}}^{(8)} + \alpha_{N_2} \Lambda(N_2 N_3) \mathbf{1} Y_{\mathbf{1}}^{(4)} + \alpha_{N_3} \Lambda(N_3 N_3) \mathbf{1}'' Y_{\mathbf{1}'}^{(4)} + \alpha_{\nu_1} (N_1 F) \mathbf{3} Y_{\mathbf{3}}^{(4)} H_5 \\ &\quad + \alpha_{\nu_2} (N_2 F) \mathbf{3} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_3} (N_3 F) \mathbf{3} Y_{\mathbf{3}}^{(2)} H_5, \\ \mathcal{W}_d &= \alpha_{d_1} (FT_1) \mathbf{3} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2) \mathbf{3} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3) \mathbf{3} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} \\ &\quad + \alpha'_{d_1} (FT_1) \mathbf{3} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2) \mathbf{3} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3) \mathbf{3} Y_{\mathbf{3}}^{(2)} H_{\bar{45}}, \\ \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2) \mathbf{1}'' Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_2} (T_1 T_3) \mathbf{1} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_3} (T_2 T_2) \mathbf{1} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_4} (T_3 T_3) \mathbf{1}'' Y_{\mathbf{1}'}^{(4)} H_5. \end{aligned} \quad (4.54)$$

which gives rise to

$$\begin{aligned} m_M^\nu &= \Lambda \begin{pmatrix} \alpha_{N_1} Y_{\mathbf{1}}^{(8)} & 0 & 0 \\ 0 & 0 & \alpha_{N_2} Y_{\mathbf{1}}^{(4)} \\ 0 & \alpha_{N_2} Y_{\mathbf{1}}^{(4)} & \alpha_{N_3} Y_{\mathbf{1}'}^{(4)} \end{pmatrix}, \\ \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{\nu_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{\nu_1} Y_{\mathbf{3},2}^{(4)} \\ \alpha_{\nu_2} Y_{\mathbf{3},2}^{(2)} & \alpha_{\nu_2} Y_{\mathbf{3},1}^{(2)} & \alpha_{\nu_2} Y_{\mathbf{3},3}^{(2)} \\ \alpha_{\nu_3} Y_{\mathbf{3},3}^{(2)} & \alpha_{\nu_3} Y_{\mathbf{3},2}^{(2)} & \alpha_{\nu_3} Y_{\mathbf{3},1}^{(2)} \end{pmatrix}. \end{aligned} \quad (4.55)$$

The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are the same as those of the model \mathcal{I}_8 as well. The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can taken to be real without loss of generality, while α_{N_2} , α_{N_3} , α'_{d_2} , α'_{d_3} , α_{u_1} , α_{u_2} and α_{u_4} are generically complex.

- Model \mathcal{V}_5 : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}'', \mathbf{1}', \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (4, 2, 2, 0, 2, 2, 4)$

The quark and lepton masses are described by the following superpotential:

$$\begin{aligned} \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1) \mathbf{1} Y_{\mathbf{1}}^{(8)} + \alpha_{N_2} \Lambda(N_2 N_3) \mathbf{1} Y_{\mathbf{1}}^{(4)} + \alpha_{N_3} \Lambda(N_3 N_3) \mathbf{1}'' Y_{\mathbf{1}'}^{(4)} + \alpha_{\nu_1} (N_1 F) \mathbf{3} Y_{\mathbf{3}}^{(4)} H_5 \\ &\quad + \alpha_{\nu_2} (N_2 F) \mathbf{3} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_3} (N_3 F) \mathbf{3} Y_{\mathbf{3}}^{(2)} H_5, \\ \mathcal{W}_d &= \alpha_{d_1} (FT_1) \mathbf{3} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2) \mathbf{3} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3) \mathbf{3} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} \\ &\quad + \alpha'_{d_1} (FT_1) \mathbf{3} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2) \mathbf{3} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3) \mathbf{3} Y_{\mathbf{3}}^{(4)} H_{\bar{45}}, \\ \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2) \mathbf{1} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_2} (T_1 T_3) \mathbf{1} Y_{\mathbf{1}}^{(6)} H_5 + \alpha_{u_3} (T_2 T_2) \mathbf{1}'' Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_4} (T_3 T_3) \mathbf{1}'' Y_{\mathbf{1}'}^{(8)} H_5. \end{aligned} \quad (4.56)$$

The matrices M_N and \mathcal{Y}_5^ν are the same as those of the model \mathcal{V}_4 as well. The \mathcal{Y}_5^d matrix is identical with the corresponding one of the model \mathcal{I}_{15} . The \mathcal{Y}_5^u matrix is in common with that of the model \mathcal{I}_{11} . The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and the other parameters are complex.

- Model \mathcal{V}_6 : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (4, 2, 2, 0, 2, 4, 2)$

The modular invariant superpotential for quark and lepton mass is given by

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_1 Y_1^{(8)} + \alpha_{N_2} \Lambda(N_2 N_3)_1 Y_1^{(4)} + \alpha_{N_3} \Lambda(N_3 N_3)_{1''} Y_{1'}^{(4)} + \alpha_{\nu_1} (N_1 F)_3 Y_3^{(4)} H_5 \\
 &\quad + \alpha_{\nu_2} (N_2 F)_3 Y_3^{(2)} H_5 + \alpha_{\nu_3} (N_3 F)_3 Y_3^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_3^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_3^{(2)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_3^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_3^{(2)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_1^{(4)} H_5 + \alpha_{u_2} (T_2 T_2)_{1'} Y_{1''}^{(8)} H_5 + \alpha_{u_3} (T_2 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_4} (T_3 T_3)_{1''} Y_{1'}^{(4)} H_5.
 \end{aligned} \tag{4.57}$$

The matrices M_N and \mathcal{Y}_5^l are identical with the corresponding ones of the model \mathcal{V}_4 as well. The \mathcal{Y}_5^d matrix coincides with that of the model \mathcal{I}_{16} . The \mathcal{Y}_5^u matrix is identical with that of the model \mathcal{I}_{12} . The phases of the parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be removed by field redefinition, and the remaining parameters are complex.

- Model \mathcal{V}_7 : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (4, 2, 2, 0, 2, 4, 4)$

The modular invariant superpotentials for quark and lepton masses are of the following form,

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_1 Y_1^{(8)} + \alpha_{N_2} \Lambda(N_2 N_3)_1 Y_1^{(4)} + \alpha_{N_3} \Lambda(N_3 N_3)_{1''} Y_{1'}^{(4)} + \alpha_{\nu_1} (N_1 F)_3 Y_3^{(4)} H_5 \\
 &\quad + \alpha_{\nu_2} (N_2 F)_3 Y_3^{(2)} H_5 + \alpha_{\nu_3} (N_3 F)_3 Y_3^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_3^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_3^{(4)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_3^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_3^{(4)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_2} (T_2 T_2)_{1'} Y_{1''}^{(8)} H_5 + \alpha_{u_3} (T_2 T_3)_1 Y_1^{(8)} H_5 + \alpha_{u_4} (T_3 T_3)_{1''} Y_{1'}^{(8)} H_5.
 \end{aligned} \tag{4.58}$$

The matrices M_N and \mathcal{Y}_5^l are identical with those of the model \mathcal{V}_4 as well. The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are the same as those of the model \mathcal{I}_{17} as well. The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and α_{N_2} , α_{N_3} , α'_{d_2} , α'_{d_3} , α_{u_1} , α_{u_2} and α_{u_4} are generically complex.

- Model \mathcal{V}_8 : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}', \mathbf{1}', \mathbf{1}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 4, 2, 0, 2, 2, 2)$

We can read out the superpotential for the fermion masses as,

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_{1''} Y_{1'}^{(4)} + \alpha_{N_2} \Lambda(N_2 N_2)_{1''} Y_{1'}^{(8)} + \alpha_{N_3} \Lambda(N_3 N_3)_1 Y_1^{(4)} + \alpha_{\nu_1} (N_1 F)_3 Y_3^{(2)} H_5 \\
 &\quad + \alpha_{\nu_2} (N_2 F)_3 Y_3^{(4)} H_5 + \alpha_{\nu_3} (N_3 F)_3 Y_3^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_3^{(2)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_3^{(2)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_{1''} Y_{1'}^{(4)} H_5 + \alpha_{u_2} (T_1 T_3)_1 Y_1^{(4)} H_5 + \alpha_{u_3} (T_2 T_2)_1 Y_1^{(4)} H_5 + \alpha_{u_4} (T_3 T_3)_{1''} Y_{1'}^{(4)} H_5.
 \end{aligned} \tag{4.59}$$

We can straightforwardly read out the right-handed neutrino mass matrix and the Yukawa matrices as follows,

$$m_M^\nu = \Lambda \begin{pmatrix} \alpha_{N_1} Y_{1'}^{(4)} & 0 & 0 \\ 0 & \alpha_{N_2} Y_{1'}^{(8)} & 0 \\ 0 & 0 & \alpha_{N_3} Y_1^{(4)} \end{pmatrix},$$

$$\mathcal{Y}_5^\nu = \begin{pmatrix} \alpha_{\nu_1} Y_{\mathbf{3},\mathbf{3}}^{(2)} & \alpha_{\nu_1} Y_{\mathbf{3},\mathbf{2}}^{(2)} & \alpha_{\nu_1} Y_{\mathbf{3},\mathbf{1}}^{(2)} \\ \alpha_{\nu_2} Y_{\mathbf{3},\mathbf{3}}^{(4)} & \alpha_{\nu_2} Y_{\mathbf{3},\mathbf{2}}^{(4)} & \alpha_{\nu_2} Y_{\mathbf{3},\mathbf{1}}^{(4)} \\ \alpha_{\nu_3} Y_{\mathbf{3},\mathbf{1}}^{(2)} & \alpha_{\nu_3} Y_{\mathbf{3},\mathbf{3}}^{(2)} & \alpha_{\nu_3} Y_{\mathbf{3},\mathbf{2}}^{(2)} \end{pmatrix}. \quad (4.60)$$

The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are identical with the corresponding ones of the model \mathcal{V}_4 as well. The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and α_{N_2} , α_{N_3} , α'_{d_2} , α'_{d_3} , α_{u_1} , α_{u_2} and α_{u_4} are generically complex.

- Model \mathcal{V}_9 : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}', \mathbf{1}', \mathbf{1}, \mathbf{3}, \mathbf{1}'', \mathbf{1}', \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 4, 2, 0, 2, 2, 4)$

The superpotentials relevant to quark and lepton masses are of the form,

$$\begin{aligned} \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} + \alpha_{N_2} \Lambda(N_2 N_2)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(8)} + \alpha_{N_3} \Lambda(N_3 N_3)_{\mathbf{1}} Y_1^{(4)} + \alpha_{\nu_1} (N_1 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 \\ &\quad + \alpha_{\nu_2} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_3} (N_3 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5, \\ \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} \\ &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}}, \\ \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_{\mathbf{1}} Y_1^{(4)} H_5 + \alpha_{u_2} (T_1 T_3)_{\mathbf{1}} Y_1^{(6)} H_5 + \alpha_{u_3} (T_2 T_2)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_4} (T_3 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(8)} H_5. \end{aligned} \quad (4.61)$$

The matrices M_N and \mathcal{Y}_5^ν are identical with the corresponding ones of the model \mathcal{V}_8 as well. The \mathcal{Y}_5^d matrix is the same as that of the model \mathcal{I}_{15} . The \mathcal{Y}_5^u matrix is identical with the corresponding one of the model \mathcal{I}_{11} . The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and α_{N_2} , α_{N_3} , α'_{d_2} , α'_{d_3} , α_{u_1} , α_{u_2} and α_{u_4} are complex.

- Model \mathcal{V}_{10} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}', \mathbf{1}', \mathbf{1}, \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 4, 2, 0, 2, 4, 2)$

We can read out the superpotential for quark and leptons as follows,

$$\begin{aligned} \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} + \alpha_{N_2} \Lambda(N_2 N_2)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(8)} + \alpha_{N_3} \Lambda(N_3 N_3)_{\mathbf{1}} Y_1^{(4)} + \alpha_{\nu_1} (N_1 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 \\ &\quad + \alpha_{\nu_2} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_3} (N_3 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5, \\ \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} \\ &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}}, \\ \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_{\mathbf{1}} Y_1^{(4)} H_5 + \alpha_{u_2} (T_2 T_2)_{\mathbf{1}'} Y_{\mathbf{1}'}^{(8)} H_5 + \alpha_{u_3} (T_2 T_3)_{\mathbf{1}} Y_1^{(6)} H_5 + \alpha_{u_4} (T_3 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5. \end{aligned} \quad (4.62)$$

The matrices M_N and \mathcal{Y}_5^ν are the same as those of the model \mathcal{V}_8 as well. The \mathcal{Y}_5^d matrix coincides with the corresponding one of the model \mathcal{I}_{16} . The \mathcal{Y}_5^u matrix is in common with that of the model \mathcal{I}_{12} . The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can taken to be real without loss of generality, while the other parameters are complex.

- Model \mathcal{V}_{11} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}', \mathbf{1}', \mathbf{1}, \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}'),$
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 4, 2, 0, 2, 4, 4)$

We can read out the superpotential for quark and leptons as follows,

$$\begin{aligned} \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} + \alpha_{N_2} \Lambda(N_2 N_2)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(8)} + \alpha_{N_3} \Lambda(N_3 N_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} + \alpha_{\nu_1} (N_1 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 \\ &\quad + \alpha_{\nu_2} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_3} (N_3 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5, \\ \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} \\ &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}}, \\ \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(6)} H_5 + \alpha_{u_2} (T_2 T_2)_{\mathbf{1}'} Y_{\mathbf{1}'}^{(8)} H_5 + \alpha_{u_3} (T_2 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(8)} H_5 + \alpha_{u_4} (T_3 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(8)} H_5. \end{aligned} \quad (4.63)$$

The matrices M_N and \mathcal{Y}_5^ν are identical with the corresponding ones of the model \mathcal{V}_8 as well. The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are identical with the corresponding ones of the model \mathcal{I}_{17} as well. The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and the remaining parameters are complex.

- Model \mathcal{V}_{12} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}'', \mathbf{1}'', \mathbf{1}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}'),$
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 4, 2, 0, 2, 2, 2)$

We can read out the superpotential for the fermion masses as,

$$\begin{aligned} \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} + \alpha_{N_2} \Lambda(N_2 N_2)_{\mathbf{1}'} Y_{\mathbf{1}'}^{(8)} + \alpha_{N_3} \Lambda(N_3 N_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} + \alpha_{\nu_1} (N_1 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 \\ &\quad + \alpha_{\nu_2} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_3} (N_3 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5, \\ \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} \\ &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}}, \\ \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_2} (T_1 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_3} (T_2 T_2)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_4} (T_3 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5, \end{aligned} \quad (4.64)$$

which lead to

$$\begin{aligned} m_M^\nu &= \Lambda \begin{pmatrix} 0 & 0 & \alpha_{N_1} Y_{\mathbf{1}'}^{(4)} \\ 0 & \alpha_{N_2} Y_{\mathbf{1}'}^{(8)} & 0 \\ \alpha_{N_1} Y_{\mathbf{1}'}^{(4)} & 0 & \alpha_{N_3} Y_{\mathbf{1}}^{(4)} \end{pmatrix}, \\ \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{\nu_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{\nu_1} Y_{\mathbf{3},3}^{(2)} \\ \alpha_{\nu_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{\nu_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{\nu_2} Y_{\mathbf{3},3}^{(4)} \\ \alpha_{\nu_3} Y_{\mathbf{3},1}^{(2)} & \alpha_{\nu_3} Y_{\mathbf{3},3}^{(2)} & \alpha_{\nu_3} Y_{\mathbf{3},2}^{(2)} \end{pmatrix}. \end{aligned} \quad (4.65)$$

The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are in common with the corresponding ones of the model \mathcal{V}_4 as well. The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can taken to be real without loss of generality, while the remaining parameters are complex.

- Model \mathcal{V}_{13} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}'', \mathbf{1}'', \mathbf{1}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 4, 2, 0, 2, 4, 4)$

We can read out the superpotential for the fermion masses as,

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_3)_{\mathbf{1}'} Y_{\mathbf{1}'}^{(4)} + \alpha_{N_2} \Lambda(N_2 N_2)_{\mathbf{1}'} Y_{\mathbf{1}''}^{(8)} + \alpha_{N_3} \Lambda(N_3 N_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} + \alpha_{\nu_1} (N_1 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 \\
 &\quad + \alpha_{\nu_2} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_3} (N_3 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(6)} H_5 + \alpha_{u_2} (T_2 T_2)_{\mathbf{1}} Y_{\mathbf{1}}^{(8)} H_5 + \alpha_{u_3} (T_2 T_3)_{\mathbf{1}'} Y_{\mathbf{1}''}^{(8)} H_5 + \alpha_{u_4} (T_3 T_3)_{\mathbf{1}''} Y_{\mathbf{1}}^{(8)} H_5.
 \end{aligned} \tag{4.66}$$

The matrices M_N and \mathcal{Y}_5^l are the same as those of the model \mathcal{V}_{12} as well. The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are identical with the corresponding ones of the model \mathcal{I}_{14} as well. The phases of the parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be removed by field redefinition, and the remaining parameters are complex.

- Model \mathcal{V}_{14} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}'', \mathbf{1}'', \mathbf{1}, \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1})$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 4, 2, 0, 2, 2, 4)$

The modular invariant superpotential for quark and lepton mass is given by

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_3)_{\mathbf{1}'} Y_{\mathbf{1}'}^{(4)} + \alpha_{N_2} \Lambda(N_2 N_2)_{\mathbf{1}'} Y_{\mathbf{1}''}^{(8)} + \alpha_{N_3} \Lambda(N_3 N_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} + \alpha_{\nu_1} (N_1 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 \\
 &\quad + \alpha_{\nu_2} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_3} (N_3 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_2} (T_2 T_2)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_3} (T_2 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(6)} H_5 + \alpha_{u_4} (T_3 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(8)} H_5.
 \end{aligned} \tag{4.67}$$

The matrices M_N and \mathcal{Y}_5^l are the same as those of the model \mathcal{V}_{12} as well. The \mathcal{Y}_5^d matrix is the same as the corresponding one of the model \mathcal{I}_1 . The \mathcal{Y}_5^u matrix is identical with that of the model \mathcal{I}_6 . The phases of the parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} are unphysical, and the other parameters are complex.

- Model \mathcal{V}_{15} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}', \mathbf{1}'', \mathbf{1}'')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 0, 0, 2, 0, 0, 4)$

The quark and lepton masses are described by the following superpotential:

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_{\mathbf{1}} + \alpha_{N_2} \Lambda(N_2 N_3)_{\mathbf{1}} + \alpha_{\nu_1} (N_1 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 \\
 &\quad + \alpha_{\nu_3} (N_3 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_{\mathbf{1}} H_5 + \alpha_{u_2} (T_1 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_{\mathbf{1}'} Y_{\mathbf{1}''}^{(8)} H_5,
 \end{aligned} \tag{4.68}$$

which lead to

$$\begin{aligned}
 m_M^\nu &= \Lambda \begin{pmatrix} \alpha_{N_1} & 0 & 0 \\ 0 & 0 & \alpha_{N_2} \\ 0 & \alpha_{N_2} & 0 \end{pmatrix}, \\
 \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{\nu_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{\nu_1} Y_{\mathbf{3},2}^{(2)} \\ \alpha_{\nu_2} Y_{\mathbf{3},2}^{(2)} & \alpha_{\nu_2} Y_{\mathbf{3},1}^{(2)} & \alpha_{\nu_2} Y_{\mathbf{3},3}^{(2)} \\ \alpha_{\nu_3} Y_{\mathbf{3},3}^{(2)} & \alpha_{\nu_3} Y_{\mathbf{3},2}^{(2)} & \alpha_{\nu_3} Y_{\mathbf{3},1}^{(2)} \end{pmatrix}, \\
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_3} Y_{\mathbf{3}I,2}^{(6)} + \alpha_{d_4} Y_{\mathbf{3}II,2}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_3} Y_{\mathbf{3}I,1}^{(6)} + \alpha_{d_4} Y_{\mathbf{3}II,1}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_3} Y_{\mathbf{3}I,3}^{(6)} + \alpha_{d_4} Y_{\mathbf{3}II,3}^{(6)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} 0 & \alpha_{u_1} & \alpha_{u_2} Y_{\mathbf{1}}^{(4)} \\ \alpha_{u_1} & 0 & 0 \\ \alpha_{u_2} Y_{\mathbf{1}}^{(4)} & 0 & \alpha_{u_3} Y_{\mathbf{1}''}^{(8)} \end{pmatrix}. \tag{4.69}
 \end{aligned}$$

The phases of the parameters $\alpha_{N_1}, \alpha_{\nu_1}, \alpha_{\nu_2}, \alpha_{\nu_3}, \alpha_{d_1}, \alpha_{d_2}, \alpha_{d_3}, \alpha'_{d_1}$ and α_{u_3} are unphysical, and the remaining parameters are complex.

- Model \mathcal{V}_{16} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}', \mathbf{1}'', \mathbf{1}),$
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 0, 0, 2, 0, 0, 4)$

The quark and lepton masses are described by the following superpotential:

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_{\mathbf{1}} + \alpha_{N_2} \Lambda(N_2 N_3)_{\mathbf{1}} + \alpha_{\nu_1} (N_1 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 \\
 &\quad + \alpha_{\nu_3} (N_3 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3}II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_{\mathbf{1}} H_5 + \alpha_{u_2} (T_2 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(8)} H_5. \tag{4.70}
 \end{aligned}$$

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\begin{aligned}
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_3} Y_{\mathbf{3}I,1}^{(6)} + \alpha_{d_4} Y_{\mathbf{3}II,1}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},2}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_3} Y_{\mathbf{3}I,3}^{(6)} + \alpha_{d_4} Y_{\mathbf{3}II,3}^{(6)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(2)} & \alpha_{d_3} Y_{\mathbf{3}I,2}^{(6)} + \alpha_{d_4} Y_{\mathbf{3}II,2}^{(6)} \end{pmatrix}, \\
 \mathcal{Y}_5^u &= \begin{pmatrix} 0 & \alpha_{u_1} & 0 \\ \alpha_{u_1} & 0 & \alpha_{u_2} Y_{\mathbf{1}'}^{(4)} \\ 0 & \alpha_{u_2} Y_{\mathbf{1}'}^{(4)} & \alpha_{u_3} Y_{\mathbf{1}}^{(8)} \end{pmatrix}. \tag{4.71}
 \end{aligned}$$

The matrices M_N and \mathcal{Y}_5^ν are identical with the corresponding ones of the model \mathcal{V}_{15} as well. The parameters $\alpha_{N_1}, \alpha_{\nu_1}, \alpha_{\nu_2}, \alpha_{\nu_3}, \alpha_{d_1}, \alpha_{d_2}, \alpha_{d_3}, \alpha'_{d_1}$ and α_{u_3} can be taken real by field redefinition, and the other parameters are complex.

- Model \mathcal{V}_{17} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}', \mathbf{1}'', \mathbf{1})$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 0, 0, 4, 0, 0, 4)$

We can read out the superpotentials relevant to quark and lepton masses

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_1 + \alpha_{N_2} \Lambda(N_2 N_3)_1 + \alpha_{\nu_1} (N_1 F)_3 Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_2} (N_2 F)_3 Y_{\mathbf{3}}^{(4)} H_5 \\
 &\quad + \alpha_{\nu_3} (N_3 F)_3 Y_{\mathbf{3}}^{(4)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_{\mathbf{3},I}^{(8)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_3 Y_{\mathbf{3},II}^{(8)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_3 Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_{\mathbf{3},I}^{(8)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_3 Y_{\mathbf{3},II}^{(8)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_1 H_5 + \alpha_{u_2} (T_2 T_3)_1 Y_{\mathbf{1}''}^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_1 Y_{\mathbf{1}}^{(8)} H_5.
 \end{aligned} \tag{4.72}$$

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\begin{aligned}
 \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{\nu_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{\nu_1} Y_{\mathbf{3},2}^{(4)} \\ \alpha_{\nu_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{\nu_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{\nu_2} Y_{\mathbf{3},3}^{(4)} \\ \alpha_{\nu_3} Y_{\mathbf{3},3}^{(4)} & \alpha_{\nu_3} Y_{\mathbf{3},2}^{(4)} & \alpha_{\nu_3} Y_{\mathbf{3},1}^{(4)} \end{pmatrix}, \\
 \mathcal{Y}_5^d &= \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,1}^{(8)} + \alpha_{d_4} Y_{\mathbf{3},II,1}^{(8)} \\ \alpha_{d_1} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,3}^{(8)} + \alpha_{d_4} Y_{\mathbf{3},II,3}^{(8)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,2}^{(8)} + \alpha_{d_4} Y_{\mathbf{3},II,2}^{(8)} \end{pmatrix}.
 \end{aligned} \tag{4.73}$$

The matrices M_N and \mathcal{Y}_5^ν are identical with those of the models \mathcal{V}_{15} and \mathcal{V}_{16} respectively. The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can taken to be real without loss of generality, while the remaining parameters α_{N_2} , α_{d_4} , α'_{d_2} , α'_{d_3} , α'_{d_4} , α_{u_1} and α_{u_2} are complex.

- Model \mathcal{V}_{18} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}', \mathbf{1}'', \mathbf{1})$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 0, 0, 2, 0, 0, 4)$

The quark and lepton masses are described by the following superpotential:

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_1 Y_{\mathbf{1}}^{(4)} + \alpha_{N_2} \Lambda(N_2 N_3)_1 + \alpha_{\nu_1} (N_1 F)_3 Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_2} (N_2 F)_3 Y_{\mathbf{3}}^{(2)} H_5 \\
 &\quad + \alpha_{\nu_3} (N_3 F)_3 Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_3 Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_3 Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_3 Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_1 H_5 + \alpha_{u_2} (T_2 T_3)_1 Y_{\mathbf{1}''}^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_1 Y_{\mathbf{1}}^{(8)} H_5,
 \end{aligned} \tag{4.74}$$

which gives rise to

$$m_M^\nu = \Lambda \begin{pmatrix} \alpha_{N_1} Y_{\mathbf{1}}^{(4)} & 0 & 0 \\ 0 & 0 & \alpha_{N_2} \\ 0 & \alpha_{N_2} & 0 \end{pmatrix}. \tag{4.75}$$

The \mathcal{Y}_5^ν matrix is identical with the corresponding one of the model \mathcal{V}_4 . The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are identical with those of the model \mathcal{V}_{16} as well. The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and the other parameters are complex.

- Model \mathcal{V}_{19} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}', \mathbf{1}'', \mathbf{1}'')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 0, 0, 4, 0, 0, 4)$

The modular invariant superpotential for quark and lepton mass is given by

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_1 + \alpha_{N_2} \Lambda(N_2 N_3)_1 + \alpha_{\nu_1} (N_1 F)_3 Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_2} (N_2 F)_3 Y_{\mathbf{3}}^{(4)} H_5 \\
 &\quad + \alpha_{\nu_3} (N_3 F)_3 Y_{\mathbf{3}}^{(4)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_{\mathbf{3},I}^{(8)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_3 Y_{\mathbf{3},II}^{(8)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_3 Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_{\mathbf{3},I}^{(8)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_3 Y_{\mathbf{3},II}^{(8)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_1 H_5 + \alpha_{u_2} (T_1 T_3)_1 Y_1^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_{1'} Y_{1''}^{(8)} H_5.
 \end{aligned} \tag{4.76}$$

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\mathcal{Y}_5^d = \begin{pmatrix} \alpha_{d_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,2}^{(8)} + \alpha_{d_4} Y_{\mathbf{3},II,2}^{(8)} \\ \alpha_{d_1} Y_{\mathbf{3},2}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,1}^{(8)} + \alpha_{d_4} Y_{\mathbf{3},II,1}^{(8)} \\ \alpha_{d_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{d_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{d_3} Y_{\mathbf{3},I,3}^{(8)} + \alpha_{d_4} Y_{\mathbf{3},II,3}^{(8)} \end{pmatrix}. \tag{4.77}$$

The matrices M_N and \mathcal{Y}_5^u are identical with the corresponding ones of the model \mathcal{V}_{15} as well. The \mathcal{Y}_5^d matrix is identical with that of the model \mathcal{V}_{17} . The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken to be real without loss of generality, while the other parameters are complex.

- Model \mathcal{V}_{20} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}', \mathbf{1}'', \mathbf{1}'')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 0, 0, 2, 0, 0, 4)$

The modular invariant superpotential is of the following form:

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_1 Y_1^{(4)} + \alpha_{N_2} \Lambda(N_2 N_3)_1 + \alpha_{\nu_1} (N_1 F)_3 Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_2} (N_2 F)_3 Y_{\mathbf{3}}^{(2)} H_5 \\
 &\quad + \alpha_{\nu_3} (N_3 F)_3 Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_3 Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_3 Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_3 Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_1 H_5 + \alpha_{u_2} (T_1 T_3)_1 Y_1^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_{1'} Y_{1''}^{(8)} H_5.
 \end{aligned} \tag{4.78}$$

The M_N matrix coincides with that of the model \mathcal{V}_{18} . The \mathcal{Y}_5^d matrix is the same as that of the model \mathcal{V}_4 . The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are identical with those of the model \mathcal{V}_{15} as well. The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and α_{N_2} , α_{d_4} , α'_{d_2} , α'_{d_3} , α'_{d_4} , α_{u_1} and α_{u_2} are complex.

- Model \mathcal{V}_{21} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}'', \mathbf{1}', \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 0, 0, 2, 2, 0, 4)$

The modular invariant superpotential is of the following form:

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_1 + \alpha_{N_2} \Lambda(N_2 N_3)_1 + \alpha_{\nu_1} (N_1 F)_3 Y_3^{(2)} H_5 + \alpha_{\nu_2} (N_2 F)_3 Y_3^{(2)} H_5 \\
 &\quad + \alpha_{\nu_3} (N_3 F)_3 Y_3^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_3^{(4)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_{3,I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_3 Y_{3,II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_3 Y_3^{(4)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_{3,I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_3 Y_{3,II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_2} (T_2 T_3)_{1''} Y_1^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_{1''} Y_1^{(8)} H_5.
 \end{aligned} \tag{4.79}$$

The matrices M_N and \mathcal{Y}_5^ν are identical with the corresponding ones of the model \mathcal{V}_{15} as well. The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are the same as those of the model \mathcal{I}_2 as well. The phases of the parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} are unphysical, and the remaining parameters α_{N_2} , α_{d_4} , α'_{d_2} , α'_{d_3} , α'_{d_4} , α_{u_1} and α_{u_2} are complex.

- Model \mathcal{V}_{22} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}'', \mathbf{1}', \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 0, 0, 2, 2, 0, 4)$

The modular invariant superpotential for quark and lepton mass is given by

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_1 Y_1^{(4)} + \alpha_{N_2} \Lambda(N_2 N_3)_1 + \alpha_{\nu_1} (N_1 F)_3 Y_3^{(4)} H_5 + \alpha_{\nu_2} (N_2 F)_3 Y_3^{(2)} H_5 \\
 &\quad + \alpha_{\nu_3} (N_3 F)_3 Y_3^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_3^{(4)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_{3,I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_3 Y_{3,II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_3 Y_3^{(4)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_{3,I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_3 Y_{3,II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_2} (T_2 T_3)_{1''} Y_1^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_{1''} Y_1^{(8)} H_5.
 \end{aligned} \tag{4.80}$$

The M_N matrix is identical with the corresponding one of the model \mathcal{V}_{18} . The \mathcal{Y}_5^ν matrix is the same as that of the model \mathcal{V}_4 . The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are identical with those of the model \mathcal{I}_2 as well. The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and the remaining parameters α_{N_2} , α_{d_4} , α'_{d_2} , α'_{d_3} , α'_{d_4} , α_{u_1} and α_{u_2} are complex.

- Model \mathcal{V}_{23} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 0, 0, 2, 0, 2, 4)$

The modular invariant superpotentials for quark and lepton masses are of the following form,

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_1 + \alpha_{N_2} \Lambda(N_2 N_3)_1 + \alpha_{\nu_1} (N_1 F)_3 Y_3^{(2)} H_5 + \alpha_{\nu_2} (N_2 F)_3 Y_3^{(2)} H_5 \\
 &\quad + \alpha_{\nu_3} (N_3 F)_3 Y_3^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_3^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_{3,I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_3 Y_{3,II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_3^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_{3,I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_3 Y_{3,II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_1^{(4)} H_5 + \alpha_{u_2} (T_2 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_3} (T_3 T_3)_{1''} Y_1^{(8)} H_5.
 \end{aligned} \tag{4.81}$$

The matrices M_N and \mathcal{Y}_5^ν are identical with those of the model \mathcal{V}_{15} as well. The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are identical with those of the model \mathcal{I}_3 as well. The phases of the parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be removed by field redefinition, and the remaining parameters are complex.

- Model \mathcal{V}_{24} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{1}', \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}'),$
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 0, 0, 2, 0, 2, 4)$

The superpotentials relevant to quark and lepton masses are of the form,

$$\begin{aligned} \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_1 Y_1^{(4)} + \alpha_{N_2} \Lambda(N_2 N_3)_1 + \alpha_{\nu_1} (N_1 F)_3 Y_3^{(4)} H_5 + \alpha_{\nu_2} (N_2 F)_3 Y_3^{(2)} H_5 \\ &\quad + \alpha_{\nu_3} (N_3 F)_3 Y_3^{(2)} H_5, \\ \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_3^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_{3,I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_3 Y_{3,II}^{(6)} H_{\bar{5}} \\ &\quad + \alpha'_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_3^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_{3,I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_3 Y_{3,II}^{(6)} H_{\bar{45}}, \\ \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_1^{(4)} H_5 + \alpha_{u_2} (T_2 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_3} (T_3 T_3)_{1''} Y_{1'}^{(8)} H_5. \end{aligned} \quad (4.82)$$

The M_N matrix is identical with that of the model \mathcal{V}_{18} . The \mathcal{Y}_5^ν matrix coincides with that of the model \mathcal{V}_4 . The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are identical with those of the model \mathcal{I}_3 as well. The phases of the parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be removed by field redefinition, and the other parameters are complex.

- Model \mathcal{V}_{25} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}', \mathbf{1}', \mathbf{1}'', \mathbf{3}, \mathbf{1}', \mathbf{1}'', \mathbf{1}),$
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 0, 2, 0, 0, 4)$

The modular invariant superpotentials for quark and lepton masses are of the following form,

$$\begin{aligned} \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_3)_1 + \alpha_{N_2} \Lambda(N_2 N_2)_{1''} Y_{1'}^{(4)} + \alpha_{\nu_1} (N_1 F)_3 Y_3^{(2)} H_5 + \alpha_{\nu_2} (N_2 F)_3 Y_3^{(4)} H_5 \\ &\quad + \alpha_{\nu_3} (N_3 F)_3 Y_3^{(2)} H_5, \\ \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_{3,I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_3 Y_{3,II}^{(6)} H_{\bar{5}} \\ &\quad + \alpha'_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_{3,I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_3 Y_{3,II}^{(6)} H_{\bar{45}}, \\ \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_1 H_5 + \alpha_{u_2} (T_2 T_3)_{1''} Y_{1'}^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_1 Y_1^{(8)} H_5. \end{aligned} \quad (4.83)$$

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\begin{aligned} m_M^\nu &= \Lambda \begin{pmatrix} 0 & 0 & \alpha_{N_1} \\ 0 & \alpha_{N_2} Y_{1'}^{(4)} & 0 \\ \alpha_{N_1} & 0 & 0 \end{pmatrix}, \\ \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{3,3}^{(2)} & \alpha_{\nu_1} Y_{3,2}^{(2)} & \alpha_{\nu_1} Y_{3,1}^{(2)} \\ \alpha_{\nu_2} Y_{3,3}^{(4)} & \alpha_{\nu_2} Y_{3,2}^{(4)} & \alpha_{\nu_2} Y_{3,1}^{(4)} \\ \alpha_{\nu_3} Y_{3,2}^{(2)} & \alpha_{\nu_3} Y_{3,1}^{(2)} & \alpha_{\nu_3} Y_{3,3}^{(2)} \end{pmatrix}. \end{aligned} \quad (4.84)$$

The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are identical with those of the model \mathcal{V}_{16} as well. The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and α_{N_2} , α_{d_4} , α'_{d_2} , α'_{d_3} , α'_{d_4} , α_{u_1} and α_{u_2} are generically complex.

- Model \mathcal{V}_{26} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}', \mathbf{1}', \mathbf{1}'', \mathbf{3}, \mathbf{1}', \mathbf{1}'', \mathbf{1}'')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 0, 2, 0, 0, 4)$

The modular invariant superpotentials for quark and lepton masses are of the following form,

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_3)_1 + \alpha_{N_2} \Lambda(N_2 N_2)_{1''} Y_{1'}^{(4)} + \alpha_{\nu_1} (N_1 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5 \\
 &\quad + \alpha_{\nu_3} (N_3 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_2)_1 H_5 + \alpha_{u_2} (T_1 T_3)_1 Y_1^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_{1'} Y_{1''}^{(8)} H_5.
 \end{aligned} \tag{4.85}$$

The matrices M_N and \mathcal{Y}_5^y are identical with those of the model \mathcal{V}_{25} as well. The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are identical with those of the model \mathcal{V}_{15} as well. The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be taken real by field redefinition, and α_{N_2} , α_{d_4} , α'_{d_2} , α'_{d_3} , α'_{d_4} , α_{u_1} and α_{u_2} are generically complex.

- Model \mathcal{V}_{27} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}', \mathbf{1}', \mathbf{1}'', \mathbf{3}, \mathbf{1}'', \mathbf{1}', \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 0, 2, 2, 0, 4)$

We can read out the superpotential for the fermion masses as,

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_3)_1 + \alpha_{N_2} \Lambda(N_2 N_2)_{1''} Y_{1'}^{(4)} + \alpha_{\nu_1} (N_1 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5 \\
 &\quad + \alpha_{\nu_3} (N_3 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_2} (T_2 T_3)_{1''} Y_{1'}^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_{1''} Y_{1'}^{(8)} H_5.
 \end{aligned} \tag{4.86}$$

The matrices M_N and \mathcal{Y}_5^y are identical with those of the model \mathcal{V}_{25} as well. The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are the same as those of the model \mathcal{I}_2 as well. The phases of the parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} are unphysical, and the remaining parameters α_{N_2} , α_{d_4} , α'_{d_2} , α'_{d_3} , α'_{d_4} , α_{u_1} and α_{u_2} are complex.

- Model \mathcal{V}_{28} : $(\rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}', \mathbf{1}', \mathbf{1}'', \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_{N_3}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 0, 2, 0, 2, 4)$

The quark and lepton masses are described by the following superpotential:

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_3)_1 + \alpha_{N_2} \Lambda(N_2 N_2)_{1''} Y_{1'}^{(4)} + \alpha_{\nu_1} (N_1 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5 \\
 &\quad + \alpha_{\nu_3} (N_3 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_1^{(4)} H_5 + \alpha_{u_2} (T_2 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_3} (T_3 T_3)_{1''} Y_{1'}^{(8)} H_5.
 \end{aligned} \tag{4.87}$$

The matrices M_N and \mathcal{Y}_5^ν are identical with those of the model \mathcal{V}_{25} as well. The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are identical with those of the model \mathcal{I}_3 as well. The phases of the parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{ν_3} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be removed by field redefinition, and α_{N_2} , α_{d_4} , α'_{d_2} , α'_{d_3} , α'_{d_4} , α_{u_1} and α_{u_2} are complex.

4.4 Type-VII

- Model $\mathcal{V}\mathcal{I}\mathcal{I}_1$: $(\rho_{N_1}, \rho_{N_2}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}'', \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}),$
 $(k_{N_1}, k_{N_2}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (1, 3, 3, 1, 1, 3)$

The modular invariant superpotentials for quark and lepton masses are of the following form,

$$\begin{aligned} \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_2)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} + \alpha_{\nu_1} (N_1 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_2} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_5 + \alpha_{\nu_3} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_5, \\ \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\ &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\ \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_2} (T_2 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(6)} H_5. \end{aligned} \quad (4.88)$$

which gives rise to

$$\begin{aligned} m_M^\nu &= \Lambda \begin{pmatrix} 0 & \alpha_{N_1} Y_{\mathbf{1}'}^{(4)} \\ \alpha_{N_1} Y_{\mathbf{1}'}^{(4)} & 0 \end{pmatrix}, \\ \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{\nu_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{\nu_1} Y_{\mathbf{3},2}^{(4)} \\ \alpha_{\nu_2} Y_{\mathbf{3},I,2}^{(6)} + \alpha_{\nu_3} Y_{\mathbf{3},II,2}^{(6)} & \alpha_{\nu_2} Y_{\mathbf{3},I,1}^{(6)} + \alpha_{\nu_3} Y_{\mathbf{3},II,1}^{(6)} & \alpha_{\nu_2} Y_{\mathbf{3},I,3}^{(6)} + \alpha_{\nu_3} Y_{\mathbf{3},II,3}^{(6)} \end{pmatrix}. \end{aligned} \quad (4.89)$$

The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are in common with the corresponding ones of the models \mathcal{I}_6 and \mathcal{I}_1 respectively. The parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can taken to be real without loss of generality, while α_{ν_3} , α_{d_4} , α'_{d_2} , α'_{d_3} , α'_{d_4} , α_{u_1} and α_{u_2} are complex.

- Model $\mathcal{V}\mathcal{I}\mathcal{I}_2$: $(\rho_{N_1}, \rho_{N_2}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}', \mathbf{1}', \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1}),$
 $(k_{N_1}, k_{N_2}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (1, 3, 3, 1, 1, 3)$

We can read out the superpotential for quark and leptons as follows,

$$\begin{aligned} \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_2)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} + \alpha_{\nu_1} (N_1 F)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_2} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_5 + \alpha_{\nu_3} (N_2 F)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_5, \\ \mathcal{W}_d &= \alpha_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\ &\quad + \alpha'_{d_1} (FT_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_{\mathbf{3}} Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\ \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} H_5 + \alpha_{u_2} (T_2 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_{\mathbf{1}} Y_{\mathbf{1}}^{(6)} H_5. \end{aligned} \quad (4.90)$$

We can straightforwardly read out the right-handed neutrino mass matrix and the Yukawa matrices as follows,

$$\mathcal{Y}_5^\nu = \begin{pmatrix} \alpha_{\nu_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{\nu_1} Y_{\mathbf{3},2}^{(4)} & \alpha_{\nu_1} Y_{\mathbf{3},1}^{(4)} \\ \alpha_{\nu_2} Y_{\mathbf{3},I,3}^{(6)} + \alpha_{\nu_3} Y_{\mathbf{3},II,3}^{(6)} & \alpha_{\nu_2} Y_{\mathbf{3},I,2}^{(6)} + \alpha_{\nu_3} Y_{\mathbf{3},II,2}^{(6)} & \alpha_{\nu_2} Y_{\mathbf{3},I,1}^{(6)} + \alpha_{\nu_3} Y_{\mathbf{3},II,1}^{(6)} \end{pmatrix}. \quad (4.91)$$

The M_N matrix is in common with that of the model $\mathcal{V}\mathcal{I}\mathcal{I}_1$. The \mathcal{Y}_5^d matrix is the same as that of the model \mathcal{I}_6 . The \mathcal{Y}_5^u matrix is in common with that of the model \mathcal{I}_1 . The phases of the parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be removed by field redefinition, and α_{ν_3} , α_{d_4} , α'_{d_2} , α'_{d_3} , α'_{d_4} , α_{u_1} and α_{u_2} are generically complex.

- Model $\mathcal{V}\mathcal{I}\mathcal{I}_3$: $(\rho_{N_1}, \rho_{N_2}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}', \mathbf{1}'', \mathbf{3}, \mathbf{1}'', \mathbf{1}, \mathbf{1})$,
 $(k_{N_1}, k_{N_2}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (1, 3, 3, 1, 1, 3)$

We can read out the superpotential for quark and leptons as follows,

$$\begin{aligned} \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_2)_1 Y_1^{(4)} + \alpha_{\nu_1} (N_1 F)_3 Y_3^{(4)} H_5 + \alpha_{\nu_2} (N_2 F)_3 Y_{3,I}^{(6)} H_5 + \alpha_{\nu_3} (N_2 F)_3 Y_{3,II}^{(6)} H_5, \\ \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_3^{(4)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_3^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_{3,I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_3 Y_{3,II}^{(6)} H_{\bar{5}} \\ &\quad + \alpha'_{d_1} (FT_1)_3 Y_3^{(4)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_3^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_{3,I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_3 Y_{3,II}^{(6)} H_{\bar{45}}, \\ \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_{1''} Y_{1'}^{(4)} H_5 + \alpha_{u_2} (T_2 T_3)_1 Y_1^{(4)} H_5 + \alpha_{u_3} (T_3 T_3)_1 Y_1^{(6)} H_5, \end{aligned} \quad (4.92)$$

which lead to

$$\begin{aligned} m_M^\nu &= \Lambda \begin{pmatrix} 0 & \alpha_{N_1} Y_1^{(4)} \\ \alpha_{N_1} Y_1^{(4)} & 0 \end{pmatrix}, \\ \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{3,3}^{(4)} & \alpha_{\nu_1} Y_{3,2}^{(4)} & \alpha_{\nu_1} Y_{3,1}^{(4)} \\ \alpha_{\nu_2} Y_{3,I,2}^{(6)} + \alpha_{\nu_3} Y_{3,II,2}^{(6)} & \alpha_{\nu_2} Y_{3,I,1}^{(6)} + \alpha_{\nu_3} Y_{3,II,1}^{(6)} & \alpha_{\nu_2} Y_{3,I,3}^{(6)} + \alpha_{\nu_3} Y_{3,II,3}^{(6)} \end{pmatrix}. \end{aligned} \quad (4.93)$$

The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u coincide with those of the model $\mathcal{V}\mathcal{I}\mathcal{I}_1$ as well. The phases of the parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} can be removed by field redefinition, and α_{ν_3} , α_{d_4} , α'_{d_2} , α'_{d_3} , α'_{d_4} , α_{u_1} and α_{u_2} are generically complex.

- Model $\mathcal{V}\mathcal{I}\mathcal{I}_4$: $(\rho_{N_1}, \rho_{N_2}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}')$,
 $(k_{N_1}, k_{N_2}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 2, 0, 2, 4)$

We can read out the superpotential for quark and leptons as follows,

$$\begin{aligned} \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_1 + \alpha_{N_2} \Lambda(N_2 N_2)_1 Y_1^{(4)} + \alpha_{\nu_1} (N_1 F)_3 Y_3^{(2)} H_5 + \alpha_{\nu_2} (N_2 F)_3 Y_3^{(4)} H_5, \\ \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_3^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_{3,I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_3 Y_{3,II}^{(6)} H_{\bar{5}} \\ &\quad + \alpha'_{d_1} (FT_1)_3 Y_3^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_3^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_{3,I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_3 Y_{3,II}^{(6)} H_{\bar{45}}, \\ \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_1^{(4)} H_5 + \alpha_{u_2} (T_2 T_3)_1 Y_1^{(6)} H_5 + \alpha_{u_3} (T_3 T_3)_{1''} Y_{1'}^{(8)} H_5. \end{aligned} \quad (4.94)$$

The right-handed neutrino mass matrix and the Yukawa coupling matrices are predicted to be

$$\begin{aligned} m_M^\nu &= \Lambda \begin{pmatrix} \alpha_{N_1} & 0 \\ 0 & \alpha_{N_2} Y_1^{(4)} \end{pmatrix}, \\ \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{3,1}^{(2)} & \alpha_{\nu_1} Y_{3,3}^{(2)} & \alpha_{\nu_1} Y_{3,2}^{(2)} \\ \alpha_{\nu_2} Y_{3,1}^{(4)} & \alpha_{\nu_2} Y_{3,3}^{(4)} & \alpha_{\nu_2} Y_{3,2}^{(4)} \end{pmatrix}. \end{aligned} \quad (4.95)$$

The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are the same as those of the model \mathcal{I}_3 as well. The phases of the parameters α_{N_1} , α_{ν_1} , α_{ν_2} , α_{d_1} , α_{d_2} , α_{d_3} , α'_{d_1} and α_{u_3} are unphysical, and the other parameters are complex.

- Model $\mathcal{V}\mathcal{L}\mathcal{L}_5$: $(\rho_{N_1}, \rho_{N_2}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}', \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}'),$
 $(k_{N_1}, k_{N_2}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (0, 2, 2, 0, 2, 4)$

We can read out the superpotentials relevant to quark and lepton masses

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_1 + \alpha_{N_2} \Lambda(N_2 N_2)_{1''} Y_{\mathbf{1}'}^{(4)} + \alpha_{\nu_1} (N_1 F)_3 Y_{\mathbf{3}}^{(2)} H_5 + \alpha_{\nu_2} (N_2 F)_3 Y_{\mathbf{3}}^{(4)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_3 Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_3 Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_3 Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_2} (T_2 T_3)_1 Y_{\mathbf{1}}^{(6)} H_5 + \alpha_{u_3} (T_3 T_3)_{1''} Y_{\mathbf{1}'}^{(8)} H_5.
 \end{aligned} \tag{4.96}$$

The right-handed neutrino mass matrix and the Yukawa matrix are given by

$$\begin{aligned}
 m_M^\nu &= \Lambda \begin{pmatrix} \alpha_{N_1} & 0 \\ 0 & \alpha_{N_2} Y_{\mathbf{1}'}^{(4)} \end{pmatrix}, \\
 \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{\mathbf{3},1}^{(2)} & \alpha_{\nu_1} Y_{\mathbf{3},3}^{(2)} & \alpha_{\nu_1} Y_{\mathbf{3},2}^{(2)} \\ \alpha_{\nu_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{\nu_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{\nu_2} Y_{\mathbf{3},1}^{(4)} \end{pmatrix}.
 \end{aligned} \tag{4.97}$$

The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u coincide with those of the model $\mathcal{V}\mathcal{L}\mathcal{L}_4$ as well. The parameters $\alpha_{N_1}, \alpha_{\nu_1}, \alpha_{\nu_2}, \alpha_{d_1}, \alpha_{d_2}, \alpha_{d_3}, \alpha'_{d_1}$ and α_{u_3} are real, and $\alpha_{N_2}, \alpha_{d_4}, \alpha'_{d_2}, \alpha'_{d_3}, \alpha'_{d_4}, \alpha_{u_1}$ and α_{u_2} are generically complex.

- Model $\mathcal{V}\mathcal{L}\mathcal{L}_6$: $(\rho_{N_1}, \rho_{N_2}, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3}) = (\mathbf{1}, \mathbf{1}', \mathbf{3}, \mathbf{1}'', \mathbf{1}'', \mathbf{1}'),$
 $(k_{N_1}, k_{N_2}, k_F, k_{T_1}, k_{T_2}, k_{T_3}) = (2, 2, 2, 0, 2, 4)$

We can read out the superpotentials relevant to quark and lepton masses

$$\begin{aligned}
 \mathcal{W}_\nu &= \alpha_{N_1} \Lambda(N_1 N_1)_1 Y_{\mathbf{1}}^{(4)} + \alpha_{N_2} \Lambda(N_2 N_2)_{1''} Y_{\mathbf{1}'}^{(4)} + \alpha_{\nu_1} (N_1 F)_3 Y_{\mathbf{3}}^{(4)} H_5 + \alpha_{\nu_2} (N_2 F)_3 Y_{\mathbf{3}}^{(4)} H_5, \\
 \mathcal{W}_d &= \alpha_{d_1} (FT_1)_3 Y_{\mathbf{3}}^{(2)} H_{\bar{5}} + \alpha_{d_2} (FT_2)_3 Y_{\mathbf{3}}^{(4)} H_{\bar{5}} + \alpha_{d_3} (FT_3)_3 Y_{\mathbf{3},I}^{(6)} H_{\bar{5}} + \alpha_{d_4} (FT_3)_3 Y_{\mathbf{3},II}^{(6)} H_{\bar{5}} \\
 &\quad + \alpha'_{d_1} (FT_1)_3 Y_{\mathbf{3}}^{(2)} H_{\bar{45}} + \alpha'_{d_2} (FT_2)_3 Y_{\mathbf{3}}^{(4)} H_{\bar{45}} + \alpha'_{d_3} (FT_3)_3 Y_{\mathbf{3},I}^{(6)} H_{\bar{45}} + \alpha'_{d_4} (FT_3)_3 Y_{\mathbf{3},II}^{(6)} H_{\bar{45}}, \\
 \mathcal{W}_u &= \alpha_{u_1} (T_1 T_3)_1 Y_{\mathbf{1}}^{(4)} H_5 + \alpha_{u_2} (T_2 T_3)_1 Y_{\mathbf{1}}^{(6)} H_5 + \alpha_{u_3} (T_3 T_3)_{1''} Y_{\mathbf{1}'}^{(8)} H_5,
 \end{aligned} \tag{4.98}$$

which gives rise to

$$\begin{aligned}
 m_M^\nu &= \Lambda \begin{pmatrix} \alpha_{N_1} Y_{\mathbf{1}}^{(4)} & 0 \\ 0 & \alpha_{N_2} Y_{\mathbf{1}'}^{(4)} \end{pmatrix}, \\
 \mathcal{Y}_5^\nu &= \begin{pmatrix} \alpha_{\nu_1} Y_{\mathbf{3},1}^{(4)} & \alpha_{\nu_1} Y_{\mathbf{3},3}^{(4)} & \alpha_{\nu_1} Y_{\mathbf{3},2}^{(4)} \\ \alpha_{\nu_2} Y_{\mathbf{3},3}^{(4)} & \alpha_{\nu_2} Y_{\mathbf{3},2}^{(4)} & \alpha_{\nu_2} Y_{\mathbf{3},1}^{(4)} \end{pmatrix}.
 \end{aligned} \tag{4.99}$$

The matrices \mathcal{Y}_5^d and \mathcal{Y}_5^u are in common with the corresponding ones of the model $\mathcal{V}\mathcal{L}\mathcal{L}_4$ as well. The phases of the parameters $\alpha_{N_1}, \alpha_{\nu_1}, \alpha_{\nu_2}, \alpha_{d_1}, \alpha_{d_2}, \alpha_{d_3}, \alpha'_{d_1}$ and α_{u_3} can be removed by field redefinition, and the other parameters are complex.

5 Numerical results

We have numerically scanned the parameter space of each model to optimize the agreement between predictions and experimental data. The dynamics determining the VEV of the complex modulus τ is an open question now, consequently we treat the VEV of τ as a free parameter to match the experimental data. As explicitly shown in previous section, the phases of some coupling constants can be removed by a field redefinition while others are generally complex. Each model depends on a set of dimensionless input parameters and three overall mass scales for the up type quark, charged lepton and neutrino mass matrices. The dimensionless input parameters include the modulus τ and the ratios of the coupling constants and they determine the mixing angles, CP violation phases and the fermion mass ratios. As can be seen from eq. (3.17), the down-type quark and charged lepton mass matrices are closely related in SU(5) and they share a same overall factor. We can use the measured values of Δm_{21}^2 , m_τ and m_t to determine the overall mass parameters of the neutrino, the charged lepton, the up quark mass matrices. As a measure of goodness of fit, we use a χ^2 function defined in the usual way,

$$\chi_{\text{total}}^2 = \sum_i \left(\frac{\xi_i - \bar{\xi}_i}{\sigma_i} \right)^2, \quad (5.1)$$

where $\bar{\xi}_i$ and σ_i refer to the global best fit values and the 1σ deviations of the corresponding observables and their values are listed in table 3.

The total χ_{total}^2 can be split into the lepton and quark contributions χ_l^2 and χ_q^2 respectively with $\chi_{\text{total}}^2 = \chi_l^2 + \chi_q^2$. We construct the lepton sector χ_l^2 function with the mass ratios m_e/m_μ , m_μ/m_τ , $\Delta m_{21}^2/\Delta m_{31}^2$ and the lepton mixing parameters $\sin^2 \theta_{12}^l$, $\sin^2 \theta_{13}^l$, $\sin^2 \theta_{23}^l$, δ_{CP}^l . The quark sector χ_q^2 function is constructed from the quark mass ratios m_u/m_c , m_c/m_t , m_d/m_s , m_s/m_b and the quark mixing parameters θ_{12}^q , θ_{13}^q , θ_{23}^q , δ_{CP}^q . The tau mass m_τ is fixed to its experimental best fit value to determine the common overall scale of the charged lepton and down quark mass matrices, the contribution from the mass ratio m_b/m_τ is included into the χ_q^2 . Notice that the overall scale factors of the mass matrices don't affect the value of χ_{total}^2 . The values of the quark and charged lepton masses and the CKM mixing parameters at the GUT scale are taken from ref. [73] with $\tan \beta = 10$ and the SUSY breaking scale $M_{\text{SUSY}} = 10$ TeV. The experimental values of the neutrino mixing parameters are taken from NuFIT v5.0 with Super-Kamiokanda atmospheric data [4]. Since the normal ordering (NO) neutrino mass spectrum is slightly preferred over the inverted ordering (IO) masses by the present data [4], we focus on the NO in the following.

For any given values of the input parameters of a model, we diagonalize the mass matrices to extract the lepton and quark masses and mixing matrices, and then the value of χ_{total}^2 is calculated. The absolute value of all dimensionless parameters are treated as random numbers varying freely between 0 and 10^8 and their phases are distributed uniformly in the region $[0, 2\pi]$, while the VEV of the modulus τ is limited in the fundamental region $\mathcal{D} = \left\{ \tau | \text{Im}(\tau) > 0, |\text{Re}(\tau)| \leq \frac{1}{2}, |\tau| \geq 1 \right\}$. We numerically search for the global minimum of χ_{total}^2 by using the minimization algorithms incorporated in the CERN developed package

Parameters	Best fit values and 1σ ranges
m_u/m_c	$(1.9286 \pm 0.6017) \times 10^{-3}$
m_c/m_t	$(2.8213 \pm 0.1195) \times 10^{-3}$
m_d/m_s	$(5.0523 \pm 0.6191) \times 10^{-2}$
m_s/m_b	$(1.8241 \pm 0.1005) \times 10^{-2}$
m_b/m_τ	$(0.7434 \pm 0.0090) \times 10^{-2}$
δ_{CP}^q	$69.213^\circ \pm 3.115^\circ$
θ_{12}^q	0.22736 ± 0.00073
θ_{13}^q	0.00349 ± 0.00013
θ_{23}^q	0.04015 ± 0.00064
m_e/m_μ	$(4.73689 \pm 0.04019) \times 10^{-3}$
m_μ/m_τ	$(5.85684 \pm 0.04654) \times 10^{-2}$
$\sin^2 \theta_{12}^l$	$0.304_{-0.012}^{+0.012}$
$\sin^2 \theta_{23}^l$	$0.573_{-0.020}^{+0.016}$
$\sin^2 \theta_{13}^l$	$0.02219_{-0.00063}^{+0.00062}$
$\delta_{CP}^l/^\circ$	197_{-24}^{+27}
$\frac{\Delta m_{21}^2}{10^{-5} \text{eV}^2}$	$7.42_{-0.20}^{+0.21}$
$\frac{\Delta m_{31}^2}{10^{-3} \text{eV}^2}$	$2.517_{-0.028}^{+0.026}$
m_t	$87.4555 \pm 2.0893 \text{ GeV}$
m_τ	$1.30234 \pm 0.0068 \text{ GeV}$

Table 3. The best fit values and 1σ errors of the mass ratios and mixing parameters of quarks and leptons. The values of the quark and charged lepton masses and the CKM parameters at the GUT scale are taken from ref. [73] with SUSY breaking scale $M_{\text{SUSY}} = 10 \text{ TeV}$ and $\tan\beta = 10$, $\bar{\eta}_b = 0.09375$ [73]. The lepton mixing parameters are taken from [4] for normal ordering neutrino masses.

TMinuit to optimize the values of the input parameters [74]. We have scanned all models with number of parameters (including the real and imaginary part of τ) less than 25. Requiring $\chi_{\text{total}}^2 < 100$, we find 17 type-I, 2 type-II, 28 type-V and 6 type-VII models are viable. The representations and weight assignments of the matter fields for these models are summarized in table 2. Moreover, we display the best fit values of the free parameters and the predictions for fermion masses and flavor mixing parameters as well as the minimal values of χ_l^2 , χ_q^2 and χ_{total}^2 in tables 4–16. The results of leptonic PMNS parameters and the quark CKM parameters are extracted in the standard way as defined in the Particle Data Group (PDG). For instance, the PDG parametrization of the lepton PMNS mixing

matrix is as follow,

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}^l c_{13}^l & s_{12}^l c_{13}^l & s_{13}^l e^{-i\delta_{CP}^l} \\ -s_{12}^l c_{23}^l - c_{12}^l s_{13}^l s_{23}^l e^{i\delta_{CP}^l} & c_{12}^l c_{23}^l - s_{12}^l s_{13}^l s_{23}^l e^{i\delta_{CP}^l} & c_{13}^l s_{23}^l \\ s_{12}^l s_{23}^l - c_{12}^l s_{13}^l c_{23}^l e^{i\delta_{CP}^l} & -c_{12}^l s_{23}^l - s_{12}^l s_{13}^l c_{23}^l e^{i\delta_{CP}^l} & c_{13}^l c_{23}^l \end{pmatrix} Q, \quad (5.2)$$

with $c_{ij}^l = \cos \theta_{ij}^l$, $s_{ij}^l = \sin \theta_{ij}^l$, δ_{CP}^l is the Dirac CP violating phase, and $Q = \text{diag}(1, e^{i\frac{\alpha_{21}}{2}}, e^{i\frac{\alpha_{31}}{2}})$ is the Majorana phase matrix factor where α_{21} and α_{31} are the so-called Majorana CP phases. If the lightest neutrino is massless $m_1 = 0$ or $m_3 = 0$, α_{31} is unphysical and the only physical Majorana phase is $\phi = \alpha_{21}$. The quark CKM mixing matrix is parameterized similar to eq. (5.2) without the Majorana phase matrix Q .

As can be seen from tables 4 to 16, there are 13 models $\mathcal{I}_4, \mathcal{I}_5, \mathcal{I}_7, \mathcal{I}_9, \mathcal{I}_{10}, \mathcal{I}_{11}, \mathcal{I}_{12}, \mathcal{I}_{15}, \mathcal{V}_5, \mathcal{V}_6, \mathcal{V}_7, \mathcal{V}_{18}$ and \mathcal{V}_{25} in good agreement with the experiment data at 1σ level, and 13 models $\mathcal{I}_6, \mathcal{I}_{13}, \mathcal{I}_{16}, \mathcal{I}_{17}, \mathcal{V}_9, \mathcal{V}_{11}, \mathcal{V}_{12}, \mathcal{V}_{15}, \mathcal{V}_{16}, \mathcal{V}_{17}, \mathcal{V}_{19}, \mathcal{V}_{20}$ and \mathcal{V}_{26} are compatible with data at 3σ level. All these models depend on 24 real parameters including the real and imaginary part of τ . In order accommodate the quark and lepton mass hierarchies, we see that hierarchical values of the coupling constants are necessary, and this is expected to be naturally explained by the weighton mechanism [30]. It is remarkable that the model \mathcal{I}_1 has only 18 real parameters including $\text{Re}(\tau)$ and $\text{Im}(\tau)$ and the predictions are in qualitative agreement with the observations, although $\sin^2 \theta_{23}^l$ and δ_{CP}^q are slightly above the 3σ allowed regions and the charm quark mass m_c is a bit smaller and the up quark mass is vanishing with $m_u = 0$. In all the 22 parameters models $\mathcal{I}_2, \mathcal{I}_3, \mathcal{V}_1, \mathcal{V}_2, \mathcal{VII}_1, \mathcal{VII}_2, \mathcal{VII}_3$ together with the models $\mathcal{V}_{21}, \mathcal{V}_{22}, \mathcal{V}_{23}, \mathcal{V}_{24}, \mathcal{V}_{27}, \mathcal{V}_{28}, \mathcal{VII}_4, \mathcal{VII}_5$ and \mathcal{VII}_6 , the top left 2×2 block of the Yukawa coupling matrix \mathcal{Y}_5^u is zero such that we have $m_u = 0$ which is compatible with the tiny despite non-vanishing up quark mass. Moreover, there are only two right-handed neutrinos in the type-VII models, and the lightest neutrino is massless.

We plot the best fit value of the modulus τ in figure 1, it can be seen that τ tends to distribute around the boundary of the fundamental domain \mathcal{D} , in particular some are clustered close to the self-dual point $\tau = i$ where S is unbroken. The value of the Dirac CP phase δ_{CP}^l is still unknown, the current and upcoming long-baseline experiments will be able to place important constraints. As can be seen from figure 2, the predictions for δ_{CP}^l mostly lie in the region of $[\pi, 3\pi/2]$ and they could be tested at future experiments. The most sensitive probe to whether neutrinos are Dirac or Majorana states is the neutrinoless double beta decay ($0\nu\beta\beta$): $(A, Z) \rightarrow (A, Z + 2) + e^- + e^-$. The decay amplitude is proportional to the effective Majorana mass

$$m_{\beta\beta} = \left| m_1 \cos^2 \theta_{12} \cos^2 \theta_{13} + m_2 \sin^2 \theta_{12} \cos^2 \theta_{13} e^{i\alpha_{21}} + m_3 \sin^2 \theta_{13} e^{i(\alpha_{31} - 2\delta_{CP}^l)} \right|, \quad (5.3)$$

which reduces to

$$m_{\beta\beta} = \left| m_2 \sin^2 \theta_{12} \cos^2 \theta_{13} e^{i\phi} + m_3 \sin^2 \theta_{13} e^{-2i\delta_{CP}^l} \right|, \quad (5.4)$$

for $m_1 = 0$. The current most stringent limit is $m_{\beta\beta} < 61$ meV from KamLAND-Zen [75]. From the predicted values of lepton mixing angles and neutrino masses, one can easily

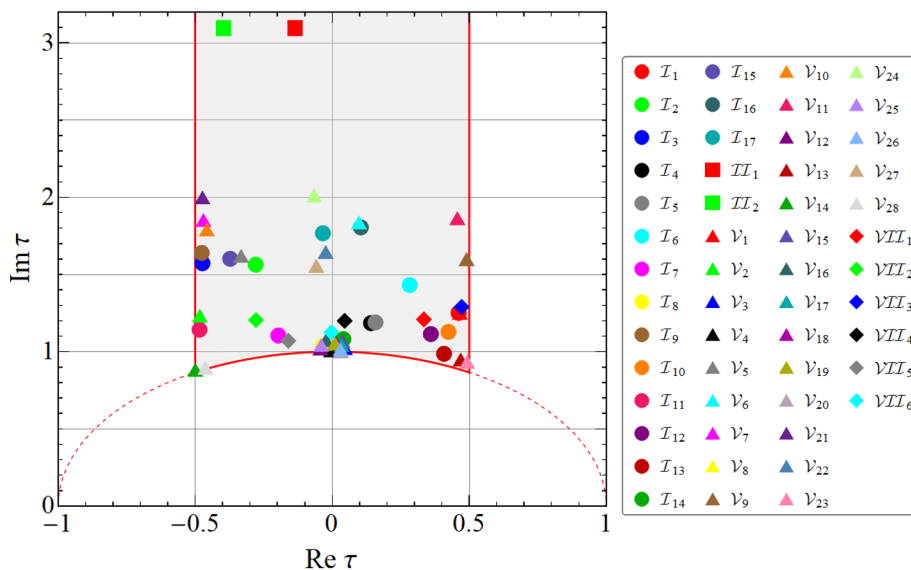


Figure 1. The best fit values of the complex modulus τ for the viable models summarized in table 2.

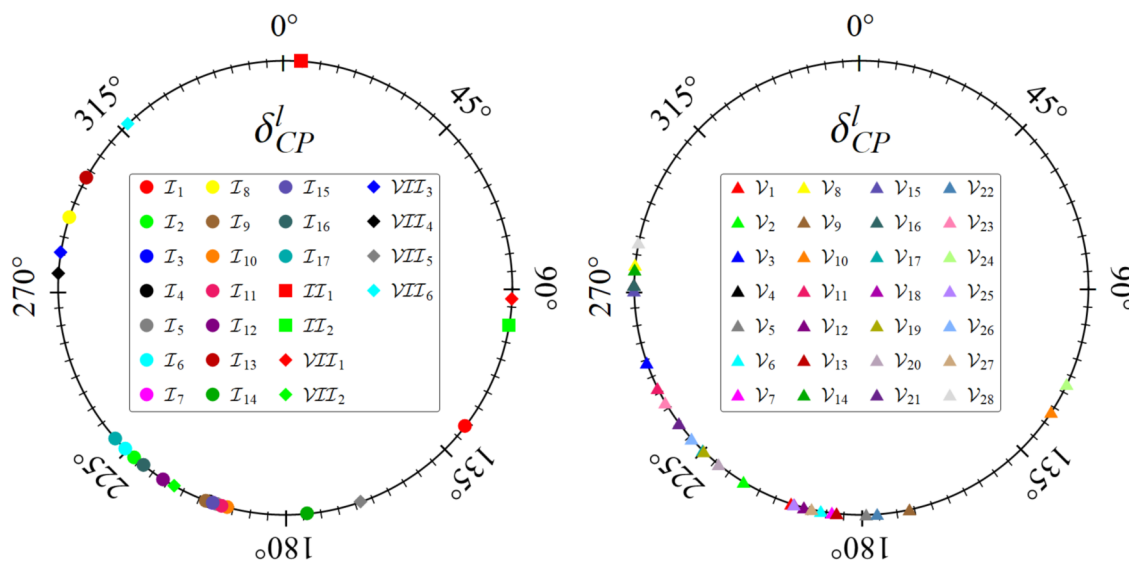


Figure 2. The predictions for the Dirac CP phase δ_{CP}^l . The left panel is for the models of type-I, type-II and type-VII, and the right is for the type-V models.

determine the effective mass $m_{\beta\beta}$. The predictions for $m_{\beta\beta}$ of the type-I, II and VII models are shown in figure 3 and those of type-V models are displayed in figure 4. There are many experiments which are in various stages of planning and construction. the new generation of $0\nu\beta\beta$ decay experiments will significantly increase the sensitivity to this rare process such that a considerable amount of our models are within the reach in foreseeable future.

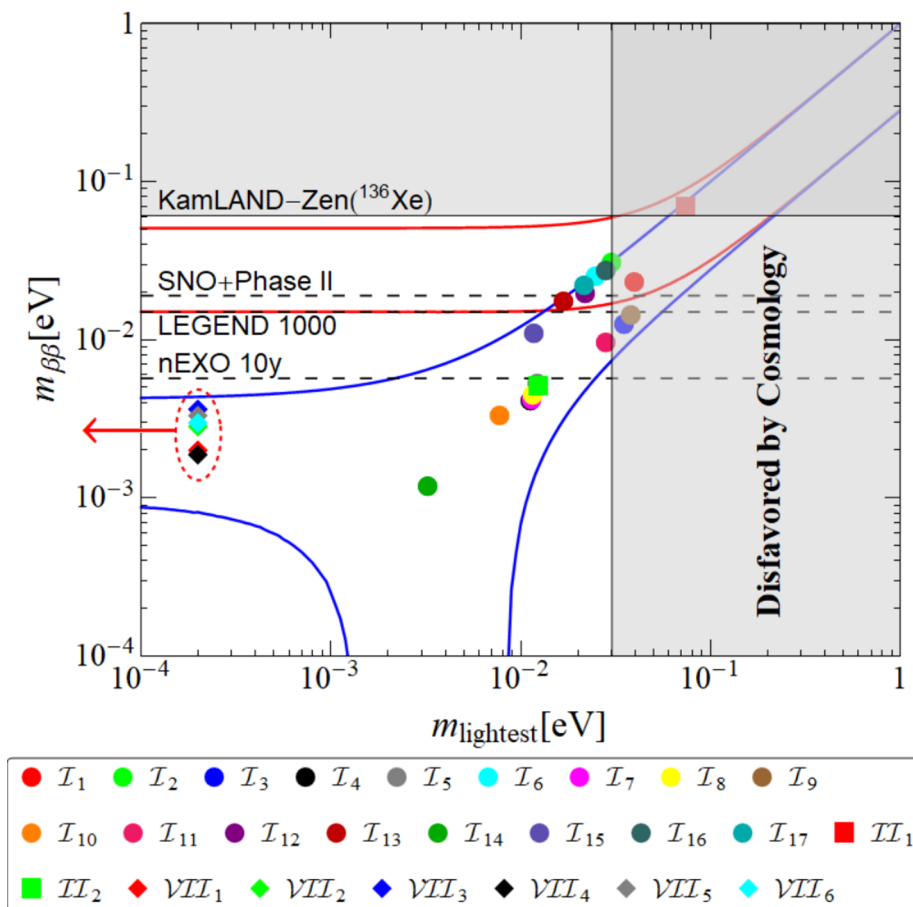


Figure 3. The predictions for the effective mass of the neutrinoless double beta decay for the models of type I, type II and type VII. Notice that the lightest neutrino is massless for type VII models since only two right-handed neutrinos are introduced. The blue (red) lines denote the most general allowed regions for NO (IO) where the neutrino oscillation parameters are freely varied in their 3σ regions [4]. The vertical grey exclusion band denotes the bound on the lightest neutrino mass coming from the cosmological data $\Sigma_i m_i < 0.120$ eV at 95% confidence level obtained by the Planck collaboration [2]. The current experimental bound from KamLAND-Zen [75] and the estimated experimental sensitivities of future $0\nu\beta\beta$ experiments are indicated by the horizontal lines.

6 Conclusion

GUTs are an important framework to understand the origin of the observed patterns of fermion masses and flavor mixing. The quark and leptons fields in each generation are embedded into a GUT multiplet so that they are closely related. It is usually assumed that the three generations of fermions are related by certain family symmetry to address the flavor mixing structure of quarks and leptons. The role of modular invariance as flavor symmetry can overcome the drawback of the conventional discrete flavor symmetry models. The modular symmetry is broken by the VEV of the complex modulus τ , and the Yukawa couplings are modular forms. Flavor models based on modular symmetry can be highly predictive, and the neutrino masses and lepton mixing can be predicted in terms of few

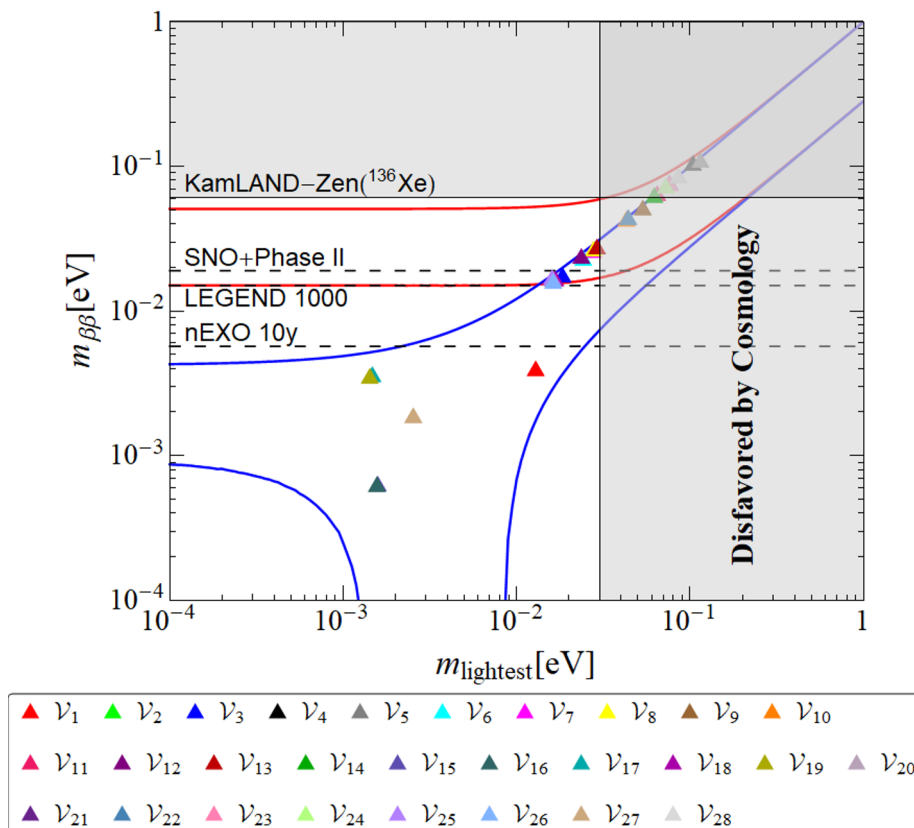


Figure 4. The predictions for the effective mass of the neutrinoless double beta decay for the type V models. We adopt the same convention as figure 3.

input parameters. Inspired by the success in the lepton sector, the modular symmetry has been extended to the quark sector [15, 18, 30, 32, 49, 50, 54]. In the present work, we imposed $\Gamma_3 \cong A_4$ modular symmetry on the SU(5) GUT to give a unified description of both quarks and leptons.

In order to account for the mass difference of charged leptons and down-type quarks, we introduced the Higgs multiplet $H_{\overline{45}}$ besides H_5 and $H_{\overline{5}}$. All the three Higgs multiplets H_5 , $H_{\overline{5}}$ and $H_{\overline{45}}$ were assumed to be invariant under A_4 with zero modular weight. The neutrino masses are generated by the type-I seesaw mechanism, and both scenarios with three and two right-handed neutrinos were considered. The matter fields N , \overline{F} and T transform as either triplets $\mathbf{3}$ or singlets $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$ under A_4 . We have given the most general form of the quark and lepton mass matrices for different possible representation assignments, as shown in section 3. All possible models were classified according to the transformation properties of the matter fields, and we found there are five types of models for three right-handed neutrinos and two types of models for two right-handed neutrinos. If N , \overline{F} and T are all singlets of A_4 , the Yukawa superpotential would be less restricted by modular symmetry and generally more free parameters would be involved, hence we did not consider these cases in the present work since we were interested in the simplest possibilities.

We have performed a numerical scan over the parameter space of each model, and searched for the minimum of the χ_{total}^2 function to optimize the agreement between predictions and data. For normal ordering neutrino masses, intensive numerical calculations revealed that there are 17 type-I models, 2 type-II models, 28 type-V and 6 type-VII models with $\chi_{\text{total}}^2 < 100$ with the number of real free parameters not larger than 24 including the real and imaginary part of τ . Out of the 54 considered models, we found that 12 type-I and 14 type-V models are in agreement with the experimental data at 3σ level, as can be seen from tables 4 to 16. All those phenomenological viable examples have 24 parameters.

The predictions for the leptonic CP violating Dirac phase, the lightest neutrino mass and the neutrinoless double beta decay parameter as shown in the figures are observed to cover a wide range of possible values, but appear to be clustered around particular regions. Consequently the forthcoming generation of neutrino oscillation experiments sensitive to the leptonic CP phase, in conjunction with the upcoming neutrinoless double beta decay experiments, will together be able to discriminate between the various types of models based on $\Gamma_3 \cong A_4$ modular symmetry with SU(5) GUTs considered here.

Acknowledgments

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A Results of the best fit for the benchmark models

In this appendix, we collect the best fit values of the input parameters and the corresponding predictions for the quark and lepton masses and mixing parameters for the phenomenologically viable models listed in table 2. Here we have considered normal ordering neutrino masses which is slightly preferred by the experimental data [4].

Model	\mathcal{I}_1	\mathcal{I}_2	\mathcal{I}_3
$\text{Re}(\tau)$	0.4623	-0.2778	-0.4732
$\text{Im}(\tau)$	1.249	1.560	1.574
$ \alpha_{\nu_2}/\alpha_{\nu_1} $	1.216	2.497	1.961
$\arg(\alpha_{\nu_2}/\alpha_{\nu_1})/\pi$	1.513	1.913	1.634
$\alpha_{d_1}/\alpha_{d_3}$	0.0275	0.00253	0.0148
$\alpha_{d_2}/\alpha_{d_3}$	0.465	2.089×10^{-6}	0.0844
$ \alpha_{d_4}/\alpha_{d_3} $	—	2.731	7.592
$\arg(\alpha_{d_4}/\alpha_{d_3})/\pi$	—	1.300	0.678
$\alpha'_{d_1}/\alpha_{d_3}$	-0.0296	0.00133	0.00692
$ \alpha'_{d_2}/\alpha_{d_3} $	0.482	0.0534	0.158
$\arg(\alpha'_{d_2}/\alpha_{d_3})/\pi$	0.979	0.203	0.625
$ \alpha'_{d_3}/\alpha_{d_3} $	0.333	0.616	0.278
$\arg(\alpha'_{d_3}/\alpha_{d_3})/\pi$	2.677×10^{-4}	1.690	0.881
$ \alpha'_{d_4}/\alpha_{d_3} $	—	0.506	3.166
$\arg(\alpha'_{d_4}/\alpha_{d_3})/\pi$	—	0.307	0.206
$ \alpha_{u_1}/\alpha_{u_3} $	0.0143	0.00421	0.00574
$\arg(\alpha_{u_1}/\alpha_{u_3})/\pi$	0.973	0.665	1.636
$ \alpha_{u_2}/\alpha_{u_3} $	0.0604	0.0514	0.0222
$\arg(\alpha_{u_2}/\alpha_{u_3})/\pi$	0.973	0.639	0.185
$(\alpha_{\nu_1}^2 v_u^2 / (\alpha_{N_1} \Lambda)) / \text{meV}$	19.224	7.806	9.643
$(\alpha_{d_3} v_d) / \text{GeV}$	0.624	0.556	0.246
$(\alpha_{u_3} v_u) / \text{GeV}$	73.304	191.031	198.532
$\sin^2 \theta_{13}^l$	0.0214	0.0218	0.0222
$\sin^2 \theta_{12}^l$	0.312	0.302	0.306
$\sin^2 \theta_{23}^l$	0.631	0.574	0.567
$\delta_{CP}^l / ^\circ$	127.565	221.626	198.745
$\alpha_{21} / ^\circ$	233.130	357.093	172.063
$\alpha_{31} / ^\circ$	247.772	22.892	171.828
m_1 / meV	39.787	30.384	35.361
m_2 / meV	40.709	31.581	36.395
m_3 / meV	64.099	58.579	61.346
$m_{\beta\beta} / \text{meV}$	23.039	30.735	12.490
θ_{12}^q	0.227	0.227	0.227
θ_{13}^q	0.00349	0.00349	0.00349
θ_{23}^q	0.0419	0.0404	0.0401
$\delta_{CP}^q / ^\circ$	80.421	70.296	69.111
m_u / m_c	0.0	0.0	0.0
m_c / m_t	0.00223	0.00273	0.00284
m_d / m_s	0.0584	0.0528	0.0518
m_s / m_b	0.0195	0.0185	0.0182
m_b / m_τ	0.737	0.743	0.743
χ_i^2	23.835	1.452	0.124
χ_q^2	58.654	11.438	10.362
χ_{total}^2	82.490	12.890	10.486

Table 4. The best fit values of the free parameters and the corresponding predictions for lepton and quark mixing parameters and fermion masses for the phenomenologically viable models of Type-I.

Model	\mathcal{I}_4	\mathcal{I}_5	\mathcal{I}_6	\mathcal{I}_7
$\text{Re}(\tau)$	0.1427	0.1601	0.2830	-0.1956
$\text{Im}(\tau)$	1.184	1.188	1.429	1.103
$ \alpha_{\nu_2}/\alpha_{\nu_1} $	2.949	2.937	2.609	3.051
$\arg(\alpha_{\nu_2}/\alpha_{\nu_1})/\pi$	4.567×10^{-8}	0.00486	0.0791	1.999
$\alpha_{d_1}/\alpha_{d_3}$	0.0523	0.0839	0.00416	0.220
$\alpha_{d_2}/\alpha_{d_3}$	0.00504	0.00463	0.0794	0.0213
$ \alpha_{d_4}/\alpha_{d_3} $	0.620	2.136	1.068	4.410
$\arg(\alpha_{d_4}/\alpha_{d_3})/\pi$	1.039	1.070	0.950	1.690
$\alpha'_{d_1}/\alpha_{d_3}$	-0.0207	-0.0285	0.00114	-0.0878
$ \alpha'_{d_2}/\alpha_{d_3} $	0.00137	0.00134	0.0421	0.00866
$\arg(\alpha'_{d_2}/\alpha_{d_3})/\pi$	0.0451	1.956	0.716	1.991
$ \alpha'_{d_3}/\alpha_{d_3} $	0.448	0.607	0.328	1.735
$\arg(\alpha'_{d_3}/\alpha_{d_3})/\pi$	0.185	0.148	0.436	0.688
$ \alpha'_{d_4}/\alpha_{d_3} $	0.324	0.245	0.427	1.263
$\arg(\alpha'_{d_4}/\alpha_{d_3})/\pi$	0.246	0.577	1.582	1.465
$ \alpha_{u_1}/\alpha_{u_3} $	5.513	3.942	0.00415	5.555
$\arg(\alpha_{u_1}/\alpha_{u_3})/\pi$	0.298	0.0437	0.647	0.238
$ \alpha_{u_2}/\alpha_{u_3} $	6.074×10^{-4}	3.998×10^{-4}	6.784×10^{-4}	6.807×10^{-4}
$\arg(\alpha_{u_2}/\alpha_{u_3})/\pi$	1.474	1.014	0.174	1.460
$ \alpha_{u_4}/\alpha_{u_3} $	88.520	64.698	19.236	82.789
$\arg(\alpha_{u_4}/\alpha_{u_3})/\pi$	0.307	1.986	0.128	1.915
$(\alpha_{\nu_1}^2 v_u^2 / (\alpha_{N_1} \Lambda)) / \text{meV}$	3.652	3.731	6.333	3.180
$(\alpha_{d_3} v_d) / \text{GeV}$	0.600	0.404	0.831	0.136
$(\alpha_{u_3} v_u) / \text{GeV}$	0.823	1.157	4.587	0.746
$\sin^2 \theta_{13}^l$	0.0222	0.0222	0.0223	0.0222
$\sin^2 \theta_{12}^l$	0.304	0.304	0.296	0.304
$\sin^2 \theta_{23}^l$	0.573	0.573	0.584	0.574
$\delta_{CP}^l / ^\circ$	199.402	197.516	224.737	198.558
$\alpha_{21} / ^\circ$	169.642	193.611	329.057	169.259
$\alpha_{31} / ^\circ$	337.832	25.864	38.097	337.628
m_1 / meV	11.357	12.222	24.989	11.391
m_2 / meV	14.254	14.953	26.432	14.281
m_3 / meV	51.447	51.635	56.040	51.437
$m_{\beta\beta} / \text{meV}$	4.118	5.278	25.027	4.169
θ_{12}^q	0.227	0.227	0.227	0.227
θ_{13}^q	0.00350	0.00349	0.00348	0.00349
θ_{23}^q	0.0405	0.0401	0.0402	0.0405
$\delta_{CP}^q / ^\circ$	69.199	69.220	68.340	69.277
m_u / m_c	0.00195	0.00193	0.00215	0.00198
m_c / m_t	0.00282	0.00282	0.00283	0.00282
m_d / m_s	0.0506	0.0505	0.0499	0.0507
m_s / m_b	0.0183	0.0182	0.0190	0.0182
m_b / m_τ	0.744	0.743	0.743	0.743
χ_1^2	0.00882	6.178×10^{-4}	2.071	0.00533
χ_q^2	0.233	8.577×10^{-4}	0.808	0.331
χ_{total}^2	0.242	0.00148	2.879	0.337

Table 5. The best fit values of the free parameters and the corresponding predictions for lepton and quark mixing parameters and fermion masses for the phenomenologically viable models of Type-I (continuation of table 4).

Model	\mathcal{I}_8	\mathcal{I}_9	\mathcal{I}_{10}	\mathcal{I}_{11}	\mathcal{I}_{12}
$\text{Re}(\tau)$	-0.03236	-0.4738	0.4264	-0.4828	0.3621
$\text{Im}(\tau)$	1.036	1.637	1.125	1.143	1.113
$ \alpha_{N_2}/\alpha_{N_1} $	1.740	—	—	—	—
$\arg(\alpha_{N_2}/\alpha_{N_1})/\pi$	0.662	—	—	—	—
$ \alpha_{N_3}/\alpha_{N_1} $	0.821	—	—	—	—
$\arg(\alpha_{N_3}/\alpha_{N_1})/\pi$	0.621	—	—	—	—
$ \alpha_{\nu_2}/\alpha_{\nu_1} $	1.717	1.930	0.683	2.273	3.224
$\arg(\alpha_{\nu_2}/\alpha_{\nu_1})/\pi$	0.273	1.629	0.930	1.624	0.0819
$\alpha_{d_1}/\alpha_{d_3}$	0.0349	0.710	0.155	0.0293	0.119
$\alpha_{d_2}/\alpha_{d_3}$	0.00446	0.0297	29.907	0.00281	12.092
$ \alpha_{d_4}/\alpha_{d_3} $	—	11.704	2.070	0.148	18.451
$\arg(\alpha_{d_4}/\alpha_{d_3})/\pi$	—	0.458	1.071	0.980	0.883
$\alpha'_{d_1}/\alpha_{d_3}$	-0.0162	-0.272	0.0455	-0.0117	0.0542
$ \alpha'_{d_2}/\alpha_{d_3} $	0.00107	0.0104	15.226	9.652×10^{-4}	23.972
$\arg(\alpha'_{d_2}/\alpha_{d_3})/\pi$	0.139	0.0678	1.570	0.0598	0.00546
$ \alpha'_{d_3}/\alpha_{d_3} $	0.0794	1.206	0.430	0.154	1.827
$\arg(\alpha'_{d_3}/\alpha_{d_3})/\pi$	1.017	1.306	1.768	1.371	0.686
$ \alpha'_{d_4}/\alpha_{d_3} $	—	15.114	0.883	0.113	20.286
$\arg(\alpha'_{d_4}/\alpha_{d_3})/\pi$	—	0.324	0.204	0.224	1.000
$ \alpha_{u_1}/\alpha_{u_3} $	10.616	80.413	0.00282	17.500	0.00311
$\arg(\alpha_{u_1}/\alpha_{u_3})/\pi$	0.273	1.502	1.945	1.436	0.322
$ \alpha_{u_2}/\alpha_{u_3} $	1798.020	9332.920	16.336	5227.610	15.062
$\arg(\alpha_{u_2}/\alpha_{u_3})/\pi$	0.511	0.766	1.009	1.804	0.675
$ \alpha_{u_4}/\alpha_{u_3} $	3.42×10^4	4.625×10^5	0.0855	1.491×10^5	0.0772
$\arg(\alpha_{u_4}/\alpha_{u_3})/\pi$	0.492	1.418	0.900	1.974	1.658
$(\alpha_{\nu_1}^2 v_u^2 / (\alpha_{N_1} \Lambda)) / \text{meV}$	17.808	10.479	11.659	5.633	3.502
$(\alpha_{d_3} v_d) / \text{GeV}$	0.839	0.0937	0.0207	1.104	0.0179
$(\alpha_{u_3} v_u) / \text{GeV}$	0.00188	4.885×10^{-4}	4.649	6.475×10^{-4}	4.204
$\sin^2 \theta'_{13}$	0.0224	0.0222	0.0222	0.0222	0.0221
$\sin^2 \theta'_{12}$	0.311	0.304	0.304	0.304	0.302
$\sin^2 \theta'_{23}$	0.561	0.572	0.570	0.573	0.565
$\delta_{CP}^l / ^\circ$	288.081	200.318	194.753	196.179	212.442
$\alpha_{21} / ^\circ$	189.550	166.202	163.174	171.051	50.977
$\alpha_{31} / ^\circ$	155.146	181.491	79.769	252.632	322.489
m_1 / meV	11.642	38.357	7.772	28.017	21.954
m_2 / meV	14.482	39.312	11.602	29.312	23.583
m_3 / meV	51.705	63.153	50.757	57.456	54.707
$m_{\beta\beta} / \text{meV}$	4.422	14.161	3.291	9.503	19.573
θ_{12}^q	0.227	0.227	0.227	0.227	0.227
θ_{13}^q	0.00349	0.00350	0.00350	0.00349	0.00350
θ_{23}^q	0.0424	0.0402	0.0401	0.0398	0.0397
$\delta_{CP}^q / ^\circ$	69.137	69.230	69.489	69.273	68.679
m_u / m_c	0.00194	0.00194	0.00189	0.00192	0.00212
m_c / m_t	0.00279	0.00282	0.00283	0.00283	0.00283
m_d / m_s	0.0519	0.0501	0.0502	0.0506	0.0512
m_s / m_b	0.0183	0.0181	0.0181	0.0182	0.0179
m_b / m_τ	0.743	0.743	0.743	0.743	0.744
χ_l^2	12.213	0.0181	0.0353	0.00136	0.556
χ_q^2	12.474	0.0286	0.0403	0.332	0.736
χ_{total}^2	24.688	0.0466	0.0756	0.333	1.293

Table 6. The best fit values of the free parameters and the corresponding predictions for lepton and quark mixing parameters and fermion masses for the phenomenologically viable models of Type-I (continuation of table 5).

Model	\mathcal{I}_{13}	\mathcal{I}_{14}	\mathcal{I}_{15}	\mathcal{I}_{16}	\mathcal{I}_{17}
$\text{Re}(\tau)$	0.4092	0.04232	-0.3716	0.1069	-0.03262
$\text{Im}(\tau)$	0.984	1.081	1.602	1.804	1.763
$ \alpha_{N_1}/\alpha_{N_3} $	6.71×10^{-4}	7.575×10^{-5}	0.800	3.402×10^{-4}	3.246×10^{-4}
$\arg(\alpha_{N_1}/\alpha_{N_3})/\pi$	1.241	1.569	0.677	0.883	1.204
$ \alpha_{N_2}/\alpha_{N_3} $	1.432	1.765	1.346	0.852	1.570
$\arg(\alpha_{N_2}/\alpha_{N_3})/\pi$	0.124	1.997	0.680	1.694	0.868
$ \alpha_{\nu_2}/\alpha_{\nu_1} $	2.149	2.181	2.782	1.542	1.534
$\arg(\alpha_{\nu_2}/\alpha_{\nu_1})/\pi$	0.221	0.0199	1.821	0.282	0.633
$\alpha_{d_1}/\alpha_{d_3}$	0.0339	0.0417	0.00262	0.00337	0.00285
$\alpha_{d_2}/\alpha_{d_3}$	0.0519	0.00306	0.465	0.0299	0.00607
$\alpha'_{d_1}/\alpha_{d_3}$	-0.0302	-0.0181	4.809×10^{-4}	0.00171	5.56×10^{-4}
$ \alpha'_{d_2}/\alpha_{d_3} $	0.0173	0.00136	0.409	0.0263	0.0228
$\arg(\alpha'_{d_2}/\alpha_{d_3})/\pi$	1.978	0.0906	0.997	0.732	0.927
$ \alpha'_{d_3}/\alpha_{d_3} $	0.541	0.440	0.267	0.566	0.106
$\arg(\alpha'_{d_3}/\alpha_{d_3})/\pi$	0.420	0.456	1.949	1.589	0.755
$ \alpha_{u_1}/\alpha_{u_3} $	0.0578	16.543	0.0605	0.0562	0.0636
$\arg(\alpha_{u_1}/\alpha_{u_3})/\pi$	0.703	0.385	0.640	0.230	1.658
$ \alpha_{u_2}/\alpha_{u_3} $	0.843	4.735×10^{-4}	1.075	1.369	0.924
$\arg(\alpha_{u_2}/\alpha_{u_3})/\pi$	1.034	1.306	0.778	0.232	1.753
$ \alpha_{u_4}/\alpha_{u_3} $	233.679	112.554	365.736	94.717	83.623
$\arg(\alpha_{u_4}/\alpha_{u_3})/\pi$	1.678	1.465	1.302	0.731	1.644
$(\alpha_{\nu_1}^2 v_u^2 / (\alpha_{N_3} \Lambda)) / \text{meV}$	6.765	2.950	5.627	14.762	11.058
$(\alpha_{d_3} v_d) / \text{GeV}$	0.490	0.661	0.746	0.752	1.033
$(\alpha_{u_3} v_u) / \text{GeV}$	0.376	0.497	0.575	3.361	3.484
$\sin^2 \theta_{13}^l$	0.0224	0.0222	0.0221	0.0225	0.0223
$\sin^2 \theta_{12}^l$	0.310	0.304	0.304	0.305	0.302
$\sin^2 \theta_{23}^l$	0.564	0.566	0.575	0.563	0.575
$\delta_{CP}^l / ^\circ$	298.913	174.478	198.634	218.550	228.392
$\alpha_{21} / ^\circ$	348.080	126.608	33.969	14.995	28.967
$\alpha_{31} / ^\circ$	154.909	253.995	191.453	210.783	162.487
m_1 / meV	16.820	3.252	11.703	28.266	21.619
m_2 / meV	18.897	9.207	14.532	29.549	23.272
m_3 / meV	53.113	50.465	51.558	56.920	54.570
$m_{\beta\beta} / \text{meV}$	17.258	1.183	10.917	27.020	21.723
θ_{12}^q	0.227	0.228	0.227	0.228	0.227
θ_{13}^q	0.00350	0.00343	0.00350	0.00350	0.00346
θ_{23}^q	0.0411	0.0423	0.0403	0.0402	0.0402
$\delta_{CP}^q / ^\circ$	69.380	68.680	69.604	69.916	69.230
m_u / m_c	0.00195	0.00208	0.00192	0.00232	0.00258
m_c / m_t	0.00282	0.00199	0.00281	0.00283	0.00284
m_d / m_s	0.0518	0.0592	0.0544	0.0569	0.0577
m_s / m_b	0.0182	0.0194	0.0179	0.0184	0.0183
m_b / m_τ	0.743	0.743	0.744	0.744	0.743
χ_l^2	14.858	1.091	0.0358	2.212	1.451
χ_q^2	2.081	64.251	0.559	1.621	2.594
χ_{total}^2	16.939	65.342	0.595	3.833	4.045

Table 7. The best fit values of the free parameters and the corresponding predictions for lepton and quark mixing parameters and fermion masses for the phenomenologically viable models of Type-I (continuation of table 6).

Model	\mathcal{II}_1	\mathcal{II}_2
$\text{Re}(\tau)$	-0.1342	-0.3947
$\text{Im}(\tau)$	3.095	3.092
$ \alpha_{\nu_2}/\alpha_{\nu_1} $	0.326	0.378
$\arg(\alpha_{\nu_2}/\alpha_{\nu_1})/\pi$	0.368	4.329×10^{-4}
$ \alpha_{\nu_3}/\alpha_{\nu_1} $	1.685	0.0148
$\arg(\alpha_{\nu_3}/\alpha_{\nu_1})/\pi$	1.992	1.721
$\alpha_{d_1}/\alpha_{d_3}$	17.909	0.0538
$\alpha_{d_2}/\alpha_{d_3}$	1.740	0.101
$ \alpha_{d_4}/\alpha_{d_3} $	1968.751	100.642
$\arg(\alpha_{d_4}/\alpha_{d_3})/\pi$	1.535	0.291
$\alpha'_{d_1}/\alpha_{d_3}$	1.370	-0.0472
$ \alpha'_{d_2}/\alpha_{d_3} $	0.605	0.0176
$\arg(\alpha'_{d_2}/\alpha_{d_3})/\pi$	0.00905	0.326
$ \alpha'_{d_3}/\alpha_{d_3} $	0.977	1.037
$\arg(\alpha'_{d_3}/\alpha_{d_3})/\pi$	1.014	0.638
$ \alpha'_{d_4}/\alpha_{d_3} $	1160.266	33.443
$\arg(\alpha'_{d_4}/\alpha_{d_3})/\pi$	1.960	0.301
$ \alpha_{u_1}/\alpha_{u_3} $	1.675	2.080
$\arg(\alpha_{u_1}/\alpha_{u_3})/\pi$	0.884	1.146
$ \alpha_{u_2}/\alpha_{u_3} $	1.674	2.082
$\arg(\alpha_{u_2}/\alpha_{u_3})/\pi$	0.884	1.146
$(\alpha_{\nu_1}^2 v_u^2 / (\alpha_{N_1} \Lambda)) / \text{meV}$	148.663	1.755×10^5
$(\alpha_{d_3} v_d) / \text{GeV}$	0.0194	0.357
$(\alpha_{u_3} v_u) / \text{GeV}$	17.414	14.004
$\sin^2 \theta_{13}^l$	0.0224	0.0224
$\sin^2 \theta_{12}^l$	0.310	0.308
$\sin^2 \theta_{23}^l$	0.553	0.564
$\delta_{CP}^l / ^\circ$	4.115	99.580
$\alpha_{21} / ^\circ$	333.672	146.058
$\alpha_{31} / ^\circ$	162.827	337.581
m_1 / meV	74.100	12.512
m_2 / meV	74.599	15.190
m_3 / meV	89.918	51.850
$m_{\beta\beta} / \text{meV}$	69.038	5.050
θ_{12}^q	0.227	0.227
θ_{13}^q	0.00342	0.00350
θ_{23}^q	0.0436	0.0380
$\delta_{CP}^q / ^\circ$	67.809	68.868
m_u / m_c	0.00162	0.00200
m_c / m_t	0.00278	0.00284
m_d / m_s	0.0491	0.0504
m_s / m_b	0.0192	0.0181
m_b / m_τ	0.743	0.743
χ^2_l	66.983	16.919
χ^2_q	31.377	10.909
χ^2_{total}	98.359	27.828

Table 8. The best fit values of the free parameters and the corresponding predictions for lepton and quark mixing parameters and fermion masses for the phenomenologically viable models of Type-II.

Model	\mathcal{V}_1	\mathcal{V}_2	\mathcal{V}_3	\mathcal{V}_4	\mathcal{V}_5
$\text{Re}(\tau)$	0.4670	-0.4818	0.04772	-0.003098	-0.3313
$\text{Im}(\tau)$	1.250	1.232	1.021	1.007	1.618
$ \alpha_{N_2}/\alpha_{N_1} $	171.272	5.059	1.917×10^7	7.571×10^5	1.181×10^7
$\arg(\alpha_{N_2}/\alpha_{N_1})/\pi$	0.203	0.723	0.157	1.146	1.987
$ \alpha_{N_3}/\alpha_{N_1} $	8.270	94.999	$9. \times 10^5$	5.692×10^5	4.117×10^6
$\arg(\alpha_{N_3}/\alpha_{N_1})/\pi$	1.857	1.081	1.156	1.120	1.924
$\alpha_{\nu_2}/\alpha_{\nu_1}$	7.255	2.597	8.209×10^4	1042.230	5765.608
$\alpha_{\nu_3}/\alpha_{\nu_1}$	367.080	26.949	1820.800	762.010	2131.981
$\alpha_{d_1}/\alpha_{d_3}$	0.00213	0.00199	0.173	0.475	0.00201
$\alpha_{d_2}/\alpha_{d_3}$	0.485	0.0221	0.00820	0.00572	0.0229
$\alpha'_{d_1}/\alpha_{d_3}$	4.719×10^{-4}	4.351×10^{-4}	-0.0782	-0.449	3.534×10^{-4}
$ \alpha'_{d_2}/\alpha_{d_3} $	0.490	0.0415	0.00203	0.00574	0.0596
$\arg(\alpha'_{d_2}/\alpha_{d_3})/\pi$	0.975	1.319	1.991	0.600	1.194
$ \alpha'_{d_3}/\alpha_{d_3} $	0.375	0.573	0.269	0.348	0.223
$\arg(\alpha'_{d_3}/\alpha_{d_3})/\pi$	1.979	0.409	1.482	0.0353	0.563
$ \alpha_{u_1}/\alpha_{u_3} $	0.0143	0.0139	3.424	44.172	0.0462
$\arg(\alpha_{u_1}/\alpha_{u_3})/\pi$	1.993	1.739	0.885	1.675	1.519
$ \alpha_{u_2}/\alpha_{u_3} $	0.0609	0.0627	231.147	1.61×10^4	0.798
$\arg(\alpha_{u_2}/\alpha_{u_3})/\pi$	0.853	1.702	0.696	1.866	0.550
$ \alpha_{u_4}/\alpha_{u_3} $	—	—	4342.700	3.252×10^5	360.682
$\arg(\alpha_{u_4}/\alpha_{u_3})/\pi$	—	—	1.608	0.0393	1.062
$(\alpha_{\nu_1}^2 v_u^2 / (\alpha_{N_1} \Lambda)) / \text{meV}$	6.693	11.659	14.799	104.887	115.765
$(\alpha_{d_3} v_d) / \text{GeV}$	0.610	0.646	0.673	0.555	0.964
$(\alpha_{u_3} v_u) / \text{GeV}$	73.344	71.808	0.0144	1.868×10^{-4}	0.602
$\sin^2 \theta_{13}^l$	0.0221	0.0222	0.0223	0.0221	0.0221
$\sin^2 \theta_{12}^l$	0.300	0.306	0.303	0.305	0.304
$\sin^2 \theta_{23}^l$	0.556	0.574	0.542	0.476	0.576
$\delta_{CP}^l / ^\circ$	197.938	211.155	250.695	270.755	178.702
$\alpha_{21} / ^\circ$	166.227	172.943	299.647	0.00786	343.091
$\alpha_{31} / ^\circ$	283.237	45.179	136.764	180.423	211.606
m_1 / meV	12.953	2.333	18.512	104.520	115.424
m_2 / meV	15.555	8.924	20.418	104.874	115.745
m_3 / meV	51.759	50.202	53.496	115.832	125.851
$m_{\beta\beta} / \text{meV}$	3.987	1.05×10^{-4}	17.629	104.875	109.795
θ_{12}^q	0.227	0.227	0.227	0.228	0.227
θ_{13}^q	0.00350	0.00350	0.00352	0.00348	0.00350
θ_{23}^q	0.0416	0.0416	0.0392	0.0411	0.0403
$\delta_{CP}^q / ^\circ$	78.674	78.216	61.068	66.222	69.271
m_u / m_c	0.0	0.0	0.00198	0.00270	0.00197
m_c / m_t	0.00227	0.00224	0.00299	0.00245	0.00282
m_d / m_s	0.0675	0.0660	0.0939	0.00730	0.0532
m_s / m_b	0.0202	0.0200	0.0191	0.0173	0.0174
m_b / m_τ	0.742	0.744	0.743	0.739	0.743
χ_l^2	0.884	0.317	6.410	31.283	0.655
χ_q^2	57.883	56.854	61.137	64.752	1.025
χ_{total}^2	58.767	57.171	67.547	96.034	1.679

Table 9. The best fit values of the free parameters and the corresponding predictions for lepton and quark mixing parameters and fermion masses for the phenomenologically viable models of Type-V.

Model	\mathcal{V}_6	\mathcal{V}_7	\mathcal{V}_8	\mathcal{V}_9	\mathcal{V}_{10}
$\text{Re}(\tau)$	0.09874	-0.4687	-0.02455	0.4915	-0.4562
$\text{Im}(\tau)$	1.835	1.850	1.049	1.598	1.790
$ \alpha_{N_2}/\alpha_{N_1} $	1619.521	2.779×10^4	7.979×10^4	4.37×10^7	2.863×10^4
$\arg(\alpha_{N_2}/\alpha_{N_1})/\pi$	1.994	1.908	0.998	1.031	1.002
$ \alpha_{N_3}/\alpha_{N_1} $	8701.305	2.797×10^4	1.405×10^4	9.499×10^6	1562.126
$\arg(\alpha_{N_3}/\alpha_{N_1})/\pi$	0.755	1.711	1.645	0.0975	0.211
$\alpha_{\nu_2}/\alpha_{\nu_1}$	32.356	374.097	255.583	6894.423	178.981
$\alpha_{\nu_3}/\alpha_{\nu_1}$	74.263	105.143	138.335	2930.630	36.906
$\alpha_{d_1}/\alpha_{d_3}$	0.00400	0.00339	0.0549	0.00204	0.00390
$\alpha_{d_2}/\alpha_{d_3}$	0.0362	0.0345	0.00608	0.406	0.0107
$\alpha'_{d_1}/\alpha_{d_3}$	0.00209	0.00179	-0.0252	0.00107	0.00198
$ \alpha'_{d_2}/\alpha_{d_3} $	0.0315	0.0250	0.00240	0.435	0.0373
$\arg(\alpha'_{d_2}/\alpha_{d_3})/\pi$	1.264	0.766	0.127	0.963	0.865
$ \alpha'_{d_3}/\alpha_{d_3} $	0.783	0.588	0.703	0.272	0.772
$\arg(\alpha'_{d_3}/\alpha_{d_3})/\pi$	0.341	1.596	0.367	0.0577	0.344
$ \alpha_{u_1}/\alpha_{u_3} $	0.0488	0.0497	16.826	0.0559	0.0607
$\arg(\alpha_{u_1}/\alpha_{u_3})/\pi$	0.746	0.230	0.668	0.0788	1.700
$ \alpha_{u_2}/\alpha_{u_3} $	1.639	1.751	4752.550	0.970	1.248
$\arg(\alpha_{u_2}/\alpha_{u_3})/\pi$	0.746	0.160	1.164	1.645	1.305
$ \alpha_{u_4}/\alpha_{u_3} $	94.958	101.803	9.489×10^4	369.490	94.852
$\arg(\alpha_{u_4}/\alpha_{u_3})/\pi$	0.128	1.864	1.438	0.131	0.841
$(\alpha_{\nu_1}^2 v_u^2 / (\alpha_{N_1} \Lambda)) / \text{meV}$	25.174	28.405	34.814	60.161	50.645
$(\alpha_{d_3} v_d) / \text{GeV}$	0.625	0.737	0.544	0.740	0.630
$(\alpha_{u_3} v_u) / \text{GeV}$	3.580	3.452	6.973×10^{-4}	0.566	3.259
$\sin^2 \theta_{13}^l$	0.0222	0.0222	0.0221	0.0222	0.0221
$\sin^2 \theta_{12}^l$	0.304	0.304	0.304	0.304	0.302
$\sin^2 \theta_{23}^l$	0.570	0.574	0.574	0.565	0.544
$\delta_{CP}^l / ^\circ$	190.204	187.373	275.792	167.688	123.193
$\alpha_{21} / ^\circ$	2.343	359.144	13.641	15.279	4.521
$\alpha_{31} / ^\circ$	196.579	200.631	66.372	222.357	182.826
m_1 / meV	24.370	27.659	27.984	53.675	43.265
m_2 / meV	25.847	28.969	29.280	54.361	44.115
m_3 / meV	55.752	57.302	57.446	73.398	66.400
$m_{\beta\beta} / \text{meV}$	23.026	26.167	26.792	51.553	43.174
θ_{12}^q	0.227	0.227	0.228	0.227	0.227
θ_{13}^q	0.00348	0.00347	0.00348	0.00352	0.00350
θ_{23}^q	0.0402	0.0407	0.0410	0.0406	0.0374
$\delta_{CP}^q / ^\circ$	69.170	67.896	67.305	69.753	68.068
m_u / m_c	0.00222	0.00208	0.00566	0.00202	0.00238
m_c / m_t	0.00283	0.00282	0.00255	0.00278	0.00282
m_d / m_s	0.0536	0.0528	0.0563	0.0535	0.0568
m_s / m_b	0.0182	0.0181	0.0186	0.0183	0.0184
m_b / m_τ	0.743	0.743	0.743	0.741	0.744
χ_l^2	0.115	0.170	8.539	1.695	11.734
χ_q^2	0.505	1.147	47.053	1.078	19.458
χ_{total}^2	0.620	1.316	55.592	2.774	31.192

Table 10. The best fit values of the free parameters and the corresponding predictions for lepton and quark mixing parameters and fermion masses for the phenomenologically viable models of Type-V (continuation of table 9).

Model	\mathcal{V}_{11}	\mathcal{V}_{12}	\mathcal{V}_{13}	\mathcal{V}_{14}
$\text{Re}(\tau)$	0.4578	-0.04235	0.4701	-0.4983
$\text{Im}(\tau)$	1.863	1.018	0.947	0.881
$ \alpha_{N_2}/\alpha_{N_1} $	4.766×10^5	67.594	3040.620	855.157
$\arg(\alpha_{N_2}/\alpha_{N_1})/\pi$	0.992	0.497	1.868	1.025
$ \alpha_{N_3}/\alpha_{N_1} $	2.698×10^7	1.746×10^4	5427.280	2436.021
$\arg(\alpha_{N_3}/\alpha_{N_1})/\pi$	1.859	0.187	1.937	0.164
$\alpha_{\nu_2}/\alpha_{\nu_1}$	707.032	777.226	1801.840	544.650
$\alpha_{\nu_3}/\alpha_{\nu_1}$	4982.583	1.362×10^4	1480.650	390.182
$\alpha_{d_1}/\alpha_{d_3}$	0.00370	0.0354	0.0123	2.46×10^{-4}
$\alpha_{d_2}/\alpha_{d_3}$	0.429	0.00487	0.0653	157.517
$\alpha'_{d_1}/\alpha_{d_3}$	0.00192	-0.0161	0.00270	-0.419
$ \alpha'_{d_2}/\alpha_{d_3} $	0.452	9.101×10^{-4}	0.0406	12.948
$\arg(\alpha'_{d_2}/\alpha_{d_3})/\pi$	1.003	0.0239	1.320	1.027
$ \alpha'_{d_3}/\alpha_{d_3} $	0.327	0.0844	1.237	2.414
$\arg(\alpha'_{d_3}/\alpha_{d_3})/\pi$	1.967	0.883	1.851	1.097
$ \alpha_{u_1}/\alpha_{u_3} $	0.0555	8.723	0.136	0.0809
$\arg(\alpha_{u_1}/\alpha_{u_3})/\pi$	1.426	0.318	0.664	0.358
$ \alpha_{u_2}/\alpha_{u_3} $	1.890	1435.860	0.806	771.945
$\arg(\alpha_{u_2}/\alpha_{u_3})/\pi$	1.119	0.583	1.787	0.358
$ \alpha_{u_4}/\alpha_{u_3} $	107.107	2.785×10^4	104.861	35.988
$\arg(\alpha_{u_4}/\alpha_{u_3})/\pi$	0.974	0.580	0.557	1.955
$(\alpha_{\nu_1}^2 v_u^2 / (\alpha_{N_1} \Lambda)) / \text{meV}$	71.415	0.00272	0.0281	0.182
$(\alpha_{d_3} v_d) / \text{GeV}$	0.723	0.825	0.268	0.00474
$(\alpha_{u_3} v_u) / \text{GeV}$	3.368	0.00223	1.191	1.318
$\sin^2 \theta_{13}^l$	0.0222	0.0222	0.0222	0.0218
$\sin^2 \theta_{12}^l$	0.304	0.304	0.303	0.303
$\sin^2 \theta_{23}^l$	0.568	0.573	0.532	0.573
$\delta_{CP}^l / ^\circ$	243.693	194.576	186.177	274.541
$\alpha_{21} / ^\circ$	7.601	323.991	357.444	7.250
$\alpha_{31} / ^\circ$	190.060	33.404	185.770	187.830
m_1 / meV	65.130	23.735	29.247	62.580
m_2 / meV	65.697	25.250	30.489	63.170
m_3 / meV	82.172	55.493	58.092	80.100
$m_{\beta\beta} / \text{meV}$	64.656	23.847	27.677	63.030
θ_{12}^q	0.228	0.227	0.227	0.227
θ_{13}^q	0.00346	0.00350	0.00329	0.00346
θ_{23}^q	0.0402	0.0407	0.0400	0.0413
$\delta_{CP}^q / ^\circ$	69.686	69.162	66.103	67.777
m_u / m_c	0.00251	0.00194	0.00361	0.00289
m_c / m_t	0.00281	0.00265	0.00295	0.00281
m_d / m_s	0.0533	0.0507	0.0488	0.00847
m_s / m_b	0.0185	0.0189	0.0258	0.0147
m_b / m_τ	0.742	0.743	0.763	0.737
χ_l^2	3.092	0.0116	6.627	9.042
χ_q^2	1.431	3.228	73.675	64.986
χ_{total}^2	4.523	3.240	80.302	74.028

Table 11. The best fit values of the free parameters and the corresponding predictions for lepton and quark mixing parameters and fermion masses for the phenomenologically viable models of Type-V (continuation of table 10).

Model	\mathcal{V}_{15}	\mathcal{V}_{16}	\mathcal{V}_{17}	\mathcal{V}_{18}
$\text{Re}(\tau)$	0.03809	-0.02129	0.03559	0.01418
$\text{Im}(\tau)$	1.067	1.074	1.041	1.056
$ \alpha_{N_2}/\alpha_{N_1} $	2.837×10^5	1.745×10^7	3.18×10^6	3.461×10^7
$\arg(\alpha_{N_2}/\alpha_{N_1})/\pi$	0.886	0.884	1.090	1.010
$\alpha_{\nu_2}/\alpha_{\nu_1}$	346.937	1.324×10^4	1.129×10^4	5535.626
$\alpha_{\nu_3}/\alpha_{\nu_1}$	2274.630	3644.926	180.796	1.08×10^4
$\alpha_{d_1}/\alpha_{d_3}$	0.0718	0.0107	0.00398	0.00370
$\alpha_{d_2}/\alpha_{d_3}$	0.00697	0.0316	0.0285	0.0161
$ \alpha_{d_4}/\alpha_{d_3} $	0.412	1.964	0.598	0.730
$\arg(\alpha_{d_4}/\alpha_{d_3})/\pi$	0.612	1.265	1.153	1.324
$\alpha'_{d_1}/\alpha_{d_3}$	-0.0342	0.00503	0.00223	0.00199
$ \alpha'_{d_2}/\alpha_{d_3} $	0.00156	0.0985	0.0323	0.0446
$\arg(\alpha'_{d_2}/\alpha_{d_3})/\pi$	0.0384	0.883	0.605	0.825
$ \alpha'_{d_3}/\alpha_{d_3} $	0.167	1.080	2.105	0.0843
$\arg(\alpha'_{d_3}/\alpha_{d_3})/\pi$	1.362	1.714	2.579×10^{-4}	0.774
$ \alpha'_{d_4}/\alpha_{d_3} $	0.617	0.171	0.231	0.105
$\arg(\alpha'_{d_4}/\alpha_{d_3})/\pi$	0.487	0.346	1.436	0.142
$ \alpha_{u_1}/\alpha_{u_3} $	2.093×10^{-4}	2.113×10^{-4}	2.506×10^{-4}	2.162×10^{-4}
$\arg(\alpha_{u_1}/\alpha_{u_3})/\pi$	1.442	1.647	0.417	1.481
$ \alpha_{u_2}/\alpha_{u_3} $	0.0659	0.0690	0.0707	0.0711
$\arg(\alpha_{u_2}/\alpha_{u_3})/\pi$	1.133	0.647	1.362	0.402
$(\alpha_{\nu_1}^2 v_u^2 / (\alpha_{N_1} \Lambda)) / \text{meV}$	8.155	8.284	54.715	13.468
$(\alpha_{d_3} v_d) / \text{GeV}$	0.483	0.261	0.557	0.598
$(\alpha_{u_3} v_u) / \text{GeV}$	53.878	52.974	48.253	50.136
$\sin^2 \theta_{13}^l$	0.0222	0.0222	0.0222	0.0222
$\sin^2 \theta_{12}^l$	0.299	0.299	0.301	0.304
$\sin^2 \theta_{23}^l$	0.570	0.570	0.571	0.573
$\delta_{CP}^l / ^\circ$	269.281	270.635	224.199	197.099
$\alpha_{21} / ^\circ$	207.895	206.929	322.312	319.494
$\alpha_{31} / ^\circ$	236.478	237.649	149.590	319.034
m_1 / meV	1.591	1.574	1.482	16.768
m_2 / meV	8.760	8.757	8.740	18.851
m_3 / meV	50.349	50.350	50.178	52.898
$m_{\beta\beta} / \text{meV}$	0.636	0.630	3.633	16.671
θ_{12}^q	0.227	0.227	0.227	0.227
θ_{13}^q	0.00350	0.00350	0.00350	0.00349
θ_{23}^q	0.0403	0.0403	0.0404	0.0402
$\delta_{CP}^q / ^\circ$	68.824	68.798	71.046	69.213
m_u / m_c	0.00219	0.00215	0.00253	0.00193
m_c / m_t	0.00275	0.00275	0.00275	0.00282
m_d / m_s	0.0607	0.0603	0.0514	0.0505
m_s / m_b	0.0185	0.0185	0.0183	0.0182
m_b / m_τ	0.743	0.743	0.743	0.743
χ_1^2	7.374	7.660	1.098	1.863×10^{-5}
χ_q^2	3.424	3.155	1.968	4.839×10^{-4}
χ_{total}^2	10.798	10.816	3.067	5.026×10^{-4}

Table 12. The best fit values of the free parameters and the corresponding predictions for lepton and quark mixing parameters and fermion masses for the phenomenologically viable models of Type-V (continuation of table 11).

Model	\mathcal{V}_{19}	\mathcal{V}_{20}	\mathcal{V}_{21}	\mathcal{V}_{22}	\mathcal{V}_{23}
$\text{Re}(\tau)$	0.01168	0.03374	-0.4722	-0.02250	0.4935
$\text{Im}(\tau)$	1.052	1.000	1.997	1.644	0.932
$ \alpha_{N_2}/\alpha_{N_1} $	3.413×10^7	1.429×10^6	1.445×10^7	2.521×10^5	1.38×10^7
$\arg(\alpha_{N_2}/\alpha_{N_1})/\pi$	1.087	1.734	1.142	1.773	0.912
$\alpha_{\nu_2}/\alpha_{\nu_1}$	7261.654	2467.673	3033.538	422.269	3529.148
$\alpha_{\nu_3}/\alpha_{\nu_1}$	3004.292	1032.908	5167.630	719.326	4260.711
$\alpha_{d_1}/\alpha_{d_3}$	5.973×10^{-6}	1.462×10^{-5}	0.00428	0.00272	4.274×10^{-6}
$\alpha_{d_2}/\alpha_{d_3}$	0.00236	0.00753	0.122	0.0821	2.754×10^{-6}
$ \alpha_{d_4}/\alpha_{d_3} $	0.317	0.687	3.278	1.146	0.252
$\arg(\alpha_{d_4}/\alpha_{d_3})/\pi$	0.472	1.160	0.811	0.445	0.759
$\alpha'_{d_1}/\alpha_{d_3}$	0.0166	0.0843	0.00224	0.00149	-0.00304
$ \alpha'_{d_2}/\alpha_{d_3} $	6.002×10^{-4}	0.00165	0.122	0.0287	0.0108
$\arg(\alpha'_{d_2}/\alpha_{d_3})/\pi$	1.823	3.134×10^{-6}	0.921	0.750	1.945
$ \alpha'_{d_3}/\alpha_{d_3} $	0.792	0.360	0.252	0.481	0.267
$\arg(\alpha'_{d_3}/\alpha_{d_3})/\pi$	0.546	1.658	1.341	0.302	0.0588
$ \alpha'_{d_4}/\alpha_{d_3} $	0.0390	0.455	1.400	1.840	0.0670
$\arg(\alpha'_{d_4}/\alpha_{d_3})/\pi$	0.497	0.657	0.689	1.979	0.00297
$ \alpha_{u_1}/\alpha_{u_3} $	2.391×10^{-4}	2.973×10^{-4}	2.922×10^{-5}	0.00289	6.914×10^{-7}
$\arg(\alpha_{u_1}/\alpha_{u_3})/\pi$	1.492	1.552	0.285	1.951	1.683
$ \alpha_{u_2}/\alpha_{u_3} $	0.0681	0.0781	0.0525	0.0531	0.0134
$\arg(\alpha_{u_2}/\alpha_{u_3})/\pi$	0.773	0.807	0.285	0.275	1.684
$(\alpha_{\nu_1}^2 v_u^2)/(\alpha_{N_1} \Lambda)/\text{meV}$	58.524	11.746	76.941	44.759	44.149
$(\alpha_{d_3} v_d)/\text{GeV}$	1.249	0.478	0.810	0.545	1.345
$(\alpha_{u_3} v_u)/\text{GeV}$	50.653	41.087	476.818	225.459	150.072
$\sin^2 \theta_{13}^l$	0.0222	0.0218	0.0219	0.0222	0.0222
$\sin^2 \theta_{12}^l$	0.301	0.304	0.307	0.304	0.303
$\sin^2 \theta_{23}^l$	0.572	0.576	0.571	0.573	0.566
$\delta_{CP}^l/^\circ$	223.846	218.864	233.313	175.930	239.595
$\alpha_{21}/^\circ$	321.694	38.491	1.592	3.722	6.776
$\alpha_{31}/^\circ$	151.525	121.364	148.807	294.686	190.708
m_1/meV	1.433	16.202	76.550	44.235	76.708
m_2/meV	8.732	18.350	77.033	45.066	77.191
m_3/meV	50.174	52.753	91.203	66.847	91.779
$m_{\beta\beta}/\text{meV}$	3.534	16.671	76.516	44.278	75.775
θ_{12}^q	0.227	0.227	0.227	0.227	0.227
θ_{13}^q	0.00350	0.00348	0.00347	0.00346	0.00354
θ_{23}^q	0.0404	0.0402	0.0415	0.0401	0.0403
$\delta_{CP}^q/^\circ$	71.153	69.952	68.660	67.833	68.989
m_u/m_c	0.00254	0.00243	0.0	0.0	0.0
m_c/m_t	0.00274	0.00283	0.00275	0.00281	0.00282
m_d/m_s	0.0511	0.0640	0.0527	0.0488	0.0628
m_s/m_b	0.0184	0.0196	0.0184	0.0183	0.0184
m_b/m_τ	0.744	0.748	0.743	0.744	0.745
χ_l^2	1.080	1.094	2.812	0.786	2.721
χ_q^2	2.048	7.551	14.996	10.627	14.460
χ_{total}^2	3.128	8.646	17.808	11.413	17.181

Table 13. The best fit values of the free parameters and the corresponding predictions for lepton and quark mixing parameters and fermion masses for the phenomenologically viable models of Type-V (continuation of table 12).

Model	\mathcal{V}_{24}	\mathcal{V}_{25}	\mathcal{V}_{26}	\mathcal{V}_{27}	\mathcal{V}_{28}
$\text{Re}(\tau)$	-0.06476	-0.03872	0.03142	-0.05690	-0.4614
$\text{Im}(\tau)$	2.009	1.042	1.013	1.554	0.894
$ \alpha_{N_2}/\alpha_{N_1} $	2.321×10^6	6260.908	423.935	2.772×10^4	1.419×10^4
$\arg(\alpha_{N_2}/\alpha_{N_1})/\pi$	1.825	0.166	1.054	1.999	1.834
$\alpha_{\nu_2}/\alpha_{\nu_1}$	1345.488	5562.698	585.205	9996.838	8367.220
$\alpha_{\nu_3}/\alpha_{\nu_1}$	1882.619	8929.744	1441.682	4377.767	5285.115
$\alpha_{d_1}/\alpha_{d_3}$	1.065×10^{-4}	0.00622	$6. \times 10^{-5}$	6.512×10^{-4}	1.667×10^{-6}
$\alpha_{d_2}/\alpha_{d_3}$	0.242	0.0396	0.00597	0.0298	8.085×10^{-4}
$ \alpha_{d_4}/\alpha_{d_3} $	13.771	0.686	0.852	1.055	0.0265
$\arg(\alpha_{d_4}/\alpha_{d_3})/\pi$	1.482	1.657	1.079	1.464	1.624
$\alpha'_{d_1}/\alpha_{d_3}$	-0.0841	0.00339	-0.0708	6.929×10^{-4}	0.00229
$ \alpha'_{d_2}/\alpha_{d_3} $	0.0785	0.0724	0.00133	0.0632	0.00823
$\arg(\alpha'_{d_2}/\alpha_{d_3})/\pi$	0.00263	1.263	1.949	1.201	1.788
$ \alpha'_{d_3}/\alpha_{d_3} $	1.131	0.770	0.0522	0.721	0.219
$\arg(\alpha'_{d_3}/\alpha_{d_3})/\pi$	1.052	0.319	0.284	0.402	1.999
$ \alpha'_{d_4}/\alpha_{d_3} $	5.016	0.592	0.158	1.405	0.106
$\arg(\alpha'_{d_4}/\alpha_{d_3})/\pi$	1.549	1.266	0.172	1.488	1.674
$ \alpha_{u_1}/\alpha_{u_3} $	0.00259	2.24×10^{-4}	2.838×10^{-4}	0.00464	1.134×10^{-4}
$\arg(\alpha_{u_1}/\alpha_{u_3})/\pi$	1.507	1.945	0.395	0.607	3.68×10^{-7}
$ \alpha_{u_2}/\alpha_{u_3} $	0.00973	0.0715	0.0757	0.0464	0.0102
$\arg(\alpha_{u_2}/\alpha_{u_3})/\pi$	0.460	0.995	1.718	1.583	1.546
$(\alpha_{\nu_1}^2 v_u^2 / (\alpha_{N_1} \Lambda)) / \text{meV}$	73.353	0.00254	0.0149	0.00556	0.00923
$(\alpha_{d_3} v_d) / \text{GeV}$	0.296	0.350	0.582	0.556	1.683
$(\alpha_{u_3} v_u) / \text{GeV}$	487.744	48.477	43.309	186.139	173.670
$\sin^2 \theta_{13}^l$	0.0211	0.0222	0.0219	0.0224	0.0223
$\sin^2 \theta_{12}^l$	0.310	0.304	0.305	0.287	0.303
$\sin^2 \theta_{23}^l$	0.570	0.573	0.574	0.562	0.546
$\delta_{CP}^l / ^\circ$	115.138	197.255	228.295	192.586	281.439
$\alpha_{21} / ^\circ$	356.402	31.817	56.081	180.297	7.680
$\alpha_{31} / ^\circ$	191.174	32.473	134.582	204.586	152.994
m_1 / meV	72.900	16.091	16.350	2.551	86.066
m_2 / meV	73.407	18.252	18.480	8.984	86.496
m_3 / meV	88.025	52.687	52.834	51.269	99.688
$m_{\beta\beta} / \text{meV}$	72.959	16.965	16.003	1.889	85.494
θ_{12}^q	0.228	0.227	0.227	0.227	0.227
θ_{13}^q	0.00350	0.00349	0.00352	0.00360	0.00347
θ_{23}^q	0.0412	0.0401	0.0402	0.0420	0.0417
$\delta_{CP}^q / ^\circ$	74.606	69.217	69.709	80.995	68.018
m_u / m_c	0.0	0.00193	0.00248	0.0	0.0
m_c / m_t	0.00315	0.00282	0.00282	0.00219	0.00282
m_d / m_s	0.0802	0.0505	0.0643	0.0213	0.0505
m_s / m_b	0.0183	0.0182	0.0193	0.0166	0.0186
m_b / m_τ	0.772	0.743	0.746	0.742	0.742
χ_l^2	20.900	9.63×10^{-5}	1.544	4.392	11.684
χ_q^2	57.067	8.846×10^{-5}	7.021	86.609	16.723
χ_{total}^2	77.966	1.848×10^{-4}	8.565	91.000	28.407

Table 14. The best fit values of the free parameters and the corresponding predictions for lepton and quark mixing parameters and fermion masses for the phenomenologically viable models of Type-V (continuation of table 13).

Model	$\mathcal{V}II_1$	$\mathcal{V}II_2$	$\mathcal{V}II_3$
$\text{Re}(\tau)$	0.3358	-0.2769	0.4734
$\text{Im}(\tau)$	1.207	1.203	1.288
$ \alpha_{\nu_3}/\alpha_{\nu_2} $	0.739	0.580	2.329
$\arg(\alpha_{\nu_3}/\alpha_{\nu_2})/\pi$	1.992	1.062	0.889
$\alpha_{d_1}/\alpha_{d_3}$	0.0544	0.00557	0.00769
$\alpha_{d_2}/\alpha_{d_3}$	0.00406	0.0476	0.101
$ \alpha_{d_4}/\alpha_{d_3} $	0.861	1.328	1.200
$\arg(\alpha_{d_4}/\alpha_{d_3})/\pi$	1.019	1.198	1.080
$\alpha'_{d_1}/\alpha_{d_3}$	-0.0292	0.00207	0.00286
$ \alpha'_{d_2}/\alpha_{d_3} $	0.00238	0.0279	0.0559
$\arg(\alpha'_{d_2}/\alpha_{d_3})/\pi$	0.205	1.222	1.176
$ \alpha'_{d_3}/\alpha_{d_3} $	0.517	0.561	1.220
$\arg(\alpha'_{d_3}/\alpha_{d_3})/\pi$	0.417	0.431	0.272
$ \alpha'_{d_4}/\alpha_{d_3} $	0.466	0.175	0.640
$\arg(\alpha'_{d_4}/\alpha_{d_3})/\pi$	1.450	1.014	1.186
$ \alpha_{u_1}/\alpha_{u_3} $	0.0622	2.64×10^{-5}	6.853×10^{-5}
$\arg(\alpha_{u_1}/\alpha_{u_3})/\pi$	0.832	1.729	1.417
$ \alpha_{u_2}/\alpha_{u_3} $	2.223×10^{-4}	0.0591	0.0658
$\arg(\alpha_{u_2}/\alpha_{u_3})/\pi$	0.0184	1.720	1.417
$(\alpha_{\nu_1}\alpha_{\nu_2}v_u^2/(\alpha_{N_1}\Lambda))/\text{meV}$	16.054	22.798	11.786
$(\alpha_{d_3}v_d)/\text{GeV}$	0.670	0.565	0.433
$(\alpha_{u_3}v_u)/\text{GeV}$	75.570	80.743	75.765
$\sin^2 \theta_{13}^l$	0.0223	0.0222	0.0222
$\sin^2 \theta_{12}^l$	0.303	0.304	0.315
$\sin^2 \theta_{23}^l$	0.586	0.569	0.514
$\delta_{CP}^l/^\circ$	92.870	209.195	278.939
$\phi/^\circ$	42.167	31.273	199.262
m_1/meV	0.0	0.0	0.0
m_2/meV	8.614	8.614	8.614
m_3/meV	50.020	50.148	49.898
$m_{\beta\beta}/\text{meV}$	1.983	2.800	3.602
θ_{12}^q	0.227	0.227	0.227
θ_{13}^q	0.00364	0.00357	0.00362
θ_{23}^q	0.0379	0.0394	0.0406
$\delta_{CP}^q/^\circ$	69.225	69.398	68.704
m_u/m_c	0.0	0.0	0.0
m_c/m_t	0.00266	0.00289	0.00280
m_d/m_s	0.0505	0.0489	0.0504
m_s/m_b	0.0146	0.0165	0.0168
m_b/m_τ	0.746	0.744	0.745
χ_I^2	19.526	0.257	18.998
χ_q^2	38.795	15.228	13.937
χ_{total}^2	58.321	15.486	32.935

Table 15. The best fit values of the free parameters and the corresponding predictions for lepton and quark mixing parameters and fermion masses for the phenomenologically viable models of Type-VII.

Model	$\mathcal{V}II_4$	$\mathcal{V}II_5$	$\mathcal{V}II_6$
$\text{Re}(\tau)$	0.04682	-0.1587	-0.001939
$\text{Im}(\tau)$	1.198	1.068	1.123
$ \alpha_{N_2}/\alpha_{N_1} $	3.397×10^4	1441.443	4902.592
$\arg(\alpha_{N_2}/\alpha_{N_1})/\pi$	0.143	0.164	0.0237
$\alpha_{\nu_2}/\alpha_{\nu_1}$	101.846	19.052	103.807
$\alpha_{d_1}/\alpha_{d_3}$	0.00964	0.00815	0.00759
$\alpha_{d_2}/\alpha_{d_3}$	0.0187	0.0538	0.0444
$ \alpha_{d_4}/\alpha_{d_3} $	2.282	1.806	1.837
$\arg(\alpha_{d_4}/\alpha_{d_3})/\pi$	0.663	1.414	0.728
$\alpha'_{d_1}/\alpha_{d_3}$	0.00384	0.00308	0.00283
$ \alpha'_{d_2}/\alpha_{d_3} $	0.0832	0.0554	0.0482
$\arg(\alpha'_{d_2}/\alpha_{d_3})/\pi$	1.116	1.290	0.678
$ \alpha'_{d_3}/\alpha_{d_3} $	0.562	0.638	0.603
$\arg(\alpha'_{d_3}/\alpha_{d_3})/\pi$	1.220	1.195	1.083
$ \alpha'_{d_4}/\alpha_{d_3} $	1.607	0.211	0.474
$\arg(\alpha'_{d_4}/\alpha_{d_3})/\pi$	0.281	1.203×10^{-6}	0.378
$ \alpha_{u_1}/\alpha_{u_3} $	0.0115	0.0132	0.0128
$\arg(\alpha_{u_1}/\alpha_{u_3})/\pi$	1.317	0.228	1.100
$ \alpha_{u_2}/\alpha_{u_3} $	0.0750	0.0842	0.119
$\arg(\alpha_{u_2}/\alpha_{u_3})/\pi$	0.809	1.981	1.710
$(\alpha_{\nu_1}^2 v_u^2 / (\alpha_{N_1} \Lambda)) / \text{meV}$	36.346	32.470	25.200
$(\alpha_{d_3} v_d) / \text{GeV}$	0.284	0.320	0.368
$(\alpha_{u_3} v_u) / \text{GeV}$	79.709	57.900	63.689
$\sin^2 \theta_{13}^l$	0.0224	0.0220	0.0224
$\sin^2 \theta_{12}^l$	0.353	0.349	0.315
$\sin^2 \theta_{23}^l$	0.498	0.458	0.588
$\delta_{CP}^l / ^\circ$	273.592	160.552	316.174
$\phi / ^\circ$	356.632	317.125	173.353
m_1 / meV	0.0	0.0	0.0
m_2 / meV	8.614	8.614	8.614
m_3 / meV	49.889	49.849	49.954
$m_{\beta\beta} / \text{meV}$	1.858	3.283	2.955
θ_{12}^q	0.227	0.227	0.227
θ_{13}^q	0.00349	0.00350	0.00350
θ_{23}^q	0.0403	0.0413	0.0413
$\delta_{CP}^q / ^\circ$	70.791	68.560	68.603
m_u / m_c	0.0	0.0	0.0
m_c / m_t	0.00270	0.00243	0.00244
m_d / m_s	0.0533	0.0551	0.0508
m_s / m_b	0.0185	0.0196	0.0194
m_b / m_τ	0.747	0.741	0.743
χ_I^2	38.651	49.559	21.435
χ_q^2	12.051	26.907	25.291
χ_{total}^2	50.703	76.466	46.727

Table 16. The best fit values of the free parameters and the corresponding predictions for lepton and quark mixing parameters and fermion masses for the phenomenologically viable models of Type-VII (continuation of table 15).

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