

On topological recursion for Wilson loops in $\mathcal{N} = 4$ SYM at strong coupling

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ABSTRACT: We consider $U(N)$ $\mathcal{N} = 4$ super Yang-Mills theory and discuss how to extract the strong coupling limit of non-planar corrections to observables involving the $\frac{1}{2}$ -BPS Wilson loop. Our approach is based on a suitable saddle point treatment of the Eynard-Orantin topological recursion in the Gaussian matrix model. Working directly at strong coupling we avoid the usual procedure of first computing observables at finite planar coupling λ , order by order in $1/N$, and then taking the $\lambda \gg 1$ limit. In the proposed approach, matrix model multi-point resolvents take a simplified form and some structures of the genus expansion, hardly visible at low order, may be identified and rigorously proved. As a sample application, we consider the expectation value of multiple coincident circular supersymmetric Wilson loops as well as their correlator with single trace chiral operators. For these quantities we provide novel results about the structure of their genus expansion at large tension, generalising recent results in [arXiv:2011.02885](https://arxiv.org/abs/2011.02885).

KEYWORDS: $1/N$ Expansion, AdS-CFT Correspondence, Wilson, 't Hooft and Polyakov loops

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1 Introduction and results

The recent papers [1–3] focused on certain features of higher genus corrections to BPS Wilson loops in dual theories related by AdS/CFT. By means of supersymmetric localization, gauge theory predictions are available as matrix model integrals that depend non-trivially on the number of colours N and 't Hooft planar coupling λ (mass deformations will not be relevant here). The large N expansion may be computed at high order starting from exact expressions in the matrix model or by perturbative loop equation methods, like topological recursion [4]. On the string side, the gauge theory parameters N, λ may be replaced by the string coupling g_s and tension T . World-sheet genus expansion is a natural perturbation

theory controlled by powers of g_s accompanied by corrections in inverse string tension, i.e. σ -model quantum corrections. The two expansions are expected to match according to AdS/CFT, but practical tests are of course non-trivial. On the gauge side, a rich set of predictions is obtained extracting the dominant strong coupling corrections order by order in $1/N$, i.e. well beyond planar level. On string side, this should reproduce the large tension limit $T \gg 1$ at specific genera, whose independent determination is obviously very hard beyond leading order. In spite of that, one can still look at manifestations of its expected structural properties in the $1/N$ gauge theory expansion.

The simplest example where this strategy may be concretely illustrated is the expectation value $\langle \mathcal{W} \rangle$ of the $\frac{1}{2}$ -BPS circular Wilson loop in $U(N)$ $\mathcal{N} = 4$ SYM. The expression for $\langle \mathcal{W} \rangle$ is known at finite N and $\lambda = N g_{\text{YM}}^2$ exactly [5–8] and is given by the Hermitian Gaussian one-matrix model average

$$\langle \mathcal{W} \rangle = \int \mathcal{D}M \operatorname{tr} e^{\frac{\sqrt{\lambda}}{2} M} e^{-\frac{N}{2} \operatorname{tr} M^2} = e^{\frac{\lambda}{8N}} L_{N-1}^1 \left(-\frac{\lambda}{4N} \right). \tag{1.1}$$

In this case, the relation among the gauge theory parameters λ, N and g_s, T in the dual $\text{AdS}_5 \times S^5$ IIB superstring is [9]

$$g_s = \frac{\lambda}{4\pi N}, \quad T = \frac{\sqrt{\lambda}}{2\pi}. \tag{1.2}$$

At large tension, (1.1) takes the following form

$$\langle \mathcal{W} \rangle = \frac{1}{2\pi} \frac{\sqrt{T}}{g_s} e^{2\pi T + \frac{\pi}{12} \frac{g_s^2}{T}} \left[1 + \mathcal{O}(T^{-1}) \right] \stackrel{T \gg 1}{\cong} e^{2\pi T} f \left(\frac{g_s^2}{T} \right), \tag{1.3}$$

$$f(x) = x^{-1/2} \exp \left(\frac{\pi}{12} x^2 \right). \tag{1.4}$$

The structure of (1.3) is consistent with the dual representation of the Wilson loop expectation value as the string path integral over world-sheets ending on a circle at ∂AdS .¹

A similar large tension analysis is presented in [2] for other quantities related again to the $\frac{1}{2}$ -BPS Wilson loop in $\mathcal{N} = 4$ SYM. In particular, one can consider the normalised ratio of n coincident Wilson loops.² This requires consideration of matrix integrals which are generalisations of (1.1), but whose $1/N$ expansion is much more difficult to extract.³ The semiclassical exponential factors $\sim e^{2\pi T}$ cancel and the ratio $\langle \mathcal{W}^n \rangle / \langle \mathcal{W} \rangle^n$ is again organised in powers of g_s^2/T , cf. (1.3),

$$\frac{\langle \mathcal{W}^n \rangle}{\langle \mathcal{W} \rangle^n} \stackrel{T \gg 1}{\cong} W_n \left(\frac{\pi g_s^2}{T} \right), \tag{1.5}$$

¹The exponential factor $\exp(2\pi T)$ comes from the AdS_2 minimal surface [10–12]. Upon expansion in g_s , the power of the string coupling is minus the Euler number of a disc with p handles ($\chi = 1 - 2p$). The fact that each power of g_s is accompanied at large tension by a factor $1/\sqrt{T}$ is non-trivial and explained in [1]. A similar structure holds for Wilson loops in ABJM theory, dual to string on $\text{AdS}_4 \times \mathbb{CP}^3$.

²See [13] for a recent application of such coincident loops in matrix models associated with JT gravity.

³Indeed, in this case one does not have a simple result like (1.1), but instead multiple finite sum of $\sim N$ terms, see for instance eq. (4.3) in [2] for $n = 2$.

where the first three terms of the scaling function W_n have been computed in [2] and read

$$W_n(x) = 1 + \frac{n(n-1)}{2}x + \frac{n(n-1)(3n-5)(n+2)}{24}x^2 + \frac{n(n-1)(15n^4 + 30n^3 - 75n^2 - 610n + 1064)}{720}x^3 + \dots \quad (1.6)$$

A third example of scaling functions emerging in the large tension limit are normalised correlators of \mathcal{W} with a single trace chiral operator $\mathcal{O}_J \sim \text{tr } \Phi^J$ [10, 14] recently reconsidered in [2]. In this case, the large tension limit is characterised by a different scaling combination

$$\frac{\langle \mathcal{W} \mathcal{O}_J \rangle}{\langle \mathcal{W} \rangle} \stackrel{T \gg 1}{\cong} J \left(\frac{\pi}{2} \right)^{J/2} T F_J \left(\frac{g_s^2}{T^2} \right), \quad F_J(x) = \frac{2}{J\sqrt{x}} \sinh \left(J \operatorname{arcsinh} \frac{\sqrt{x}}{2} \right), \quad (1.7)$$

where we draw attention to the non-trivial dependence of $F_J(x)$ on the R-charge J .⁴

Beyond proving general structures as in (1.3), (1.5), and (1.7), it is important to develop methods to determine the detailed form of scaling functions like f , W_n and F_J . A common approach is to compute the $1/N$ expansion at finite planar coupling λ in the Hermitian Gaussian one-matrix model, and then take the strong coupling limit $\lambda \gg 1$. For instance, in the case of $\langle \mathcal{W} \rangle$, one has the exact representation at finite λ [16]

$$\langle \mathcal{W} \rangle = \frac{2N}{\sqrt{\lambda}} \operatorname{Res}_{x=0} \left[e^{\frac{\lambda}{4N} H \left(\frac{\sqrt{\lambda}}{4N} x \right)} \sum_{n=0}^{\infty} I_n(\sqrt{\lambda}) x^{-n} \right], \quad H(x) \equiv \frac{1}{2} \left(\coth x - \frac{1}{x} \right). \quad (1.8)$$

From (1.8), we get all coefficients of the $1/N$ power series in terms of explicit combinations of modified Bessel functions ($I_n \equiv I_n(\sqrt{\lambda})$), see also [6],

$$\langle \mathcal{W} \rangle = \frac{2N I_1}{\sqrt{\lambda}} + \frac{\lambda I_2}{48N} + \frac{1}{N^3} \left(\frac{\lambda^{5/2} I_3}{9216} - \frac{\lambda^2 I_4}{11520} \right) + \frac{1}{N^5} \left(\frac{\lambda^4 I_4}{2654208} - \frac{\lambda^{7/2} I_5}{1105920} + \frac{\lambda^3 I_6}{1935360} \right) + \dots \quad (1.9)$$

When each term of this expression is expanded at large λ , the result takes the simple exponential form (1.3). Of course, the case of $\langle \mathcal{W} \rangle$ is particularly simple because of the compact closed formula (1.1) leading to (1.8). Somehow, a similar situation occurs in the case of the scaling function F_J in (1.7). Indeed, the correlator $\langle \mathcal{W} \mathcal{O}_J \rangle$ admits the representation [17]

$$\langle \mathcal{W} \mathcal{O}_J \rangle = \left(\frac{2}{\pi} \right)^{1-J/2} \frac{N^2}{\sqrt{\lambda}} e^{\frac{\lambda}{8N}} \oint \frac{dz}{2\pi i} z^J e^{\frac{\sqrt{\lambda}}{2} z} \left(1 + \frac{\sqrt{\lambda}}{2Nz} \right)^N \left[\left(1 + \frac{\sqrt{\lambda}}{2Nz} \right)^J - 1 \right], \quad (1.10)$$

and one can prove (1.7) from this formula, which is exact at finite N and λ [2, 15].

However, as soon as the observables under study become more complicated, it is increasingly difficult to extract the genus expansion order by order in $1/N$ at finite λ . An example are multiple coincident Wilson loops $\langle W^n \rangle$ — not to be confused with multiply wound loops — or multi-trace chiral operators [18]. In this case, exact expressions are not

⁴Through an analytical continuation it is possible to capture F_J by a D3-brane calculation, see [15].

available or are too cumbersome to be useful. Toda recursion relations [19–23] are a possible method to determine the $1/N$ expansion, but work well only for simple observables [2] (and their scope is limited to the Gaussian matrix model). A more general approach is to take advantage of topological recursion [24, 25] which is an efficient way to organise the hierarchy the matrix model loop equations.⁵ In practice, a serious bottleneck in applying this method is the rapid increase of computational complexity at higher genus, see for instance [31]. For these reasons, it seems important to devise a version of topological recursion suitable for strong coupling directly.

In this paper, we take a first step in this direction. We illustrate a practical approach to work out topological recursion at strong coupling by isolating dominant contributions at large tension. Despite its simplicity, the method turns out to be rather effective. As an illustration, we present an algorithm for computing the function $W_n(x)$ in (1.5) at any desired order with minor effort, and we illustrate remarkable exponentiation properties of the dominant terms at large n . This result will be cross checked by means of an extension to all n of the Toda recursion method used in [2] for $n = 2, 3$. As a second application, we shall prove that the structure of (1.7) is rather special and does not extend to the normalized correlators of a chiral primary single trace operator with multiple coinciding Wilson loops, i.e. ratios $\langle W^n \mathcal{O}_J \rangle / \langle W^n \rangle$ when $n > 1$. Instead, we prove that the relevant scaling variable is g_s^2/T and that the dependence on the R-charge is

$$\frac{\langle W^n \mathcal{O}_J \rangle}{\langle W^n \rangle} \stackrel{T \gg 1}{\cong} J \left(\frac{\pi}{2} \right)^{J/2} n \left[T + (J^2 - 1) H_n \left(\frac{\pi g_s^2}{T} \right) \right], \quad (1.11)$$

where the function $H_n(x)$ is independent of J and may be computed in terms of W_n by the relation

$$H_n(x) = \frac{x}{2\pi} \left[\frac{1}{12} + \frac{1}{n} (\log W_n(x))' \right]. \quad (1.12)$$

The derivation of these results is straightforward in the framework of the strong coupling version of topological recursion, and far from trivial by other methods. A similar approach is expected to be useful and apply in harder cases with separated Wilson loops or more local operator insertions. Some of these problems can be mapped to multi-matrix models calculations [32] that would be interesting to study by a suitable strong coupling limit of more general topological recursions [33].

The detailed plan of the paper is as follows. In section 2 we briefly recall the structure of topological recursion for $\mathcal{N} = 4$ SYM and its application to the evaluation of $\langle W \rangle$. In section 3 we show how to perform a saddle point expansions at strong coupling in the considered problems. We clarify what are the relevant features of resolvents in that regime. Section 4 presents the strong coupling version of topological recursion, capturing the reduced resolvents. In section 5 we apply this formalism to our first application, i.e. the computation of $\langle W^n \rangle$ at large tension. In section 5.1, as a non-trivial check of our approach, the same results are obtained by solving in the strong coupling limit a suitable Toda recursion for correlators of traced exponentials in the Gaussian matrix model. Finally, in section 6 we discuss the correlators $\langle W^n \mathcal{O}_J \rangle$ between coincident Wilson loops and a single

⁵See also [26–30] for other recent applications of topological recursion to $\mathcal{N} = 4$ SYM.

trace chiral operator. The relation with the scaling function characterising $\langle W^n \rangle$ is proved in section 6.4.

2 Topological recursion for the Gaussian matrix model

For a Hermitian one-matrix model with potential V , the spectral curve is defined by [25, 34]

$$y^2 - \frac{1}{2}V'(x)y + P(x) = 0, \quad P(x) = \frac{1}{N} \left\langle \text{tr} \frac{V'(x) - V'(M)}{x - M} \right\rangle, \quad (2.1)$$

where $\langle \mathcal{O}(M) \rangle = \int DM e^{-N \text{tr} V(M)} \mathcal{O}(M)$ and normalization is fixed by $\langle 1 \rangle = 1$. In the Gaussian case, $V(M) = \frac{1}{2}M^2$, cf. (1.1), and the curve (2.1) takes the form

$$y^2 - xy + 1 = 0, \quad (2.2)$$

admitting the rational (complex) parametrization

$$x = z + \frac{1}{z}, \quad y = \frac{1}{z}. \quad (2.3)$$

The n -point resolvent is defined as the connected correlator⁶

$$W_n(x_1, \dots, x_n) = \left\langle \text{tr} \frac{1}{x_1 - M} \cdots \text{tr} \frac{1}{x_n - M} \right\rangle_c, \quad (2.4)$$

and admits the following genus expansion at large N

$$W_n(x_1, \dots, x_n) = \sum_{g=0}^{\infty} \frac{1}{N^{n-2+2g}} W_{n,g}(x_1, \dots, x_n). \quad (2.5)$$

The functions $W_n(x_1, \dots, x_n)$ may be traded by multi-differentials on the algebraic curve (2.2)

$$\omega_{n,g}(z_1, \dots, z_n) = W_{n,g}(x(z_1), \dots, x(z_n)) dx(z_1) \cdots dx(z_n). \quad (2.6)$$

Multi-trace connected correlators may be computed as contour integrals around the cut

$$\left\langle \prod_i \text{tr} \mathcal{O}_i(M) \right\rangle_c = \sum_{g=0}^{\infty} \frac{1}{N^{n-2+2g}} \frac{1}{(2\pi i)^n} \oint \omega_{n,g}(z_1, \dots, z_n) \prod_i \mathcal{O}_i(x(z_i)). \quad (2.7)$$

Higher genus resolvents obey the topological recursion

$$\begin{aligned} \omega_{1,0}(z) &= \frac{1}{z} \left(1 - \frac{1}{z^2} \right) dz, & \omega_{2,0}(z_1, z_2) &= \frac{dz_1 dz_2}{(z_1 - z_2)^2}, \\ \omega_{n,g}(z_1, \mathbf{z}) &= \text{Res}_{\zeta=1, -1} K(z_1, \zeta) \left[\omega_{n+1, g-1}(\zeta, \zeta^{-1}, \mathbf{z}) + \sum_{h \leq g} \sum_{\mathbf{w} \subset \mathbf{z}} \omega_{|w|, h+1}(\zeta, \mathbf{w}) \omega_{n-|w|, g-h}(\zeta^{-1}, \mathbf{z} \setminus \mathbf{w}) \right], \\ K(z, w) &= \frac{w^3}{2(w^2-1)(z-w)(zw-1)} \frac{dz}{dw}, \end{aligned} \quad (2.8)$$

⁶Connected correlators $\langle X_1 X_2 \cdots \rangle_c$ are functional derivatives of the logarithm of the generating function of correlators with respect to sources coupled to X_i operators.

where $\mathbf{z} = (z_2, \dots, z_n)$, \mathbf{w} is a subset of \mathbf{z} (preserving the order of the variables), $|\mathbf{w}|$ is the number of elements of \mathbf{w} , and $\mathbf{z} \setminus \mathbf{w}$ is the complement of \mathbf{w} in \mathbf{z} . In the double sum we exclude the two cases $(h, \mathbf{w}) = (0, \emptyset)$ and $(h, \mathbf{w}) = (g, \mathbf{z})$. The recursion (2.8) allows to compute the following quantities in triangular sequence (the number under brace is the total weight $g + n$)

$$\underbrace{\omega_{1,1}}_2 \rightarrow \underbrace{\omega_{3,0} \rightarrow \omega_{2,1} \rightarrow \omega_{1,2}}_3 \rightarrow \underbrace{\omega_{4,0} \rightarrow \omega_{3,1} \rightarrow \omega_{2,2} \rightarrow \omega_{1,3}}_4 \rightarrow \dots \quad (2.9)$$

Apart from the seeds $\omega_{1,0}$ and $\omega_{2,0}$, all other resolvents have poles in the z_i variables only at the special points ± 1 . The first entries in (2.9) read (omitting the $dz_1 \cdots dz_n$ differentials)

$$\begin{aligned} \omega_{1,1}(z) &= \frac{z^3}{(z^2-1)^4}, \\ \omega_{3,0}(z_1, z_2, z_3) &= -\frac{1}{2(z_1-1)^2(z_2-1)^2(z_3-1)^2} + \frac{1}{2(z_1+1)^2(z_2+1)^2(z_3+1)^2}, \\ \omega_{2,1}(z_1, z_2) &= \frac{1}{4(z_1^2-1)^6(z_2^2-1)^6} \left[4z_1^3 z_2 (1+z_2^2)^2 (1-7z_2^2+z_2^4) + 4z_1^7 z_2 (1+z_2^2)^2 (1-7z_2^2+z_2^4) \right. \\ &\quad + 5(z_2^4+z_2^6) + 5z_1^{10}(z_2^4+z_2^6) + 4z_1(z_2^3+3z_2^5+z_2^7) + 4z_1^9(z_2^3+3z_2^5+z_2^7) \\ &\quad + 3z_1^2(z_2^2-6z_2^4-6z_2^6+z_2^8) + 3z_1^8(z_2^2-6z_2^4-6z_2^6+z_2^8) \\ &\quad + 12z_1^5(z_2-4z_2^3+16z_2^5-4z_2^7+z_2^9) + z_1^4(5-18z_2^2+23z_2^4+23z_2^6-18z_2^8+5z_2^{10}) \\ &\quad \left. + z_1^6(5-18z_2^2+23z_2^4+23z_2^6-18z_2^8+5z_2^{10}) \right], \\ \omega_{1,2}(z) &= -\frac{21 z^7 (1+3z^2+z^4)}{(-1+z^2)^{10}}, \end{aligned} \quad (2.10)$$

and so on. The expression of $\omega_{2,1}$ shows how explicit results become quickly unwieldy.

Analysis of the simple loop $\langle \mathcal{W} \rangle$. It is useful illustrate how resolvents are used to compute the genus expansion of the simple loop expectation value $\langle \mathcal{W} \rangle$. We have

$$\langle \mathcal{W} \rangle = \int \mathcal{D}M \operatorname{tr} e^{\frac{\sqrt{\lambda}}{2} M} e^{-\frac{N}{2} \operatorname{tr} M^2} \xrightarrow{N \rightarrow \infty} N \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \langle \mathcal{W} \rangle_g, \quad \langle \mathcal{W} \rangle_g = \frac{1}{2\pi i} \oint \omega_{1,g}(z) e^{\frac{\sqrt{\lambda}}{2}(z+1/z)}. \quad (2.11)$$

The leading term is simply⁷

$$\langle \mathcal{W} \rangle_0 = \oint \frac{dz}{2\pi i} \frac{1}{z} \left(1 - \frac{1}{z^2} \right) e^{\frac{\sqrt{\lambda}}{2}(z+1/z)} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}), \quad (2.12)$$

in agreement with the well known planar result. The next-to-leading term is

$$\langle \mathcal{W} \rangle_1 = \oint \frac{dz}{2\pi i} \frac{z^3}{(z^2-1)^4} e^{\frac{\sqrt{\lambda}}{2}(z+1/z)}. \quad (2.13)$$

⁷We use the generating function $e^{\frac{x}{2}(z+1/z)} = \sum_{n=-\infty}^{\infty} I_n(x) z^n$ and the identity $I_0(x) - I_2(x) = \frac{2}{x} I_1(x)$.

The contour encircles all three singular points, but one can check that there are no residues from $z = \pm 1$. Thus, integrating by parts two times gives

$$\langle W \rangle_1 = \frac{\lambda}{48} \oint \frac{dz}{2\pi i} \frac{1}{z^3} e^{\frac{\sqrt{\lambda}}{2}(z+1/z)} = \frac{\lambda}{48} I_2(\sqrt{\lambda}), \quad (2.14)$$

which is the well known $1/N^2$ correction. A similar manipulation can be repeated for the next order. Integrating by parts five times gives

$$\begin{aligned} \langle W \rangle_2 &= \oint \frac{dz}{2\pi i} \frac{21 z^7 (1 + 3z^2 + z^4)}{(z^2 - 1)^{10}} e^{\frac{\sqrt{\lambda}}{2}(z+1/z)} = \frac{\lambda^{5/2}}{92160} \oint \frac{dz}{2\pi i} \left(\frac{1}{z^6} + \frac{9}{z^4} \right) e^{\frac{\sqrt{\lambda}}{2}(z+1/z)} \\ &= \frac{\lambda^{5/2}}{92160} \left[I_5(\sqrt{\lambda}) + 9 I_3(\sqrt{\lambda}) \right] = \left[\frac{\lambda^{5/2}}{9216} I_3(\sqrt{\lambda}) - \frac{\lambda^2}{11520} I_4(\sqrt{\lambda}) \right], \end{aligned} \quad (2.15)$$

in agreement with the $1/N^3$ term in (1.9).

In the case of $\langle W \rangle$, this method may be extended to all orders in the $1/N$ expansion, and can also be generalized to give explicit Bessel function combinations for higher point resolvents at finite λ , see for instance [31]. Nevertheless, the calculation quickly becomes impractical at higher orders due to the very involved expressions that are generated going recursively through the chain of evaluations (2.9). Also, as we explained in the introduction, we are ultimately interested in extracting the large tension limit and want to bypass the cumbersome procedure of first obtaining exact expressions at finite λ , and then expand them at $\lambda \gg 1$. For instance, in the above genus-two contribution both Bessel functions give a similar leading asymptotic contribution due to the expansion

$$I_n(\sqrt{\lambda}) = \frac{1}{\sqrt{2\pi}} \lambda^{-1/4} \left(1 + \frac{4n^2 - 1}{8} \frac{1}{\sqrt{\lambda}} + \dots \right) e^{\sqrt{\lambda}} + \dots, \quad (2.16)$$

and it would be desirable to pin the total contribution in a more direct way. To this aim, one needs to study (2.8) working at strong coupling from the beginning and making more transparent the origin of the dominant terms. The next section will be devoted to this problem.

3 Saddle point methods for Wilson loops

In this section, we discuss how to extract dominant terms from integrals like (2.12) by saddle point evaluation. Although this is a fairly well known topic, we want to emphasize some specific technical issues that are relevant in the calculations we are interested in. To this aim, we consider the large $\sigma \rightarrow +\infty$ expansion of a contour integral of the form

$$I(\sigma) = \oint dz g(z) e^{-\sigma f(z)}. \quad (3.1)$$

Suppose that $f(z)$ has a critical point \bar{z} where $f'(\bar{z}) = 0$. Deforming the contour such that it passes through \bar{z} with constant $\text{Im}f(z)$ along the contour locally around \bar{z} , we write

$$I(\sigma) = e^{-\sigma \bar{f}} \int_{-\infty}^{\infty} dt \frac{dz}{dt} g(z(t)) e^{-\sigma \frac{1}{2} \bar{f}'' t^2 + \dots}, \quad (3.2)$$

where t parameterises the contour such that $z(0) = \bar{z}$ and $\bar{f} = f(\bar{z})$. Lastly, $\bar{f}'' = f''(\bar{z})$ is the second derivative w.r.t. t at the critical point. If $g(\bar{z})$ is finite, we simply extract it from the integral and perform the Gaussian integral. In the following, we shall be interested in the case when g has *an odd zero or an even pole* around the saddle point. In the case of a zero with

$$g \stackrel{t \rightarrow 0}{\simeq} A t^{2m-1} + B t^{2m} + \dots, \tag{3.3}$$

we just include it in the Gaussian integration and get

$$\begin{aligned} I(\sigma) &= e^{-\sigma \bar{f}} \int_{-\infty}^{\infty} dt [A t^{2m-1} z'(0) + (B z'(0) + A z''(0)) t^{2m} + \dots] e^{-\sigma \frac{1}{2} \bar{f}'' t^2 + \dots} \\ &= \sqrt{2\pi} [B z'(0) + A z''(0)] e^{-\sigma \bar{f}} (2m-1)!! (\sigma \bar{f}'')^{-m-\frac{1}{2}} + \dots. \end{aligned} \tag{3.4}$$

In the case of a pole with

$$g \stackrel{t \rightarrow 0}{\simeq} A t^{-2m} + \dots, \tag{3.5}$$

we compute the finite quantity⁸

$$\begin{aligned} \frac{d^m}{d\sigma^m} [e^{\sigma \bar{f}} I(\sigma)] &= \left(-\frac{\bar{f}''}{2}\right)^m \int_{-\infty}^{\infty} dt \frac{dz}{dt} g(z(t)) t^{2m} e^{-\sigma \frac{1}{2} \bar{f}'' t^2 + \dots} \\ &= \left(-\frac{\bar{f}''}{2}\right)^m z'(0) A \sqrt{\frac{2\pi}{\sigma \bar{f}''}} + \dots. \end{aligned} \tag{3.6}$$

Integrating back in σ gives then

$$I(\sigma) = \pi A z'(0) e^{-\sigma \bar{f}} \frac{(-1)^m}{\Gamma(m + \frac{1}{2})} \left(\frac{\sigma \bar{f}''}{2}\right)^{m-\frac{1}{2}} + \dots = \sqrt{2\pi} A z'(0) e^{-\sigma \bar{f}} (-1)^m \frac{(\sigma \bar{f}'')^{m-\frac{1}{2}}}{(2m-1)!!} + \dots. \tag{3.7}$$

Revisiting $\langle \mathcal{W} \rangle$ at strong coupling. These formulas may be applied to contour integrals involving Wilson loops and higher order resolvents. Let us illustrate this once again in the case of the simple Wilson loop (1.1). The planar contribution in (2.12) has $\sigma = \sqrt{\lambda}$, $f(z) = -\frac{1}{2}(z + 1/z)$ and $g(z) = \frac{1}{z}(1 - \frac{1}{z^2})$. The dominant contribution at large λ comes from the saddle point at $z = 1$ which is a zero of $g(z)$ of linear order. The parametrization is $z(t) = e^{it}$ thus $\bar{f}'' = 1$. Expanding $g(z)$ around the zero and taking the first even term gives (3.3) with $A = 2i$ and $B = 4$ and $m = 1$. Evaluation of (3.4) gives then

$$\langle \mathcal{W} \rangle_0 = \sqrt{\frac{2}{\pi}} \lambda^{-3/4} e^{\sqrt{\lambda}} + \dots, \tag{3.8}$$

in agreement with (1.3). All the higher genus corrections have even poles at $z = \pm 1$. Again, the leading contribution comes from $z = 1$ and may be computed using (3.7). For

⁸This is equivalent to an implicit integration by parts. In both cases we have to be careful about the poles at $t = 0$ since a non-zero residue for the pole causes a discontinuity in the contour. In our discussion, this will not matter because topological recursion ensures that this residue is always zero, when computing expectation values of functions of the matrix model variable. See last section for examples and appendix C for general details.

instance, at genus one we have

$$g(z) = \frac{1}{2\pi i} \frac{z^3}{(z^2 - 1)^4}, \quad g(z(t)) = -\frac{i}{32\pi} \frac{1}{t^4} + \dots \rightarrow A = -\frac{i}{32\pi}, \quad m = 2, \quad (3.9)$$

and

$$\langle W \rangle_1 = \sqrt{2\pi} \frac{-i}{32\pi} i e^{\sqrt{\lambda}} \frac{(\sqrt{\lambda})^{2-\frac{1}{2}}}{3!!} + \dots = \frac{\lambda^{3/4}}{48\sqrt{2\pi}} e^{\sqrt{\lambda}} + \dots. \quad (3.10)$$

Similarly at genus 2 and higher we can check that this procedure reproduces the expansion (1.3). Higher order corrections in $1/\sqrt{\lambda}$ may also be computed in the same way just by doing Gaussian integration with more accuracy. For instance, we know that (up to exponentially suppressed terms)

$$\langle W \rangle_1 = \frac{\lambda}{48} I_2(\sqrt{\lambda}) = e^{\sqrt{\lambda}} \left(\frac{\lambda^{3/4}}{48\sqrt{2\pi}} - \frac{5\lambda^{1/4}}{128\sqrt{2\pi}} + \dots \right) \quad (3.11)$$

and we reproduce this expansion by the convenient change of parametrization

$$z + \frac{1}{z} = 2 - u^2. \quad (3.12)$$

Using again $z = e^{it}$, this gives $u = 2 \sin \frac{t}{2}$ and one gets

$$\begin{aligned} \langle W \rangle_1 &= \frac{e^{\sqrt{\lambda}}}{\pi} \int_{-\infty}^{\infty} du \frac{1}{u^4(4-u^2)^{5/2}} e^{-\frac{\sqrt{\lambda}}{2}u^2} = \frac{e^{\sqrt{\lambda}}}{\pi} \int_{-\infty}^{\infty} du \left(\frac{1}{32u^4} + \frac{5}{256u^2} + \dots \right) e^{-\frac{\sqrt{\lambda}}{2}u^2} \\ &= \frac{e^{\sqrt{\lambda}}}{\pi} \pi \left[\frac{1}{32} \frac{1}{\Gamma\left(2 + \frac{1}{2}\right)} \left(\frac{\sqrt{\lambda}}{2}\right)^{2-1/2} - \frac{5}{256} \frac{1}{\Gamma\left(1 + \frac{1}{2}\right)} \left(\frac{\sqrt{\lambda}}{2}\right)^{1-1/2} + \dots \right] \\ &= \frac{\lambda^{3/4}}{48\sqrt{2\pi}} - \frac{5\lambda^{1/4}}{128\sqrt{2\pi}} + \dots, \end{aligned} \quad (3.13)$$

in agreement with (3.11).

Remark. The integrals in (3.13) are apparently divergent, even in Cauchy prescription. Actually, they are evaluated by formulas as (3.7) that hide their original definition as finite contour integrals.

4 Topological recursion for dominant strong coupling poles

We now look for a simplification of topological recursion (2.8) based on considering the principal part of resolvents at $z_i = 1$, i.e. the terms that dominate at strong coupling. Let us denote the highest pole part by $\hat{\omega}_{n,g}$. Introducing $\Delta_i = z_i - 1$, the resolvents in (2.10)

reduce to the compact expressions

$$\begin{aligned}
 \hat{\omega}_{1,1}(\Delta) &= \frac{1}{16\Delta^4}, \\
 \hat{\omega}_{3,0}(\Delta_1, \Delta_2, \Delta_3) &= -\frac{1}{2\Delta_1^2\Delta_2^2\Delta_3^2}, \\
 \hat{\omega}_{2,1}(\Delta_1, \Delta_2) &= \frac{5\Delta_1^4 + 3\Delta_1^2\Delta_2^2 + 5\Delta_2^4}{32\Delta_1^6\Delta_2^6}, \\
 \hat{\omega}_{1,2}(\Delta) &= -\frac{105}{1024\Delta^{10}}.
 \end{aligned} \tag{4.1}$$

The (total) degree of the pole terms is $6(g-1) + 4n$. In general, only even powers of Δ_i appear. If such an Ansatz is plugged into the topological recursion, one can compute the associated resolvent and project onto the maximal pole part. For instance, the last four resolvents in (2.9) become, after projection,

$$\begin{aligned}
 \hat{\omega}_{4,0}(\Delta_1, \Delta_2, \Delta_3, \Delta_4) &= -\frac{3(\Delta_1^2\Delta_2^2\Delta_3^2 + \Delta_1^2\Delta_2^2\Delta_4^2 + \Delta_1^2\Delta_3^2\Delta_4^2 + \Delta_2^2\Delta_3^2\Delta_4^2)}{4\Delta_1^4\Delta_2^4\Delta_3^4\Delta_4^4}, \\
 \hat{\omega}_{3,1}(\Delta_1, \Delta_2, \Delta_3) &= \frac{1}{64\Delta_1^8\Delta_2^8\Delta_3^8} \left[35\Delta_2^6\Delta_3^6 + 30\Delta_1^2\Delta_2^4\Delta_3^4(\Delta_2^2 + \Delta_3^2) \right. \\
 &\quad \left. + 5\Delta_1^6(7\Delta_2^6 + 6\Delta_2^4\Delta_3^2 + 6\Delta_2^2\Delta_3^4 + 7\Delta_3^6) + 6\Delta_1^4(5\Delta_2^6\Delta_3^2 + 3\Delta_2^4\Delta_3^4 + 5\Delta_2^2\Delta_3^6) \right], \\
 \hat{\omega}_{2,2}(\Delta_1, \Delta_2) &= -\frac{35(33\Delta_1^{10} + 27\Delta_1^8\Delta_2^2 + 29\Delta_1^6\Delta_2^4 + 29\Delta_1^4\Delta_2^6 + 27\Delta_1^2\Delta_2^8 + 33\Delta_2^{10})}{2048\Delta_1^{12}\Delta_2^{12}}, \\
 \hat{\omega}_{1,3}(\Delta) &= \frac{25025}{32768\Delta^{16}},
 \end{aligned} \tag{4.2}$$

which are very compact expressions, compared with the full resolvents. Being symmetric functions, we can further simplify in terms of elementary symmetric polynomials

$$e_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} \cdots x_{i_k}, \tag{4.3}$$

where $x = \frac{1}{\Delta^2}$. One finds indeed the concise expressions

$$\begin{aligned}
 \hat{\omega}_{4,0} &= -\frac{3}{4}e_1e_4, \\
 \hat{\omega}_{3,1} &= \frac{35}{64}e_1^3e_3 - \frac{75}{64}e_1e_2e_3 + \frac{33}{64}e_3^2, \\
 \hat{\omega}_{2,2} &= -\frac{1155}{2048}e_1^5e_2 + \frac{2415}{1024}e_1^3e_2^2 - \frac{3955}{2048}e_1e_2^3.
 \end{aligned} \tag{4.4}$$

Further results are collected in appendix C.3.

Remark. Of course, the key point of the method is to use $\hat{\omega}$ projected resolvent in the topological recursion and never using the full ω 's.

5 Large tension analysis of coincident Wilson loops

As a first application, we consider the large tension limit of $\langle \mathcal{W}^n \rangle$ and, in particular, the ratio (1.5). As an illustration of the our strategy, we will begin with the doubly coincident Wilson loop, i.e. the case $n = 2$. Later, we shall extend the analysis to a generic number n of coinciding loops. For $n = 2$, the $1/N$ expansion of $\langle \mathcal{W}^2 \rangle$ has been considered in [2, 31, 35, 36] and its first terms read

$$\begin{aligned} \frac{1}{N^2} \langle \mathcal{W}^2 \rangle &= \left(\frac{2}{\sqrt{\lambda}} I_1 \right)^2 + \frac{\sqrt{\lambda}}{2N^2} \left(I_0 I_1 + \frac{1}{6} I_1 I_2 \right) \\ &+ \frac{1}{N^4} \left[\frac{37\lambda^2}{2304} I_0^2 - \frac{\sqrt{\lambda}(24 + 131\lambda)}{2880} I_0 I_1 + \frac{192 + 332\lambda + 185\lambda^2}{11520} I_1^2 \right] + \dots, \end{aligned} \tag{5.1}$$

where $I_n \equiv I_n(\sqrt{\lambda})$. The associated connected correlator is

$$\langle \mathcal{W}^2 \rangle_c = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \langle \mathcal{W}^2 \rangle_{c,g} = \frac{\sqrt{\lambda}}{2} I_0 I_1 + \frac{1}{N^2} \left[\frac{\lambda^2}{64} I_0^2 - \frac{\lambda^{3/2}}{24} I_0 I_1 + \frac{\lambda(4 + 3\lambda)}{192} I_1^2 \right] + \dots \tag{5.2}$$

Expanding at large λ and keeping the leading contribution at each order in $1/N$ gives

$$\langle \mathcal{W}^2 \rangle_{c,0} = \frac{1}{4\pi} e^{2\sqrt{\lambda}} \left(1 - \frac{1}{4\sqrt{\lambda}} + \dots \right), \quad \langle \mathcal{W}^2 \rangle_{c,1} = \frac{\lambda^{3/2}}{64\pi} e^{2\sqrt{\lambda}} \left(1 - \frac{19}{12\sqrt{\lambda}} + \dots \right). \tag{5.3}$$

Let us show how these contributions can be easily recovered from the “maximal poles” topological recursion. We start from the 2-point formula

$$\langle \mathcal{W}^2 \rangle_{c,g}^{\lambda \gg 1} = \frac{1}{(2\pi i)^2} \oint \hat{\omega}_{2,g}(z_1, z_2) e^{\frac{\sqrt{\lambda}}{2}(z_1+1/z_1)} e^{\frac{\sqrt{\lambda}}{2}(z_2+1/z_2)}. \tag{5.4}$$

The genus 0 contribution is special being related to the universal Bargmann kernel and having no poles at $z_{1,2} = 1$. It is

$$\langle \mathcal{W}^2 \rangle_{c,0} = \oint \frac{dz_1 dz_2}{2\pi i 2\pi i} \frac{1}{(z_1 - z_2)^2} e^{\frac{\sqrt{\lambda}}{2}(z_1+1/z_1)} e^{\frac{\sqrt{\lambda}}{2}(z_2+1/z_2)}. \tag{5.5}$$

After changing variable $z_2 = w z_1$, we get

$$\langle \mathcal{W}^2 \rangle_{c,0} = \oint \frac{dz_1 dw}{2\pi i 2\pi i} \frac{1}{z_1(1-w)^2} e^{\frac{\sqrt{\lambda}}{2} \left[(1+w)z_1 + (1+\frac{1}{w})\frac{1}{z_1} \right]}. \tag{5.6}$$

The integral over z_1 gives the Bessel function I_0 and we obtain

$$\begin{aligned} \langle \mathcal{W}^2 \rangle_{c,0} &= \oint \frac{dw}{2\pi i} \frac{1}{(1-w)^2} I_0 \left(\frac{1+w}{\sqrt{w}} \sqrt{\lambda} \right) = \text{Res}_{w=0} \sum_{n=0}^{\infty} \frac{1}{n!^2} \frac{1}{(1-w)^2} \left(\frac{1+w}{2\sqrt{w}} \sqrt{\lambda} \right)^{2n} \\ &= \sum_{n=1}^{\infty} \frac{\Gamma\left(n + \frac{1}{2}\right)}{2\sqrt{\pi} \Gamma(n) \Gamma(n+1)^2} \lambda^n. \end{aligned} \tag{5.7}$$

On the other hand, from [37] (see paragraphs 13.72–73) we have

$$I_\mu(x)I_\nu(x) = \frac{2}{\pi} \int_0^{\pi/2} d\theta I_{\mu+\nu}(2x \cos \theta) \cos((\mu - \nu)\theta). \quad (5.8)$$

Hence, with $\mu = \nu = 0$,

$$I_0(x)^2 = \frac{2}{\pi} \int_0^{\pi/2} I_0(2x \cos \theta) d\theta = \frac{2}{\pi} \int_0^{\pi/2} d\theta \sum_{n=0}^{\infty} \frac{1}{n!^2} (x \cos \theta)^{2n} = \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+1)^3} x^{2n}. \quad (5.9)$$

Finally,

$$\frac{x}{2} I_0(x) I_1(x) = \frac{x}{4} \frac{d}{dx} I_0(x)^2 = \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{1}{2}\right)}{2\sqrt{\pi}\Gamma(n)\Gamma(n+1)^2} x^{2n} \quad (5.10)$$

in agreement with the r.h.s. of (5.7), after setting $x = \sqrt{\lambda}$.

Starting at genus 1 we can apply the formula (3.7) for the factorized poles. For instance, the first correction is

$$\begin{aligned} \langle \mathcal{W}^2 \rangle_{c,1}^{\lambda \gg 1} = & \oint \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{1}{32} \left[\frac{5}{(z_1 - 1)^2 (z_2 - 1)^6} + \frac{3}{(z_1 - 1)^4 (z_2 - 1)^4} \right. \\ & \left. + \frac{5}{(z_1 - 1)^6 (z_2 - 1)^2} \right] e^{\frac{\sqrt{\lambda}}{2}(z_1+1/z_1)} e^{\frac{\sqrt{\lambda}}{2}(z_2+1/z_2)} \stackrel{(3.7)}{\rightarrow} e^{2\sqrt{\lambda}} \left[\frac{5}{32}(h_1 h_3 + h_3 h_1) + \frac{3}{32} h_2^2 \right], \end{aligned} \quad (5.11)$$

where the numerical constants h_m are

$$h_m = \frac{(-1)^m}{\sqrt{2\pi}} \frac{\lambda^{\frac{2m-1}{4}}}{(2m-1)!}. \quad (5.12)$$

Replacing (5.12) in (5.11) reproduces the leading term in the second expression in (5.3).

Extension to $\langle \mathcal{W}^n \rangle$ and high order calculation. Similarly to (5.11), we can exploit the resolvents in (4.1) and (4.2) (together with other ones in appendix C) to evaluate the saddle point integrals needed to compute $\langle \mathcal{W}^n \rangle$ at high order in the genus expansion. Remarkably, this can be done for a generic n . To this aim, we introduce the variable

$$\xi = \frac{\lambda^{\frac{3}{2}}}{8N^2} = \frac{\pi g_s^2}{T}, \quad (5.13)$$

and the connected correlators

$$\begin{aligned} \langle \mathcal{W}^2 \rangle_c &= -\langle \mathcal{W} \rangle^2 + \langle \mathcal{W}^2 \rangle, & \langle \mathcal{W}^3 \rangle_c &= 2\langle \mathcal{W} \rangle^3 - 3\langle \mathcal{W} \rangle \langle \mathcal{W}^2 \rangle + \langle \mathcal{W}^3 \rangle, \\ \langle \mathcal{W}^4 \rangle_c &= -6\langle \mathcal{W} \rangle^4 + 12\langle \mathcal{W} \rangle^2 \langle \mathcal{W}^2 \rangle - 3\langle \mathcal{W}^2 \rangle^2 - 4\langle \mathcal{W} \rangle \langle \mathcal{W}^3 \rangle + \langle \mathcal{W}^4 \rangle, \text{ etc.} \end{aligned} \quad (5.14)$$

Normalizing by suitable powers of the simple Wilson loop, we obtain the following results valid up to order $\mathcal{O}(\xi^8)$:

$$\begin{aligned}
 \frac{\langle \mathcal{W}^2 \rangle_c}{\langle \mathcal{W} \rangle^2} &\stackrel{T \gg 1}{\cong} \xi + \frac{\xi^2}{3} + \frac{\xi^3}{15} + \frac{\xi^4}{105} + \frac{\xi^5}{945} + \frac{\xi^6}{10395} + \frac{\xi^7}{135135} + \frac{\xi^8}{2027025} + \dots, \\
 \frac{\langle \mathcal{W}^3 \rangle_c}{\langle \mathcal{W} \rangle^3} &\stackrel{T \gg 1}{\cong} 4\xi^2 + \frac{14\xi^3}{3} + \frac{16\xi^4}{5} + \frac{169\xi^5}{105} + \frac{3046\xi^6}{4725} + \frac{532\xi^7}{2475} + \frac{290492\xi^8}{4729725} + \dots, \\
 \frac{\langle \mathcal{W}^4 \rangle_c}{\langle \mathcal{W} \rangle^4} &\stackrel{T \gg 1}{\cong} 32\xi^3 + 84\xi^4 + \frac{1936\xi^5}{15} + \frac{138722\xi^6}{945} + \frac{635672\xi^7}{4725} + \frac{1078888\xi^8}{10395} + \dots, \\
 \frac{\langle \mathcal{W}^5 \rangle_c}{\langle \mathcal{W} \rangle^5} &\stackrel{T \gg 1}{\cong} 400\xi^4 + \frac{5776\xi^5}{3} + 5420\xi^6 + \frac{2145776\xi^7}{189} + \frac{7815796\xi^8}{405} + \dots, \\
 \frac{\langle \mathcal{W}^6 \rangle_c}{\langle \mathcal{W} \rangle^6} &\stackrel{T \gg 1}{\cong} 6912\xi^5 + 54080\xi^6 + 247504\xi^7 + \frac{53289280\xi^8}{63} + \dots, \\
 \frac{\langle \mathcal{W}^7 \rangle_c}{\langle \mathcal{W} \rangle^7} &\stackrel{T \gg 1}{\cong} 153664\xi^6 + 1804128\xi^7 + \frac{185939824\xi^8}{15} + \dots, \\
 \frac{\langle \mathcal{W}^8 \rangle_c}{\langle \mathcal{W} \rangle^8} &\stackrel{T \gg 1}{\cong} 4194304\xi^7 + \frac{209380864\xi^8}{3} + \dots, \\
 \frac{\langle \mathcal{W}^9 \rangle_c}{\langle \mathcal{W} \rangle^9} &\stackrel{T \gg 1}{\cong} 136048896\xi^8 + \dots.
 \end{aligned} \tag{5.15}$$

From connected correlators we obtain correlators of n coincident Wilson loops using the combinatorial formula

$$\frac{\langle \mathcal{W}^n \rangle}{\langle \mathcal{W} \rangle^n} = 1 + \sum_{k < n} \sum_{\pi \in P(k, n)} \frac{n!}{S(\pi)(n - k - |\pi|)! \prod_{p \in \pi} (p + 1)!} \prod_{p \in \pi} \frac{\langle \mathcal{W}^{p+1} \rangle_c}{\langle \mathcal{W} \rangle^{p+1}}, \tag{5.16}$$

where $P(k, n)$ is the set of integer partitions π of k satisfying $k + |\pi| \leq n$ where $|\pi|$ is the number of elements of π , and $S(\pi)$ is the symmetry factor of partition π given by products of $m!$ for each group of m equal elements in π . This expression follows from the fact that $\langle \mathcal{W}^n \rangle$ can be written as a sum over just the partitions of n . Since we divide by $\langle \mathcal{W} \rangle^n$, all the parts of a given partition that are 1 disappear, leaving the partitions of n with every part at least 2. Such partitions can be seen to be in one to one correspondence with the partitions of integers $k \leq n$ such that $k + |\pi| \leq n$, giving the expression above.

Since on general grounds $\frac{\langle \mathcal{W}^m \rangle_c}{\langle \mathcal{W} \rangle^m} = \mathcal{O}(\xi^{m-1})$, to obtain the expansion of $\frac{\langle \mathcal{W}^n \rangle}{\langle \mathcal{W} \rangle^n}$ up to ξ^m we can restrict the sum in (5.16) to $k < m$. Furthermore, taking into account that a part p enters the above expression as $\frac{\langle \mathcal{W}^{p+1} \rangle_c}{\langle \mathcal{W} \rangle^{p+1}}$, we need the terms corresponding to all the partitions of m to obtain the result up to ξ^m . Let's consider a couple of examples. For $m = 1$ the only possible partition is 1 and we obtain

$$\frac{\langle \mathcal{W}^n \rangle}{\langle \mathcal{W} \rangle^n} = 1 + \frac{n!}{(n-2)!2!} \frac{\langle \mathcal{W}^2 \rangle_c}{\langle \mathcal{W} \rangle^2} + \mathcal{O}(\xi^2) = 1 + \frac{n(n-1)}{2} \frac{\langle \mathcal{W}^2 \rangle_c}{\langle \mathcal{W} \rangle^2} + \mathcal{O}(\xi^2). \tag{5.17}$$

For $m = 2$ we have three partitions, i.e. 1 and 2 and (1, 1). The last one has a symmetry factor of two. So we obtain

$$\begin{aligned} \frac{\langle \mathcal{W}^n \rangle}{\langle \mathcal{W} \rangle^n} &= 1 + \frac{n!}{(n-2)!2!} \frac{\langle \mathcal{W}^2 \rangle_c}{\langle \mathcal{W} \rangle^2} + \frac{n!}{(n-3)!3!} \frac{\langle \mathcal{W}^3 \rangle_c}{\langle \mathcal{W} \rangle^3} + \frac{n!}{(n-4)!2!2!} \left(\frac{\langle \mathcal{W}^2 \rangle_c}{\langle \mathcal{W} \rangle^2} \right)^2 + O(\xi^3) \\ &= 1 + \frac{n(n-1)}{2} \frac{\langle \mathcal{W}^2 \rangle_c}{\langle \mathcal{W} \rangle^2} + \frac{n(n-1)(n-2)}{6} \frac{\langle \mathcal{W}^3 \rangle_c}{\langle \mathcal{W} \rangle^3} + \frac{n(n-1)(n-2)(n-3)}{8} \left(\frac{\langle \mathcal{W}^2 \rangle_c}{\langle \mathcal{W} \rangle^2} \right)^2 + O(\xi^3). \end{aligned} \quad (5.18)$$

In a similar way, to obtain the result up to ξ^3 we will have to add all terms corresponding to the partitions of three to the above results and so on. We now have all the ingredients needed to evaluate the above expression to ξ^8 . The final result is, cf. (1.5)

$$\begin{aligned} W_n(\xi) &= 1 + \frac{1}{2}(n-1)n\xi + \frac{1}{24}n(3n^3 - 2n^2 - 11n + 10)\xi^2 \\ &+ \frac{n}{720}(15n^5 + 15n^4 - 105n^3 - 535n^2 + 1674n - 1064)\xi^3 \\ &+ \frac{n}{40320}(105n^7 + 420n^6 - 70n^5 - 13440n^4 - 44303n^3 + 401772n^2 - 731028n + 386544)\xi^4 \\ &+ \frac{n}{241920}(63n^9 + 525n^8 + 1890n^7 - 10710n^6 - 177401n^5 - 169715n^4 + 8836872n^3 \\ &+ 33525316n^2 + 47031760n - 21987968)\xi^5 \\ &+ \frac{n}{159667200}(3465n^{11} + 48510n^{10} + 363825n^9 + 596750n^8 - 23242065n^7 - 242099550n^6 + 717147915n^5 \\ &+ 18424615770n^4 - 131848499156n^3 + 354391190648n^2 - 421025682592n + 179605556480)\xi^6 \\ &+ \frac{n}{4151347200}(6435n^{13} + 135135n^{12} + 1606605n^{11} + 10385375n^{10} - 33778745n^9 - 1632646015n^8 \\ &- 11172952505n^7 + 124368186085n^6 + 1239293411642n^5 - 17142200059556n^4 + 77299998069320n^3 \\ &- 168644525652096n^2 + 177551331490176n - 70415438201856)\xi^7 \\ &+ \frac{n}{1394852659200}(135135n^{15} + 3963960n^{14} + 66846780n^{13} + 734894160n^{12} + 3276739466n^{11} \\ &- 61066061056n^{10} - 1348672905580n^9 - 4090445158800n^8 + 176081159789535n^7 + 702493835315272n^6 \\ &- 24840194890564872n^5 + 176744402329465152n^4 - 616283230677646864n^3 + 1159320084247595136n^2 \\ &- 1109932678089579264n + 414118538187171840)\xi^8 + \dots, \end{aligned} \quad (5.19)$$

where the large tension limit is understood. This is the extension to order ξ^8 of the cubic result in eq. (1.17) of [2]. The special cases $n = 2$ and $n = 3$ are

$$\begin{aligned} W_2(\xi) &= 1 + \xi + \frac{\xi^2}{3} + \frac{\xi^3}{15} + \frac{\xi^4}{105} + \frac{\xi^5}{945} + \frac{\xi^6}{10395} + \frac{\xi^7}{135135} + \frac{\xi^8}{2027025} + \dots, \\ W_3(\xi) &= 1 + 3\xi + 5\xi^2 + \frac{73\xi^3}{15} + \frac{113\xi^4}{35} + \frac{508\xi^5}{315} + \frac{33521\xi^6}{51975} + \frac{16139\xi^7}{75075} + \frac{8803\xi^8}{143325} + \dots, \end{aligned} \quad (5.20)$$

and agree with the exact expressions [2],

$$\begin{aligned}
 W_2(\xi) &= 1 + e^{\frac{\xi}{2}} \sqrt{\frac{\pi \xi}{2}} \operatorname{erf} \left(\sqrt{\frac{\xi}{2}} \right), \\
 W_3(\xi) &= 1 + 3e^{\frac{\xi}{2}} \sqrt{\frac{\pi \xi}{2}} \operatorname{erf} \left(\sqrt{\frac{\xi}{2}} \right) + \frac{4\pi}{3\sqrt{3}} \xi e^{2\xi} \left[1 - 12 T \left(\sqrt{3\xi}, \frac{1}{\sqrt{3}} \right) \right],
 \end{aligned} \tag{5.21}$$

where

$$T(h, a) = \frac{1}{2\pi} \int_0^a \frac{dx}{1+x^2} e^{-\frac{h^2}{2}(1+x^2)}, \tag{5.22}$$

is the Owen T-function. The coefficient of ξ^k is a polynomial in n of degree $2k$. A remarkable simplification is achieved by writing (5.19) in exponential form

$$\log W_n(\xi) = \sum_{k=1}^{\infty} n(n-1) P_k(n) \xi^k, \tag{5.23}$$

since P_k turns out to be a polynomial of (approximately half) degree $k-1$. Explicitly, one finds

$$\begin{aligned}
 P_1 &= \frac{1}{2}, & P_2 &= -\frac{5}{12} + \frac{n}{6}, & P_3 &= \frac{133}{90} - \frac{19n}{18} + \frac{n^2}{6}, \\
 P_4 &= -\frac{8053}{840} + \frac{3373n}{360} - \frac{67n^2}{24} + \frac{n^3}{4}, \\
 P_5 &= \frac{171781}{1890} - \frac{34313n}{315} + \frac{821n^2}{18} - \frac{118n^3}{15} + \frac{7n^4}{15}, \\
 P_6 &= -\frac{20045263}{17820} + \frac{7383058n}{4725} - \frac{773086n^2}{945} + \frac{6047n^3}{30} - \frac{419n^4}{18} + n^5, \\
 P_7 &= \frac{694596731}{40950} - \frac{1177098997n}{44550} + \frac{230808223n^2}{14175} - \frac{4806629n^3}{945} + \frac{534613n^4}{630} - \frac{3001n^5}{42} + \frac{33n^6}{14}, \\
 P_8 &= -\frac{45388623451}{152880} + \frac{2353546004957n}{4633200} - \frac{177458006053n^2}{498960} + \frac{2001333893n^3}{15120} - \frac{183361073n^4}{6480} \\
 &\quad + \frac{2499191n^5}{720} - \frac{3609n^6}{16} + \frac{143n^7}{24},
 \end{aligned} \tag{5.24}$$

with leading terms at large n following the pattern

$$P_k = \frac{4^{k-1} \Gamma\left(k - \frac{1}{2}\right)}{k\sqrt{\pi} \Gamma(k+2)} n^{k-1} + \frac{4^{k-2}}{3k} \left[1 - \frac{6(k-1)(2k-3)\Gamma\left(k - \frac{1}{2}\right)}{\sqrt{\pi}\Gamma(k+2)} \right] n^{k-2} + \dots \tag{5.25}$$

5.1 Solution by Toda recursion

The genus expansion of (1.3) is efficiently computed by exploiting the Toda integrability of the 1-matrix Hermitian Gaussian model [23]. In general, correlators in this model are constrained by integrable differential equations [20–22] that in Gaussian case take the Toda form [19]. Notice that in [23] the matrix model measure is $\exp\left(-\frac{1}{2} \operatorname{tr} \widetilde{M}^2\right)$ without explicit

N factor. This will be the convention throughout this section. After defining the connected correlators

$$e(x_1, \dots, x_k) = \left\langle \text{tr } e^{x_1 \tilde{M}} \dots \text{tr } e^{x_k \tilde{M}} \right\rangle_c, \quad (5.26)$$

one has

$$e_{N+1}(x) + e_{N-1}(x) = 2e_N(x) + \frac{x^2}{N}e_N(x), \quad (5.27)$$

$$e_{N+1}(x, y) + e_{N-1}(x, y) = 2e_N(x, y) + \frac{(x+y)^2}{N}e_N(x, y) - \frac{x^2 y^2}{N^2}e_N(x)e_N(y). \quad (5.28)$$

The general structure is

$$e_{N+1}(x_1, \dots, x_k) + e_{N-1}(x_1, \dots, x_k) - \left[2 + \frac{1}{N}(x_1 + \dots + x_k)^2 \right] e_N(x_1, \dots, x_k) = g_N(x_1, \dots, x_k), \quad (5.29)$$

where g_N may be read from the non-leading terms of the cumulant expansion of $\langle X_1 \dots X_k \rangle_c$ and replacing

$$\langle X_{I_1} \dots X_{I_p} \rangle \rightarrow \frac{(x_{I_1} + \dots + x_{I_p})^2}{N} e_N(x_{I_1}, \dots, x_{I_p}). \quad (5.30)$$

Here we are interested in the specialization to $x_i = \sqrt{\frac{\lambda}{4N}}$. Hence, defining

$$e^{(k)}(\lambda, N) = e_N \left(\underbrace{\sqrt{\frac{\lambda}{4N}}, \dots, \sqrt{\frac{\lambda}{4N}}}_{k \text{ terms}} \right), \quad (5.31)$$

we have the equations

$$e^{(k)} \left(\frac{N+1}{N} \lambda, N+1 \right) + e^{(k)} \left(\frac{N-1}{N} \lambda, N-1 \right) - \left(2 + \frac{k^2 \lambda}{4N^2} \right) e^{(k)}(\lambda, N) = g^{(k)}(\lambda, N), \quad (5.32)$$

where $g^{(k)}(\lambda, N)$ is obtained from the non-leading terms of the cumulant expansion of $\langle X^k \rangle_c$ and replacing

$$\langle X^p \rangle \rightarrow \frac{p^2 \lambda}{4N^2} e^{(p)}(\lambda, N). \quad (5.33)$$

The explicit coefficients of the cumulant expansion of $\langle X^p \rangle_c$ may be expressed in terms of integer partitions $\pi = (1^{m_1} 2^{m_2} \dots)$ of p

$$\langle X^p \rangle_c = \sum_{\pi \in \mathcal{P}(p)} (-1)^{|\pi|-1} (|\pi|-1)! \sigma(\pi) \prod_r \langle X^r \rangle_c^{m_r}, \quad \sigma(\pi) = \frac{p!}{\prod_r (r!)^{m_r} m_r!}. \quad (5.34)$$

Hence, the equations are

$$\begin{aligned} & e^{(k)} \left(\frac{N+1}{N} \lambda, N+1 \right) + e^{(k)} \left(\frac{N-1}{N} \lambda, N-1 \right) - \left(2 + \frac{k^2 \lambda}{4N^2} \right) e^{(k)}(\lambda, N) \\ &= \sum_{\substack{\pi \in \mathcal{P}(p) \\ \pi \neq (p)}} (-1)^{|\pi|-1} (|\pi|-1)! \sigma(\pi) \prod_r \left(\frac{r^2 \lambda}{4N^2} e^{(r)}(\lambda, N) \right)^{m_r}. \end{aligned} \quad (5.35)$$

The large tension scaling Ansatz is

$$e^{(k)}(\lambda, N) = e^{k\sqrt{\lambda}} F_k(\xi), \quad \xi = \frac{\lambda^{3/2}}{8N^2}. \quad (5.36)$$

Replacing in the Toda equations gives

$$\begin{aligned} & e^{k\sqrt{\lambda}} \left[\frac{k}{12\xi} \xi^{1/3} (k^3\xi - 6) F_k(\xi) - k \xi^{4/3} F'_k(\xi) \right] N^{-4/3} + \dots \\ &= \sum_{\substack{\pi \in \mathcal{P}(p) \\ \pi \neq (p)}} (-1)^{|\pi|-1} (|\pi| - 1)! \sigma(\pi) \prod_r \left(r^2 \xi^{2/3} N^{-2/3} e^{r\sqrt{\lambda}} F_r(\xi) \right)^{m_r}. \end{aligned} \quad (5.37)$$

The only partitions that may give a contribution have $|\pi| = \sum m_r = 2$. One case is when k is even and then the partition is $\pi = \left(\frac{M}{2}, \frac{M}{2}\right)$, or when k is split into the sum of two different parts $\pi = (q, (k - q))$ with $q \neq k/2$.⁹ Denoting by an apex such partitions, we have (using $\sum_r r m_r = k$)

$$\frac{k}{12\xi} (k^3\xi - 6) F_k(\xi) - k F'_k(\xi) = \sum_{\substack{\pi \in \mathcal{P}(p) \\ \pi \neq (p)}} (-1)^{|\pi|-1} (|\pi| - 1)! \sigma(\pi) \prod_r \left(r^2 F_r(\xi) \right)^{m_r}. \quad (5.38)$$

Finally, evaluating the r.h.s. for the two relevant kinds of partitions, we obtain the differential equation

$$\frac{k}{12\xi} (k^3\xi - 6) F_k(\xi) - k F'_k(\xi) = -\frac{1}{2} \sum_{p=1}^{k-1} \binom{k}{p} p^2 (k-p)^2 F_p(\xi) F_{k-p}(\xi), \quad (5.39)$$

that we rearrange in the form

$$F'_k(\xi) + \frac{1}{12\xi} (6 - k^3\xi) F_k(\xi) = \frac{1}{2k} \sum_{p=1}^{k-1} \binom{k}{p} p^2 (k-p)^2 F_p(\xi) F_{k-p}(\xi). \quad (5.40)$$

The first instance $k = 1$ gives

$$F'_1(\xi) + \frac{1}{12\xi} (6 - \xi) F_1(\xi) = 0 \quad \rightarrow \quad F_1(\xi) = C \xi^{-1/2} e^{\frac{\xi}{12}}. \quad (5.41)$$

The constant is fixed by (1.3) and gives

$$F_1(\xi) = \frac{1}{2\sqrt{\pi}} \xi^{-1/2} e^{\frac{\xi}{12}}. \quad (5.42)$$

We shall be interested in the ratios

$$R_k(\xi) = \frac{F_k(\xi)}{F_1(\xi)^k}. \quad (5.43)$$

⁹Notice that distinct partitions are ordered.

They obey

$$R'_k(\xi) - \frac{k-1}{12\xi} [6 + k(k+1)\xi] R_k(\xi) = \frac{1}{2k} \sum_{p=1}^{k-1} \binom{k}{p} p^2 (k-p)^2 R_p(\xi) R_{k-p}(\xi). \quad (5.44)$$

We also know that $R_k(\xi) = \mathcal{O}(\xi^{k-1})$. This gives the integration constant and the explicit recurrence relation

$$R_1(\xi) = 1, \\ R_k(\xi) = e^{\frac{k(k^2-1)}{12}\xi} \xi^{\frac{k-1}{2}} \int_0^\xi dz e^{-\frac{k(k^2-1)}{12}z} z^{\frac{1-k}{2}} \frac{1}{2k} \sum_{p=1}^{k-1} \binom{k}{p} p^2 (k-p)^2 R_p(z) R_{k-p}(z). \quad (5.45)$$

This recursion provides the expressions in (5.15) to be plugged into (5.16) in order to compute the scaling functions $W_n(\xi)$. Just to give an example, using (5.45) one may easily extend the last line in (5.15) and find

$$\begin{aligned} \frac{\langle W^9 \rangle_c}{\langle W \rangle^9} \stackrel{T \gg 1}{\approx} & 136048896 \xi^8 + 3073838592 \xi^9 + \frac{203451048576}{5} \xi^{10} + \frac{14167449806208}{35} \xi^{11} \\ & + \frac{579258140455408}{175} \xi^{12} + \frac{26870941356966016}{1155} \xi^{13} + \frac{487595584858030932224}{3378375} \xi^{14} \\ & + \frac{1672326501977995232}{2079} \xi^{15} + \frac{140563211710013333686888}{34459425} \xi^{16} + \frac{24326465124436842115824}{1280125} \xi^{17} \\ & + \frac{13943786316178585925001407879336}{170147718365625} \xi^{18} + \frac{22599997060403576152832986605968}{68656096884375} \xi^{19} + \dots \end{aligned} \quad (5.46)$$

This allows to compute the polynomials in (5.24) at higher order. For instance

$$\begin{aligned} P_9 = & \frac{4199422654038881}{723647925} - \frac{1832589301073441}{170270100} n + \frac{3565245577877609}{425675250} n^2 - \frac{372258099643}{103950} n^3 \\ & + \frac{26122674017}{28350} n^4 - \frac{828493349}{5670} n^5 + \frac{3760909}{270} n^6 - \frac{19652}{27} n^7 + \frac{143}{9} n^8, \\ P_{10} = & -\frac{3360311016680854273}{27498621150} + \frac{21169992523320924583}{86837751000} n - \frac{118100410140883837}{567567000} n^2 \\ & + \frac{85134261154188137}{851350500} n^3 - \frac{27943401961331}{935550} n^4 + \frac{81563996453}{14175} n^5 - \frac{225150754}{315} n^6 \\ & + \frac{4961927}{90} n^7 - \frac{14341}{6} n^8 + \frac{221}{5} n^9, \end{aligned} \quad (5.47)$$

and so on. Further expressions of P_k for k up to 20 are collected in appendix B.

Remark. Of course, one can also use (5.45) without expanding. This gives exact expressions for R_k as iterated integrals. The first two cases are

$$\begin{aligned} R_2(\xi) &= \sqrt{\frac{\pi\xi}{2}} e^{\xi/2} \operatorname{erf}\left(\sqrt{\frac{\xi}{2}}\right), \\ R_3(\xi) &= -\frac{4\pi}{3\sqrt{3}} \xi e^{2\xi} \left[-1 + 12 T\left(\sqrt{3\xi}, \frac{1}{\sqrt{3}}\right) \right], \end{aligned} \quad (5.48)$$

where T is the Owen function, cf. (5.21). The expression for R_4 may be obtained by continuing the iteration but will involve integrals of the T function. A simple general feature of the functions R_k is that they are all entire in ξ . Hence, the radius of convergence of (5.19) is infinite for all n .

Remark. One has to keep in mind that Toda recursion methods are not suitable to treat insertions of local chiral operators, see the discussion in appendix A. In this case, one has to keep using topological recursion, as discussed in the next section.

6 Correlator of coincident Wilson loops and a chiral operator

In this section, we address the problem of computing the correlator between multiple coincident Wilson loops \mathcal{W}^n and a single trace chiral operator. In other words, we want to generalize (1.7) and prove (1.11), (1.12). To properly define chiral primaries let us recall that the $\frac{1}{2}$ -BPS Wilson loop, associated with $\text{tr}\left(e^{\frac{\sqrt{\lambda}}{2}M}\right)$ in the Gaussian matrix model, cf. (1.1), stands for the operator

$$\mathcal{W} = \text{tr} \mathcal{P} \exp \left\{ g_{\text{YM}} \int_{\mathcal{C}} d\sigma [i A_\mu(x) \dot{x}^\mu(\sigma) + R \Phi_1(x)] \right\}, \quad (6.1)$$

where \mathcal{C} is a circle of radius R (set to unity in the following), and Φ_1 is one of the six real scalars $\{\Phi_I\}_{I=1,\dots,6}$ in $\mathcal{N} = 4$ SYM. Single trace chiral operators take the general form $\mathcal{O}_J = \text{tr}(u_I \Phi_I(x))^J$ where u_I is a complex null 6-vector obeying $u^2 = 0$ [14]. The dependence of the correlator $\langle \mathcal{W} \mathcal{O}_J \rangle$ on u_I and the choice of coupling between the loop and the scalars factorizes and will be absorbed in the operator normalization [38]. With the same conventions as in [2], the matrix model representative for the chiral operator \mathcal{O}_J is

$$\mathcal{O}_J = \frac{N}{2} \left(\frac{\pi}{2} \right)^{J/2-1} : \text{tr} M^J :, \quad (6.2)$$

where normal ordering subtracts self-contractions and is necessary to map matrix model correlators to \mathbb{R}^4 quantum expectation values [39, 40].¹⁰ At leading order in large tension, the correlator between a single Wilson loop and the chiral operator \mathcal{O}_J obeys (1.7) in terms of a scaling function that depends on the specific ratio g_s^2/T^2 and has a non-trivial dependence on J . The most natural scaling dependence is actually on g_s^2/T as in (1.5). Several cancellations occur and are responsible for the relevant variable being g_s^2/T^2 . We shall show that this pattern changes in the case of the correlator between multiple coincident Wilson loops and one chiral operator. The above mentioned cancellations do not occur anymore and one has instead the structure (1.11). Besides, the function \mathcal{H}_n can be computed explicitly in terms of \mathcal{W}_n , cf. (1.12). To derive such a result, we will conveniently use the strong coupling version of topological recursion. As we remarked previously, Toda recursion is rather cumbersome for these purposes, as illustrated in the example $(n, J) = (1, 2)$ in appendix A.

¹⁰The choice of normalization in (6.2), and in particular the overall power of N , is dictated by string theory and makes direct contact with the associated natural vertex operators [1]. Another standard choice is to require a fixed normalization of the chiral operators 2-point functions as in [14].

6.1 Contribution from multi-trace operators in normal ordering

As a preliminary step we first address the issue of the effects of normal ordering in (6.2) and the role of multi-trace operators. It is instructive to look at the first cases at low J . A straightforward explicit calculation gives (we restrict to even J for the purpose of illustration)

$$\begin{aligned}
 : \text{tr } M^2 : &:= \text{tr } M^2 - \frac{N}{2}, & : \text{tr } M^4 : &:= \text{tr } M^4 - 2 \text{tr } M^2 - \frac{1}{N} (\text{tr } M)^2 + \frac{N}{2} + \frac{1}{4N}, \\
 : \text{tr } M^6 : &:= \text{tr } M^6 - 3 \text{tr } M^4 + \left(\frac{15}{4} + \frac{15}{4N^2} \right) \text{tr } M^2 \\
 &+ \frac{1}{N} \left(-\frac{3}{2} (\text{tr } M^2)^2 - 3 \text{tr } M \text{tr } M^3 + \frac{15}{4} (\text{tr } M)^2 \right) - \frac{5N}{8} - \frac{5}{4N}, \tag{6.3}
 \end{aligned}$$

and so on. In general, terms involving products of k traces have a coefficient which is $\sim 1/N^{k-1}$ at large N . Thus, we will write

$$: \text{tr } M^J : = \sum_{k \geq 0} \frac{1}{N^{k-1}} \left[: \text{tr } M^J : \right]_k, \tag{6.4}$$

where the operators in $\left[: \text{tr } M^J : \right]_k$ have coefficients $\mathcal{O}(1)$ at large N .¹¹ Now, let us consider the genus g contribution to the connected correlator $\left\langle \mathcal{W}^n \mathcal{O}^{(k)} \right\rangle_c$ where $\mathcal{O}^{(k)}$ is any arbitrary k -trace operator. We can write, cf. (2.5),

$$\begin{aligned}
 &\left\langle \mathcal{W}^n \mathcal{O}^{(k)} \right\rangle_c \Big|_{\text{genus } g} \tag{6.5} \\
 &= \frac{1}{N^{2g+k+n-2} (2\pi i)^{k+n}} \oint \omega_{n,g}(z_1, \dots, z_{n+k}) \mathcal{O}^{(k)}(x(z_1), \dots, x(z_k)) \exp \left(\frac{\lambda}{2} \sum_{l=1}^n x(z_{k+l}) \right).
 \end{aligned}$$

In the case of $\mathcal{O}^{(k)} = \left[: \text{tr } M^J : \right]_k$, taking into account the extra factor $\frac{1}{N^{k-1}}$ in (6.4), we find that $\left\langle \mathcal{W}^n \left[: \text{tr } M^J : \right]_k \right\rangle_c \Big|_{\text{genus } g}$ scales as $N^{-(2g+2k+n-3)}$. Finally, let us pin the dependence on $\lambda \gg 1$. Since the operator $: \text{tr } M^J :$ does not depend on λ explicitly, the strong coupling limit of the expectation value of Wilson loop with chiral operators corresponds to maximizing the order of poles of variables corresponding to Wilson loop or conversely minimizing the order of the poles of the variables that correspond to the $: \text{tr } M^J :$ operator. The total order of the poles of $\omega_{n,g}$ is $6(g-1) + 4n$, as discussed in section 4. According to the saddle point analysis this implies the final scaling behaviour

$$\left\langle \mathcal{W}^n \left[: \text{tr } M^J : \right]_k \right\rangle_c \Big|_{\text{genus } g} \sim \frac{\lambda^{\frac{1}{4}(6g+3n+2k-6)}}{N^{2g+2k+n-3}}. \tag{6.6}$$

This gives the leading power at large λ for all genera. In particular, a term with an overall $\frac{1}{N^P}$ factor will be accompanied by the following powers of λ

$$\lambda^{\frac{3}{4}-k} \left(\frac{\lambda^{\frac{3}{4}}}{N} \right)^P, \tag{6.7}$$

¹¹The k -trace part may have an explicit N dependence as in $\left[: \text{tr } M^6 : \right]_1$ which has a piece $\frac{15}{4}(1 + 1/N^2) \text{tr } M^2$ whose $N \rightarrow \infty$ limit is finite.

showing that multiple trace contributions are suppressed. Besides, since the saddle point expansion has relative corrections in powers $\lambda^{-1/2} \sim 1/T$, double trace corrections to normal ordering cannot be seen even at first subleading order in large λ .

Let us see this explicitly in the simplest case of a single Wilson loop keeping only up to double trace operators. At the planar level, as is well known, the double trace part doesn't contribute. At $1/N^2$ level there are two relevant cumulants corresponding to $(k, g) = (1, 1)$ and $(2, 0)$. Their λ dependence can be obtained using the explicit strong coupling resolvents given (4.2) as respectively,

$$\frac{1}{N^2} \oint_{z_2} \exp\left(\frac{\sqrt{\lambda}}{2}x(z_2)\right) \hat{\omega}_{2,1}(z_1, z_2) \sim \frac{\lambda^{\frac{5}{4}}}{N^2}, \quad \frac{1}{N^2} \oint_{z_3} \exp\left(\frac{\sqrt{\lambda}}{2}x(z_3)\right) \hat{\omega}_{3,0}(z_1, z_2, z_3) \sim \frac{\lambda^{\frac{1}{4}}}{N^2}. \tag{6.8}$$

The first contribution is dominant in the large tension limit and in fact we would have to expand it to three orders in λ before the second one becomes effective. Since, the rate of growth of the exponent of λ is 6 for g but only 2 for k , as g and k increase or as cumulants are multiplied, the gap between the contributions of single and higher trace operators only increases.¹²

6.2 $\langle \mathcal{W}^n \mathcal{O}_J \rangle$ at leading order

As a result of the above discussion, we can restrict ourselves to the single trace part of normal ordering, i.e. the planar approximation. According to [41, 42], it may be written in terms of Chebyshev polynomials and, in the z variable, it reads

$$: \text{tr } M^J : \rightarrow z^J + \frac{1}{z^J} + 2\delta_{j,2} + \dots \tag{6.9}$$

The relevant connected correlators $\langle \mathcal{W}^n : \text{tr } M^J : \rangle_{c,g}$ are¹³

$$\langle \mathcal{W}^n \mathcal{O} \rangle_{c,g} = \frac{1}{(2\pi i)^{n+1}} \oint \omega_{n,g}(z_1, \dots, z_{n+1}) z_1^{-J} \exp\left[\frac{\sqrt{\lambda}}{2}(x(z_2) + \dots + x(z_{n+1}))\right]. \tag{6.10}$$

In the strong coupling limit we use the strong coupling resolvent $\hat{\omega}_{n,g}(z)$ and keep in it only those terms that minimize the order of the poles of z_1 at 1. This can be done by going through one step of topological recursion. This corresponds to starting with $\omega_{g,n-1}(z_1, \dots, z_n)$ and using:

$$\prod_{i=2}^{n+1} \frac{1}{(z_i - 1)^{2k_i}} \rightarrow \sum_{j=2}^n \frac{2k_j + 1}{4(z_1 - 1)^2 (z_j - 1)^2} \prod_{i=2}^{n+1} \frac{1}{(z_i - 1)^{2k_i}} + \dots \tag{6.11}$$

¹²This is of course contingent on these first two contributions from single trace operators not vanishing once we take residue integrals corresponding to the chiral operator. As we now show this is indeed the case. This remark will be important later.

¹³We deform the z_1 integration shrinking it around $z_1 = 0$. This is possible because residues vanish at $z_i = \pm 1$ and, in particular, we can drop the parts that are not singular at $z_1 = 0$.

We can now integrate over z_2, \dots, z_n in the saddle point approximation. The above factor of $2k_j + 1$ ensures that the result has a very simple relation to $\langle \mathcal{W}^n \rangle_{c,g}$ in the strong coupling limit, i.e.¹⁴

$$\langle \mathcal{W}^n : \text{tr } M^J : \rangle_{c,g}^{\lambda \gg 1} = \frac{n\lambda}{2} \langle \mathcal{W}^n \rangle_{c,g}^{\lambda \gg 1} \text{Res}_{z_1=0} \frac{dz_1}{(z_1 - 1)^2 z_1^J} + \dots = \frac{J n \sqrt{\lambda}}{2} \langle \mathcal{W}^n \rangle_{c,g}^{\lambda \gg 1} + \dots \tag{6.12}$$

The same is true for the full correlator, after expanding into connected correlators, i.e.

$$\frac{\langle \mathcal{W}^n : \text{tr } M^J : \rangle}{\langle \mathcal{W}^n \rangle} \stackrel{\lambda \gg 1}{=} \frac{J n \sqrt{\lambda}}{2} + \dots \tag{6.13}$$

This is of course expected from the known results for $n = 1$ and $n = 2$, see [2].

6.3 Subleading corrections

To go beyond leading order we need to carry out topological recursion with poles of one subleading order included. It is convenient to change variables from z to u , cf. (3.12), and write

$$\omega_{n,g}(u_1, \dots, u_n) = \hat{\omega}_{n,g} + \delta\omega_{n,g} + \dots \tag{6.14}$$

Where $\delta\omega_{n,g}$ includes the poles of total degree $6g + 4n - 8$, see appendix C for full details of the procedure. Using (6.14) we can compute the one-variable resolvents obtained after integration of all but one variable,¹⁵

$$\bar{\omega}_{n,g}(z) = \frac{1}{(2\pi i)^n} \oint \omega_{n+1,g}(u(z), u_1, \dots, u_n) \tag{6.15}$$

Due to our previous discussion, cf. (6.6), the first two orders in the $1/\sqrt{\lambda}$ expansion at large λ can be computed by ignoring mixing with multi-trace operators and using the simple correspondence in (6.9). Thus, we simply obtain

$$\langle \mathcal{W}^n : \text{tr } M^J : \rangle_{c,g} = \text{Res}_{z \rightarrow 0} \frac{\bar{\omega}_{n,g}(z)}{z^J} \tag{6.16}$$

This computes the connected part of the correlator but we can also define a function that similarly computes the full correlator, i.e.

$$\bar{\Omega}_{n,g}(z) = \sum_{k=1}^n \sum_{h=0}^g \binom{n}{k} \langle \mathcal{W}^{n-k} \rangle_{g-h} \bar{\omega}_{k,h}(z) \rightarrow \langle \mathcal{W}^n : \text{tr } M^J : \rangle = \text{Res}_{z \rightarrow 0} \sum_{g=0}^{\infty} \frac{1}{N^{2g-1}} \frac{\bar{\Omega}_{n,g}(z)}{z^J} \tag{6.17}$$

¹⁴Let us remind that the presence of dz_1 in the residue is formal at this level and could be omitted. Nevertheless, it is convenient to keep it to emphasize transformation properties under change of variables.

¹⁵We change back to z -coordinates for the free variable because this is convenient to compute expectation values with a chiral operator.

To compute $\langle \mathcal{W}^n : \text{tr } M^J : \rangle / \langle \mathcal{W}^n \rangle$ it is convenient to expand $\bar{\Omega}_{n,g}(z)$ as:

$$\bar{\Omega}_{n,g}(z) = U_{n,0}(z) \langle \mathcal{W}^n \rangle_{c,g} + U_{n,1}(z) \langle \mathcal{W}^n \rangle_{c,g-1} + \dots + U_{n,g}(z) \langle \mathcal{W}^n \rangle_{c,0} + \dots, \quad (6.18)$$

where each $U_{n,g}(z)$ is determined recursively genus by genus and final dots stand for a correction of order $\mathcal{O}((1/\sqrt{\lambda})^{g+2})$ relative to the leading order. Then it can be seen that,

$$\frac{\langle : \text{tr } M^J : \mathcal{W}^n \rangle}{\langle \mathcal{W}^n \rangle} = \text{Res}_{z \rightarrow 0} \sum_{g=0}^{\infty} \frac{1}{N^{2g-1}} \frac{U_{n,g}(z)}{z^J}. \quad (6.19)$$

The functions $U_{n,g}(z)$ depend also on λ . To the leading order in λ , $U_{n,0}(z)$ can be read from (6.12). To get a non-vanishing result for all other $U_{n,g}$ we need to go beyond $\hat{\omega}_{n,g}$ and include $\delta\omega_{n,g}$. Restricting ourselves to two leading term in λ , the most general structure possible for $U_{n,g}$ is:

$$\begin{aligned} U_{n,0}(z) &= dz \left(\frac{n\sqrt{\lambda}}{2(z-1)^2} + \frac{f_{n,0}(z)}{(z-1)^4} \right) + \dots, \\ U_{n,g}(z) &= \frac{dz f_{n,g}(z) \lambda^{\frac{3g}{2}}}{(z-1)^4} + \dots \end{aligned} \quad (6.20)$$

Where $f_{n,g}(z)$ are polynomials of degree at most 3 and independent of λ . Two out of the 4 free coefficients are determined by the requirement from topological recursion that $U_{n,g}(\frac{1}{z}) = -U_{n,g}(z)$. Another one can be fixed by requiring that $\langle : \text{tr } M : \mathcal{W}^n \rangle_{c,g}$ vanishes for $g > 0$.¹⁶ Combining these two requirements we obtain

$$U_{n,g} = dz \frac{c_{n,g} \lambda^{\frac{3g}{2}} z}{(z-1)^4}. \quad (6.21)$$

Explicit results. After having clarified the general structure of topological recursion for the quantities we need, let us present explicit results. For the ‘critical’ case $n = 1$ we find¹⁷

$$\begin{aligned} U_{1,0}(z) &= dz \left(\frac{\sqrt{\lambda}}{2(z-1)^2} + \frac{3z}{2(z-1)^4} \right) + \dots, \\ U_{1,1}(z) &= dz \frac{\lambda^{3/2} z}{32(z-1)^4} + \dots, \end{aligned} \quad (6.22)$$

while the higher $U_{1,g}(z)$ vanish i.e. $c_{1,g} = 0$ for $g > 1$. This can be seen as consistency check and is a result of the cancellations required to reorganize the series for $\langle \mathcal{W} : \mathcal{O}_J : \rangle$ as in (1.7). To calculate non-vanishing terms in $U_{1,g}(z)$ for $g > 1$ we will need to keep more than 2 leading terms in $\omega_{n,g}$.

¹⁶This may be shown by explicit splitting of $U(N)$ into $U(1) \times SU(N)$, see appendix D.

¹⁷Recall that we expect a major change of features when moving from $n = 1$ to $n > 1$.

These peculiar cancellations do not occur for $n > 1$ and make the calculation of subleading corrections possible with our level of accuracy. We find

$$\begin{aligned}
 \sum_g \frac{1}{N^{2g-1}} U_{2,g}(z) &= \frac{1}{N} \left(\frac{\sqrt{\lambda}}{(z-1)^2} - \frac{3z}{(z-1)^4} + \frac{z}{(z-1)^4} \left[\frac{7\lambda^{3/2}}{16N^2} - \frac{\lambda^3}{64N^4} + \frac{3\lambda^{9/2}}{2560N^6} \right. \right. \\
 &\quad \left. \left. - \frac{37\lambda^6}{430080N^8} + \frac{13\lambda^{15/2}}{2064384N^{10}} - \frac{299\lambda^9}{648806400N^{12}} \right] \right) + \dots \\
 \sum_g \frac{1}{N^{2g-1}} U_{3,g}(z) &= \frac{3}{2N} \left(\frac{\sqrt{\lambda}}{(z-1)^2} - \frac{3z}{(z-1)^4} + \frac{z}{(z-1)^4} \left[\frac{13\lambda^{3/2}}{16N^2} + \frac{\lambda^3}{32N^4} \right. \right. \\
 &\quad \left. \left. - \frac{17\lambda^{9/2}}{1280N^6} + \frac{123\lambda^6}{71680N^8} - \frac{\lambda^{15/2}}{516096N^{10}} \right] \right) + \dots, \\
 \sum_g \frac{1}{N^{2g-1}} U_{4,g}(z) &= \frac{2}{N} \left(\frac{\sqrt{\lambda}}{(z-1)^2} - \frac{3z}{(z-1)^4} + \frac{z}{(z-1)^4} \left[\frac{19\lambda^{3/2}}{16N^2} + \frac{9\lambda^3}{64N^4} - \frac{21\lambda^{9/2}}{2560N^6} - \frac{391\lambda^6}{28672N^8} \right] \right) + \dots, \\
 \sum_g \frac{1}{N^{2g-1}} U_{5,g}(z) &= \frac{5}{2N} \left(\frac{\sqrt{\lambda}}{(z-1)^2} - \frac{3z}{(z-1)^4} + \frac{z}{(z-1)^4} \left[\frac{25\lambda^{3/2}}{16N^2} + \frac{5\lambda^3}{16N^4} + \frac{33\lambda^{9/2}}{640N^6} \right] \right) + \dots \quad (6.23)
 \end{aligned}$$

As a result of this, the dependence on J in $\frac{\langle W^n \cdot \text{tr } M^j \cdot \rangle}{\langle W^n \rangle}$ is much simpler for $n > 1$ than in the $n = 1$ case, cf. (1.7). Indeed, from the above, it has to be proportional to

$$\text{Res}_{z=0} \frac{dz}{(z-1)^4 z^{J-1}} = \frac{1}{6} J (J^2 - 1). \quad (6.24)$$

This means in that the structure of large tension limit of $\frac{\langle W^n \mathcal{O}_J \rangle}{\langle W^n \rangle}$ is given by (1.11). The first few terms of $H_n(x)$ can be calculated from (6.23) and read

$$\begin{aligned}
 H_2(x) &= \frac{7x}{24\pi} - \frac{x^2}{12\pi} + \frac{x^3}{20\pi} - \frac{37x^4}{1260\pi} + \frac{13x^5}{756\pi} - \frac{299x^6}{29700\pi} + \dots, \\
 H_3(x) &= \frac{13x}{24\pi} + \frac{x^2}{6\pi} - \frac{17x^3}{30\pi} + \frac{41x^4}{70\pi} - \frac{x^5}{189\pi} + \dots, \\
 H_4(x) &= \frac{19x}{24\pi} + \frac{3x^2}{4\pi} - \frac{7x^3}{20\pi} - \frac{391x^4}{84\pi} + \dots, \\
 H_5(x) &= \frac{25x}{24\pi} + \frac{5x^2}{3\pi} + \frac{11x^3}{5\pi} + \dots \quad (6.25)
 \end{aligned}$$

6.4 Relating H_n to W_n

The discussion in previous section has led to the expansion (6.25) for the scaling functions H_n . Most importantly, we could prove the general structure (1.11), with its peculiar dependence on the J parameter. In this section we show how this can be exploited to express H_n in terms of W_n . To this aim we take $J = 2$ in the topological recursion result (1.11) and write

$$\frac{\langle W^n \mathcal{O}_2 \rangle}{\langle W^n \rangle} \stackrel{T \gg 1}{\simeq} \pi n (T + 3 H_n). \quad (6.26)$$

The l.h.s. may be traded for a logarithmic derivative of $\langle \mathcal{W}^n \rangle$ due to the matrix model identity

$$\langle \mathcal{W}^n \mathcal{O}_2 \rangle = \lambda \frac{d}{d\lambda} \langle \mathcal{W}^n \rangle. \tag{6.27}$$

Hence we have

$$\lambda \frac{d}{d\lambda} \log \langle \mathcal{W}^n \rangle \stackrel{T \gg 1}{\cong} \pi n (T + 3 \mathbf{H}_n), \tag{6.28}$$

and a short calculation gives the relation

$$\mathbf{H}_n(x) = \frac{x}{2\pi} \left[\frac{1}{12} + \frac{1}{n} \frac{d}{dx} \log W_n(x) \right]. \tag{6.29}$$

Replacing W_n by its evaluation by means of (5.45) and using the series expansion (5.19), we get

$$\begin{aligned} \mathbf{H}_n(x) = & \frac{-5 + 6n}{24\pi} x + \frac{(-1 + n)(-5 + 2n)}{12\pi} x^2 + \frac{(-1 + n)(133 - 95n + 15n^2)}{60\pi} x^3 \\ & + \frac{(-1 + n)(-24159 + 23611n - 7035n^2 + 630n^3)}{1260\pi} x^4 + \dots, \end{aligned} \tag{6.30}$$

in agreement with (6.25). Of course, the exact determination of W_n by Toda recursion means that we can provide easily all order expansion of the \mathbf{H}_n function by means of (6.29).

6.5 A few sample calculations

Let us give some examples of (1.12) by explicit computations. For $n = 2$ we need the explicit exact expansion

$$\begin{aligned} \frac{1}{N^2} \langle \mathcal{W}^2 \rangle = & \left[\frac{2}{\sqrt{\lambda}} I_1 \right]^2 + \frac{\sqrt{\lambda}}{2N^2} \left[I_0 I_1 + \frac{1}{6} I_1 I_2 \right] + \frac{1}{N^4} \left[\frac{37\lambda^2}{2304} I_0^2 - \frac{\sqrt{\lambda}(24 + 131\lambda)}{2880} I_0 I_1 \right. \\ & + \left. \frac{192 + 332\lambda + 185\lambda^2}{11520} I_1^2 \right] + \frac{1}{N^6} \left[-\frac{\lambda^2(62 + 37\lambda)}{23040} I_0^2 \right. \\ & + \left. \frac{\sqrt{\lambda}(23040 + 56160\lambda + 40920\lambda^2 + 6209\lambda^3)}{5806080} I_0 I_1 \right. \\ & \left. - \frac{92160 + 111168\lambda + 85440\lambda^2 + 24857\lambda^3}{11612160} I_1^2 \right] + \mathcal{O} \left(\frac{1}{N^8} \right). \end{aligned} \tag{6.31}$$

Using (6.27) we work out the case $(n, J) = (2, 2)$

$$\frac{\langle \mathcal{W}^2 \mathcal{O}_2 \rangle}{\langle \mathcal{W}^2 \rangle} = \sqrt{\lambda} + \dots + \frac{1}{N^2} \left[\frac{7}{32} \lambda^{3/2} + \dots \right] + \frac{1}{N^4} \left[-\frac{1}{128} \lambda^3 + \dots \right] + \frac{1}{N^6} \left[\frac{3}{5120} \lambda^{9/2} + \dots \right] + \dots \tag{6.32}$$

Comparing with (1.11) gives the first terms

$$\mathbf{H}_2(x) = \frac{1}{6\pi} \left(\frac{7}{4} x - \frac{1}{2} x^2 + \frac{3}{10} x^3 + \dots \right), \tag{6.33}$$

in agreement with (6.30). In this case we can give the exact expression in a reasonable compact form using the first equation in (5.21)

$$H_2(x) = \frac{3}{2} + 2x - \frac{3}{2 + e^{x/2} \sqrt{2\pi x} \operatorname{erf}\left(\sqrt{\frac{x}{2}}\right)}. \quad (6.34)$$

A similar calculation can be repeated for $n = 3$. In this case we have

$$\begin{aligned} \frac{1}{N^3} \langle \mathcal{W}^3 \rangle &= \left[\frac{2}{\sqrt{\lambda}} I_1 \right]^3 + \frac{1}{N^2} \left(\frac{13}{4} I_1^2 I_0 - \frac{1}{2\sqrt{\lambda}} I_1^3 \right) \\ &+ \frac{1}{N^4} \left[\frac{193}{384} \lambda^{3/2} I_0^2 I_1 - \frac{6+79\lambda}{240} I_0 I_1^2 + \frac{192+592\lambda+845\lambda^2}{3840\sqrt{\lambda}} I_1^3 \right] \\ &+ \frac{1}{N^6} \left[\frac{2557\lambda^3}{110592} I_0^3 - \frac{\lambda^{3/2}(1776+7865\lambda)}{92160} I_0^2 I_1 + \frac{92160+474624\lambda+878688\lambda^2+572537\lambda^3}{7741440} I_0 I_1^2 \right. \\ &\left. - \frac{23040+46944\lambda+64396\lambda^2+52073\lambda^3}{967680\sqrt{\lambda}} I_1^3 \right] + \dots \end{aligned} \quad (6.35)$$

This gives

$$\frac{\langle \mathcal{W}^3 \mathcal{O}_2 \rangle}{\langle \mathcal{W}^3 \rangle} = \frac{3}{2} \sqrt{\lambda} + \dots + \frac{1}{N^2} \left[\frac{39}{64} \lambda^{3/2} + \dots \right] + \frac{1}{N^4} \left[\frac{3}{128} \lambda^3 + \dots \right] + \frac{1}{N^6} \left[-\frac{51}{5120} \lambda^{9/2} + \dots \right] + \dots \quad (6.36)$$

Comparing with (1.11) we obtain

$$H_3(x) = \frac{1}{9\pi} \left(\frac{39}{8} x + \frac{3}{2} x^2 - \frac{51}{10} x^3 + \dots \right), \quad (6.37)$$

in agreement with (1.12). As in (6.34), one can give a closed formula for this function in terms of the special error and Owen-T functions. As a final check, probing the peculiar simple J dependence in (1.11), we consider the case $(n, J) = (2, 3)$. To analyze this case by expansion of exact expressions at finite λ we need the Bessel function expansion of $\langle \mathcal{W}^2 \mathcal{O}_3 \rangle$ where $\mathcal{O}_3 = \frac{N}{2} \sqrt{\frac{\pi}{2}} : \operatorname{tr} M^3 :$ and $: \operatorname{tr} M^3 := \operatorname{tr} M^3 - 3 \operatorname{tr} M$. By matching a large number of weak coupling perturbative coefficients, we find

$$\begin{aligned} \langle \mathcal{W}^2 \mathcal{O}_3 \rangle &= N^2 \sqrt{\frac{\pi}{2}} \left\{ -\frac{24I_0 I_1}{\lambda} + \frac{6(8+\lambda)I_1^2}{\lambda^{3/2}} \right. \\ &+ \frac{1}{N^2} \left[-\frac{1}{4} \sqrt{\lambda} I_0^2 + \frac{1}{8} (8+7\lambda) I_0 I_1 - \frac{I_1^2}{\sqrt{\lambda}} \right] \\ &+ \frac{1}{N^4} \left[\frac{\sqrt{\lambda}(192+48\lambda+185\lambda^2)I_0^2}{7680} + \frac{(-192-72\lambda+239\lambda^2)I_0 I_1}{1920} + \frac{(768+384\lambda-240\lambda^2+185\lambda^3)I_1^2}{7680\sqrt{\lambda}} \right] \\ &+ \frac{1}{N^6} \left[\frac{\sqrt{\lambda}(-5760-1440\lambda-528\lambda^2+1939\lambda^3)I_0^2}{483840} + \frac{(184320+69120\lambda+8544\lambda^2-39902\lambda^3+6209\lambda^4)I_0 I_1}{3870720} \right. \\ &\left. + \frac{(-368640-184320\lambda-6144\lambda^2+41128\lambda^3+24815\lambda^4)I_1^2}{7741440\sqrt{\lambda}} \right] + \dots \left. \right\}. \end{aligned} \quad (6.38)$$

This gives

$$\frac{\langle \mathcal{W}^2 \mathcal{O}_3 \rangle}{\langle \mathcal{W}^2 \rangle} = \sqrt{\frac{\pi}{2}} \left\{ \frac{3}{2} \sqrt{\lambda} + \dots + \frac{1}{N^2} \left[\frac{7}{8} \lambda^{3/2} + \dots \right] + \frac{1}{N^4} \left[-\frac{1}{32} \lambda^3 + \dots \right] + \frac{1}{N^6} \left[\frac{3}{1280} \lambda^{9/2} + \dots \right] + \dots \right\}. \quad (6.39)$$

This expansion should be compared with the $n = 2$ $J = 3$ case of (1.12), i.e.

$$3 \left(\frac{\pi}{2} \right)^{3/2} 2(T + 8\mathcal{H}_2(x)) = \sqrt{\frac{\pi}{2}} \left[\frac{3}{2} \times 2\pi T + 7x - 2x^2 + \frac{6}{5}x^3 + \dots \right], \quad (6.40)$$

and indeed we find that this is equivalent to the previous expansion (6.33).

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A Toda recursion for correlators with chiral primaries

The genus expansion of the ratio $\langle \mathcal{W} \mathcal{O}_2 \rangle / \langle \mathcal{W} \rangle$ may be computed by (6.27) in terms of $\langle \mathcal{W} \rangle$. Alternatively, it is equivalent to use the integral representation (1.10) derived in [17]. Here, we want to show how such correlators may be treated by Toda recursion, as an illustration, generalizing the treatment in appendix B.3 of [2]. From

$$e_N(x, y) = \left\langle \text{tr} e^{x \sqrt{\frac{4N}{\lambda}} M} \text{tr} e^{y \sqrt{\frac{4N}{\lambda}} M} \right\rangle - \left\langle \text{tr} e^{x \sqrt{\frac{4N}{\lambda}} M} \right\rangle \left\langle \text{tr} e^{y \sqrt{\frac{4N}{\lambda}} M} \right\rangle, \quad (A.1)$$

we have

$$\frac{\partial_x^2 e_N \left(x, \sqrt{\frac{\lambda}{4N}} \right) \Big|_{x=0}}{e_N \left(\sqrt{\frac{\lambda}{4N}} \right)} = \frac{4N}{\lambda} \frac{\langle \mathcal{W} : \text{tr} M^2 : \rangle}{\langle \mathcal{W} \rangle} = 2 \frac{\langle \mathcal{W} : \text{tr} a^2 : \rangle}{\langle \mathcal{W} \rangle}, \quad (A.2)$$

where

$$M = \sqrt{\frac{\lambda}{2N}} a \quad \left[\text{tr} e^M e^{-\frac{2N}{\lambda} \text{tr} M^2} = \text{tr} e^{\sqrt{\frac{\lambda}{2N}} a} e^{-\text{tr} a^2} \right], \quad (A.3)$$

to make contact with the expressions in [2]. The relevant Toda equation is (5.28). Taking two derivatives involves the auxiliary quantity

$$\frac{\partial_x e_N \left(x, \sqrt{\frac{\lambda}{4N}} \right) \Big|_{x=0}}{e_N \left(\sqrt{\frac{\lambda}{4N}} \right)} = \sqrt{\frac{4N}{\lambda}} \frac{\langle \mathcal{W} \text{tr} M \rangle}{\langle \mathcal{W} \rangle} = \sqrt{2} \frac{\langle \mathcal{W} \text{tr} a \rangle}{\langle \mathcal{W} \rangle}. \quad (A.4)$$

To continue, we need the correct Ansatz for the r.h.s. of (A.2) and (A.4) at large tension. This is

$$\frac{\langle \mathcal{W} : \text{tr} a^J : \rangle}{\langle \mathcal{W} \rangle} = N^{\frac{J}{2}-1} \sqrt{\lambda} C_J(\zeta), \quad \zeta = \frac{g_s^2}{T^2} = \frac{\lambda}{4N^2}. \quad (A.5)$$

The Toda recursion takes the form

$$4\sqrt{N(N-1)}\sqrt{\zeta}C_2\left(\frac{N}{N-1}\zeta\right)\frac{e_{N-1}(\sqrt{N\zeta})}{e_N(\sqrt{N\zeta})}+4\sqrt{N(N+1)}\sqrt{\zeta}C_2\left(\frac{N}{N+1}\zeta\right)\frac{e_{N+1}(\sqrt{N\zeta})}{e_N(\sqrt{N\zeta})}-4N\sqrt{\zeta}(2+\zeta)C_2(\zeta)-8\sqrt{2}\zeta C_1(\zeta)+2\zeta=0 \tag{A.6}$$

The expansion at large N with fixed ζ require to study the asymptotic behaviour of $e_N(\sqrt{N}\mu)$ at fixed μ . Recall that

$$e_N(x) = e^{\frac{x^2}{2}}L_{N-1}^1(-x^2), \quad e_N''(x) + \frac{3}{x}e_N'(x) - (4N + x^2)e_N(x) = 0. \tag{A.7}$$

Setting $x = \sqrt{N}\mu$ and expanding the differential equation gives

$$e_N(\sqrt{N}\mu) = N^{-1/2} \exp\left[Nf_0(\mu) + f_1(\mu) + \frac{1}{N}f_2(\mu) + \dots\right], \tag{A.8}$$

with

$$f_0(\mu) = \frac{\mu}{2}\sqrt{\mu^2 + 4} + 2\operatorname{arcsinh}\frac{\mu}{2}, \quad f_1(\mu) = -\frac{3}{2}\log\mu - \frac{1}{4}\log(\mu^2 + 4). \tag{A.9}$$

This is enough to derive the relevant terms in the expansion

$$\frac{e_{N\pm 1}(\sqrt{N\zeta})}{e_N(\sqrt{N\zeta})} = e^{\pm 2\operatorname{arcsinh}\frac{\sqrt{\zeta}}{2}} \left[1 + \frac{1}{N} \frac{\pm(2 + \zeta) - \sqrt{\zeta}\sqrt{4 + \zeta}}{2(4 + \zeta)} + \mathcal{O}\left(\frac{1}{N^2}\right) \dots\right]. \tag{A.10}$$

Using this in the expansion of (A.6) gives

$$C_2'(\zeta) - \frac{1}{2\zeta}C_2(\zeta) + \frac{-1 + 4\sqrt{2}C_1(\zeta)}{2\zeta\sqrt{4 + \zeta}} = 0. \tag{A.11}$$

It is easy to check that $C_1(\zeta) \equiv \frac{1}{2\sqrt{2}}$, so that

$$C_2(\zeta) = \frac{1}{4}\sqrt{4 + \zeta} + k\sqrt{\zeta}, \tag{A.12}$$

where k is a constant that we set to zero by analyticity. The result agrees with [2], see eqs. (2.34), (2.35), (2.40) there.

B The polynomial P_k for $k = 11, \dots, 20$

The polynomials $P_k(n)$ have been defined in (5.23) and their expression for k up to 10 have been given in (5.24) and (5.47). The expressions for $k = 11, \dots, 20$ are given below.

$$P_{11} = \frac{4199n^{10}}{33} - \frac{87577n^9}{11} + \frac{21375883n^8}{99} - \frac{209156735279n^7}{62370} + \frac{5161998742529n^6}{155925} - \frac{101671734522896n^5}{467775} + \frac{77972872201319n^4}{81081} - \frac{120508596162974836n^3}{42567525} + \frac{2558119973701481627n^2}{482431950} - \frac{335724211283681053843n}{58925616750} + \frac{156764630068025273339}{58925616750},$$

$$\begin{aligned}
 P_{12} = & \frac{2261n^{11}}{6} - \frac{965975n^{10}}{36} + \frac{10060585n^9}{12} - \frac{172514522407n^8}{11340} + \frac{30353622743141n^7}{170100} \\
 & - \frac{483860927926589n^6}{340200} + \frac{40171770380189773093n^5}{5108103000} - \frac{51369864627494293667n^4}{1702701000} \\
 & + \frac{102472685620882550488819n^3}{1302566265000} - \frac{574763444256918348659002n^2}{4331032831125} \\
 & + \frac{393843049171726870141793n}{3024848326500} - \frac{98527083191201048311039181}{1753202090039400}, \\
 P_{13} = & \frac{14858n^{12}}{13} - \frac{3561572n^{11}}{39} + \frac{1890539597n^{10}}{585} - \frac{824252140702n^9}{12285} + \frac{11192496489158n^8}{12285} \\
 & - \frac{942785947894636n^7}{110565} + \frac{14402080240084077257n^6}{255405150} - \frac{203271501770547142769n^5}{766215450} \\
 & + \frac{9555769999611687383321n^4}{10854718875} - \frac{11588871456123457506361969n^3}{5774710441500} \\
 & + \frac{2838595185137082339418889941n^2}{952827222847500} - \frac{51452536482602877650860472n}{19922751023175} \\
 & + \frac{3608383690059034257720530341}{3652504354248750}, \\
 P_{14} = & \frac{74290n^{13}}{21} - \frac{2193609n^{12}}{7} + \frac{3898791707n^{11}}{315} - \frac{54577362356n^{10}}{189} + \frac{62912522293682n^9}{14175} \\
 & - \frac{671658180402872n^8}{14175} + \frac{212496790875244595807n^7}{589396500} - \frac{30206956766158029102587n^6}{15324309000} \\
 & + \frac{2004508256000686158596273n^5}{260513253000} - \frac{726580590679310973992290361n^4}{34648262649000} \\
 & + \frac{2908859723836817593513494211n^3}{76226177827800} - \frac{46704914627229276739182458986n^2}{1095751306274625} \\
 & + \frac{6980915896986492598862795960273n}{284895339631402500} - \frac{745725550174967494764854054797}{170937203778841500}, \\
 P_{15} = & \frac{22287n^{14}}{2} - \frac{97455637n^{13}}{90} + \frac{471242699n^{12}}{10} - \frac{7296074705n^{11}}{6} + \frac{1762900954888741n^{10}}{85050} \\
 & - \frac{229802660901854531n^9}{935550} + \frac{29028607670070832589n^8}{14033250} - \frac{4742825903093184375173n^7}{383107725} \\
 & + \frac{116822823990093464083591n^6}{2277213750} - \frac{17658116093652216575266660279n^5}{129930984933750} \\
 & + \frac{3019686155476492526344171776962n^4}{17865510428390625} + \frac{59133660041425626175154220297689n^3}{328725391882387500} \\
 & - \frac{40902420138582238361745904216740547n^2}{38460870850239337500} + \frac{5184745996967930004978479698356554n}{3205072570853278125} \\
 & - \frac{8779046910792332017382938446939173}{9783905742604743750}, \\
 P_{16} = & \frac{570285n^{15}}{16} - \frac{361327963n^{14}}{96} + \frac{256924672387n^{13}}{1440} - \frac{151766993879233n^{12}}{30240} + \frac{938030587963673n^{11}}{10080} \\
 & - \frac{5938730301115621841n^{10}}{4989600} + \frac{44402659987134144011n^9}{4191264} - \frac{776491656618909257262217n^8}{12259447200} \\
 & + \frac{675670971369618816638500351n^7}{3126159036000} + \frac{65379058846039414975807037n^6}{24457597164000} \\
 & - \frac{101233943008886150795055382519397n^5}{22867853348340000} + \frac{13481261415837935476957015174331149n^4}{525960627011820000}
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{11425876041556661448955324338319179059n^3}{143587251174226860000} + \frac{24484874250586308873555049657424059483n^2}{165677597508723300000} \\
 & - \frac{9643906311965688880889035950642704853709n}{62460454260788684100000} + \frac{1802190759535306926724608738299225695879}{25816987761125989428000}, \\
 P_{17} = & \frac{1964315n^{16}}{17} - \frac{39445288n^{15}}{3} + \frac{60467476843n^{14}}{90} - \frac{115225766352631n^{13}}{5670} + \frac{253254621872909n^{12}}{630} \\
 & - \frac{5026692736385852671n^{11}}{935550} + \frac{13913460599257648861603n^{10}}{294698250} - \frac{122986340264675433569593n^9}{547296750} \\
 & - \frac{2027150038463820624702058n^8}{6512831325} + \frac{1295892244324167362329025833687n^7}{86620656622500} \\
 & - \frac{57860508893407313308984093326059n^6}{433103283112500} + \frac{1873177157824570617487083559682467n^5}{2629803135059100} \\
 & - \frac{27442715333021284930815598305084080569n^4}{10769043838067014500} + \frac{1970688218629393121039936070672626597n^3}{318610764439852500} \\
 & - \frac{452466495402865130794144735387180208039n^2}{46198560843778612500} + \frac{883182453823558336713057441231757494529669n}{96813704104222460355000} \\
 & - \frac{4841495934337212459267479837954600070947}{1280604551643154237500}, \\
 P_{18} = & \frac{1137235n^{17}}{3} - \frac{2490913949n^{16}}{54} + \frac{679389195281n^{15}}{270} - \frac{228968967202858n^{14}}{2835} \\
 & + \frac{2027925651493864n^{13}}{1215} - \frac{33176267023544737n^{12}}{1485} + \frac{18475026421421474122847n^{11}}{109459350} \\
 & + \frac{32427853962749456647193n^{10}}{1532430900} - \frac{141562392656331388912303109n^9}{7662154500} \\
 & + \frac{4407051037515992567242285006784n^8}{16705412348625} - \frac{12998232848938898943788373084481291n^7}{5846894322018750} \\
 & + \frac{1920684931913550119678771491316267836n^6}{147926426347074375} - \frac{5256663538285399899069978418495823491471n^5}{96152177125598343750} \\
 & + \frac{16669479504189008529752425764572885471n^4}{100332706565841750} \\
 & - \frac{16671094370233566929477876372974768312804309n^3}{46845340695591513075000} \\
 & + \frac{100433411639637043975757841635773385645900627n^2}{197213100953045752575000} \\
 & - \frac{43847679911162088556712200641949149804409228141n}{100578681486053333813250000} \\
 & + \frac{1986475693390240135403726810007174861110233163}{11807062609232347882425000}, \\
 P_{19} = & \frac{23881935n^{18}}{19} - \frac{9257432861n^{17}}{57} + \frac{8012771594381n^{16}}{855} - \frac{754041922791055n^{15}}{2394} \\
 & + \frac{595825494782601934n^{14}}{89775} - \frac{35206774729032743512n^{13}}{423225} + \frac{188028571662687223154182n^{12}}{577702125} \\
 & + \frac{356196397870788389963150374n^{11}}{36395233875} - \frac{26744467018972588685277494738n^{10}}{109185701625} \\
 & + \frac{123556902465696451688500797402787n^9}{38979295480125} - \frac{164109225413861098250930345185156307n^8}{5846894322018750} \\
 & + \frac{76973564904301165571543256581512685027n^7}{422646932420212500}
 \end{aligned}$$

$$\begin{aligned}
& - \frac{1190462126419784655208489507867266436505147n^6}{1346130479758376812500} \\
& + \frac{4326837524673482906294836489485405025912703n^5}{1346130479758376812500} \\
& - \frac{31045843858795962745573700604629252477544851n^4}{3603487745814731775000} \\
& + \frac{175717184221761997495399027443863716431266200813n^3}{10649507451464470639050000} \\
& - \frac{11573462752723826199667926549477988549478191522987n^2}{543124880024688002591550000} \\
& + \frac{2046376907527838309214791130013642260390433408739n}{123437472732883636952625000} \\
& - \frac{77748831089365050929680848793957593192606798599}{13290754339228476253893750} , \\
P_{20} = & \frac{8415539n^{19}}{2} - \frac{34412953709n^{18}}{60} + \frac{6246399383567n^{17}}{180} - \frac{13656186763905823n^{16}}{11340} \\
& + \frac{1422187058979010861n^{15}}{56700} - \frac{45793882693341165107n^{14}}{178200} - \frac{1234280007795319427540237n^{13}}{729729000} \\
& + \frac{357438485323679466115258847n^{12}}{3064861800} - \frac{545681680470218713526170387867n^{11}}{229864635000} \\
& + \frac{138672908598687244744136052146053n^{10}}{4476091347000} - \frac{376350262572215475642638310861327499277n^9}{1286316750844125000} \\
& + \frac{120017260584537152890452186439443309163n^8}{57671121382875000} \\
& - \frac{77980193487962152970270963662541582475107n^7}{6903233229530137500} \\
& + \frac{9322371447260705245309378650978156112491889n^6}{199426737741981750000} \\
& - \frac{22752890372440420457183359365326085913312317041n^5}{156151135651971710250000} \\
& + \frac{5928819160164250201043568590148571162631031954661n^4}{17749179085774117731750000} \\
& - \frac{10281819624013490020658374119628597074265484293037677n^3}{19009370800864080090704250000} \\
& + \frac{10982787902444574382251854966765664605681046163557857n^2}{19009370800864080090704250000} \\
& - \frac{1795787195145485845354622910580596712088982772028004517n}{5011345377377793113911907906250} \\
& + \frac{9747831016605627462652486155608940497922541559209369}{102796828253903448490500675000}
\end{aligned} \tag{B.1}$$

C Some details about topological recursion at large tension

Here we summarize some details about topological recursion that are relevant to the strong coupling limit of correlation functions studied in the main text. Our presentation will be for the Gaussian matrix model although most of the statements have straightforward generalizations to a general genus 0 spectral curve. See [25, 34] for pedagogical details and general treatment.

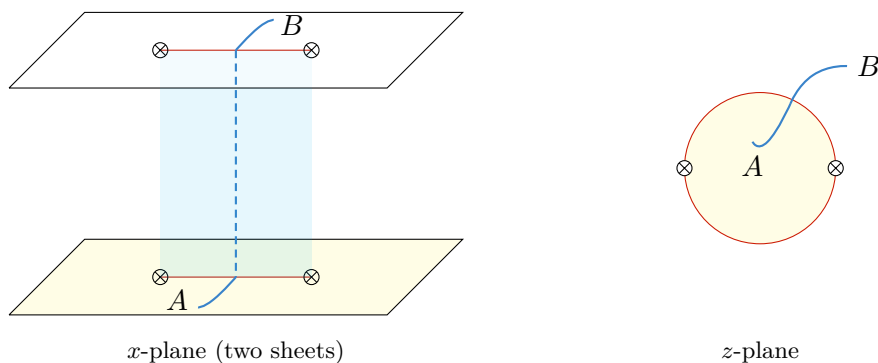


Figure 1. Illustration of the spectral curve. *Left:* the two x -sheets connected by the red cut. The points A and B are in different sheets. *Right:* in the z -plane the circle is formed from two copies of the cut and separates the two sheets whose images are the outer/inner parts. In particular, the (image of the) point A is inside the circle.

C.1 Spectral curve, resolvents, and residues

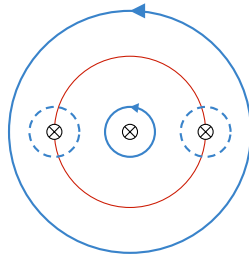
For the Gaussian matrix model the spectral curve is a two-sheeted cover of the complex plane.¹⁸ The two sheets are glued along the cut on which the eigenvalues condense in the large N limit. The coordinate z defined in (2.3) maps these two sheets to the Riemann sphere as shown in figure 1. A generic value of x has two preimages since $x(z) = x\left(\frac{1}{z}\right)$, if $|z| \neq 1$ then for one of these preimages $|z| > 1$ and for other $|z| < 1$. These are the two sheets which have been mapped to the exterior and interior respectively of unit circle on the z -plane. Let's now focus on the unit circle itself on which we write $z = \exp(it)$. Then $x(z(t)) = 2 \cos t$. So as z goes from 0 to π , $x(z(t))$ goes from 2 to -2 . This is one copy of the cut while the other copy is corresponds to t going from $\pi \rightarrow 2\pi \sim 0$. The two copies of the cut are joined at $z = 1$ and $z = -1$ which correspond to $x = 2$ and $x = -2$ i.e. the end points of the cut. These are the only two values of x which have a single preimage. These are the zeroes of the differential dx . Lastly, notice that although y is not a single valued function of x , it is a single valued function of z . Note that the unit circle is also the contour for the saddle point approximation, the saddle point integral is actually done over a double copy of the cut.

The resolvents $\omega_{n,g}(z_1, \dots, z_n)$ are all meromorphic multi-differentials on the z -plane which poles only at the branch point $z = \pm 1$. One of results of the topological recursion is the antisymmetry property:

$$\omega_{n,g}\left(\frac{1}{z_1}, z_2, \dots, z_n\right) = -\omega_{n,g}(z_1, \dots, z_n). \quad (\text{C.1})$$

As a consequence, correlation functions of the polynomials formed from the trace of the matrix M don't receive any contribution from the poles of the resolvents. This follows from the fact that these matrix observables map to polynomials $f(x(z_1), \dots, x(z_n))$. And since $x(z) = x\left(\frac{1}{z}\right)$, the same is true of f too. Since $z \rightarrow \frac{1}{z}$ leaves ± 1 fixed, changing variables

¹⁸This holds more generally in 1-cut cases, not necessarily Gaussian. The spectral curve has genus s when the large N limit is associated with $s + 1$ disconnected cuts.



z -plane

Figure 2. The contour integral for computing the matrix correlator $f(x)$ in z -plane (the big circle) gets contribution only from the pole of $f(x(z))$ at $z = 0$. The residue at poles of resolvent (dashed circles) vanish.

to $\frac{1}{z}$, we obtain

$$\begin{aligned} \text{Res}_{z_1=\pm 1} \omega_{n,g}(z_1, \dots, z_n) f(x(z_1), \dots, x(z_n)) &= \text{Res}_{z_1=\pm 1} \omega_{n,g}\left(\frac{1}{z_1}, z_2, \dots, z_n\right) f\left(x\left(\frac{1}{z_1}\right), \dots, x(z_n)\right) \\ &= - \text{Res}_{z_1=\pm 1} \omega_{n,g}(z_1, \dots, z_n) f(x(z_1), \dots, x(z_n)) = 0. \end{aligned} \quad (\text{C.2})$$

As a result the sole contribution to the correlation function of f comes from its own poles, which for a polynomial of x_i are at $z(x_i) = 0$ (inside the contour) and $z(x_i) \rightarrow \infty$ (outside the contour). These correspond to $x \rightarrow \infty$ in the two sheets. Hence, we can write

$$\oint_{z_i} \omega_{n,g}(z_1, \dots, z_n) f(x(z_1), \dots, x(z_n)) = \text{Res}_{z_i=0} \omega_{n,g}(z_1, \dots, z_n) f(x(z_1), \dots, x(z_n)). \quad (\text{C.3})$$

The same logic works for any holomorphic function of x among them the Wilson loop. As we have seen in practice, the contour integral is more convenient for the strong coupling expansion of Wilson loops while the residue at 0 is simpler for chiral operators. Nevertheless this vanishing of residues at ± 1 ensures that there is no ambiguity in the saddle point prescription, since we can smoothly deform the contour past the branch points, as illustrated in figure 2.

C.2 Topological recursion at subleading order

The coordinate u defined in (3.13) which is convenient for extending the saddle point approximation to subleading orders can be seen as a reparameterization of spectral curve as

$$z(u) = \exp\left(2i \arcsin\left(\frac{u}{2}\right)\right). \quad (\text{C.4})$$

This change of variables maps the z -plane to a cylinder $u = \phi + ir$ where ϕ parameterizes a circle of radius 4 extending from -2 to 2 while r a real line. In this manner u is the local complex coordinate on an infinite cylinder. This cylinder is compactified to a sphere by identifying the circle at $u = i\infty$ with one point and the circle at $u = -i\infty$ with another

point. In the u -coordinate the branch point are mapped to 0 and $-2 \sim 2$. In the strong coupling limit the dominant contribution to the expectation value of Wilson loops comes from $u = 0$ while the contour for saddle point integral is the circle $r = 0$.

These coordinates also turns out to be somewhat simpler for carrying out topological recursion. Changing variables, the topological recursion formula (2.8) becomes

$$\omega_{n,g}(u_1, \mathbf{u}) = \underset{v \rightarrow 0, \pm 2}{\text{Res}} K(u_1, v) \left[\omega_{g-1, n+1}(v, -v, \mathbf{u}) + \sum_{h \leq g} \sum_{\mathbf{r} \subset \mathbf{u}} \omega_{h, |\mathbf{r}|}(v, \mathbf{r}) \omega_{g-h, n-|\mathbf{r}|}(-v, \mathbf{u}/\mathbf{r}) \right]. \tag{C.5}$$

In terms of these variables the recursion Kernel K is

$$K(u, v) = -\frac{i du}{2v\sqrt{4-u^2}(u^2-v^2) dv}. \tag{C.6}$$

Apart from the factor $\sqrt{u^2-4}$ which is independent of v and as a result gives an overall multiplicative factor, the kernel is homogeneous in these coordinates if we only keep the residue at $u = 0$. This makes it easier to separate out the contribution of different orders. Indeed defining $\hat{K}(u, v) = -i\sqrt{4-u^2}K(u, v)$, we see that

$$\begin{aligned} \underset{v \rightarrow 0}{\text{Res}} \hat{K}(u, v) \frac{du}{v^{2k}} &= -\frac{du}{2u^{2k+2}}, \\ \underset{v \rightarrow 0}{\text{Res}} \hat{K}(u, v) \frac{dv}{(v-w)^2 v^{2k}} &= -\frac{du}{2} \sum_{i=0}^k \frac{2i+1}{w^{2i+2} u^{2k-2i+2}}. \end{aligned} \tag{C.7}$$

So $\hat{K}(u, v)$ uniformly increases the degree of the poles of differential it acts on by 2. This simplification in the recursion kernel is a trade off due to the fact that the starting point of the recursion $\omega_{2,0}(u, v)$ is now more complicated being given by

$$w_{2,0}(u, v) = \frac{8du dv}{\sqrt{4-u^2}\sqrt{4-v^2} \left(2u^2 + 2v^2 - u^2v^2 - uv\sqrt{4-u^2}\sqrt{4-v^2} \right)}, \tag{C.8}$$

and, for the purposes of carrying out topological recursion, it will be expanded into a double power series easily. Another simplification is that in these coordinates the antisymmetry property (C.1) reads

$$\omega_{n,g}(-u_1, \dots, u_n) = \omega_{n,g}(u_1, \dots, u_n). \tag{C.9}$$

This means in particular that

$$\omega_{n,g}(u_1, \dots, u_n) = i^n du_1 \dots du_n f_{n,g} \left(\frac{1}{u_1^2}, \dots, \frac{1}{u_n^2} \right), \tag{C.10}$$

for some symmetric polynomials $f_{n,g}$. As a result the poles encountered in the saddle point integrals are always of even order. Finally, we observe that all $\omega_{g,n}$ computed through the topological recursion have poles of order at least 4 at $u = 0$ and as a result for the first two orders of poles that we need we can ignore the residues at ± 2 in (C.5).

C.3 Expressions for resolvents at leading and first subleading order of poles

Now we present some of the resolvents needed to compute various explicit expansions presented in the main text (5.15), (6.22), (6.23). We do this by presenting $f_{n,g} \left(\frac{1}{u_1^2}, \dots, \frac{1}{u_n^2} \right)$ as defined above in (C.10). These are symmetric polynomials of their arguments $\frac{1}{u_i^2}$ and to keep the expression relatively compact we present them in terms of elementary symmetric polynomials, cf. (4.3). Similarly to our decomposition of $\omega_{n,g} = \hat{\omega}_{n,g} + \delta\omega_{n,g}$ we divide $f_{n,g}$ into leading $\hat{f}_{n,g}$ and subleading $\delta f_{n,g}$ pieces.

Leading order $\hat{\omega}_{g,n}$:

$n = 1$:

$$\begin{aligned}
 \hat{f}_{1,1} &= \frac{e_1^2}{16}, \\
 \hat{f}_{1,2} &= -\frac{105e_1^5}{1024}, \\
 \hat{f}_{1,3} &= \frac{25025e_1^8}{32768}, \\
 \hat{f}_{1,4} &= -\frac{56581525e_1^{11}}{4194304}, \\
 \hat{f}_{1,5} &= \frac{58561878375e_1^{14}}{134217728}, \\
 \hat{f}_{1,6} &= -\frac{193039471750125e_1^{17}}{8589934592}, \\
 \hat{f}_{1,7} &= \frac{464259929559050625e_1^{20}}{274877906944}, \\
 \hat{f}_{1,8} &= -\frac{12277353837189093778125e_1^{23}}{70368744177664}.
 \end{aligned} \tag{C.11}$$

$n = 2$:

$$\begin{aligned}
 \hat{f}_{1,1} &= \frac{5}{32}e_1^2e_2 - \frac{7e_2^2}{32}, \\
 \hat{f}_{2,2} &= -\frac{1155e_2e_1^5}{2048} + \frac{2415e_2^2e_1^3}{1024} - \frac{3955e_2^3e_1}{2048}, \\
 \hat{f}_{2,3} &= \frac{425425e_2e_1^8}{65536} - \frac{3028025e_2^2e_1^6}{65536} + \frac{6641635e_2^3e_1^4}{65536} - \frac{4582655e_2^4e_1^2}{65536} + \frac{119665e_2^5}{16384}, \\
 \hat{f}_{2,4} &= -\frac{1301375075e_2e_1^{11}}{8388608} + \frac{1640864225e_2^2e_1^9}{1048576} - \frac{23885486625e_2^3e_1^7}{4194304} + \frac{18836542725e_2^4e_1^5}{2097152} \\
 &\quad - \frac{47950777875e_2^5e_1^3}{8388608} + \frac{2103075975e_2^6e_1}{2097152}, \\
 \hat{f}_{2,5} &= \frac{1698294472875e_2e_1^{14}}{268435456} - \frac{22194951904125e_2^2e_1^{12}}{268435456} + \frac{56694405142375e_2^3e_1^{10}}{134217728} - \frac{142801188354825e_2^4e_1^8}{134217728} \\
 &\quad + \frac{366814122449775e_2^5e_1^6}{268435456} - \frac{222563485169025e_2^6e_1^4}{268435456} + \frac{6268140053175e_2^7e_1^2}{33554432} - \frac{113818730025e_2^8}{16777216},
 \end{aligned}$$

$$\begin{aligned}
 \hat{f}_{2,6} = & -\frac{6756381511254375e_2e_1^{17}}{17179869184} + \frac{54244091561785125e_2^2e_1^{15}}{8589934592} - \frac{714860261976485625e_2^3e_1^{13}}{17179869184} \\
 & + \frac{622461720688933125e_2^4e_1^{11}}{4294967296} - \frac{4917560490983765625e_2^5e_1^9}{17179869184} + \frac{2741103726364128125e_2^6e_1^7}{8589934592} \\
 & - \frac{3222371840821508375e_2^7e_1^5}{17179869184} + \frac{105576169391317625e_2^8e_1^3}{2147483648} - \frac{4002195436113875e_2^9e_1}{1073741824}, \\
 \hat{f}_{2,7} = & \frac{19034657111921075625e_2e_1^{20}}{549755813888} - \frac{362587004985618538125e_2^2e_1^{18}}{549755813888} + \frac{2928158800333479838125e_2^3e_1^{16}}{549755813888} \\
 & - \frac{13056036458334748003125e_2^4e_1^{14}}{549755813888} + \frac{35071726847099021454375e_2^5e_1^{12}}{549755813888} \\
 & - \frac{58109227982455794519375e_2^6e_1^{10}}{549755813888} + \frac{58386905310562738284375e_2^7e_1^8}{549755813888} \\
 & - \frac{33548300609678088609375e_2^8e_1^6}{549755813888} + \frac{2434905970723398046875e_2^9e_1^4}{137438953472} \\
 & - \frac{68144533015364430625e_2^{10}e_1^2}{34359738368} + \frac{312467552127355625e_2^{11}}{8589934592}. \tag{C.12}
 \end{aligned}$$

$n = 3$:

$$\begin{aligned}
 \hat{f}_{3,0} = & -\frac{e_3}{2}, \\
 \hat{f}_{3,1} = & \frac{35}{64}e_3e_1^3 - \frac{75}{64}e_2e_3e_1 + \frac{33e_3^2}{64}, \\
 \hat{f}_{3,2} = & -\frac{15015e_3e_1^6}{4096} + \frac{38115e_2e_3e_1^4}{2048} - \frac{29925e_2^2e_3e_1^3}{2048} - \frac{93555e_2^2e_3e_1^2}{4096} + \frac{46095e_2e_3^2e_1}{2048} + \frac{3955e_3^2e_3}{1024} - \frac{14595e_3^3}{4096}, \\
 \hat{f}_{3,3} = & \frac{8083075e_3e_1^9}{131072} - \frac{65090025e_2e_3e_1^7}{131072} + \frac{56531475e_2^2e_3e_1^6}{131072} + \frac{172297125e_2^2e_3e_1^5}{131072} - \frac{122207085e_2e_3^2e_1^4}{65536} \\
 & + \frac{80627085e_3^3e_1^3}{131072} - \frac{165840675e_2^3e_3e_1^3}{131072} + \frac{247328235e_2^2e_3^2e_1^2}{131072} + \frac{10481625e_2^4e_3e_1}{32768} \\
 & - \frac{97970985e_2e_3^3e_1}{131072} + \frac{7978285e_3^4}{131072} - \frac{8327655e_2^3e_3^2}{32768}, \\
 \hat{f}_{3,4} = & -\frac{32534376875e_3e_1^{12}}{16777216} + \frac{89794880175e_2e_3e_1^{10}}{4194304} - \frac{81307651425e_2^2e_3e_1^9}{4194304} - \frac{737879667525e_2^2e_3e_1^8}{8388608} \\
 & + \frac{588019457025e_2e_3^2e_1^7}{4194304} + \frac{692291124525e_2^2e_3e_1^6}{4194304} - \frac{455097968325e_2^3e_1^6}{8388608} - \frac{1379583079875e_2^2e_3^2e_1^5}{4194304} \\
 & + \frac{858144412125e_2e_3^3e_1^4}{4194304} - \frac{2322086641875e_2^4e_3e_1^4}{16777216} + \frac{1157389608375e_2^3e_3^2e_1^3}{4194304} - \frac{162384146925e_2^4e_3^3e_1^3}{4194304} \\
 & + \frac{87699236625e_2^5e_3e_1^2}{2097152} - \frac{1481627572425e_2^2e_3^3e_1^2}{8388608} + \frac{163545349275e_2e_3^4e_1}{4194304} - \frac{62462324925e_2^4e_3^4e_1}{1048576} \\
 & + \frac{41443427025e_2^3e_3^3}{2097152} - \frac{2103075975e_2^6e_3}{1048576} - \frac{32314471875e_3^5}{16777216}, \\
 \hat{f}_{3,5} = & \frac{52647128659125e_3e_1^{15}}{536870912} - \frac{738758095700625e_2e_3e_1^{13}}{536870912} + \frac{684295548811875e_2^2e_3e_1^{12}}{536870912} \\
 & - \frac{1741727866003125e_2e_3^2e_1^{10}}{134217728} + \frac{1447231892030925e_2^3e_1^9}{268435456} - \frac{5837805914565625e_2^3e_3e_1^9}{268435456} \\
 & + \frac{13104215838338625e_2^2e_3^3e_1^8}{268435456} + \frac{17576000871053625e_2^4e_3e_1^7}{536870912} - \frac{9503237789320275e_2e_3^3e_1^7}{268435456}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{2209625052609975e_3^4e_1^6}{268435456} - \frac{11160180446790375e_2^3e_3^2e_1^6}{134217728} + \frac{20033877534345675e_2^2e_3^3e_1^5}{268435456} \\
& - \frac{13477372127344125e_2^5e_3e_1^5}{536870912} + \frac{33627317776927875e_2^4e_3^2e_1^4}{536870912} - \frac{3697210384993125e_2e_3^4e_1^4}{134217728} \\
& + \frac{1840228057292625e_3^5e_1^3}{536870912} + \frac{564221631348375e_2^5e_3e_1^3}{67108864} - \frac{14908243313588625e_2^3e_3^3e_1^3}{268435456} \\
& + \frac{5572189269021375e_2^2e_3^4e_1^2}{268435456} - \frac{1126515991726125e_2^5e_3^2e_1^2}{67108864} + \frac{702424017169875e_2^4e_3e_1}{67108864} \\
& - \frac{27007478623125e_2^7e_3e_1}{33554432} - \frac{1584673523658225e_2e_3^5e_1}{536870912} + \frac{23592916722375e_2^6e_3^2}{33554432} \\
& + \frac{2063427784543125e_2^2e_3e_1^1}{268435456} + \frac{52435195988175e_3^6}{536870912} - \frac{133329976054275e_2^3e_3^4}{67108864}. \tag{C.13}
\end{aligned}$$

$n = 4$:

$$\begin{aligned}
\hat{f}_{4,0} &= -\frac{3}{4}e_1e_4, \\
\hat{f}_{4,1} &= \frac{315}{128}e_4e_1^4 - \frac{945}{128}e_2e_4e_1^2 + \frac{615}{128}e_3e_4e_1 + \frac{75}{32}e_2^2e_4 - \frac{159e_4^2}{64}, \\
\hat{f}_{4,2} &= -\frac{225225e_4e_1^7}{8192} + \frac{675675e_2e_4e_1^5}{4096} - \frac{557865e_3e_4e_1^4}{4096} + \frac{111825e_4^2e_1^3}{1024} - \frac{2248785e_2^2e_4e_1^3}{8192} + \frac{1335285e_2e_3e_4e_1^2}{4096} \\
& - \frac{41895}{256}e_2e_4^2e_1 + \frac{222705e_2^3e_4e_1}{2048} - \frac{655305e_3^2e_4e_1}{8192} + \frac{109095e_3e_4^2}{2048} - \frac{40845}{512}e_2^2e_3e_4, \\
\hat{f}_{4,3} &= \frac{169744575e_4e_1^{10}}{262144} - \frac{1527701175e_2e_4e_1^8}{262144} + \frac{1351575225e_3e_4e_1^7}{262144} + \frac{4751571825e_2^2e_4e_1^6}{262144} \\
& - \frac{3545266725e_2e_3e_4e_1^5}{131072} + \frac{1267160895e_2e_4^2e_1^4}{65536} + \frac{2514637125e_3^2e_4e_1^4}{262144} - \frac{5933552625e_2^3e_4e_1^4}{262144} \\
& + \frac{10111025925e_2^2e_3e_4e_1^3}{262144} - \frac{104757345e_3e_4^2e_1^3}{8192} + \frac{125697285e_4^3e_1^2}{32768} + \frac{316891575e_2^4e_4e_1^2}{32768} \\
& - \frac{5013144675e_2e_3^2e_4e_1^2}{262144} + \frac{250441065e_2e_3e_4^2e_1}{16384} + \frac{659630475e_3^3e_4e_1}{262144} - \frac{837819675e_2^3e_3e_4e_1}{65536} \\
& + \frac{247868775e_2^2e_3^2e_4}{65536} - \frac{10481625e_2^5e_4}{16384} - \frac{62462295e_2e_4^3}{32768} - \frac{246043245e_2^2e_4^2}{131072} \\
& - \frac{591666075e_4^2e_1^6}{131072} - \frac{2529529695e_2^2e_4^2e_1^2}{131072} + \frac{83600685e_2^3e_4^2}{32768}. \tag{C.14}
\end{aligned}$$

$n = 5, 6, 7, 8, 9$:

$$\begin{aligned}
\hat{f}_{5,0} &= \frac{3e_2e_5}{2} - \frac{15}{8}e_1^2e_5, \\
\hat{f}_{5,1} &= \frac{3465}{256}e_5e_1^5 - \frac{13545}{256}e_2e_5e_1^3 + \frac{9975}{256}e_3e_5e_1^2 + \frac{2415}{64}e_2^2e_5e_1 - \frac{3255}{128}e_4e_5e_1 + \frac{1347e_5^2}{128} - \frac{1515}{64}e_2e_3e_5, \\
\hat{f}_{6,0} &= -\frac{105}{16}e_6e_1^3 + \frac{45}{4}e_2e_6e_1 - \frac{9e_3e_6}{2}, \\
\hat{f}_{6,1} &= \frac{45045}{512}e_6e_1^6 - \frac{218295}{512}e_2e_6e_1^4 + \frac{171045}{512}e_3e_6e_1^3 + \frac{31185}{64}e_2^2e_6e_1^2 - \frac{62685}{256}e_4e_6e_1^2 - \frac{59535}{128}e_2e_3e_6e_1 \\
& + \frac{39285}{256}e_5e_6e_1 - \frac{2415}{32}e_2^3e_6 + \frac{4545}{64}e_2^2e_6 + \frac{9315}{64}e_2e_4e_6 - \frac{2385e_6^2}{32}, \\
\hat{f}_{7,0} &= -\frac{945}{32}e_7e_1^4 + \frac{315}{4}e_2e_7e_1^2 - 45e_3e_7e_1 - \frac{45}{2}e_2^2e_7 + 18e_4e_7
\end{aligned}$$

$$\begin{aligned}
 \hat{f}_{7,1} &= \frac{675675e_7e_1^7}{1024} - \frac{3918915e_2e_7e_1^5}{1024} + \frac{3191265e_3e_7e_1^4}{1024} + \frac{779625}{128}e_2^2e_7e_1^3 - \frac{1248345}{512}e_4e_7e_1^3 - \frac{898695}{128}e_2e_3e_7e_1^2 \\
 &+ \frac{901845}{512}e_5e_7e_1^2 - \frac{146475}{64}e_2^3e_7e_1 + \frac{52605}{32}e_3^2e_7e_1 + \frac{428715}{128}e_2e_4e_7e_1 - \frac{141705}{128}e_6e_7e_1 \\
 &+ \frac{55845e_7^2}{128} + \frac{103005}{64}e_2^2e_3e_7 - \frac{64305}{64}e_3e_4e_7 - \frac{132435}{128}e_2e_5e_7, \\
 \hat{f}_{8,0} &= -\frac{10395}{64}e_8e_1^5 + \frac{4725}{8}e_2e_8e_1^3 - \frac{1575}{4}e_3e_8e_1^2 - \frac{1575}{4}e_2^2e_8e_1 + 225e_4e_8e_1 + 225e_2e_3e_8 - 90e_5e_8, \\
 \hat{f}_{8,1} &= \frac{11486475e_8e_1^8}{2048} - \frac{77702625e_2e_8e_1^6}{2048} + \frac{64999935e_3e_8e_1^5}{2048} + \frac{39864825}{512}e_2^2e_8e_1^4 - \frac{26496855e_4e_8e_1^4}{1024} \\
 &- \frac{25623675}{256}e_2e_3e_8e_1^3 + \frac{20600055e_5e_8e_1^3}{1024} - \frac{6288975}{128}e_2^3e_8e_1^2 + \frac{3642975}{128}e_3^2e_8e_1^2 + \frac{7396515}{128}e_2e_4e_8e_1^2 \\
 &- \frac{1848735}{128}e_6e_8e_1^2 + \frac{7158375}{128}e_2^2e_3e_8e_1 - \frac{427455}{16}e_3e_4e_8e_1 - \frac{7017885}{256}e_2e_5e_8e_1 + \frac{2263095}{256}e_7e_8e_1 \\
 &+ \frac{146475}{32}e_2^4e_8 - \frac{414225}{32}e_2e_3^2e_8 + \frac{64305}{16}e_4^2e_8 - \frac{840735}{64}e_2^2e_4e_8 + \frac{1040355}{128}e_3e_5e_8 \\
 &+ \frac{269505}{32}e_2e_6e_8 - \frac{545175e_8^2}{128}, \\
 \hat{f}_{9,0} &= -\frac{135135}{128}e_9e_1^6 + \frac{155925}{32}e_2e_9e_1^4 - \frac{14175}{4}e_3e_9e_1^3 - \frac{42525}{8}e_2^2e_9e_1^2 + \frac{4725}{2}e_4e_9e_1^2 \\
 &+ 4725e_2e_3e_9e_1 - 1350e_5e_9e_1 + \frac{1575}{2}e_2^3e_9 - 675e_3^2e_9 - 1350e_2e_4e_9 + 540e_6e_9.
 \end{aligned} \tag{C.15}$$

First subleading order $\delta\omega_{n,g}$:

$n = 1$:

$$\begin{aligned}
 \delta f_{1,1} &= \frac{5e_1}{128}, \\
 \delta f_{1,2} &= -\frac{483e_1^4}{8192}, \\
 \delta f_{1,3} &= \frac{137137e_1^7}{262144}, \\
 \delta f_{1,4} &= -\frac{370204835e_1^{10}}{33554432}, \\
 \delta f_{1,5} &= \frac{448974400875e_1^{13}}{1073741824}.
 \end{aligned} \tag{C.16}$$

$n = 2$:

$$\begin{aligned}
 \delta f_{2,1} &= \frac{5e_1^3}{256} + \frac{3e_2e_1}{256}, \\
 \delta f_{2,2} &= -\frac{1155e_1^6}{16384} + \frac{819e_2e_1^4}{8192} + \frac{5817e_2^2e_1^2}{16384} - \frac{833e_2^3}{4096}, \\
 \delta f_{2,3} &= \frac{425425e_1^9}{524288} - \frac{1396395e_2e_1^7}{524288} - \frac{3252249e_2^2e_1^5}{524288} + \frac{12156375e_2^3e_1^3}{524288} - \frac{1591065e_2^4e_1}{131072}, \\
 \delta f_{2,4} &= -\frac{1301375075e_1^{12}}{67108864} + \frac{1663496835e_2e_1^{10}}{16777216} + \frac{5426976555e_2^2e_1^8}{33554432} - \frac{27126714615e_2^3e_1^6}{16777216} \\
 &+ \frac{184265806725e_2^4e_1^4}{67108864} - \frac{11560844295e_2^5e_1^2}{8388608} + \frac{427732305e_2^6}{4194304}.
 \end{aligned} \tag{C.17}$$

$n = 3$:

$$\begin{aligned}
 \delta f_{3,0} &= -\frac{e_2}{16}, \\
 \delta f_{3,1} &= \frac{35}{512}e_2e_1^3 + \frac{15}{256}e_3e_1^2 - \frac{75}{512}e_2^2e_1 + \frac{21e_2e_3}{512}, \\
 \delta f_{3,2} &= -\frac{15015e_2e_1^6}{32768} - \frac{7623e_3e_1^5}{8192} + \frac{38115e_2^2e_1^4}{16384} + \frac{29925e_2e_3e_1^3}{16384} - \frac{21945e_3^2e_1^2}{8192} - \frac{93555e_2^3e_1^2}{32768} \\
 &\quad + \frac{3255e_2^2e_3e_1}{16384} + \frac{3955e_2^4}{8192} + \frac{37485e_2e_3^2}{32768}, \\
 \delta f_{3,3} &= \frac{8083075e_2e_1^9}{1048576} + \frac{13018005e_3e_1^8}{524288} - \frac{65090025e_2^2e_1^7}{1048576} - \frac{125450325e_2e_3e_1^6}{1048576} + \frac{172297125e_3^2e_1^5}{1048576} \\
 &\quad + \frac{38657619e_2^3e_1^5}{262144} + \frac{18063045e_2^2e_3e_1^4}{131072} - \frac{165840675e_2^4e_1^3}{1048576} - \frac{432151335e_2e_3^2e_1^3}{1048576} + \frac{75661425e_3^3e_1^2}{524288} \\
 &\quad - \frac{10481625e_2^3e_3e_1^2}{1048576} + \frac{10481625e_2^5e_1}{262144} + \frac{205872975e_2^2e_3^2e_1}{1048576} - \frac{1963395e_2^4e_3}{262144} - \frac{65346575e_2e_3^3}{1048576}.
 \end{aligned} \tag{C.18}$$

$n = 4, 5$:

$$\begin{aligned}
 \delta f_{4,0} &= \frac{3e_4}{16} - \frac{3e_1e_3}{32}, \\
 \delta f_{4,1} &= \frac{315e_3e_1^4}{1024} - \frac{945e_2e_3e_1^2}{1024} + \frac{615e_3^2e_1}{1024} + \frac{75}{512}e_2e_4e_1 + \frac{75}{256}e_2^2e_3 - \frac{111e_3e_4}{256}, \\
 \delta f_{4,2} &= -\frac{225225e_3e_1^7}{65536} - \frac{153153e_4e_1^6}{32768} + \frac{675675e_2e_3e_1^5}{32768} + \frac{183645e_2e_4e_1^4}{8192} - \frac{557865e_3^2e_1^4}{32768} - \frac{31815e_3e_4e_1^3}{8192}, \\
 &\quad - \frac{2248785e_2^2e_3e_1^3}{65536} + \frac{1335285e_2e_3^2e_1^2}{32768} + \frac{106785e_4^2e_1^2}{8192} - \frac{823095e_2^2e_4e_1^2}{32768} + \frac{222705e_2^3e_3e_1}{16384}, \\
 &\quad + \frac{50295e_2e_3e_4e_1}{16384} - \frac{655305e_3^3e_1}{65536} + \frac{30975e_2^3e_4}{8192} + \frac{105735e_3^2e_4}{32768} - \frac{40845e_2^2e_3^2}{4096} - \frac{61635e_2e_4^2}{8192}, \\
 \delta f_{5,0} &= -\frac{15}{64}e_4e_1^2 + \frac{21e_5e_1}{32} + \frac{3e_2e_4}{16}, \\
 \delta f_{5,1} &= \frac{3465e_4e_1^5}{2048} - \frac{315}{256}e_5e_1^4 - \frac{13545e_2e_4e_1^3}{2048} + \frac{9975e_3e_4e_1^2}{2048} + \frac{4305e_2e_5e_1^2}{1024} + \frac{2415}{512}e_2^2e_4e_1 \\
 &\quad - \frac{3255e_4^2e_1}{1024} - \frac{3225e_3e_5e_1}{1024} - \frac{1515}{512}e_2e_3e_4 - \frac{375}{256}e_2^2e_5 + \frac{3123e_4e_5}{1024}.
 \end{aligned} \tag{C.19}$$

D The correlation function $\langle : \text{tr} M : \mathcal{W}^n \rangle$

In the main text, to prove (6.21), we exploited the fact that $\langle : \text{tr} M : \mathcal{W}^n \rangle$ has no higher genus corrections beyond the leading order. This can be easily proved by starting from the following splitting of M in the $U(N)$ theory

$$M = \tilde{M} + \frac{m}{N}, \quad \tilde{M} = M - \frac{1}{N} \text{tr} M, \quad m = \text{tr} M, \tag{D.1}$$

where \tilde{M} is the traceless part. The matrix model partition function becomes

$$Z = \int_{-\infty}^{\infty} dm \int d\tilde{M} \delta(\text{tr} \tilde{M}) \exp\left(-\frac{N}{2} \text{tr} \tilde{M}^2 - \frac{m^2}{2}\right). \tag{D.2}$$

For the Wilson loop operator, the splitting (D.1) implies

$$\text{tr exp} \left(\frac{\sqrt{\lambda}}{2} M \right) = \exp \left(\frac{\sqrt{\lambda}}{2N} m \right) \text{tr exp} \left(\frac{\sqrt{\lambda}}{2} \tilde{M} \right). \quad (\text{D.3})$$

As a result the expectation value of n coincident Wilson loops takes the form

$$\begin{aligned} \langle \mathcal{W}^n \rangle &= \langle \mathcal{W}^n \rangle_{\text{traceless}} \int_{-\infty}^{\infty} dm \exp \left(-\frac{m^2}{2} + \frac{n\lambda}{2N} m \right), \\ \langle \mathcal{W}^n \rangle_{\text{traceless}} &= \int d\tilde{M} \delta(\text{tr } \tilde{M}) \left[\text{tr exp} \left(\frac{\lambda}{2} \tilde{M} \right) \right]^n \exp \left(-\frac{N}{2} \text{tr } \tilde{M}^2 \right). \end{aligned} \quad (\text{D.4})$$

In the case of $\langle m\mathcal{W}^n \rangle$, we obtain the same integral for \tilde{M} with an extra insertion of m in the m -integral. As a result, the “traceless” part $\langle \mathcal{W}^n \rangle_{\text{traceless}}$ cancels and we obtain

$$\frac{\langle m\mathcal{W}^n \rangle}{\langle \mathcal{W}^n \rangle} = \frac{\int_{-\infty}^{\infty} dm m \exp \left(-\frac{m^2}{2} + \frac{n\sqrt{\lambda}}{2N} m \right)}{\int_{-\infty}^{\infty} dm \exp \left(-\frac{m^2}{2} + \frac{n\sqrt{\lambda}}{2N} m \right)} = \frac{n\sqrt{\lambda}}{2N}. \quad (\text{D.5})$$

This is just the leading order result obtained in (6.13) and specialized to $J = 1$. The above discussion shows that it is in fact exact.

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