

Modular zero modes and sewing the states of QFT

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ABSTRACT: We point out an important difference between continuum relativistic quantum field theory (QFT) and lattice models with dramatic consequences for the theory of multipartite entanglement. On a lattice given a collection of density matrices $\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(n)}$ there is no guarantee that there exists an n -partite pure state $|\Omega\rangle_{12\dots n}$ that reduces to these marginals. The state $|\Omega\rangle_{12\dots n}$ exists only if the eigenvalues of the density matrices $\rho^{(i)}$ satisfy certain polygon inequalities. We show that in QFT, as opposed to lattice systems, splitting the space into n non-overlapping regions any collection of local states $\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(n)}$ come from the restriction of a global pure state. The reason is that rotating any local state $\omega^{(i)}$ by unitary U_i localized in the i^{th} region we come arbitrarily close to any other local state $\psi^{(i)}$. We construct explicit examples of such local unitaries using the cocycle.

KEYWORDS: Field Theories in Higher Dimensions, Space-Time Symmetries

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1 Introduction

Consider the Hilbert space of an n -partite lattice quantum system $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_n$, where \mathcal{H}_i is the Hilbert space of site A_i . We call a pure state of the n -partite system a global state, and the density matrices $\rho^{(i)}$ at site A_i local states. Given a collection of local states $\rho^{(1)}$ to $\rho^{(n)}$ we say we can *sew* them if there exists a global pure state $|\Omega\rangle$ that reduces to $\rho^{(i)}$ at each site A_i ; e.g. $\rho^{(i)} = \text{tr}_{j \neq i} |\Omega\rangle \langle \Omega|$ for all i . Physically, the collection of density matrices on A_i are a mean field approximation to a pure global state. In finite quantum systems, there are constraints on the density matrices that can be sewn together. For instance, in a bipartite system, the density matrices $\rho^{(1)}$ and $\rho^{(2)}$ can be sewn if and only if they have the same eigenvalues. These eigenvalues are the Schmidt coefficients of the pure global state $|\Omega\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. The constraints become more complicated for $n > 2$, and imply that the eigenvalues of $\rho^{(i)}$ should satisfy polygon inequalities [1, 2]. The knowledge of $\rho^{(j)}$ for $j \neq i$ and the fact that the global state is pure implies some information about $\rho^{(i)}$,

however this information is partial because it does not fix $\rho^{(i)}$. A local unitary rotation U_i at site A_i does not affect the state on the other sites, therefore $\rho^{(i)}$ can always be replaced with $U_i \rho^{(i)} U_i^\dagger$.¹

Consider a pure state of a Poincare invariant QFT in Minkowski spacetime of arbitrary dimensions on a constant time-slice. Split the time-slice into n non-overlapping regions A_1 to A_n such that $\cup_i A_i$ is the whole time-slice. For instance, in two dimensional quantum field theory consider the intervals $(-\infty, a_1)$, (a_1, a_2) , (a_2, a_3) , up to (a_{n-1}, ∞) . Each region A_i has a local algebra of observables \mathcal{A}_i .² The analog of local density matrices are the restrictions of the global state to the local algebras:

$$\omega_i(a_i) = \langle \Omega | a_i \Omega \rangle \quad \forall a_i \in \mathcal{A}_i . \quad (1.1)$$

Since the global algebra of QFT is not the tensor product $\mathcal{A}_1 \otimes \dots \mathcal{A}_n$ we cannot sew all states perfectly. However, in this work we argue that as opposed to lattice systems, if we allow for arbitrarily small errors it is possible to sew any collection of ω_i s in a pure state. As we show, the reason is that, in QFT, for any local state ψ_i there exists a unitary U_i in \mathcal{A}_i such that $\psi_i(a_i) \simeq \omega_i(U_i^\dagger a_i U_i)$ for all $a_i \in \mathcal{A}_i$. We use \simeq as opposed to equality to indicate that the unitarily rotated states approximate ω_i arbitrarily well in the norm topology of operators. For instance, this procedure allows us to start with an arbitrary global state $|\Psi\rangle$ and act on A with a unitary to prepare a state that is the same as vacuum on region A and the same as $|\Psi\rangle$ outside of A . Similarly, one can sew states of different QFTs together. The conventional formulation of quantum entanglement theory is designed for lattice systems. To study entanglement in QFT, one needs to replace the conventional formulation with a more general algebraic framework called Modular theory that applies to any quantum system from qubits to QFT. We use modular theory to construct the local unitaries that sew states to the vacuum in QFT. This unitary was recently used to derive a surprising inequality that holds in any QFT called Quantum Null Energy Condition (QNEC), but has no analog in lattice systems [3].

2 Sewing qudits

It is worthwhile to analyze the sewing problem on a lattice first to identify the origin of the obstruction to sew arbitrary states. We focus on the bipartite case, and come back to the general n -partite case in section 8. Every density matrix of system A written in

¹If the local Hilbert space \mathcal{H}_i is infinite dimensional it suffices for U_i to be an isometry, i.e. $U_i^\dagger U_i = 1_i$ and $U_i U_i^\dagger$ a projection in \mathcal{A}_i . In finite dimensions, there are no distinctions between unitaries and isometries. In infinite dimensional systems every local isometry has a unitary arbitrarily close to it in the strong operator topology. We prove this in appendix D. We thank Martin Argerami and Roberto Longo for discussion on this.

²A subtle assumption we make is that the algebras of disconnected regions of quantum field theory are additive. That is if $A_1 \cap A_2 = 0$ then the algebra $\mathcal{A}_{12} = \mathcal{A}_1 \cup \mathcal{A}_2$. This assumption is violated in systems with conserved charges if we take the local algebra to be comprised of only charge neutral operators. However, in such system the local algebra of all charged operators (algebra of fields) is still additive. We are also assuming that the local algebra of quantum field theory is type III₁, see appendix C. This excludes models such as generalized free fields.

its eigenbasis $\rho = \sum_k p_k |k\rangle \langle k|$ has a canonical purification in a pure global state $|\Omega\rangle = \sum_k \sqrt{p_k} |k, k\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, where $\mathcal{H}_{\bar{A}}$ is isomorphic to \mathcal{H}_A and the reduced state on \bar{A} is also ω . The probability p_k corresponds to the projections $e_k = |k\rangle \langle k|$ and if all $p_k > 0$ we say the state has the Reeh-Schlieder property. Reeh-Schlieder states have invertible density matrices and form a dense set of states. In the remainder of this work, we restrict to the Reeh-Schlieder states. The generalization of our discussion to non-Reeh-Schlieder states is straightforward. The canonical purification of a density matrix ρ sews it to a second identical copy on \bar{A} . This symmetry of the canonical purification $|\Omega\rangle$ is captured by its invariance under the action of the anti-linear operator J_Ω that swaps the eigenbasis of ρ_A and $\rho_{\bar{A}}$: $J_\Omega(c|k, k'\rangle) = c^* |k', k\rangle$. This operator is called the modular conjugation and depends on the state $|\Omega\rangle$ through the choice of the eigenbasis of ρ . Given a second density matrix $\psi = \sum_k q_k |\alpha_k\rangle \langle \alpha_k|$ there is a unique purification $|\Psi_\Omega\rangle = (\psi^{1/2} \otimes \rho^{-1/2}) |\Omega\rangle$ that is symmetric under the swap $J_\Omega |\Psi_\Omega\rangle = |\Psi_\Omega\rangle$. We call $\Delta_{\Psi|\Omega} \equiv \psi \otimes \rho^{-1}$ the relative modular operator and write $|\Psi_\Omega\rangle = \Delta_{\Psi|\Omega}^{1/2} |\Omega\rangle$. If both density matrices are the same we write $\Delta_\rho = \rho \otimes \rho^{-1}$ and call it the modular operator. These operators act in the global Hilbert space of $A\bar{A}$. For two vectors $|\Psi_\Omega\rangle$ and $|\Phi_\Omega\rangle$ both invariant under J_Ω we have

$$\Delta_{\Psi|\Omega}^{1/2} |\Phi_\Omega\rangle = |\Psi_\Omega\rangle . \tag{2.1}$$

The adjoint action of the modular conjugation on operators in A sends them to \bar{A} and vice versa:

$$J_\Omega(|k\rangle \langle k'|_A \otimes 1_{\bar{A}}) J_\Omega = 1_A \otimes |k\rangle \langle k'|_{\bar{A}} . \tag{2.2}$$

If a is in the algebra of A then we define $a_J = J_\Omega a J_\Omega$ to be its corresponding operator in \bar{A} . Hereafter, we suppress the subscript Ω of J , and where it is clear from the context suppress the identity operator $1_{\bar{A}}$ to simplify notation.

In the bipartite case, the necessary and sufficient condition for sewing two density matrices is that they have the same eigenvalues. If ψ does not have the same eigenvalue as ρ there might still exist a density matrix very close to ψ that can be sewn to ρ . We would like to find the closest density matrix to ψ that can be sewn to ρ . The only density matrices that can be sewn to ρ are $U\rho U^\dagger$, therefore we have to introduce a distance measure $d(\rho, \psi)$ on the space of density matrices and taking an infimum over all unitaries: $\inf_U d(U\rho U^\dagger, \psi)$. The trace distance is a distance measure on the space of density matrices defined to be

$$\|\rho - \psi\|_1^2 = \text{tr}(|\rho - \psi|) = \sup_{\|C\|_\infty=1} \text{tr}(C(\rho - \psi)) \tag{2.3}$$

where $\|C\|_\infty$ is the operator norm of C that is its largest eigenvalue. It is sometimes more convenient to work with the following distance measure on the space of density matrices:

$$d_F(\rho, \psi)^2 = \frac{1}{2} \|\sqrt{\rho} - \sqrt{\psi}\|_F^2 = 1 - \text{tr}(\sqrt{\rho}\sqrt{\psi}) \tag{2.4}$$

where $\|X\|_F^2 = \text{tr}(X^\dagger X)$ is the Frobenius norm of a matrix. If two density matrices are close in the Frobenius distance they are also close in trace distance; see appendix A. The

advantage of the Frobenius distance is that it matches the Hilbert space distance of the canonical purifications:

$$d_F(\rho, \psi) = \| |\Psi_\Omega\rangle - |\Omega\rangle \| . \quad (2.5)$$

If we think of $|\chi\rangle = |\Psi_\Omega\rangle - |\Omega\rangle$ as an unnormalized vector in the Hilbert space we have

$$d_F(\rho, \psi) = \|\chi\| = \sup_{\|\Phi\|=1} |\langle \Phi | \chi \rangle| . \quad (2.6)$$

The vector $|\Phi\rangle$ that achieves the supremum is parallel to $|\chi\rangle$ and saturates the Cauchy-Schwarz inequality. On a lattice, the infimum distance $\inf_U d_F(U\rho U^\dagger, \psi)$ can be calculated explicitly. The unitary $U_0 = \sum_k |\alpha_k\rangle \langle k|$ that rotates the eigenbasis of ρ to that of ψ sets an upper bound on this infimum distance:

$$\inf_U d(U\rho U^\dagger, \psi) \leq d(p, q) \quad (2.7)$$

where $d(p, q)$ is the classical analog of our distance measure for the probability distributions $\{p_k\}$ and $\{q_k\}$, i.e. $d(p, q)^2 = \frac{1}{2} \sum_k (\sqrt{q_k} - \sqrt{p_k})^2 = 1 - \sum_k \sqrt{p_k q_k}$. In general, this is not a tight bound because the unitaries that relabel the basis of ρ can further reduce the distance. Each permutation in the symmetric group $\sigma \in S_d$, where d is the dimension of \mathcal{H}_A , gives a relabelling unitary $U_\sigma = \sum_k |\alpha_{\sigma(k)}\rangle \langle \alpha_k|$. The action of these unitaries is equivalent to keeping the eigenvectors fixed and permuting the eigenvalues. We tighten our bound to

$$\begin{aligned} \inf_U d(\rho, \psi) &\leq \inf_{\sigma \in S_d} d(p, \sigma(q)) \\ \sigma(q)_k &= q_{\sigma(k)} . \end{aligned} \quad (2.8)$$

The classical distance $d(p, \sigma(q))$ is minimized if the fidelity $\sum_k \sqrt{p_k q_k}$ is maximized. If we order p_k in decreasing order $p_1 \geq p_2 \geq \dots \geq p_d$ this maximum is achieved by the relabeling unitary that orders q in decreasing order: $q_1 \geq q_2 \geq \dots \geq q_d$, and then matches them by $\sigma(k) = k$:

$$\inf_{\sigma \in S_d} d_F^2(p, \sigma(q)) = 1 - \sum_k \sqrt{p_k q_k} = \sum_k q_k \left(1 - \frac{p_k}{q_k}\right), \quad (2.9)$$

The terms p_k/q_k above are the eigenvalues of the relative modular operator. The Frobenius distance is small if the spectrum of the relative modular operator is limited to a small range near one. In fact, for finite d the inequality in (2.8) is an equality [4], and the minimum of the classical distance we found is the same as the minimum of the quantum distance.

In infinite dimensions, the spectrum of the modular operator can become continuous and notion of eigenvalues might stop making sense. In such a case, there can exist relabelling unitaries that squeeze the spectrum of the relative modular operator to a small neighborhood of one. These are the unitaries that we allow us to sew any two local states in QFT. We come back to this in section 4.

3 Infinite dimensions

The generalization of the trace distance in (2.3) to local states in QFT is

$$\|\omega - \psi\|_1^2 = \sup_{\|a\|=1} |\omega(a) - \psi(a)| \tag{3.1}$$

with $\|a\|$ is the operator norm and ω is the restriction of the global state to a local algebra as in (1.1). Let us call the set of states $U\omega U^\dagger$ for U unitaries in \mathcal{A} the unitary orbit of ω , and denote by $[\omega]$ the equivalence class of ω under unitary rotations. We define the strong unitary orbit distance to be

$$d_s([\omega], [\psi]) = \inf_U \sup_{\|a\|=1} |\omega(a) - \psi(U^\dagger a U)| \tag{3.2}$$

where U are unitaries in the local algebra. As in the case of density matrices, it is more convenient to work the Frobenius distance in (2.5) of the canonical purifications and use the inequality³

$$\| |\Omega\rangle - |\Psi_\Omega\rangle \|^2 \leq \|\omega - \psi\|^2 \leq 2 \| |\Omega\rangle - |\Psi_\Omega\rangle \|^2 \tag{3.3}$$

and consider the unitary orbit distance measure

$$d_F([\omega], [\psi]) = \inf_U \| |U_\Omega\rangle - |\Psi_\Omega\rangle \|^2 \tag{3.4}$$

Both $|U_\Omega\rangle = UJU |\Omega\rangle = UU_J |\Omega\rangle$ and $|\Psi_\Omega\rangle$ are the canonical purifications of local states $U^\dagger \omega U$ and ψ , respectively, and are both invariant under the action of J . We apply (2.1) to rewrite the Frobenius distance in terms of the relative modular operator:

$$\begin{aligned} d_F([\omega], [\psi]) &= \inf_{U \in \mathcal{A}} \left\| \left(\Delta_{\Psi_\Omega|U_\Omega}^{1/2} - 1 \right) |U_\Omega\rangle \right\| \\ &= \inf_{U \in \mathcal{A}} \| (\Delta_{\Psi|_\Omega}^{1/2} - 1) U |\Omega\rangle \| \end{aligned} \tag{3.5}$$

The second line can be checked explicitly in the case of density matrices, or more formally using the equations $|\Psi_\Omega\rangle = V' |\Psi\rangle$ with V' and U' unitaries in the algebra of \bar{A} and $\Delta_{V'\Psi|UU'\Omega} = U' \Delta_{\Psi|_\Omega} U$ that can be shown using equation (6.4). The unitary orbit distances is zero if and only if the state $|U_\Omega\rangle$ is invariant under the action of the relative modular operator. This ties the problem of sewing local states ψ of A to ω of its complementary region \bar{A} to the problem of locally preparing an invariant state of their relative modular operator.

In infinite dimensions, the infimum and supremum above need not commute. Therefore, we define a second distance measure that we call the weak unitary orbit distance to be

$$d_w([\omega], [\psi]) = \sup_{\|a\|=1} \inf_U |\omega(a) - \psi(U^\dagger a U)| \tag{3.6}$$

To obtain intuition about the difference between the weak and the strong distance consider the following example. Take the Hilbert space of a particle on a line and the set

³See theorem 2.2.3 (vi) of [5].

of unitaries $U_x = e^{ixP}$ with P the translation operator and the matrix element $\langle \Phi | U_x \Psi \rangle$. For generic $|\Phi\rangle$ and $|\Psi\rangle$ we have

$$\lim_{x \rightarrow \infty} \sup_{|\Phi\rangle \in \mathcal{H}} |\langle \Phi | e^{ixP} \Psi \rangle| = \lim_{x \rightarrow \infty} \|e^{ixP} |\Psi\rangle\| = 1 \quad (3.7)$$

whereas if we commute the sup and lim we find

$$\sup_{|\Phi\rangle \in \mathcal{H}} \lim_{x \rightarrow \infty} |\langle \Phi | e^{ixP} \Psi \rangle| = \langle \Phi | \Pi_0 \Psi \rangle < 1 \quad (3.8)$$

where Π_0 is the projection to the subspace of translation-invariant states: $P\Pi_0 = 0$. Before proving this statement, let us explain why it holds in physics terms. For vectors $|\Phi\rangle$ and $|\Psi\rangle$ that are the wave packets of two particles localized at some point in space with some width and the operator e^{ixP} moves them away from each other. In the limit of infinite distance, the overlap between the wave-packets decays to zero. Now, if we replace $|\Psi\rangle$ with the delta-function normalizable momentum eigenstate $|p\rangle$ then $\langle \Phi | e^{ixP} |p\rangle = \langle \Phi | p \rangle e^{ixp}$ that oscillates erratically at large t with no limit. The point is that to make physical states with unit norm we need wave-packets for which the limit in (3.8) exists. In other words, we have the limit in (3.8) because the translation operator has an entirely continuum spectrum with no normalizable eigenvectors. The momentum eigenstate $|p\rangle$ is not normalizable.

If we call $f(x) = |\langle \Phi | e^{ixP} \Psi \rangle|$ then the large time limit in (3.8) is the infinite Fourier mode of $f(x)$. The spectrum of the momentum operator is continuous, therefore

$$\lim_{x \rightarrow \infty} |\langle \Phi | e^{ixP} \Psi \rangle| = \int_{-\infty}^{\infty} e^{ixp} \langle \Phi | \Pi_p(dp) \Psi \rangle \quad (3.9)$$

where $\Pi_p(dp)$ is a projection operator valued measure. In the language of delta-function normalizable vectors $\Pi_p(dp) = dp |p\rangle \langle p|$. Set $|\Phi\rangle = |\Psi\rangle$ so that $\hat{f}(p)$ is positive and

$$\int |\hat{f}(p)| dp = \langle \Psi | \Psi \rangle = 1. \quad (3.10)$$

Then, equation (3.8) for $|\Phi\rangle = |\Psi\rangle$ follows from the Riemann-Lebesgue lemma that says if $\int |\hat{f}(p)| dp < \infty$ for function $\hat{f}(p)$ its infinite Fourier mode $\lim_{x \rightarrow \infty} f(x)$ vanishes. The equation (3.8) for all matrix elements follows from considering the diagonal terms in states $|\Phi\rangle \pm |\Psi\rangle$.

One can ask how large is the subspace of translation invariant states $\Pi_0 \mathcal{H}$. If there is an operator h that commutes with the group e^{itP} and $|\Omega\rangle$ is a translation invariant state then $h|\Omega\rangle$ is also translation invariant, and any translation invariant state can be written as $h|\Omega\rangle$ for some h that commutes with the group e^{itP} . If we consider the unitary flow $U_t = e^{itK_x}$ where K_x is boost the only invariant state is the vacuum, and Π_0 is the projection to the vacuum $|\Omega\rangle \langle \Omega|$.

To come closer to the expression in (3.6) consider the expectation value $\lim_{x \rightarrow \infty} \langle \Psi | a(x) \Psi \rangle$ where $a(x) = e^{ixP} a e^{-ixP}$ is the translation of the operator a and the excited state is $|\Psi\rangle = b|\Omega\rangle$. Then,

$$\begin{aligned} \lim_{x \rightarrow \infty} \langle \Psi | a(x) \Psi \rangle &= \lim_{x \rightarrow \infty} \langle \Psi | a(x) b \Omega \rangle \\ &= \lim_{x \rightarrow \infty} \langle \Psi | b e^{ixP} a \Omega \rangle = \langle \Psi | b \Pi_0 a \Omega \rangle \end{aligned} \quad (3.11)$$

where we have used (3.8) the fact that the matrix elements of $[a(x), b]$ vanish in the limit $x \rightarrow \infty$ by causality. Consider the decomposition of the projection Π_0 into an orthonormal set of translation-invariant states $\Pi_0 = \sum_h |h\rangle \langle h|$ with $|h\rangle = h|\Omega\rangle$. Then, the clustering property of correlation functions in state $|\Omega\rangle$ fails

$$\lim_{x \rightarrow \infty} \langle \Omega | a(x) b \Omega \rangle \neq \langle \Omega | a \Omega \rangle \langle \Omega | b \Omega \rangle \quad (3.12)$$

unless $|\Omega\rangle$ is the unique translation-invariant state in which case

$$\lim_{x \rightarrow \infty} \langle \Psi | a(x) \Psi \rangle = \langle \Omega | a \Omega \rangle \quad (3.13)$$

and the matrix elements of the operator $\lim_{a \rightarrow \infty} a(x)$ are the same as the identity operator multiplied by $\langle \Omega | a \Omega \rangle$.

We are now ready to argue qualitatively why in QFT the weak unitary orbit distance vanishes. The rigorous proof is postponed to section 6. We use the intuition of density matrices and consider the unitary operators $\Delta_\Omega^{it} = \omega^{it} \otimes \omega^{-it}$, $\Delta_\Psi^{it} = \psi^{it} \otimes \psi^{-it}$ for density matrices ω and ψ . We define the unitary flow $u_{\omega|\psi}(t) = \omega^{it} \psi^{-it} \in \mathcal{A}$ called the *cocycle* [6]. The claim is that the cocycle in the limit of $t \rightarrow \infty$ is the unitary that makes the weak unitary orbit distance defined in (3.6) vanish. To simplify the notation, we write $u_t = u_{\omega|\psi}(t)$ so that

$$\begin{aligned} \psi(u_t^\dagger a u_t) &= \langle \Psi | (u_t^\dagger \otimes u_{-t}^\dagger) a (u_t \otimes u_{-t}) | \Psi \rangle \\ &= \langle \Psi | \Delta_\Omega^{-it} a \Delta_\Omega^{it} | \Psi \rangle \end{aligned} \quad (3.14)$$

where we have used

$$\begin{aligned} (u_t \otimes u_{-t}) | \Psi \rangle &= (\omega^{it} \otimes \omega^{-it}) (\psi^{-it} \otimes \psi^{it}) | \Psi \rangle \\ &= (\omega^{it} \otimes \omega^{-it}) | \Psi \rangle = \Delta_\Omega^{it} | \Psi \rangle . \end{aligned} \quad (3.15)$$

The eigenvalues of the modular operator $\Delta_\omega = \sum_{kk'} \frac{p_k}{p_{k'}} |kk'\rangle \langle kk'|$ are $p_k/p_{k'}$. The defining feature of the algebra of QFT is that the spectrum of the modular operator Δ_Ψ for any state is entirely continuous with no eigenvalues. Then, from (3.8) it follows that if $|\Omega\rangle$ is the only invariant state of Δ_Ω we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \psi(u_t^\dagger a u_t) &= \lim_{t \rightarrow \infty} \langle \Psi | \Delta_\Omega^{-it} a \Delta_\Omega^{it} | \Psi \rangle = \lim_{t \rightarrow \infty} \langle \Psi | \Delta_\Omega^{-it} a \Delta_\Omega^{it} b' | \Omega \rangle \\ &= \lim_{t \rightarrow \infty} \langle \Psi | b' \Delta_\Omega^{-it} a | \Omega \rangle = \langle \Omega | a \Omega \rangle = \omega(a) . \end{aligned}$$

where we have used $|\Psi\rangle = b'|\Omega\rangle$ where b' in the commutant algebra. Therefore, $d_w([\omega], [\psi]) = 0$. In the vacuum of QFT, the modular operator of half-space is the boost, which has only one invariant state. Note that the cocycle cannot make the strong distance small because $(u_t \otimes u_{-t}) | \Psi \rangle = \Delta_\Omega^{it} | \Psi \rangle$ and as a result

$$\begin{aligned} \|(u_t \otimes u_{-t}) | \Psi \rangle - |\Omega\rangle\|^2 &= 2 \left(1 - \langle \Omega | \Delta_\Omega^{it} | \Psi \rangle - \langle \Psi | \Delta_\Omega^{-it} | \Omega \rangle \right) \\ &= \| |\Psi\rangle - |\Omega\rangle \|^2 . \end{aligned} \quad (3.16)$$

If the strong distance is small the weak distance is also small, whereas as we saw above the opposite is incorrect. The equation (3.5) says that the unitary U that sews local states ω and ψ is the unitary that rotates the purification of ω to an invariant vector of the relative modular operator. In the next section, we study the invariant states of relative modular operator.

4 Invariant states of relative modular operator

To make the strong unitary orbit distance small one needs to find unitaries that prepare invariant states of the relative modular operator. To find these invariant states we look at its spectrum of the relative modular operator in the case density matrices exist:

$$\Delta_{\Psi|\Omega} \equiv \sum_{\lambda} e^{\lambda} P_{\lambda} = \sum_{kk'} \frac{q_{k'}}{p_k} |\alpha_{k'}k\rangle \langle \alpha_{k'}k|, \tag{4.1}$$

where e^{λ} are the eigenvalues $q_{k'}/p_k$ for any pair (k, k') and P_{λ} is the projection to the subspace of eigenvalue e^{λ} . There can be degeneracies for each λ :

$$P_{\lambda} = \sum_{\gamma=1}^{d_{\lambda}} |\lambda; \gamma\rangle \langle \lambda; \gamma|. \tag{4.2}$$

The invariant states of the relative modular operator are in the subspace $P_0\mathcal{H}$. If there are no $q_{k'} = p_k$ the projection $P_0 = 0$, and there are no invariant states. If S_0 is the subset of pairs (k, k') with $p_k = q_{k'}$ then

$$P_0 = \sum_{(k,k') \in S_0} |\alpha_{k'}, k\rangle \langle \alpha_{k'}, k|. \tag{4.3}$$

The local operator $v = \sum_{(k,k') \in S_0} |\alpha_{k'}\rangle \langle k|$ is a partial isometry⁴ of system A that acts on $|\Omega\rangle$ and prepares invariant states of the relative modular operator

$$\Delta_{\Psi|\Omega} (v \otimes 1) |\Omega\rangle = (v \otimes 1) |\Omega\rangle. \tag{4.4}$$

The operator v rotates the eigenvectors of ω to those of ψ and further relabels them such that $q_{k'} = p_k$. To simplify the discussion, let us order p_k in descending order. The operator v is unitary if and only if for every eigenvalue p_k there exists a k' such that $q_{k'} = p_k$, in which case, v is the unitary that sorts $q_{k'}$ in descending order. We recover the statement that one can sew two density matrices ω and ψ if and only if they have the same eigenvalues. As we saw in the last section, when the eigenvalues do not match, the unitary v that sorts $q_{k'}$ still minimizes the unitary orbit distance.

In infinite systems, the eigenvalues $p_{k'}/p_k$ of the modular operator can become unbounded for small enough p_k . Let us assume that our states of interest, $|\Psi\rangle$ and $|\Omega\rangle$, both

⁴A partial isometry v is an operator that satisfies $v^{\dagger}v = \pi_1$ and $vv^{\dagger} = \pi_2$ where π_1 and π_2 are projections. An operator π is a projection if $\pi = \pi^{\dagger}$ and $\pi^2 = \pi$.

have the property that the spectra of their modular operators are the entire continuum $[0, \infty)$ and the number of degeneracies at each λ goes to infinity:

$$\begin{aligned}\Delta_\Omega &= \sum_{kk'} p_k p_{k'}^{-1} |kk'\rangle \langle kk'| = \int d\lambda_1 e^{\lambda_1} \sum_\gamma |\lambda_1; \gamma\rangle \langle \lambda_1, \gamma| \\ \Delta_\Psi &= \sum_{kk'} q_k q_{k'}^{-1} |\alpha_k \alpha_{k'}\rangle \langle \alpha_k \alpha_{k'}| = \int d\lambda_2 e^{\lambda_2} \sum_\gamma |\lambda_2; \gamma\rangle \langle \lambda_2, \gamma|,\end{aligned}\quad (4.5)$$

keeping in mind that the vectors $|\lambda; \gamma\rangle$ are generalized eigenvectors that cannot be normalized: $\langle \lambda; \gamma | \lambda'; \gamma' \rangle = \delta_{\gamma\gamma'} \delta(\lambda - \lambda')$. In analogy with (4.4), we act on $|\Omega\rangle$ with the partial isometry $v_{k'k} = |\alpha_{k'}\rangle \langle k|$ and prepare an eigenstate of the relative modular operator

$$\Delta_{\Psi|\Omega} (v_{k'k} |\Omega\rangle) = q_{k'} p_k^{-1} (v_{k'k} |\Omega\rangle) . \quad (4.6)$$

The partial isometry $f_{lk'} = |\alpha_l\rangle \langle \alpha_{k'}|$ that relabels $q_{k'}$ to q_l acts on $v_{k'k} |\Omega\rangle$ and creates a new eigenstate

$$\Delta_{\Psi|\Omega} (f_{lk'} v_{k'k} |\Omega\rangle) = q_l p_k^{-1} (f_{lk'} v_{k'k} |\Omega\rangle) . \quad (4.7)$$

This partial isometry sends an eigenspaces of relative modular operator to another one simply by changing the label:

$$f_{lk} P_\lambda \mathcal{H} \in P_{\lambda_{q_l/q_k}} \mathcal{H} . \quad (4.8)$$

Since we assumed that the spectrum of Δ_Ψ is continuous and entire in $[0, \infty)$ we can tune the partial isometry f such that it brings any eigenspace P_λ to an eigenspace P_ϵ with ϵ near zero. Take two partial isometries f_1 and f_2 that, respectively, map the eigenspace P_{λ_1} to P_{ϵ_1} and P_{λ_2} to P_{ϵ_2} such that $\lambda_1 \neq \lambda_2$ and $\epsilon_1 \neq \epsilon_2$ then the operator $v_1 + v_2$ is also partial isometry that maps $P_{\lambda_1} + P_{\lambda_2}$ to $P_{\epsilon_1} + P_{\epsilon_2}$ with ϵ 's in a small neighborhood of zero. Adding such partial isometries and using a bijection from the eigenspace with $\lambda \in (-\Lambda, \Lambda)$ to the one with $\lambda \in (-\epsilon, \epsilon)$ we construct a partial isometry v_Λ that compresses the spectrum of the relative modular operator from $(e^{-\Lambda}, e^\Lambda)$ to a narrow interval near one $(e^{-\epsilon}, e^\epsilon)$. Denote by $\Pi_\Lambda = \int_{-\epsilon}^\epsilon d\lambda P_\lambda$ the projection in the spectrum to $(e^{-\epsilon}, e^\epsilon)$. The state $P_0 |\Omega\rangle$ is invariant under the action of the relative modular operator and the state $\Pi_\epsilon |\Omega\rangle$ for small ϵ is almost invariant in the sense that

$$\|(\Delta_{\Psi|\Omega}^{1/2} - 1) \Pi_\epsilon |\Omega\rangle\| \leq (e^{\epsilon/2} - 1) . \quad (4.9)$$

At large Λ , we can approximate $|\Omega\rangle \simeq \Pi_\Lambda |\Omega\rangle$, therefore the state $v_\Lambda |\Omega\rangle$ is also almost invariant under the action of relative modular operator:

$$\begin{aligned}\|(\Delta_{\Psi|\Omega}^{1/2} - 1) v_\Lambda |\Omega\rangle\| &\leq \|(\Delta_{\Psi|\Omega}^{1/2} - 1) v_\Lambda \Pi_\Lambda |\Omega\rangle\| + \|(1 - \Pi_\Lambda) (\Delta_{\Psi|\Omega}^{1/2} - 1) |\Omega\rangle\| \\ &\leq (e^{\epsilon/2} - 1) + 2\|1 - \Pi_\Lambda\| .\end{aligned}\quad (4.10)$$

Intuitively, one expects that in the limit $\Lambda \rightarrow \infty$ the sequence of v_Λ tends to a unitary operator as the right-hand-side of the inequality above becomes vanishingly small. We

postpone a discussion of this limit to section 5 and conclude that in infinite dimensional systems, the states $|\Omega\rangle$ and $|\Psi\rangle$ for which the modular operator has an entire continuous spectrum there exists a local unitary such that $\| |\Psi\rangle - U \otimes U_J |\Omega\rangle \| \leq \epsilon$ for ϵ arbitrarily small.⁵ Since any purification of local state ψ is related to the canonical one by a unitary in $\bar{\mathcal{A}}$ we have

$$\inf_{U \in \mathcal{A}, U' \in \bar{\mathcal{A}}} \| UU' |\Omega\rangle - |\Psi\rangle \| = 0 \tag{4.11}$$

for $|\Omega\rangle$ and $|\Psi\rangle$ any two vectors in the Hilbert space.

The local algebra of QFT is the limit of a finite quantum system where for every state the individual probabilities p_k go to zero and only the ratio $p_{k'}/p_k$ makes sense.⁶ Every eigenvalue $p_{k'}/p_k$ repeats infinite times, therefore the sum over γ in (4.5) runs to infinity. In the continuum limit, it is more precise to replace the expression in these parenthesis in (4.1) with a projection operator valued measure $P(d\lambda)$ and write

$$\Delta_{\Psi|\Omega} = \int e^{\lambda} P(d\lambda) . \tag{4.12}$$

In [8] Connes and Størmer used a generalization of the argument above to prove that for every pair of states $|\Omega_U\rangle$ and $|\Psi\rangle$ there exists a local unitary U that satisfies $\| UU_J |\Omega\rangle - |\Psi\rangle \| \leq \epsilon$ for ϵ arbitrarily small. In other words, they showed that the strong distance of unitary orbits of any two local states is zero, i.e. $d_s([\omega], [\psi]) = 0$. We review the steps needed for the generalization in section 5. Unfortunately the argument in [8] just proves the existence of a unitary U that sews states. In the remainder of this section, we focus on the weak distance $d_w([\omega], [\psi])$ so that we can explicitly construct examples of unitaries that sew states of QFT.

5 Connes-Størmer result

In QFT, there are no local Hilbert spaces analogous to \mathcal{H}_A . Splitting a time-slice into two complementary regions A and \bar{A} there are local algebras on each region that we denote by \mathcal{A} and $\bar{\mathcal{A}}$. They are isomorphic and together they generate the global algebra. In the problem of sewing local states we are only interested in pure global states that we refer to as vectors in the Hilbert space. The local state on \mathcal{A} is the restriction of the expectation values in the global vector $|\Omega\rangle$ to the local algebra $\omega(a) = \langle \Omega | a \Omega \rangle$ with $a \in \mathcal{A}$. We can sew local states ω on A to ψ on \bar{A} if there exists a global vector $|\Phi\rangle$ such that

$$\begin{aligned} \forall a \in \mathcal{A} : \quad & \omega(a) = \langle \Phi | a \Phi \rangle \\ \forall a' \in \bar{\mathcal{A}} : \quad & \psi(a') = \langle \Phi | a' \Phi \rangle . \end{aligned} \tag{5.1}$$

⁵Such states can be constructed in infinite dimensional systems with density matrices (so-called type I systems). However, not all states have the property above. We thank Roberto Longo for pointing this out to us.

⁶See [7] for the construction of the local algebra of quantum fields as the limit of finite quantum systems.

We say an algebra $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ if for any local states ω on \mathcal{A}_1 and ψ on \mathcal{A}_2 there exists a state ϕ on \mathcal{A} such that

$$\begin{aligned} \forall a \in \mathcal{A}_1 : \quad & \omega(a) = \phi(a) \\ \forall b \in \mathcal{A}_2 : \quad & \psi(b) = \phi(b) . \end{aligned} \tag{5.2}$$

In QFT, as opposed to finite quantum systems, if \bar{A} is the complement of A the global algebra does not split into a tensor product $\mathcal{A} \otimes \bar{\mathcal{A}}$. The absence of tensor product between the algebra of region A and that of \bar{A} implies that it is impossible to sew all states perfectly. However, as we argue below it is possible to sew any two states with arbitrary precision.

Consider the set of vectors in the Hilbert space that reduce to ω on \mathcal{A} . Analogous to the case of qudits, there exists a unique canonical global vector $|\Omega\rangle$ that is invariant under the anti-linear J_Ω that swaps A and \bar{A} [5]. Given a second local state ψ on \mathcal{A} there exists a unique global vector $|\Psi_\Omega\rangle$ that reduces to ψ and is invariant under J_Ω . We postpone a definition of J_Ω to the next section. In (2.5), we expressed the Frobenius distance of local states on \mathcal{A} in terms of the Hilbert space distance of their corresponding global vectors. We take the Hilbert space distance as the definition of distance between local states

$$d_F(\omega, \psi) = \| |\Psi_\Omega\rangle - |\Omega\rangle \| . \tag{5.3}$$

Rotating the local state by a local unitary corresponds to considering a new local state with $\omega_U(a) = \omega(UaU^\dagger)$. We are interested in the minimum distance in the unitary orbit distance of ω :

$$\inf_{U \in \mathcal{A}} d(\omega_U, \psi) = \inf_{U \in \mathcal{A}} \| |\Psi_\Omega\rangle - |U\Omega\rangle \| . \tag{5.4}$$

where $|U\Omega\rangle = UU_J |\Omega\rangle$ is the canonical purification of ω_U . In the last section, we argued that if the modular operators of both $|\Psi_\Omega\rangle$ and $|\Omega\rangle$ have an entirely continuous spectrum the infimum above is zero. We now generalize that argument to show that in QFT the infimum above is zero for any pair of states. The key observation is that QFT is a special type of infinite quantum system for which the spectrum of the modular operator of any state is the entire $[0, \infty)$.⁷ If U is the local unitary that makes the infimum in (5.4) vanishingly small the state $U|\Omega\rangle$ reduced to \mathcal{A} is arbitrarily close to ψ , whereas outside of \mathcal{A} the state is the same as ω . The seeming contradiction with finite quantum systems has to do with the fact there is no notion in QFT analogous to the eigenvalues of local states. There is only the modular operator whose spectrum is entirely continuous with no eigenvalues.⁸ That is the reason underlying the divergences that appear in a naive calculation of unitary invariant measures of local states such as entanglement or Renyi entropies.

An important step in the argument of the last section was the construction of the partial isometry v_A . There is no local Hilbert space \mathcal{H}_A in QFT, therefore one needs to approximate the partial isometry $f_{kk'} = |\alpha_k\rangle \langle \alpha_{k'}|$ that relates the projections f_{kk} and $f_{k'k'}$.

⁷A quantum system with this property is called type III₁. The local algebra of QFT in any dimension is a type III₁ system. Here, we are excluding theories such as generalized free field where the split property fails.

⁸We require an eigenvector to be normalizable.

The projection f_{kk} is special in that it commutes with the density matrix ψ . In QFT, the projections in the local algebra that commute with the modular operator play the role of f_{kk} , i.e. $[\Delta_\Psi, f_{kk}] = 0$. The other special feature of f_{kk} is that it is the smallest projection in the sense that there are no other non-zero projection f' such that $f'\mathcal{H} \subset f_{kk}\mathcal{H}$. This aspect cannot be generalized to QFT.⁹ To run the argument in QFT we start with two projections f_k and $f_{k'}$ that commute with the modular operator Δ_Ψ and take the partial isometry that relates them as a replacement of f_{kk} .¹⁰ We repeat the argument of the last section by considering two partial isometries of this type v_1 and v_2 and their corresponding projections, respectively, $f_i = v_i^\dagger v_i$ and $f'_i = v_i v_i^\dagger$ for $i = 1, 2$. If $f_1 \perp f_2$ and $f'_1 \perp f'_2$ the sum $v_1 + v_2$ is also partial isometry with extended domain $(f_1 + f_2)\mathcal{H}$ and range $(f'_1 + f'_2)\mathcal{H}$. We continue adding such partial isometries until either the domain or the range of the partial isometry becomes the whole Hilbert space, at which point the resulting operator is either an isometry $v^\dagger v = 1$ or a co-isometry $vv^\dagger = 1$.¹¹ The isometry v has the property that $\| |\Psi_\Omega\rangle - vv_J |\Omega\rangle \| \simeq 0$. The final step is to notice that in QFT any isometry is the limit of sequence of unitaries in the strong operator topology; see appendix D. This finishes the proof that in QFT any state ω on A can be sewn to any ψ on \bar{A} .

6 Modular theory and the cocycle

In this section, our goal is to go beyond systems with density matrices and define the appropriate generalizations of the relative modular operator, modular conjugation and the cocycle in a general quantum system. The mathematical framework that accomplishes this is called the Tomita-Takesaki modular theory. See [7, 9] for a review of the modular theory. The remainder of this section applies to any general quantum system, from qubits to QFT.

Consider a local quantum system A and the auxiliary quantum system \bar{A} that purifies its state. Their corresponding algebras \mathcal{A} and $\bar{\mathcal{A}}$ together generate a global algebra with a representation on a separable Hilbert space \mathcal{H} . The pure global states are vectors $|\Omega\rangle \in \mathcal{H}$. There are many vectors in the Hilbert space that reduce to ω on \mathcal{A} . Acting on a vector with any isometry $W' \in \bar{\mathcal{A}}$, i.e. $W'^\dagger W' = 1_{\bar{A}}$ and $W'W'^\dagger = \pi \in \bar{\mathcal{A}}$ a projection, leaves the local state invariant

$$\langle \Omega | a \Omega \rangle = \langle W' \Omega | a W' \Omega \rangle . \tag{6.1}$$

A vector $|\Omega\rangle$ has the Reeh-Schlieder property if the set of vector $a|\Omega\rangle$ is dense in the Hilbert space. Every local operator $a \in \mathcal{A}$ has a hermitian conjugate operator $a^\dagger \in \mathcal{A}$. We define

⁹A quantum system is called type I if and only if it has minimal projections in the sense defined here. QFT is not a type I system.

¹⁰It is possible that there are no f in the local algebra that commutes with the modular operator. We consider that case in section 7.

¹¹Since the spectrum of Δ_Ψ is the uncountable set $[0, \infty)$ the rigorous argument is as follows: if we have two sets of projections $\tilde{f} - f > 0$ and $\tilde{f}' - f' > 0$ and partial isometries $v^\dagger v = f$ and $vv^\dagger = f'$ and $\tilde{v}^\dagger \tilde{v} = \tilde{f}$ and $\tilde{v}\tilde{v}^\dagger = \tilde{f}'$. we say $\tilde{v} \succ v$ if $\tilde{v}f = v$ and $\tilde{v}^\dagger f' = v^\dagger$. This define a partial order among partial isometries. Since every chain in this partially ordered set has a maximal element, by Zorn's lemma, there exists a maximal partial isometry that one proves to be either an isometry or a co-isometry [8]. A rigorous treatment of this step turns the argument presented here to a proof.

the anti-linear operator S_Ω^A using its action on vectors

$$\forall a \in \mathcal{A} : \quad S_\Omega^A a |\Omega\rangle = a^\dagger |\Omega\rangle . \quad (6.2)$$

For vectors with the Reeh-Schlieder property S_Ω^A in (6.2) is densely defined. Hereafter, we refer to S_Ω^A as the closure of the operator above. The S_Ω^A is called the Tomita operator and is the starting point of modular theory. The modular operator is the norm of the Tomita operator: $\Delta_\Omega = S_\Omega^\dagger S_\Omega$ where we have suppressed the region index A . The modular conjugation is the anti-linear operator $J_\Omega = \Delta_\Omega^{1/2} S_\Omega$. If $|\Omega\rangle$ has the Reeh-Schlieder property J_Ω is an anti-unitary. The results of the modular theory that we need here are the following two facts:¹²

1. The operator Δ_Ω^{it} generates a unitary flow in the algebra

$$a \in \mathcal{A} \rightarrow a_\Omega(t) \equiv \Delta_\Omega^{it} a \Delta_\Omega^{-it} \in \mathcal{A} \quad (6.3)$$

2. The modular conjugation is an anti-linear map that sends every operator in the \mathcal{A} to $\bar{\mathcal{A}}$: $a_J \equiv J_\Omega a J_\Omega \in \bar{\mathcal{A}}$. Every local state ω on A has a unique canonical purification $|\Psi_\Omega\rangle$ satisfying: $J_\Omega |\Psi_\Omega\rangle = |\Psi_\Omega\rangle$.

In the case of a density matrix ω and its canonical purification $|\Omega\rangle$, the modular conjugation is the anti-linear swap operator we discussed earlier and the modular operator is $\Delta_\Omega = \omega \otimes \omega^{-1}$. The modular flow of operators in the algebra is given by $\omega^{it} a \omega^{-it}$. The main difference in QFT is that, as opposed to the case of density matrices, the modular flow is not generated by unitaries in the algebra, but with the modular operator that is an operator in the global Hilbert space.

For two vectors $|\Omega\rangle$ and $|\Psi\rangle$ we define the relative Tomita operator and the relative modular operator as its norm

$$S_{\Psi|\Omega} a |\Omega\rangle = a^\dagger |\Psi\rangle, \quad \Delta_{\Psi|\Omega} = S_{\Psi|\Omega}^\dagger S_{\Psi|\Omega} . \quad (6.4)$$

If we choose the canonical purification $|\Psi_\Omega\rangle$ then $\Delta_{\Psi_\Omega|\Omega}^{1/2} |\Omega\rangle = |\Psi_\Omega\rangle$. The definition of the weak and strong distance in (3.5) and definition of the cocycle $u_{\Psi|\Omega}(t) = \Delta_{\Psi|\Omega}^{it} \Delta_\Omega^{-it}$ generalize trivially. However, we need to show that the cocycle is inside the algebra \mathcal{A} . To prove this, we use a trick introduced by Connes in [6]. We add a qubit to the local algebra $\mathcal{A} \equiv \mathcal{A}_A$ so that the new algebra \mathcal{A}_{AQ} is generated by $a \otimes |i\rangle \langle j|$ with $|i\rangle$ a vector in the Hilbert space of the qubit Q . We also add a qubit \bar{Q} to the algebra of the complementary region $\bar{\mathcal{A}} \equiv \mathcal{A}_{\bar{A}}$ so that we can purify local states of AQ in $AQ\bar{A}\bar{Q}$. Consider the global state

$$|\Theta\rangle_{AQ\bar{A}\bar{Q}} = |\Omega\rangle_{A\bar{A}} \otimes |00\rangle_{Q\bar{Q}} + |\Psi\rangle_{A\bar{A}} \otimes |11\rangle_{Q\bar{Q}} \quad (6.5)$$

where $|\Omega\rangle$ and $|\Psi\rangle$ are global field theory states which we assume to be Reeh-Schlieder.¹³ The combined local state of AQ is $\theta_{AQ} = \omega_A \otimes |0\rangle \langle 0|_Q + \psi_A \otimes |1\rangle \langle 1|_A$. The Tomita

¹²See [10] for proofs of these statements.

¹³It is straightforward to relax this assumption. In QFT, the set of states with the Reeh-Schlieder property is dense in the Hilbert space and includes the vacuum [5, 7].

operator for this state acts as $S_\Theta x |\Theta\rangle = x^\dagger |\Theta\rangle$ with $x \in \mathcal{A}_{AQ}$. The Tomita operator and the modular conjugation of this state decompose according to

$$S_\Theta = \begin{pmatrix} S_\Omega & & & \\ & 0 & S_{\Psi|\Omega} & \\ & S_{\Omega|\Psi} & 0 & \\ & & & S_\Psi \end{pmatrix} \quad (6.6)$$

and

$$\Delta_\Theta = \begin{pmatrix} \Delta_\Omega & & & \\ & \Delta_{\Omega|\Psi} & & \\ & & \Delta_{\Psi|\Omega} & \\ & & & \Delta_\Psi \end{pmatrix}. \quad (6.7)$$

The cocycle $u_{\Psi|\Omega}(t)$ is simply the modular flow of the operator $1_A \otimes |0\rangle \langle 1|_Q$ in state $|\Theta\rangle$:

$$\Delta_\Theta^{it} (1 \otimes |0\rangle \langle 1|_Q \otimes 1_{\bar{A}Q}) \Delta_\Theta^{-it} = \Delta_\Omega^{it} \Delta_{\Psi|\Omega}^{-it} \otimes |00\rangle \langle 10| + \Delta_{\Omega|\Psi}^{it} \Delta_\Psi^{-it} \otimes |01\rangle \langle 11|.$$

Since the modular flow has to remain inside \mathcal{A}_{AQ} we find that the cocycle belongs to the local algebra of field theory in A and satisfies $u_{\Psi|\Omega}(t) = \Delta_{\Psi|\Omega}^{it} \Delta_\Omega^{-it} = \Delta_\Psi^{it} \Delta_{\Omega|\Psi}^{-it}$.¹⁴ More generally, the modular flow of the operator $V \otimes |0\rangle \langle 1|$ is

$$\Delta_\Theta^{it} (V \otimes |0\rangle \langle 1|_Q) \Delta_\Theta^{-it} = V_\Omega(t) u_{\Psi|\Omega}(t)^\dagger \otimes |0\rangle \langle 1|,$$

which is the conjugate of the operator we found in section 4 to sew states in the large t limit. If both states are Reeh-Schlieder the cocycle generates a unitary flow in the algebra. For an operator in the complementary region $a' \in \mathcal{A}_{\bar{A}}$ we have $u_{\Psi|\Omega}^\dagger a' u_{\Psi|\Omega} = a'$ which further implies

$$\begin{aligned} \Delta_{\Psi|\Omega}^{it} a' \Delta_{\Psi|\Omega}^{-it} &= \Delta_\Omega^{it} a' \Delta_\Omega^{-it} \\ \Delta_{\Psi|\Omega}^{it} a \Delta_{\Psi|\Omega}^{-it} &= \Delta_\Psi^{it} a \Delta_\Psi^{-it} \end{aligned} \quad (6.8)$$

where in the second line we have used the relation $\Delta_{\Omega|\Psi}^{\bar{A}} = (\Delta^A)_{\Psi|\Omega}^{-1}$ [9].

As an example, consider state $u_{\Psi|\Omega}(t) |\Omega\rangle$ in the large t limit. From (6) we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \langle \Omega | u(t)^\dagger a' u(t) | \Omega \rangle &= \langle \Omega | a \Omega \rangle \\ \lim_{t \rightarrow \infty} \langle \Omega | u(t)^\dagger a u(t) | \Omega \rangle &= \lim_{t \rightarrow \infty} \omega_{u(t)}(a). \end{aligned} \quad (6.9)$$

In section 4 we found that the weak distance $\lim_{t \rightarrow \infty} d_w(\omega_{u(t)}, \psi) = 0$. That is the sense in which the state is the same as ψ on A .

Consider the algebra of half-space ($x > |t|$) in the vacuum state of QFT in any dimension. The modular operator is $e^{-2\pi K_x}$ where K_x is the boost operator in the x direction [11]. If we pick $|\Psi\rangle$ to be the local state of the vacuum QFT on half space the modular flow Δ_Ψ^{it}

¹⁴If we start with a three-level system for Q instead of a qubit the same argument establishes the so-called cocycle chain rule: $u_{\Psi|\Omega}(t) u_{\Omega|\Phi}(t) = u_{\Psi|\Phi}(t)$ [6] which in the case $\Phi = \Psi$ gives $\Delta_{\Psi|\Omega}^{it} \Delta_\Omega^{-it} = \Delta_\Psi^{it} \Delta_{\Omega|\Psi}^{-it}$.

is the boost operation. The authors of [12] argued that $\Delta_{\Psi}^{it} |\chi\rangle$ for some $|\chi\rangle$ is an infinitely boosted state that should look like vacuum. Here, we presented a generalization of that argument to an arbitrary state $|\Psi\rangle$.

A benefit of enlarging the local algebra by a qubit is that it allows us to write the Frobenius distance measure in equation (2.4) in terms of the norm of a commutator between the local density matrix and an observable. In the next section, we use this rewriting to study invariant states of relative modular operator.

7 Commutators and invariant states

In section 5, the projections that commute with the modular operator and the invariant states of relative modular operator played an important role in our construction of the unitary that sew states. Here, we discuss such projections in more detail. Let us go back to the Hilbert space of system AQ and consider the density matrix $\theta_{AQ} = \omega_A \otimes |0\rangle\langle 0|_Q + \psi_A \otimes |1\rangle\langle 1|_Q$. For any $a \in \mathcal{A}_A$ we can write

$$(\sqrt{\omega}a - a\sqrt{\psi}) \otimes |0\rangle\langle 1|_Q = \left[\sqrt{\theta_{AQ}}, a \otimes |0\rangle\langle 1|_Q \right]. \quad (7.1)$$

We rewrite the Frobenius distance of density matrices as

$$\begin{aligned} d_F(\omega, \psi)^2 &= \frac{1}{2} \left\| \left[\sqrt{\theta_{AQ}}, 1_A \otimes \sigma_Q^X \right] \right\|_F^2 \\ &= -\frac{1}{2} \text{tr} \left(\left[\sqrt{\theta_{AQ}}, 1_A \otimes \sigma_Q^X \right]^2 \right), \end{aligned} \quad (7.2)$$

where $\sigma_Q^X = |0\rangle\langle 1| + |1\rangle\langle 0|$ is the X Pauli matrix in the algebra of the qubit Q . More generally, for an arbitrary operator $a \in \mathcal{A}$ we have

$$\begin{aligned} \frac{1}{2} \left\| \sqrt{\omega}a - a\sqrt{\psi} \right\|_F^2 &= \|a |\Psi_\Omega\rangle - a^\dagger |\Omega\rangle\| \\ &= -\frac{1}{2} \text{tr} \left(\left[\sqrt{\theta}, x_a \right]^2 \right) = \left\| \left(\Delta_\Theta^{1/2} - 1 \right) x_a |\Theta\rangle \right\|, \end{aligned} \quad (7.3)$$

where $x_a \equiv a \otimes |0\rangle\langle 1| + a^\dagger \otimes |1\rangle\langle 0|$ is in the algebra of AQ . Our distance measure is the Frobenius norm of the commutator of density matrix θ and an observable x_a , otherwise known as the Wigner-Yanase skew information [13, 14]. For unitary orbits we rewrite the infimum distance as

$$d_F([\omega], [\psi])^2 = -\frac{1}{2} \inf_U \text{tr} \left(\left[\sqrt{\theta}, x_U \right]^2 \right), \quad (7.4)$$

where $x_U = U \otimes |0\rangle\langle 1| + U^\dagger \otimes |1\rangle\langle 0|$ is a self-adjoint unitary operators. If there exists an x_U that commutes with θ the density matrices ω and ψ are simultaneously diagonalizable. If x_U almost commutes with θ_{AQ} then $U\omega U^\dagger$ becomes close to ψ . The almost invariant states of relative modular operator are prepared by x_U that almost commute with θ .

Consider the density matrix $\omega = \sum_k e^{-\lambda_k} |k\rangle\langle k|$ and its canonical purification $|\Omega\rangle$. The orthonormal projection $e_k = |k\rangle\langle k|$ commute with ω . If there are degenerate eigenvalues, i.e. $\lambda_k = \lambda_{k'}$, then the partial isometry $|k\rangle\langle k'|$ also commutes with ω . The sub-algebra of

all operators that commute with ω is called the centralizer of ω . If the algebra has a center Z , the center is in the centralizer of all density matrices. Using the cyclicity of trace we find that any operator h that commutes with ω satisfies: $\text{tr}([\omega, h]a) = \text{tr}(\omega[h, a]) = 0$ for all $a \in \mathcal{A}$. In QFT, we define the centralizer as the set of h such that $\omega([h, a]) = 0$ for all $a \in \mathcal{A}$. Any operator h in the centralizer of ω commutes with the modular operator, and is invariant under modular flow [15]:

$$\begin{aligned} (\Delta_\Omega^{1/2} - 1)h|\Omega\rangle &= 0 \\ \Delta_\Omega^{it}h\Delta_\Omega^{-it} &= h. \end{aligned} \tag{7.5}$$

Acting with h in the centralizer on $|\Omega\rangle$ creates a new invariant state of Δ_Ω . Moreover, every invariant vector $|h\rangle$ is $h|\Omega\rangle$ for some h affiliated with the centralizer [15].¹⁵ Therefore, to understand invariant states of modular operator it suffices to study the centralizer of the state.

So far we have only discussed the invariant states of the relative modular operator $\Delta_{\Psi|\Omega}$ that are locally prepared as $u|\Omega\rangle$. It is natural to ask what does a general invariant state of the relative modular operator look like. Consider the state $UU'|\Omega\rangle$ with $U \in \mathcal{A}$ and $U' \in \bar{\mathcal{A}}$. Its relative modular operator is $\Delta_{\Omega|UU'\Omega} = U'\Delta_\Omega(U')^\dagger$ [16], therefore $U'|\Omega\rangle$ is its invariant state. It is a state that looks like $|\Omega\rangle$ on A and $|UU'\Omega\rangle$ on \bar{A} . As we showed in this work, every state $|\Psi\rangle$ is well-approximated by some $UU'|\Omega\rangle$. Hence, it is tempting to think that all states $|\Omega_\Psi\rangle$ that are invariant under $\Delta_{\Omega|\Psi}$ look like $|\Omega\rangle$ on A and $|\Psi\rangle$ on \bar{A} , and only differ by their long-range correlations across $A\bar{A}$. We show below that this is incorrect if the centralizers of the states are non-trivial.

Every operator h that commutes with Δ_Ω also commutes with $\Delta_{\Omega|\Psi}$:

$$\Delta_{\Omega|\Psi}^{it}h\Delta_{\Omega|\Psi}^{-it} = \Delta_\Omega^{it}h\Delta_\Omega^{-it} = h \tag{7.6}$$

where we used (6.8). If $|\Omega_\Psi\rangle$ is an invariant state of relative modular flow so is $h|\Omega_\Psi\rangle$. This implies that if $UU_J|\Omega\rangle \simeq |\Psi_\Omega\rangle$ then $(hU)(hU)_J|\Omega\rangle \simeq hh_J|\Psi_\Omega\rangle$. The physical interpretation of an arbitrary invariant state $|\Omega_\Psi\rangle$ is that it is a state that is the same as $h|\Omega\rangle$ with respect to \mathcal{A} and $\tilde{h}|\Psi\rangle$ with respect to $\bar{\mathcal{A}}$ where h is in the centralizer of $|\Omega\rangle$ and \tilde{h} is in the centralizer of $|\Psi\rangle$. In the state $|\Omega_\Psi\rangle$ the flow Δ_Ω^{it} generates a symmetry of \mathcal{A} and Δ_Ψ^{it} generates a symmetry of $\bar{\mathcal{A}}$:

$$\begin{aligned} \Delta_{\Omega|\Psi}|\Omega_\Psi\rangle &= |\Omega_\Psi\rangle \\ \langle\Omega_\Psi|a\Omega_\Psi\rangle &= \langle\Omega_\Psi|\Delta_{\Omega|\Psi}^{it}a\Delta_{\Omega|\Psi}^{-it}\Omega_\Psi\rangle = \langle\Omega_\Psi|\Delta_\Omega^{it}a\Delta_\Omega^{-it}\Omega_\Psi\rangle \\ \langle\Omega_\Psi|a'\Omega_\Psi\rangle &= \langle\Omega_\Psi|\Delta_{\Omega|\Psi}^{it}a'\Delta_{\Omega|\Psi}^{-it}\Omega_\Psi\rangle = \langle\Omega_\Psi|\Delta_\Psi^{it}a'\Delta_\Psi^{-it}\Omega_\Psi\rangle \end{aligned}$$

for all $a \in \mathcal{A}$ and $a' \in \bar{\mathcal{A}}$. From the point of view of \mathcal{A} the state $|\Omega_\Psi\rangle$ is $h|\Omega\rangle$ and from the point of view of the algebra $\bar{\mathcal{A}}$ the state is $\tilde{h}|\Psi\rangle$.

In the vacuum of QFT restricted to half-space the modular operator is the boost. Vacuum is the only boost invariant state, therefore its centralizer is trivial. The modular

¹⁵Affiliated with \mathcal{A} means that it commutes with $\bar{\mathcal{A}}$. If $\bar{\mathcal{A}}$ has trivial center any bounded operator that is affiliated with \mathcal{A} is in \mathcal{A} .

flow in such states is ergodic in the sense that in the large time limit $\Delta_{\Omega}^{it}|\chi\rangle$ converges to vacuum in the weak norm for any state $|\chi\rangle$. See [17] for a review. The algebra of QFT is not the tensor product of $\mathcal{A} \otimes \bar{\mathcal{A}}$ which means that not all $|\Omega\rangle$ and $|\Psi\rangle$ can have exact invariant states of their relative modular operator $|\Omega_{\Psi}\rangle$. However, there are always states that are almost invariant with arbitrary precision. We take this as evidence that, in general, the exact state $|\Omega_{\Psi}\rangle$ that sews ω on the left with ψ on the right has a singularity (shockwave) at the boundary where the states are sewn together. As we saw this singular state $|\Omega_{\Psi}\rangle$ can be approximated arbitrarily well with normalizable states by acting with unitaries from the algebra. The approximation states are invariant under Δ_{Ψ}^{it} in $\bar{\mathcal{A}}$ and almost invariant under Δ_{Ω}^{it} in \mathcal{A} .

The centralizer of the vacuum of QFT is trivial. To apply the argument of section 4 we need to find projections that are approximately in the centralizer of the vacuum. In this case, we need to find approximately boost-invariant projections in the algebra of the half-space. Once again, we start with finite quantum systems to obtain intuition. It is straightforward to construct operators that commute with density matrix ω using the modular flow $a_{\omega}(t) = \omega^{it}a\omega^{-it}$. The idea is to integrate $a_{\omega}(t)$ over t to kill the off-diagonal terms:

$$\begin{aligned} \left[\sqrt{\omega}, \int_{-\infty}^{\infty} dt a_{\omega}(t) \right] &= \sum_{k,k'} \int dt e^{it(\lambda_k - \lambda_{k'})} a_{kk'} [\sqrt{\omega}, |k\rangle \langle k'|] \\ &= \sum_k a_{kk} [\sqrt{\omega}, |k\rangle \langle k|] = 0 . \end{aligned} \tag{7.7}$$

The operator $\int_{-\infty}^{\infty} dt a_{\omega}(t)$ is the zero Fourier mode of $a_{\omega}(t)$. A general Fourier mode $\hat{a}_{\omega}(l) = \int_{-\infty}^{\infty} dt e^{itl} a_{\omega}(t)$ satisfies

$$\begin{aligned} [\sqrt{\omega}, \hat{a}_{\omega}(l)] &= \sum_k a_{k(k-l)} [\sqrt{\omega}, |k\rangle \langle k-l|] \\ &= \sum_k a_{k(k-l)} e^{-\lambda_k/2} (1 - e^{l/2}) |k\rangle \langle k-l| , \end{aligned}$$

where $|k_l\rangle$ is the eigenvector with $\lambda_{k_l} = \lambda_k - l$. If there are no such eigenvector $\hat{a}_{\omega}(l) = 0$. In a finite quantum system the set of non-zero frequency modes is discrete, and the smallest frequency l corresponds to is the smallest eigenvalue gap. If the spectrum of the modular operator we have chosen is the full positive line all Fourier frequencies are non-zero and one can choose the Fourier mode $\hat{a}_{\omega}(\epsilon)$ for some $\epsilon \ll 1$. Such a mode almost commutes with ω :

$$\|[\sqrt{\omega}, \hat{a}_{\omega}(\epsilon)]\|_F \leq \epsilon \|a\| \| \sqrt{\omega} |k\rangle \langle k_{\epsilon}| \|_F \leq \epsilon \|a\| . \tag{7.8}$$

If we take a set of Fourier modes $|l| < \epsilon$ and $g(l)$ independent of ϵ the operator $\int_{|l| \leq \epsilon} dl g(l) \hat{a}_{\omega}(l)$ almost commutes with ω . Fourier transforming back we find that for any square-integrable function $g(t)$ whose Fourier transform $g(l)$ is restricted to $|l| \leq \epsilon$ the operator $g_{\omega}(a) = \int dt g(t) a_{\omega}(t)$ almost commutes with $\sqrt{\omega}$. It is self-adjoint if $g(t) = g(-t)$. The Fourier modes of modular flow have been discussed in connection with the holographic duality in [18].

8 Sewing multiple states

Finally, we come to the case of n subsystems. Consider a collection of n density matrices $\{\omega^{(1)}, \dots, \omega^{(n)}\}$ each corresponding to a qubit. The qubits can be sewn together if there exists a global pure state $|\Omega\rangle$ such that $\omega^{(i)} = \text{tr}_{i \neq j} |\Omega\rangle \langle \Omega|$ for all i . Local unitaries that rotate each qubit $U_i \omega^{(i)} U_i^\dagger$ have no effect on whether or not the density matrices can be sewn. The constraints on sewing $\omega^{(i)}$ only depend on their eigenvalues. Expand each density matrix in its diagonal eigenbasis:

$$\omega^{(i)} = \sum_i \lambda_i |0\rangle_i \langle 0|_i + (1 - \lambda_i) |1\rangle_i \langle 1|_i \tag{8.1}$$

with $0 \leq \lambda_i \leq 1/2$. The obstruction to sew density matrices is a constraint on the set of eigenvalues λ_i . We briefly review the derivation of these constraints in [1].

In the case of two qubits, one can sew the qubits if and only if they have the same spectrum, i.e. $\lambda_1 = \lambda_2$. Next, consider the case of three qubits. The projection $e_i = |0\rangle \langle 0|_i$ satisfy the identity: $e_1 + e_2 = 1_{12} + e_1 e_2 - (1 - e_1)(1 - e_2)$. This identity implies that if $|\Omega\rangle$ is a global vectors that sews them together

$$\lambda_1 + \lambda_2 = \langle \Omega | e_1 + e_2 | \Omega \rangle \geq 1 - \text{tr} \left(\omega^{(12)} (1 - e_1)(1 - e_2) \right),$$

where we have discarded the positive term $\langle \Omega | e_1 e_2 | \Omega \rangle$. Since $|\Omega\rangle$ is pure the eigenvalues of $\omega^{(12)}$ are the same as those of $\omega^{(3)}$. Its largest eigenvalue is $1 - \lambda_3$ which means $\langle \Phi | \omega^{(12)} | \Phi \rangle \leq 1 - \lambda_3$ for all $|\Phi\rangle$ in the Hilbert space of three qubits. The projector $(1 - e_1)(1 - e_2)$ is rank one, therefore $\text{tr}(\omega^{(12)}(1 - e_1)(1 - e_2)) \leq 1 - \lambda_3$. Plugging this back into (8) we find

$$\lambda_1 + \lambda_2 \geq \lambda_3 . \tag{8.2}$$

Permuting the qubits and repeating the same argument gives two more constraints:

$$\sum_{i \neq j} \lambda_j \geq \lambda_i . \tag{8.3}$$

Any three density matrices with $\{\lambda_1, \lambda_2, \lambda_3\}$ that satisfy the above inequality can be sewn together in a global vector that is non-unique.

This argument generalizes to n -qubits in a straightforward manner, and the final constraint on the eigenvalues λ_i are the same as (8.3) but with $i = 1, \dots, n$. The eigenvalues of local density matrices of an n -partite system that can be sewn together form a convex polytope. See [19] for a review of the generalization to arbitrary finite dimensional systems.

We now turn to quantum field theory. Starting with any global state $|\Psi\rangle$ acting with local unitaries in region A_1 we can prepare an arbitrary local state ω_1 on A_1 . Then, we act locally on A_2 and prepare ω_2 , and repeat this for all subregions to A_n to obtain a global state that sews local states ω_i for $i = 1, \dots, n$. There are no constraints on sewing ω_i . For instance, using local unitaries we can make every state look like vacuum on each A_i . Every state $|\Psi\rangle$ has the same entanglement structure as $U_1 U_2 \dots U_n |\Psi\rangle$. By classifying all states that sew vacuum reduced states ω_i together we learn about various forms of multi-partite entanglement than can appear in QFT.

Consider the case $n = 3$ in QFT with the region A_2 separating A_1 and A_3 . We take the mutual information between A_1 and A_3 in the vacuum as a measure of entanglement between A_1 and A_3 . This mutual information is the same as the relative entropy of ω_{13} with respect to ω_1 and ω_3 :

$$I(A_1 : A_3) = S(\omega_{13} || \omega_1 \otimes \omega_3) . \tag{8.4}$$

Since the relative entropy on $A_1 A_3$ does not depend on the purification we write

$$S_{vac}(\omega_{13} || \omega_1 \otimes \omega_3) = - \langle \Omega | \log \Delta_{\omega_1 \otimes \omega_3 | \omega_{13}} | \Omega \rangle . \tag{8.5}$$

Any other global state that sews ω_i s has the form $|\tilde{\Omega}\rangle \simeq U_2 U_{13} |\Omega\rangle$. The mutual information in this state is

$$S(\tilde{\omega}_{13} || \omega_1 \otimes \omega_3) = - \langle \Omega | U_{13}^\dagger \log \Delta_{\omega_1 \otimes \omega_3 | \omega_{13}} U_{13} | \Omega \rangle . \tag{8.6}$$

The vacuum relative modular operator $\Delta_{\omega_1 \otimes \omega_3 | \omega_{13}}$ teaches us about the mutual information in a dense set of states $|\tilde{\Omega}\rangle$. The same principle extends to tripartite entanglement and other measures of multi-partite entanglement for larger n that are invariant under local unitaries.

It is tempting to conclude from the discussion above that the vacuum relative modular operators $\Delta_{\omega_1 \otimes \dots \otimes \omega_n | \omega_{1\dots n}}$ contain the information about the multi-partite entanglement of all states. This is incorrect because the relative modular operator and its logarithm are unbounded and if we have a sequence of states $\psi_n \rightarrow \psi$ the relative modular operators $\lim_n \Delta_{\psi_n | \psi}$ need not converge. In fact, the relative entropy is not continuous, but just lower semi-continuous [20]:

$$S(\phi || \psi) \leq \lim_n S(\phi || \psi_n) . \tag{8.7}$$

We postpone the study of the implications of sewing for the theory of multi-partite entanglement in QFT to future work.

9 Discussion

In this work, we showed that in quantum field theory any collection of local reduced states in non-overlapping regions can be sewn together with arbitrary precision. We argued that the local unitary that acts on $|\Psi\rangle$ and creates the vacuum state ω is the cocycle $u_{\Omega|\Psi}(t)$ in the large t limit. Ideas similar to sewing states of quantum field theory have appeared in various context, recently [21, 22]. It would be interesting to explore the connection between our algebraic sewing prescription and sewing Euclidean path-integrals discussed in the literature. Here, we addressed the problem of sewing one-body local states of in a global pure state. The collection of one-body density matrices can be understood as a mean-field approximation to the state [1]. The problem can be generalized to sewing multi-body local states with overlapping regions, called the quantum marginal problem. See [19] for a discussion of the quantum marginal problem in the general setup. It is interesting to study the generalized marginal problem in QFT [23]. It is worthwhile to note that the sewing argument presented in this paper applied generically to any two states of a type III₁

von Neumann algebra, independent of which quantum field theory they belong to. It would be interesting to study the physics of sewing states of different QFT and its connection with boundary states in a QFT.

In a QFT with a global symmetry the algebra of charge-neutral operators can be enlarged by the generator of the symmetry group on A [24]. The modular operator for the charge-neutral subalgebra has non-trivial centralizers that correspond to $\oplus \lambda_r 1_r$ where 1_r is the identity operator in the irreducible representation r of the symmetry group. From our work here, it follows that if we take the infinite time limit of the cocycle $u_{\Psi|\Omega}(t)$ computed with the relative entropies defined with respect to the invariant algebra any excitation can be washed away (the state ψ becomes locally like the vacuum ω) except for centralizers that correspond to the charges. It is tempting to speculate that this procedure can be used to formally define a theory of hydrodynamics where all local physics is averaged out except for the conserved charges.

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A Distance measures and unitary orbits

There are various distance measures one can introduce on the space of density matrices ρ and ω . In this work, we primarily used the distance $d_F(\rho, \omega)$. In this work, we primarily used the distance $d_F(\rho, \omega)$. The physical interpretation of this measure comes from (2.5) that equates it with the distance in the Hilbert space. For completeness, here we compare this measure with some other measures commonly used in information theory. The trace distance d_T , the Bures distance d_B and the distance d_F are symmetric in their arguments. A few well-known distance measures commonly used in information theory are:

$$\begin{aligned}
 d_T(\rho, \omega)^2 &= \frac{1}{2} \|\rho - \omega\|^2 \\
 d_B(\rho, \omega)^2 &= 1 - \|\sqrt{\rho}\sqrt{\omega}\| \\
 d_F(\rho, \omega)^2 &= \frac{1}{2} \|\sqrt{\omega} - \sqrt{\rho}\|_F^2 \\
 d_R(\rho, \omega)^2 &= 1 - e^{-\frac{1}{2}S(\rho|\omega)} \\
 S(\rho|\omega) &= \text{tr}(\rho \log \rho) - \text{tr}(\omega \log \rho)
 \end{aligned}
 \tag{A.1}$$

where $\|X\| = \text{tr}(|X|)$. Bures distance is a metric, and is continuous with respect to the trace distance.¹⁶ It is closely related to quantum fidelity $F(\omega, \rho) = \|\sqrt{\omega}\sqrt{\rho}\|$. All four distance

¹⁶That is to say it can be bounded from above by trace distance.

measures vanish if and only if $\rho = \omega$, and remain invariant under simultaneous rotation of both states, i.e. $d(\rho, \omega) = d(U\rho U^\dagger, U\omega U^\dagger)$. Their relationship can be summarized with the inequality:

$$0 \leq d_T \leq d_B \leq d_F \leq d_R \leq 1 \quad (\text{A.2})$$

The first inequality is the Fuchs–van de Graaf inequality [25]. The second inequality is due to the fact that $\text{tr}(\sqrt{\rho}\sqrt{\omega}) \leq \|\sqrt{\rho}\sqrt{\omega}\|$. The last inequality uses $S(\rho\|\omega) \geq -2 \ln \text{tr}(\sqrt{\rho}\sqrt{\omega})$ [26].

The inequality (A.2) implies in our analysis we could have used d_T or d_B and the result would have been the same.

B Skew information and relative entropy

In defining the skew information in (7.3) we can use p -norms. This generalization of the Wigner-Yanase information is due to Dyson. For a self-adjoint operator x and ω a density matrix purified canonically in $|\Omega\rangle$ the Wigner-Yanase-Dyson p -skew information is

$$I_p(\omega, x) = -\text{tr}([\sigma^p, x][\sigma^{1-p}, x]) \quad (\text{B.1})$$

In terms of the canonical purification $|\Omega\rangle$ of the density matrix the skew information is

$$I_p(x, \Omega) = \langle \Omega | x (\Delta_\Omega^p + \Delta_\Omega^{1-p} - 1 - \Delta_\Omega) x | \Omega \rangle \quad (\text{B.2})$$

The derivative at $p = 0$ is

$$\lim_{p \rightarrow 0} I_p(x, \Omega) = 2 \langle \Omega | x \log \Delta_\Omega x | \Omega \rangle \quad (\text{B.3})$$

If x is a unitary operator then this is the relative entropy

$$S(u^\dagger \Omega | \Omega) = -\langle \Omega | \log \Delta_{u^\dagger \Omega} | \Omega \rangle \quad (\text{B.4})$$

The p -skew information is symmetric under $p \rightarrow 1 - p$ with the symmetric point corresponding to the Wigner-Yanase measure of section 7.

C Classification of quantum systems

Every quantum mechanical system has an algebra of bounded observables \mathcal{A} . To every operator $a \in \mathcal{A}$ corresponds a unique $a^* \in \mathcal{A}$ that satisfies $(a^*)^* = a$. Every operator also has a norm $\|a\|$ that satisfies $\|a^*a\| = \|a\|^2$. Such an algebra is called a C^* algebra. In physics, we often work with a representation of the C^* algebra of observables in a Hilbert space \mathcal{H} . In the Hilbert space, a^* is represented by the complex conjugate a^\dagger , and the norm of an operator is

$$\|a\| = \sup_{|\Psi\rangle \in \mathcal{H}} \frac{\|a|\Psi\rangle\|}{\| |\Psi\rangle \|} \quad (\text{C.1})$$

The algebra of observables of a d -level quantum system (qudit) is the algebra of $d \times d$ complex matrices $M_d(\mathbb{C})$. It has a trivial center $\lambda \mathbb{I}$. A C^* algebra with a trivial center is

called a factor. Every finite dimensional C^* algebra is a direct sum of a finite number of C^* factors, therefore it suffices to classify C^* factors. The qudit algebra $M_d(\mathbb{C})$ is the unique C^* factor of dimension d .

In infinite dimensions we need to decide what limits of a sequences of operators a_n to include in the algebra. From the physics stand-point the observables of the theory are the overlaps $\langle \phi|a|\psi \rangle$ for two states $|\phi \rangle$ and $|\psi \rangle$ in the Hilbert space. Therefore, it is natural to close the algebra by including its weak limits. That is to say if for a sequence of operators a_n and the matrix elements $\lim_n \langle \psi|a_n|\phi \rangle = \lambda_a$ we include in the algebra an operator a with matrix elements $\langle \psi|a|\phi \rangle = \lambda_a$. A C^* algebra that is closed in the weak topology is called a *von Neumann algebra*. A general quantum system is a von Neumann algebra. Note that all finite dimensional C^* algebras are von Neumann algebras.

To classify all quantum systems we need to classify all von Neumann factors. Von Neumann divided all von Neumann factors into three types: type I, II and III. The only infinite dimensional Type I algebra is the algebra of bounded operators in a separable infinite dimensional Hilbert space. Any other infinite dimensional algebra that admits a trace is called type II. A trace is a positive linear map from the algebra to complex numbers $\text{tr} : \mathcal{A} \rightarrow \mathbb{C}$ that satisfies $\text{tr}(a_1 a_2) = \text{tr}(a_2 a_1)$ for all $a_1, a_2 \in \mathcal{A}$.

The insight of Connes was that the key to the classification of type III algebras is the spectrum of modular operator. Consider the intersection of the spectrum of the modular operator Δ_Ψ for all states $|\Psi \rangle \in \mathcal{H}$. If this set is λ^n for some positive λ and integer n the von Neumann factor is called type III_n . If the intersection of the modular spectra is the full positive line it is type III_1 .

D Isometries as a limit of unitaries

In this section, we argue that in QFT the unitary orbit of any state passes through subspaces $\pi\mathcal{H}$ for any projection $\pi \in \mathcal{A}$. There are two parts to the argument. First, we observe that in QFT (in general, any type III von Neumann algebra) for every projection π there exists an isometry w such that $ww^\dagger = \pi$ and $w^\dagger w = 1$ [5]. The state $|\Phi_w \rangle = w|\Phi \rangle$ is an eigenstate of π for any $|\Phi \rangle$. Therefore, by acting with an isometry we can bring any state to the subspace $\pi\mathcal{H}$. Second, we notice that in QFT (any type III algebra) any isometry w is a limit of unitary operators u_n in strong operator norm, i.e. $\lim_n w = u_n$.¹⁷ To see this, consider a sequence of projections π_n in the algebra that converge to identity: $\lim_n \pi_n = 1$. For any n the two projections $1 - \pi_n$ and $1 - w\pi_n w^\dagger$ belong to the algebra. A type III algebra is one in which for any two projections π and $\tilde{\pi}$ there exists an operator v such that $v^\dagger v = \pi$ and $vv^\dagger = \tilde{\pi}$. Therefore, there exists a partial isometry v_n such that $v_n^\dagger v_n = 1 - \pi_n$ and $v_n v_n^\dagger = 1 - w\pi_n w^\dagger$. The operator $\pi_n w^\dagger v_n = 0$ is zero because its norm vanishes: $(\pi_n w^\dagger v_n)(v_n^\dagger w\pi_n) = 0$. Similarly, we have $v_n \pi_n = 0$. If we define the operator $u_n = w\pi_n + v_n$ we find that it is a unitary operator for all n :

$$\begin{aligned} u_n^\dagger u_n &= 1 + \pi_n w^\dagger v_n + v_n^\dagger w\pi_n = 1 \\ u_n u_n^\dagger &= 1 + w\pi_n v_n^\dagger + v_n \pi_n w^\dagger = 1 . \end{aligned} \tag{D.1}$$

The sequence of unitaries u_n tends to w in strong operator topology, i.e. $\lim_n w = u_n$.

¹⁷We learned the argument presented here from Martin Argerami.

We have established that there exists a sequence of unitaries that bring any state to the eigenspace of any projector π in the algebra: $\pi(\lim_n u_n |\Phi\rangle) = \lim_n u_n |\Phi\rangle$. Since this can be done for any two states, one might wonder if any two states can be brought arbitrarily close with unitaries. This happens only in a type III₁ algebra.

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