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# Soliton, breather and shockwave solutions of the Heisenberg and the $T\bar{T}$ deformations of scalar field theories in 1+1 dimensions

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**ABSTRACT:** In this note we study soliton, breather and shockwave solutions in certain two dimensional field theories. These include: (i) Heisenberg's model suggested originally to describe the scattering of high energy nucleons (ii)  $T\bar{T}$  deformations of certain canonical scalar field theories with a potential. We find explicit soliton solutions of these models with sine-Gordon and Higgs-type potentials. We prove that the  $T\bar{T}$  deformation of a theory of a given potential does not correct the mass of the soliton of the undeformed one. We further conjecture the form of breather solutions of these models. We show that certain  $T\bar{T}$  deformed actions admit shockwave solutions that generalize those of Heisenberg's Lagrangian.

**KEYWORDS:** Field Theories in Lower Dimensions, Solitons Monopoles and Instantons

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## 1 Introduction

The DBI action, which has been known as the action describing D-branes, had been used much earlier. In fact, Heisenberg implemented it in 1952 when he proposed a very simple model for high energy nucleon-nucleon scattering [1]. He wrote down a massive DBI action to describe an interacting “pion field” that mediates the interaction between nucleons of the following form,

$$S = \int d^{3+1}x l^{-4} \left[ 1 - \sqrt{1 + l^4 [(\partial_\mu \phi)^2 + m^2 \phi^2]} \right]. \quad (1.1)$$

Remarkably, Heisenberg’s model reproduces the saturation of the Froissart bound [2, 3],

$$\sigma_{\text{tot}}(s) \leq C \ln^2 \frac{s}{s_0}, \quad C \leq \frac{\pi}{m_1^2}, \quad (1.2)$$

even though the model was proposed not just before Froissart, but even before QCD. In the model, Heisenberg first dimensionally reduced the Lagrangian to 1+1 dimensions (time

and direction of propagation of the pion field) and then discovered a shock wave solution of the equation of motion of this model defined perturbatively. An analysis of the Heisenberg model, uniqueness properties, and its generalizations was done in [4].

Recently there has been a very important development in deciphering the space of field theories in 1+1 dimensions in the form of what is referred to as the  $T\bar{T}$  deformations [5, 6]. In particular the Lagrangian density for the deformations of canonical scalar action with a potential  $V$ , was determined in [7–9]. The interest in these deformations comes from the fact that they are among very few exactly known quantum deformations. While new deformations of non-integrable models were also considered, the original motivation to study these deformations was to explore the subspace of field theories in two dimensions which are integrable.  $T\bar{T}$  deformations have been explored in relation to holography, gravity and string theories. In addition they have served as laboratories to investigate various aspects of field theory, for instance in [10–15].

Key players of integrable field theories are solitons, anti-solitons and their breather bound states. Thus, it is important to study the properties of the latter in the  $T\bar{T}$  deformed theories. This is the task taken in this paper. We determine explicit soliton solutions of the  $T\bar{T}$  deformed theories as well as solitons of the Heisenberg model. In particular we prove that the classical mass of the soliton of the deformed theory is the same as the mass of the soliton of the undeformed theory. In route to these results we also write down the soliton solution of the Heisenberg model. We further write down explicit shockwave solutions of both theories.

The application of these solutions to the Heisenberg model of nucleon scattering in 3+1 dimensions will be done in a separate publication [16].

The paper is organized as follows. In section 2 we define the Lagrangians, (generalized) Heisenberg and  $T\bar{T}$  deformations. In section 3, considering the Heisenberg and  $T\bar{T}$  deformed scalar actions, we find soliton and single shockwave solutions for them. In section 4 we consider perturbative shockwave solutions for them, and in section 5 we conclude. In appendix A we consider extensions of the pure DBI action and Heisenberg actions to some cases with arbitrary powers.

## 2 $T\bar{T}$ deformed scalar fields and Heisenberg model: actions and equations of motion

The  $T\bar{T}$  deformation of a Lagrangian, proposed by Zamolodchikov [5, 6] is that, for a 1+1 dimensional theory Wick rotated to Euclidean space, the deformation with parameter  $\lambda$  is

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -4 \left[ T_{zz}^\lambda T_{\bar{z}\bar{z}}^\lambda - (T_{z\bar{z}}^\lambda)^2 \right]. \quad (2.1)$$

where  $\mathcal{L}(\lambda = 0)$  is the original undeformed Lagrangian density,  $T_{\mu\nu}^\lambda$  are the components of the energy-momentum tensor of the theory deformed by  $\lambda$  and the composite operator on the right-hand side is defined via point splitting.

The interesting property of this deformation is that it is of a full quantum theory, and all objects are renormalized and UV finite. For our purposes we will treat this as simply a certain (effective) classical Lagrangian.

One can solve the above equation by expanding in a series in  $\lambda$ , if we give a starting point (an unperturbed Lagrangian  $\mathcal{L}_0$ ) and a perturbation parameter  $\lambda$ . It follows from (2.1), that the first order Lagrangian density,  $\mathcal{L}_1$  is given by the unperturbed determinant of the energy-momentum tensor,  $\det T_{\mu\nu}$ , above, etc. Consider as a starting point a canonical real scalar field with a potential  $V$  (in Euclidean space),

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 + V, \tag{2.2}$$

and a perturbation parameter  $\lambda$ .

Then, at first one obtained a complicated expression, with an infinite series of complicated hypergeometric functions (see eq. 6.34 in [7]), but the series can be summed, to obtain a simple expression [8], giving

$$\begin{aligned} \mathcal{L}(\lambda, X) &= \frac{-(1 - 2\lambda V) + \sqrt{1 + 2\bar{\lambda}(\partial_\mu\phi)^2}}{2\bar{\lambda}} \\ &\equiv \tilde{V}(\phi) + \frac{1}{2\bar{\lambda}}\sqrt{1 + 2\bar{\lambda}(\partial_\mu\phi)^2}, \end{aligned} \tag{2.3}$$

where we can denote  $X = (\partial_\mu\phi)^2$ , and we have defined

$$\bar{\lambda} \equiv \lambda(1 - \lambda V). \tag{2.4}$$

Note that when  $\lambda V \rightarrow 0$ ,  $\tilde{V} \rightarrow V - 1/(2\lambda)$ , which cancels the a constant coming from the square root, and gives the unperturbed potential. This can then also be written as in [9] (since, in Euclidean complex coordinates,  $4\partial\phi\bar{\partial}\phi = (\partial_\mu\phi)^2$ ),

$$\mathcal{L}_E = \frac{V}{1 - \lambda V} + \frac{-1 + \sqrt{1 + 8\bar{\lambda}\partial\phi\bar{\partial}\phi}}{2\bar{\lambda}}. \tag{2.5}$$

This is the Lagrangian in Euclidean space. Going back to the Minkowski signature, we obtain

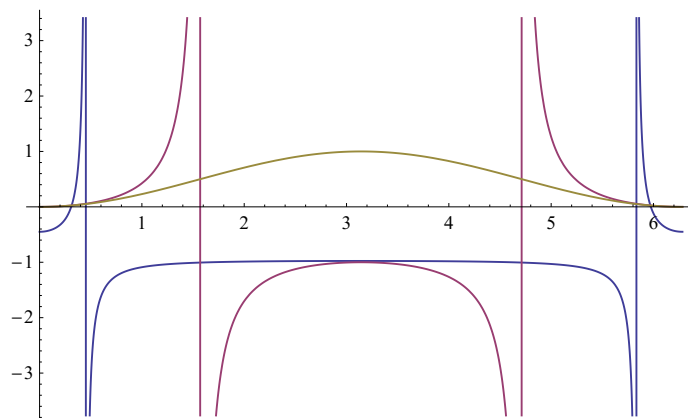
$$\mathcal{L}_M = -\frac{V}{1 - \lambda V} + \frac{1 - \sqrt{1 + 2\bar{\lambda}\partial_\mu\phi\partial^\mu\phi}}{2\bar{\lambda}} = -\tilde{V} - \frac{\sqrt{1 + 2\bar{\lambda}\partial_\mu\phi\partial^\mu\phi}}{2\bar{\lambda}}, \tag{2.6}$$

with  $\partial_\mu\phi\partial^\mu\phi = -(\dot{\phi})^2 + (\phi')^2 \equiv X$  and

$$\tilde{V} \equiv -\frac{1 - 2\lambda V}{2\lambda(1 - \lambda V)}. \tag{2.7}$$

It is thus obvious that, depending on the potential and  $\lambda$ ,  $\tilde{V}$  can be singular if there is a value of  $\phi$  for which  $V(\phi) = \frac{1}{\lambda}$ . Notice that when we expand  $\tilde{V}$  to leading order in  $\lambda$ , namely,  $\lambda X \ll 1$  and  $\lambda V \ll 1$ , we get  $\tilde{V} \sim V - \frac{1}{2\lambda}$ . But when we expand the full Lagrangian density, we indeed get  $V$  and the term of order  $\lambda^{-1}$  cancels out as can be seen from

$$\mathcal{L}_M = \frac{1}{2} \left[ (\dot{\phi})^2 - (\phi')^2 \right] - V + \lambda \left[ \frac{1}{4} \left[ (\dot{\phi})^2 - (\phi')^2 \right]^2 - V^2 \right] + \mathcal{O}(\lambda^2). \tag{2.8}$$



**Figure 1.** The potential of the deformed sine-Gordon model with  $\mu = \beta = 1$  for  $\lambda = 0$ . The undeformed case is in brown,  $\lambda = 1$  is in blue and  $\lambda = 10$  is in red.

In figure 1 we draw the deformed potential  $\tilde{V} - \frac{1}{2\lambda}$  for the sine-Gordon model, for which the undeformed potential is given in (3.1).

The other Lagrangian we will be interested in is the Heisenberg Lagrangian, with a generic potential  $V(\phi)$  inside the square root, and dimensionally reduced to 1+1 dimensions, so

$$\mathcal{L} = l^{-4} \left[ 1 - \sqrt{1 + l^4 [(\partial_\mu \phi)^2 + 2V(\phi)]} \right]. \quad (2.9)$$

When expanding in small  $l^4 [(\partial_\mu \phi)^2 + 2V(\phi)]$  the lagrangian density takes the form

$$-\frac{1}{2} (\partial_\mu \phi)^2 - V(\phi) + \frac{l^4}{8} \left( (\partial_\mu \phi)^2 + 2V \right)^2 + \mathcal{O}(l^8) \quad (2.10)$$

### 3 Soliton and breather solutions

We now examine whether these  $T\bar{T}$  and Heisenberg Lagrangians (for various possible potentials  $V(\phi)$ ) admit soliton, breather and shockwave solutions. We first briefly remind the reader the form of these solutions in the ordinary sine-Gordon model.<sup>1</sup> We then analyze the Heisenberg model with a generic potential and then the  $T\bar{T}$  deformed action.

#### 3.1 The sine-Gordon model

We start with a review of an un-deformed model with solitons in 1+1 dimensions.<sup>2</sup>

The most well known 1+1 dimensions scalar field theory that admits soliton solutions is the sine-Gordon model which is defined by the potential

$$V_{sG} = -\frac{\mu^2}{\beta} [\cos(\beta\phi) - 1]. \quad (3.1)$$

<sup>1</sup>The deformed sine-Gordon action has been discussed in several publication for instance [6, 7].

<sup>2</sup>For a review of these issues see for instance [17].

The corresponding equation of motion is

$$\phi'' - \ddot{\phi} - \partial_\phi V = 0. \quad (3.2)$$

For static solution we multiply the equation of motion with  $\phi'$  to find

$$\phi' \phi'' - \phi' \partial_\phi V = 0 \rightarrow \partial_x \left[ \frac{1}{2} (\phi')^2 - V \right] = 0 \rightarrow \frac{1}{2} (\phi')^2 - V = C, \quad (3.3)$$

where  $C$  is a constant. This is the “virial theorem”. For  $C = 0$  the equation takes the form

$$x - x_0 = \frac{\sqrt{\beta}}{\sqrt{2\mu}} \int \frac{d\phi}{\sqrt{1 - \cos(\beta\phi)}}, \quad (3.4)$$

which yields the soliton solution

$$\phi_{\text{sol}} = \frac{4}{\beta} \arctan \left[ e^{\pm \mu \sqrt{\beta}(x-x_0)} \right]. \quad (3.5)$$

The mass of the soliton, i.e., its energy is

$$\begin{aligned} E_{\text{sol}} &= \int dx T_{00} = \int dx \left[ \frac{1}{2} (\phi')^2 + V \right] = \int dx (\phi')^2 = \int d\phi \phi' \\ &= \int d\phi \sqrt{2V(\phi)} = \frac{8\mu}{\beta^{3/2}}. \end{aligned} \quad (3.6)$$

Next we would like to check whether the equations of motion admit also time dependent exact solutions. Another question seemingly unrelated is, what is the interaction between two solitons and between a soliton and an anti-soliton, and whether there are bound states. In fact the two questions are very related.

It is easy to check that performing the following transformation

$$\phi_{\text{sol}} = \frac{4}{\beta} \arctan \left[ e^{\pm \mu \sqrt{\beta}(x-x_0)} \right] \rightarrow \phi_{\text{bre}} = \frac{4}{\beta} \arctan \left[ \frac{\eta \sin(\eta t)}{\cosh(\eta w x)} \right], \quad (3.7)$$

where  $\eta = \tan(\psi) = \frac{\sqrt{\mu^2 \beta - w^2}}{w}$ , yields a solution of the equation of motion. To see that this solution describes a bound state, consider first  $w \ll \mu \sqrt{\beta}$ . When  $x \rightarrow -\infty$  we can approximate

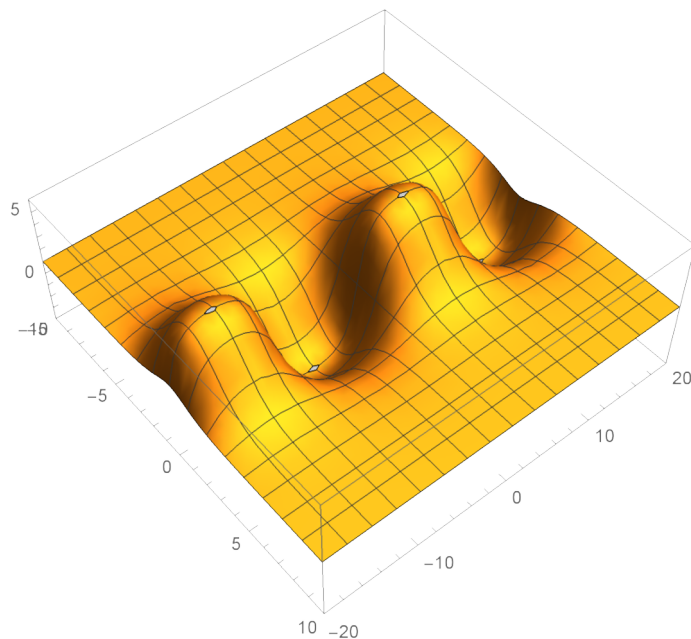
$$\phi_{\text{bre}} \sim \frac{4}{\beta} \arctan \left[ e^{\mu \sqrt{\beta}(x-x_0) + \log \left( \frac{\mu \sqrt{\beta}}{w} \sin(\eta t) \right)} \right], \quad (3.8)$$

which looks like a soliton to the left. Similarly the solution looks like an anti-soliton to the right. The breather soliton is drawn in figure 2.

Altogether the breather solution describe an oscillating bound state. Indeed we can determine the mass of the breather by computing it at  $t = 0$ , for which  $\dot{\phi} = V = 0$ , so that

$$E_{\text{bre}} = \int dx \left[ \frac{1}{2} (\dot{\phi})^2 \right] = \frac{8(\eta w)^2}{\beta^2} \int dx \frac{1}{\cosh^2(\eta w x)} = 2M_{\text{sol}} \sqrt{1 - \left( \frac{w}{\mu \sqrt{\beta}} \right)^2}, \quad (3.9)$$

which is obviously smaller than twice the mass of the soliton, namely there is a non-vanishing bin.



**Figure 2.** The breather solution for  $\mu = 1$ ,  $\beta = 1 \tan \psi = 2\pi/5$ .

### 3.2 The Heisenberg model with a generic potential

Since it is hard to find soliton solutions of the  $T\bar{T}$  deformations, we start by first finding solutions of the Heisenberg model modified with a generic potential, instead of just a mass term, i.e., the Minkowski space Lagrangian in (2.9), namely

$$\mathcal{L}_M = 1 - \sqrt{1 + (\partial_\mu \phi)^2 + 2V} , \tag{3.10}$$

where we have put  $l = 1$  for simplicity (to reinstate it, we can just replace  $\phi = l^2 \tilde{\phi}$ ,  $V = l^4 \tilde{V}$  and multiply the Lagrangian by  $l^{-d}$ ), and  $(\partial_\mu \phi)^2 = -\dot{\phi}^2 + \phi'^2$ , as before. The Hamiltonian is

$$\mathcal{H} = \dot{\phi} \frac{\partial \mathcal{L}_M}{\partial \dot{\phi}} - \mathcal{L}_M = \frac{1 + \phi'^2 + 2V}{\sqrt{1 - \dot{\phi}^2 + \phi'^2 + 2V}} - 1. \tag{3.11}$$

For static solutions

$$\mathcal{H} = -\mathcal{L}_M = \sqrt{1 + \phi'^2 + 2V} - 1. \tag{3.12}$$

Consider a static solution,  $\dot{\phi} = 0$ , so  $(\partial_\mu \phi)^2 = \phi'^2$ . Then the equation of motion is

$$\frac{d}{dx} \left( \frac{\phi'}{\sqrt{1 + \phi'^2 + 2V}} \right) = \frac{\frac{dV}{d\phi}}{\sqrt{1 + \phi'^2 + 2V}}. \tag{3.13}$$

Multiply it with  $\phi'$ , so that on the right-hand side we have  $(dV/d\phi)(d\phi/dx) = dV/dx$ .

Then note the identities

$$\begin{aligned} \frac{d}{dx} \left( \frac{\phi'^2/2}{\sqrt{1+\phi'^2+2V}} \right) &= \phi' \frac{d}{dx} \left( \frac{\phi'}{\sqrt{1+\phi'^2+2V}} \right) - \frac{\phi'^2}{2} \frac{d}{dx} \left( \frac{1}{\sqrt{1+\phi'^2+2V}} \right) \\ \frac{d}{dx} \left( \frac{V}{\sqrt{1+\phi'^2+2V}} \right) &= \frac{dV/dx}{\sqrt{1+\phi'^2+2V}} + V \frac{d}{dx} \frac{1}{\sqrt{1+\phi'^2+2V}}. \end{aligned} \quad (3.14)$$

Subtract them and use the equation of motion above, to obtain

$$\begin{aligned} \frac{d}{dx} \left( \frac{V - \phi'^2/2}{\sqrt{1+\phi'^2+2V}} \right) &= (2V + \phi'^2) \frac{d}{dx} \left( \frac{1}{2\sqrt{1+\phi'^2+2V}} \right) \\ &= \frac{d}{dx} \left( \frac{V + \phi'^2/2}{\sqrt{1+\phi'^2+2V}} \right) - \frac{d}{dx} \sqrt{1+\phi'^2+2V}, \end{aligned} \quad (3.15)$$

which finally implies the conservation equation

$$\frac{d}{dx} \left( -\frac{1+2V}{\sqrt{1+2V+\phi'^2}} \right) = 0, \quad (3.16)$$

with general solution

$$\frac{1+2V}{\sqrt{1+2V+\phi'^2}} = C, \quad (3.17)$$

for an arbitrary  $C$ . Solving it for  $\phi'^2$ , we get

$$\phi'^2 = (1+2V) \left( \frac{1+2V}{C^2} - 1 \right). \quad (3.18)$$

Then the general solution is written implicitly as

$$x - x_0 = \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{(1+2V) \left[ \frac{1+2V}{C^2} - 1 \right]}}. \quad (3.19)$$

For comparison, consider first the canonical Lagrangian in 1+1 dimensions,  $\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - V$ . The virial theorem (3.3) is the conservation equation  $\frac{d}{dx}(\phi'^2/2 - V) = 0$ , in the case of *finite energy* soliton solutions, for which the integration constant  $\tilde{C}$  vanishes, so the solution is

$$x - x_0 = \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{2V(\phi)}}. \quad (3.20)$$

We see that our case is obtained for  $C = 1$  and in the  $l \rightarrow 0$  limit. Keeping  $l$  finite (=1 in our convention), but  $C = 1$ , we obtain

$$x - x_0 = \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{2V(1+2V)}}, \quad (3.21)$$

corresponding to the equality (deformed virial theorem)

$$\phi'^2 = 2V(1+2V). \quad (3.22)$$



Using this deformed virial theorem we can now compute the mass of the soliton

$$\begin{aligned} M_{sol} &= \int dx \mathcal{H} = \int dx [\sqrt{1 + \phi'^2 + 2V} - 1] = \int dx 2V \\ &= \int d\phi \frac{2V}{\phi'} = \int d\phi \frac{2V}{\sqrt{2V(1+2V)}} = \int d\phi \frac{\sqrt{2V}}{\sqrt{(1+2V)}}. \end{aligned}$$

We can check now what is the soliton solution and what is its corresponding mass for the case of the sine-Gordon potential (3.1). Substituting it in (3.21) we get that the solution for  $\phi$  reads

$$x - x_0 = \frac{1}{\mu\sqrt{\beta}} \tanh^{-1} \left[ \frac{\cos \frac{\beta\phi}{2}}{\sqrt{1 + A \sin^2 \frac{\beta\phi}{2}}} \right]_{\phi}^{\frac{\pi}{\beta}}, \quad (3.23)$$

or, inverting the formula,

$$\begin{aligned} \phi(x) &= \frac{2}{\beta} \arcsin \left( \sqrt{\frac{1 - \tanh^2[-\mu\sqrt{\beta}(x-x_0)]}{1 + A \tanh^2[-\mu\sqrt{\beta}(x-x_0)]}} \right) \\ &= \frac{2}{\beta} \arctan \left( \sqrt{\frac{1 - \tanh^2[-\mu\sqrt{\beta}(x-x_0)]}{(1+A) \tanh^2[-\mu\sqrt{\beta}(x-x_0)]}} \right), \end{aligned} \quad (3.24)$$

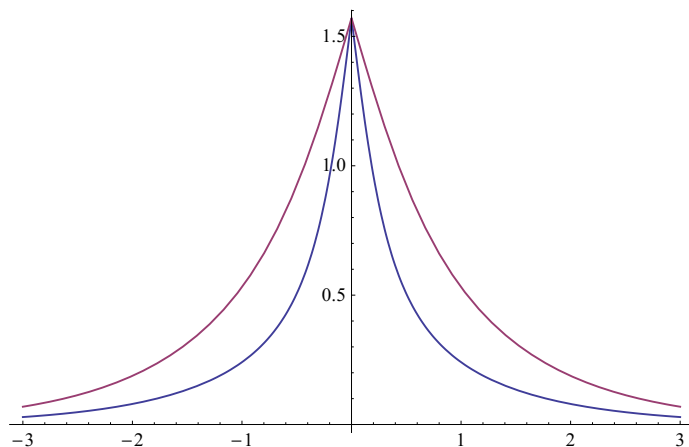
where  $A = \frac{4l^2\mu^2}{\beta}$ . However, from (3.23) note that  $\tanh[-\mu\sqrt{\beta}(x-x_0)]$  takes values between  $+1$  (for  $x = -\infty$ ) and  $-1$  (for  $x = +\infty$ ), corresponding to  $\frac{\beta\phi}{2}$  taking values between  $0$  (for  $x = -\infty$ ) and  $\pi$  (for  $x = +\infty$ ), while  $x = x_0$  is mapped to  $\frac{\beta\phi}{2} = \frac{\pi}{2}$ , just like the topological sine-Gordon soliton, whereas arcsin and arctan have range  $[-\pi/2, \pi/2]$ , so (3.24) produces a result in  $[0, \pi/2]$ . Then, the topological soliton that is a deformation of the sine-Gordon soliton (and so must go between  $0$  at  $-\infty$  and  $\frac{2}{\beta}\pi$  at  $+\infty$ ) is defined as (3.24) for  $x \leq x_0$ , and as  $\pi - (3.24)$  for  $x \geq x_0$ , which ensures also that  $\phi'$  is continuous at  $x_0$ . On the other hand, (3.24) can be taken to be a solution for all  $x \in \mathbb{R}$ . The solution defined in (3.24) is drawn in figure 3 and the corresponding  $(\phi')^2$  in figure 4.

Note also that the limit  $l \rightarrow 0$  of the deformed topological soliton, which should give back the undeformed soliton, means  $A \rightarrow 0$ , in which case we can simplify (3.24) using the fact that  $\tanh^{-1}(\cos 2x) = y$  implies  $x = \arctan e^{\pm y}$ . Then, indeed, the implicit solution on the first line of (3.24) becomes  $\phi = \frac{4}{\beta} \arctan[e^{\pm\mu\sqrt{\beta}(x-x_0)}]$ , which is just the undeformed sine-Gordon soliton.

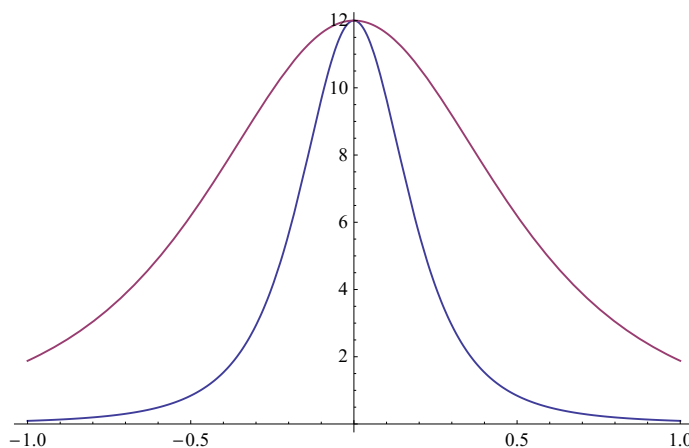
We can also determine the mass associated with this solution and check in particular if it is finite, and hence indeed a soliton. We find

$$M_{solsG} = \int d\phi \frac{\sqrt{2V}}{\sqrt{(1+2V)}} = \frac{4}{l\beta} \arctan \left( \frac{2l\mu}{\sqrt{\beta}} \right), \quad (3.25)$$

where we used that  $\frac{\beta\phi}{2}$  changes in the interval  $[0, \pi]$  (note that the variation of  $\frac{\beta\phi}{2}$  in figure 3 is from  $0$  to  $\pi/2$  to  $0$ , but as we said, the topological soliton corresponds to  $\pi - (3.24)$  on the positive real axis). This is indeed a finite result, and in the limit  $l\mu \rightarrow 0$ , it goes back to the mass of the ordinary sine-Gordon soliton.



**Figure 3.** The soliton (3.24) of the Heisenberg model with a sine-Gordon potential. The colours correspond to:  $A=0.1$  for blue and  $A=10$  for purple.  $\frac{\beta\phi}{2}$  on the vertical varies from 0 at infinity to  $\pi/2$  at the origin, corresponding to  $x = x_0$ .



**Figure 4.**  $(\phi')^2$  for (3.24) for the Heisenberg model with a sine-Gordon potential. The colors correspond to:  $A=0.1$  for blue and  $A=5$  for purple.

### 3.2.1 Pure DBI action

As another comparison, for the pure DBI action, at  $V(\phi) = 0$ , we obtain

$$x - x_0 = \int_{\phi(x_0)}^{\phi(x)} \frac{2d\phi}{\sqrt{\frac{1}{C^2} - 1}} = K(\phi(x) - \phi(x_0)), \tag{3.26}$$

where  $K$  is another constant, related to  $C$ . Note that we can glue together two such solutions to obtain the solution to the Poisson equation in one dimension,

$$\phi(x) = \phi(x_0) + K^{-1}|x - x_0|. \tag{3.27}$$

Note that if we boost this solution, we obtain

$$\phi(x) = \phi(x'_0) + K^{-1}\gamma|x - x'_0 - vt|, \tag{3.28}$$

or, in the ultrarelativistic limit  $v \rightarrow 1$ , with  $\phi(x'_0) = 0$  (since it becomes anyway negligible with respect to the second term in the  $\gamma \rightarrow \infty$  limit) and  $K^{-1}\gamma \equiv \tilde{K}$ ,

$$\phi(x) = \tilde{K}|x^-|. \quad (3.29)$$

This is clearly not an soliton since it does not have finite mass.

### 3.3 Soliton solutions of the $T\bar{T}$ deformed scalar field

We are finally in a position to write the same for the  $T\bar{T}$  deformed Lagrangian.

For a static solution, the equation of motion of the Lagrangian (2.6) is

$$\begin{aligned} & \frac{d}{dx} \left( \frac{\phi'}{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} \right) \\ &= -\frac{\lambda}{2(1-\lambda V)} \frac{\frac{dV}{d\phi}\phi'^2}{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} + \frac{\lambda \frac{dV}{d\phi}}{2(1-\lambda V)^2} \sqrt{1+2\lambda(1-\lambda V)\phi'^2} + \frac{d\tilde{V}}{d\phi}. \end{aligned} \quad (3.30)$$

As before, we multiply the equation by  $\phi'$ , and find on the right-hand side  $(dV/d\phi)(d\phi/dx) = dV/dx$  and  $(d\tilde{V}/d\phi)(d\phi/dx) = d\tilde{V}/dx$ . We then note the identities

$$\begin{aligned} & \frac{d}{dx} \left( \frac{\phi'^2/2}{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} \right) = \phi' \frac{d}{dx} \left( \frac{\phi'}{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} \right) - \frac{\phi'^2}{2} \frac{d}{dx} \left( \frac{1}{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} \right) \\ & \frac{d}{dx} \left( -\frac{\lambda V}{2(1-\lambda V)} \frac{\phi'^2}{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} + \frac{\lambda V}{2(1-\lambda V)^2} \sqrt{1+2\lambda(1-\lambda V)\phi'^2} + \tilde{V} \right) \\ &= -\frac{\lambda}{2(1-\lambda V)} \frac{\frac{dV}{dx}\phi'^2}{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} + \frac{\lambda \frac{dV}{dx}}{2(1-\lambda V)^2} \sqrt{1+2\lambda(1-\lambda V)\phi'^2} + \frac{d\tilde{V}}{dx} \\ &+ V \frac{d}{dx} \left[ -\frac{\lambda}{2(1-\lambda V)} \frac{\phi'^2}{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} + \frac{\lambda}{2(1-\lambda V)^2} \sqrt{1+2\lambda(1-\lambda V)\phi'^2} \right], \end{aligned} \quad (3.31)$$

and by subtracting them and using the equations of motion above, we get

$$\begin{aligned} & \frac{d}{dx} \left( -\frac{\lambda V}{2(1-\lambda V)} \frac{\phi'^2}{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} + \frac{\lambda V \sqrt{1+2\lambda(1-\lambda V)\phi'^2}}{2(1-\lambda V)^2} + \tilde{V} - \frac{\phi'^2/2}{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} \right) \\ &= \frac{\phi'^2}{2} \frac{d}{dx} \frac{1}{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} + V \frac{d}{dx} \left[ -\frac{\lambda \phi'^2}{2(1-\lambda V)\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} + \frac{\lambda \sqrt{1+2\lambda(1-\lambda V)\phi'^2}}{2(1-\lambda V)^2} \right] \\ &= \frac{d}{dx} \left( \frac{\phi'^2}{2} \frac{1}{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} + V \left[ -\frac{\lambda \phi'^2}{2(1-\lambda V)\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} + \frac{\lambda \sqrt{1+2\lambda(1-\lambda V)\phi'^2}}{2(1-\lambda V)^2} \right] \right) \\ & \quad - \left( \frac{d\phi'^2}{dx} \frac{\partial}{\partial \phi'^2} + \frac{dV}{dx} \frac{\partial}{\partial V} \right) \left( \frac{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}}{2\lambda(1-\lambda V)} \right) \\ &= \frac{d}{dx} \left( \frac{\phi'^2}{2} \frac{1}{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} + V \left[ -\frac{\lambda \phi'^2}{2(1-\lambda V)\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} + \frac{\lambda \sqrt{1+2\lambda(1-\lambda V)\phi'^2}}{2(1-\lambda V)^2} \right] \right) \\ & \quad - \frac{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}}{2\lambda(1-\lambda V)}, \end{aligned} \quad (3.32)$$

so that finally we have the conservation equation

$$\frac{d}{dx} \left[ \frac{1}{2\lambda(1-\lambda V)\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} + \tilde{V} \right] = 0. \quad (3.33)$$

It is solved by

$$\frac{1}{2\lambda(1-\lambda V)\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} + \tilde{V} = C, \quad (3.34)$$

for a general  $C$ .

The implicit general solution for  $\phi(x)$  is then

$$x - x_0 = \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi \sqrt{2\lambda(1-\lambda V)}}{\sqrt{\frac{1}{[2\lambda(1-\lambda V)(C-\tilde{V})]^2} - 1}}. \quad (3.35)$$

Consider the case  $C = 0$ , like in the case of the canonical scalar with potential. Then then solution becomes much simpler:

$$x - x_0 = \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi |1 - 2\lambda V|}{\sqrt{2V}}. \quad (3.36)$$

We can now determine the mass of the soliton of the deformed  $T\bar{T}$  action. We first calculate the Hamiltonian density on the static solution (soliton),

$$\mathcal{H} = T_{00} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} = -\mathcal{L}. \quad (3.37)$$

Using the condition for a soliton solution of the equation of motion (3.36) inside  $\mathcal{H} = -\mathcal{L}(\dot{\phi} = 0)$ , we find that the *on-shell* Hamiltonian is

$$\phi' = \frac{\sqrt{2V}}{|1 - 2\lambda V|} \Rightarrow \mathcal{H} = \frac{2V}{1 - 2\lambda V}, \quad (3.38)$$

where we have substituted the expression for  $\phi$  on the solution in the Lagrangian.

Recall now the derivation of the soliton mass of the undeformed theory

$$\begin{aligned} M_s &= \int_{-\infty}^{+\infty} dx T_{00} = \int_{-\infty}^{+\infty} dx \left[ \frac{1}{2}(\phi')^2 + V \right] = \int_{-\infty}^{+\infty} dx 2V \\ &= \int_{-\infty}^{+\infty} dx \phi' \sqrt{2V} = \int_0^\pi d\phi \sqrt{2V}, \end{aligned} \quad (3.39)$$

For the deformed solution we use the above on-shell Hamiltonian, and from (3.36) we replace  $\sqrt{2V}/(1 - 2\lambda V)$  with  $\phi'$ , so the mass is given by

$$M_{s,T\bar{T}} = \int_{-\infty}^{+\infty} dx \mathcal{H} = \int_{-\infty}^{+\infty} dx \frac{2V}{1 - 2\lambda V} = \int_0^\pi d\phi \sqrt{2V} = M_{s,undef.}, \quad (3.40)$$

so we find a general statement that:

*there is no correction to the soliton mass at all!*

This is, no doubt, due to the special nature of the  $T\bar{T}$  deformation. Note also that we have been able to calculate the mass of the deformed soliton, even though we do not have an explicit analytic expression for  $\phi(x)$ , only an implicit one, because we have traded the  $x$  integral for the  $\phi$  integral, and we know the values of  $\phi$  at  $x = \pm\infty$ , where the soliton is unmodified.

The reason that we know the solution for  $\phi$  near  $x = \pm\infty$ , namely the un-deformed solution, is the following. Consider the implicit deformed solution (3.36) near  $x = \pm\infty$ . Assume that the undeformed solution  $\phi$  at  $x = \pm\infty$  is finite, which is true for most cases of solitons. Moreover, since the mass of these solitons must be finite, then  $V(\phi)$  must be finite (actually, must be going to zero) near  $\phi(x = \pm\infty)$ . Then the deformation term in (3.36),  $-\lambda \int d\phi \sqrt{2V}$ , is also finite, so can be ignored with respect to the first term, which is infinite (because the left-hand side of (3.36) is infinite). It follows then that the soliton solution near  $x = \pm\infty$  is the undeformed soliton one.

### 3.3.1 Deformation of sine-Gordon

For the  $T\bar{T}$  deformation of the sine-Gordon potential (3.1), we obtain the following solution

$$\begin{aligned} \frac{\mu}{\sqrt{\beta}}(x - x_0) &= \int_{\phi(x_0)}^{\phi(x)} d\phi \frac{|1 - 4\frac{\lambda\mu^2}{\beta} \sin^2 \beta\phi/2|}{2|\sin \beta\phi/2|} \\ &= \pm \frac{1}{\beta} \left[ \ln \left( \tan \frac{\beta\phi}{4} \right) + 4\frac{\lambda\mu^2}{\beta} \cos \frac{\beta\phi}{2} \right] \Bigg|_{\phi(x_0)}^{\phi(x)}, \end{aligned} \quad (3.41)$$

or

$$\pm \mu\sqrt{\beta}(x - x_0) = \left[ \ln \left( \tan \frac{\beta\phi}{4} \right) + 4\frac{\lambda\mu^2}{\beta} \cos \frac{\beta\phi}{2} \right] \Bigg|_{\phi(x_0)}^{\phi(x)}. \quad (3.42)$$

We note that near  $\phi = 0$  or  $\phi = +\infty$ , corresponding to  $\pm x = -\infty$  or  $+\infty$ , respectively, the solution is the sine-Gordon soliton, as expected from the general analysis above. It is only near  $x = x_0$  that the soliton is changed. So we can think of it as a modification of the sine-Gordon soliton. Moreover, the limit  $\lambda \rightarrow 0$  exactly gives the sine-Gordon soliton, as it should, since in this limit the action goes to the sine-Gordon soliton action as well.

Following the result in the general case, the mass of the soliton of the deformed sine-Gordon action is unmodified, namely

$$M_{s,T\bar{T}sg} = \frac{8\mu}{\beta^{3/2}} \quad (3.43)$$

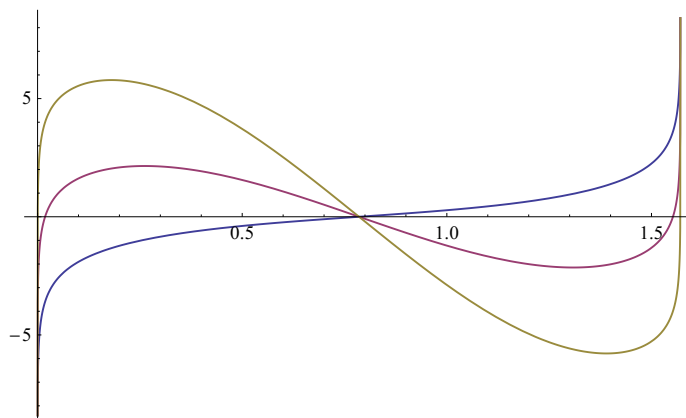
### 3.3.2 Higgs-type potential

We can also consider the Higgs-like potential that gives (in the canonical scalar case) the kink solution,

$$V(\phi) = \alpha(\phi^2 - a^2)^2. \quad (3.44)$$

The usual kink solution is

$$\phi = a \tanh \left( \pm a\sqrt{2\alpha}(x - x_0) \right), \quad (3.45)$$



**Figure 5.** The profile of the modified Sine-Gordon soliton.  $\mu\sqrt{\beta}(x - x_0)$  as a function of  $\beta\phi/4$  for  $\frac{\lambda\mu^2}{\beta}=0.1, 1, 2$  in blue, purple and brown, respectively.

or

$$x - x_0 = \pm \frac{1}{a\sqrt{2\alpha}} \tanh^{-1} \frac{\phi}{a}. \tag{3.46}$$

In our case, we obtain

$$x - x_0 = \int \frac{d\phi |1 - 4\lambda\alpha(\phi^2 - a^2)^2|}{\sqrt{2\alpha} |a^2 - \phi^2|}, \tag{3.47}$$

implying

$$x - x_0 = \pm \frac{1}{a\sqrt{2\alpha}} \left[ \tanh^{-1} \frac{\phi}{a} - 4\lambda\alpha a^3 \left| \frac{\phi^3}{3a^3} - \frac{\phi}{a} \right| \right], \tag{3.48}$$

which again can be thought of as a modification of the kink solution, once we realize that we need to restrict to  $\phi \in [-a, +a]$ , since  $\phi(\pm a) = \pm\infty$ . Then near  $x = \pm\infty$ , the solution is unmodified, as expected from the general analysis, while it is modified near  $x = x_0$ , though this time we still have  $\phi(x_0) = 0$  for the modified kink. The profile of the soliton is shown in figure 6.

If we can neglect the potential  $V(\phi)$  altogether, for instance if we consider a small  $x - x_0$ , near a point where  $\phi(x_0)$  is also small (so that  $V(\phi)$  is also small), then we find again (as in the previous subsections) a linear solution  $x - x_0 = K(\phi(x) - \phi(x_0))$ , and by gluing two of those, we can again find a solution to the Poisson equation in one dimension, the same

$$\phi(x) = \phi(x_0) + K^{-1}|x - x_0| \tag{3.49}$$

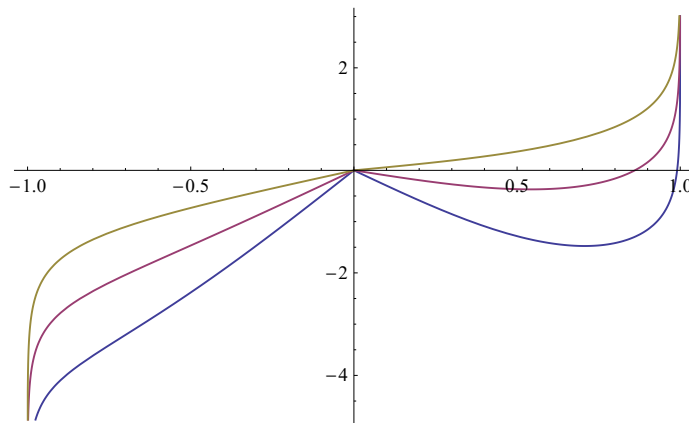
as before, which after an infinite boost goes over to the same

$$\phi(x) = \tilde{K}|x^-|. \tag{3.50}$$

### 3.4 Breathers of the $T\bar{T}$ system

The construction of breather solutions to the sine-Gordon model (3.7) was based on performing a transformation of the form

$$\left[ e^{\pm\mu\sqrt{\beta}(x-x_0)} \right] \rightarrow \left[ \frac{\eta \sin(wt)}{\cosh(\eta wx)} \right]. \tag{3.51}$$



**Figure 6.** The profile of the modified Higgs-like soliton.  $a\sqrt{2\alpha}(x - x_0)$  as a function of  $\phi/a$  for  $a = 1$  and  $\lambda\alpha=0.1, 0.5, 1$  in brown, purple and blue, respectively.

This naturally leads us to conjecture that the breather solution for the  $T\bar{T}$  deformed sine-Gordon system takes the form

$$\ln \left[ \frac{\eta \sin(\eta wt)}{\cosh(\eta wx)} \right] = \left[ \ln \left( \tan \frac{\beta\phi}{4} \right) + 4 \frac{\lambda\mu^2}{\beta} \cos \frac{\beta\phi}{2} \right] \Bigg|_{\phi(x_0)}^{\phi(x)}. \quad (3.52)$$

For a scalar field that depends on both  $x$  and  $t$ , the equation of motion reads (remember that  $X = (\partial_\mu\phi)^2 = -\dot{\phi}^2 + \phi'^2$ )

$$\begin{aligned} & -\frac{\lambda}{2(1-\lambda V)} \frac{\frac{dV}{d\phi} X}{\sqrt{1+2\lambda(1-\lambda V)X}} + \frac{\frac{dV}{d\phi}}{2(1-\lambda V)^2} \sqrt{1+2\lambda(1-\lambda V)X} + \frac{d\tilde{V}}{d\phi} \\ & = \partial_\mu \left( \frac{\partial^\mu \phi}{\sqrt{1+2\lambda(1-\lambda V)X}} \right). \end{aligned} \quad (3.53)$$

Normally, we should check our conjectured breather solution (3.52) on the above equation of motion, but that is increasingly complicated. Instead, we note that when we differentiate (3.52) with respect to either  $t$  or  $x$ , we obtain on the right-hand side

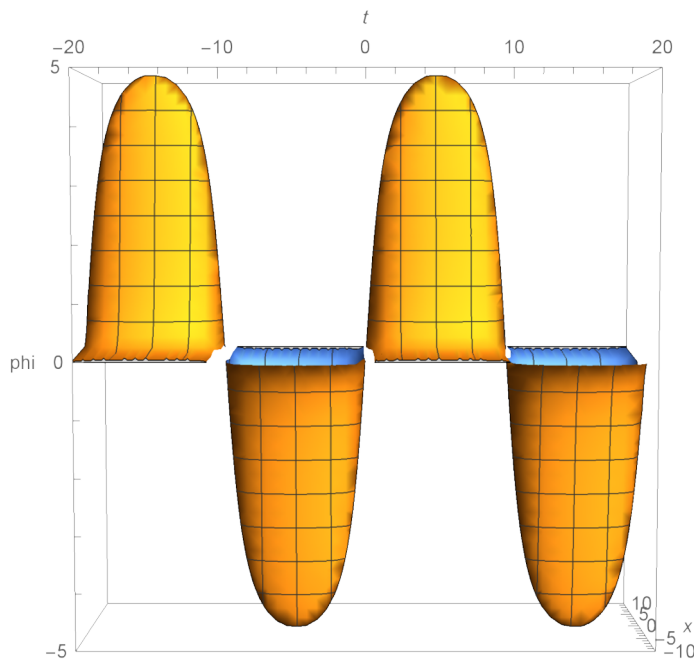
$$\frac{\beta}{2 \sin \frac{\beta\phi}{2}} (1 - 2\lambda V(\phi)) (\dot{\phi} \text{ or } \phi'). \quad (3.54)$$

Then the soliton solution satisfies

$$\phi' = \frac{1}{1-2\lambda V(\phi)} \frac{d}{dx} \frac{4}{\beta} \arctan \left[ e^{\pm\mu\sqrt{\beta}(x-x_0)} \right], \quad (3.55)$$

besides satisfying the sine-Gordon equation,  $\phi'' - \ddot{\phi} = \mu^2 \sin(\beta\phi)$ . The conjectured breather solution would satisfy

$$\begin{aligned} \phi' &= \frac{1}{1-2\lambda V(\phi)} \frac{d}{dx} \frac{4}{\beta} \arctan \left[ \frac{\eta \sin(\eta wt)}{\cosh(\eta wx)} \right] \\ \dot{\phi} &= \frac{1}{1-2\lambda V(\phi)} \frac{d}{dt} \frac{4}{\beta} \arctan \left[ \frac{\eta \sin(\eta wt)}{\cosh(\eta wx)} \right], \end{aligned} \quad (3.56)$$



**Figure 7.** The undeformed breather solution for  $\mu = 1$ ,  $\beta = 1$ ,  $\tan \psi = 2\pi/5$ .

which is true for the usual breather (at  $\lambda = 0$ ). Thus in effect we have  $\dot{\phi} \rightarrow \dot{\phi}/(1 - 2\lambda V)$  and  $\phi' \rightarrow \phi'/(1 - 2\lambda V)$ , resulting in  $X = -\dot{\phi}^2 + \phi'^2 \rightarrow X/(1 - 2\lambda V)^2$ , both for the soliton and for the breather, which is why our conjectured solution is most likely correct.

Similar to the way that we have shown that the breather is indeed a boundstate of a soliton anti-soliton (3.8) we can check now the deformed breather for  $w \ll \mu\sqrt{\beta}$  and in the limit  $x \rightarrow -\infty$ , which takes the form

$$\pm \mu\sqrt{\beta}(x - x_0) \log \left( \frac{\mu\sqrt{\beta}}{w} \sin(wt) \right) = \left[ \ln \left( \tan \frac{\beta\phi}{4} \right) + 4 \frac{\lambda\mu^2}{\beta} \cos \frac{\beta\phi}{2} \right] \Big|_{\phi(x_0)}^{\phi(x)}. \quad (3.57)$$

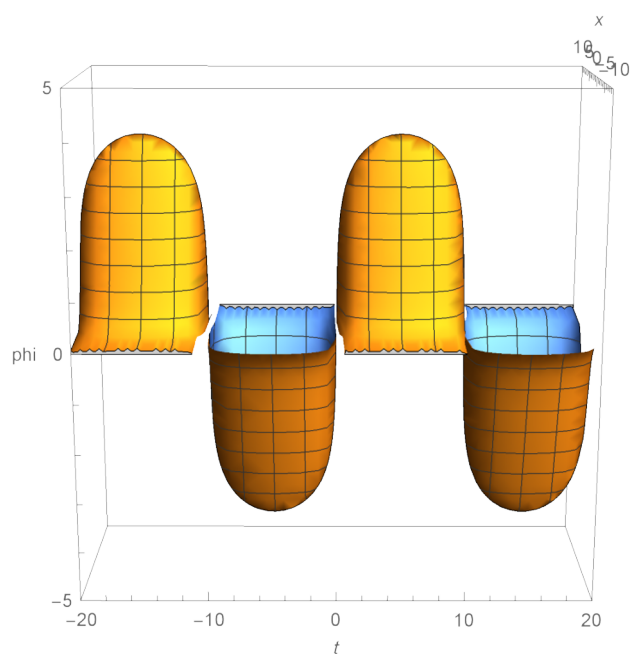
which indeed looks like the deformed soliton to the left. In figure 7 we re-draw the breather of the undeformed theory and then in figure 8 and figure 9 the breathers of the same parameters with  $\lambda = -1/2$  and  $\lambda = 1/10$ , respectively.

Furthermore, we can use the same logic as in the undeformed case to calculate the mass of the deformed breather. Namely, we first calculate the Hamiltonian,

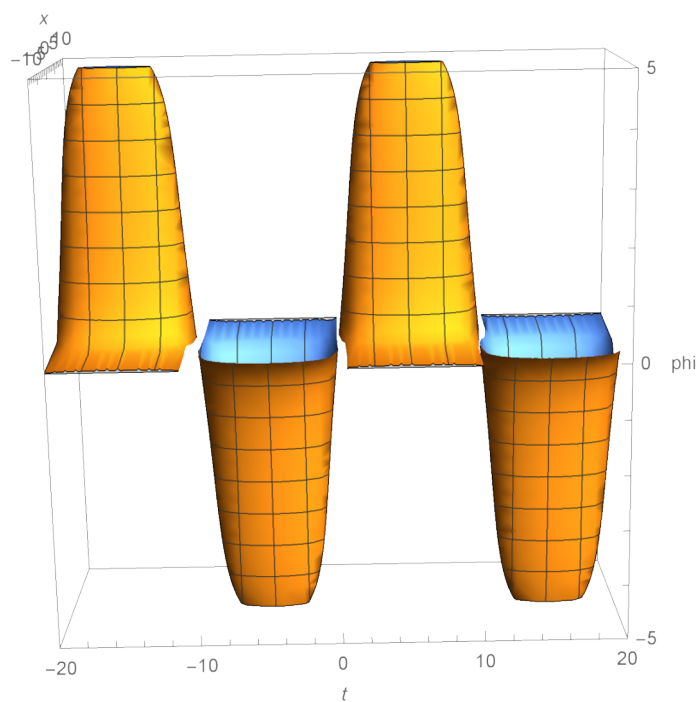
$$\begin{aligned} \mathcal{H} &= \dot{\phi} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} \\ &= \frac{(\dot{\phi})^2}{\sqrt{1 + 2\bar{\lambda}(-\dot{\phi}^2 + \phi'^2)}} + \tilde{V} + \frac{1 + 2\bar{\lambda}(-\dot{\phi}^2 + \phi'^2)}{2\bar{\lambda}}, \end{aligned} \quad (3.58)$$

and then we calculate the mass at  $t = 0$ , when we observe that  $\phi'(t = 0) = 0$  and  $\phi(t = 0) = 0$  using (3.56) and (3.52), so also  $V(\phi(t = 0)) = 0$ , like for the undeformed





**Figure 8.** The deformed breather solution for  $\mu = 1$ ,  $\beta = 1$ ,  $\tan \psi = 2\pi/5$ ,  $\lambda = -1/2$ .



**Figure 9.** The breather solution for  $\mu = 1$ ,  $\beta = 1$ ,  $\tan \psi = 2\pi/5$ ,  $\lambda = 1/10$ .

breather. This implies  $\bar{\lambda} = \lambda$  and  $\tilde{V} = -1/(2\lambda)$ . Then we find

$$\begin{aligned}
 M_{bre,T\bar{T}} &= \int_{-\infty}^{+\infty} dx \mathcal{H}(t=0) = \frac{1}{2\lambda} \int_{-\infty}^{+\infty} dx \left( \frac{1}{\sqrt{1-2\lambda\dot{\phi}^2}} - 1 \right) \\
 &= \int_{-\infty}^{+\infty} dx \sum_{n=0}^{\infty} \frac{(-1/2)(-1/2-1)\dots(-1/2-n)}{1 \cdot 2 \cdot \dots \cdot (n+1)} (2\lambda)^n \dot{\phi}^{2(n+1)} \\
 &\simeq \int_{-\infty}^{+\infty} dx \left[ \frac{\dot{\phi}^2}{2} + \frac{3}{8} 2\lambda \dot{\phi}^4 + \dots \right].
 \end{aligned} \tag{3.59}$$

Since, as in the undeformed case,

$$\dot{\phi}^2(t=0) = \left( \frac{4\eta\omega}{\beta} \right)^2 \frac{1}{\cosh^2(\eta\omega x)}, \tag{3.60}$$

doing the integrals we find

$$M_{bre,T\bar{T}} = M_{bre} \left[ 1 + \sum_{n=1}^{\infty} \frac{(-1/2)(-1/2-1)\dots(-1/2-n)}{1 \cdot 2 \cdot \dots \cdot (n+1)} B^n \sum_{k=0}^n \frac{n!}{(2k+1)k!(n-k)!} \right], \tag{3.61}$$

where

$$B \equiv 2\lambda \left( \frac{4\eta\omega}{\beta} \right)^2 = 32\lambda \frac{\mu^2\beta - \omega^2}{\beta^2}. \tag{3.62}$$

We see that the mass of the conjectured breather solution increases with a small  $\lambda$  from the mass of the undeformed solution.

## 4 Shockwave solutions

In this section, we consider the perturbative solutions of the shockwave type, more precisely shockwaves depending on  $s \equiv t^2 - x^2 = x^+x^-$  ( $x^\pm = t \pm x$ ), such that  $\phi = 0$  for  $s < 0$  (outside the light-cone coming out of  $(x, t) = (0, 0)$ ), and nontrivial only inside the light-cone, i.e., for  $s \geq 0$ . This is the case considered by Heisenberg for the action (1.1).

### 4.1 Shockwaves of the Heisenberg model

The first case is of the Heisenberg model, with a general potential  $V$ , as in (2.9). This has been considered in [4], and we just review it here. On the ansatz  $\phi = \phi(s)$ , we have

$$X = (\partial_\mu \phi)^2 = -4s \left( \frac{d\phi}{ds} \right)^2 \lambda, \tag{4.1}$$

so the Lagrangian on the ansatz is

$$l^2 \mathcal{L} = 1 - \sqrt{1 - l^4 4s \left( \frac{d\phi}{ds} \right)^2 + l^4 2V}. \tag{4.2}$$

We will see that on the solution near  $s = 0$ , which is the only one we will consider, the potential is negligible, so we will drop it for the moment. Afterwards we will see that this was self-consistent, since it is irrelevant for the solution.

The equation of motion of the Lagrangian on the ansatz is

$$\begin{aligned}
 0 &= 2l^4 \frac{d}{ds} \left[ \frac{1}{\sqrt{1 - 4l^4 s \left(\frac{d\phi}{ds}\right)^2}} s \left(\frac{d\phi}{ds}\right)^2 \right] \\
 &= \frac{2l^4 \left(1 - 2l^4 s \left(\frac{d\phi}{ds}\right)^2\right)}{\left(1 - 4l^4 s \left(\frac{d\phi}{ds}\right)^2\right)^{3/2}} \frac{d}{ds} \left[ s \left(\frac{d\phi}{ds}\right)^2 \right].
 \end{aligned}
 \tag{4.3}$$

We see that the equation of motion reduces, near  $s = 0$ , to

$$\frac{d}{ds} \left[ s \left(\frac{d\phi}{ds}\right)^2 \right] = 0,
 \tag{4.4}$$

solved by

$$\phi = l^{-2} \sqrt{s} + \mathcal{O}(s^{3/2}).
 \tag{4.5}$$

Then, *a posteriori*, we can check that indeed, if  $V(\phi = 0) = 0$ ,  $V(\phi(s)) \simeq 0$  near  $s = 0$ , so it can be neglected in the Heisenberg Lagrangian on the ansatz, and the solution above is still valid for  $V \neq 0$ .

Moreover, we note that  $X = (\partial_\mu \phi)^2 = -4s \left(\frac{d\phi}{ds}\right)^2$  jumps from 0 at  $s = 0^-$  to  $-l^{-4}$  at  $s = 0^+$ , even though  $\phi(s)$  is continuous.

In [4], several further generalizations of this model have been considered, for instance with a function of  $\phi$  in front of the  $(\partial_\mu \phi)^2$  inside the square root. There, however, we have seen that we cannot truncate the square root to any finite order; if we do so, we have no solution (4.5) anymore. This, together with the fact that a canonical scalar with a potential also results in no solution (4.5) was described as a certain “uniqueness” of the Heisenberg Lagrangian.

We will show in appendix A that, in fact, we can also exchange the square root for a fractional power smaller than 1, as well as consider  $[(\partial_\mu \phi)^2]^{2k+1}$  inside the square root, and an overall power  $1/(2k + 1)$ , and the nontrivial shockwave (4.5) is still a solution. However, only the first case is physically interesting, since in the second we obtain a complex on-shell Lagrangian.

Next, however, we consider the case of our  $T\bar{T}$  deformation.

## 4.2 Shockwaves of the $T\bar{T}$ deformed model

Consider then, like in the case of [4], that the scalar field  $\phi$  is only a function of  $s = t^2 - x^2 = (t + x)(t - x)$ ,  $\phi = \phi(s)$ , and also  $\phi = 0$  for  $s \leq 0$ , which means a propagating shockwave solution (instead of the general behaviour of both  $x + t$  and  $x - t$ , now we have only the dependence on their product). Then, as in the previous subsection,

$$4\partial\phi\bar{\partial}\phi = -4s \left(\frac{d\phi}{ds}\right)^2,
 \tag{4.6}$$

and for simplicity we use the definition from (2.3),

$$\tilde{V} \equiv \frac{2\lambda V - 1}{2\lambda(1 - \lambda V)}. \tag{4.7}$$

Then, the Lagrangian on the ansatz is

$$-\mathcal{L} = \tilde{V}(\phi(s)) + \frac{\sqrt{1 + 8\lambda(1 - \lambda V(\phi(s)))(-4s) \left(\frac{d\phi}{ds}\right)^2}}{2\lambda(1 - \lambda V(\phi(s)))}. \tag{4.8}$$

Its equation of motion is

$$\begin{aligned} & \tilde{V}'(\phi(s)) + \frac{\sqrt{1 + 8\lambda(1 - \lambda V(\phi(s)))(-4s) \left(\frac{d\phi}{ds}\right)^2}}{2\lambda(1 - \lambda V(\phi(s)))^2} \lambda V'(\phi(s)) \\ & + \frac{-1/2}{2\lambda(1 - \lambda V)} \frac{-8\lambda^2 V'(\phi(s))(-4s) \left(\frac{d\phi}{ds}\right)^2}{\sqrt{1 + 8\lambda(1 - \lambda V(\phi(s)))(-4s) \left(\frac{d\phi}{ds}\right)^2}} \\ & + \frac{d}{ds} \left[ \frac{+8\lambda(1 - \lambda V)(-8s) \frac{d\phi}{ds}}{2\lambda(1 - \lambda V) \sqrt{1 + 8\lambda(1 - \lambda V(\phi(s)))(-4s) \left(\frac{d\phi}{ds}\right)^2}} \right] = 0. \end{aligned} \tag{4.9}$$

We want to see if the same solution near  $s = 0$ ,  $\phi \simeq l^{-2}\sqrt{s}$ , is valid here (the coefficient of  $X = (\partial_\mu \phi)^2$  inside the square root is defined to be  $l^4$ ). We note that then,  $s(d\phi/ds)^2 \sim \mathcal{O}(1)$ , whereas, assuming that  $V(\phi)$  has only positive powers of  $\phi$  (and no linear one, since that would be a tadpole in QFT, and is any way not something we want),  $V(\phi)$  and  $V'(\phi)$  go to zero on the solution near  $s = 0$ . Then, it follows (as we can easily check) that the first two lines in the equation of motion above are subleading with respect to the third one. Moreover, in the third line, we can put  $V$  to zero for the leading behaviour, which finally just leaves the equation of motion of the Heisenberg model, so indeed its solution near  $s = 0$ ,  $\phi \simeq l^{-2}\sqrt{s}$  (here  $l^4 = 2\lambda$ ), is also a solution near  $s = 0$  here. We could have made a simpler argument: since, as we saw,  $V(\phi)$  on the solution near  $s = 0$  vanishes, we could put that directly in the  $T\bar{T}$  deformed action (2.3), which directly gives the Heisenberg DBI action (at  $m = 0$ , since near  $s = 0$ , Heisenberg already noted that the solution is independent of  $m$  being or not zero), hence its solution, too.

## 5 Conclusions

In this paper we have found solutions of the  $T\bar{T}$  deformations and the Heisenberg deformation of the canonical scalar with a potential  $V$ .

We have first found that the 1+1 dimensional  $T\bar{T}$  deformation of a canonical scalar has soliton solutions, as well as shockwave solutions, that could be used in the Heisenberg model. We have written explicitly the static soliton solutions of the  $T\bar{T}$  deformation action, in the case of sine-Gordon and Higgs-type potential  $V$ , where we have seen that

the solitons are deformations of the solitons in canonical case. We have also argued for the existence of breather solutions in the sine-Gordon case, also as deformations of the breather solutions in the canonical case. In the generic case of the  $T\bar{T}$  deformation solitons, we found the remarkable property that the mass of the solitons is undeformed, as is the solution near  $\pm\infty$ .

In the case of shockwave solutions, we have shown that the Heisenberg perturbative shockwave solution  $\phi(s) \simeq l^{-2}\sqrt{s}$  near  $s = 0$ , with  $\phi(s < 0) = 0$ , is still valid for the  $T\bar{T}$  case, as well as in other cases. Also a generic solution  $\phi(x) = K|x - x_0|$  can be (infinitely) boosted to a solution  $\phi(x^-) = \tilde{K}|x^-|$ .

The application of these solutions to the understanding of the Heisenberg model will be described in a separate publication [16].

There are several open questions that could be addressed as a continuation of this research work. There include

- The fact that the soliton mass of any deformed theory is the same as that of the undeformed theory deserves further investigation. The question is whether there is some physical reason for that property and whether it has implications about other properties.
- This note includes a conjectured solution for the breather mode. Further work is needed to verify or falsify this conjecture.
- We have seen that there is a shock wave solution for the  $T\bar{T}$  system has the same behavior as the solution of Heisenberg's model close to the origin. It will be interesting to further explore possible differences between the solutions in the other parts of space-time.
- A very natural generalization of this investigation is about solutions of scalar field theories in higher space-time dimensions.

**Note added.** After the paper was first posted on the arXiv, we became aware of the papers [18, 19], which have some overlap with the current paper, as they also discuss solitons, breathers and shockwaves in the context of  $T\bar{T}$  deformations of the sine-Gordon model. According to the anonymous referee's suggestion, we have drawn the figures 2 and 7, 8, 9, in order to facilitate comparison with the breather solution in Fig.4 of [19] (in their case, as in ours, there is no analytical solution possible for  $\phi(x, t)$ , only a numerical one; in our case the implicit analytical solution in (3.52) cannot be inverted).

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## A Shockwave solutions in generalizations of the DBI and Heisenberg Lagrangians, with different powers

In this appendix we find that two possible modifications of the DBI and Heisenberg Lagrangians preserve the perturbative shockwave solution (4.5). As in the main text, since we assume  $V(\phi = 0) = 0$ , and because (4.5) is approximately zero near  $s = 0$ , we neglect the potential, so we treat the modification of the Heisenberg Lagrangian as a modification of just the DBI Lagrangian.

The first modification we analyze is

$$-\mathcal{L}l^4 = \left[ \sqrt{1 + [l^4(\partial_\mu\phi)^2]^{2k+1}} - 1 \right]^{\frac{1}{2k+1}}. \quad (\text{A.1})$$

We note that the overall power was chosen so that, for  $l \rightarrow 0$ , we obtain the canonical Lagrangian,  $\mathcal{L} = -(\partial_\mu\phi)^2/2$ . The power of  $X$  inside the Lagrangian was chosen such that  $(-1)^{2k+1} = -1$ .

As before, for  $\phi = \phi(s)$ ,  $X \equiv (\partial_\mu\phi)^2 = -4s \left(\frac{d\phi}{ds}\right)^2$ , so the Lagrangian on the ansatz is

$$-\mathcal{L} = \left[ \sqrt{1 + \left[ -l^4 4s \left(\frac{d\phi}{ds}\right)^2 \right]^{2k+1}} - 1 \right]^{\frac{1}{2k+1}}. \quad (\text{A.2})$$

The equation of motion is

$$\frac{d}{ds} \left\{ \left[ \sqrt{1 + \left[ -l^4 4s \left(\frac{d\phi}{ds}\right)^2 \right]^{2k+1}} - 1 \right]^{\frac{1}{2k+1}-1} \left\{ \frac{2k+1}{\sqrt{1 + \left[ -l^4 4s \left(\frac{d\phi}{ds}\right)^2 \right]^{2k+1}}} \times \right. \right. \\ \left. \left. \times \left[ -4s \left(\frac{d\phi}{ds}\right)^2 \right]^{2k} \left( -8s \frac{d\phi}{ds} \right) \right\} \right\} = 0. \quad (\text{A.3})$$

Next, we want to check whether the solution (4.5) near  $s = 0$  is still valid. As a first step, we note that, due to the properly chosen power inside the square root, the square root still vanishes for  $\phi(s) = l^{-2}\sqrt{s}$ . That means that, from among the terms in the equation of motion (A.3), the dominant one will be the one where the external  $\frac{d}{ds}$  acts on the square root in the denominator, which means the equation of motion

$$\frac{d}{ds} \left[ s \left(\frac{d\phi}{ds}\right)^2 \right] = 0, \quad (\text{A.4})$$

indeed solved by (4.5).

The only problem with this Lagrangian and its solution is that, on the solution, the Lagrangian is complex, since the square root vanishes, so

$$\mathcal{L}_{\text{on-shell}} = -(-1)^{\frac{1}{2k+1}} = -e^{\frac{\pi i}{2k+1}} \in \mathbb{C}. \quad (\text{A.5})$$

The second modification is more physical,

$$-2\frac{p}{q}l^4\mathcal{L}' = \left[1 + l^4(\partial_\mu\phi)^2\right]^{\frac{p}{q}} - 1, \quad (\text{A.6})$$

where the overall constant in the Lagrangian was chosen such that the first term in the expansion in  $l^4$  is the canonical kinetic term  $-\frac{1}{2}(\partial_\mu\phi)^2$ . In order for the Lagrangian to appear to a negative power in the equation of motion, we choose  $p/q < 1$ , i.e.,  $p < q$ .

On the ansatz  $\phi = \phi(s)$ , the Lagrangian is

$$-2\frac{p}{q}l^4\mathcal{L}' = \left[1 - l^4 4s \left(\frac{d\phi}{ds}\right)^2\right]^{\frac{p}{q}} - 1, \quad (\text{A.7})$$

and the equation of motion is

$$\frac{d}{ds} \left\{ \left[1 - l^4 4s \left(\frac{d\phi}{ds}\right)^2\right]^{\frac{p}{q}-1} \left(-8s \frac{d\phi}{ds}\right) \right\} = 0. \quad (\text{A.8})$$

As for the first Lagrangian, the square root vanishes on the solution (4.5), and since in the equation of motion above it appears to the negative power  $p/q - 1$ , the leading term in the equation on the solution is the one where the overall  $d/ds$  acts on the square root, namely (A.4), which does indeed have (4.5) as a solution.

Thus this Lagrangian still has the same perturbative shockwave solution, and this time the on-shell Lagrangian is actually real (and positive).

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