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(m, n)-string and D1-brane in stringy Newton-Cartan background

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Abstract: This paper is devoted to the analysis of (m, n)-string in stringy Newton-Cartan background. We start with the Hamiltonian constraint for (m, n)-string in general background and perform limiting procedure on metric and NSNS and Ramond-Ramond two form background that leads to stringy Newton-Cartan gravity. We also analyze conditions that these background fields have to obey in order to define consistent world-sheet $(m, n)-$theory. We also discuss D1-brane with dynamical electric field in stringy Newton-Cartan gravity.

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Contents

1 Introduction

Recently new interesting generalization of Newton-Cartan (NC) gravity [1] was proposed in [2]. In standard NC gravity there is one-dimensional foliation direction of space-time corresponding to the absolute time direction that is longitudinal to the world-line of particle. The stringy NC gravity is generelization of this picture in the sense that one dimensional foliation is replaced by a two-dimensional foliation with one time-like and spatial foliation directions that are longitudinal to the world-sheet of the string. Further, stringy NC gravity is related to non-relativistic strings [3, 4] in the same way as general relativity is related to relativistic string theory as was recently discussed in [5].[1] This paper also analyzed T-duality properties of non-relativistic strings in stringy Newton-Cartan background that are very interesting when it was shown that T-duality along the longitudinal direction of the stringy Newton-Cartan geometry describes relativistic string theory on a Lorentzian geometry with a compact lightlike isometry, which is otherwise only defined by a subtle infinite boost limit. This fact was further confirmed in [9] when T-duality properties of non-relativistic string in stringy NC background was analyzed with the help of canonical formulation of this string sigma model found in [10].

Since the proposal suggested in [5] is very interesting we mean that it is natural to study further aspects of stringy NC. In particular, we would like to see whether it is possible to define another extended objects known in string theory, as for example D-branes [11] in this background. It is natural to begin with D1-brane which is two dimensional object with gauge field propagating on its world-sheet. This action couples to Ramond-Ramond (RR) two form and one form together to gravity, NSNS two form and dilation that are background fields of type IIB supegravity. Such an action can be written in manifestly $\mathrm{SL}(2,\mathbb{Z})$ invariant form [12, 13] that reflects $\mathrm{SL}(2,\mathbb{Z})$ duality of type IIB theory. We

[1] For alternative definition of non-relativistic strings with AdS/CFT aplications, see [6, 7]. The Hamiltonian analysis of this model was performed in [8].

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consider Hamiltonian for this object and analyze non-relativistic limit of the background metric as was proposed in [14] and that was used for the definition of non-relativistic string in stringy NC background in [15]. In case of D1-brane as a probe the situation is more interesting since there is an electric flux on the world-sheet of D1-brane. We firstly consider D1-brane with fixed gauge invariance when the momentum conjugate to A_σ is equal to some integer number m that counts the number of fundamental strings in the bound state with D1-brane. We call resulting object as (m, n)-string [12, 13]. If we start with such a probe we should define background RR and NSNS two form fields in such a way that the limiting procedure [14] leads to finite Hamiltonian constraint. It turns out that this procedure is similar to the case of fundamental string [15]. However due to the non-trivial structure of the background fields that define stringy Newton-Cartan geometry we have to check that they solve the background equations of motion of type IIB gravity. More explicitly, we consider type IIB gravity equations of motion and study their solutions for the metric ansatz proposed in [14] together with another fields that appear in (m, n)-string action. Inserting the ansatz [14] into definition of the relativistic Christoffel connection we derive condition when this connection is finite even if the parameter that defines non-relativistic limit goes to infinity. We restrict ourselves to the case of zero torsion condition [16] even if certainly more general situations are possible, see for example [17]. Now since the Christoffel connection is finite we also find that Ricci tensor is finite and hence in order to have well defined non-relativistic limit in the background equations of motion we have to demand that the stress energy tensor for the background fields is finite as well. It turns out that this condition has important consequence on the dilaton and RR zero form which now have to be constant. Then the equation of motion for dilaton RR and zero form implies that field strengths of RR and NSNS two form fields have to vanish. We also discuss more interesting case of the fundamental string when now dilaton does not have to be constant. In this case however we find that requirement that Christoffel symbols for Einstein frame metric is finite implies that spatial projection of derivative of dilaton should be zero.

In the second part of the paper we focus on the problem of the definition of the action for n coincident D1-branes in the stringy NC background when the gauge symmetry on the world-sheet is unfixed. Then in order to have finite and non-trivial theory we have to scale RR two form and zero form field in appropriate way as well. As a result we obtain finite Hamiltonian constraint for D1-brane in stringy NC background. Then following standard procedure we find corresponding Lagrangian density. Finally we also discuss the equations of motion for the background fields and we find that they have to obey the same conditions as in case of the stringy NC gravity.

This paper is organized as follows. In the next section 2 we introduce an action and Hamiltonian for n coincident D1-branes in general background. Then in section 3 we consider gauge fixed theory that corresponds to (m, n)-string and find its non-relativistic action in NC background. In section 4 we analyze equations of motion for background fields. In section 5 we study D1-brane in stringy Newton-Cartan background when the gauge symmetry is not fixed and we find corresponding Lagrangian density. Finally in conclusion 6 we outline our results and suggest possible extension of this work.

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2 Hamiltonian formulation of D1-brane in general background

In this section we review basic facts about D1-brane in general background and its Hamiltonian formulation. As the starting point we consider an action for n coincident D1-branes in general background

$$\begin{aligned} S &= -nT_{D1}\int d\tau d\sigma e^{-\Phi}\sqrt{-\det \mathbf{A}}+ \\ &\quad + nT_{D1}\int d\tau d\sigma((b_{\tau\sigma}+2\pi\alpha'\mathcal{F}_{\tau\sigma})\chi + c_{\tau\sigma})\,, \\ \mathbf{A}_{\alpha\beta} &= G_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu + 2\pi\alpha'\mathcal{F}_{\alpha\beta} + B_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu\,, \\ \mathcal{F}_{\alpha\beta} &= \partial_\alpha A_\beta - \partial_\beta A_\alpha\,, \end{aligned} \tag{2.1}$$

where $x^\mu, \nu = 0, 1, \ldots, 9$ are embedding coordinates of D1-brane in the background that is specified by the metric $G_{\mu\nu}$ and NSNS two form $B_{\mu\nu} = -B_{\nu\mu}$ together with Ramond-Ramond two form $C^{(2)}_{\mu\nu} = -C^{(2)}_{\nu\mu}$. We further consider background with non-trivial dilaton Φ and RR zero form χ. Further, $\sigma^\alpha = (\tau, \sigma)$ are world-sheet coordinates and $b_{\tau\sigma}, c_{\tau\sigma}$ are pull-backs of $B_{\mu\nu}$ and $C_{\mu\nu}$ to the world-volume of D1-brane. Explicitly,

$$b_{\alpha\beta} \equiv B_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu\,, \quad c_{\alpha\beta} = C^{(2)}_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu\,. \tag{2.2}$$

Finally $T_{D1} = \frac{1}{2\pi\alpha'}$ is D1-brane tension and $A_\alpha, \alpha = \tau, \sigma$ is two dimensional gauge field that propagates on the world-sheet of D1-brane.

It is useful to rewrite the action (2.1) into the form

$$\begin{aligned} S &= -nT_{D1}\int d\tau d\sigma e^{-\Phi}\sqrt{-\det g - (2\pi\alpha'\mathcal{F}_{\tau\sigma}+b_{\tau\sigma})^2} \\ &\quad + nT_{D1}\int d\tau d\sigma((b_{\tau\sigma}+2\pi\alpha'\mathcal{F}_{\tau\sigma})\chi + c_{\tau\sigma})\,, \end{aligned} \tag{2.3}$$

where $g_{\alpha\beta} = G_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu, \det g = g_{\tau\tau}g_{\sigma\sigma} - (g_{\tau\sigma})^2$. Now we proceed to the Hamiltonian formulation of the theory defined by the action (2.3). First of all we derive conjugate momenta to x^μ and A_α from (2.3)

$$\begin{aligned} p_\mu &= \frac{\delta L}{\delta\partial_\tau x^\mu} = nT_{D1}\frac{e^{-\Phi}}{\sqrt{-\det g - (2\pi\alpha' F_{\tau\sigma}+b_{\tau\sigma})^2}}(G_{\mu\nu}\partial_\alpha x^\nu g^{\alpha\tau}\det g + \\ &\quad + (2\pi\alpha' F_{\tau\sigma}+b_{\tau\sigma})B_{\mu\nu}\partial_\sigma x^\nu) + nT_{D1}(\chi B_{\mu\nu}\partial_\sigma x^\nu + C^{(2)}_{\mu\nu}\partial_\sigma x^\nu)\,, \\ \pi^\sigma &= \frac{\delta L}{\delta\partial_\tau A_\sigma} = \frac{ne^{-\Phi}(2\pi\alpha' F_{\tau\sigma}+b_{\tau\sigma})}{\sqrt{-\det g - (2\pi\alpha' F_{\tau\sigma}+b_{\tau\sigma})^2}} + n\chi\,, \quad \pi^\tau = \frac{\delta L}{\delta\partial_\tau A_\tau} \approx 0 \end{aligned} \tag{2.4}$$

and hence

$$\begin{aligned} \Pi_\mu &\equiv p_\mu - T_{D1}\pi^\sigma B_{\mu\nu}\partial_\sigma x^\nu - nT_{D1}C^{(2)}_{\mu\nu}\partial_\sigma x^\nu = \\ &= nT_{D1}\frac{e^{-\Phi}}{\sqrt{-\det g - (2\pi\alpha' F_{\tau\sigma}+b_{\tau\sigma})^2}}G_{\mu\nu}\partial_\alpha x^\nu g^{\alpha\tau}\det g\,. \end{aligned} \tag{2.5}$$

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Using these relations it is easy to see that the bare Hamiltonian is equal to

$$H_B = \int d\sigma (p_\mu \partial_\tau x^\mu + \pi^\sigma \partial_\tau A_\sigma - \mathcal{L}) = \int d\sigma \pi^\sigma \partial_\sigma A_\tau \tag{2.6}$$

while we have three primary constraints

$$\begin{aligned} &\pi^\tau \approx 0\,, \quad \mathcal{H}_\sigma \equiv p_\mu \partial_\sigma x^\mu \approx 0\,, \\ &\mathcal{H}_\tau \equiv \Pi_\mu G^{\mu\nu} \Pi_\nu + T_{D1}^2 \left(n^2 e^{-2\Phi} + (\pi^\sigma - n\chi)^2 \right) g_{\sigma\sigma} \approx 0\,. \end{aligned} \tag{2.7}$$

Including these primary constraints to the definition of the Hamiltonian we obtain an extended Hamiltonian in the form

$$H = \int d\sigma (N^\tau \mathcal{H}_\tau + N^\sigma \mathcal{H}_\sigma - A_\tau \partial_\sigma \pi^\sigma + v_\tau \pi^\tau)\,, \tag{2.8}$$

where N^τ, N^σ, v_τ are Lagrange multipliers corresponding to the primary constraints $\mathcal{H}_\tau \approx 0$, $\mathcal{H}_\sigma \approx 0$, $\pi^\tau \approx 0$. Now we have to check the stability of all constraints. The requirement of the preservation of the primary constraint $\pi^\tau \approx 0$ implies the secondary constraint

$$\mathcal{G} = \partial_\sigma \pi^\sigma \approx 0\,. \tag{2.9}$$

Now we are ready to proceed to the analysis of (m, n)-string in stringy NC background.

3 (m, n)-string in stringy NC background

(m, n)-string is defined as D1-brane where the gauge symmetry is fixed so that the electric flux π^σ is constant and counts the number of fundamental strings. Explicitly, we fix the gauge generated by $\mathcal{G}$ with the gauge fixing function $A_\sigma = \text{const}$. Then the fixing of the gauge implies that $\pi^\sigma = f(\tau)$ but the equation of motion for π^σ implies that $\partial_\tau \pi^\sigma = 0$ and hence $\pi^\sigma = m$, where m is integer that counts the number of fundamental strings bound to n D1-branes.

In order to define stringy NC background we follow [14] and introduce 10− dimensional vierbein $E_\mu{}^A$ so that the metric components have the form

$$G_{\mu\nu} = E_\mu{}^A E_\nu{}^B \eta_{AB}\,, \eta_{AB} = \text{diag}(-1, \ldots, 1)\,. \tag{3.1}$$

Note that the metric inverse $G^{\mu\nu}$ is defined with the help of the inverse vierbein $E^\mu{}_B$ that obeys the relation

$$E_\mu{}^A E^\mu{}_B = \delta^A_B\,, \quad E_\mu{}^A E^\nu{}_A = \delta^\mu_\nu\,. \tag{3.2}$$

As the next step we split target-space indices A into $A = (a', a)$ where now $a' = 0, 1$ and $a = 2, \ldots, d-1$. Then we introduce longitudinal vielbein $\tau_\mu{}^a$ so that we write

$$\tau_{\mu\nu} = \tau_\mu{}^a \tau_\nu{}^b \eta_{ab}\,, \quad a, b = 0, 1\,. \tag{3.3}$$

In the same way we introduce vierbein $e_\mu{}^{a'}, a = 2, \ldots, d-1$ and also introduce gauge field $m_\mu{}^a$. The $\tau_\mu{}^a$ can be interpreted as the gauge fields of the longitudinal translations while

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$e_\mu{}^{a'}$ as the gauge fields of the transverse translations [2]. Then we can also introduce their inverses with respect to their longitudinal and transverse translations

$$e_\mu{}^{a'}e^\mu{}_{b'} = \delta^{a'}_{b'}\,, \quad e_\mu{}^{a'}e^\nu{}_{a'} = \delta^\nu_\mu - \tau_\mu{}^a\tau^\nu{}_a\,, \quad \tau^\mu{}_a\tau_\mu{}^b = \delta^b_a\,, \quad \tau^\mu{}_a e_\mu{}^{a'} = 0\,, \quad \tau_\mu{}^a e^\mu{}_{a'} = 0\,. \tag{3.4}$$

In [14] specific form of the relativistic vielbein was proposed that in the limit $\omega \to \infty$ gives Newton-Cartan gravity. It is important to stress that this form of the vielbein was determined by contraction of Poincare algebra with abelian extension that gives Bergmann algebra. In case of the stringy NC gravity such a contraction is not known however we mean that it is natural to generalize construction introduced in [14] when we introduce longitudinal indices. Explicitly, we propose following parameterization of relativistic vierbein [14]

$$E_\mu{}^a = \omega\tau_\mu{}^a + \frac{1}{2\omega}m_\mu{}^a\,, \quad E_\mu{}^{a'} = e_\mu{}^{a'}\,, \tag{3.5}$$

where ω is free parameter that we take to infinity when we define non-relativistic limit. It is important to stress that justification of this approach can be also found in the construction of non-relativistic string in stringy NC background [15] when it was shown in [10] that it agrees with non-relativistic sigma model [5]. Note that the inverse vierbein to (3.5) has the form (up to terms of order ω^{-3})

$$E^\mu_a = \frac{1}{\omega}\tau^\mu_a - \frac{1}{2\omega^3}\tau^\mu_b m_\rho{}^b\tau^\rho_a\,, \quad E^\mu_{a'} = e^\mu{}_{a'} - \frac{1}{2\omega^2}\tau^\mu_a m_\rho{}^a e^\rho{}_{a'}\,. \tag{3.6}$$

Then with the help of (3.5) and (3.6) we obtain following form of the metric

$$\begin{aligned}
G_{\mu\nu} &= \omega^2\tau_{\mu\nu} + h_{\mu\nu} + \frac{1}{2}\tau_\mu{}^a m_\nu{}^b\eta_{ab} + \frac{1}{2}m_\mu{}^a\tau_\nu{}^b\eta_{ab} + \frac{1}{4\omega^2}m_\mu{}^a m_\nu{}^b\eta_{ab}\,,\\
G^{\mu\nu} &= \frac{1}{\omega^2}\tau^{\mu\nu} + h^{\mu\nu} - \frac{1}{2\omega^2}(\tau^\nu_b m_\rho{}^b h^{\rho\mu} + \tau^\mu_b m_\rho{}^b h^{\rho\nu}) -\\
&\quad - \frac{1}{2\omega^4}(\tau^\mu_c m^c{}_\rho\tau^{\rho\nu} + \tau^\nu_d m^d{}_\rho\tau^{\rho\mu}) + \frac{1}{4\omega^4}\tau^\mu_a m_\rho{}^a h^{\rho\sigma}\tau^\nu_b m_\sigma{}^b\,.
\end{aligned} \tag{3.7}$$

As the next step we have to introduce an appropriate parameterization of RR and NSNS two forms. Following analysis presented [18] we suggest that they have the form[2]

$$\begin{aligned}
B_{\mu\nu} &= \mathbf{X}\left(\omega\tau_\mu{}^a - \frac{1}{2\omega}m_\mu{}^a\right)\left(\omega\tau_\nu{}^b - \frac{1}{2\omega}m_\nu{}^b\right)\epsilon_{ab} + b_{\mu\nu} =\\
&= \mathbf{X}\left(\omega^2\tau_\mu{}^a\tau_\nu{}^b\epsilon_{ab} - \frac{1}{2}(m_\mu{}^a\tau_\nu{}^b + \tau_\mu{}^a m_\nu{}^b)\epsilon_{ab} + \frac{1}{4\omega^2}m_\mu{}^a m_\mu{}^b\epsilon_{ab}\right) + b_{\mu\nu}\,,\\
C_{\mu\nu} &= \mathbf{Y}\left(\omega\tau_\mu{}^a - \frac{1}{2\omega}m_\mu{}^a\right)\left(\omega\tau_\nu{}^b - \frac{1}{2\omega}m_\nu{}^b\right)\epsilon_{ab} + b_{\mu\nu} =\\
&= \mathbf{Y}\left(\omega^2\tau_\mu{}^a\tau_\nu{}^b\epsilon_{ab} - \frac{1}{2}(m_\mu{}^a\tau_\nu{}^b + \tau_\mu{}^a m_\nu{}^b)\epsilon_{ab} + \frac{1}{4\omega^2}m_\mu{}^a m_\mu{}^b\epsilon_{ab}\right) + c_{\mu\nu}\,,
\end{aligned} \tag{3.8}$$

[2]This proposed form of NSNS and RR two forms is again generalization of the parameterization of U(1) gauge field introduced in [14] to the case of NC gravity with two longitudinal directions where we have to introduce antisymmetric symbol ϵ_{ab} in order to have antisymmetric $B_{\mu\nu}$ and $R_{\mu\nu}$. This proposal can be again justified by an agreement between non-relativistic string found in this way [15] and an action introduced in [15].

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where

$$\epsilon_{ab} = -\epsilon_{ba}\,, \quad \epsilon_{01} = 1\,,$$

and where $\mathbf{A}, \mathbf{B}$ are factors that can depend on space-time fields and m, n that we choose in such a way to make Hamiltonian constraint finite in the limit $\omega \to \infty$. In fact it turns out that this requirement implies

$$m\mathbf{X} + n\mathbf{Y} = \sqrt{n^2 e^{-2\Phi} + (m - n\chi)^2}\,. \tag{3.9}$$

Then using (3.8) and also (3.9) we can now perform the limit $\omega \to \infty$ in the Hamiltonian constraint and we find that it is equal to

$$\begin{aligned}\mathcal{H}_\tau = {} & \Pi_\mu h^{\mu\nu}\Pi_\nu - 2T_{D1}\sqrt{n^2 e^{-2\Phi} + (m - n\chi)^2}\,\Pi_\mu \hat{\tau}^\mu_{\ a}\eta^{ab}\epsilon_{bc}\tau_\rho{}^c\partial_\sigma x^\rho + \\ & + T_{D1}^2(n^2 e^{-2\Phi} + (m - n\chi)^2)\partial_\sigma x^\mu \bar{h}_{\mu\nu}\partial_\sigma x^\nu - \\ & - T_{D1}^2(n^2 e^{-2\Phi} + (m - n\chi)^2)\partial_\sigma x^\nu \tau_\nu{}^c \epsilon_{cd}\Phi^{da}\epsilon_{ab}\tau_\nu{}^f\partial_\sigma x^\nu\,,\end{aligned} \tag{3.10}$$

where we introduced following fields

$$\begin{aligned}\hat{\tau}^\mu_{\ a} &= \tau^\mu_{\ a} - h^{\mu\nu}m_\nu{}^b\eta_{ba}\,, \quad \hat{e}_\mu{}^{a'} = e_\mu{}^{a'} + m_\nu{}^a e^\nu_{\ c'}\delta^{c'a'}\tau_\mu{}^b\eta_{ba}\,, \\ \Phi^{ab} &= -\tau^\mu_{\ d}\eta^{da}m_\mu{}^b - m_\mu{}^a\tau^\mu_{\ d}\eta^{db} + m_\mu{}^a h^{\mu\nu}m_\nu{}^b\,, \\ \bar{h}_{\mu\nu} &= h_{\mu\nu} + m_\mu{}^a\tau_\nu{}^b\eta_{ab} + \tau_\mu{}^a m_\nu{}^b\eta_{ab}\,,\end{aligned} \tag{3.11}$$

and where

$$\Pi_\mu = p_\mu - T_{D1}mb_{\mu\rho}\partial_\sigma x^\rho - T_{D1}nc_{\mu\rho}\partial_\sigma x^\rho\,. \tag{3.12}$$

Finally using the fact that

$$\begin{aligned}&-\partial_\sigma x^\mu \tau_\mu{}^a\epsilon_{ab}\Phi^{bc}\epsilon_{cd}\tau_\nu{}^d\partial_\sigma x^\nu = \tau_\sigma{}^c\eta_{ca}\Phi^{ab}\eta_{bd}\tau_\sigma{}^d - \mathbf{a}_{\sigma\sigma}\Phi^{ab}\eta_{ba}\,, \\ &\tau_\alpha{}^a = \tau_\mu{}^a\partial_\alpha x^\mu\,, \qquad \mathbf{a}_{\alpha\beta} = \partial_\alpha x^\mu \tau_{\mu\nu}\partial_\beta x^\nu\end{aligned} \tag{3.13}$$

we can rewrite the Hamiltonian constraint into the form

$$\begin{aligned}\mathcal{H}_\tau = {} & \Pi_\mu h^{\mu\nu}\Pi_\nu - 2T_{D1}\sqrt{n^2 e^{-2\Phi} + (m - n\chi)^2}\,\Pi_\mu \hat{\tau}^\mu_{\ a}\eta^{ab}\epsilon_{bc}\tau_\rho{}^c\partial_\sigma x^\rho + \\ & + T_{D1}^2(n^2 e^{-2\Phi} + (m - n\chi)^2)\partial_\sigma x^\mu \bar{h}_{\mu\nu}\partial_\sigma x^\nu + \\ & + T_{D1}^2(n^2 e^{-2\Phi} + (m - n\chi)^2)\left(\tau_\sigma{}^c\eta_{ca}\Phi^{ab}\eta_{bd}\tau_\sigma{}^d - \mathbf{a}_{\sigma\sigma}\Phi^{ab}\eta_{ba}\right)\,.\end{aligned} \tag{3.14}$$

This is the Hamiltonian constraint for (m, n)-string in stringy NC background.

Let us now proceed to the Lagrangian form of this theory. Note that the Hamiltonian is equal to

$$H = \int d\sigma(N^\tau\mathcal{H}_\tau + N^\sigma\mathcal{H}_\sigma)\,, \tag{3.15}$$

where $\mathcal{H}_\tau$ is given in (3.14). From (3.15) we obtain

$$\dot{x}^\mu = \{x^\mu, H\} = 2Nh^{\mu\nu}\Pi_\nu - 2NT_{D1}\sqrt{n^2 e^{-2\Phi} + (m - n\chi)^2}\,\hat{\tau}^\mu_{\ a}\eta^{ab}\epsilon_{bc}\tau_\rho{}^c\partial_\sigma x^\rho + N^\sigma\partial_\sigma x^\mu\,. \tag{3.16}$$

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To proceed further we use the fact that

$$\hat{e}_\mu{}^{a'}\tau^\mu_{\ b}=0\,,\quad \hat{e}_\mu{}^{a'}h^{\mu\nu}=e^\nu_{\ b'}\delta^{b'a'} \tag{3.17}$$

and hence

$$(\dot{x}^\mu-N^\sigma x'^\mu)\hat{e}_\mu{}^{a'}\delta_{a'b'}\hat{e}_\nu{}^{b'}(\dot{x}^\nu-N^\sigma x'^\nu)=4(N^\tau)^2\Pi_\mu h^{\mu\nu}\Pi_\nu\,. \tag{3.18}$$

Then the Lagrangian density has the form

$$\begin{aligned}\mathcal{L}=\dot{x}^\mu p_\mu-\mathcal{H}=(\dot{x}^\mu-N^\sigma x'^\mu)p_\mu-N\mathcal{H}_\tau=\\ =\frac{1}{4N^\tau}(\dot{x}^\mu-N^\sigma x'^\mu)\hat{e}_\mu{}^{a'}\delta_{a'b'}\hat{e}_\nu{}^{b'}(\dot{x}^\nu-N^\sigma x'^\nu)-\\ -T_{D1}^2N^\tau(n^2e^{-2\Phi}+(m-n\chi)^2)\partial_\sigma x^\mu\bar{h}_{\mu\nu}\partial_\sigma x^\nu-\\ -N^\tau T_{D1}^2(n^2e^{-2\Phi}+(m-n\chi)^2)\left(\tau_\sigma{}^c\eta_{ca}\Phi^{ab}\eta_{bd}\tau_\sigma{}^d-\mathbf{a}_{\sigma\sigma}\Phi^{ab}\eta_{ba}\right)\\ +T_{D1}(m\dot{x}^\mu b_{\mu\nu}x'^\nu+n\dot{x}^\mu c_{\mu\nu}x'^\nu)\,.\end{aligned} \tag{3.19}$$

To proceed further we use the fact that

$$\hat{e}_\mu{}^{a'}\delta_{a'b'}\hat{e}_\nu{}^{b'}=\bar{h}_{\mu\nu}+\tau_\mu{}^c\eta_{ca}\Phi^{ab}\eta_{bd}\tau_\nu{}^d\,,\quad \bar{h}_{\mu\nu}=h_{\mu\nu}+m_\mu{}^a\tau_\nu{}^b\eta_{ab}+m_\nu{}^a\tau_\mu{}^b\eta_{ab}\,. \tag{3.20}$$

Further, the special property of non-relativistic string and (m,n)-string is that N^τ, N^σ are determined by the equations of motion for x^μ. To see this, we multiply the equation of motion for x^μ by $\tau_{\mu\nu}$ and we obtain

$$\tau_{\mu\nu}(\dot{x}^\nu-N^\sigma x'^\nu)=-2N^\tau T_{D1}\sqrt{n^2e^{-2\Phi}+(m-n\chi)^2}\tau_{\mu\nu}\hat{\tau}^\nu_{\ a}\eta^{ab}\epsilon_{bc}\hat{\tau}_\rho{}^c\partial_\sigma x^\rho\,. \tag{3.21}$$

If we multiply this equation with $\partial_\sigma x^\mu$ we get

$$N^\sigma=\frac{\mathbf{a}_{\tau\sigma}}{\mathbf{a}_{\sigma\sigma}}\,. \tag{3.22}$$

In the similar way we obtain

$$(\partial_\tau x^\mu-N^\sigma\partial_\sigma x^\mu)\tau_{\mu\nu}(\partial_\tau x^\nu-N^\sigma\partial_\sigma x^\nu)=-4(N^\tau)^2T_{D1}^2(n^2e^{-2\Phi}+(m-n\chi)^2)\mathbf{a}_{\sigma\sigma} \tag{3.23}$$

that can be solved for N^τ as

$$N^\tau=\frac{1}{2\mathbf{a}_{\sigma\sigma}\sqrt{n^2e^{-2\Phi}+(m-n\chi)^2}}\sqrt{-\det\mathbf{a}}\,. \tag{3.24}$$

Inserting (3.22) and (3.24) into (3.19) we obtain Lagrangian density in the form

$$\begin{aligned}\mathcal{L}=-\frac{1}{2}\sqrt{n^2e^{-2\Phi}+(m-n\chi)^2}\sqrt{-\det\mathbf{a}}\mathbf{a}^{\alpha\beta}\bar{h}_{\alpha\beta}+\\ +T_{D1}(m\dot{x}^\mu b_{\mu\nu}x'^\nu+n\dot{x}^\mu c_{\mu\nu}x'^\nu)\,,\end{aligned} \tag{3.25}$$

where we used the result found in [15] that two terms proportional to Φ^{ab} cancel each other as

$$\mathbf{a}^{\alpha\beta}\tau_\alpha{}^a\eta_{ab}\Phi^{bc}\eta_{cd}\tau_\beta{}^d-\Phi^{ab}\eta_{ba}=0\,. \tag{3.26}$$

The Lagrangian density (3.25) is the final form of the Lagrangian density for (m,n)-string in stringy NC background. However in order to be our approach self-consistent we have to check the equations of motion for the background fields that were used in the construction of the Lagrangian density (3.25).

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4 Background equation of motion

In order to derive form of the non-relativistic (m,n)-string we presumed special form of the background field. Our goal is to check the consistency of the equations of motion for the background fields when the limiting procedure is performed.

We start with Christoffel symbols of parent relativistic theory

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}G^{\rho\sigma}(\partial_\mu G_{\sigma\nu} + \partial_\nu G_{\sigma\mu} - \partial_\sigma G_{\mu\nu})\,. \tag{4.1}$$

Now recall that the metric components that define stringy NC background have the form

$$\begin{aligned} G_{\mu\nu} &= \omega^2\tau_{\mu\nu} + h_{\mu\nu} + \frac{1}{2}\tau_\mu{}^a m_\nu{}^b\eta_{ab} + \frac{1}{2}m_\mu{}^a\tau_\nu{}^b\eta_{ab} + \frac{1}{4\omega^2}m_\mu{}^a m_\nu{}^b\eta_{ab}\,,\\ G^{\mu\nu} &= \frac{1}{\omega^2}\tau^{\mu\nu} + h^{\mu\nu} - \frac{1}{2\omega^2}(\tau^\nu{}_b m_\rho{}^b h^{\rho\mu} + \tau^\mu{}_b m_\rho{}^b h^{\rho\nu}) -\\ &\quad - \frac{1}{2\omega^4}(\tau^\mu{}_c m^c{}_\rho\tau^{\rho\nu} + \tau^\nu{}_d m^d{}_\rho\tau^{\rho\mu}) + \frac{1}{4\omega^4}\tau^\mu{}_a m_\rho{}^a h^{\rho\sigma}\tau^\nu{}_b m_\sigma{}^b\,. \end{aligned} \tag{4.2}$$

Inserting (4.2) into (4.1) we obtain in the limit $\omega\to\infty$

$$\begin{aligned} \Gamma^{\rho}_{\mu\nu} &= \frac{\omega^2}{2}h^{\mu\sigma}(\partial_\mu\tau_{\sigma\nu} + \partial_\nu\tau_{\mu\sigma} - \partial_\sigma\tau_{\mu\nu}) +\\ &\quad + \frac{1}{2}h^{\mu\sigma}(\partial_\mu\hat{h}_{\sigma\nu} + \partial_\nu\hat{h}_{\sigma\mu} - \partial_\sigma\hat{h}_{\mu\nu}) + \frac{1}{2}\tau^{\rho\sigma}(\partial_\mu\tau_{\sigma\nu} + \partial_\nu\tau_{\sigma\mu} - \partial_\sigma\tau_{\mu\nu}) -\\ &\quad - \frac{1}{4}(\tau^\rho{}_b m_\omega h^{\omega\sigma} + \tau^\sigma{}_b m_\omega{}^b h^{\omega\rho})(\partial_\mu\tau_{\sigma\nu} + \partial_\nu\tau_{\sigma\mu} - \partial_\sigma\tau_{\mu\nu})\,, \end{aligned} \tag{4.3}$$

where

$$\hat{h}_{\mu\nu} = h_{\mu\nu} + \frac{1}{2}\tau_\mu{}^a m_\nu{}^b\eta_{ab} + \frac{1}{2}m_\mu{}^a\tau_\nu{}^b\eta_{ab}\,. \tag{4.4}$$

Let us now focus on the divergent term that can be rewritten into the form

$$\begin{aligned} &h^{\rho\sigma}(\partial_\mu\tau_{\sigma\nu} + \partial_\nu\tau_{\mu\sigma} - \partial_\sigma\tau_{\mu\nu}) =\\ &\quad = h^{\rho\sigma}[(\partial_\mu\tau_\sigma{}^a - \partial_\sigma\tau_\mu{}^a)\eta_{ab}\tau_\nu{}^b + \tau_\mu{}^a(\partial_\nu\tau_\sigma{}^b - \partial_\sigma\tau_\nu{}^b)\eta_{ab}]\,. \end{aligned} \tag{4.5}$$

We see that this term vanishes when we impose zero torsion condition

$$\partial_\mu\tau_\nu{}^a - \partial_\nu\tau_\mu{}^a = 0\,. \tag{4.6}$$

In what follows we will presume this condition. Then after some calculation we find that (4.3) simplifies considerably and has the form[3]

$$\begin{aligned} \Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}\tau^\rho{}_a(\partial_\mu\tau_\nu{}^a + \partial_\nu\tau_\mu{}^a) + \frac{1}{2}h^{\mu\sigma}(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}) +\\ &\quad + \frac{1}{4}h^{\rho\sigma}((\partial_\mu m_\sigma{}^a - \partial_\sigma m_\mu{}^a)\eta_{ab}\tau_\nu{}^b + (\partial_\nu m_\sigma{}^a - \partial_\sigma m_\nu{}^a)\eta_{ab}\tau_\mu{}^b)\,. \end{aligned} \tag{4.7}$$

[3]Note that this Christoffel symbols differ from the connection found [2] since the expression on the second line contain ordinary derivative of $m_\mu{}^a$ instead of the covariant one. We mean that this is a consequence of the condition (4.6) which is rather strong and could be replaced by weaker one asi in [2]. We hope to return to the analysis of non-relativistic limit with this weaker condition in future.

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As a check we can easily find that $\tau_\mu{}^a$ and $h^{\mu\nu}$ are covariantly constant

$$\nabla_\mu \tau_\nu{}^a = 0\,, \quad \nabla_\mu h^{\nu\rho} = 0\,. \tag{4.8}$$

Now we are ready to proceed to the analysis of type IIB supergravity equations of motion. We start with the Type IIB string effective action in the form

$$S = \frac{1}{2\tilde{\kappa}_{10}^2}\int d^{10}x\sqrt{-G_E}\left(R(G_E) - \frac{\partial_\mu \tau G_E^{\mu\nu}\partial_\nu \bar{\tau}}{2(Im\tau)^2} - \frac{1}{2}\frac{|G_3|^2}{Im\tau}\right)\,, \tag{4.9}$$

where

$$\tau = \chi + ie^{-\Phi} \equiv \tau_R + i\tau_I\,, G_3 = dC_2 - \chi H - ie^{-\Phi}H = dC_2 - \tau H\,, \tag{4.10}$$

and where the action is written in the Einstein frame where the metric components are related to the string components metric as

$$G^E_{\mu\nu} = e^{-\Phi/2}G_{\mu\nu} \tag{4.11}$$

Finally note that we use the convention

$$|G_3|^2 = \frac{1}{3!}G_{\mu\nu\rho}G_E^{\mu\mu_1}G_E^{\nu\nu_1}G^{\rho\rho_1}G^*_{\mu_1\nu_1\rho_1}\,. \tag{4.12}$$

The variation of the action with respect to $G_E^{\mu\nu}$ gives following equations of motion for metric

$$R_{\mu\nu}(G_E) - \frac{1}{2}G^E_{\mu\nu}R(G_E) = T_{\mu\nu}\,, \tag{4.13}$$

where $T_{\mu\nu}$ is stress energy tensor whose exact definition will be given below. Now taking the trace of the equation (4.13) with $G_E^{\mu\nu}$ we obtain

$$R(G_E) = -4T\,, T \equiv G_E^{\mu\nu}T_{\mu\nu}\,. \tag{4.14}$$

Inserting this result into (4.13) we find an alternative form of the equation of motion

$$R_{\mu\nu}(G_E) = T_{\mu\nu} - \frac{1}{8}G^E_{\mu\nu}T \tag{4.15}$$

that is suitable for non-relativistic limit. Before we proceed to this point we should analyze one subtle point which is the relation between Einstein frame and string frame connection that generally has the form

$$\Gamma^\rho_{\mu\nu}(G_E) = \Gamma^\rho_{\mu\nu}(G) - \frac{1}{4}(\delta^\rho_\nu\partial_\mu\Phi + \delta^\rho_\mu\partial_\nu\Phi - G^{\rho\sigma}\partial_\sigma\Phi G_{\mu\nu}) \tag{4.16}$$

Then for the metric ansatz (4.2) we obtain in the limit $\omega \to \infty$

$$\begin{aligned}\Gamma^\rho_{\mu\nu}(G_E) = \Gamma^\rho_{\mu\nu}(G) &- \frac{1}{4}(\delta^\rho_\nu\partial_\mu\Phi + \delta^\rho_\mu\partial_\nu\Phi) + \\ &+ \frac{1}{4}\left(\tau^{\rho\sigma} - \frac{1}{2}\tau^\rho{}_b m_\omega{}^b h^{\rho\sigma} - \frac{1}{2}\tau^\sigma{}_b m_\omega{}^b h^{\omega\rho}\right)\partial_\sigma\Phi\tau_{\mu\nu} + \\ &+ \frac{1}{4}h^{\rho\sigma}\partial_\sigma\Phi\left(h_{\mu\nu} + \frac{1}{2}\tau_\mu{}^a m_\nu{}^b\eta_{ab} + \frac{1}{2}m_\mu{}^a\tau_\nu{}^b\eta_{ab}\right) + \omega^2\frac{1}{4}h^{\rho\sigma}\partial_\sigma\Phi\tau_{\mu\nu}\,.\end{aligned} \tag{4.17}$$

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We see that there is a divergent contribution proportional to $h^{\rho\sigma}\partial_\sigma\Phi$. Then in order to have finite Christoffel connection in Einstein frame as well we impose following condition on the dilaton

$$h^{\mu\nu}\partial_\nu\Phi = 0\,. \tag{4.18}$$

We will discuss this restriction in more details below.

Let us now return to the equation of motion (4.15). Since its left side is finite the right side has to be finite too in the limit $\omega \to \infty$. This fact implies additional conditions on the background fields. To see this let us focus on the matter part of the action that depends on $G_{\mu\nu\rho}$ that has following stress energy tensor defined as

$$T^{G_3}_{\mu\nu} = -\frac{1}{\sqrt{-G_E}}\frac{\delta S_{G_3}}{\delta G^{\mu\nu}_E} = -\frac{1}{24}G^E_{\mu\nu}G_{\rho\sigma\omega}G^{\rho\rho_1}G^{\sigma\sigma_1}G^{\omega\omega_1}G^*_{\rho_1\sigma_1\omega_1} + \frac{1}{4}G_{\mu\sigma\omega}G^{\sigma\sigma_1}_E G^{\omega\omega_1}_E G^*_{\nu\sigma_1\omega_1}\,. \tag{4.19}$$

We see that necessary condition to have finite stress energy tensor we have to demand that $G_{\mu\nu\rho}$ has to be finite in the limit $\omega \to \infty$. Let us analyse this requirement in more details and start for simplicity of notation with the field strength of $B_{\mu\nu}$. Inserting (3.8) into definition of $H_{\mu\nu\rho}$ we obtain

$$\begin{aligned}
H_{\rho\mu\nu} &= \partial_\rho B_{\mu\nu} + \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} = \\
&= \omega^2\mathbf{X}[(\partial_\rho\tau_\mu{}^a - \partial_\mu\tau_\rho{}^a)\tau_\nu{}^b\epsilon_{ab} + (\partial_\nu\tau_\rho{}^a - \partial_\rho\tau_\nu{}^a)\tau_\mu{}^b\epsilon_{ab} + (\partial_\mu\tau_\nu{}^a - \partial_\nu\tau_\mu{}^a)\epsilon_{ab}\tau_\rho{}^b] - \\
&\quad - \frac{1}{2}\mathbf{X}[(\partial_\mu m_\nu{}^a - \partial_\nu m_\mu{}^a)\tau_\rho{}^b\epsilon_{ab} + (\partial_\nu m_\rho{}^a - \partial_\rho m_\nu{}^a)\tau_\mu{}^b\epsilon_{ab} + (\partial_\rho m_\mu{}^a - \partial_\mu m_\rho{}^a)\tau_\nu{}^b\epsilon_{ab} + \\
&\quad + (\partial_\mu\tau_\nu{}^a - \partial_\nu\tau_\mu{}^a)\epsilon_{ab}m_\rho{}^b\epsilon_{ab} + (\partial_\rho\tau_\mu{}^a - \partial_\mu\tau_\rho{}^a)m_\nu{}^b\epsilon_{ab} + (\partial_\nu\tau_\rho{}^a - \partial_\rho\tau_\nu{}^a)m_\mu{}^b\epsilon_{ab}] \\
&\quad + \partial_\rho\mathbf{X}\mathbf{B}_{\mu\nu} + \partial_\mu\mathbf{X}\mathbf{B}_{\nu\rho} + \partial_\nu\mathbf{X}\mathbf{B}_{\rho\mu} + h_{\rho\mu\nu}\,.
\end{aligned} \tag{4.20}$$

First of all we see that the first divergent contribution proportional to ω^2 vanishes when we impose zero torsion condition. On the other hand there is still divergent contribution coming from the terms on the last line that are proportional to derivative of $\mathbf{X}$. Since $\mathbf{B}_{\mu\nu}$ does not vanish and it is proportional to ω^2 we see that in order to ensure finetness of $H_{\rho\mu\nu}$ we have to presume that Φ and χ are constant and hence $\partial_\rho\mathbf{X} = 0$. Note that this is much stronger condition than (4.18). Then however the equations of motion for τ implies that G_3 has to be zero that also solves the equation of motion for G_3. Further, the stress energy tensor for complex scalar τ vanishes for constant τ. As a result we find that the equations of motion for stringy NC gravity has simple form

$$R_{\mu\nu}(G_E) = 0\,. \tag{4.21}$$

Thanks to the fact that Φ is constant this equation also implies

$$R_{\mu\nu}(G) = 0\,, \tag{4.22}$$

where $R_{\mu\nu}$ depends on Christoffel symbols given in (4.7). It is important to stress that (4.22) corresponds to the standard equation of motion of stringy Newton-Cartan gravity.

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4.1 Special case $n = 0, m = 1$

There is one important special case when $n = 0, m = 1$ that corresponds to the situation when the probe that defines stringy Newton-Cartan gravity is fundamental string. This case is special since we have $\mathbf{X} = 1$. For simplicity we further presume that the background RR fields are zero. In this case the field strength for $B_{\mu\nu}$ has the form

$$H_{\rho\mu\nu} = \partial_\rho B_{\mu\nu} + \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} = h_{\mu\nu\rho} - \\ - \frac{1}{2}[(\partial_\mu m_\nu{}^a - \partial_\nu m_\mu{}^a)\tau_\rho{}^b\epsilon_{ab} + (\partial_\nu m_\rho{}^a - \partial_\rho m_\nu{}^a)\tau_\mu{}^b\epsilon_{ab} + (\partial_\rho m_\mu{}^a - \partial_\mu m_\rho{}^a)\tau_\nu{}^b]\,, \tag{4.23}$$

where we imposed zero torsion condition $\partial_\mu\tau_\nu{}^a - \partial_\nu\tau_\mu{}^a = 0$ and where $h = db$. Let us again study the equation of motion for the background fields. Note that the relativistic theory is governed by the action

$$S = \frac{1}{2\tilde{\kappa}_{10}^2}\int d^{10}x\sqrt{-G_E}\left[R - \frac{1}{2}\partial_\mu\Phi G_E^{\mu\nu}\partial_\nu\Phi - \frac{1}{12}e^{-\Phi}H_{\mu\nu\rho}G_E^{\mu\mu_1}G_E^{\nu\nu_1}G_E^{\rho\rho_1}H_{\mu_1\nu_1\rho_1}\right], \tag{4.24}$$

$$G^E_{\mu\nu} = e^{-\Phi/2}G_{\mu\nu}\,. \tag{4.25}$$

Let us start with the equation of motion for Φ

$$\frac{1}{\sqrt{-G_E}}\partial_\mu\left[\sqrt{-G_E}G_E^{\mu\nu}\partial_\nu\Phi\right] + + \frac{1}{12}e^{-\Phi}H_{\mu\nu\rho}G_E^{\mu\mu_1}G_E^{\nu\nu_1}G^{\rho\rho_1}H_{\mu_1\nu_1\rho_1} = 0 \tag{4.26}$$

and find its non-relativistic limit. First of all we have

$$\det G^E_{\mu\nu} = e^{-5\Phi}\det G_{\mu\nu} = e^{-5\Phi}(\det E_\mu{}^A)^2\,. \tag{4.27}$$

Since $E_\mu{}^a = \omega\tau_\mu{}^a + \frac{1}{2\omega}m_\mu{}^a\,, E_\mu{}^{a'} = e_\mu{}^{a'}$ we find

$$\det E_\mu{}^A = \omega\det(\tau_\mu{}^a, e_\mu{}^{a'}) + \frac{1}{2\omega}\det(m_\mu{}^a, e_\mu{}^{a'})\,. \tag{4.28}$$

Then the equation of motion for Φ reduces into following form in the limit $\omega \to \infty$

$$\frac{1}{\det(\tau_\mu{}^a, e_\mu{}^{a'})}\partial_\mu[e^{-2\Phi}\det(\tau_\mu{}^a, e_\mu{}^{a'})h^{\mu\nu}\partial_\nu\Phi] + \frac{1}{12}e^{3\Phi}H_{\mu\nu\rho}h^{\mu\mu_1}h^{\nu\nu_1}h^{\rho\rho_1}H_{\mu_1\nu_1\rho_1} = 0\,, \tag{4.29}$$

where $H_{\mu\nu\rho}$ is given in (4.23). Further, the equation of motion for $H_{\mu\nu\rho}$ takes the form

$$\partial_\mu\left[\sqrt{-\det G_E}e^{-\Phi}G_E^{\mu\mu_1}G_E^{\nu\nu_1}G_E^{\rho\rho_1}H_{\mu_1\nu_1\rho_1}\right] = 0 \tag{4.30}$$

that in the non-relativistic limit reduces into

$$\partial_\mu\left[e^{-2\Phi}\det(\tau_\mu{}^a, e_\mu{}^{a'})h^{\mu\mu_1}h^{\nu\nu_1}h^{\rho\rho_1}H_{\mu_1\nu_1\rho_1}\right] = 0\,, \tag{4.31}$$

where $H_{\mu\nu\rho}$ is again given in (4.23). Let us now discuss consequence of the condition (4.18). Imposing this condition in (4.29) we obtain following condition on $H_{\mu\nu\rho}$

$$h^{\mu\mu_1}h^{\nu\nu_1}h^{\rho\rho_1}H_{\mu_1\nu_1\rho_1} = 0\,. \tag{4.32}$$

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Note that in this case the equation of motion (4.31) is satisfied as well. Note that in adapted coordinates when $e_\mu{}^{a'} = e_i{}^{a'}, i = 2, \ldots, 9$ and hence $h^{\mu\nu}$ has non-zero components h^{ij} we find that the condition above implies

$$H_{ijk} = 0\,. \tag{4.33}$$

Let us finally discuss the equation of motion for $G^E_{\mu\nu}$

$$R_{\mu\nu}(G_E) = T_{\mu\nu} - \frac{1}{8} G^E_{\mu\nu} T\,. \tag{4.34}$$

As we argued above The left side of this equation is finite in the limit $\omega \to \infty$. Then on the right side we should have finite contribution as well. In case of the scalar field we obtain

$$T^{\rm scal}_{\mu\nu} = -\frac{1}{\sqrt{-\det G_E}} \frac{\delta S^{\rm scal}}{\delta G^{\mu\nu}_E} = -\frac{1}{4} G^E_{\mu\nu} G^{\rho\sigma}_E \partial_\rho \Phi \partial_\sigma \Phi + \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi \tag{4.35}$$

and hence

$$T^{\rm scal}_{\mu\nu} - \frac{T^{\rm scal}}{D-2} G^E_{\mu\nu} = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi\,. \tag{4.36}$$

On the other hand in case of $B_{\mu\nu}$ we obtain

$$T^B_{\mu\nu} = -\frac{1}{24} G^E_{\mu\nu} H_{\rho\sigma\omega} G^{\rho\rho_1}_E G^{\sigma\sigma_1}_E G^{\omega\omega_1}_E H_{\rho_1\sigma_1\omega_1} + \frac{1}{4} H_{\mu\sigma\omega} G^{\sigma\sigma_1}_E G^{\omega\omega_1}_E H_{\nu\sigma_1\omega_1} \tag{4.37}$$

so that

$$T^B = G^{\mu\nu}_E T^B_{\mu\nu} = -\frac{1}{6} H^2 \tag{4.38}$$

which implies

$$T^B_{\mu\nu} - \frac{1}{8} G^E_{\mu\nu} T^B = -\frac{1}{48} G^E_{\mu\nu} H^2 + \frac{1}{4} H_{\mu\rho\sigma} G^{\rho\rho_1}_E G^{\sigma\sigma_1}_E H_{\nu\sigma_1\omega_1}\,.$$

Note that this expression is finite as well when we impose the condition (4.32). As a result we obtain final form of the equation of motion of stringy Newton-Cartan gravity

$$R_{\mu\nu}(G_E) = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi + T^B_{\mu\nu}\,, \tag{4.39}$$

where Φ has to obey the condition (4.18) that in adapted coordinates $\tau_\mu{}^a = \delta_\mu{}^a$, $e_\mu{}^{a'} = e_i{}^{a'}$, $i = 2, \ldots, 9$ implies $\partial_i \Phi = 0$ and hence $\Phi = \Phi(t)$.

5 D1-brane with unfixed gauge symmetry in NC background

In this section we would consider situation when we have n coincident D1-branes and take the non-relativistic limit on metric and NSNS two form field that defines stringy NC gravity. We again start with the Hamiltonian given in (2.6) and (2.7) where the metric is given in (3.7) and where NSNS two form field has the form As the next step we have to

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introduce an appropriate parameterization of NSNS two form. Since we want to discuss D1-brane in stringy NC background we consider following ansatz for NSNS two form

$$B_{\mu\nu} = \left(\omega^2 \tau_\mu{}^a \tau_\nu{}^b \epsilon_{ab} - \frac{1}{2}(m_\mu{}^a \tau_\nu{}^b + \tau_\mu{}^a m_\nu{}^b)\epsilon_{ab} + \frac{1}{4\omega^2} m_\mu{}^a m_\mu{}^b \epsilon_{ab}\right) + b_{\mu\nu}\,. \tag{5.1}$$

With such a choice of the NSNS field we find that divergent contribution in the Hamiltonian constraint that are proportional to $\tau_{\mu\nu}$ cancel for any π^σ. However there are now several possibilities how to take scaling limit of RR fields. For example, if we demand that $C_{\mu\nu}$ has similar form as $B_{\mu\nu}$, i.e. $C_{\mu\nu} = \mathbf{B}_{\mu\nu} + c_{\mu\nu}$ the requirement of the finitness of the Hamiltonian constraint implies following condition

$$2\pi^\sigma n = -2\pi^\sigma n\chi\,, \quad e^{-2\Phi} + \chi^2 = 1\,. \tag{5.2}$$

The first equation implies $\chi = -1$ that together with the second one gives $e^{-2\Phi} = 0$. However this is the realm of the infinite coupled string theory which is clearly rather subtle limit. For that reason we consider another scaling limit when

$$n = \frac{1}{\omega}\tilde{n}\,, \quad \chi = \frac{1}{\omega}\tilde{\chi}\,, \quad C_{\mu\nu} = \omega\tilde{C}_{\mu\nu}\,. \tag{5.3}$$

Now the Hamiltonian constraint is finite and has the form

$$\begin{aligned}\mathcal{H}_\tau = {} & \Pi_\mu h^{\mu\nu}\Pi_\nu - 2T_{D1}\pi^\sigma \Pi_\mu \hat{\tau}^\mu_a \eta^{ab}\epsilon_{bc}\tau_\sigma{}^c - T^2_{D1}(\pi^\sigma)^2 \tau_\sigma{}^a \epsilon_{ab}\Phi^{bc}\epsilon_{cd}\tau_\sigma{}^d + \\ & + T^2_{D1}(\pi^\sigma)^2 \partial_\sigma x^\mu \bar{h}_{\mu\nu}\partial_\sigma x^\nu + T^2_{D1} e^{-2\Phi}\tilde{n}^2 \partial_\sigma x^\mu \tau_{\mu\nu}\partial_\sigma x^\nu\,,\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}\Pi_\mu &= p_\mu - T_{D1}(\pi^\sigma b_{\mu\rho} + \tilde{n}\tilde{C}_{\mu\rho})\partial_\sigma x^\rho\,, \\ \bar{h}_{\mu\nu} &= h_{\mu\nu} + m_\mu{}^a \tau_\nu{}^b \eta_{ab} + \tau_\mu{}^a m_\nu{}^b \eta_{ab}\,.\end{aligned} \tag{5.5}$$

Note that π^σ is still dynamical field which is a crucial difference when we compare with the analysis presented in 3 section.

As the next step we determine corresponding Lagrangian. From the Hamiltonian

$$H = \int d\sigma (N^\tau \mathcal{H}_\tau + N^\sigma \mathcal{H}_\sigma + \partial_\sigma A_\tau \pi^\sigma) \tag{5.6}$$

we obtain

$$\begin{aligned}\partial_\tau x^\mu = \{x^\mu, H\} &= 2N^\tau h^{\mu\nu}\Pi_\nu - 2N^\tau T_{D1}\pi^\sigma \hat{\tau}^\mu_a \eta^{ab}\epsilon_{bc}\tau_\sigma{}^c + N^\sigma \partial_\sigma x^\mu\,, \\ \partial_\tau A_\sigma = \{A_\sigma, H\} &= \partial_\sigma A_\tau - 2N^\tau T_{D1} b_{\mu\rho}\partial_\sigma x^\rho h^{\mu\nu}\Pi_\nu - 2N^\tau T_{D1}\Pi_\mu \hat{\tau}^\mu_a \eta^{ab}\epsilon_{bc}\tau_\sigma{}^c + \\ & + 2N^\tau T^2_{D1} b_{\mu\rho}\partial_\sigma x^\rho \pi^\sigma \hat{\tau}^\mu_a \eta^{ab}\epsilon_{bc}\tau_\sigma{}^c - 2N^\tau T^2_{D1}\pi^\sigma \tau_\sigma{}^a \epsilon_{ab}\Phi^{bc}\epsilon_{cd}\tau_\sigma{}^d + \\ & + 2N^\tau T^2_{D1}\pi^\sigma \partial_\sigma x^\mu \bar{h}_{\mu\nu}\partial_\sigma x^\nu\,.\end{aligned} \tag{5.7}$$

Then the Lagrangian density has the form

$$\begin{aligned}\mathcal{L} = {} & N^\tau \Pi_\mu h^{\mu\nu}\Pi_\nu + T_{D1}\tilde{n}X^\mu \tilde{C}_{\mu\nu}\partial_\sigma x^\nu - 2N^\tau \pi^\sigma T_{D1}\Pi_\mu \hat{\tau}^\mu_a \eta^{ab}\epsilon_{bc}\tau_\sigma{}^c + \\ & + N^\tau T^2_{D1}(\pi^\sigma)^2 \tau_\sigma{}^a \epsilon_{ab}\Phi^{bc}\epsilon_{cd}\tau_\sigma{}^d - N^\tau T^2_{D1}(\pi^\sigma)^2 \partial_\sigma x^\mu \bar{h}_{\mu\nu}\partial_\sigma x^\nu - e^{-2\Phi}T^2_{D1}N^\tau \tilde{n}^2 \mathbf{a}_{\sigma\sigma}\,.\end{aligned} \tag{5.8}$$

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To proceed further we use the fact that

$$\hat{e}_{\mu}^{\ a'}\tau_{b}^{\mu}=0 \tag{5.9}$$

and we again obtain

$$X^{\mu}\hat{e}_{\mu}^{\ a'}\delta_{a'b'}\hat{e}_{\nu}^{\ b'}X^{\nu}=4(N^{\tau})^{2}\Pi_{\mu}h^{\mu\nu}\Pi_{\nu}\,. \tag{5.10}$$

As the next step we express $\hat{\tau}_{a}^{\mu}\Pi_{\mu}$ with the help of the equation of motion for A_{σ} as

$$\begin{aligned}&F_{\tau\sigma}+T_{D1}X^{\mu}b_{\mu\nu}\partial_{\sigma}x^{\nu}+T_{D1}\frac{\sqrt{-\det\mathbf{a}}}{\mathbf{a}_{\sigma\sigma}}\tau_{\sigma}^{\ a}\epsilon_{ab}\Phi^{bc}\epsilon_{cd}\tau_{\sigma}^{\ d}-T_{D1}\frac{\sqrt{-\det\mathbf{a}}}{\mathbf{a}_{\sigma\sigma}}\partial_{\sigma}x^{\mu}\bar{h}_{\mu\nu}\partial_{\sigma}x^{\nu}=\\&\quad=-2N^{\tau}T_{D1}\Pi_{\mu}\hat{\tau}_{a}^{\mu}\eta^{ab}\epsilon_{bc}\tau_{\sigma}^{\ c}\end{aligned} \tag{5.11}$$

where we also used the fact equation of motion for x^{μ} implies

$$N^{\sigma}=\frac{\mathbf{a}_{\tau\sigma}}{\mathbf{a}_{\sigma\sigma}}\,,\quad \pi^{\sigma}N^{\tau}=\frac{\sqrt{-\det\mathbf{a}}}{2\mathbf{a}_{\sigma\sigma}T_{D1}} \tag{5.12}$$

and hence we expressed π^{σ} with the help of the second equation in (5.12). Collecting all these terms together and after some calculations we obtain Lagrangian density in the form

$$\begin{aligned}\mathcal{L}=&\frac{\det\mathbf{a}}{4N^{\tau}\mathbf{a}_{\sigma\sigma}}\left(\mathbf{a}^{\alpha\beta}\bar{h}_{\alpha\beta}+\mathbf{a}^{\alpha\beta}\tau_{\alpha}^{\ a}\eta_{ab}\Phi^{bc}\eta_{cd}\tau_{\beta}^{\ d}-\Phi^{ab}\eta_{ba}-\frac{2}{\sqrt{-\det\mathbf{a}}}(F_{\tau\sigma}+T_{D1}b_{\tau\sigma})\right)-\\&-e^{-2\Phi}N^{\tau}T_{D1}^{2}\tilde{n}^{2}\mathbf{a}_{\sigma\sigma}+T_{D1}\tilde{n}\tilde{C}_{\tau\sigma}\,.\end{aligned} \tag{5.13}$$

Finally solving equation of motion for N^{τ} and inserting the result into (5.13) we obtain final form of the Lagrangian density for D1-brane in stringy NC background

$$\mathcal{L}=-\tilde{n}e^{-\Phi}T_{D1}\sqrt{-\det\mathbf{a}}\sqrt{\mathbf{a}^{\alpha\beta}\bar{h}_{\alpha\beta}-\frac{1}{\sqrt{-\det\mathbf{a}}}\epsilon^{\alpha\beta}(F_{\alpha\beta}+T_{D1}b_{\alpha\beta})}+T_{D1}\tilde{n}\tilde{C}_{\tau\sigma} \tag{5.14}$$

This is the Lagrangian density for $\tilde{n}-$ D1-branes in stringy NC background.

Analysis of the consistency of the background fields is the same as in section 4 since we consider stringy NC background. On the other hand RR two form $C^{(2)}$ scales as ω we should demand that its field strength vanishes in order to have finite stress energy tensor. Then clearly the equations of motion for $C^{(2)}$ are obeyed as well.

6 Conclusion

Let us outline results derived in this paper and suggest possible extension of this work. The main goal of this paper was to analyze properties of D1-brane theory in the limit that defines stringy NC background from relativistic theory [14]. We firstly considered D1-brane theory with fixed gauge invariance that defines (m,n)-string and we found Lagrangian density for (m,n)-string in stringy NC background. Then we studied conditions that the background fields have to obey in order to define consistent theory. We also found that the (m,n)-string in stringy NC background is manifestly $\mathrm{SL}(2,\mathbb{Z})$ invariant as a consequence of $\mathrm{SL}(2,\mathbb{Z})$ invariance of the parent Type IIB theory.

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We further analyzed the problem of general D1-brane in stringy NC background. We showed that in order to have non-trivial D1-brane theory we should scale the number of D1-branes and RR two form in an appropriate way. We find corresponding Hamiltonian for D1-brane with dynamical gauge field and corresponding Lgrangian density that is manifestly gauge invariant. We mean that this is nice result that brings new insight into the definition of the action for D1-branes in stringy NC gravity.

This work can be extended in several directions. For example, it would be nice to analyze limit when the string coupling goes to infinity in more details. Further, we would like to analyze T-duality along world-volume direction of D1-brane in NC theory. With analogy with standard relativistic string theory we should expect that resulting object is D0-brane in T-dual theory. It would be nice to analyze similar situation in stringy NC theory in more details. We hope to return to this problem in future.

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