Published for SISSA by Springer

Received: February 6, 2019
Accepted: April 3, 2019
Published: April 8, 2019

OPE selection rules for Schur multiplets in 4D $\mathcal{N} = 2$ superconformal field theories

Kazuki Kiyoshige[a] **and Takahiro Nishinaka**[b]

[a] *Department of Physics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi, Osaka 558-8585, Japan*
[b] *Department of Physical Sciences, College of Science and Engineering, Ritsumeikan University, Shiga 525-8577, Japan*

E-mail: kiyoshig@sci.osaka-cu.ac.jp, nishinak@fc.ritsumei.ac.jp

Abstract: We compute general expressions for two types of three-point functions of (semi-)short multiplets in four-dimensional $\mathcal{N} = 2$ superconformal field theories. These (semi-)short multiplets are called "Schur multiplets" and play an important role in the study of associated chiral algebras. The first type of the three-point functions we compute involves two half-BPS Schur multiplets and an arbitrary Schur multiplet, while the second type involves one stress tensor multiplet and two arbitrary Schur multiplets. From these three-point functions, we read off the corresponding OPE selection rules for the Schur multiplets. Our results particularly imply that there are non-trivial selection rules on the quantum numbers of Schur operators in these multiplets. We also give a conjecture on the selection rules for general Schur multiplets.

Keywords: Conformal Field Theory, Extended Supersymmetry, Superspaces

ArXiv ePrint: 1812.06394

https://doi.org/10.1007/JHEP04(2019)060

Contents

---

1 Introduction

The space of four-dimensional $\mathcal{N}=2$ superconformal field theories (SCFTs) has a rich structure. The best known $\mathcal{N}=2$ SCFTs are the $\mathrm{SU}(N_c)$ gauge theories with N_f fundamental matter hypermultiplets where $N_f = 2N_c$ is satisfied so that the beta function vanishes. While these theories are well-described by Lagrangian, there are many $\mathcal{N}=2$ SCFTs whose Lagrangian description is not known, such as Argyres-Douglas SCFTs [1–3],[1] Minahan-Nemeschanskey theories [14, 15], and an infinite series of non-Lagrangian SCFTs of class $\mathcal{S}$ [16]. To study general $\mathcal{N}=2$ SCFTs including these non-Lagrangian theories, we need a technique that relies only on the symmetry and unitarity of SCFTs.

Recently there was important progress in this direction. The authors of [17] showed that the operator product expansions (OPEs) of a special class of BPS local operators are naturally encoded in a two-dimensional chiral algebra. These BPS operators are called "Schur operators" since they contribute to the Schur limit of the superconformal index [17–19]. We call the superconformal multiplets including a Schur operator "Schur multiplets."

The existence of the associated chiral algebra implies, along with the four-dimensional unitarity and superconformal symmetry, that the "c central charge" of any interacting four-dimensional $\mathcal{N}=2$ SCFTs is constrained by $c \geq \frac{11}{30}$ [20], which is saturated by the minimal Argyres-Douglas SCFT.[2] Moreover, a similar analysis for $\mathcal{N}=2$ SCFTs with a flavor symmetry leads to a universal bound involving c and the flavor central charge [17, 21]. For more recent works on the associated chiral algebras, see [22–50].

Let us briefly sketch how these bounds on the central charges were derived from the chiral algebra analysis. In four-dimensional $\mathcal{N}=2$ SCFTs, the superconformal symmetry and the unitarity impose strong constraints on superconformal multiplets appearing in

[1] For recent discussions on $\mathcal{N}=1$ Lagrangians that flow to the Argyres-Douglas SCFTs, see [4–13].

[2] Our normalization of the four-dimensional central charge is such that a free hypermultiplet has $c=\frac{1}{12}$ and $a=\frac{1}{24}$.

OPEs, which we call "selection rules" in the following. In particular, the selection rules

$$\begin{aligned}
\widehat{\mathcal{B}}_1 \times \widehat{\mathcal{B}}_1 &\sim \widehat{\mathcal{B}}_1 + \widehat{\mathcal{B}}_2 + \sum_{\ell=0}^{\infty} \widehat{\mathcal{C}}_{0(\frac{\ell}{2},\frac{\ell}{2})} + \sum_{\ell=0}^{\infty} \widehat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})} + \cdots\,,\\
\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{C}}_{0(0,0)} &\sim \sum_{\ell=0}^{\infty} \widehat{\mathcal{C}}_{0(\ell,\ell)} + \sum_{\ell=0}^{\infty} \widehat{\mathcal{C}}_{1(\ell+\frac{1}{2},\ell+\frac{1}{2})} + \cdots\,,\\
\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{B}}_1 &\sim \widehat{\mathcal{B}}_1 + \sum_{\ell=0}^{\infty} \widehat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})} + \widehat{\mathcal{C}}_{0(\frac{\ell+1}{2},\frac{\ell+1}{2})} + \cdots\,,
\end{aligned} \tag{1.1}$$

were used to derive the central charge bounds mentioned above. Here, $\widehat{\mathcal{B}}_R$ and $\widehat{\mathcal{C}}_{R(j,\bar{j})}$ are two types of Schur multiplets labeled by the $SU(2)_R$ charge R and the spin $(j,\bar{j})$ of the superconformal primary field, and the ellipses stand for non-Schur multiplets.[3] In particular, the $\widehat{\mathcal{B}}_1$ multiplet is a Schur multiplet including a flavor current, and the $\widehat{\mathcal{C}}_{0(0,0)}$ multiplet is the stress tensor multiplet. The above selection rules (and unitarity) are crucial in deriving the central charge bounds. For example, the bound $c \geq \frac{11}{30}$ was derived by interpreting the reality of the OPE coefficients for the second selection rule in (1.1) in terms of the two-dimensional chiral algebra.[4] This implies that identifying the selection rules for Schur multiplets provides a powerful tool to reveal universal constraints on general $\mathcal{N}=2$ SCFTs.

Moreover, the selection rules are also important in recovering four-dimensional OPEs from the two-dimensional chiral algebra. Indeed, the 4d/2d correspondence of [17] implies that Schur operators with different quantum numbers could correspond to two-dimensional operators with the same quantum numbers. Therefore, it is generically non-trivial to recover four-dimensional OPEs from two-dimensional OPEs. The selection rules, however, strongly constrain Schur multiplets appearing in the four-dimensional OPEs and therefore will be useful for reconstructing the four-dimensional OPEs from the associated chiral algebra.

In this paper, we study the selection rules for

$$\widehat{\mathcal{C}}_{0(0,0)} \times \mathcal{O}^{\text{Schur}}\,, \tag{1.2}$$

up to non-Schur multiplets, where $\mathcal{O}^{\text{Schur}}$ is an arbitrary Schur multiplet. Since the Schur operator in the stress tensor multiplet $\widehat{\mathcal{C}}_{0(0,0)}$ maps to the Virasoro stress tensor in the associated chiral algebra, the selection rules for (1.2) are particularly important in the study of the 4d/2d correspondence. In particular, they reveal how the four-dimensional operator associated with a Virasoro primary is related to those of the Virasoro descendants. Indeed we find that, when four-dimensional Schur operators $\mathcal{O}$ and $\mathcal{O}'$ respectively correspond to a Virasoro primary and its descendant in the associated chiral algebra, the $SU(2)_R$ charge of $\mathcal{O}$ is always smaller than or equal to that of $\mathcal{O}'$ (see for example (4.31)). Note that the selection rules for $\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{C}}_{0(0,0)}$ and $\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{B}}_1$ were already identified respectively in [20] and [51], which we generalize to (1.2) for all Schur multiplets $\mathcal{O}^{\text{Schur}}$ in this paper.

[3] For the precise definitions of $\widehat{\mathcal{B}}_R$ and $\widehat{\mathcal{C}}_{R(j,\bar{j})}$, see section 2.

[4] In this interpretation, it was assumed that $\widehat{\mathcal{C}}_{0(\frac{\ell}{2},\frac{\ell}{2})}$ for $\ell > 0$ are absent in interacting $\mathcal{N}=2$ SCFTs since they involve higher spin currents.

JHEP04(2019)060

To derive the above selection rules, we study three-point functions of the form $\langle\widehat{\mathcal{C}}_{0(0,0)}\mathcal{O}_1\mathcal{O}_2\rangle$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are arbitrary Schur multiplets. Our strategy is to write down the most general ansatz for the three-point functions and then impose the (semi-)shortening conditions corresponding to the Schur multiplets. The same strategy was employed in [20, 51, 52] to compute several three-point functions. We stress that, since our analysis relies only on the (semi-)shortening conditions which purely follow from the superconformal algebra, our results are applicable to any four-dimensional $\mathcal{N}=2$ SCFT.

Before studying the selection rules for $\widehat{\mathcal{C}}_{0(0,0)}\times\mathcal{O}^{\text{Schur}}$, we first apply our strategy to the selection rules for

$$\widehat{\mathcal{B}}_{R_1}\times\widehat{\mathcal{B}}_{R_2}\,, \tag{1.3}$$

as a warm-up. While these rules were already identified in [53],[5] we believe it is worth showing an explicit derivation of the rules. Moreover, we evaluate the most general expressions for the three-point functions $\langle\widehat{\mathcal{B}}_{R_1}\widehat{\mathcal{B}}_{R_2}\mathcal{O}\rangle$ with $\mathcal{O}$ being an arbitrary Schur multiplet, which contain more information than the selection rules.

Let us here make an observation on our OPE selection rules. For some of the OPEs we study in this paper, the three-point function of the corresponding superconformal primary fields turns out to vanish even though three-point functions involving their descendants do not. This reflects the fact that the sum of the $\mathrm{U}(1)_r$ charges of the superconformal primary fields in three-point functions is non-vanishing. On the other hand, we find that the sum of the $\mathrm{U}(1)_r$ charges of *Schur operators* in these multiplets always vanishes, which suggests that the Schur operators play a central role in Schur multiplets. With this observation, we give a conjecture on the OPE selection rules for general Schur multiplets in section 5.

The outline of this paper is as follow. In section 2, we review the four-dimensional $\mathcal{N}=2$ superconformal algebra and the (semi-)shortening conditions for the Schur multiplets, and also introduce a useful formalism [52, 54, 55] to analyze the superconformal three-point functions. In sections 3 and 4, we derive the two types of three-point functions $\langle\widehat{\mathcal{B}}_{R_1}\widehat{\mathcal{B}}_{R_2}\mathcal{O}^{\mathcal{I}}\rangle$ and $\langle\widehat{\mathcal{C}}_{0(0,0)}\mathcal{O}_1^{\mathcal{I}_1}\mathcal{O}_2^{\mathcal{I}_2}\rangle$ respectively. From these correlation functions, we present the $\widehat{\mathcal{B}}_{R_1}\times\widehat{\mathcal{B}}_{R_2}$ selection rule and the $\widehat{\mathcal{C}}_{0(0,0)}\times\mathcal{O}^{\text{Schur}}$ selection rules. section 5 is devoted to conclusions and discussions, where we conjecture more general selection rules between Schur multiplets as a natural generalization of our results. In appendix A, we summarize the nilpotent structure of the Grassmann variables what we call Fierz identities, and appendices B, C, and D are the details of our calculations.

2 (Semi-)shortening conditions and three-point functions

In this section, we review the four-dimensional $\mathcal{N}=2$ superconformal algebra and the short multiplets following [56] and introduce a useful formalism constructed in [52, 54, 55] for the computations of correlation functions of SCFTs. We follow the convention of [17] unless otherwise stated.

[5] See eq. (3.44) in particular.

JHEP04(2019)060

2.1 Superconformal shortening and semi-shortening conditions

The four-dimensional $\mathcal{N}=2$ superconformal algebra is the superalgebra $\mathfrak{su}(2,2|2)$, whose generators are the dilatation, translations, special conformal transformations, Lorentz transformations, Poincaré supercharges Q^i_α and $\widetilde{Q}_{\dot\alpha i}$, conformal supercharges S^α_i and $\widetilde{S}^{\dot\alpha i}$, and $\mathrm{SU}(2)_R\times \mathrm{U}(1)_r$ charges $\mathcal{R}^i{}_j$ and r. Here $\alpha=\pm$ and $\dot\alpha=\dot\pm$ are the Weyl spinor indices and $i,j=1,2$ are $\mathrm{SU}(2)_R$ indices.

A general long multiplet of the four-dimensional $\mathcal{N}=2$ superconformal algebra is labeled by five eigenvalues of the Cartan subalgebra for the primary state, namely, the conformal dimension Δ, the Lorentz spin $(j,\jmath)$, the irreducible representation R of $\mathrm{SU}(2)_R$, and the $\mathrm{U}(1)_r$ charge r.[6] Here, the superconformal primary field is defined as a state annihilated by all conformal supercharges, S^α_i and $\widetilde{S}^{\dot\alpha i}$. We denote the superconformal primary field by $|\Delta,r\rangle^{(i_1\cdots i_{2R})}_{(\alpha_1\cdots\alpha_{2j})(\dot\alpha_1\cdots\dot\alpha_{2\jmath})}$, where the parentheses in the scripts such as $(i_1\cdots i_{2R_1})$ denote the total symmetrization of the indices. Acting Q^i_α and $\widetilde{Q}_{\dot\alpha i}$ on the primary, we can generate $256(2R+1)(2j+1)(2\jmath+1)$ components of the long multiplet. These long multiplets satisfy unitarity bounds,

$$\Delta\geq E_i\,,\qquad\qquad \text{if}\quad j_i\neq 0\,,\tag{2.1}$$

$$\Delta=E_i-2\quad\text{or}\quad \Delta\geq E_i\,,\qquad \text{if}\quad j_i=0\,,\tag{2.2}$$

where E_i and j_i are defined by

$$E_1:=2R+2+2j_1+r\,,\qquad E_2:=2R+2+2j_2-r\,,\qquad j_1:=j\,,\qquad j_2:=\jmath\,.\tag{2.3}$$

An $\mathcal{N}=2$ Poincaré supersymmetric field theory has a unitarity bound called the BPS bound, and if a long multiplet saturates the bound it becomes a short multiplet whose number of components is half the original one. The above unitarity bounds for $\mathcal{N}=2$ SCFTs play a similar role; when the eigenvalues $(\Delta,R,r,j,\jmath)$ saturate the unitarity bounds (2.1) or (2.2), a long multiplet is shortened. If we consider the case of $j=0,\Delta=E_1-2$, the superconformal primary field $|\Delta\rangle^{(i_1\cdots i_{2R})}_{(\dot\alpha_1\cdots\dot\alpha_{2\jmath})}$ satisfies the condition

$$\mathcal{B}^1:\ Q^{(i}_\alpha|\Delta\rangle^{i_1\cdots i_{2R})}_{(\dot\alpha_1\cdots\dot\alpha_{2\jmath})}=0\,,\quad \text{for}\quad \alpha=\pm.\tag{2.4}$$

Similarly, for $\jmath=0,\Delta=E_2-2$, the superconformal primary field satisfies the condition

$$\bar{\mathcal{B}}^2:\widetilde{Q}^{(i}_{\dot\alpha}|\Delta\rangle^{i_1\cdots i_{2R})}_{(\alpha_1\cdots\alpha_{2j})}=0\,,\quad \text{for}\quad \dot\alpha=\dot\pm.\tag{2.5}$$

These two conditions $\mathcal{B}^1$ and $\bar{\mathcal{B}}^2$ are called shortening conditions. On the other hand, for $\Delta=E_1$, the following condition is possible:

$$\mathcal{C}^1:\begin{cases}\epsilon^{\alpha\beta}Q^{(i}_\alpha|\Delta\rangle^{i_1\cdots i_{2R})}_{(\beta\alpha_2\cdots\alpha_{2j})}=0\,, & \text{for}\quad j>0\,,\\[2ex] \epsilon^{\alpha\beta}Q^{(i}_\alpha Q^{i'}_\beta|\Delta\rangle^{i_1\cdots i_{2R})}=0\,, & \text{for}\quad j=0\,.\end{cases}\tag{2.6}$$

[6]We take R so that the Dynkin label for the irreducible $\mathrm{SU}(2)_R$ representation is $2R$.

JHEP04(2019)060

Multiplet	Condition	Conformal dimension and $\mathrm{U}(1)_r$ charge of the primary
$\widehat{\mathcal{B}}_R$	$\mathcal{B}^1 \cap \bar{\mathcal{B}}^2$	$\Delta = 2R, \qquad r = 0,$
$\mathcal{D}_{R(0,\bar{\jmath})}$	$\mathcal{B}^1 \cap \bar{\mathcal{C}}^2$	$\Delta = 2R + \bar{\jmath} + 1, \qquad r = \bar{\jmath} + 1,$
$\bar{\mathcal{D}}_{R(j,0)}$	$\mathcal{C}^1 \cap \bar{\mathcal{B}}^2$	$\Delta = 2R + j + 1, \qquad r = -j - 1,$
$\widehat{\mathcal{C}}_{R(j,\bar{\jmath})}$	$\mathcal{C}^1 \cap \bar{\mathcal{C}}^2$	$\Delta = 2R + j + \bar{\jmath} + 2, \qquad r = \bar{\jmath} - j,$

Table 1. The list of Schur multiplets. Their shortening conditions are shown in the middle column. The rightmost column denotes the conformal dimension and the $\mathrm{U}(1)_r$ charge of the superconformal primary field. Here $(j, \bar{\jmath})$ is the Lorentz spin of the superconformal primary field field.

Similarly for $\Delta = E_2$, we can impose

$$\bar{\mathcal{C}}^2 : \begin{cases} \epsilon^{\dot{\alpha}\dot{\beta}} \widetilde{Q}^{(i}_{\dot{\alpha}} |\Delta\rangle^{i_1 \cdots i_{2R})}_{(\dot{\beta}\dot{\alpha}_2 \cdots \dot{\alpha}_{2\bar{\jmath}})} = 0\,, & \text{for} \quad \bar{\jmath} > 0\,, \\ \epsilon^{\dot{\alpha}\dot{\beta}} \widetilde{Q}^{(i}_{\dot{\alpha}} \widetilde{Q}^{i'}_{\dot{\beta}} |\Delta\rangle^{i_1 \cdots i_{2R})} = 0\,, & \text{for} \quad \bar{\jmath} = 0. \end{cases} \tag{2.7}$$

These two conditions $\mathcal{C}^1$ and $\bar{\mathcal{C}}^2$ are called semi-shortening conditions.

The Schur multiplets are defined as multiplets satisfying a shortening condition or a semi-shortening condition for each of the chiralities. There are four types of Schur multiplet, which are denoted as $\widehat{\mathcal{B}}_R$, $\mathcal{D}_{R(0,\bar{\jmath})}$, $\bar{\mathcal{D}}_{R(j,0)}$, and $\widehat{\mathcal{C}}_{R(j,\bar{\jmath})}$ in the notation of [56]. See table 1 for the definition of the four types and the relations among the quantum numbers of their superconformal primary states.

Some of these Schur multiplets contain important operators. For example, the $\widehat{\mathcal{C}}_{0(0,0)}$ multiplet contains the stress-tensor operator and the $\mathrm{SU}(2)_R \times \mathrm{U}(1)_r$ conserved current operator. We can regard the semi-shortening conditions as the conservation equations of these operators. In this paper, we assume that the theory we are considering has a unique stress-tensor multiplet $\widehat{\mathcal{C}}_{0(0,0)}$. This assumption leads to a constraint on the three-point functions involving two stress tensor multiplets, as discussed later. The $\widehat{\mathcal{C}}_{0(j,\bar{\jmath})}$ multiplet is a higher-spin generalization of the stress-tensor multiplet $\widehat{\mathcal{C}}_{0(0,0)}$ and contains a higher spin current operator. It is expected that interacting (S)CFTs containing such a higher spin current have a decoupled free sector [57–59]. On the other hand, the $\widehat{\mathcal{B}}_R$ multiplets are half-BPS multiplets whose superconformal primary field is annihilated by both ${Q_\alpha}^1$ and $\widetilde{Q}_{\dot{\alpha}2}$. As mentioned in section 1, the $\widehat{\mathcal{B}}_1$ multiplet particularly contains a conserved flavor current. Finally, the $\mathcal{D}_{0(0,0)} \oplus \bar{\mathcal{D}}_{0(0,0)}$ multiplet is an $\mathcal{N} = 2$ free vector multiplet, whose superconformal primary field has the conformal dimension 1, and its (semi-)shortening conditions imply the massless equation of motion.

2.2 Superspace formalism

In this subsection, we review a useful superspace formalism following [52, 54, 55]. We denote by $z := (x^\mu, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i})$ the coordinate of the $\mathcal{N} = 2$ superspace. We then define chiral/anti-chiral variables $x_\pm^\mu$ and derivatives $D_\alpha^i, \bar{D}_{\dot\alpha i}$ as[7]

$$x_\pm^\mu := x^\mu \pm \mathrm{i}\theta_i^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha i}\,,$$
$$D_\alpha^i := \frac{\partial}{\partial\theta_i^\alpha} + \mathrm{i}(\sigma^\mu\bar\theta^i)_\alpha \frac{\partial}{\partial x^\mu}\,, \qquad \bar{D}_{\dot\alpha i} := -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} - \mathrm{i}(\theta_i\sigma^\mu)_{\dot\alpha}\frac{\partial}{\partial x^\mu}\,.$$

In the superspace formalism, the (semi-)shortening conditions reviewed above are expressed in terms of the covariant derivatives D_α^i and $\bar{D}_{\dot\alpha i}$. For example, let $\mathcal{L}_{(i_1\cdots i_{2R})}(z)$ be a superfield for the $\widehat{\mathcal{B}}_R$ multiplet. Then the shortening conditions for $\widehat{\mathcal{B}}_R$ imply that any correlation function involving $\mathcal{L}_{(i_1\cdots i_{2R})}(z_1)$ satisfies

$$\begin{aligned} D^\alpha{}_{(i}\left\langle \mathcal{L}_{i_1\cdots i_{2R})}(z)\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\cdots\right\rangle &= 0\,,\\ \bar{D}^{\dot\alpha}{}_{(i}\left\langle \mathcal{L}_{i_1\cdots i_{2R})}(z)\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\cdots\right\rangle &= 0\,, \end{aligned} \tag{2.8}$$

where $\mathcal{O}_1(z_1)$, $\mathcal{O}_2(z_2)$, $\cdots$ are superfields for general superconformal multiplets. In the rest of this section, we review basic techniques to solve differential equations of this form. We use them in sections 3 and 4 to derive selection rules for the Schur multiplets.

To that end, we first introduce the following chiral and anti-chiral variables for given two points z_1, z_2 in the superspace:

$$x^\mu_{\bar{1}2} := -x^\mu_{2\bar{1}} := x^\mu_{1-} - x^\mu_{2+} + 2\mathrm{i}\theta_{1i}\sigma^\mu\bar\theta_2^i\,,$$
$$\theta_{12} := \theta_1 - \theta_2\,, \qquad \bar\theta_{12} := \bar\theta_1 - \bar\theta_2\,,$$

where $x^\mu_{\bar{1}2}$ is anti-chiral for z_1 and chiral for z_2. Next, we introduce superconformal covariant $\mathrm{U}(2)_R$ and $\mathrm{SU}(2)_R$ matrices respectively as

$$u_i{}^j(z_{12}) := \delta_i^j + 4\mathrm{i}\theta_{12i}\tilde{x}^{-1}_{\bar{1}2}\bar\theta^j_{12}\,, \qquad \widehat{u}_i{}^j(z_{12}) := \left(\frac{x^2_{\bar{2}1}}{x^2_{\bar{1}2}}\right)^{\frac{1}{2}} u_i{}^j(z_{12})\,, \qquad \det\widehat{u}_i{}^j(z_{12}) = 1\,, \tag{2.9}$$

where $z_{12} := (x^\mu_{\bar{1}2}, \theta^\alpha_{12\,i}, \bar\theta^{\dot\alpha i}_{12})$. These matrices satisfy the relations

$$u_i{}^j(z_{12})^\dagger u_j{}^k(z_{12}) = u_i{}^j(z_{21})u_j{}^k(z_{12}) = \delta_i^k\,, \tag{2.10}$$
$$\hat{u}_{ij}(z_{12}) = -\hat{u}_{ji}(z_{21})\,, \tag{2.11}$$

where $\hat{u}_{ij}(z_{12}) := \widehat{u}_i{}^k(z_{12})\epsilon_{kj}$,[8] with ϵ_{kj} being the $\mathrm{SU}(2)_R$ invariant tensor. We also define a spin $\mathrm{SL}(2,\mathbb{C})$ covariant matrix

$$I_{\alpha\dot\alpha}(x_{1\bar{2}}) := \mathrm{i}\frac{x_{1\bar{2}\alpha\dot\alpha}}{\sqrt{x^2_{1\bar{2}}}}. \tag{2.12}$$

These matrices (2.9) and (2.12) are building blocks of superconformal two-point functions.

[7] Following the notation of [60], we define spinorial variables $x_{\alpha\dot\alpha} = x^\mu(\sigma_\mu)_{\alpha\dot\alpha}\,, \tilde{x}^{\dot\alpha\alpha} = x^\mu(\bar\sigma_\mu)^{\dot\alpha\alpha}$ and the short-handed notations $x^2 = x^\mu x_\mu = -\frac{1}{2}\mathrm{tr}(x\tilde{x})\,, (\tilde{x}^{-1})_{\alpha\dot\alpha} = -\frac{1}{x^2}x_{\alpha\dot\alpha}$. We also use $\theta = (\theta^\alpha)$, $\tilde\theta = (\theta_\alpha) = \epsilon_{\alpha\beta}\theta^\beta$, $\bar\theta = (\bar\theta^{\dot\alpha})$ and $\tilde{\bar\theta} = (\bar\theta_{\dot\alpha}) = \epsilon^{\dot\alpha\dot\beta}\bar\theta_{\dot\beta}$. Complex conjugate of Grassmann variable is $(\theta_i^\alpha)^* = \bar\theta^{\dot\alpha i}$.

[8] Except these $\hat{u}_{ij}(z_{12})$, all other $\mathrm{SU}(2)_R$ indices are raised and lowered as $C^i = \epsilon^{ij}C_j$, $C_i = \epsilon_{ij}C^j$.

Next, for given different three points z_1, z_2, and z_3 in superspace, we introduce the superconformally covariant variable $\boldsymbol{Z}_1 = (X_{1\alpha\dot{\alpha}}, \Theta_1^{i\alpha}, \bar{\Theta}_{1i}^{\dot{\alpha}})$ by

$$X_1 := \tilde{x}_{1\bar{2}}^{-1}\tilde{x}_{\bar{2}3}\tilde{x}_{3\bar{1}}^{-1}\,, \qquad \bar{X}_1 := X_1^\dagger = -\tilde{x}_{1\bar{3}}^{-1}\tilde{x}_{\bar{3}2}\tilde{x}_{2\bar{1}}^{-1}\,, \tag{2.13}$$

$$\tilde{\Theta}_{1\alpha}^i := \mathrm{i}\left((\tilde{x}_{1\bar{3}}^{-1})_{\alpha\dot{\alpha}}\bar{\theta}_{13}^{\dot{\alpha}i} - (\tilde{x}_{1\bar{2}}^{-1})_{\alpha\dot{\alpha}}\bar{\theta}_{12}^{\dot{\alpha}i}\right)\,, \tag{2.14}$$

$$\tilde{\bar{\Theta}}_{1\dot{\alpha}i} := \mathrm{i}\left(\theta_{12i}^{\alpha}(\tilde{x}_{\bar{1}2}^{-1})_{\alpha\dot{\alpha}} - \theta_{13i}^{\alpha}(\tilde{x}_{\bar{1}3}^{-1})_{\alpha\dot{\alpha}}\right)\,. \tag{2.15}$$

The variable $\boldsymbol{Z}_1$ transforms similarly to z_1. In particular, the following identity will be important in our calculations below:

$$X_{\alpha\dot{\alpha}} - \bar{X}_{\alpha\dot{\alpha}} = 4\mathrm{i}\tilde{\Theta}_\alpha^i\tilde{\bar{\Theta}}_{\dot{\alpha}i}\,. \tag{2.16}$$

Similar variables $\boldsymbol{Z}_2$ and $\boldsymbol{Z}_3$ are defined as cyclically permuting z_1, z_2 and z_3 in the above definition.[9] These $\boldsymbol{Z}$ variables play central role in expressing three-point functions. We also define $\mathrm{SU}(2)_R$ matrices using $\boldsymbol{Z}_1$ as

$$\boldsymbol{u}_i{}^j(\boldsymbol{Z}_1) := u_i{}^k(z_{12})u_k{}^l(z_{23})u_l{}^j(z_{31}) = \delta_i^j - 4\mathrm{i}\tilde{\bar{\Theta}}_{1i}X_1^{-1}\tilde{\Theta}_1^j\,, \tag{2.17}$$

$$\boldsymbol{u}_i{}^{j\dagger}(\boldsymbol{Z}_1) = \boldsymbol{u}^{-1}(\boldsymbol{Z}_1) = \delta_i^j + 4\mathrm{i}\tilde{\bar{\Theta}}_{1i}\bar{X}_1^{-1}\tilde{\Theta}_1^j\,, \tag{2.18}$$

$$\hat{\boldsymbol{u}}_i{}^j(\boldsymbol{Z}_1) := \left(\frac{\bar{X}_1^2}{X_1^2}\right)^{1/2}\boldsymbol{u}_i{}^j(\boldsymbol{Z}_1)\,. \tag{2.19}$$

2.3 (Semi-)shortening conditions for three-point functions

Let us now consider a three-point function of three quasi superfields $\Phi_{\mathcal{I}_i}(z_i)$ for $i = 1, 2, 3$. We denote their conformal dimension and $\mathrm{U}(1)_r$ charge by $(q_i+\bar{q}_i)$ and $(\bar{q}_i-q_i)$ respectively. The subscript $\mathcal{I}_i$ expresses $\mathrm{SU}(2)_R$ and $\mathrm{SL}(2,\mathbb{C})$ indices collectively. A general three-point function $\langle\Phi_{\mathcal{I}_1}(z_1)\Phi_{\mathcal{I}_2}(z_2)\Phi_{\mathcal{I}_3}(z_3)\rangle$ is written as

$$\langle\Phi_{\mathcal{I}_1}(z_1)\Phi_{\mathcal{I}_2}(z_2)\Phi_{\mathcal{I}_3}(z_3)\rangle = \frac{T_{\mathcal{I}_1}^{\mathcal{J}_1}\left[\hat{u}(z_{13}), I(x_{1\bar{3}}, x_{3\bar{1}})\right]T_{\mathcal{I}_2}^{\mathcal{J}_2}\left[\hat{u}(z_{23}), I(x_{2\bar{3}}, x_{3\bar{2}})\right]}{\left(x_{1\bar{3}}^2\right)^{q_1}\left(x_{\bar{1}3}^2\right)^{\bar{q}_1}\left(x_{2\bar{3}}^2\right)^{q_2}\left(x_{\bar{2}3}^2\right)^{\bar{q}_2}}H_{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}(\boldsymbol{Z}_3)\,, \tag{2.20}$$

where $T_{\mathcal{I}_1}^{\mathcal{J}_1}$ and $T_{\mathcal{I}_2}^{\mathcal{J}_2}$ are some functions composed of (2.9) and (2.12) in the representation of $\mathrm{SU}(2)_R \times \mathrm{SL}(2,\mathbb{C})$ specified by $\mathcal{I}_1$ and $\mathcal{I}_2$. On the other hand, the function $H(\boldsymbol{Z}_3)$ satisfies following homogeneity property

$$H_{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}(\lambda\bar{\lambda}X, \lambda\Theta, \bar{\lambda}\bar{\Theta}) = \lambda^{2a}\bar{\lambda}^{2\bar{a}}H_{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}(X, \Theta, \bar{\Theta})\,, \tag{2.21}$$

where a and $\bar{a}$ are fixed by

$$a - 2\bar{a} = \bar{q}_1 + \bar{q}_2 - q_3\,, \qquad \bar{a} - 2a = q_1 + q_2 - \bar{q}_3\,. \tag{2.22}$$

[9]The conformal dimension of $(X_{\alpha\dot{\alpha}}, \Theta^{i\alpha}, \bar{\Theta}_i^{\dot{\alpha}})$ is $(1, \frac{1}{2}, \frac{1}{2})$, and the $\mathrm{U}(1)_r$ charge is $(0, \frac{1}{2}, -\frac{1}{2})$, respectively.

JHEP04(2019)060

Therefore $H(\boldsymbol{Z}_3)$ has the conformal dimension $q_3+\bar{q}_3-(q_1+\bar{q}_1)-(q_2+\bar{q}_2)$ and the $\mathrm{U}(1)_r$ charge $(\bar{q}_3-q_3+\bar{q}_1-q_1+\bar{q}_2-q_2)$. The function $H(\boldsymbol{Z}_3)$ is not fully determined by the global superconformal symmetry. When some of the $\Phi_{\mathcal{I}_i}(z_i)$ correspond to a (semi-)short multiplet, their shortening conditions restrict the form of $H(\boldsymbol{Z}_3)$. For example in [20, 51, 52], $H(\boldsymbol{Z}_3)$ is determined in such cases, up to an overall constant.

The formalism (2.20) is very useful when we consider the (semi-)shortening conditions such as (2.8). Since the prefactor in (2.20)

$$\frac{T_{\mathcal{I}_1}^{\mathcal{J}_1}\left[\hat{u}(z_{13}),I(x_{1\bar{3}},x_{3\bar{1}})\right]T_{\mathcal{I}_2}^{\mathcal{J}_2}\left[\hat{u}(z_{23}),I(x_{2\bar{3}},x_{3\bar{2}})\right]}{\left(x_{\bar{1}3}^2\right)^{q_1}\left(x_{\bar{1}\bar{3}}^2\right)^{\bar{q}_1}\left(x_{2\bar{3}}^2\right)^{q_2}\left(x_{\bar{2}3}^2\right)^{\bar{q}_2}}\,, \tag{2.23}$$

can be factorized into two-point functions $\langle\Phi_{\mathcal{I}_1}(z_1)\bar{\Phi}^{\mathcal{J}_1}(z_3)\rangle$ and $\langle\Phi_{\mathcal{I}_2}(z_2)\bar{\Phi}^{\mathcal{J}_2}(z_3)\rangle$, it trivially satisfies the (semi-)shortening conditions. This implies that the (semi-)shortening conditions only constrain the function $H(\boldsymbol{Z}_3)$. Therefore, hereafter, we focus on $H(\boldsymbol{Z}_3)$. It is easy to find that D_1, $\bar{D}_1$, D_2 and $\bar{D}_2$ act on $H(\boldsymbol{Z}_3)$ as

$$\begin{aligned}
D^i_{1\alpha}H(\boldsymbol{Z}_3)&=-\mathrm{i}(\tilde{x}^{-1}_{\bar{3}1})_{\alpha\dot{\beta}}u_j{}^i(z_{31})\bar{\mathcal{D}}^{\dot{\beta}j}H(\boldsymbol{Z}_3)\,,\\
\bar{D}_{1\dot{\alpha}i}H(\boldsymbol{Z}_3)&=\mathrm{i}(\tilde{x}^{-1}_{\bar{1}3})_{\beta\dot{\alpha}}u_i{}^j(z_{13})\mathcal{D}^{\beta}_j H(\boldsymbol{Z}_3)\,,\\
D^i_{2\alpha}H(\boldsymbol{Z}_3)&=\mathrm{i}(\tilde{x}^{-1}_{\bar{3}2})_{\alpha\dot{\beta}}u_j{}^i(z_{32})\bar{\mathcal{Q}}^{\dot{\beta}j}H(\boldsymbol{Z}_3)\,,\\
\bar{D}_{2\dot{\alpha}i}H(\boldsymbol{Z}_3)&=-\mathrm{i}(\tilde{x}^{-1}_{\bar{2}3})_{\beta\dot{\alpha}}u_i{}^j(z_{23})\mathcal{Q}^{\beta}_j H(\boldsymbol{Z}_3)\,,
\end{aligned} \tag{2.24}$$

where derivatives $\bar{\mathcal{D}}^{j\dot{\beta}},\mathcal{D}^\beta_j,\mathcal{Q}^\alpha_i$ and $\bar{\mathcal{Q}}^{\dot{\alpha}i}$ are defined respectively as

$$\begin{aligned}
\bar{\mathcal{D}}^{\dot{\alpha}i}&:=\frac{\partial}{\partial\tilde{\bar{\Theta}}_{3\dot{\alpha}i}}\,,\quad \mathcal{D}^\alpha_i:=\frac{\partial}{\partial\tilde{\Theta}^i_{3\alpha}}+4\mathrm{i}\tilde{\bar{\Theta}}_{3i\dot{\alpha}}\frac{\partial}{\partial X_{3\alpha\dot{\alpha}}}\,,\\
\mathcal{Q}^\alpha_i&:=\frac{\partial}{\partial\tilde{\Theta}^i_{3\alpha}}\,,\quad \bar{\mathcal{Q}}^{\dot{\alpha}i}:=\frac{\partial}{\partial\tilde{\bar{\Theta}}_{3\dot{\alpha}i}}-4\mathrm{i}\tilde{\Theta}^i_{3\alpha}\frac{\partial}{\partial X_{3\alpha\dot{\alpha}}}.
\end{aligned} \tag{2.25}$$

Moreover, quadratic derivatives such as $D^{(i}_{1\alpha}D^{i')\alpha}_1H(\boldsymbol{Z}_3)$ are also concisely written in terms of $\bar{\mathcal{D}}^{j\dot{\beta}},\mathcal{D}^\beta_j,\mathcal{Q}^\alpha_i$ and $\bar{\mathcal{Q}}^{\dot{\alpha}i}$. For instance,

$$D^{(i}_{1\alpha}D^{i')\alpha}_1H(\boldsymbol{Z}_3)=-\frac{u_j{}^i(z_{31})u_{j'}{}^{i'}(z_{31})}{x^2_{\bar{3}1}}\bar{\mathcal{D}}_{\dot{\beta}}{}^{j'}\bar{\mathcal{D}}^{\dot{\beta}j}H(\boldsymbol{Z}_3)\,. \tag{2.26}$$

Therefore, the (semi-)shortening conditions are now translated into partial differential equations of $H(\boldsymbol{Z}_3)$ with respect to $\boldsymbol{Z}_3$.

While the (semi-)shortening conditions of the first and second superfields, $\Phi_{\mathcal{I}_1}(z_1)$ and $\Phi_{\mathcal{I}_2}(z_2)$, are easily expressed as partial differential equations for $H(\boldsymbol{Z}_3)$, it is not straightforward to translate the conditions for the third superfield $\Phi_{\mathcal{I}_3}(z_3)$ into a similar equation for $H(\boldsymbol{Z}_3)$. To consider the (semi-)shortening conditions of the third superfield, we change the variable from $\boldsymbol{Z}_3$ to $\boldsymbol{Z}_2$.[10] Indeed, using the cyclicity of z_1,z_2 and z_3, the correlation function (2.20) is also expressed as

$$\frac{T'^{\mathcal{J}_1}_{\mathcal{I}_1}\left[\hat{u}(z_{12}),I(x_{1\bar{2}},x_{2\bar{1}})\right]T'^{\mathcal{J}_3}_{\mathcal{I}_3}\left[\hat{u}(z_{32}),I(x_{3\bar{2}},x_{3\bar{2}})\right]}{\left(x_{\bar{1}2}^2\right)^{q_1}\left(x_{\bar{1}\bar{2}}^2\right)^{\bar{q}_1}\left(x_{3\bar{2}}^2\right)^{q_3}\left(x_{\bar{3}2}^2\right)^{\bar{q}_3}}G_{\mathcal{J}_1\mathcal{I}_2\mathcal{J}_3}(\boldsymbol{Z}_2)\,, \tag{2.27}$$

[10] Here, we can also use $\boldsymbol{Z}_1$ instead of $\boldsymbol{Z}_2$.

JHEP04(2019)060

for some function $G(\boldsymbol{Z}_2)$. The action of $D^i_{3\alpha}$ and $\bar{D}_{3\dot{\alpha}i}$ on the $G(\boldsymbol{Z}_2)$ are given by

$$D^i_{3\alpha}G(\boldsymbol{Z}_2) = -\mathrm{i}(\tilde{x}^{-1}_{\bar{2}3})_{\alpha\dot{\beta}}u_j{}^i(z_{23})\bar{\mathcal{S}}^{\dot{\beta}j}G(\boldsymbol{Z}_2)\,, \tag{2.28}$$

$$\bar{D}_{3\dot{\alpha}i}G(\boldsymbol{Z}_2) = \mathrm{i}(\tilde{x}^{-1}_{\bar{3}2})_{\beta\dot{\alpha}}u_i{}^j(z_{32})\mathcal{S}^{\beta}_j G(\boldsymbol{Z}_2)\,, \tag{2.29}$$

where the derivatives are now defined by

$$\mathcal{S}^{\alpha}_i := \frac{\partial}{\partial\tilde{\Theta}^i_{2\,\alpha}} + 4\mathrm{i}\tilde{\bar{\Theta}}_{i\dot{\alpha}}\frac{\partial}{\partial X_{2\,\alpha\dot{\alpha}}}\,, \quad \bar{\mathcal{S}}^{\dot{\alpha}i} := \frac{\partial}{\partial\tilde{\bar{\Theta}}_{2\,\dot{\alpha}i}} + 4\mathrm{i}\tilde{\Theta}^i_{2\,\alpha}\frac{\partial}{\partial\bar{X}_{2\,\alpha\dot{\alpha}}}\,. \tag{2.30}$$

As shown in [52, 55], $\boldsymbol{Z}_3$ and $\boldsymbol{Z}_2$ are related as

$$\begin{aligned} \tilde{x}_{\bar{2}3}X_3\tilde{x}_{\bar{3}2} &= -(\bar{X}_2)^{-1}\,, & \tilde{x}_{\bar{2}3}\bar{X}_3\tilde{x}_{\bar{3}2} &= -(X_2)^{-1}\,, \\ \tilde{x}_{\bar{3}2}\tilde{\Theta}^i_2 u_i{}^j(z_{23}) &= -X_3^{-1}\tilde{\Theta}^j_3\,, & u_i{}^j(z_{32})\tilde{\bar{\Theta}}_{2j}\tilde{x}_{\bar{2}3} &= \tilde{\bar{\Theta}}_{3i}\bar{X}_3^{-1}\,. \end{aligned} \tag{2.31}$$

Using these relations, we see that the function $G_{\mathcal{J}_1\mathcal{I}_2\mathcal{J}_3}(\boldsymbol{Z}_2)$ is related to $H_{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}(\boldsymbol{Z}_3)$ by

$$G_{\mathcal{J}_1\mathcal{I}_2\mathcal{J}_3}(\boldsymbol{Z}_2) = \frac{T^{\mathcal{L}}_{\mathcal{J}_1}\left[\hat{\boldsymbol{u}}^{\dagger}(Z_2), I(X_2,\bar{X}_2)\right]}{\left(\bar{X}_2^2\right)^{\bar{q}_1}\left(X_2^2\right)^{q_1}}H_{\mathcal{L}\mathcal{I}_2\mathcal{J}_3}\left(\bar{X}_2^{-1}, -\mathrm{i}\bar{X}_2^{-1}\tilde{\Theta}_2, \mathrm{i}\tilde{\bar{\Theta}}_2 X_2^{-1}\right). \tag{2.32}$$

It is important to consider the third superfield conditions since it is insufficient to fix the function $H(\boldsymbol{Z}_3)$ only considering the first and second superfields of the (semi-)shortening conditions in section 4.2.

In the following sections, we will use the above formalism and techniques to study the three-point functions of Schur multiplets.

3 $\widehat{\boldsymbol{\mathcal{B}}}_{\boldsymbol{R_1}} \times \widehat{\boldsymbol{\mathcal{B}}}_{\boldsymbol{R_2}}$ fusion

In this section, we study the most general expressions for three-point functions of two half-BPS Schur multiplets $\widehat{\mathcal{B}}_R$ and an arbitrary Schur multiplet $\mathcal{O}^{\mathcal{I}}$.[11] Our result is particularly consistent with the fusion rules for $\widehat{\mathcal{B}}_{R_1} \times \widehat{\mathcal{B}}_{R_2}$ which were first obtained in [53].

The general expression for the three-point function $\langle\widehat{\mathcal{B}}_{R_1}\widehat{\mathcal{B}}_{R_2}\mathcal{O}^{\mathcal{I}}\rangle$ is given by

$$\begin{aligned} \langle\mathcal{L}_{(i_1\cdots i_{2R_1})}(z_1)\mathcal{L}_{(j_1\cdots j_{2R_2})}(z_2)\mathcal{O}^{\mathcal{I}}(z_3)\rangle = &\frac{\widehat{u}_{i_1}{}^{l_1}(z_{13})\cdots\widehat{u}_{i_{2R_1}}{}^{l_{2R_1}}(z_{13})\widehat{u}_{j_1}{}^{m_1}(z_{23})\cdots\widehat{u}_{j_{2R_2}}{}^{m_{2R_2}}(z_{23})}{\left(x^2_{\bar{3}1}x^2_{\bar{1}3}\right)^{R_1}\left(x^2_{\bar{3}2}x^2_{\bar{2}3}\right)^{R_2}} \\ &\times H^{\mathcal{I}}_{(l_1\cdots l_{2R_1})(m_1\cdots m_{2R_2})}(\boldsymbol{Z}_3)\,, \end{aligned} \tag{3.1}$$

where $\mathcal{L}_{(i_1\cdots i_{2R})}(z)$ is the superfield of $\widehat{\mathcal{B}}_R$ multiplet, and the parentheses denote the total symmetrization of the indices. Hereafter, we will often omit the parentheses with the understanding that the indices associated with the same Latin and Greek alphabet letters are always totally symmetrized.

[11] By definition, $\mathrm{SU}(2)_R$ irreducible representation R of $\widehat{\mathcal{B}}_R$ must be $R \geq \frac{1}{2}$.

Each of the $\widehat{\mathcal{B}}_{R_1}$ and $\widehat{\mathcal{B}}_{R_2}$ multiplets satisfies two shortening conditions as shown in table 1. As we have mentioned in the previous section, the shortening conditions are translated into differential equations for $H(\boldsymbol{Z}_3)$. For the two $\widehat{\mathcal{B}}_R$ multiplets, the differential equations are written as

$$\mathcal{D}^{\alpha}_{(l} H^{\mathcal{I}}_{l_1\cdots l_{2R_1})(m_1\cdots m_{2R_2})}(\boldsymbol{Z}_3) = 0\,, \tag{3.2}$$

$$\bar{\mathcal{D}}^{\dot\alpha}_{(l} H^{\mathcal{I}}_{l_1\cdots l_{2R_1})(m_1\cdots m_{2R_2})}(\boldsymbol{Z}_3) = 0\,, \tag{3.3}$$

$$\mathcal{Q}^{\alpha}_{(m} H^{\mathcal{I}}_{m_1\cdots m_{2R_2})(l_1\cdots l_{2R_1})}(\boldsymbol{Z}_3) = 0\,, \tag{3.4}$$

$$\bar{\mathcal{Q}}^{\dot\alpha}_{(m} H^{\mathcal{I}}_{m_1\cdots m_{2R_2})(l_1\cdots l_{2R_1})}(\boldsymbol{Z}_3) = 0\,. \tag{3.5}$$

It is easy to solve (3.3) and (3.4), since these are merely first-order linear equations for Θ or $\bar\Theta$. In contrast, (3.2) and (3.5) contain both X and Θ (or $\bar\Theta$) derivations and therefore are more complicated. However, if we use the $\bar{\boldsymbol{Z}}_3 := (\bar X_3, \Theta_3, \bar\Theta_3)$ coordinate instead of $\boldsymbol{Z}_3 = (X_3, \Theta_3, \bar\Theta_3)$, the two equations (3.2) and (3.5) become simpler, because $\mathcal{D}^{\alpha}_i$ and $\bar{\mathcal{Q}}^{\dot\alpha i}$ are expressed in terms of $\bar{\boldsymbol{Z}}_3$ as

$$\mathcal{D}^{\alpha}_i = \frac{\partial}{\partial \tilde\Theta^i_{3\alpha}}\,, \qquad \bar{\mathcal{Q}}^{\dot\alpha i} = \frac{\partial}{\partial \tilde{\bar\Theta}_{3\dot\alpha i}}\,. \tag{3.6}$$

Indeed, the most general solution to (3.3) and (3.4) is simply expressed in terms of $\boldsymbol{Z}_3$ as

$$\begin{aligned} H^{\mathcal{I}}_{(l_1\cdots l_{2R_1})(m_1\cdots m_{2R_2})}(\boldsymbol{Z}_3) = {}& f^{\mathcal{I}}_{l_1\cdots l_{2R_1},m_1\cdots m_{2R_2}}(X_3) + \Theta^{\alpha}_{3m_1}\bar\Theta^{\dot\alpha}_{3l_1} g^{\mathcal{I}}_{l_2\cdots l_{2R_1},m_2\cdots m_{2R_2},\alpha\dot\alpha}(X_3) \\ & + \Theta_{3m_1}\Theta_{3m_2}\bar\Theta_{3l_1}\bar\Theta_{3l_2} h^{\mathcal{I}}_{l_3\cdots l_{2R_1},m_3\cdots m_{2R_2}}(X_3)\,, \end{aligned} \tag{3.7}$$

while that of (3.2) and (3.5) is written in terms of $\bar{\boldsymbol{Z}}_3$ as

$$\begin{aligned} H^{\mathcal{I}}_{(l_1\cdots l_{2R_1})(m_1\cdots m_{2R_2})}(\bar{\boldsymbol{Z}}_3) = {}& \bar f^{\mathcal{I}}_{l_1\cdots l_{2R_1},m_1\cdots m_{2R_2}}(\bar X_3) + \bar\Theta^{\dot\alpha}_{3m_1}\Theta^{\alpha}_{3l_1} \bar g^{\mathcal{I}}_{l_2\cdots l_{2R_1},m_2\cdots m_{2R_2},\alpha\dot\alpha}(\bar X_3) \\ & + \bar\Theta_{3m_1}\bar\Theta_{3m_2}\Theta_{3l_1}\Theta_{3l_2} \bar h^{\mathcal{I}}_{l_3\cdots l_{2R_1},m_3\cdots m_{2R_2}}(\bar X_3)\,. \end{aligned} \tag{3.8}$$

Here, we use the short-hand notation

$$\Theta_{3m_1}\Theta_{3m_2} := \Theta^{\alpha}_{3m_1}\epsilon_{\alpha\beta}\Theta^{\beta}_{3m_2}\,, \qquad \bar\Theta_{3m_1}\bar\Theta_{3m_2} := \bar\Theta^{\dot\alpha}_{3m_1}\epsilon_{\dot\alpha\dot\beta}\bar\Theta^{\dot\beta}_{3m_2}\,. \tag{3.9}$$

The above two expressions, (3.8) and (3.7), must be equal under the relation (2.16). Therefore, our strategy is to rewrite (3.8) in terms of $\boldsymbol{Z}_3$ by using (2.16) and restrict the parameters in the expression to be consistent with (3.7). This gives us the most general solution to the equations (3.2)–(3.5).

After solving (3.2)–(3.5), we have to check it also satisfies the (semi-)shortening conditions of the third superfield $\mathcal{O}^{\mathcal{I}}(z_3)$. In this process, we relate $H(\boldsymbol{Z}_3)$ to $G(\boldsymbol{Z}_2)$ using (2.32) and see if the $G(\boldsymbol{Z}_2)$ satisfies the differential equations corresponding to the third set of (semi-) shortening conditions.

Below, we apply this strategy to evaluate the most general expression for the three-point function $\langle \widehat{\mathcal{B}}_{R_1}\widehat{\mathcal{B}}_{R_2}\mathcal{O}^{\mathcal{I}}\rangle$.

3.1 $\langle \widehat{\mathcal{B}}_{R_1}\widehat{\mathcal{B}}_{R_2}\widehat{\mathcal{B}}_{R_3}\rangle$

Let us first consider the case of $\mathcal{O}^{\mathcal{I}}$ in the $\widehat{\mathcal{B}}_{R_3}$ multiplet. In this case, the function $H(\boldsymbol{Z}_3)$ has dimension $-2R := 2R_3 - 2R_1 - 2R_2$ and vanishing $\mathrm{U}(1)_r$ charge. Note here that R_1, R_2, and R_3 are constrained by the inequalities $0 \le R_1 + R_3 - R_2$ and $0 \le R_1 + R_2 - R_3$. In other words, R_3 must be such that

$$|R_1 - R_2| \le R_3 \le R_1 + R_2. \tag{3.10}$$

Moreover, since the primary of $\widehat{\mathcal{B}}_R$ is a scalar, so is the $H(\boldsymbol{Z}_3)$. Therefore, the function $H(\boldsymbol{Z}_3)$ only carries $\mathrm{SU}(2)_R$ indices. The most general ansatz for $H(\boldsymbol{Z}_3)$ is then written as[12]

$$\begin{aligned} H(\boldsymbol{Z}_3) = \frac{1}{(X_3^2)^R}\left(A'\epsilon_{l_1m_1}\epsilon_{l_2m_2} + B'\frac{M_{3m_1l_1}}{X_3^2}\epsilon_{l_2m_2} + C'\frac{M_{3m_1l_1}}{X_3^2}\frac{M_{3m_2l_2}}{X_3^2}\right)\epsilon_{l_3m_3}\cdots\epsilon_{l_Rm_R} \\ \times\left(\epsilon_{l_{R+1}k_1}\cdots\epsilon_{l_{2R_1}k_{2R_1-R}}\right)\left(\epsilon_{m_{R+1}k_{2R_1-R+1}}\cdots\epsilon_{m_{2R_2}k_{2R_3}}\right), \end{aligned} \tag{3.11}$$

where $R \in \mathbb{Z}_{\ge 0}$, and $k_1\cdots k_{2R_3}$ are the $\mathrm{SU}(2)_R$ indices associated with $\widehat{\mathcal{B}}_{R_3}$. As mentioned above, indices associated with the same Latin and Greek alphabet letters, such as $l_1\cdots l_{2R_1}$, are totally symmetrized. On the other hand, the same function should also be written in terms of $\bar{\boldsymbol{Z}}_3$. The most general expression in terms of $\bar{\boldsymbol{Z}}_3$ is given by

$$\begin{aligned} H(\bar{\boldsymbol{Z}}_3) = \frac{1}{(\bar{X}_3^2)^R}\left(A\epsilon_{l_1m_1}\epsilon_{l_2m_2} + B\frac{\bar{M}_{3l_1m_1}}{\bar{X}_3{}^2}\epsilon_{l_2m_2} + C\frac{\bar{M}_{3l_1m_1}}{\bar{X}_3{}^2}\frac{\bar{M}_{3l_2m_2}}{\bar{X}_3{}^2}\right)\epsilon_{l_3m_3}\cdots\epsilon_{l_Rm_R} \\ \times\left(\epsilon_{l_{R+1}k_1}\cdots\epsilon_{l_{2R_1}k_{2R_1-R}}\right)\left(\epsilon_{m_{R+1}k_{2R_1-R+1}}\cdots\epsilon_{m_{2R_2}k_{2R_3}}\right). \end{aligned} \tag{3.12}$$

For the above two expressions to be consistent, the coefficients A, B and C have to satisfy some conditions. To identify the conditions, we change the variables from $\bar{\boldsymbol{Z}}_3$ to $\boldsymbol{Z}_3$ in (3.12) by using (2.16). Using the Fierz identities summarized in appendix A, we see that the conditions are

$$B = 4\mathrm{i}RA\,, \quad C = -16\frac{R(R-1)}{2}A\,, \tag{3.13}$$

for an arbitrary constant A. Up to an overall constant, the function $H(\boldsymbol{Z}_3)$ is written as

$$H(\boldsymbol{Z}_3) = \frac{\boldsymbol{u}(\boldsymbol{Z}_3)_{l_1m_1}\cdots\boldsymbol{u}(\boldsymbol{Z}_3)_{l_Rm_R}}{(X_3^2)^R}\left(\epsilon_{l_{R+1}k_1}\cdots\epsilon_{l_{2R_1}k_{2R_1-R}}\right)\left(\epsilon_{m_{R+1}k_{2R_1-R+1}}\cdots\epsilon_{m_{2R_2}k_{2R_3}}\right). \tag{3.14}$$

This is the most general expression for $H(\boldsymbol{Z}_3)$ satisfying (3.2)–(3.5).

Although it satisfies the shortening conditions of the $\widehat{\mathcal{B}}_{R_1}$ and $\widehat{\mathcal{B}}_{R_2}$ multiplets, it is non-trivial whether the expression (3.14) satisfies the shortening conditions of the third multiplet $\widehat{\mathcal{B}}_{R_3}$. To check the shortening conditions for $\widehat{\mathcal{B}}_{R_3}$, let us relate the $H(\boldsymbol{Z}_3)$ to $G(\boldsymbol{Z}_2)$ using (2.32). Indeed, as reviewed above, the correlation function (3.1) is also written as

$$\frac{\widehat{u}_{i_1}{}^{l_1}(z_{12})\cdots\widehat{u}_{i_{2R_1}}{}^{l_{2R_1}}(z_{12})\widehat{u}_{k_1}{}^{n_1}(z_{32})\cdots\widehat{u}_{k_{2R_3}}{}^{n_{2R_3}}(z_{32})}{(x_{1\bar{2}}^2x_{2\bar{1}}^2)^{R_1}(x_{3\bar{2}}^2x_{2\bar{3}}^2)^{R_3}}G_{(l_1\cdots l_{2R_1})(j_1\cdots j_{2R_2})(n_1\cdots n_{2R_3})}(\boldsymbol{Z}_2). \tag{3.15}$$

[12] $M_{ml} := \Theta_m^{\ \alpha}X_{\alpha\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}{}_l$.

JHEP04(2019)060

The function $G(\boldsymbol{Z}_2)$ is uniquely fixed by $H(\boldsymbol{Z}_3)$ via (2.32), i.e.,

$$G(\boldsymbol{Z}_2) = \frac{\boldsymbol{u}^\dagger_{l_1 n_1}(\boldsymbol{Z}_2)\cdots\boldsymbol{u}^\dagger_{l_{R'}n_{R'}}(\boldsymbol{Z}_2)}{(\bar{X}_2^2)^{R'}}(\epsilon_{l_{R'+1}j_1}\cdots\epsilon_{l_{2R_1}j_R})(\epsilon_{j_{R+1}n_{R'+1}}\cdots\epsilon_{j_{2R_2}n_{2R_3}})\,, \tag{3.16}$$

where $R' := R_1 + R_3 - R_2$. Now, the shortening conditions for $\widehat{\mathcal{B}}_{R_3}$ are written as

$$\mathcal{S}^\alpha_{(n}G_{n_1\cdots n_{2R_3})}(\boldsymbol{Z}_2) = 0\,, \qquad \bar{\mathcal{S}}^{\dot\alpha}_{(n}G_{n_1\cdots n_{2R_3})}(\boldsymbol{Z}_2) = 0\,, \tag{3.17}$$

with $\mathcal{S}^\alpha_n$ and $\bar{\mathcal{S}}^{\dot\alpha n}$ defined by (2.30). Since (3.14) trivially satisfies the above two equations, the expression (3.16) also satisfies the shortening conditions for the third Schur multiplet $\widehat{\mathcal{B}}_{R_3}$.

In the rest of this paper, we omit the subscript 3 of X_3, Θ_3 and $\bar\Theta_3$ in the expression for $H(\boldsymbol{Z}_3)$. Similarly, we omit the subscript 2 of X_2, Θ_2 and $\bar\Theta_2$ in the expression for $G(\boldsymbol{Z}_2)$.

3.2 $\langle\widehat{\mathcal{B}}_{R_1}\widehat{\mathcal{B}}_{R_2}\bar{\mathcal{D}}_{R_3(j,0)}\rangle$

Let us turn to the case of $\mathcal{O}^{\mathcal{I}}$ in the $\bar{\mathcal{D}}_{R(j,0)}$ multiplet.[13] In this case, the function $H(\boldsymbol{Z}_3)$ has dimension $-2R+j+1$ and $\mathrm{U}(1)_r$ charge $-j-1$. Since the highest possible degree of $\bar\Theta$ in $H(\boldsymbol{Z}_3)$ is two (see (3.7)), the only possible value of j is $j=0$, and therefore the $\mathrm{U}(1)_r$ charge of $H(\boldsymbol{Z}_3)$ is indeed -1. From (3.7), we see that the most general expression for such $H(\boldsymbol{Z}_3)$ is given by

$$H(\boldsymbol{Z}_3) = A\frac{\bar\Theta_{l_1}\bar\Theta_{l_2}}{(X^2)^R}(\epsilon_{l_3m_1}\cdots\epsilon_{l_{R+1}m_{R-1}})(\epsilon_{l_{R+2}k_1}\cdots\epsilon_{l_{2R_1}k_{2R_1-R-1}})(\epsilon_{m_Rk_{2R_1-R}}\cdots\epsilon_{m_{2R_2}k_{2R_3}})\,, \tag{3.18}$$

where $k_1\cdots k_{2R_3}$ are the $\mathrm{SU}(2)_R$ indices for $\bar{\mathcal{D}}_{R_3(0,0)}$. As mentioned at the end of the previous subsection, $(X,\Theta,\bar\Theta)$ stands for $(X_3,\Theta_3,\bar\Theta_3)$ here. However, when we change the variables from $\boldsymbol{Z}_3$ to $\bar{\boldsymbol{Z}}_3$, this expression cannot be written in the form of (3.8).[14] This means that there are no solutions to (3.2)–(3.5). Therefore, the $\bar{\mathcal{D}}_{R(j,0)}$ multiplet does not appear in the $\widehat{\mathcal{B}}_{R_1}\times\widehat{\mathcal{B}}_{R_2}$ selection rule. Note that its conjugate implies that the $\mathcal{D}_{R(0,\bar\jmath)}$ multiplet also does not appear in the $\widehat{\mathcal{B}}_{R_1}\times\widehat{\mathcal{B}}_{R_2}$ fusion. By using the same argument, we can extend our results to that $H(\boldsymbol{Z}_3)$ for a non-vanishing correlation function $\langle\widehat{\mathcal{B}}_{R_1}\widehat{\mathcal{B}}_{R_2}\mathcal{O}\rangle$ must be $\mathrm{U}(1)_r$ neutral.

3.3 $\langle\widehat{\mathcal{B}}_{R_1}\widehat{\mathcal{B}}_{R_2}\widehat{\mathcal{C}}_{R_3(j,\bar\jmath)}\rangle$

Let us finally consider the case of $\mathcal{O}^{\mathcal{I}}$ in the $\widehat{\mathcal{C}}_{R_3(j,\bar\jmath)}$. By using the similar argument in previous section 3.2, $\bar\jmath = j$ is necessary for the three-point function to be non-vanishing. Therefore the function $H(\boldsymbol{Z}_3)$ has dimension $2-2R+2j$ and $\mathrm{U}(1)_r$ neutral, where we

[13] Note here that the result for $\mathcal{O}^{\mathcal{I}}$ in the $\mathcal{D}_{R(0,j)}$ multiplet follows from this case by CPT.
[14] In particular, $\bar\Theta_{l_1}\bar\Theta_{l_2}$ cannot be mapped to $\bar\Theta_{m_1}\bar\Theta_{m_2}$.

recall that $R := R_1 + R_2 - R_3$. The most general solution to (3.3) and (3.4) is written as

$$
\begin{aligned}
H(\boldsymbol{Z}_3) = \frac{1}{X^{2(R-1)}}\bigg[&\left(A'\epsilon_{l_1 m_1}\epsilon_{l_2 m_2} + B'\frac{M_{m_1 l_1}}{X^2}\epsilon_{l_2 m_2} + C'\frac{M_{m_1 l_1}}{X^2}\frac{M_{m_2 l_2}}{X^2}\right)X_{\beta_1\dot\beta_1}\\
&+ D'\tilde\Theta_{m_1\beta_1}\tilde{\bar\Theta}_{\dot\beta_1 l_1}\epsilon_{l_2 m_2}\bigg](\epsilon_{l_3 m_3}\cdots\epsilon_{l_R m_R})(\epsilon_{l_{R+1}k_1}\cdots\epsilon_{l_{2R_1}k_{2R_1-R}})\\
&\times(\epsilon_{m_{R+1}k_{2R_1-R+1}}\cdots\epsilon_{m_{2R_2}k_{2R_3}})\left(X_{\beta_2\dot\beta_2}\cdots X_{\beta_{2j}\dot\beta_{2j}}\right)\,,
\end{aligned}
\tag{3.19}
$$

where k_i and $(\beta_i, \dot\beta_i)$ are the $\mathrm{SU}(2)_R$ and $\mathrm{SL}(2,\mathbb{C})$ indices associated with $\widehat{\mathcal{C}}_{R_3(j,j)}$. On the other hand, the most general solution to (3.2) and (3.5) is written as

$$
\begin{aligned}
H(\bar{\boldsymbol{Z}}_3) = \frac{1}{\bar X^{2(R-1)}}\bigg[&\left(A\epsilon_{l_1 m_1}\epsilon_{l_2 m_2} + B\frac{\bar M_{l_1 m_1}}{\bar X^2}\epsilon_{l_2 m_2} + C\frac{\bar M_{l_1 m_1}}{\bar X^2}\frac{\bar M_{l_2 m_2}}{\bar X^2}\right)\bar X_{\beta_1\dot\beta_1}\\
&+ D\tilde\Theta_{l_1\beta_1}\tilde{\bar\Theta}_{\dot\beta_1 m_1}\epsilon_{l_2 m_2}\bigg](\epsilon_{l_3 m_3}\cdots\epsilon_{l_R m_R})(\epsilon_{l_{R+1}k_1}\cdots\epsilon_{l_{2R_1}k_{2R_1-R}})\\
&\times(\epsilon_{m_{R+1}k_{2R_1-R+1}}\cdots\epsilon_{m_{2R_2}k_{2R_3}})\left(\bar X_{\beta_2\dot\beta_2}\cdots \bar X_{\beta_{2j}\dot\beta_{2j}}\right)\,.
\end{aligned}
\tag{3.20}
$$

For the above two expressions to be consistent, the coefficients have to satisfy some conditions. Unless $R = 0$, the conditions are

$$
B = 4\mathrm{i}(R-1)A\,,\quad C = -16\frac{(R-1)(R-2-2j)}{2}A\,,\quad D = 4\mathrm{i}(2j)A\,. \tag{3.21}
$$

On the other hand, for $R = 0$ or equivalently $R_3 = R_1 + R_2$, all the coefficients have to vanish. Therefore, the three-point function $\langle\widehat{\mathcal{B}}_{R_1}\widehat{\mathcal{B}}_{R_2}\widehat{\mathcal{C}}_{R_3(j,j)}\rangle$ vanishes if $R_3 = R_1 + R_2$.

Next, we consider the semi-shortening conditions for $\widehat{\mathcal{C}}_{R_3(j,\bar j)}$. For that purpose, we relate $H(\boldsymbol{Z}_3)$ to $G(\boldsymbol{Z}_2)$ via (2.32). Indeed, the three-point function (3.1) can be rewritten as

$$
\begin{aligned}
&\frac{\left(\widehat{u}_{i_1}^{\ l_1}\cdots\widehat{u}_{i_{2R_1}}^{\ l_{2R_1}}(z_{12})\right)\left(\widehat{u}_{k_1}^{\ n_1}\cdots\widehat{u}_{k_{2R_3}}^{\ n_{2R_3}}(z_{32})\right)\left(I_{\delta_1\dot\beta_1}\cdots I_{\delta_{2j}\dot\beta_{2j}}(x_{3\bar2})\right)\left(I_{\beta_1\dot\delta_1}\cdots I_{\beta_{2j}\dot\delta_{2j}}(x_{2\bar3})\right)}{(x_{1\bar2}^2 x_{2\bar1}^2)^{R_1}(x_{3\bar2}^2 x_{2\bar3}^2)^{R_3+j+1}}\\
&\quad\times G^{(\delta_1\cdots\delta_{2j})(\dot\delta_1\cdots\dot\delta_{2j})}_{(l_1\cdots l_{2R_1})(j_1\cdots j_{2R_2})(n_1\cdots n_{2R_3})}(\boldsymbol{Z}_2)\,.
\end{aligned}
\tag{3.22}
$$

and the explicit form of $G(\boldsymbol{Z}_2)$ becomes

$$
\begin{aligned}
G(\boldsymbol{Z}_2) = \bigg[\bigg(&\boldsymbol{u}^\dagger_{l_1 j_1}\boldsymbol{u}^\dagger_{l_2 j_2} + 4\mathrm{i}(R-1)\frac{\bar M_{j_1 l_1}}{\bar X^2}\boldsymbol{u}^\dagger_{l_2 j_2}\\
&- \frac{16(R-1)(R-2-2j)}{2}\frac{\bar M_{j_1 l_1}}{\bar X^2}\frac{\bar M_{j_2 l_2}}{\bar X^2}\bigg)(\bar X^{-1})^{\dot\delta_1\delta_1}\\
&+ 4\mathrm{i}(2j)(\bar X^{-1}\tilde\Theta_{j_1})^{\dot\delta_1}(\tilde{\bar\Theta}_{l_1}\bar X^{-1})^{\delta_1}\boldsymbol{u}^\dagger_{l_2 j_2}\bigg]\\
&\times(\epsilon_{l_3 j_3}\cdots\epsilon_{l_R j_R})(\bar X^{-1})^{\dot\delta_2\delta_2}\cdots(\bar X^{-1})^{\dot\delta_{2j}\delta_{2j}}\\
&\times\frac{\boldsymbol{u}^\dagger_{l_{R+1}n_1}\cdots\boldsymbol{u}^\dagger_{l_{2R_1}n_{2R_1-R}}(\boldsymbol{Z}_2)}{(\bar X_2)^{2R_1-R+1}}(\epsilon_{j_{R+1}n_{2R_1-R+1}}\cdots\epsilon_{j_{2R_2}n_{2R_3}})\,.
\end{aligned}
\tag{3.23}
$$

In terms of $G(\boldsymbol{Z}_2)$, the semi-shortening conditions for $\widehat{\mathcal{C}}_{R_3(j,j)}$ are written as

$$\mathcal{S}^{\delta_1}_{(n}G_{n_1\cdots n_{2R_3})(\delta_1\cdots\delta_{2j})(\dot\delta_1\cdots\dot\delta_{2j})}=0\,,\qquad \bar{\mathcal{S}}^{\dot\delta_1}_{(n}G_{n_1\cdots n_{2R_3})(\delta_1\cdots\delta_{2j})(\dot\delta_1\cdots\dot\delta_{2j})}=0\,, \tag{3.24}$$

for $j>0$ and

$$\epsilon_{\alpha\beta}\mathcal{S}^{\alpha}_{(n}\mathcal{S}^{\beta}_{n'}\,G_{n_1\cdots n_{2R_3})}=0\,,\qquad \bar{\mathcal{S}}_{\dot\alpha(n}\bar{\mathcal{S}}^{\dot\alpha}_{n'}\,G_{n_1\cdots n_{2R_3})}=0\,, \tag{3.25}$$

for $j=0$. It is straightforward to check (3.23) satisfies these conditions.

In summary, the function $H(\boldsymbol{Z}_3)$ in $\langle\widehat{\mathcal{B}}_{R_1}\widehat{\mathcal{B}}_{R_2}\widehat{\mathcal{C}}_{R_3(j,j)}\rangle$ is given by, up to an overall constant

$$\begin{aligned}H(\boldsymbol{Z}_3)=\frac{1}{X^{2(R-1)}}\Bigg(&\epsilon_{l_1m_1}\epsilon_{l_2m_2}X_{\beta_1\dot\beta_1}+4\mathrm{i}(R-1)\frac{M_{m_1l_1}}{X^2}\epsilon_{l_2m_2}X_{\beta_1\dot\beta_1}\\&-16\frac{(R-1)(R-2-2j)}{2}\frac{M_{m_1l_1}}{X^2}\frac{M_{m_2l_2}}{X^2}X_{\beta_1\dot\beta_1}+4\mathrm{i}(2j)\tilde\Theta_{m_1\beta_1}\bar{\tilde\Theta}_{\dot\beta_1l_1}\epsilon_{l_2m_2}\Bigg)\\&\times(\epsilon_{l_3m_3}\cdots\epsilon_{l_Rm_R})(\epsilon_{l_{R+1}k_1}\cdots\epsilon_{l_{2R_1}k_{2R_1-R}})(\epsilon_{m_{R+1}k_{2R_1-R+1}}\cdots\epsilon_{m_{2R_2}k_{2R_3}})\\&\times\left(X_{\beta_2\dot\beta_2}\cdots X_{\beta_{2j}\dot\beta_{2j}}\right)\,,\end{aligned} \tag{3.26}$$

for any integer or half-integer $j\geq 0$ and $R_3<R_1+R_2$. Therefore, the $\widehat{\mathcal{C}}_{R_3(j,j)}$ multiplet appears in $\widehat{\mathcal{B}}_{R_1}\times\widehat{\mathcal{B}}_{R_2}$ fusion for $|R_1-R_2|\leq R_3\leq R_1+R_2-1$.

3.4 Selection rule

In the above subsections, we have computed the most general expression for non-vanishing three-point functions of the form $\langle\widehat{\mathcal{B}}_{R_1}\widehat{\mathcal{B}}_{R_2}\mathcal{O}\rangle$. From these results, we see that the selection rules for two $\widehat{\mathcal{B}}_R$ multiplets are written as

$$\widehat{\mathcal{B}}_{R_1}\times\widehat{\mathcal{B}}_{R_2}\sim\sum_{R=|R_1-R_2|>0}^{R_1+R_2}\widehat{\mathcal{B}}_R+\sum_{R=|R_1-R_2|}^{R_1+R_2-1}\sum_{\ell=0}^{\infty}\widehat{\mathcal{C}}_{R(\frac{\ell}{2},\frac{\ell}{2})}\,, \tag{3.27}$$

up to non-Schur multiplets. This is particularly consistent with eq. (3.44) of [53]. Especially, for $R_1=R_2=1$, the selection rule is written as

$$\widehat{\mathcal{B}}_1\times\widehat{\mathcal{B}}_1\sim\widehat{\mathcal{B}}_1+\widehat{\mathcal{B}}_2+\sum_{\ell=0}^{\infty}\left[\widehat{\mathcal{C}}_{0(\frac{\ell}{2},\frac{\ell}{2})}+\widehat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})}\right]\,, \tag{3.28}$$

which is consistent with the harmonic superspace analysis in [61].

4 $\widehat{\mathcal{C}}_{0(0,0)}\times\mathcal{O}^{\mathrm{Schur}}$ fusion

In this section, we turn to the selection rules for $\widehat{\mathcal{C}}_{0(0,0)}\times\mathcal{O}^{\mathrm{Schur}}$ for an arbitrary Schur multiplet $\mathcal{O}^{\mathrm{Schur}}$. These selection rules are important in studying the corresponding two-dimensional chiral algebra, since the highest weight component of the $\mathrm{SU}(2)_R$ current operator in the stress-tensor multiplet $\widehat{\mathcal{C}}_{0(0,0)}$ is mapped to the Virasoro stress-tensor operator in the chiral algebra [17]. To derive the selection rules, we compute the three-point functions of the form $\langle\widehat{\mathcal{C}}_{0(0,0)}\mathcal{O}^{\mathcal{I}_1}\mathcal{O}^{\mathcal{I}_2}\rangle$ for two Schur multiplets $\mathcal{O}^{\mathcal{I}_1}$ and $\mathcal{O}^{\mathcal{I}_2}$.

JHEP04(2019)060

Recall here that the stress-tensor multiplet $\widehat{\mathcal{C}}_{0(0,0)}$ has two semi-shortening conditions

$$\bar{\mathcal{D}}_{\dot\alpha}^{\;i}\bar{\mathcal{D}}^{\dot\alpha i'}H(\boldsymbol{Z}_3)=0\,, \tag{4.1}$$
$$\epsilon_{\alpha\beta}\mathcal{D}^{i\alpha}\mathcal{D}^{i'\beta}H(\boldsymbol{Z}_3)=0\,. \tag{4.2}$$

The most general solution to these two equations are respectively written as

$$H(\boldsymbol{Z}_3)=f(X,\Theta)+g^k_{\dot\alpha}(X,\Theta)\bar\Theta^{\dot\alpha}_k+h_{(\dot\alpha\dot\alpha')}(X,\Theta)\bar\Theta^{\dot\alpha\dot\alpha'}\,, \tag{4.3}$$
$$H(\bar{\boldsymbol{Z}}_3)=\tilde f(\bar X,\bar\Theta)+\tilde g_{k,\alpha}(\bar X,\bar\Theta)\Theta^{k\alpha}+\tilde h_{(\alpha\alpha')}(\bar X,\bar\Theta)\Theta^{\alpha\alpha'} \tag{4.4}$$

where

$$\Theta^{\alpha\alpha'}:=\Theta^{i\alpha}\epsilon_{ij}\Theta^{j\alpha'}\,,\quad \bar\Theta^{\dot\alpha\dot\alpha'}:=\bar\Theta^{\dot\alpha}{}_i\epsilon^{ij}\bar\Theta^{\dot\alpha'}{}_j\,. \tag{4.5}$$

For the above two expressions to be consistent, the functions $f,g,h,\tilde f,\tilde g$, and $\tilde h$ have to satisfy some conditions. Moreover, they are also constrained by the (semi-)shortening conditions associated with $\mathcal{O}^{\mathcal{I}_1}$ and $\mathcal{O}^{\mathcal{I}_2}$. Below, we solve all these conditions to find general expressions for $H(\boldsymbol{Z}_3)$. Since the concrete calculations are highly involved, we here write the results and, details of the computations are in appendices B, C, and D.

4.1 $\langle\widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{B}}_R\mathcal{O}^{\mathcal{I}}\rangle$

Let us first consider the three-point function $\langle\widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{B}}_R\mathcal{O}^{\mathcal{I}}\rangle$. We denote by $\mathcal{J}(z)$ the superfield of the stress-tensor multiplet $\widehat{\mathcal{C}}_{0(0,0)}$. The three-point function is then written as

$$\langle\mathcal{J}(z_1)\mathcal{L}_{(j_1\cdots j_{2R})}(z_2)\mathcal{O}^{\mathcal{I}}(z_3)\rangle=\frac{\widehat{u}_{j_1}^{\;m_1}(z_{23})\cdots\widehat{u}_{j_{2R}}^{\;m_{2R}}(z_{23})}{\left(x_{\bar31}^2x_{\bar13}^2\right)\left(x_{\bar32}^2x_{\bar23}^2\right)^R}\;H^{\mathcal{I}}_{(m_1\cdots m_{2R})}(\boldsymbol{Z}_3)\,. \tag{4.6}$$

The (semi-)shortening conditions for $\widehat{\mathcal{C}}_{0(0,0)}$ and $\widehat{\mathcal{B}}_R$ are encoded in (3.4), (3.5), (4.1), and (4.2). The three-point function is consistent with these four conditions only when $\mathcal{O}^{\mathcal{I}}$ is $\widehat{\mathcal{B}}_R$, $\widehat{\mathcal{C}}_{R(j,j)}$, or $\widehat{\mathcal{C}}_{R-1(j,j)}$. Up to an overall constant, the expressions for $H(\boldsymbol{Z}_3)$ in these three cases are written as

$$\widehat{\mathcal{B}}_R\;:\quad\frac{\boldsymbol{u}_{m_1k_1}(\boldsymbol{Z}_3)}{X^2}(\epsilon_{m_2k_2}\cdots\epsilon_{m_{2R}k_{2R}}), \tag{4.7}$$
$$\widehat{\mathcal{C}}_{R(j,j)}\;:\quad\left(\epsilon_{m_1k_1}X_{\beta_1\dot\beta_1}-4\mathrm{i}(2j)\tilde\Theta_{m_1\beta_1}\tilde{\bar\Theta}_{\dot\beta_1k_1}\right)(\epsilon_{m_2k_2}\cdots\epsilon_{m_{2R}k_{2R}})X_{\beta_2\dot\beta_2}\cdots X_{\beta_{2j}\dot\beta_{2j}}\,, \tag{4.8}$$
$$\widehat{\mathcal{C}}_{R-1(j,j)}\;:\quad\frac{1}{X^2}\left(\frac{M_{m_1m_2}}{X^2}X_{\beta_1\dot\beta_1}-2j\Theta_{m_1\beta_1}\tilde{\bar\Theta}_{\dot\beta_1m_2}+4\mathrm{i}j\frac{\Theta_{m_1}\Theta_{m_2}}{X^2}\tilde{\bar\Theta}_{\dot\beta_1i}(X\tilde{\bar\Theta}^i)_{\beta_1}\right)$$
$$\times(\epsilon_{m_3k_1}\cdots\epsilon_{m_{2R}k_{2R-2}})X_{\beta_2\dot\beta_2}\cdots X_{\beta_{2j}\dot\beta_{2j}}\,, \tag{4.9}$$

where k_i and $(\beta_i,\dot\beta_i)$ are respectively the $\mathrm{SU}(2)_R$ and $\mathrm{SL}(2,\mathbb{C})$ indices associated with the third multiplet. The derivations of these functions are given in appendix B.

We see that these are also consistent with the (semi-)shortening conditions for the third Schur multiplet. Indeed, using (2.32), we see that the function $G(\boldsymbol{Z}_2)$ corresponding to the above $H(\boldsymbol{Z}_3)$ is written as

$$\widehat{\mathcal{B}}_R \ : \ \frac{\boldsymbol{u}_{j_1 n_1}(\boldsymbol{Z}_2)}{X^2}(\epsilon_{j_2 n_2}\cdots\epsilon_{j_{2R}n_{2R}}), \tag{4.10}$$

$$\begin{aligned}\widehat{\mathcal{C}}_{R(j,j)} \ &: \ \left(\frac{\epsilon_{j_1 n_1}\tilde{\bar{X}}^{-1}_{\delta_1\dot{\delta}_1}}{X^2\bar{X}^2} + \frac{4\mathrm{i}(2j)(\Theta_{j_1}\tilde{\bar{X}}^{-1})_{\dot{\delta}_1}(\tilde{\bar{X}}^{-1}\bar{\Theta}_{n_1})_{\delta_1}}{\bar{X}^4}\right) \\ &\quad \times(\epsilon_{j_2 n_2}\cdots\epsilon_{j_{2R}n_{2R}})\tilde{\bar{X}}^{-1}_{\delta_2\dot{\delta}_2}\cdots\tilde{\bar{X}}^{-1}_{\delta_{2j}\dot{\delta}_{2j}}\,,\end{aligned} \tag{4.11}$$

$$\begin{aligned}\widehat{\mathcal{C}}_{R-1(j,j)} \ &: \ \frac{1}{\bar{X}^2}\left(\frac{\bar{M}_{j_1 j_2}}{\bar{X}^2}\bar{X}_{\delta_1\dot{\delta}_1} - 2j\tilde{\Theta}_{j_1\delta_1}\tilde{\bar{\Theta}}_{\dot{\delta}_1 j_2} + 4\mathrm{i}j\frac{\Theta_{j_1}\Theta_{j_2}}{\bar{X}^2}\tilde{\bar{\Theta}}_{\dot{\delta}_1 i}\left(\bar{X}\bar{\Theta}^i\right)_{\delta_1}\right) \\ &\quad \times(\epsilon_{j_3 n_1}\cdots\epsilon_{j_{2R}n_{2R-2}})\tilde{\bar{X}}^{-1}_{\delta_2\dot{\delta}_2}\cdots\tilde{\bar{X}}^{-1}_{\delta_{2j}\dot{\delta}_{2j}}\,.\end{aligned} \tag{4.12}$$

Here n_i and $(\delta_i, \dot{\delta}_i)$ are respectively the $SU(2)_R$ and $SL(2,\mathbb{C})$ indices associated with the third multiplet $\mathcal{O}^{\mathcal{I}}$. These equations are all consistent with the (semi-)shortening conditions for the third multiplet.

Let us briefly comment on the case of $\mathcal{O}^{\mathcal{I}}$ in $\widehat{\mathcal{C}}_{0(0,0)}$. When we assume that there is only one stress tensor in the theory, the corresponding three-point function $\langle\widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{B}}_1\widehat{\mathcal{C}}_{0(0,0)}\rangle$ has to be symmetric under the action of $\mathbb{Z}_2$ exchanging the first and the third multiplets. This $\mathbb{Z}_2$ symmetry implies that the function $G(\boldsymbol{Z}_2)$ is invariant under $(X_2, \Theta_2, \bar{\Theta}_2) \leftrightarrow (-\bar{X}_2, -\Theta_2, -\bar{\Theta}_2)$. However, the expression (4.12) is not invariant under this $\mathbb{Z}_2$ action. Therefore, in an SCFT with unique stress tensor multiplet, the three-point function $\langle\widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{B}}_1\widehat{\mathcal{C}}_{0(0,0)}\rangle$ must vanish [52].

Before closing this subsection, let us also make a quick comment on the correlation function $\langle\widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{B}}_R\widehat{\mathcal{C}}_{R-1(j,j)}\rangle$. In CFTs, any correlation function of *conformal* descendant fields is obtained by differentiating the correlation function of the conformal primary fields. This particularly implies that, when a correlation function of conformal primary fields vanishes, the corresponding descendant correlators also vanish. This, however, is not the case for *superconformal* descendants in SCFTs. Indeed, when we set all Grassmann variables, $\theta_{1,2,3}, \bar{\theta}_{1,2,3}$, to zero in (4.9), the correlation function vanishes. This shows that the correlator of the three superconformal primary fields in $\widehat{\mathcal{C}}_{0(0,0)}$, $\widehat{\mathcal{B}}_R$, and $\widehat{\mathcal{C}}_{R-1(j,j)}$ vanishes, while there are non-vanishing correlators involving superconformal descendants. This is a common feature of SCFTs [62].

4.2 $\langle\widehat{\mathcal{C}}_{0(0,0)}\bar{\mathcal{D}}_{R(j,0)}\mathcal{O}^{\mathcal{I}}\rangle$

Let us next consider $\langle\widehat{\mathcal{C}}_{0(0,0)}\bar{\mathcal{D}}_{R(j,0)}\mathcal{O}^{\mathcal{I}}\rangle$. We denote by $\bar{\mathcal{N}}_{(j_1\cdots j_{2R})(\alpha_1\cdots\alpha_{2j})}(z)$ the superfield of a $\bar{\mathcal{D}}_{R(j,0)}$ multiplet. The three-point function is written as

$$\begin{aligned}&\langle\mathcal{J}(z_1)\bar{\mathcal{N}}_{(j_1\cdots j_{2R})(\alpha_1\cdots\alpha_{2j})}(z_2)\mathcal{O}^{\mathcal{I}}(z_3)\rangle \\ &\qquad = \frac{(\widehat{u}_{j_1}{}^{m_1}\cdots\widehat{u}_{j_{2R}}{}^{m_{2R}}(z_{23}))(I_{\alpha_1\dot{\gamma}_1}\cdots I_{\alpha_{2j}\dot{\gamma}_{2j}}(x_{2\bar{3}}))}{x_{1\bar{3}}^2 x_{3\bar{1}}^2 (x_{2\bar{3}}^2)^{R+j+1}(x_{3\bar{2}}^2)^R} H^{(\dot{\gamma}_1\cdots\dot{\gamma}_{2j})\mathcal{I}}_{(m_1\cdots m_{2R})}(\boldsymbol{Z}_3)\,.\end{aligned} \tag{4.13}$$

In this case, the function $H(\boldsymbol{Z}_3)$ has to satisfy the semi-shortening conditions (4.1) and (4.2) associated with the stress tensor multiplet $\widehat{\mathcal{C}}_{0(0,0)}$, the shortening condition (3.4) of $\bar{\mathcal{D}}_{R(j,0)}$, and the following semi-shortening condition of $\bar{\mathcal{D}}_{R(j,0)}$:

$$\bar{\mathcal{Q}}_{\dot{\gamma}(m} H^{(\dot{\gamma}\dot{\gamma}_2\cdots\dot{\gamma}_{2j})\mathcal{I}}_{m_1\cdots m_{2R})}(\boldsymbol{Z}_3) = 0\,, \quad \text{for} \quad j>0\,, \tag{4.14}$$

$$\bar{\mathcal{Q}}_{\dot{\alpha}(m} \bar{\mathcal{Q}}^{\dot{\alpha}}_{m'} H^{\mathcal{I}}_{m_1\cdots m_{2R})}(\boldsymbol{Z}_3) = 0\,, \quad \text{for} \quad j=0. \tag{4.15}$$

We see that a non-vanishing solution to (4.1), (4.2), (3.4) and (4.14)/(4.15) exists if and only if $\mathcal{O}^{\mathcal{I}}$ is in the $\mathcal{D}_{R-\frac{1}{2}(0,j-\frac{1}{2})}$, $\mathcal{D}_{R(0,j)}$, $\mathcal{D}_{R-1(0,j)}$, $\widehat{\mathcal{C}}_{R-1(j_1,j+j_1+1)}$, $\widehat{\mathcal{C}}_{R+\frac{1}{2}(j_1,j+j_1+\frac{1}{2})}$, or $\widehat{\mathcal{C}}_{R-\frac{1}{2}(j_1,j+j_1+\frac{1}{2})}$ multiplets. Moreover, considering the (semi-)shortening conditions for each of these third multiplets, we find that the only possible third multiplets $\mathcal{O}^{\mathcal{I}}$ which can have a non-vanishing $H(\boldsymbol{Z}_3)$ are $\mathcal{D}_{R(0,j)}$, $\widehat{\mathcal{C}}_{R+\frac{1}{2}(j_1,j+j_1+\frac{1}{2})}$, and $\widehat{\mathcal{C}}_{R-\frac{1}{2}(j_1,j+j_1+\frac{1}{2})}$. The explicit expressions for $H(\boldsymbol{Z}_3)$ for these three cases are written, up to an overall constant, as

$$\begin{aligned}
\mathcal{D}_{R(0,j)} \;&:\; \frac{\boldsymbol{u}_{m_1k_1}(\boldsymbol{Z}_3)}{X^2}(\epsilon_{m_2k_2}\cdots\epsilon_{m_{2R}k_{2R}})(\epsilon_{\dot{\gamma}_1\dot{\beta}_1}\cdots\epsilon_{\dot{\gamma}_{2j}\dot{\beta}_{2j}})\,, \quad \text{for} \quad j\neq\frac{1}{2}\,,\\
\widehat{\mathcal{C}}_{R+\frac{1}{2}(j_1,j+j_1+\frac{1}{2})} \;&:\; \bar{\Theta}_{\dot{\beta}_1k_1}(\epsilon_{m_1k_2}\cdots\epsilon_{m_{2R}k_{2R+1}})(\epsilon_{\dot{\gamma}_1\dot{\beta}_2}\cdots\epsilon_{\dot{\gamma}_{2j}\dot{\beta}_{2j+1}})\\
&\quad\times X_{\beta_1\dot{\beta}_{2j+2}}\cdots X_{\beta_{2j_1}\dot{\beta}_{2j+2j_1+1}}\,,\\
\widehat{\mathcal{C}}_{R-\frac{1}{2}(j_1,j+j_1+\frac{1}{2})} \;&:\; \frac{\tilde{\bar{\Theta}}_{\dot{\beta}_1k}\boldsymbol{u}^k{}_{m_1}(\boldsymbol{Z}_3)}{X^2}(\epsilon_{m_2k_1}\cdots\epsilon_{m_{2R}k_{2R-1}})(\epsilon_{\dot{\gamma}_1\dot{\beta}_2}\cdots\epsilon_{\dot{\gamma}_{2j}\dot{\beta}_{2j+1}})\\
&\quad\times X_{\beta_1\dot{\beta}_{2j+2}}\cdots X_{\beta_{2j_1}\dot{\beta}_{2j+2j_1+1}}\,,
\end{aligned} \tag{4.16}$$

where $j_1 \geq 0$ is an integer or a half-integer, and k_i and $(\beta_i,\dot{\beta}_i)$ are respectively the $\mathrm{SU}(2)_R$ and $\mathrm{SL}(2,\mathbb{C})$ indices for the third multiplet. Note that the first expression for the case of $\mathcal{O}^{\mathcal{I}}$ in the $\mathcal{D}_{R(0,j)}$ multiplet is only for $j\neq 1/2$. In the case $j=1/2$, the function $H(\boldsymbol{Z}_3)$ has two independent terms as

$$\begin{aligned}
H(\boldsymbol{Z}_3) = \frac{1}{X^2}\Bigg(&A\boldsymbol{u}_{m_1k_1}(\boldsymbol{Z}_3)\epsilon_{\dot{\gamma}_1\dot{\beta}_1} + \frac{B}{X^2}\left((\Theta_{m_1}X)_{\dot{\gamma}_1}\tilde{\bar{\Theta}}_{\dot{\beta}_1k_1} - M_{m_1k_1}\epsilon_{\dot{\gamma}_1\dot{\beta}_1}\right)\Bigg)\\
&\times(\epsilon_{m_2k_2}\cdots\epsilon_{m_{2R}k_{2R}})\,,
\end{aligned} \tag{4.17}$$

where A and B are arbitrary constants. For the detail of derivations of (4.16) and (4.17), see appendix C.

Note here that the second and third lines of (4.16) are proportional to $\bar{\Theta}$, which means that the three-point functions of the superconformal primaries vanish. This can also be seen from the fact that the sum of the $\mathrm{U}(1)_r$ charges of the superconformal primaries does not vanish.

4.3 $\langle\widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{C}}_{R(j,\bar{\jmath})}\mathcal{O}^{\mathcal{I}}\rangle$

Let us finally consider the correlation function $\langle\widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{C}}_{R(j,\bar{\jmath})}\mathcal{O}^{\mathcal{I}}\rangle$. Since we have already studied the cases of $\mathcal{O}^{\mathcal{I}} = \widehat{\mathcal{B}}_{R'}$, $\mathcal{D}_{R'(0,\bar{\jmath})}$, and $\bar{\mathcal{D}}_{R'(j,0)}$, the only remaining case we have to study here is $\mathcal{O}^{\mathcal{I}} = \widehat{\mathcal{C}}_{R'(j_2,\bar{j}_2)}$, in which case the three-point function is written as

$$\begin{aligned}\Big\langle \mathcal{J}(z_1)\mathcal{J}^{(\alpha_1\cdots\alpha_{2j_1}),(\dot\alpha_1\cdots\dot\alpha_{2\bar{j}_1})}_{(j_1\cdots j_{2R})}(z_2)\mathcal{J}^{(\beta_1\cdots\beta_{2j_2}),(\dot\beta_1\cdots\dot\beta_{2\bar{j}_2})}_{(k_1\cdots k_{2R'})}(z_3)\Big\rangle \\ = \frac{\Big(\widehat{u}_{j_1}^{\ m_1}\cdots\widehat{u}_{j_{2R}}^{\ m_{2R}}(z_{23})\Big)(I_{\alpha_1\dot\gamma_1}\cdots I_{\alpha_{2j_1}\dot\gamma_{2j_1}}(x_{2\bar{3}}))\,(I_{\gamma_1\dot\alpha_1}\cdots I_{\gamma_{2\bar{j}_1}\dot\alpha_{2\bar{j}_1}}(x_{3\bar{2}}))}{x_{1\bar{3}}^2x_{3\bar{1}}^2(x_{2\bar{3}}^2)^{q_2}(x_{3\bar{2}}^2)^{\bar{q}_2}} \\ \times H^{(\gamma_1\cdots\gamma_{2\bar{j}_1})(\dot\gamma_1\cdots\dot\gamma_{2j_1})(\beta_1\cdots\beta_{2j_2})(\dot\beta_1\cdots\dot\beta_{2\bar{j}_2})}_{(m_1\cdots m_{2R})(k_1\cdots k_{2R'})}(\boldsymbol{Z}_3)\,,\end{aligned} \tag{4.18}$$

where $\mathcal{J}^{(\alpha_1\cdots\alpha_{2j}),(\dot\alpha_1\cdots\dot\alpha_{2\bar{j}})}_{(j_1\cdots j_{2R})}(z)$ is the superfield in the $\widehat{\mathcal{C}}_{R(j,\bar{j})}$ multiplet. In this case, the $H(\boldsymbol{Z}_3)$ has to satisfy the semi-shortening conditions (4.1) and (4.2) associated with $\widehat{\mathcal{C}}_{0(0,0)}$ and the semi-shortening conditions for the other two $\widehat{\mathcal{C}}_{R(j_1,\bar{j}_1)}$ multiplets. The semi-shortening conditions for the second multiplet, $\widehat{\mathcal{C}}_{R(j_1,\bar{j}_1)}$, are written as

$$\mathcal{Q}_{\gamma(m}H^{(\gamma\gamma_2\cdots\gamma_{2\bar{j}_1})(\dot\gamma_1\cdots\dot\gamma_{2j_1})\mathcal{I}}_{m_1\cdots m_{2R})} = 0 \quad \text{for} \quad \bar{j}_1 > 0\,, \tag{4.19a}$$

$$\mathcal{Q}_{\alpha(m}\mathcal{Q}^{\alpha}_{m'}H^{\mathcal{I}}_{m_1\cdots m_{2R})} = 0 \qquad \text{for} \quad \bar{j}_1 = 0\,, \tag{4.19b}$$

and

$$\bar{\mathcal{Q}}_{\dot\gamma(m}H^{(\gamma_1\cdots\gamma_{2\bar{j}_1})(\dot\gamma\dot\gamma_2\cdots\dot\gamma_{2j_1})\mathcal{I}}_{m_1\cdots m_{2R})} = 0 \quad \text{for} \quad j_1 > 0\,, \tag{4.20a}$$

$$\bar{\mathcal{Q}}_{\dot\alpha(m}\bar{\mathcal{Q}}^{\dot\alpha}_{m'}H^{\mathcal{I}}_{m_1\cdots m_{2R})} = 0 \qquad \text{for} \quad j_1 = 0\,, \tag{4.20b}$$

while the semi-shortening conditions for the third multiplet, $\widehat{\mathcal{C}}_{R'(j_2,\bar{j}_2)}$, are similarly expressed in terms of $G(\boldsymbol{Z}_2)$.

We have found that a non-vanishing $H(\boldsymbol{Z}_3)$ in (4.18) satisfying all these semi-shortening conditions is possible only for the following two types of correlator:[15]

$$\langle\widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{C}}_{R(j+\ell_1,j)}\widehat{\mathcal{C}}_{R(j+\ell_2,j+\ell_1+\ell_2)}\rangle\,, \tag{4.21}$$

$$\langle\widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{C}}_{R(j+\ell_1,j)}\widehat{\mathcal{C}}_{R+1(j+\ell_2,j+\ell_1+\ell_2)}\rangle\,, \tag{4.22}$$

up to charge conjugation, where ℓ_1 and ℓ_2 are non-negative (half-)integers. Note that the function $H(\boldsymbol{Z}_3)$ is $\mathrm{U}(1)_r$ neutral for all these cases. Since general solutions for $H(\boldsymbol{Z}_3)$ in these two cases are highly involved, it is beyond the scope of this paper to identify the most general expression for the allowed $H(\boldsymbol{Z}_3)$. However, we find a special solution for each of the above two types of correlators,[16] which is sufficient to identify the selection rule.

[15] We will show other type correlation functions does not satisfy semi-shortening conditions in appendix D.

[16] Our method is as follows. We first solve all the semi-shortening conditions for smaller spins $j = 1/2, 1$ and so on to find the explicit expression for $H(\boldsymbol{Z}_3)$. We then guess an ansatz (4.23) and (4.29) for general j, and check that the ansatz satisfies the (semi-)shortening conditions for arbitrary j.

Let us first focus on (4.21). Unless $j = \ell_1 = 0$, our special solution is written as

$$
\begin{aligned}
H(\boldsymbol{Z}_3) = &\frac{1}{X^2}(\epsilon_{m_2k_2}\cdots\epsilon_{m_{2R}k_{2R}})(\tilde{X}^{\dot\beta_{2j+1}\beta_{2j+1}}\ldots\tilde{X}^{\dot\beta_{2j+2\ell_2}\beta_{2j+2\ell_2}})\\
&\times(\epsilon^{\dot\gamma_{2j+1}\dot\beta_{2j+2\ell_2+1}}\ldots\epsilon^{\dot\gamma_{2j+2\ell_1}\dot\beta_{2j+2\ell_1+2\ell_2}})\left(\sum_{k=0}^{2j}(-1)^k\frac{(2\ell_2+2)_k}{(2\ell_1+2)_k}\binom{2j}{k}\right.\\
&\times\left[\epsilon_{m_1k_1}+4\mathrm{i}(2j+1)\left(\epsilon_{m_1k_1}(\bar{\tilde\Theta}_{\dot\gamma m}\tilde\Theta^m_\gamma)+\frac{2R}{2R+1}(\bar{\tilde\Theta}_{\dot\gamma k_1}\tilde\Theta_{m_1\gamma})\right)\frac{\tilde{X}^{\dot\gamma\gamma}}{X^2}\right.\\
&\left.-4(2j+1)(2j+2)\epsilon_{m_1k_1}\frac{\Theta_{\gamma\gamma'}\bar\Theta_{\dot\gamma\dot\gamma'}\tilde{X}^{\dot\gamma\gamma}\tilde{X}^{\dot\gamma'\gamma'}}{X^4}\right]\\
&\left.\times\left(\frac{\tilde{X}^{\dot\gamma_1\gamma_1}\tilde{X}^{\dot\beta_1\beta_1}}{X^2}\right)\cdots\left(\frac{\tilde{X}^{\dot\gamma_k\gamma_k}\tilde{X}^{\dot\beta_k\beta_k}}{X^2}\right)\left(\epsilon^{\gamma_{k+1}\beta_{k+1}}\epsilon^{\dot\gamma_{k+1}\dot\beta_{k+1}}\ldots\epsilon^{\gamma_{2j}\beta_{2j}}\epsilon^{\dot\gamma_{2j}\dot\beta_{2j}}\right)\right).
\end{aligned}
\tag{4.23}
$$

Here $(2\ell+2)_k := \frac{(2\ell+2+k-1)!}{(2\ell+2-1)!}$ is the Pochhammer symbol, and γ (and γ' when present) are contracted after totally symmetrized as $(\gamma\gamma'\gamma_1\cdots)$. Similarly, m is also contracted after totally symmetrized as $(mm_1m_2\cdots)$. On the other hand, in the case $j = \ell_1 = 0$, we find a solution with several free parameters:

$$
\begin{aligned}
H(\boldsymbol{Z}_3) = \frac{\epsilon_{m_1k_1}\cdots\epsilon_{m_{2R}k_{2R}}}{X^2}&\left[\left(A - B\frac{m_1}{X^2}+C\frac{m_1^2-m_2}{X^4}\right)\tilde{X}^{\dot\beta_1\beta_1}\tilde{X}^{\dot\beta_2\beta_2}\right.\\
&+2\ell_2\left((4\mathrm{i}A-B)\bar\Theta^{\dot\beta_1}\Theta^{\beta_1}+(4\mathrm{i}B-2C)\frac{\bar\Theta^{\dot\beta_1}_j M^j{}_l\Theta^{l\beta_1}-\bar\Theta^{\dot\beta_1}\Theta^{\beta_1}m_1}{X^2}\right)\tilde{X}^{\dot\beta_2\beta_2}\\
&\left.-\frac{2\ell_2(2\ell_2-1)}{2}(16A+8\mathrm{i}B-2C)\bar\Theta^{\dot\beta_1}\Theta^{\beta_1}\bar\Theta^{\dot\beta_2}\Theta^{\beta_2}\right]\left(\tilde{X}^{\dot\beta_3\beta_3}\ldots\tilde{X}^{\dot\beta_{2\ell_2}\beta_{2\ell_2}}\right),
\end{aligned}
\tag{4.24}
$$

where A, B, and C are arbitrary constants, and m_1 and m_2 are combinations of variables defined in (A.2).

Note that, for $j = \ell_1 = R = 0$, the three-point function $\langle\widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{C}}_{0(\ell_2,\ell_2)}\rangle$ has a $\mathbb{Z}_2$-symmetry, under the assumption of uniqueness of the stress tensor multiplet. The $\mathbb{Z}_2$ symmetry implies that the function $H(\boldsymbol{Z}_3)$ has to be invariant under $(X_3,\Theta_3,\bar\Theta_3)\leftrightarrow(-\bar{X}_3,-\Theta_3,-\bar\Theta_3)$. This condition constrains (4.24) as follows.

- If ℓ_2 is a half-integer such that $\ell_2 \geq \frac{3}{2}$, the $\mathbb{Z}_2$ symmetry implies $A = B = C = 0$. Therefore, no $\widehat{\mathcal{C}}_{0(\ell_2,\ell_2)}$ for such ℓ_2 appears in the $\widehat{\mathcal{C}}_{0(0,0)}\times\widehat{\mathcal{C}}_{0(0,0)}$ fusion.
- If ℓ_2 is an integer such that $\ell_2 \geq 1$, the $\mathbb{Z}_2$ symmetry implies $B = 2\mathrm{i}A, C = 0$, and therefore our solution (4.24) reduces to up to an over all constant

$$
\begin{aligned}
H(\boldsymbol{Z}_3) = \frac{A}{X^2}&\left[\left(1-2\mathrm{i}\frac{m_1}{X^2}\right)\tilde{X}^{\dot\beta_1\beta_1}\right.\\
&\left.+2\mathrm{i}(2\ell_2)\left(\bar\Theta^{\dot\beta_1}\Theta^{\beta_1}-4\mathrm{i}\frac{\bar\Theta^{\dot\beta_1}_j M^j{}_l\Theta^{l\beta_1}-\bar\Theta^{\dot\beta_1}\Theta^{\beta_1}m_1}{X^2}\right)\right]\\
&\times(\tilde{X}^{\dot\beta_2\beta_2}\ldots\tilde{X}^{\dot\beta_{2\ell_2}\beta_{2\ell_2}}).
\end{aligned}
\tag{4.25}
$$

- If $\ell_2 = \frac{1}{2}$, the $\mathbb{Z}_2$ symmetry implies $A = 0, C = 2\mathrm{i}B$, which reduces our solution (4.24) to

$$H(\boldsymbol{Z}_3) = \frac{A}{X^2}\left(\left(\frac{m_1}{X^2} - 2\mathrm{i}\frac{m_2 - m_1^2}{X^2}\right)\tilde{X}^{\dot{\beta}_1\beta_1} + \bar{\Theta}^{\dot{\beta}_1}\Theta^{\beta_1}\right). \tag{4.26}$$

- If $\ell_2 = 0$, the $\mathbb{Z}_2$ symmetry implies $B = 2\mathrm{i}A$. Therefore our solution (4.24) reduces to

$$H(\boldsymbol{Z}_3) = \frac{A}{X^2}\left(1 - 2\mathrm{i}\frac{m_1}{X^2}\right) + \frac{C}{X^2}\left(\frac{m_1^2 - m_2}{X^4}\right). \tag{4.27}$$

  This corresponds to the three-point functions of the stress tensor multiplet and is consistent with [52] (see also [20] for its implication in the associated two-dimensional algebras). As shown in the paper, the two constants A and C are related to the conformal anomalies, a and c, of the four-dimensional SCFTs as

$$A = \frac{3}{32\pi^6}(4a - c)\,, \quad C = \frac{1}{8\pi^6}(4a - 5c)\,. \tag{4.28}$$

Let us now turn to the second type of correlator, (4.22). In this case, the function $H(\boldsymbol{Z}_3)$ has dimension $2\ell_2$ and vanishing $\mathrm{U}(1)_r$ charge. We see that there is no non-trivial solution for $\ell_2 = 0$. For $\ell_2 > 0$, we find the following special solution:

$$\begin{aligned} H(\boldsymbol{Z}_3) = \bar{\Theta}^{\dot{\beta}_1}_{k_1}\Theta^{\beta_1}_{k_2}\left(\tilde{X}^{\dot{\beta}_2\beta_2}\cdots\tilde{X}^{\dot{\beta}_{2\ell_2}\beta_{2\ell_2}}\right)(\epsilon_{m_1k_3}\cdots\epsilon_{m_{2R}k_{2R+2}}) \\ \times(\epsilon^{\gamma_1\beta_{2\ell_2+1}}\cdots\epsilon^{\gamma_{2j}\beta_{2j+2\ell_2}})(\epsilon^{\dot{\gamma}_1\dot{\beta}_{2j+2\ell_2+1}}\cdots\epsilon^{\dot{\gamma}_{2j+2\ell_1}\dot{\beta}_{2j+2\ell_1+2\ell_2}})\,, \end{aligned} \tag{4.29}$$

up to a constant prefactor.[17] Note here that, for $R = j = \ell_1 = 0$, the $\mathbb{Z}_2$ symmetry discussed above constrains (4.29) as

$$H(-\bar{X}_3, -\Theta_3, -\bar{\Theta}_3) = (-1)^{2\ell_2-1}\left(\bar{\Theta}^{\dot{\beta}_1}_{k_1}\Theta^{\beta_1}_{k_2}\right)\left(\tilde{X}^{\dot{\beta}_2\beta_2}\cdots\tilde{X}^{\dot{\beta}_{2\ell_2}\beta_{2\ell_2}}\right). \tag{4.30}$$

It must be equal to $H(X_3, \Theta_3, \bar{\Theta}_3)$, and therefore, ℓ_2 must be a half-integer, otherwise correlation function should vanish.

4.4 Selection rules

We here write down the selection rules for $\widehat{\mathcal{C}}_{0(0,0)} \times \mathcal{O}^{\mathrm{Schur}}$ read off from the three-point functions we computed above. Note that all the following rules are only up to non-Schur multiplets.

[17] It has very recently been shown in [63] that the square of the OPE coefficient of $\widehat{\mathcal{C}}_{0(0,0)}\times\widehat{\mathcal{C}}_{n-1(\frac{n-1}{2},\frac{n-1}{2})} \supset \widehat{\mathcal{C}}_{n(\frac{n}{2},\frac{n}{2})}$ for $n \in \mathbb{N}$ is proportional to $\prod_{i=1}^{n}(c-c_i)$ with $c_n \equiv \frac{n(6n+5)}{6(2n+3)}$. This implies that the constant prefactor of $H(\boldsymbol{Z}_3)$ for this channel vanishes when $c = c_i$ for $i = 1, 2, \cdots, n$.

4.4.1 $\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{B}}_R$ fusion

The $\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{B}}_R$ selection rules are as follows.

- For $R > 1$, the selection rule is

$$\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{B}}_R \sim \widehat{\mathcal{B}}_R + \sum_{\ell=0}^{\infty} \left[\widehat{\mathcal{C}}_{R(\frac{\ell}{2},\frac{\ell}{2})} + \widehat{\mathcal{C}}_{R-1(\frac{\ell}{2},\frac{\ell}{2})} \right] . \tag{4.31}$$

- For $R = 1$, because of the $\mathbb{Z}_2$-symmetry, the selection rule is

$$\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{B}}_1 \sim \widehat{\mathcal{B}}_1 + \sum_{\ell=0}^{\infty} \left[\widehat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})} + \widehat{\mathcal{C}}_{0(\frac{\ell+1}{2},\frac{\ell+1}{2})} \right] , \tag{4.32}$$

  which is consistent with [51].

- For $R = \frac{1}{2}$, the rule is

$$\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{B}}_{\frac{1}{2}} \sim \widehat{\mathcal{B}}_{\frac{1}{2}} + \sum_{\ell=0}^{\infty} \widehat{\mathcal{C}}_{\frac{1}{2}(\frac{\ell}{2},\frac{\ell}{2})} . \tag{4.33}$$

4.4.2 $\widehat{\mathcal{C}}_{0(0,0)} \times \bar{\mathcal{D}}_{R(j,0)}$ fusion

The selection rules for $\widehat{\mathcal{C}}_{0(0,0)} \times \bar{\mathcal{D}}_{R(j,0)}$ are written as follows.

- For $R > 0$,

$$\widehat{\mathcal{C}}_{0(0,0)} \times \bar{\mathcal{D}}_{R(j,0)} \sim \bar{\mathcal{D}}_{R(j,0)} + \sum_{\ell=0}^{\infty} \left[\widehat{\mathcal{C}}_{R+\frac{1}{2}(j+\frac{\ell}{2}+\frac{1}{2},\frac{\ell}{2})} + \widehat{\mathcal{C}}_{R-\frac{1}{2}(j+\frac{\ell}{2}+\frac{1}{2},\frac{\ell}{2})} \right] . \tag{4.34}$$

- For $R = 0$,

$$\widehat{\mathcal{C}}_{0(0,0)} \times \bar{\mathcal{D}}_{0(j,0)} \sim \bar{\mathcal{D}}_{0(j,0)} + \sum_{\ell=0}^{\infty} \left[\widehat{\mathcal{C}}_{\frac{1}{2}(j+\frac{\ell}{2}+\frac{1}{2},\frac{\ell}{2})} \right] . \tag{4.35}$$

Note here that, for the $\widehat{\mathcal{C}}_{R,(j,\bar{j})}$ type multiplets on the right-hand sides, the corresponding three-point functions of the superconformal primaries vanishes. This reflects the fact that the sum of the $\mathrm{U}(1)_r$ charges of the primaries in three-point function is non-vanishing. However, the sum of the $\mathrm{U}(1)_r$ charges of the *Schur operators* in the same multiplets vanishes, which implies that the three-point functions of the Schur operators can be non-trivial. Note also that the selection rules for $\widehat{\mathcal{C}}_{0(0,0)} \times \mathcal{D}_{R(0,j)}$ are obtained by taking the charge conjugate of (4.34) and (4.35).

JHEP04(2019)060

4.4.3 $\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{C}}_{R(j,\bar{\jmath})}$ fusion

The selection rules for $\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{C}}_{R(j,\bar{\jmath})}$ are written as follows.

- For $j-\bar{\jmath}=\ell_1 \geq \frac{1}{2}$, $R>\frac{1}{2}$,

$$\begin{aligned}\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{C}}_{R(\bar{\jmath}+\ell_1,\bar{\jmath})} \sim \bar{\mathcal{D}}_{R+\frac{1}{2}(\ell_1-\frac{1}{2},0)} + \bar{\mathcal{D}}_{R-\frac{1}{2}(\ell_1-\frac{1}{2},0)} + \sum_{\bar{\jmath}+\frac{\ell}{2}=0}^{\infty}\left[\widehat{\mathcal{C}}_{R(\bar{\jmath}+\frac{\ell}{2}+\ell_1,\bar{\jmath}+\frac{\ell}{2})}\right] \\ + \sum_{\ell=1}^{\infty}\left[\widehat{\mathcal{C}}_{R+1(\bar{\jmath}+\frac{\ell}{2}+\ell_1,\bar{\jmath}+\frac{\ell}{2})}\right] + \sum_{\ell=1}^{2\bar{\jmath}}\left[\widehat{\mathcal{C}}_{R-1(\bar{\jmath}-\frac{\ell}{2}+\ell_1,\bar{\jmath}-\frac{\ell}{2})}\right].\end{aligned} \tag{4.36}$$

- For $j-\bar{\jmath}=\ell_1 \geq \frac{1}{2}$, $R=\frac{1}{2}$,

$$\begin{aligned}\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{C}}_{\frac{1}{2}(\bar{\jmath}+\ell_1,\bar{\jmath})} \sim \bar{\mathcal{D}}_{1(\ell_1-\frac{1}{2},0)} + \bar{\mathcal{D}}_{0(\ell_1-\frac{1}{2},0)} \\ + \sum_{\bar{\jmath}+\frac{\ell}{2}=0}^{\infty}\left[\widehat{\mathcal{C}}_{\frac{1}{2}(\bar{\jmath}+\frac{\ell}{2}+\ell_1,\bar{\jmath}+\frac{\ell}{2})}\right] + \sum_{\ell=1}^{\infty}\left[\widehat{\mathcal{C}}_{\frac{3}{2}(\bar{\jmath}+\frac{\ell}{2}+\ell_1,\bar{\jmath}+\frac{\ell}{2})}\right].\end{aligned} \tag{4.37}$$

- For $j-\bar{\jmath}=\ell_1 \geq \frac{1}{2}$, $R=0$,

$$\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{C}}_{0(\bar{\jmath}+\ell_1,\bar{\jmath})} \sim \bar{\mathcal{D}}_{\frac{1}{2}(\ell_1-\frac{1}{2},0)} + \sum_{\bar{\jmath}+\frac{\ell}{2}=0}^{\infty}\left[\widehat{\mathcal{C}}_{0(\bar{\jmath}+\frac{\ell}{2}+\ell_1,\bar{\jmath}+\frac{\ell}{2})}\right] + \sum_{\ell=1}^{\infty}\left[\widehat{\mathcal{C}}_{1(\bar{\jmath}+\frac{\ell}{2}+\ell_1,\bar{\jmath}+\frac{\ell}{2})}\right]. \tag{4.38}$$

- For $\bar{\jmath}=j>0$, $R>1$,

$$\begin{aligned}\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{C}}_{R(j,j)} \sim \widehat{\mathcal{B}}_R + \widehat{\mathcal{B}}_{R+1} + \sum_{j+\frac{\ell}{2}=0}^{\infty}\left[\widehat{\mathcal{C}}_{R(j+\frac{\ell}{2},j+\frac{\ell}{2})}\right] \\ + \sum_{\ell=1}^{\infty}\left[\widehat{\mathcal{C}}_{R+1(j+\frac{\ell}{2},j+\frac{\ell}{2})}\right] + \sum_{\ell=1}^{2j}\left[\widehat{\mathcal{C}}_{R-1(j-\frac{\ell}{2},j-\frac{\ell}{2})}\right].\end{aligned} \tag{4.39}$$

- For $\bar{\jmath}=j>0$, $R=1$,

$$\begin{aligned}\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{C}}_{1(j,j)} \sim \widehat{\mathcal{B}}_1 + \widehat{\mathcal{B}}_2 + \sum_{j+\frac{\ell}{2}=0}^{\infty}\left[\widehat{\mathcal{C}}_{1(j+\frac{\ell}{2},j+\frac{\ell}{2})}\right] \\ + \sum_{\ell=1}^{\infty}\left[\widehat{\mathcal{C}}_{2(j+\frac{\ell}{2},j+\frac{\ell}{2})}\right] + \sum_{\ell=1}^{2j}\left[\widehat{\mathcal{C}}_{0(j-\frac{\ell}{2},j-\frac{\ell}{2})}\right].\end{aligned} \tag{4.40}$$

  When j is an integer, the stress-tensor multiplet $\widehat{\mathcal{C}}_{0(0,0)}$ in the last term on the right-hand side must be excluded by the $\mathbb{Z}_2$-symmetry, under the assumption of uniqueness of the stress tensor.

- For $\bar{\jmath} = j > 0$, $R = \frac{1}{2}$,

$$\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{C}}_{\frac{1}{2}(j,j)} \sim \widehat{\mathcal{B}}_{\frac{1}{2}} + \widehat{\mathcal{B}}_{\frac{3}{2}} + \sum_{j+\frac{\ell}{2}=0}^{\infty} \left[\widehat{\mathcal{C}}_{\frac{1}{2}(j+\frac{\ell}{2},j+\frac{\ell}{2})}\right] + \sum_{\ell=1}^{\infty} \left[\widehat{\mathcal{C}}_{\frac{3}{2}(j+\frac{\ell}{2},j+\frac{\ell}{2})}\right]. \tag{4.41}$$

- For $\bar{\jmath} = j = 0, R = 0$,

$$\widehat{\mathcal{C}}_{0(0,0)} \times \widehat{\mathcal{C}}_{0(0,0)} \sim \sum_{\ell=0}^{\infty} \left[\widehat{\mathcal{C}}_{0(\ell,\ell)} + \widehat{\mathcal{C}}_{1(\ell+\frac{1}{2},\ell+\frac{1}{2})}\right]. \tag{4.42}$$

The cases of $j - \bar{\jmath} < 0$ are obtained by the charge conjugates of the above ones.

5 Conclusions and discussions

In this paper, we have computed the three-point functions of the form $\langle \widehat{\mathcal{B}}_{R_1} \widehat{\mathcal{B}}_{R_2} \mathcal{O}\rangle$ and $\langle \widehat{\mathcal{C}}_{0(0,0)} \mathcal{O}_1 \mathcal{O}_2\rangle$ for arbitrary Schur multiplets $\mathcal{O}, \mathcal{O}_1$ and $\mathcal{O}_2$. We have obtained the most general expressions for these three-point functions, except for the two types correlators in (4.21) and (4.22). For the two correlators in (4.21) and (4.22), we have found special solutions to the semi-shortening conditions. From these results, we have derived the OPE selection rules for $\widehat{\mathcal{C}}_{0(0,0)} \times \mathcal{O}^{\text{Schur}}$ up to non-Schur multiplets, where $\mathcal{O}^{\text{Schur}}$ is an arbitrary Schur multiplet. Our selection rules are listed in sub-section 4.4. We have also shown in sub-section 3.4 that our results on the three-point functions $\langle \widehat{\mathcal{B}}_{R_1} \widehat{\mathcal{B}}_{R_2} \mathcal{O}\rangle$ are consistent with the selection rules for $\widehat{\mathcal{B}}_{R_1} \times \widehat{\mathcal{B}}_{R_2}$ obtained in [53]. We emphasize that our analysis relies only on the shortening conditions for Schur multiplets and therefore does not depend on any detail of four-dimensional $\mathcal{N} = 2$ SCFTs.

Let us here discuss an interesting constraint appearing in the selection rules for $\widehat{\mathcal{C}}_{0(0,0)} \times \mathcal{O}^{\text{Schur}}$. Suppose that $\mathcal{O}$ and $\mathcal{O}'$ are two Schur multiplets so that $\mathcal{O}'$ appears in the OPE of $\widehat{\mathcal{C}}_{0(0,0)} \times \mathcal{O}$, i.e., $\widehat{\mathcal{C}}_{0(0,0)} \times \mathcal{O} \supset \mathcal{O}'$. We denote by $R^{(s)}$ and $R'^{(s)}$ the $SU(2)_R$ charges of the Schur operators in $\mathcal{O}$ and $\mathcal{O}'$, respectively. We also denote respectively by h and h' the holomorphic dimensions of the two-dimensional operators associated with the Schur operators in $\mathcal{O}$ and $\mathcal{O}'$. See table 2 for the relation between the two-dimensional holomorphic dimension and four-dimensional quantum numbers of operators. Now, we see that our selection rules imply

$$R^{(\mathrm{s})} < R'^{(\mathrm{s})} \implies h < h', \qquad R^{(\mathrm{s})} > R'^{(\mathrm{s})} \implies h > h'. \tag{5.1}$$

In the 4d/2d correspondence of [17], this means that the $SU(2)_R$ charge of the four-dimensional ancestor of a two-dimensional operator is always smaller than or equal to those of its Virasoro descendants. Since the $SU(2)_R$ symmetry is broken in the associated chiral algebra [17], this relation between the $SU(2)_R$ charge and the holomorphic dimension is surprising.[18] See also [64] for a remarkable discussion on reconstructing the $SU(2)_R$-filtration of the chiral algebra.

[18] Note that, since h is related to the $SU(2)_R$ charge $R^{(s)}$ and the spin $(j^{(s)}, \bar{\jmath}^{(s)})$ of the Schur operator by $h = R^{(s)} + j^{(s)} + \bar{\jmath}^{(s)}$, the constraint (5.1) can also be regarded as a constraint on the $SU(2)_R$ charges and spins of the Schur operators in $\mathcal{O}$ and $\mathcal{O}'$.

Multiplet	Schur operator	$U(1)_r$ charge	$SU(2)_R$	h
$\widehat{\mathcal{B}}_R$	$\Phi^{(i_1\cdots i_{2R})}$	$r=0$	R	R
$\mathcal{D}_{R(0,\bar{\jmath})}$	$\bar{\mathcal{Q}}^{(i}_{(\dot\alpha}\Phi^{i_1\cdots i_{2R})}_{\dot\alpha_1\cdots\dot\alpha_{2\bar{\jmath}})}$	$r=\bar{\jmath}+\frac{1}{2}$	$R+\frac{1}{2}$	$R+j+1$
$\bar{\mathcal{D}}_{R(j,0)}$	$\mathcal{Q}^{(i}_{(\alpha}\Phi^{i_1\cdots i_{2R})}_{\alpha_1\cdots\alpha_{2j})}$	$r=-j-\frac{1}{2}$	$R+\frac{1}{2}$	$R+\bar{\jmath}+1$
$\widehat{\mathcal{C}}_{R(j,\bar{\jmath})}$	$\bar{\mathcal{Q}}^{(i}_{(\dot\alpha\vert}\mathcal{Q}^{i'}_{(\alpha}\Phi^{i_1\cdots i_{2R})}_{\alpha_1\cdots\alpha_{2j})\vert\dot\alpha_1\cdots\dot\alpha_{2\bar{\jmath}})}$	$r=\bar{\jmath}-j$	$R+1$	$R+j+\bar{\jmath}+2$

Table 2. Schur operators in Schur multiplets with their $U(1)_r$ and $SU(2)_R$ charges. Here Φ is the superconformal primary field of the multiplet. The rightmost column shows the holomorphic dimension of the corresponding operator in the two-dimensional chiral algebra.

Another interesting observation is that, in some of the OPE channels allowed by our selection rules, the three-point function of the superconformal primaries vanishes even though those of their descendants do not. As mentioned already, this is a common feature of SCFTs [62]. Indeed, the vanishing of the three-point function of the primaries reflects the fact that the sum of their $U(1)_r$ charges is non-vanishing. Therefore, our selection rules for Schur multiplets do not imply the non-vanishing of the three-point function of the corresponding superconformal primaries.

On the other hand, when we focus on the *Schur operator* in each Schur multiplet, we see that the sum of their $U(1)_r$ charges vanishes whenever the corresponding three Schur multiplets are allowed by the selection rules. This seems to suggest that the three-point functions of Schur *operators* are always non-vanishing whenever the corresponding Schur *multiplets* have non-vanishing three-point functions.[19] This observation leads us to a conjecture on BPS selection rules of general Schur multiplets. Suppose that a Schur multiplet $\mathcal{O}_3^{\text{Schur}}$ appears in the OPE of $\mathcal{O}_1^{\text{Schur}}$ and $\mathcal{O}_2^{\text{Schur}}$, i.e., $\mathcal{O}_1^{\text{Schur}}\times\mathcal{O}_2^{\text{Schur}}\supset\mathcal{O}_3^{\text{Schur}}$. Let us denote the $SU(2)_R$ and $U(1)_r$ charges of the Schur operators in the multiplets respectively by $R_i^{(s)}$ and $r_i^{(s)}$ for $i=1,2$, and 3. Then we conjecture that the following two conditions are satisfied:

$$r_1^{(s)}+r_2^{(s)}=r_3^{(s)}, \qquad |R_1^{(s)}-R_2^{(s)}|\le R_3^{(s)}\le R_1^{(s)}+R_2^{(s)}\,. \tag{5.2}$$

Note that these conditions are necessary for the three-point functions of the Schur operators to be non-vanishing. Recognizing (5.1) and (5.2) as principle of selection rule related to four-dimensional $\mathcal{N}=2$ SCFT whose stress tensor is unique, we recover our all selection rules in section 3.4 and 4.4. We leave the detailed study of this conjecture to future work.

Acknowledgments

The authors are particularly grateful to Sanefumi Moriyama for collaboration at the early stage of this work as well as for various useful discussions during this project. The au-

[19] While this could in principle be checked by using the three-point functions we computed, it is not straightforward to extract the correlators of Schur components from our superfield correlators. We leave it to future work.

thors thank Matthew Buican for illuminating discussions. The authors are grateful to Hiroshi Itoyama, Kazunobu Maruyoshi, Yutaka Matsuo, and Satoshi Yamaguchi for discussions. The authors are also grateful to Nobuhito Maru for his encouragement. Most of our computations are done with the Mathematica package "grassmann.m" provided by M. Headrick to whom the authors are grateful. The work of T. N. is partially supported by JSPS Grant-in-Aid for Early-Career Scientists 18K13547. The work of K. K is supported by JSPS KAKENHI Grant Number JP18J22009.

A Fierz identities

In this appendix, we summarize useful identities for Grassmann variables $\Theta^{i\alpha}$ and $\bar{\Theta}^{\dot{\alpha}}{}_i$, which we call Fierz identities. We first introduce the following variables:

$$M^i{}_j \equiv \Theta^{i\alpha} X_{\alpha\dot{\alpha}} \bar{\Theta}^{\dot{\alpha}}{}_j \,, \tag{A.1}$$

$$m_l \equiv \mathrm{tr} M^l \,, \tag{A.2}$$

$$H^{\dot{\alpha}\alpha}_{ij} \equiv \bar{\Theta}^{\dot{\alpha}}{}_j (\epsilon\Theta)_i{}^{\alpha}. \tag{A.3}$$

From the nilpotent structure of Θ and $\bar{\Theta}$, we see that the following Fierz identities hold[20]

$$\begin{gathered} m_1 m_2 = 0 \,, \qquad m_2^2 = m_1^4 \,, \qquad (\epsilon M)_{(ii')}(m_2 + 2m_1^2) = 0 \,, \\ (\epsilon M)_{(ii')}(\epsilon M)_{(jj')} m_1 = \frac{1}{12}(\epsilon_{ij}\epsilon_{i'j'} + \epsilon_{ij'}\epsilon_{i'j}) m_1^3 \,. \end{gathered} \tag{A.4}$$

Moreover, by using the variables m_l, we can expand powers of $\bar{X}^2$ as [20]

$$\frac{1}{(\bar{X}^2)^\Delta} = \frac{1}{(X^2)^\Delta}\left(1 - 4\mathrm{i}\Delta\frac{m_1}{X^2} + 8\Delta\frac{m_2}{X^4} - 8\Delta^2\frac{m_1^2}{X^4} + \frac{32i}{3}\Delta(\Delta^2-1)\frac{m_1^3}{X^6} + \frac{32}{3}\Delta^2(\Delta^2-1)\frac{m_1^4}{X^8}\right). \tag{A.5}$$

We can also derive[21]

$$\frac{M_{(ii')}}{X^4} = \frac{\bar{M}_{(ii')}}{\bar{X}^4} \,, \qquad \frac{m_1^2 - m_2}{X^6} = \frac{\bar{m}_1^2 - \bar{m}_2}{\bar{X}^6} = \frac{\Theta^{\alpha\alpha'} X_{\alpha\dot{\alpha}} X_{\alpha'\dot{\alpha}'} \bar{\Theta}^{\dot{\alpha}\dot{\alpha}'}}{X^6} \,. \tag{A.6}$$

The following Fierz identities are derived by using the Mathematica package `grassmann.m` [65]:

$$(\Theta^{i\alpha}\Theta^{j\beta}\epsilon_{\alpha\beta})\bar{\Theta}\Theta^{\dot{\alpha}_1\alpha_1}\bar{\Theta}\Theta^{\dot{\alpha}_2\alpha_2} = 0 \,, \tag{A.7}$$

$$(m_1^2 + m_2)\bar{\Theta}\Theta^{\dot{\alpha}_1\alpha_1}\bar{\Theta}\Theta^{\dot{\alpha}_2\alpha_2} = 0 \,, \tag{A.8}$$

$$m_1(\epsilon M)_{(ij)}\bar{\Theta}\Theta^{\dot{\alpha}_1\alpha_1}\bar{\Theta}\Theta^{\dot{\alpha}_2\alpha_2} = 0 \,, \tag{A.9}$$

$$m_1\bar{\Theta}\Theta^{\dot{\alpha}_1\alpha_1}\bar{\Theta}\Theta^{\dot{\alpha}_2\alpha_2} = \frac{1}{3X^2}(m_1^2 + m_2)\bar{\Theta}\Theta^{\dot{\alpha}_1\alpha_1}\tilde{X}^{\dot{\alpha}_2\alpha_2} \,, \tag{A.10}$$

[20] $(\epsilon M)_{ij} = \epsilon_{ik} M^k{}_j$.

[21] $\bar{M}^i{}_j := \Theta^{i\alpha}\bar{X}_{\alpha\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}{}_j \,, \bar{m}_l := \mathrm{tr}\bar{M}^l$.

JHEP04(2019)060

$$(\epsilon M)_{(ij)}\bar{\Theta}\Theta^{\dot{\alpha}_1\alpha_1}\bar{\Theta}\Theta^{\dot{\alpha}_2\alpha_2} = \frac{m_1}{X^2}(\epsilon M)_{(ij)}\bar{\Theta}\Theta^{\dot{\alpha}_1\alpha_1}\tilde{X}^{\dot{\alpha}_2\alpha_2}\,, \tag{A.11}$$

$$(\epsilon M)_{(ij)}\bar{\Theta}\Theta^{\dot{\alpha}\alpha}m_1^2 = (\epsilon M)_{(ij)}\bar{\Theta}\Theta^{\dot{\alpha}\alpha}m_2 = 0\,, \tag{A.12}$$

$$(2m_1^2 - m_2)\bar{\Theta}\Theta^{\dot{\alpha}\alpha} = \frac{m_1^3}{X^2}\tilde{X}^{\dot{\alpha}\alpha}\,, \tag{A.13}$$

$$m_1^3\bar{\Theta}\Theta^{\dot{\alpha}\alpha} = \frac{m_1^4}{2X^2}\tilde{X}^{\dot{\alpha}\alpha}\,, \tag{A.14}$$

$$(\epsilon M)_{(ij')}(\epsilon M)_{(i'j')}\bar{\Theta}\Theta^{\dot{\alpha}\alpha} = \frac{(\epsilon_{ij}\epsilon_{i'j'})}{12}\left((m_1^2+m_2)\bar{\Theta}\Theta^{\dot{\alpha}\alpha} - \frac{m_1^3}{X^2}\tilde{X}^{\dot{\alpha}\alpha}\right), \tag{A.15}$$

$$(\epsilon M)_{(ij)}(\epsilon M)_{(i'j')}\bar{\Theta}\Theta^{\dot{\alpha}_1\alpha_1}\bar{\Theta}\Theta^{\dot{\alpha}_2\alpha_2} = -\frac{m_1^4(\epsilon_{ij}\epsilon_{i'j'})}{24(X^2)^2}\tilde{X}^{\dot{\alpha}_1\alpha_1}\tilde{X}^{\dot{\alpha}_2\alpha_2}\,, \tag{A.16}$$

$$H_{(ij)}^{\dot{\alpha}_1\alpha_1}\bar{\Theta}\Theta^{\dot{\alpha}_2\alpha_2} = 0\,, \tag{A.17}$$

$$m_1^3 H_{(ij)}^{\dot{\alpha}\alpha} = 0\,, \tag{A.18}$$

$$m_1 H_{(ij)}^{\dot{\alpha}\alpha} + (\epsilon M)_{(ij)}\bar{\Theta}\Theta^{\dot{\alpha}\alpha} = \frac{m_1}{X^2}(\epsilon M)_{(ij)}\tilde{X}^{\dot{\alpha}\alpha}\,, \tag{A.19}$$

$$\frac{m_2}{X^2}H_{(ij)}^{\dot{\alpha}\alpha} = -\frac{m_1^2}{X^2}\frac{\epsilon M_{(ij)}}{X^2}\tilde{X}^{\dot{\alpha}\alpha}\,, \tag{A.20}$$

$$\frac{(m_1^2+m_2)}{X^2}H_{(ij)}^{\dot{\alpha}\alpha} = -\frac{m_1}{X^2}(\epsilon M)_{(ij)}\bar{\Theta}\Theta^{\dot{\alpha}\alpha}. \tag{A.21}$$

B $\langle \widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{B}}_R\mathcal{O}^{\mathcal{I}}\rangle$

In this appendix, we describe a derivation of the most general expression for the correlation function $\langle \widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{B}}_R\mathcal{O}^{\mathcal{I}}\rangle$ for various $\mathcal{O}^{\mathcal{I}}$. Since the case of $\mathcal{O}^{\mathcal{I}} = \widehat{\mathcal{B}}_{R'}$ has been analyzed in section 3.3, we here focus on $\mathcal{O}^{\mathcal{I}} = \bar{\mathcal{D}}_{R'(j,0)}$, $\mathcal{D}_{R'(0,j)}$, and $\widehat{\mathcal{C}}_{R'(j,\bar{\jmath})}$.

B.1 $\langle \widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{B}}_R\bar{\mathcal{D}}_{R'(j,0)}\rangle$

Let us first consider the case of $\mathcal{O}^{\mathcal{I}} = \bar{\mathcal{D}}_{R'(j,0)}$. In this case, $H(\bar{\boldsymbol{Z}}_3)$ has dimension $2(R'-R)+j-1$ and $\mathrm{U}(1)_r$ charge $-j-1$. Since the absolute value of the $\mathrm{U}(1)_r$ charge is at most one, the spin j must be zero. Then the only possibility for R' is $R' = R-1$, which implies

$$H(\bar{\boldsymbol{Z}}_3) = \frac{A}{\bar{X}^4}\bar{\Theta}_{m_1}\bar{\Theta}_{m_2}\epsilon_{m_3k_1}\cdots\epsilon_{m_{2R}k_{2R-2}}\,. \tag{B.1}$$

However, this is not consistent with (4.3) unless $A = 0$, and therefore the correlation function $\langle \widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{B}}_R\bar{\mathcal{D}}_{R'(j,0)}\rangle$ has to vanish. From charge conjugation, we see that $\langle \widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{B}}_R\mathcal{D}_{R'(0,j)}\rangle$ also vanishes.

B.2 $\langle \widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{B}}_R\widehat{\mathcal{C}}_{R'(j,\bar{\jmath})}\rangle$

Let us next turn to the case of $\mathcal{O}^{\mathcal{I}} = \widehat{\mathcal{C}}_{R'(j,\bar{\jmath})}$. In this case, the function $H(\boldsymbol{Z}_3)$ has dimension $2(R'-R)+j+\bar{\jmath}$ and $\mathrm{U}(1)_r$ charge $\bar{\jmath}-j$. Recall that $|\bar{\jmath}-j|$ is at most one. Moreover, since charge conjugation exchanges j and $\bar{\jmath}$, we only need to study the cases of $j \geq \bar{\jmath}$. Therefore, we assume $-1 \leq \bar{\jmath}-j \leq 0$. Then the possible combinations of R' and $\bar{\jmath}-j$ are $(R', \bar{\jmath}-j) = (R,-1)$, $(R+\frac{1}{2},-\frac{1}{2})$, $(R-\frac{1}{2},-\frac{1}{2})$, $(R+1,0)$, $(R,0)$, and $(R-1,0)$. We study the most general expression for $H(\boldsymbol{Z}_3)$ in each of these cases below.

B.2.1 $\bar{\jmath} = j - 1 \geq 0\,, R' = R$

In this case, the only possible candidate for $H(\boldsymbol{Z}_3)$ is

$$H(\boldsymbol{Z}_3) = \frac{A}{X^2} \epsilon_{m_1 k_1} \cdots \epsilon_{m_{2R} k_{2R}} X_{\beta_1 \dot{\beta}_1} \cdots X_{\beta_{2j} \dot{\beta}_{2j}} \tilde{\bar{\Theta}}_{\dot{\beta}_{2j+1} \dot{\beta}_{2j+2}} . \tag{B.2}$$

This is not consistent with (4.4) unless $A = 0$.

B.2.2 $\bar{\jmath} = j - \frac{1}{2} \geq 0\,, R' = R + \frac{1}{2}$

In this case, the only possible candidate is

$$H(\boldsymbol{Z}_3) = A \epsilon_{m_1 k_1} \cdots \epsilon_{m_{2R} k_{2R}} X_{\beta_1 \dot{\beta}_1} \cdots X_{\beta_{2j-1} \dot{\beta}_{2j-1}} X_{\beta_{2j} \dot{\alpha}} \bar{\Theta}^{\dot{\alpha}}_{k_{2R+1}} , \tag{B.3}$$

which is not consistent with (3.5) unless $A = 0$.

B.2.3 $\bar{\jmath} = j - \frac{1}{2} \geq 0\,, R' = R - \frac{1}{2}$

In this case, there are three possible terms in $H(\boldsymbol{Z}_3)$;

$$\begin{aligned} H(\boldsymbol{Z}_3) = & \frac{1}{X^2} \epsilon_{m_1 k_1} \cdots \epsilon_{m_{2R-1} k_{2R-1}} \\ & \times \left(A \bar{\Theta}^{\dot{\alpha}'}_{m_{2R}} X_{\beta_1 \dot{\beta}_1} + \Theta^{\alpha}_{m_{2R}} \bar{\Theta}^{\dot{\alpha}\dot{\alpha}'} \left(B \frac{X_{\alpha\dot{\alpha}} X_{\beta_1 \dot{\beta}_1}}{X^2} + C \epsilon_{\dot{\beta}_1 \dot{\alpha}} \epsilon_{\beta_1 \alpha} \right) \right) \\ & \times X_{\beta_2 \dot{\beta}_2} \cdots X_{\beta_{2j-1} \dot{\beta}_{2j-1}} X_{\beta_{2j} \dot{\alpha}'} . \end{aligned} \tag{B.4}$$

However, this is not consistent with (4.4) unless $A = B = C = 0$.

B.2.4 $\bar{\jmath} = j \geq 0\,, R' = R$

The only possible $H(Z_3)$ is of the form

$$\begin{aligned} H(\boldsymbol{Z}_3) = & \left(A \epsilon_{m_1 k_1} X_{\beta_1 \dot{\beta}_1} + B X_{\beta_1 \dot{\beta}_1} \frac{M_{m_1 k_1}}{X^2} + C \tilde{\Theta}_{m_1 \beta_1} \tilde{\bar{\Theta}}_{\dot{\beta}_1 k_1} \right) \\ & \times \epsilon_{m_2 k_2} \cdots \epsilon_{m_{2R} k_{2R}} X_{\beta_2 \dot{\beta}_2} \cdots X_{\beta_{2j} \dot{\beta}_{2j}} . \end{aligned} \tag{B.5}$$

The constraint (3.4) is then expressed as

$$0 = B \left(\epsilon_{m_1 k_1} \cdots \epsilon_{m_{2R} k_{2R}} \right) X_{\beta_1 \dot{\beta}_1} \cdots X_{\beta_{2j} \dot{\beta}_{2j}} \frac{\Theta^{\alpha}_m X_{\alpha\dot{\alpha}}}{X^2} , \tag{B.6}$$

which implies $B = 0$. Moreover, for (B.5) to be consistent with (4.4), we should set $C = -4\mathrm{i}(2j)A$. Indeed, with this condition imposed, the function (B.5) can be rewritten as

$$H(\bar{\boldsymbol{Z}}_3) = A(\epsilon_{m_1 k_1} \bar{X}_{\beta_1 \dot{\beta}_1} - 4\mathrm{i}(2j) \tilde{\Theta}_{k_1 \beta_1} \tilde{\bar{\Theta}}_{\dot{\beta}_1 m_1})(\epsilon_{m_2 k_2} \cdots \epsilon_{m_{2R} k_{2R}}) \bar{X}_{\beta_2 \dot{\beta}_2} \cdots \bar{X}_{\beta_{2j} \dot{\beta}_{2j}} , \tag{B.7}$$

and therefore satisfies (4.2). We also see that this expression satisfies (3.5).

B.2.5 $\bar{j} - j = 0\,, R' = R - 1$

In this case, the possible $H(\boldsymbol{Z}_3)$ for $j > 0$ is of the form

$$H(\boldsymbol{Z}_3) = \frac{1}{X^2}\left(\left(AX_{\beta_1\dot{\beta}_1}\frac{M_{m_1m_2}}{X^2} + B\tilde{\Theta}_{m_1\beta_1}\tilde{\bar{\Theta}}_{\dot{\beta}_1m_2}\right) + \frac{C}{X^2}\Theta_{m_1}\Theta_{m_2}\bar{\Theta}^{\dot{\alpha}\dot{\alpha}'}X_{\beta_1\dot{\alpha}}\epsilon_{\dot{\beta}_1\dot{\alpha}'}\right) \\ \times\left(\epsilon_{m_3k_1}\cdots\epsilon_{m_{2R}k_{2R-2}}\right)X_{\beta_2\dot{\beta}_2}\cdots X_{\beta_{2j}\dot{\beta}_{2j}}. \tag{B.8}$$

For this to be consistent with (3.5) and (4.2), we must impose $B = -2jA$ and $C = -4\mathrm{i}jA$. On the other hand, for $j = 0$, $H(\boldsymbol{Z}_3)$ is given by

$$H(\boldsymbol{Z}_3) = A\frac{M_{m_1m_2}}{X^4}\left(\epsilon_{m_3k_1}\cdots\epsilon_{m_{2R}k_{2R-2}}\right). \tag{B.9}$$

Using the Fierz identity (A.6), we see that this satisfies all the (semi-)shortening conditions.

C $\langle\widehat{\mathcal{C}}_{0(0,0)}\bar{\mathcal{D}}_{R'(j,0)}\mathcal{O}^{\mathcal{I}}\rangle$

In this appendix, we describe the details of our computations of $\langle\widehat{\mathcal{C}}_{0(0,0)}\bar{\mathcal{D}}_{R'(j,0)}\mathcal{O}^{\mathcal{I}}\rangle$. We solve the equations (4.1), (4.2), (3.4), and (4.14)/(4.15) together with the (semi-)shortening conditions associated with the third Schur multiplet $\mathcal{O}^{\mathcal{I}}$. We note here that the most general solution to (4.14) or (4.15) is written as

$$\begin{aligned} H = {} & f^{(\dot{\gamma}_1\cdots\dot{\gamma}_{2j}),\mathcal{I}}_{1(m_1\cdots m_{2R})} + f^{(\dot{\gamma}_1\cdots\dot{\gamma}_{2j}\dot{\gamma}),\mathcal{I}}_{2(m_1\cdots m_{2R}m)}\tilde{\bar{\Theta}}_{\dot{\gamma}}{}^{m} + f^{(\dot{\gamma}_1\cdots\dot{\gamma}_{2j}\dot{\gamma}),\mathcal{I}}_{3(m_1\cdots m_{2R-1}|}\tilde{\bar{\Theta}}_{\dot{\gamma}|m_{2R})} + f^{\mathcal{I}(\dot{\gamma}_1\cdots\dot{\gamma}_{2j-1}}_{4(m_1\cdots m_{2j-1}}\bar{\Theta}^{\dot{\gamma}_{2j})}_{m_{2R})} \\ & + f^{(\dot{\gamma}_1\cdots\dot{\gamma}_{2j}\dot{\gamma}\dot{\gamma}'),\mathcal{I}}_{5(m_1\cdots m_{2R})}\tilde{\bar{\Theta}}_{\dot{\gamma}\dot{\gamma}'} + f^{(\dot{\gamma}_1\cdots\dot{\gamma}_{2j}),\mathcal{I}}_{6(m_1\cdots m_{2R-2}}\bar{\Theta}_{m_{2R-1}}\bar{\Theta}_{m_{2R})} \\ & + \left(2j\epsilon^{\dot{\beta}(\dot{\gamma}_1}f^{\dot{\gamma}_2\cdots\dot{\gamma}_{2j})\dot{\beta}',\mathcal{I}}_{7(m_1\cdots m_{2R})}\tilde{\bar{\Theta}}_{\dot{\beta}\dot{\beta}'} + (2j+2)\bar{\Theta}^{(k}\bar{\Theta}^{k')}\epsilon_{k(m_1}f^{(\dot{\gamma}_1\cdots\dot{\gamma}_{2j}),\mathcal{I}}_{7|m_2\cdots m_{2R})k'}\right) \\ & + \bar{\Theta}^{(\dot{\gamma}|}_{j}\tilde{\bar{\Theta}}^{j}_{\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}_{(m}f^{|\dot{\gamma}_1\cdots\dot{\gamma}_{2j}),\mathcal{I}}_{8m_1\cdots m_{2R})} + \tilde{\bar{\Theta}}_{\dot{\gamma}j}\tilde{\bar{\Theta}}^{j}_{\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}_{(m_1}f^{(\dot{\gamma}_1\cdots\dot{\gamma}_{2j}\dot{\gamma}),\mathcal{I}}_{9m_2\cdots m_{2R})}. \end{aligned} \tag{C.1}$$

Here f_i are functions of X and Θ. Note in particular that the superscripts $\dot{\gamma}_i$ in $f^{(\dot{\gamma}_1\cdots\dot{\gamma}_{2j}),\mathcal{I}}_{7(m_1\cdots m_{2R})}$ are totally symmetric, which is implicit in the first term in the bracket in (C.1).[22]

The function $H(\boldsymbol{Z}_3)$ has dimension $\Delta_3 - 2R - j - 3$ and $\mathrm{U}(1)_r$ charge $r_3 - j - 1$, where Δ_3 and r_3 are the dimension and the $\mathrm{U}(1)_r$ charge of the third multiplet $\mathcal{O}^{\mathcal{I}}$ respectively. The case of $\mathcal{O}^{\mathcal{I}} = \widehat{\mathcal{B}}_{R'}$ has already been studied in section B.1. Moreover, $H(\boldsymbol{Z}_3)$ turns out to vanish for $\mathcal{O}^{\mathcal{I}} = \bar{\mathcal{D}}_{R'(j',0)}$. Indeed, for $\mathcal{O}^{\mathcal{I}} = \bar{\mathcal{D}}_{R'(j',0)}$, the $\mathrm{U}(1)_r$ charge of $H(\boldsymbol{Z}_3)$ is $-j - j' - 2$, which is not possible since the $\mathrm{U}(1)_r$ charge is bounded from below by -1. Therefore, $\bar{\mathcal{D}}_{R'(j',0)}$ dose not appear in the OPE of $\widehat{\mathcal{C}}_{0(0,0)} \times \bar{\mathcal{D}}_{R(j,0)}$. In the rest of this appendix, we consider the remaining cases $\mathcal{O}^{\mathcal{I}} = \mathcal{D}_{R'(0,\bar{j})}$ and $\widehat{\mathcal{C}}_{R'(j',\bar{j}')}$.

C.1 $\langle\widehat{\mathcal{C}}_{0(0,0)}\bar{\mathcal{D}}_{R(j,0)}\mathcal{D}_{R'(0,\bar{j})}\rangle$

For $\mathcal{O}^{\mathcal{I}} = \mathcal{D}_{R'(0,\bar{j})}$, $H(\boldsymbol{Z}_3)$ has dimension $2(R'-R) + (\bar{j} - j) - 2$ and $\mathrm{U}(1)_r$ charge $\bar{j} - j$. Up to charge conjugation, the possible values of the $\mathrm{U}(1)_r$ charge is 0, $\frac{1}{2}$, and 1. Therefore

[22] For example in the case of $j = 1$, f_7 is defined with some function f by $f_7^{(\gamma_1\gamma_2)} = \frac{1}{2}\left(f^{\gamma_1\gamma_2} + f^{\gamma_2\gamma_1}\right)$ while in (C.1) we further symmetrize as in $\epsilon^{\beta(\gamma_1}f_7^{\gamma_2)\beta'} = \frac{1}{2}\left(\epsilon^{\beta\gamma_1}f_7^{(\gamma_2\beta')} + \epsilon^{\beta\gamma_2}f_7^{(\gamma_1\beta')}\right)$ using the function f_7 we have just defined, and $\mathrm{SU}(2)_R$ indices are symmetrized similarly.

JHEP04(2019)060

the possible combinations of R' and $\bar{\jmath}$ are $(R', \bar{\jmath}) = (R, j-1)$, $(R-\frac{1}{2}, j-\frac{1}{2})$, $(R+\frac{1}{2}, j-\frac{1}{2})$, (R, j), and $(R-1, j)$. The shortening condition of the third multiplet $\mathcal{D}_{R'(0,\bar{\jmath})}$ is

$$\bar{\mathcal{S}}^{\dot{\alpha}}_{(n} G(\boldsymbol{Z}_2)_{n_1 \cdots n_{2R'})} = 0\,, \tag{C.2}$$

and the semi-shortening condition is

$$\mathcal{S}^{\delta}_{(n} G_{n_1 \cdots n_{2R'})(\delta\delta_2 \cdots \delta_{2\bar{\jmath}})}(\boldsymbol{Z}_2) = 0\,, \quad \text{for} \quad j > 0\,, \tag{C.3}$$

$$\epsilon_{\alpha\beta} \mathcal{S}^{\alpha}_{(n} \mathcal{S}^{\beta}_{n'} G_{n_1 \cdots n_{2R'})}(\boldsymbol{Z}_2) = 0\,, \quad \text{for} \quad j = 0\,, \tag{C.4}$$

where indices n_i, δ_i and $\dot{\delta}_i$ are related with the $\mathcal{D}_{R'(0,\bar{\jmath})}$ multiplet. Below, we solve these equations together with (4.1), (4.2), (3.4), and (4.14)/(4.15), for all possible values of R' and $\bar{\jmath}$.

C.1.1 $\bar{\jmath} = j - 1 \geq 0\,, R' = R$

In this case, $H(\boldsymbol{Z}_3)$ is

$$H(\boldsymbol{Z}_3) = \frac{A}{X^4} (\epsilon_{m_1 k_1} \cdots \epsilon_{m_{2R} k_{2R}}) \bar{\Theta}^{\dot{\gamma}_1 \dot{\gamma}_2} \epsilon^{\dot{\gamma}_3 \dot{\beta}_1} \cdots \epsilon^{\dot{\gamma}_{2j} \dot{\beta}_{2j-2}}\,, \tag{C.5}$$

which is inconsistent with (4.14) unless $A = 0$.

C.1.2 $\bar{\jmath} = j - \frac{1}{2} \geq 0\,, R' = R - \frac{1}{2}$

In this case, the only possible $H(\boldsymbol{Z}_3)$ is of the form[23]

$$\begin{aligned} H(\boldsymbol{Z}_3) = \frac{1}{X^4} \bigg(& A \epsilon_{\dot{\gamma}_1 \dot{\beta}_1} \tilde{\bar{\Theta}}_{\dot{\gamma}_2 m_1} + B \frac{\epsilon_{\dot{\gamma}_1 \dot{\beta}_1}}{X^2} (\Theta_{m_1} X)_{\dot{\alpha}} \bar{\Theta}^{\dot{\alpha}}{}_{\dot{\gamma}_2} + \frac{C}{X^2} (\Theta_{m_1} X)_{\dot{\beta}_1} \tilde{\bar{\Theta}}_{\dot{\gamma}_1 \dot{\gamma}_2} \bigg) \\ & \times (\epsilon_{m_2 k_1} \cdots \epsilon_{m_{2R} k_{2R-1}}) (\epsilon_{\dot{\gamma}_3 \dot{\beta}_2} \cdots \epsilon_{\dot{\gamma}_{2j} \dot{\beta}_{2j-1}}). \end{aligned} \tag{C.6}$$

For this to be consistent with (4.1) and (4.14), we must set $B = -\frac{4\mathrm{i}}{j+1} A\,, C = -\frac{2\mathrm{i}(2j-1)}{j+1} A$. Then the function $G(\boldsymbol{Z}_2)$ given by (2.32) is now written as

$$\begin{aligned} G(\boldsymbol{Z}_2) = \; & -\mathrm{i}A \bigg((j+1) \epsilon_{\alpha_1 \delta_1} (\tilde{\bar{X}}^{-1} \bar{\Theta}^i_2)_{\alpha_2} \boldsymbol{u}^{\dagger}_{ij_1}(\boldsymbol{Z}_2) - 4\mathrm{i} \epsilon_{\alpha_1 \delta_1} (\Theta_{j_1} \tilde{\bar{X}}^{-1} \bar{\Theta}_i)(\tilde{\bar{X}}^{-1} \bar{\Theta}^i)_{\alpha_2} \\ & \qquad - 2\mathrm{i}(2j-1) \tilde{\Theta}_{j_1 \delta_1} (\tilde{\bar{X}}^{-1} \bar{\Theta}_i)_{\alpha_1} (\tilde{\bar{X}}^{-1} \bar{\Theta}^i)_{\alpha_2} \bigg) \\ & \times (\epsilon_{j_2 n_1} \cdots \epsilon_{j_{2R} n_{2R-1}}) (\epsilon_{\alpha_3 \delta_2} \cdots \epsilon_{\alpha_{2j} \delta_{2j-1}})\,. \end{aligned} \tag{C.7}$$

However, we see that this does not satisfy the shortening condition (C.2) unless $A = 0$.

C.1.3 $\bar{\jmath} = j - \frac{1}{2} \geq 0\,, R' = R + \frac{1}{2}$

In this case, the possible solution is given by

$$H(\boldsymbol{Z}_3) = \frac{A}{X^2} \bar{\Theta}^{\dot{\gamma}_1}_{k_1} (\epsilon_{m_1 k_2} \cdots \epsilon_{m_{2R} k_{2R+1}}) (\epsilon^{\dot{\gamma}_2 \dot{\beta}_1} \cdots \epsilon^{\dot{\gamma}_{2j} \dot{\beta}_{2j-1}})\,, \tag{C.8}$$

which is not consistent with (4.14) unless $A = 0$.

[23] Recall here that $\bar{\Theta}^{\dot{\alpha}}{}_{\dot{\gamma}_{2j}} = \bar{\Theta}^{\dot{\alpha}}{}_i \epsilon^{ik} \tilde{\bar{\Theta}}_{\dot{\gamma}_{2j} k}$.

C.1.4 $\bar{j} = j \geq 0\,, R' = R$

In this case, $H(\boldsymbol{Z}_3)$ is given by

$$H(\boldsymbol{Z}_3) = \frac{1}{X^2}\left(\epsilon^{\dot{\gamma}_1\dot{\beta}_1}\left(A\epsilon_{m_1k_1} + C\frac{M_{m_1k_1}}{X^2}\right) + B\frac{(\Theta_{m_1}X)_{\dot{\gamma}_1}\tilde{\bar{\Theta}}_{\dot{\beta}_1k_1}}{X^2}\right) \times (\epsilon_{m_2k_2}\cdots\epsilon_{m_{2R}k_{2R}})(\epsilon^{\dot{\gamma}_2\dot{\beta}_2}\cdots\epsilon^{\dot{\gamma}_{2j}\dot{\beta}_{2j}})\,. \tag{C.9}$$

It is straightforward to show that this satisfies (4.2). Let us next consider the semi-shortening condition for the second multiplet. For $j > 0$, the condition (4.14) reads

$$0 = 4\mathrm{i}\Theta^{\alpha}_{(m}\epsilon_{m_1)k_1}\frac{(B + 8\mathrm{i}jA + 2jC)\,X_{\alpha\dot{\beta}}}{X^4}(\epsilon_{m_2k_2}\cdots)(\epsilon^{\dot{\gamma}\dot{\beta}}\cdots)\,, \tag{C.10}$$

which implies $B = -4\mathrm{i}(2j)A - 2jC$. On the other hand, for $j = 0$, the condition (4.15) implies $C = -4\mathrm{i}A$. Note that the term proportional to B in (C.10) does not exist for $j = 0$.

To solve the conditions associated with the third multiplet, let us relate the above $H(\boldsymbol{Z}_3)$ to $G(\boldsymbol{Z}_2)$ via (2.32). The result is generally written as

$$G(\boldsymbol{Z}_2) = \frac{A}{X^2}\left(\left(\epsilon_{j_1n_1} - (4\mathrm{i} + jC)\frac{M_{j_1n_1}}{X^2}\right)\epsilon_{\alpha_1\delta_1} - 2j^2C\tilde{\Theta}_{j_1\alpha_1}(\tilde{X}^{-1}\bar{\Theta}_{n_1})_{\delta_1}\right) \times (\epsilon_{j_2n_2}\cdots\epsilon_{j_{2R}n_{2R}})(\epsilon_{\alpha_2\delta_2}\cdots\epsilon_{\alpha_{2j}\delta_{2j}})\,, \tag{C.11}$$

where C is a free parameter that is present only in the case of $j > 0$. We see that this expression satisfies the shortening condition (C.2) for arbitrary C. On the other hand, the semi-shortening condition implies $C = 0$ unless $j = \frac{1}{2}$. For $j = \frac{1}{2}$, the semi-shortening condition does not restrict the value of C.

C.1.5 $\bar{j} = j\,, R' = R - 1$

In this case, the possible $H(\boldsymbol{Z}_3)$ is of the form

$$H(\boldsymbol{Z}_3) = \frac{1}{X^6}\left(\Theta^{\alpha}_{m_1}\left(AX_{\alpha\dot{\gamma}_1}\tilde{\bar{\Theta}}_{\dot{\beta}_1m_2} + BX_{\alpha\dot{\beta}_1}\tilde{\bar{\Theta}}_{\dot{\gamma}_1m_2}\right) + C\epsilon_{\dot{\gamma}_1\dot{\beta}_1}M_{m_1m_2} + D\Theta_{m_1}\Theta_{m_2}\tilde{\bar{\Theta}}_{\dot{\gamma}_1\dot{\beta}_1}\right) \times (\epsilon_{m_3k_1}\cdots\epsilon_{m_{2R}k_{2R-2}})(\epsilon_{\dot{\gamma}_2\dot{\beta}_1}\cdots\epsilon_{\dot{\gamma}_{2j}\dot{\beta}_{2j}}). \tag{C.12}$$

Note that this expression vanishes if $j = 0$. Therefore there is no non-trivial solution for $j = 0$.

For $j > 0$, the above expression is consistent with (4.2) and (4.14) if and only if $C = \frac{A-B}{2}$ and $D = \mathrm{i}(A + B)$. With these conditions imposed, the function $G(\boldsymbol{Z}_2)$ given by (2.32) is written as

$$G(\boldsymbol{Z}_2) = \Bigg(\left(A\tilde{\Theta}_{j_1\alpha_1}(\tilde{\bar{X}}^{-1}\bar{\Theta}^k)_{\delta_1}\boldsymbol{u}^{\dagger}_{kj_2} + B\tilde{\Theta}_{j_1\delta_1}(\tilde{\bar{X}}^{-1}\bar{\Theta}^k)_{\alpha_1}\boldsymbol{u}^{\dagger}_{kj_2}\right) + \frac{A-B}{2}\epsilon_{\alpha_1\delta_1}\frac{\bar{M}_{j_1j_2}}{\bar{X}_2^2} - \mathrm{i}(A+B)\Theta_{j_1}\Theta_{j_2}(\tilde{\bar{X}}^{-1}\bar{\Theta}_i)_{\alpha_1}(\tilde{\bar{X}}^{-1}\bar{\Theta}^i)_{\delta_1}\Bigg) \times (\epsilon_{j_3n_1}\cdots\epsilon_{j_{2R}n_{2R-2}})(\epsilon_{\alpha_2\delta_2}\cdots\epsilon_{\alpha_{2j}\delta_{2j}})\,. \tag{C.13}$$

It is then straightforward to see that $A = B = 0$ is necessary for this to be consistent with (C.2) and (C.3). Therefore, no non-trivial solution exists in this case.

C.2 $\langle \widehat{\mathcal{C}}_{0(0,0)} \bar{\mathcal{D}}_{R(j,0)} \widehat{\mathcal{C}}_{R'(j_1,\bar{\jmath}_2)} \rangle$

Let us now turn to the case of $\mathcal{O}^{\mathcal{I}} = \widehat{\mathcal{C}}_{R'(j_1,\bar{\jmath}_2)}$. The function $H(\boldsymbol{Z}_3)$ now has dimension $2(R'-R)+(j_1+\bar{\jmath}_2-j)-1$ and $\mathrm{U}(1)_r$ charge $\bar{\jmath}_2-j_1-j-1$. Since $|\bar{\jmath}_2-j_1-j-1| \leq 1$, the possible values of $\bar{\jmath}_2-j_1$ are j, $j+\frac{1}{2}$, $j+1$, $j+\frac{3}{2}$, and $j+2$. It is straightforward to see that no non-trivial $H(\boldsymbol{Z}_3)$ is possible for $\bar{\jmath}_2-j_1{=}j+\frac{3}{2}$ and $j+2$. Therefore, the only possible values of R' and $\bar{\jmath}_2-j_1$ are $(R', \bar{\jmath}_2-j_1)=(R,j+1)$, $(R-1,j+1)$, $(R+\frac{1}{2},j+\frac{1}{2})$, $(R-\frac{1}{2},j+\frac{1}{2})$, and (R,j). The semi-shortening conditions for the third multiplet are (3.24) and (3.25). Below, we solve all the (semi-)shortening conditions for each of the possible values of $\bar{\jmath}_2-j_1$ and R'.

C.2.1 $\bar{\jmath}_2 - j_1 = j+1\,, R' = R$

In this case, the function $H(\boldsymbol{Z}_3)$ is of the form

$$H(\boldsymbol{Z}_3) = A\frac{(\Theta_{m_1}X)_{\dot\beta_1}\tilde{\bar{\Theta}}_{\dot\beta_2 k_1}}{X^2}(\epsilon_{m_2k_2}\cdots\epsilon_{m_{2R}k_{2R}})(\epsilon_{\dot\gamma_1\dot\beta_3}\cdots\epsilon_{\dot\gamma_{2j}\dot\beta_{2j}})X_{\beta_1\dot\beta_{2j+3}}\cdots X_{\beta_{2j_1}\dot\beta_{2j+2j_1+2}}\,, \tag{C.14}$$

which is not consistent with the condition (4.2) unless $A=0$.

C.2.2 $\bar{\jmath}_2 - j_1 = j+1\,, R' = R-1 \geq 0$

In this case, the possible $H(\boldsymbol{Z}_3)$ is

$$\begin{aligned} H(\boldsymbol{Z}_3) = \frac{1}{X^4}&\left(A(\Theta_{m_1}X)_{\dot\beta_1}\tilde{\bar{\Theta}}_{\dot\beta_2 m_2} + B\,\Theta_{m_1}\Theta_{m_2}\tilde{\bar{\Theta}}_{\dot\beta_1\dot\beta_2}\right)(\epsilon_{m_3k_1}\cdots\epsilon_{m_{2R}k_{2R-2}}) \\ &\times(\epsilon_{\dot\gamma_1\dot\beta_3}\cdots\epsilon_{\dot\gamma_{2j}\dot\beta_{2j+2}})X_{\beta_1\dot\beta_{2j+3}}\cdots X_{\beta_{2j_1}\dot\beta_{2j_1+2j+2}}\,, \end{aligned} \tag{C.15}$$

where $j\geq 0$, $j_1 \geq 0$, $R\geq 1$. We see that (4.2) implies $B=-\mathrm{i}(2j_1)A$. Then the function $G(\boldsymbol{Z}_2)$ is written as

$$\begin{aligned} G(\boldsymbol{Z}_2) = \frac{-A}{\bar{X}_2^2}&\left(\tilde{\Theta}_{j_1\delta_1}\left(\tilde{\bar{X}}^{-1}\bar{\Theta}^k\right)_{\delta_2}\boldsymbol{u}^\dagger_{kj_2}(\boldsymbol{Z}_2) - \mathrm{i}(2j_1)\Theta_{j_1}\Theta_{j_2}\left(\tilde{\bar{X}}^{-1}\bar{\Theta}_k\right)_{\delta_1}\left(\tilde{\bar{X}}^{-1}\bar{\Theta}^k\right)_{\delta_2}\right) \\ &\times(\epsilon_{j_3n_1}\cdots\epsilon_{j_{2R}n_{2R-2}})(\epsilon_{\alpha_1\delta_3}\cdots\epsilon_{\alpha_{2j}\delta_{2j+2}})\tilde{\bar{X}}^{-1}_{\delta_1\dot\delta_{2j+3}}\cdots\tilde{\bar{X}}^{-1}_{\delta_{2j_1}\dot\delta_{2j+2j_1+2}}\,, \end{aligned} \tag{C.16}$$

which does not satisfy the conditions (3.24) and (3.25) unless $A=0$.

C.2.3 $\bar{\jmath}_2 - j_1 = j+\frac{1}{2}\,, R' = R+\frac{1}{2}$

In this case, the only possible candidate for $H(\boldsymbol{Z}_3)$ is of the form

$$H(\boldsymbol{Z}_3) = A\tilde{\bar{\Theta}}_{\dot\beta_1 k_1}X_{\beta_1\dot\beta_2}\cdots X_{\beta_{2j_1}\dot\beta_{2j_1+1}}(\epsilon_{\dot\gamma_1\dot\beta_{2j_1+2}}\cdots\epsilon_{\dot\gamma_{2j}\dot\beta_{2j+2j_1+1}})(\epsilon_{m_1k_2}\cdots\epsilon_{m_{2R}k_{2R+1}})\,. \tag{C.17}$$

By using (2.16), it is rewritten as

$$\begin{aligned} H(\bar{\boldsymbol{Z}}_3) = A\tilde{\bar{\Theta}}_{\dot\beta_1 k_1}&\left(\bar{X}_{\beta_1\dot\beta_2} - 4\mathrm{i}(2j_1)\tilde{\bar{\Theta}}_{\dot\beta_2}\tilde{\Theta}_{\beta_1}\right)\bar{X}_{\beta_2\dot\beta_3}\cdots\bar{X}_{\beta_{2j_1}\dot\beta_{2j_1+2}}(\epsilon_{m_1k_2}\cdots\epsilon_{m_{2R}k_{2R+1}}) \\ &\times(\epsilon_{\dot\gamma_1\dot\beta_{2j_1+2}}\cdots\epsilon_{\dot\gamma_{2j}\dot\beta_{2j+2j_1+1}})\,. \end{aligned} \tag{C.18}$$

We see that this is of the form of (4.4) and (C.1) and therefore satisfies (4.14)/(4.15) and (4.2). On the other hand, the function $G(\boldsymbol{Z}_2)$ is now evaluated as

$$G(\boldsymbol{Z}_2) = \mathrm{i}\frac{\boldsymbol{u}^\dagger_{kn_1}(\boldsymbol{Z}_2)}{\bar{X}^4}\left(\bar{X}\bar{\Theta}^k\right)_{\delta_1}(\epsilon_{j_1n_2}\cdots\epsilon_{j_{2R}n_{2R+1}})(\epsilon_{\alpha_1\delta_2}\cdots\epsilon_{\alpha_{2j}\delta_{2j+1}}) \\ \times\tilde{\bar{X}}^{-1}_{\delta_{2j+2}\dot{\delta}_1}\cdots\tilde{\bar{X}}^{-1}_{\delta_{2j+2j_1+1}\dot{\delta}_{2j_1}}\,. \tag{C.19}$$

We see that this expression satisfies the semi-shortening conditions associated with the third multiplet for any $j\geq 0$, $j_1\geq 0$ and $R\geq 0$.

C.2.4 $\bar{\jmath}_2 - j_1 = j + \frac{1}{2}\,, R' = R - \frac{1}{2}$

The only possible $H(\boldsymbol{Z}_3)$ is of the form

$$H(\boldsymbol{Z}_3) = \frac{1}{X^2}\Bigg(A\epsilon_{\dot{\gamma}_1\dot{\beta}_1}\tilde{\bar{\Theta}}_{\dot{\beta}_2m_1} \\ +\frac{\Theta^{\alpha}_{m_1}\bar{\Theta}^{\dot{\alpha}\dot{\alpha}'}}{X^2}\Big(B\epsilon_{\dot{\gamma}_1\dot{\beta}_1}\epsilon_{\dot{\beta}_2\dot{\alpha}'}X_{\alpha\dot{\alpha}}+CX_{\alpha\dot{\gamma}_1}\epsilon_{\dot{\beta}_1\dot{\alpha}}\epsilon_{\dot{\beta}_2\dot{\alpha}'}+DX_{\alpha\dot{\beta}_1}\epsilon_{\dot{\beta}_2\dot{\alpha}}\epsilon_{\dot{\gamma}_1\dot{\alpha}'}\Big)\Bigg) \\ \times(\epsilon_{m_2k_1}\cdots\epsilon_{m_{2R}k_{2R-1}})(\epsilon_{\dot{\gamma}_2\dot{\beta}_3}\cdots\epsilon_{\dot{\gamma}_{2j}\dot{\beta}_{2j+1}})X_{\beta_1\dot{\beta}_{2j+3}}\cdots X_{\beta_{2j_1}\dot{\beta}_{2j+2j_1+1}}. \tag{C.20}$$

In the case of $j>0$, the constraints (4.2) and (4.14) imply that $B=-4\mathrm{i}A+C$ and $D=-C$. On the other hand, for $j=0$, the constraints (4.2) and (4.14) imply $B=-4\mathrm{i}A$. Note that C and D do not exist for $j=0$. The function $G(\boldsymbol{Z}_2)$ is now evaluated as

$$G(\boldsymbol{Z}_2) = \frac{1}{\bar{X}^2}\Bigg(A\epsilon_{\alpha_1\delta_1}\left(\tilde{\bar{X}}\bar{\Theta}^k\right)_{\delta_2}+\Bigg((-4\mathrm{i}A+jC)\epsilon_{\alpha_1\delta_1}(\Theta_{j_1}\tilde{\bar{X}}^{-1}\bar{\Theta}_k)(\tilde{\bar{X}}^{-1}\bar{\Theta}^k)_{\delta_2} \\ +jC\left(\tilde{\Theta}_{j_1\alpha_1}(\tilde{\bar{X}}^{-1}\bar{\Theta}_k)_{\delta_1}(\tilde{\bar{X}}^{-1}\bar{\Theta}^k)_{\delta_2}-\tilde{\Theta}_{j_1\delta_1}(\tilde{\bar{X}}^{-1}\bar{\Theta}_k)_{\delta_2}(\tilde{\bar{X}}^{-1}\bar{\Theta}^k)_{\alpha_1}\right)\Bigg)\Bigg) \\ \times(\epsilon_{j_2n_1}\cdots\epsilon_{j_{2R}n_{2R-1}})(\epsilon_{\alpha_2\delta_3}\cdots\epsilon_{\alpha_{2j}\delta_{2j+2}})\tilde{\bar{X}}_{\delta_1\dot{\delta}_{2j+3}}\cdots\tilde{\bar{X}}_{\delta_{2j_1}\dot{\delta}_{2j+2j_1+2}}\,, \tag{C.21}$$

which satisfies (3.24) and (3.25) if and only if $C=0$.

C.2.5 $\bar{\jmath}_2 - j_1 = j\,, R' = R$

In this case, the only possible $H(\boldsymbol{Z}_3)$ is

$$H(\boldsymbol{Z}_3) = \frac{(\epsilon_{m_1k_1}\cdots\epsilon_{m_{2R}k_{2R}})}{X^2}\left(AX_{\beta_1\dot{\beta}_1}\tilde{\bar{\Theta}}_{\dot{\gamma}_1\dot{\beta}_2}+BX_{\beta_1\dot{\alpha}}\epsilon_{\dot{\beta}_1\dot{\alpha}'}\bar{\Theta}^{\dot{\alpha}\dot{\alpha}'}\epsilon_{\dot{\gamma}_1\dot{\beta}_2}\right) \\ \times(\epsilon_{\dot{\gamma}_2\dot{\beta}_3}\cdots\epsilon_{\dot{\gamma}_{2j}\dot{\beta}_{2j+1}})X_{\beta_2\dot{\beta}_{2j+2}}\cdots X_{\beta_{2j_1}\dot{\beta}_{2j+2j_1}}\,. \tag{C.22}$$

Since $\bar{\Theta}_{(i}\bar{\Theta}_{j)}\bar{\Theta}_{\dot{\gamma}_1\dot{\beta}_2}=\bar{\Theta}_{(i}\bar{\Theta}_{j)}\bar{\Theta}^{\dot{\alpha}\dot{\alpha}'}=0$, the constraint (4.2) is trivially satisfied. On the other hand, the constraint (4.14) implies $A=B=0$.

D $\langle\widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{C}}_{R(j_1,\bar{\jmath}_1)}\widehat{\mathcal{C}}_{R'(j_2,\bar{\jmath}_2)}\rangle$

In this appendix, we show that the only non-vanishing three-point functions of the form $\langle\widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{C}}_{R(j_1,\bar{\jmath}_1)}\widehat{\mathcal{C}}_{R'(j_2,\bar{\jmath}_2)}\rangle$ are those listed in (4.21) and (4.22). To that end, in sub-section D.1, we will show that the three point function vanishes when the corresponding $H(\boldsymbol{Z}_3)$ has non-vanishing $\mathrm{U}(1)_r$ charge. In sub-section D.2, we show that $\langle\widehat{\mathcal{C}}_{0(0,0)}\widehat{\mathcal{C}}_{R(j+\ell_2,j+\ell_1+\ell_2)}\widehat{\mathcal{C}}_{R+1(j+\ell_1,j)}\rangle$ vanishes for any $j,\ell_1\geq 0$ and $\ell_2>0$.

$(j_1, \bar{\jmath}_1, j_2, \bar{\jmath}_2)$	$(\Delta, L_1, L_2, L_3, L_4)$
$(j+1, j+\ell_1, j+\ell_1+\ell_2, j+\ell_2)$	$(k+4, 2\ell_2+k, k+2, 2j-k, 2j+2\ell_1-k-2)$,
$(j+1+\ell_1, j, j+\ell_2, j+\ell_1+\ell_2)$	$(k+4, 2\ell_2+k, k+2, 2j+2\ell_1-k, 2j-k-2)$,
$(j+1+\ell_2, j+\ell_1+\ell_2, j+\ell_1, j)$	$(2\ell_2+k+4, k, 2\ell_2+k+2, 2j-k, 2j-k+2\ell_1-2)$,
$(j+1+\ell_1+\ell_2, j+\ell_2, j, j+\ell_1)$	$(2\ell_2+k+4, k, 2\ell_2+k+2, 2j+2\ell_1-k, 2j-k-2)$,

Table 3. The left column shows possible spin eigenvalues, and the right column shows the corresponding values of parameters appearing in (D.5).

D.1 Non-vanishing $\mathrm{U}(1)_r$ charge

We first show that $\langle \widehat{\mathcal{C}}_{0(0,0)} \widehat{\mathcal{C}}_{R(j_1,\bar{\jmath}_1)} \widehat{\mathcal{C}}_{R'(j_2,\bar{\jmath}_2)} \rangle$ vanishes when $H(\boldsymbol{Z}_3)$ has $\mathrm{U}(1)_r$ charge ± 1. It is straightforward to generalize it to the case of $\mathrm{U}(1)_r$ charge $\pm\frac{1}{2}$. While $\mathrm{U}(1)_r$ charge vanishes, we can also show $\langle \widehat{\mathcal{C}}_{0(0,0)} \widehat{\mathcal{C}}_{R(j+\ell_1+\ell_2,j+\ell_2)} \widehat{\mathcal{C}}_{R+1(j,j+\ell_1)} \rangle$ is zero with same argument for $\ell_2 > 0$.

Since charge conjugation flips the sign of the $\mathrm{U}(1)_r$ charge, we focus on the case of $\mathrm{U}(1)_r$ charge -1. Then we see from (4.3) that $R' = R$ must hold for a non-trivial solution to exist. All possible combinations of four spin eigenvalues $(j_1, \bar{\jmath}_1, j_2, \bar{\jmath}_2)$ are listed in table 3. From (4.3), we see that the general expression for $H(\boldsymbol{Z}_3)$ is of the form

$$H(\boldsymbol{Z}_3) = (\epsilon_{m_1k_1} \cdots \epsilon_{m_{2R}k_{2R}})\, \tilde{\bar{\Theta}}_{\dot\gamma\dot\gamma'} f^{(\dot\gamma\dot\gamma')(\dot\gamma_1\cdots\dot\gamma_{2j_1})(\gamma_1\cdots\gamma_{2\bar{\jmath}_1})\mathcal{I}}(X)\,, \tag{D.1}$$

where $\mathcal{I}$ stands for the Lorentz indices associated with the third multiplet, and we write $(X, \Theta, \bar\Theta)$ for $(X_3, \Theta_3, \bar\Theta_3)$. Since $j_1 > 0$, the semi-shortening conditions for the second multiplet contain (4.20a). The condition (4.20a) particularly implies

$$2\,(\epsilon_{m_1k_1} \cdots \epsilon_{m_{2R}k_{2R}})\, \tilde{\bar{\Theta}}^{\,i}_{\dot\gamma'} \epsilon_{\dot\gamma\dot\gamma_1} f^{(\dot\gamma\dot\gamma')(\dot\gamma_1\cdots\dot\gamma_{2j_1})(\gamma_1\cdots\gamma_{2\bar{\jmath}_1})\mathcal{I}} = 0\,, \tag{D.2}$$

which means that $f^{(\dot\gamma\dot\gamma')(\dot\gamma_1\cdots\dot\gamma_{2j_1})(\gamma_1\cdots\gamma_{2\bar{\jmath}_1})\mathcal{I}} = f^{(\dot\gamma\dot\gamma'\dot\gamma_1\cdots\dot\gamma_{2j_1})(\gamma_1\cdots\gamma_{2\bar{\jmath}_1})\mathcal{I}}$. Then the remaining constraint arising from (4.20a) is expressed as

$$\epsilon_{\dot\alpha\dot\gamma_1} \frac{\partial}{\partial X_{\alpha\dot\alpha}} f^{(\dot\gamma\dot\gamma'\dot\gamma_1\cdots\dot\gamma_{2j_1})(\gamma_1\cdots\gamma_{2\bar{\jmath}_1})\mathcal{I}}(X) = 0\,. \tag{D.3}$$

To show that there is no non-trivial solution to this equation, let us first write down the most general expression for $f^{(\dot\gamma\dot\gamma'\dot\gamma_1\cdots\dot\gamma_{2j_1})(\gamma_1\cdots\gamma_{2\bar{\jmath}_1})\mathcal{I}}(X_3)$. Recall first that $\mathcal{I}$ stands for the Lorentz indices associated with $\widehat{\mathcal{C}}_{R(j_2,\bar{\jmath}_2)}$. Therefore, the most general ansatz for $f^{(\dot\gamma\dot\gamma'\dot\gamma_1\cdots\dot\gamma_{2j_1})(\gamma_1\cdots\gamma_{2\bar{\jmath}_1})\mathcal{I}}$ is written as

$$f(X) = \sum_{k=0}^{2j-2} \lambda_k f_k(X)\,, \tag{D.4}$$

$$f_k(X) \equiv \frac{1}{(X^2)^\Delta} (\tilde{X}^{\dot\beta\beta})^{L_1} (\tilde{X}^{\dot\gamma\gamma})^{L_2} (\epsilon^{\dot\gamma\dot\beta})^{L_3} (\epsilon^{\gamma\beta})^{L_4} (\tilde{X}^{\dot\gamma\beta})^2\,, \tag{D.5}$$

where we use the short-hand notation $(\tilde{X}^{\dot\gamma\gamma})^L \equiv \tilde{X}^{\dot\gamma_1\gamma_1}\cdots\tilde{X}^{\dot\gamma_L\gamma_L}$ with $\dot\gamma_{2j_1+1} \equiv \dot\gamma$ and $\dot\gamma_{2j_1+2} \equiv \dot\gamma'$. The parameters Δ, L_1, L_2, L_3, and L_4 are fixed by $j_1, \bar{\jmath}_1, j_2, \bar{\jmath}_2$ and k as in table 3.[24] Substituting this general ansatz into the constraint (D.3), we obtain

$$0 = \sum_{k=0}^{2j-2} \lambda_k \Bigg[\epsilon^{\alpha\gamma} \frac{L_2(\Delta - L_2 - L_3 - 3)}{(X^2)^\Delta} (\tilde{X}^{\dot\beta\beta})^{L_1} (\tilde{X}^{\dot\gamma\gamma})^{L_2-1} (\epsilon^{\dot\gamma\dot\beta})^{L_3} (\epsilon^{\gamma\beta})^{L_4} (\tilde{X}^{\dot\gamma\beta})^2 + \tilde{X}^{\dot\beta\alpha} \frac{\Delta L_3}{(X^2)^{\Delta+1}} (\tilde{X}^{\dot\beta\beta})^{L_1} (\tilde{X}^{\dot\gamma\gamma})^{L_2} (\epsilon^{\dot\gamma\dot\beta})^{L_3-1} (\epsilon^{\gamma\beta})^{L_4} (\tilde{X}^{\dot\gamma\beta}) + (\text{linearly independent terms}) \Bigg] . \tag{D.6}$$

This particularly implies that, for $f(X)$ to be non-vanishing, $L_2(\Delta - L_2 - L_3 - 3) = \Delta L_3 = 0$ must hold for some k. However, this last set of equations has no solution for $l_1 \geq 0$, $l_2 \geq 0$, $j \geq 1$, and $2j - 2 \geq k \geq 0$. Therefore there is no non-trivial solution to the constraint (D.3). In other words, when $H(\boldsymbol{Z}_3)$ has $\mathrm{U}(1)_r$ charge -1, the correlation function $\langle \widehat{\mathcal{C}}_{0(0,0)} \widehat{\mathcal{C}}_{R(j_1,\bar{\jmath}_1)} \widehat{\mathcal{C}}_{R'(j_2,\bar{\jmath}_2)} \rangle$ vanishes.

D.2 Vanishing $\mathrm{U}(1)_r$ charge

In this sub-section we show that, even though the corresponding $H(\boldsymbol{Z}_3)$ is neutral under $\mathrm{U}(1)_r$, the three-point function $\langle \widehat{\mathcal{C}}_{0(0,0)} \widehat{\mathcal{C}}_{R(j+\ell_2,j+\ell_1+\ell_2)} \widehat{\mathcal{C}}_{R+1(j+\ell_1,j)} \rangle$ must vanish for $j, \ell_1 \geq 0$ and $\ell_2 > 0$.

In the case of $\langle \widehat{\mathcal{C}}_{0(0,0)} \widehat{\mathcal{C}}_{R(j+\ell_2,j+\ell_1+\ell_2)} \widehat{\mathcal{C}}_{R+1(j+\ell_1,j)} \rangle$, (C.1) and the fact that the third multiplet has two more $\mathrm{SU}(2)_R$ indices than the second one imply that

$$H(\boldsymbol{Z}_3) = \bar\Theta_{\dot\gamma k_1} \Theta_{\gamma k_2} \epsilon_{m_1 k_3} \cdots \epsilon_{m_{2R} k_{2R+2}} g^{(\gamma\gamma_1\cdots\gamma_{2j+2\ell_1+2\ell_2})\dot\gamma(\dot\gamma_1\cdots\dot\gamma_{2j+2\ell_2})}_{(\beta_1\cdots\beta_{2j+2\ell_1})(\dot\beta_1\cdots\dot\beta_{2j})}(X) \tag{D.7}$$

is the most general expression for $H(\boldsymbol{Z}_3)$, where g is a function of X, $\{m_i, \gamma_j, \dot\gamma_j\}$ are indices associated with $\widehat{\mathcal{C}}_{R(j+\ell_2,j+\ell_1+\ell_2)}$, and $\{k_i, \beta_j, \dot\beta_j\}$ are indices associated with $\widehat{\mathcal{C}}_{R+1(j+\ell_1,j)}$. In the rest of this sub-section, we write γ_0 and $\dot\gamma_0$ for γ and $\dot\gamma$, respectively. The function g has many possible terms corresponding to different assignments of the spinor indices to $\delta^{\gamma_i}_{\beta_j}$, $\delta^{\dot\gamma_i}_{\dot\beta_j}$, $X^{\gamma_i\dot\gamma_j}$, $X_{\beta_i\dot\beta_j}$, $X^{\gamma_i}{}_{\dot\beta_j}$ and $X_{\beta_i}{}^{\dot\gamma_j}$. However, since the number of γ_i minus that of β_i is at least $2\ell_2 + 1 \geq 2$, every term contains at least one of the following factors

$$X^{\gamma_{i_1}\dot\gamma_{j_1}} X^{\gamma_{i_2}\dot\gamma_{j_2}} , \qquad X^{\gamma_{i_1}\dot\gamma_{j_1}} X^{\gamma_{i_2}}{}_{\dot\beta_{j_2}} , \qquad X^{\gamma_{i_1}}{}_{\dot\beta_{j_1}} X^{\gamma_{i_2}}{}_{\dot\beta_{j_2}} . \tag{D.8}$$

Let us focus on a k-th term in g, whose coefficient we denote by λ_k. Since all indices $\gamma_0, \cdots, \gamma_{2j+2\ell_1+2\ell_2}$ are symmetric, the k-th term in g involves a piece proportional to $\lambda_k X^{\gamma_0\dot\gamma_{i+1}}$ or $\lambda_k X^{\gamma_0}{}_{\beta_i}$ for some $i \geq 0$. This means that $H(\boldsymbol{Z}_3)$ involves a term proportional to one of the following

$$\lambda_k \bar\Theta_{\dot\gamma_0 k_1} \Theta_{\gamma_0 k_2} X^{\gamma_0\dot\gamma_{i+1}} , \qquad \lambda_k \bar\Theta_{\dot\gamma_0 k_1} \Theta_{\gamma_0 k_2} X^{\gamma_0}{}_{\dot\beta_i} , \tag{D.9}$$

[24] Note that j, ℓ_1, and ℓ_2 are fixed by $j_1, \bar{\jmath}_1, j_2$, and $\bar{\jmath}_2$ as in the left column of the table.

for some $i \geq 0$. When we change the variables from $(X, \Theta, \bar{\Theta})$ to $(\bar{X}, \Theta, \bar{\Theta})$, it gives rise to a term proportional to one of the following

$$\lambda_k \bar{\Theta}_{\dot{\gamma}_0 k_1} \bar{\Theta}^{k\dot{\gamma}_{i+1}} \Theta_{k_2} \Theta_k \,, \qquad \lambda_k \bar{\Theta}_{\dot{\gamma}_0 k_1} \bar{\Theta}^{k}{}_{\dot{\beta}_i} \Theta_{k_2} \Theta_k \,, \tag{D.10}$$

which is prohibited by (4.2) since the most general solution to (4.2) is written as (4.4).[25] Therefore we must impose $\lambda_k = 0$ for (D.7) to satisfy the semi-shortening conditions. Since k is arbitrary, this means that $g = 0$ as a function of X. Therefore, there is no non-trivial solution to the semi-shortening conditions in the case of $\langle \widehat{\mathcal{C}}_{0(0,0)} \widehat{C}_{R(j+\ell_2, j+\ell_1+\ell_2)} \widehat{\mathcal{C}}_{R+1(j+\ell_1,j)} \rangle$ for $j, \ell_1 \geq 0$ and $\ell_2 > 0$.

References

[1] P.C. Argyres and M.R. Douglas, *New phenomena in* SU(3) *supersymmetric gauge theory*, *Nucl. Phys.* **B 448** (1995) 93 [hep-th/9505062] [INSPIRE].

[2] P.C. Argyres, M.R. Plesser, N. Seiberg and E. Witten, *New $N = 2$ superconformal field theories in four-dimensions*, *Nucl. Phys.* **B 461** (1996) 71 [hep-th/9511154] [INSPIRE].

[3] T. Eguchi, K. Hori, K. Ito and S.-K. Yang, *Study of $N = 2$ superconformal field theories in four-dimensions*, *Nucl. Phys.* **B 471** (1996) 430 [hep-th/9603002] [INSPIRE].

[4] K. Maruyoshi and J. Song, *Enhancement of supersymmetry via renormalization group flow and the superconformal index*, *Phys. Rev. Lett.* **118** (2017) 151602 [arXiv:1606.05632] [INSPIRE].

[5] K. Maruyoshi and J. Song, *$\mathcal{N} = 1$ deformations and RG flows of $\mathcal{N} = 2$ SCFTs*, *JHEP* **02** (2017) 075 [arXiv:1607.04281] [INSPIRE].

[6] P. Agarwal, K. Maruyoshi and J. Song, *$\mathcal{N} = 1$ deformations and RG flows of $\mathcal{N} = 2$ SCFTs, part II: non-principal deformations*, *JHEP* **12** (2016) 103 [arXiv:1610.05311] [INSPIRE].

[7] K. Maruyoshi, E. Nardoni and J. Song, *Landscape of simple superconformal field theories in* 4*d*, *Phys. Rev. Lett.* **122** (2019) 121601 [arXiv:1806.08353] [INSPIRE].

[8] P. Agarwal, A. Sciarappa and J. Song, *$\mathcal{N} = 1$ Lagrangians for generalized Argyres-Douglas theories*, *JHEP* **10** (2017) 211 [arXiv:1707.04751] [INSPIRE].

[9] S. Benvenuti and S. Giacomelli, *Abelianization and sequential confinement in* $2 + 1$ *dimensions*, *JHEP* **10** (2017) 173 [arXiv:1706.04949] [INSPIRE].

[10] S. Benvenuti and S. Giacomelli, *Lagrangians for generalized Argyres-Douglas theories*, *JHEP* **10** (2017) 106 [arXiv:1707.05113] [INSPIRE].

[11] S. Giacomelli, *RG flows with supersymmetry enhancement and geometric engineering*, *JHEP* **06** (2018) 156 [arXiv:1710.06469] [INSPIRE].

[12] S. Giacomelli, *Infrared enhancement of supersymmetry in four dimensions*, *JHEP* **10** (2018) 041 [arXiv:1808.00592] [INSPIRE].

[25] Recall that $\Theta_k \Theta_l$ and $\Theta^{\alpha\beta}$ are linearly independent.

[13] F. Carta, S. Giacomelli and R. Savelli, *SUSY enhancement from T-branes*, *JHEP* **12** (2018) 127 [arXiv:1809.04906] [INSPIRE].

[14] J.A. Minahan and D. Nemeschansky, *An $N = 2$ superconformal fixed point with E_6 global symmetry*, *Nucl. Phys.* **B 482** (1996) 142 [hep-th/9608047] [INSPIRE].

[15] J.A. Minahan and D. Nemeschansky, *Superconformal fixed points with E_n global symmetry*, *Nucl. Phys.* **B 489** (1997) 24 [hep-th/9610076] [INSPIRE].

[16] D. Gaiotto, *$N = 2$ dualities*, *JHEP* **08** (2012) 034 [arXiv:0904.2715] [INSPIRE].

[17] C. Beem et al., *Infinite chiral symmetry in four dimensions*, *Commun. Math. Phys.* **336** (2015) 1359 [arXiv:1312.5344] [INSPIRE].

[18] A. Gadde, L. Rastelli, S.S. Razamat and W. Yan, *The 4d superconformal index from q-deformed 2d Yang-Mills*, *Phys. Rev. Lett.* **106** (2011) 241602 [arXiv:1104.3850] [INSPIRE].

[19] A. Gadde, L. Rastelli, S.S. Razamat and W. Yan, *Gauge theories and Macdonald polynomials*, *Commun. Math. Phys.* **319** (2013) 147 [arXiv:1110.3740] [INSPIRE].

[20] P. Liendo, I. Ramirez and J. Seo, *Stress-tensor OPE in $\mathcal{N} = 2$ superconformal theories*, *JHEP* **02** (2016) 019 [arXiv:1509.00033] [INSPIRE].

[21] M. Lemos and P. Liendo, *$\mathcal{N} = 2$ central charge bounds from 2d chiral algebras*, *JHEP* **04** (2016) 004 [arXiv:1511.07449] [INSPIRE].

[22] C. Beem, W. Peelaers, L. Rastelli and B.C. van Rees, *Chiral algebras of class S*, *JHEP* **05** (2015) 020 [arXiv:1408.6522] [INSPIRE].

[23] M. Lemos and W. Peelaers, *Chiral algebras for trinion theories*, *JHEP* **02** (2015) 113 [arXiv:1411.3252] [INSPIRE].

[24] C. Beem, M. Lemos, P. Liendo, L. Rastelli and B.C. van Rees, *The $\mathcal{N} = 2$ superconformal bootstrap*, *JHEP* **03** (2016) 183 [arXiv:1412.7541] [INSPIRE].

[25] M. Buican and T. Nishinaka, *On the superconformal index of Argyres–Douglas theories*, *J. Phys.* **A 49** (2016) 015401 [arXiv:1505.05884] [INSPIRE].

[26] C. Cordova and S.-H. Shao, *Schur indices, BPS particles and Argyres-Douglas theories*, *JHEP* **01** (2016) 040 [arXiv:1506.00265] [INSPIRE].

[27] S. Cecotti, J. Song, C. Vafa and W. Yan, *Superconformal index, BPS monodromy and chiral algebras*, *JHEP* **11** (2017) 013 [arXiv:1511.01516] [INSPIRE].

[28] C. Beem, W. Peelaers and L. Rastelli, *Deformation quantization and superconformal symmetry in three dimensions*, *Commun. Math. Phys.* **354** (2017) 345 [arXiv:1601.05378] [INSPIRE].

[29] T. Nishinaka and Y. Tachikawa, *On 4d rank-one $\mathcal{N} = 3$ superconformal field theories*, *JHEP* **09** (2016) 116 [arXiv:1602.01503] [INSPIRE].

[30] M. Buican and T. Nishinaka, *Conformal manifolds in four dimensions and chiral algebras*, *J. Phys.* **A 49** (2016) 465401 [arXiv:1603.00887] [INSPIRE].

[31] D. Xie, W. Yan and S.-T. Yau, *Chiral algebra of Argyres-Douglas theory from M5 brane*, arXiv:1604.02155 [INSPIRE].

[32] C. Cordova, D. Gaiotto and S.-H. Shao, *Infrared computations of defect Schur indices*, *JHEP* **11** (2016) 106 [arXiv:1606.08429] [INSPIRE].

JHEP04(2019)060

[33] M. Lemos, P. Liendo, C. Meneghelli and V. Mitev, *Bootstrapping $\mathcal{N} = 3$ superconformal theories*, JHEP **04** (2017) 032 [arXiv:1612.01536] [INSPIRE].

[34] C. Beem, L. Rastelli and B.C. van Rees, *More $\mathcal{N} = 4$ superconformal bootstrap*, Phys. Rev. **D 96** (2017) 046014 [arXiv:1612.02363] [INSPIRE].

[35] F. Bonetti and L. Rastelli, *Supersymmetric localization in AdS_5 and the protected chiral algebra*, JHEP **08** (2018) 098 [arXiv:1612.06514] [INSPIRE].

[36] J. Song, *Macdonald index and chiral algebra*, JHEP **08** (2017) 044 [arXiv:1612.08956] [INSPIRE].

[37] L. Fredrickson, D. Pei, W. Yan and K. Ye, *Argyres-Douglas theories, chiral algebras and wild hitchin characters*, JHEP **01** (2018) 150 [arXiv:1701.08782] [INSPIRE].

[38] C. Cordova, D. Gaiotto and S.-H. Shao, *Surface defects and chiral algebras*, JHEP **05** (2017) 140 [arXiv:1704.01955] [INSPIRE].

[39] J. Song, D. Xie and W. Yan, *Vertex operator algebras of Argyres-Douglas theories from $M5$-branes*, JHEP **12** (2017) 123 [arXiv:1706.01607] [INSPIRE].

[40] M. Buican, Z. Laczko and T. Nishinaka, *$\mathcal{N} = 2$ S-duality revisited*, JHEP **09** (2017) 087 [arXiv:1706.03797] [INSPIRE].

[41] K. Ito and H. Shu, *ODE/IM correspondence and the Argyres-Douglas theory*, JHEP **08** (2017) 071 [arXiv:1707.03596] [INSPIRE].

[42] C. Beem and L. Rastelli, *Vertex operator algebras, Higgs branches and modular differential equations*, JHEP **08** (2018) 114 [arXiv:1707.07679] [INSPIRE].

[43] A. Neitzke and F. Yan, *Line defect Schur indices, Verlinde algebras and $\mathrm{U}(1)_r$ fixed points*, JHEP **11** (2017) 035 [arXiv:1708.05323] [INSPIRE].

[44] Y. Pan and W. Peelaers, *Chiral algebras, localization and surface defects*, JHEP **02** (2018) 138 [arXiv:1710.04306] [INSPIRE].

[45] M. Fluder and J. Song, *Four-dimensional lens space index from two-dimensional chiral algebra*, JHEP **07** (2018) 073 [arXiv:1710.06029] [INSPIRE].

[46] J. Choi and T. Nishinaka, *On the chiral algebra of Argyres-Douglas theories and S-duality*, JHEP **04** (2018) 004 [arXiv:1711.07941] [INSPIRE].

[47] M. Buican and Z. Laczko, *Nonunitary Lagrangians and unitary non-Lagrangian conformal field theories*, Phys. Rev. Lett. **120** (2018) 081601 [arXiv:1711.09949] [INSPIRE].

[48] K. Ito and H. Shu, *Massive ODE/IM correspondence and non-linear integral equations for $A_r^{(1)}$-type modified affine Toda field equations*, J. Phys. **A 51** (2018) 385401 [arXiv:1805.08062] [INSPIRE].

[49] Y. Wang and D. Xie, *Codimension-two defects and Argyres-Douglas theories from outer-automorphism twist in 6d $(2,0)$ theories*, arXiv:1805.08839 [INSPIRE].

[50] T. Nishinaka, S. Sasa and R.-D. Zhu, *On the correspondence between surface operators in Argyres-Douglas theories and modules of chiral algebra*, JHEP **03** (2019) 091 [arXiv:1811.11772] [INSPIRE].

[51] I.A. Ramírez, *Mixed OPEs in $\mathcal{N} = 2$ superconformal theories*, JHEP **05** (2016) 043 [arXiv:1602.07269] [INSPIRE].

[52] S.M. Kuzenko and S. Theisen, *Correlation functions of conserved currents in $N = 2$ superconformal theory*, Class. Quant. Grav. **17** (2000) 665 [hep-th/9907107] [INSPIRE].

[53] M. Nirschl and H. Osborn, *Superconformal Ward identities and their solution*, Nucl. Phys. **B 711** (2005) 409 [hep-th/0407060] [INSPIRE].

[54] H. Osborn, *$N = 1$ superconformal symmetry in four-dimensional quantum field theory*, Annals Phys. **272** (1999) 243 [hep-th/9808041] [INSPIRE].

[55] J.-H. Park, *Superconformal symmetry and correlation functions*, Nucl. Phys. **B 559** (1999) 455 [hep-th/9903230] [INSPIRE].

[56] F.A. Dolan and H. Osborn, *On short and semi-short representations for four-dimensional superconformal symmetry*, Annals Phys. **307** (2003) 41 [hep-th/0209056] [INSPIRE].

[57] J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a higher spin symmetry*, J. Phys. **A 46** (2013) 214011 [arXiv:1112.1016] [INSPIRE].

[58] V. Alba and K. Diab, *Constraining conformal field theories with a higher spin symmetry in $d = 4$*, arXiv:1307.8092 [INSPIRE].

[59] V. Alba and K. Diab, *Constraining conformal field theories with a higher spin symmetry in $d > 3$ dimensions*, JHEP **03** (2016) 044 [arXiv:1510.02535] [INSPIRE].

[60] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton U.S.A. (1992).

[61] G. Arutyunov, B. Eden and E. Sokatchev, *On nonrenormalization and OPE in superconformal field theories*, Nucl. Phys. **B 619** (2001) 359 [hep-th/0105254] [INSPIRE].

[62] J.-F. Fortin, K. Intriligator and A. Stergiou, *Current OPEs in superconformal theories*, JHEP **09** (2011) 071 [arXiv:1107.1721] [INSPIRE].

[63] P. Agarwal, S. Lee and J. Song, *Vanishing OPE coefficients in 4d $N = 2$ SCFTs*, arXiv:1812.04743 [INSPIRE].

[64] F. Bonetti, C. Meneghelli and L. Rastelli, *VOAs labelled by complex reflection groups and 4d SCFTs*, arXiv:1810.03612 [INSPIRE].

[65] http://people.brandeis.edu/ headrick/Mathematica/