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Holomorphic classical limit for spin effects in gravitational and electromagnetic scattering

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Abstract: We provide universal expressions for the classical piece of the amplitude given by the graviton/photon exchange between massive particles of arbitrary spin, at both tree and one loop level. In the gravitational case this leads to higher order terms in the post-Newtonian expansion, which have been previously used in the binary inspiral problem. The expressions are obtained in terms of a contour integral that computes the Leading Singularity, which was recently shown to encode the relevant information up to one loop. The classical limit is performed along a holomorphic trajectory in the space of kinematics, such that the leading order is enough to extract arbitrarily high multipole corrections. These multipole interactions are given in terms of a recently proposed representation for massive particles of any spin by Arkani-Hamed et al. This explicitly shows universality of the multipole interactions in the effective potential with respect to the spin of the scattered particles. We perform the explicit match to standard EFT operators for $S = \frac{1}{2}$ and $S = 1$. As a natural byproduct we obtain the classical pieces up to one loop for the bending of light.

Keywords: Scattering Amplitudes, Black Holes, Effective Field Theories

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1 Introduction

Since the early days of QFT, the use of effective methods to describe the low energy regime of more fundamental theories [1–3] has proven extremely successful [4–6]. One of the most powerful applications of Effective Field Theories (EFTs) is the case where the high energy completion of the underlying theory is unknown. In this direction, the problem of General Relativity as an EFT has been studied as a tool for obtaining predictions whenever the relevant scales are much smaller than M_{Planck} [7, 8]. For this regime the methods of QFT can be safely applied to compute both classical and quantum long range observables. In this context, the motivation for these problems stems from the always increasing interest in the measurement of gravitational waves as definitive tests of GR, which has led to the acclaimed first detection by LIGO in 2016 [9, 10]. Specifically, the binary inspiral stage, defined by the characteristic scale $v^2 \sim Gm/r$, has been the subject of extensive research since it can be addressed with analytical methods [11–13]. The late inspiral stage and transition to unstable plunge have been also successfully adressed by the Effective One Body (EOB) approach [14–17].

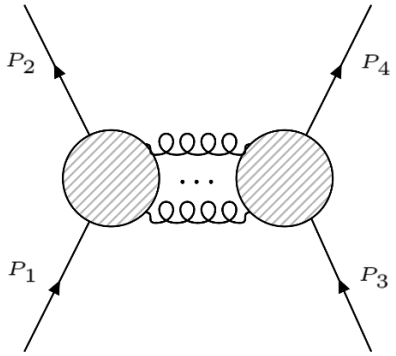

Figure 1. A typical scattering process contributing to the effective potential. In this case two massive particles represented by $P_1^2 = P_2^2 = m_a^2$ and $P_3^2 = P_4^2 = m_b^2$ exchange several gravitons. The momentum transfer is given by $K = P_1 - P_2 = (0, \vec{q})$ in the COM frame.

The key object in the study of the binary inspiral problem is the effective potential associated to a two-body system. This potential admits a non-relativistic expansion in powers of $v^2 \sim Gm/r$, known as the post-Newtonian (PN) expansion. Pioneered by the seminal work of Einstein-Infeld-Hoffman long ago [18], several attempts have been made to evaluate the potential at higher PN orders. The EFT approach uses diagrammatic techniques for classical sources combined with power-counting schemes to track the relevant scales [7, 19]. From a QFT side these computations can be done by treating the sources as particles and the PN expansion as contained in the loop expansion [20–23]. The setup is the $2 \to 2$ scattering of massive objects m_a and m_b, interacting through the exchange of multiple gravitons (figure 1). In this case the classical potential can be obtained from the long range behavior of the amplitude after implementing the Born approximation [24–26]. This classical piece is in turn extracted by setting the COM (Center of Mass) frame, in which the momentum transfer reads $|\vec{q}| = \sqrt{-t}$ and corresponds to the Fourier conjugate of the distance r. Calculations in this framework have proven very tedious, even though there have been remarkable simplifications in the context of non-relativistic approaches [13, 27–30]. In addition, the electromagnetic analog of the effective potential has been also discussed in [24, 25, 31, 32] in the context of classical corrections to Coulomb scattering. As expected the long range behavior of this potential, i.e. the $\frac{1}{r^n}$ falloff, is identical to the gravitational case. The computations are simpler in general and thus it also serves as a toy model for the PN problem.

One of the distinctive characteristics of the PN expansion is the treatment of the binary system as localized sources endowed with a tower of multipole moments. The evaluation of higher multipole moments starting at 1.5PN requires the incorporation of spin into the massive particles involved in the scattering process [26, 29], along with radiative corrections and finite size effects, see e.g. [33–38]. These spin contributions account for the internal angular momentum of the objects in the macroscopic setting [26, 39]. The universality of the gravitational coupling implies that it is enough to consider massive particles of spin S to evaluate the spin multipole effects up to order $2S$ in the spin vector $|\vec{S}|$. Such computation

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was first done up to 1-loop by Holstein and Ross in [40] and then by Vaidya in [26], leading to $|\vec{S}|^2$ and $|\vec{S}|^4$ results, respectively. The electromagnetic counterpart has also been discussed up to $|\vec{S}|^2$ [25]. Higher spin multipole moments are characterized by containing higher powers of the momentum transfer $|\vec{q}|$ and $|\vec{S}|$. Thus, in order to evaluate classical spin effects an expansion of the amplitude to arbitrarily subleading orders in $|\vec{q}| = \sqrt{-t}$ is required. This, together with the natural increase in difficulty for manipulating higher spin degrees of freedom in loop QFT processes [40], renders the computation virtually doable only within the framework of non-relativistic approaches or with the aid of a computer for higher PN orders [33, 35, 37, 41–48]. One of the main current tools for the derivation of higher PN results is the EFT formalism developed along [27, 29, 38, 49–51], see also [52] for a review of this line of work.

In this paper we find that the combination of several new tools can bypass some of the aforementioned difficulties. We provide a fully relativistic method to compute the classical part of the amplitude at any spin for both tree and 1-loop level. The difficulty in extracting arbitrarily subleading momentum powers is avoided by noting that the $|\vec{q}| \to 0$ and $t \to 0$ expansions can be disentangled outside the COM frame. We shall see that in the former case the constituents of the amplitude become degenerate whereas the latter is non-degenerate and natural from an on-shell perspective. In fact at tree-level one simply extracts the classical contribution as a t-channel residue. We will implement this both at tree and 1-loop level by providing careful definition of what we call the *holomorphic classical limit* (HCL). Using complexified momenta, we find that the *leading* order in the limit $t = K^2 \to 0$ is in fact enough to obtain the classical spin-multipoles in a covariant form. We can go back from the HCL to the COM frame (with real momenta) by analytically continuing $K^\mu \to (0, \vec{q})$ in our results.

To bypass the intrinsic complications due to the evaluation of higher spin loop processes we draw upon a battery of modern techniques based on the analytic structure of scattering amplitudes. In fact, techniques such as spinor helicity formalism, on-shell recursion relations (BCFW), and unitarity cuts have proven extremely fruitful for both computations of gravity and gauge theory amplitudes [53–58]. In this context, several simplifications in the computation of the 1-loop potential have already been found for scalar particles in [26, 31, 59, 60]. Recently, these methods have also been applied to the light-bending case [61–66], where one of the external particle carries helicity $|h| \in \{0, \frac{1}{2}, 1\}$, and universality with respect to $|h|$ was found. Here we extend these approaches by considering two more techniques, both very recently developed as a natural evolution of the previously mentioned. The first one appeared in [67], where Cachazo and the author proposed to use a generalized form of unitarity cuts, known as the Leading Singularity (LS), in order to extract the classical part of gravitational amplitudes leading to the effective potential. It was shown that while at tree level this simply corresponds to computing the t channel residue, at 1-loop the LS associated to the triangle diagram leads to a fully relativistic form containing the 1PN correction for scalar particles, through a multidispersive treatment in the t channel. The second technique was proposed by Arkani-Hamed et al. in [68] and gives a representation for *massive* states of arbitrary spin completely built from spinor helicity variables. Hence we use such construction to compute the LS associated to both the grav-

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itational and electromagnetic triangle diagram as well as the respective tree level residues, this time including higher spin in the external particles. Both contributions are suitable simplified by implementing the leading order Holomorphic Classical Limit in massive spinor variables. The result is an expansion in spinor operators, which are then matched to more standard EFT operators. As a crosscheck we recover the classical results for both gravity and EM presented in [25, 26, 40, 59] for $S \leq 1$. By suitably defining a massless limit, we are also able to address the light-like scattering situation and check the aforementioned universality of light bending phenomena.

As an important remark, in this work we restrict our attention to spinning particles minimally coupled to gravity or EM. This is what is needed to reproduce the effective potential and intrinsic multipole corrections associated to point-like sources, corresponding to black hole systems. As a consequence we find various universalities with respect to spin which are manifest in spinor helicity variables, and were previously argued in [26, 40, 60]. The non-minimal extension, phenomenologically relevant for evaluating finite size effects [34, 37, 38], is left for future work.

This paper is organized as follows. In section 2 we review the kinematics and spin considerations associated to the $2 \to 2$ process, which motivates the holomorphic classical limit. We then proceed to give a short overview of the notation and conventions used along the work, specifically those regarding manipulations of spinor helicity variables. Next, in section 3 we review scalar scattering and implement the HCL to extract the electromagnetic and gravitational classical part from leading singularities at tree and 1-loop level, including the light bending case. Next, in section 4 we introduce the new spinor helicity representation for massive kinematics, leaving the details to appendix A, and use it to extend the previous computations to spinning particles. In section 5 we discuss the applications of these results as well as possible future directions. Finally, in appendix B we provide a prescription to match our results to the standard form of EFT operators appearing in the effective potential for the cases $S = \frac{1}{2}, 1$.

2 Preliminaries

2.1 Kinematical considerations and the HCL

In the EFT framework, the off-shell effective potential can be extracted from the S-matrix element associated to the process depicted in figure 1, see e.g. [59]. The standard kinematical setup for this computation is given by the Center of Mass (COM) coordinates, which are defined by $\vec{P}_1 + \vec{P}_3 = 0$. We can check that 4-particle kinematics for this setup imply

$$(P_1 + P_3) \cdot (P_1 - P_2) = 0\,, \tag{2.1}$$

which means that the momentum transfer vector $K := (P_1 - P_2)$ has the form

$$K = (0, \vec{q})\,, \quad t = K^2 = -\vec{q}^2\,, \tag{2.2}$$

in the COM frame. For completeness, we also define here the average momentum $\vec{p}$ as

$$\frac{P_1 + P_2}{2} = (E_a, \vec{p})\,, \quad \frac{P_3 + P_4}{2} = (E_b, -\vec{p})\,, \tag{2.3}$$

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where E_a,E_b are the respective energies for the COM frame, while $\vec{p}^2 \propto v^2$ gives the characteristic velocity of the problem. From these definitions we can solve for the explicit form of the momenta P_i, $i \in \{1,2,3,4\}$, and also easily check the transverse condition $\vec{p}\cdot\vec{q}=0$. In the non-relativistic limit $\frac{\sqrt{-t}}{m} = \frac{|\vec{q}|}{m} \to 0$, the center of mass energy $\sqrt{s}$ can be parametrized as a function of $\vec{p}^2$. In fact,

$$\begin{aligned} s &= (P_1+P_3)^2\,, \\ &= (E_a+E_b)^2 \\ &= (m_a+m_b)^2\left(1+\frac{\vec{p}^2}{m_a m_b}+O(\vec{p}^4)\right)+O(\vec{q}^2) \end{aligned} \tag{2.4}$$

Note that the remaining kinematic invariant may be obtained as $u = 2(m_a^2+m_b^2)-t-s$. In practice, we regard the amplitude for figure 1 as a function $M(t,s)$, which may contain poles and branch cuts in both variables. At this point we can also introduce the spin vector S^μ, which will be in general constructed from polarization tensors associated to the spinning particles, see e.g. [26]. Suppose for instance that the particle m_b carries spin, then the spin vector satisfies the transversal condition

$$S^\mu (P_3+P_4)_\mu = 0\,, \tag{2.5}$$

implying that in the non-relativistic regime $\vec{p}\to 0$ the 4-vector becomes purely spatial, i.e. $S^\mu \to (0,\vec{S})$.

The PN expansion and the corresponding EM analog then proceed by extracting the classical (i.e. $\hbar$-independent) part of the scattering amplitude $M(t,s)$ expressed in these coordinates. This is done by selecting the lowest order in $|\vec{q}|$ for fixed powers of G, spin $|\vec{S}|$ and $\vec{p}^2$ [26, 59]. This claim is argued by dimensional analysis, where it is clear that for a given order in G each power of $|\vec{q}|$ carries a power of $\hbar$ unless a spin factor $|\vec{S}|$ is attached [29, 59, 69]. Here G is equivalent to 1PN order and acts as a loop counting parameter, while the latter quantities can be counted as 1PN corrections each [26]. For a given number of loops and fixed value of the COM energy s, the expansion around $t=-\vec{q}^2=0$ used to select the classical pieces coincides with the non-relativistic limit $\frac{\vec{q}}{m}\to 0$. Additionally, in the COM frame the 2^{2n}-pole and 2^{2n-1}-pole interactions due to spin emerge in the form [25, 26, 29]

$$V_S = c_1(|\vec{p}|)S_1^{i_1\ldots i_{2n}} q_{i_1}\ldots q_{i_{2n}} + c_2(|\vec{p}|)S_2^{i_1\ldots i_{2n}} q_{i_1}\ldots q_{i_{2n-1}} p_{i_{2n}} = O(|\vec{q}|^{2n-1})\,, \tag{2.6}$$

where $S_j^{i_1\ldots i_{2n}}$, $j=1,2$, are constructed from polarization tensors of the scattered particles in such a way that the powers of $|\vec{S}|$ exactly match the powers of $|\vec{q}|$ in V_S. They are, in consequence, classical contributions and correspond to the so-called mass ($j=1$) and current ($j=2$) multipoles [70]. These terms arise in the scattering amplitude when one of the external particles, for instance the one with mass m_a, carries spin $S_a \geq n$. Two difficulties arise when trying to extract them directly: first, they require to expand the scattering amplitude to subleading orders in $|\vec{q}|$, which becomes cumbersome as one increases the powers of spin [25, 40]. Second, these orders generically involve pure quantum mechanical contributions which need to be discarded from the result. In an EFT setup

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these difficulties can be sorted out via a power-counting scheme [7, 19]. We now propose a different approach which is natural from the on-shell point of view. Inspection of the amplitude shows that the structures (2.6) arise in the COM frame from generic contractions of the form

$$S^{\mu_1\cdots\mu_m} K_{\mu_1}\cdots K_{\mu_k}(P_{a_{k+1}})_{\mu_{k+1}}\cdots(P_{a_m})_{\mu_m}\,, \quad a_i\in\{1,3\}. \tag{2.7}$$

after the non-relativistic limit is taken. Here we have $k=2n$ for mass multipoles and $k=2n+1$ for current multipoles, while the tensor S is the relativistic version of (2.6).[1] The advantage of writing the multipole terms in the covariant form (2.7) is that, being frame-independent, they are non-degenerate in the limit $t=K^2\to 0$. This means they can be computed directly at leading order in a t-expansion of the amplitude, which we call the Holomorphic Classical Limit (HCL). All the neglected pieces, i.e. subleading orders in t, simply correspond to quantum corrections. These operators are the same at both tree and loop level, and in a slight abuse of terminology we refer to them as EFT operators.[2] Thus our strategy is to compute the coefficients associated to them from the evaluation of the amplitude (more precisely, the Leading Singularity) in the HCL. In the next section we give a systematic implementation of the limit in terms of spinor-helicity variables, which we use to construct 'spinor' operators to be matched to the EFT operators in appendix B. Both of these are well defined in the HCL whereas (2.6) is degenerate. In fact, the limit is not compatible with the COM frame (2.2), thus we require an analytic continuation to the region $t\neq 0$: we simply set $K=(0,\vec{q})$ in (2.7) (and the corresponding expressions for P_i) after the matching.

Some further remarks regarding the HCL are in order. First, as anticipated the term holomorphic stems from the on-shell condition $P_i\cdot K=\pm K^2$, $i\in\{1,\ldots,4\}$, which for $t\to 0$ yields $P_i\cdot K\to 0$. In turn, this implies that the external momenta P_i must be complexified. Hence, in order to reach the $t=0$ configuration we must consider an analytic trajectory in the kinematic space, which we can parametrize in terms of a complex variable β. We introduce such trajectory explicitly in section 3.2, where we also evaluate the amplitude as $\beta\to 1$. Second, we stress that just the HCL is enough to recover the classical potential with arbitrary multipole corrections. The complete non-relativistic limit can be further obtained by expanding around $s\to(m_a+m_b)^2$, i.e. expanding in $\vec{p}^2$ for a given power of $|\vec{q}|$. These corrections in $\vec{p}^2$ account for higher PN corrections when implemented through the Born approximation, which at 1-loop also requires one to subtract the iterated tree level potential. We perform the procedure only at the level of the amplitude and refer to [24, 26, 40, 59] for details on iterating higher PN corrections. As the expressions we provide for the classical piece correspond to all the orders in $\vec{p}^2$ encoded in a covariant way, we regard the HCL output as a fully relativistic form of the classical potential. The fact that the construction is covariant will also prove useful when defining the massless limit of external particles for addressing light-like scattering. Finally, the soft behavior of

[1] More precisely, it is made from powers of Pauli-Lubanski spin vector and as such satisfies some general constraints: it vanishes when the momentum transfer K is orthogonal to the polarization tensors $K_{\mu_1}\epsilon^{\mu_1\cdots\mu_S}=0$ (see i.e. [26]), or equivalently, when it is aligned with the spin vector.

[2] This is to contrast them with the spinor operators defined in section 4, which are to be matched to EFT operators.

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the momentum transfer $K \to 0$, which is the equivalent of $\frac{\vec{q}}{m} \to 0$ for COM coordinates, is not needed and we find that it does not lead to further insights on the behavior of the potential.

2.2 Conventions

Before proceeding to the computation of scattering processes, we set the conventions that will be used extensively throughout the paper. The constructions are based on the acclaimed spinor helicity variables, see e.g. [66, 71] for a review. Here we just stress some of the notation.

Using a mostly minus $(+,-,-,-)$ signature, a generic 4-momentum P^μ, with $P^2 = m^2$, can be written as

$$P_{\alpha\dot{\alpha}} = P^\mu(\sigma_\mu)_{\alpha\dot{\alpha}}, \quad \bar{P}^{\dot{\alpha}\alpha} = P^\mu(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}\,, \tag{2.8}$$

where $\sigma^\mu = (\mathbb{I}, \sigma^i)$ and indices are lowered/raised from the left via the ϵ tensor,[3] for instance $\bar{P}^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}P_{\dot{\beta}\beta}$ or simply $\bar{P} = \epsilon P \epsilon^T$. We will also use P to refer both to the 4-vector P^μ and the bispinor $P_{\alpha\dot{\alpha}}$. For instance,

$$\begin{aligned} P_{\alpha\dot{\alpha}}\bar{P}^{\dot{\alpha}\beta} &= m^2\delta_\alpha^\beta\,, \quad \text{or} \quad P\bar{P} = m^2\mathbb{I}\,, \\ P_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}\alpha} &= \mathrm{Tr}(P\bar{Q}) = 2P\cdot Q\,. \end{aligned} \tag{2.9}$$

A massless momentum satisfies $\det(K) = 0$ and hence can be written as

$$K_{\alpha\dot{\alpha}} = |\lambda]_{\dot{\alpha}}\langle\lambda|_\alpha\,, \quad \text{or simply} \quad K = |\lambda]\langle\lambda|. \tag{2.10}$$

The conjugates are defined by $[\lambda| = \epsilon|\lambda]$ and $|\lambda\rangle = \langle\lambda|\epsilon^T$. With these definitions $\bar{K} = |\lambda\rangle[\lambda|$. The bilinears $[\lambda\eta] = [\lambda|^{\dot{\alpha}}|\eta]_{\dot{\alpha}}$ and $\langle\lambda\eta\rangle = \langle\lambda|_\alpha|\eta\rangle^\alpha$ are then naturally defined as the corresponding contractions. From eq. (2.9) we have

$$[\lambda\eta]\langle\eta\lambda\rangle = 2K\cdot R\,, \tag{2.11}$$

where $R = |\eta]\langle\eta|$. This also motivates the notation

$$[\lambda|P|\lambda\rangle = \langle\lambda|\bar{P}|\lambda]\,, \tag{2.12}$$

for the contraction $[\lambda|^{\dot{\alpha}}P_{\beta\dot{\alpha}}|\lambda\rangle^\beta$. In the following we may omit the spinor indices $(\alpha, \dot{\alpha})$ when possible and deal with 2×2 operators. In appendix A we use these variables to construct the representation for massive states of arbitrary spin, first introduced in [68].

3 Scalar scattering

In this section we recompute the Leading Singularity for gravitational scattering of both tree and 1-loop level amplitudes for the non-spinning case, as first presented in [67]. This time we embed the computation into the framework of the HCL, which will lead directly to

[3] Such that $\epsilon^{12} = -\epsilon_{12} = 1$.

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the classical contribution. We also present, without additional effort, the analogous results for the EM case. Along the way we introduce new variables which will prove helpful for the next sections.

Let us first introduce a dimensionless variable which will be well suited to describe the internal helicity structure of the scattering. Motivated by the $2 \to 2$ process described in section 2.1, we start by considering two massive particles interacting with a massless one. If both massive particles have the same mass m, the on-shell condition for the process implies $[k|P|k\rangle = 0$, where P is one of the (incoming) massive momenta and $K = |k]\langle k|$ corresponds to the momenta of the massless particle. Thus, as proposed in [68], it is natural to introduce dimensionless variables x and $\bar{x}$ such that

$$[k|P = mx\langle k|\,, \quad \langle k|\bar{P} = m\bar{x}[k|\,. \tag{3.1}$$

The condition $P\bar{P} = m^2$ yields $x\bar{x} = 1$. Note that x carries helicity weight $h = +1$ under the little group transformations of K. Furthermore, mx precisely corresponds to the stripped 3pt amplitude for the case in which the massive particle is a scalar and the massless particle has $h = 1$.[4] For higher helicity one simply finds $(h > 0)$ [68]

$$A^{(+h)}_{\text{scalar}} = \alpha(mx)^h\,, \qquad A^{(-h)}_{\text{scalar}} = \alpha(m\bar{x})^h\,. \tag{3.2}$$

The (minimal) coupling constant α has to be chosen according to the theory under consideration, determined once the helicity $|h|$ is given, i.e. $h = \pm 1$ for EM and $h = \pm 2$ for gravity. Regarding the gravitational interaction, its universal character allows us to fix the coupling by $\alpha = \frac{\kappa}{2} = \sqrt{8\pi G}$ irrespective of the particle type, whereas for EM it will depend on the electric charge carried by such particle.

3.1 Tree amplitude

Let us start by computing the tree level contributions to the classical potential. As explained in [67], these can be directly obtained from the Leading Singularity, which for tree amplitudes is simply the residue at $t = 0$. Here, it is transparent that the analytic expansion around such pole will yield subleading terms t^n, $n \geq 0$, which are ultralocal (e.g. quantum) once the Fourier transform is implemented in COM coordinates $t = -\vec{q}^2$ [40]. Furthermore, by unitarity this residue precisely corresponds to the product of on-shell 3pt amplitudes (see figure 2), that is to say, we can use the leading term in the HCL to evaluate the classical potential. Note that, even though there exist different couplings contributing to the s and u channel, these correspond to contact interactions between the different particles and do not lead to a long-range potential [67].

With these considerations we proceed to compute the leading contribution to the Coulomb potential by considering the one-photon exchange diagram. Summing over both helicities and using (3.2) we find

$$M^{(0)}_{(0,0,1)} = \frac{1}{t}\left(A_3^{(+1)}(P_1)A_3^{(-1)}(P_3) + A_3^{(-1)}(P_1)A_3^{(+1)}(P_3)\right) = \alpha^2\frac{m_a m_b}{t}\left(x_1\bar{x}_3 + \bar{x}_1 x_3\right)\,. \tag{3.3}$$

[4] For real momenta we find that x corresponds to a phase. It also induces non-local behavior in the 3pt amplitudes [68]. However, we ignore these physical restrictions for now since we are describing generic 3pt amplitudes which will be used to construct the leading singularities.

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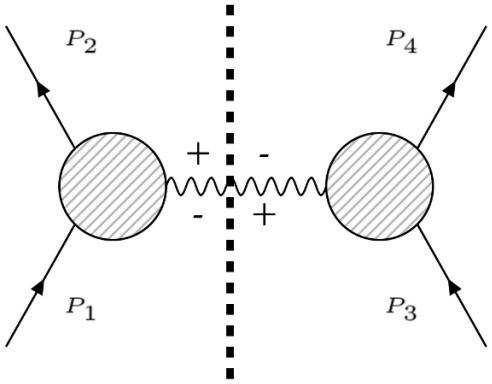

Figure 2. A one photon/graviton exchange process. In the HCL the internal particle is on-shell and the two polarizations need to be considered.

Here we have used $M^{(0)}_{(S_a,S_b,|h|)}$ to denote the classical piece of the $2 \to 2$ amplitude, as opposed to the notation $A_n(P_i)$ which we reserve for the n pt amplitudes used as building blocks. The index (0) indicates leading order (tree level), which will be equivalent to 0PN for the gravitational case. The subindex $(S_a, S_b, |h|) = (0, 0, 1)$ denotes spinless particles exchanging a photon.

The variables $x_1(\bar{x}_1)$ and $x_3(\bar{x}_3)$ are now associated to P_1 and P_3 respectively, through (3.1). An explicit form can be obtained in terms of the null momentum transfer $K = P_4 - P_3 = |k]\langle k|$, but it is not needed here. At this stage we introduce the kinematic variables

$$u := m_a m_b x_1 \bar{x}_3 \,, \quad v := m_a m_b \bar{x}_1 x_3 . \tag{3.4}$$

Note that these variables are defined only in the HCL, i.e. for $t = 0$. Each of these carries no helicity, i.e. it is invariant under little group transformations of the internal particle. However, they represent the contribution from the two polarizations in the exchange of figure 2, and as such they are swapped under parity. In appendix B we give explicit expressions for u and v in terms of their parity even and odd parts. Nevertheless, we stress that for this and the remaining sections the only identities which are needed can be stated as

$$uv = m_a^2 m_b^2 \,, \quad u + v = 2P_1 \cdot P_3 \,, \tag{3.5}$$

and readily follow from their definition and (3.1). We then regard the new variables as a (parity sensitive) parametrization of the s channel emerging in the HCL. Further expanding in the non-relativistic limit yields $u, v \to m_a m_b$.

With these definitions, we can now proceed to write the result in a parity invariant form as

$$M^{(0)}_{(0,0,1)} = \alpha^2 \frac{u+v}{t} = \alpha^2 \frac{s - m_a^2 - m_b^2}{t} \,. \tag{3.6}$$

After implementing COM coordinates and including the proper relativistic normalization, this leads to the Coulomb potential in Fourier space, which can be expanded in the limit $s \to (m_a + m_b)^2$. In fact, assuming both particles carry the same electric charge $e = \frac{\alpha}{\sqrt{2}}$,

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we can use (2.2), (2.4) to write

$$\frac{M^{(0)}_{(0,0,1)}}{4E_aE_b} = -\frac{e^2}{\vec{q}^2}\left(1+\frac{\vec{p}^2}{m_am_b}+\dots\right). \tag{3.7}$$

We are now in position to easily compute the one-graviton exchange diagram. The answer is again given by the parity invariant expression

$$M^{(0)}_{(0,0,2)} = \alpha^2\frac{u^2+v^2}{t} = \frac{\kappa^2}{4}\frac{(s-m_a^2-m_b^2)^2-2m_a^2m_b^2}{t}. \tag{3.8}$$

Again, this leads to a relativistic expression for the Newtonian potential, and can be put into the standard form by using the dictionary provided in subsection 2.1

$$\frac{M^{(0)}_{(0,0,2)}}{4E_aE_b} = -4\pi G\frac{m_am_b}{\vec{q}^2}\left(1+\frac{(3m_a^2+8m_am_b+3m_b^2)}{2m_a^2m_b^2}\vec{p}^2+\dots\right), \tag{3.9}$$

in agreement with the computations in [20, 59, 67, 72].

Two final remarks are in order. First, it is interesting that the gravitational result can be directly obtained by squaring the u, v variables, i.e. squaring both contributions from the EM case. This will be a general property that we will encounter again for the discussion of the Compton amplitude in the next section, as was already pointed out in [60] in relation with the double-copy construction. Second, it is worth noting that up to this point no parametrization of the external momenta was needed. In other words, the tree level computation was done solely in terms of (pseudo)scalar variables. As we will see now, the 1-loop case can be addressed with the help of an external parametrization specifically designed for the HCL. This parametrization will provide an extension of the variables u and v in a sense that will become clear.

3.2 1-loop amplitude: triangle leading singularity

Here we proceed to compute the triangle LS [67] in order to obtain the first classical correction to the potential. This leading singularity is associated to the 1-loop diagram arising from two photons/gravitons exchange, figure 3. As explained in the previous work, the LS of the triangle diagram captures the second discontinuity of the amplitude as a function of t, which is precisely associated to the non-analytic behavior $\frac{1}{\sqrt{-t}} = \frac{1}{|\vec{q}|}$. In the gravitational case this accounts for G^2 corrections or equivalently 1PN. In order to track exclusively this contribution we proceed to discard higher (analytic and non-analytic) powers of t by appealing to the HCL. This can be implemented to any order in t by means of the following parametrization of the external kinematics

$$\begin{aligned} P_3 &= |\eta]\langle\lambda| + |\lambda]\langle\eta|\,, \\ P_4 &= \beta|\eta]\langle\lambda| + \frac{1}{\beta}|\lambda]\langle\eta| + |\lambda]\langle\lambda|\,, \\ \frac{t}{m_b^2} &= \frac{(\beta-1)^2}{\beta}\,, \\ \langle\lambda\eta\rangle &= [\lambda\eta] = m_b\,. \end{aligned} \tag{3.10}$$

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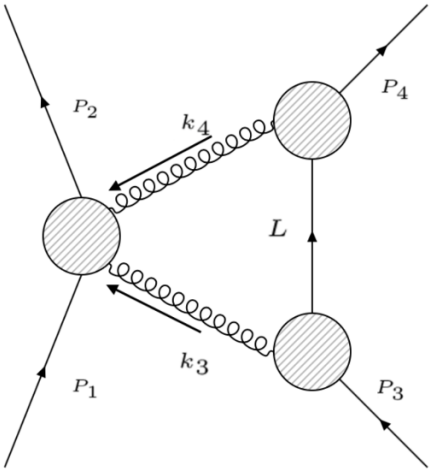

Figure 3. The triangle diagram used to compute the leading singularity, corresponding to the b-topology. The a-topology is obtained by reflection, i.e. by appropriately exchanging the external particles.

The parametrization is constructed by first defining a complex null vector $K = |\lambda]\langle\lambda|$ orthogonal to P_3 and P_4. Then the bispinors $(P_3)_{\alpha\dot{\alpha}}$ and $(P_4)_{\alpha\dot{\alpha}}$ are expanded in a suitably constructed basis, which also provides the definition of $|\eta]_{\dot{\alpha}}$ and $\langle\eta|_{\alpha}$ up to a scale which is fixed by the fourth condition. As explained in appendix A (following the lines of [68]) this basis also provides a representation for the little group associated to massive states. The dimensionless parameter β was called x in [67] and was introduced as a natural description of the t channel. In this sense, this parametrization should be regarded as an extension of the one presented there, which can be recovered for $\beta^2 \neq 1$ by means of the shift

$$|\eta] \to |\eta] + \frac{\beta}{1-\beta^2}|\lambda]\,, \quad \langle\eta| \to \langle\eta| - \frac{\beta}{1-\beta^2}\langle\lambda|\,. \tag{3.11}$$

However, in this case we are precisely interested in the degenerate point $\beta = 1$, i.e. $t = 0$, in order to define the HCL. For this point we have $P_4 - P_3 = K = |\lambda]\langle\lambda|$ as the null momentum transfer. As opposed to the tree level case, such momentum is not associated to any particle in the exchange of figure 3, but distributed between the internal photons/gravitons. In general for $\beta \neq 1$, K is just an auxiliary vector and thus we need not to consider little group transformations for $|\lambda], \langle\lambda|$, i.e. these are fixed spinors. Finally, we also provide a parametrization for the s channel by extending the definitions (3.4) for $t \neq 0$

$$u = [\lambda|P_1|\eta\rangle\,, \quad v = [\eta|P_1|\lambda\rangle\,, \tag{3.12}$$

such that $u + v = 2P_1 \cdot P_3$ and $uv \to m_a^2 m_b^2 \quad \text{as} \quad \beta \to 1$.

We are now well equipped to evaluate the triangle Leading Singularity. Here we sketch the computation of the contour integral and refer to [67] for further details. It is given by

$$\begin{aligned} M^{(1,b)}_{(0,0,|h|)} = \frac{1}{4} \sum_{h_3,h_4=\pm|h|} \int_{\Gamma_{\rm LS}} d^4L\, \delta(L^2 - m_b^2)\, \delta(k_3^2)\, \delta(k_4^2) \\ \times A_4(P_1, -P_2, k_3^{h_3}, k_4^{h_4}) \times A_3(P_3, -L, -k_3^{-h_3}) \times A_3(-P_4, L, -k_4^{-h_4})\,, \end{aligned} \tag{3.13}$$

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where the superscript $(1,b)$ denotes the (1-loop) triangle b-topology depicted in figure 3. The a-topology is simply obtained by exchanging particles m_a and m_b: we leave the explicit procedure for the appendix and in the following we deal only with $M^{(1,b)}_{(0,0,|h|)}$. In (3.13) we denote by A_3 and A_4 to the respective tree level amplitudes entering the diagram (note the minus sign for *outgoing* momenta), and

$$k_3 = -L + P_3\,, \quad k_4 = L - P_4\,. \tag{3.14}$$

The sum is performed over propagating internal states and enforces matching polarizations between the 3pt and 4pt amplitudes. Γ_{LS} is a complex contour defined to enclose the emerging pole in (3.13). This pole will be explicit after a parametrization for the loop momenta L is implemented and the triple-cut corresponding to the three delta functions is performed. This will leave only a 1-dimensional contour integral for a suitably defined $z \in \mathbb{C}$, where $L = L(z)$. We now use the previously defined basis of spinors to parametrize

$$\begin{aligned} L(z) &= z\ell + \omega K\,, \\ \ell &= A|\eta]\langle\lambda| + B|\lambda]\langle\eta| + AB|\lambda]\langle\lambda| + |\eta]\langle\eta|\,, \end{aligned} \tag{3.15}$$

where one scale in ℓ has been absorbed into z and we have further imposed the condition $\ell^2 = 0$. Using eqs. (3.10), we find that implementing the triple-cut in (3.13) fixes $\omega(z) = -\frac{1}{z}$, while $A(z), B(z)$ become simple rational functions of z and β. The integral then takes the form

$$\begin{aligned} M^{(1,b)}_{(0,0,|h|)} = \sum_{h_3,h_4} \frac{\beta}{16(\beta^2-1)m_b^2} \int_{\Gamma_{\rm LS}} \frac{dy}{y} A_4(P_1, -P_2, k_3^{h_3}(y), k_4^{h_4}(y)) \\ \times A_3(P_3, -L(y), -k_3^{-h_3}(y)) \times A_3(-P_4, L(y), -k_4^{-h_4}(y))\,, \end{aligned} \tag{3.16}$$

where $y := -\frac{z}{(\beta-1)^2}$ and we now define the contour to enclose the emergent pole at $y = \infty$, i.e. $\Gamma_{\rm LS} = S^1_\infty$.[5] The internal massless momenta are given by

$$\begin{aligned} k_3(y) &= \underbrace{\frac{1}{\beta+1}\left(|\eta](\beta^2-1)y + |\lambda](1+\beta y)\right)}_{|k_3]} \underbrace{\frac{1}{\beta+1}\left(\langle\eta|(\beta^2-1) - \frac{1}{y}\langle\lambda|(1+\beta y)\right)}_{\langle k_3|}\,, \\ k_4(y) &= \underbrace{\frac{1}{\beta+1}\left(-\beta|\eta](\beta^2-1)y + |\lambda](1-\beta^2 y)\right)}_{|k_4]} \underbrace{\frac{1}{\beta+1}\left(\frac{1}{\beta}\langle\eta|(\beta^2-1) + \frac{1}{y}\langle\lambda|(1-y)\right)}_{\langle k_4|}\,. \end{aligned} \tag{3.17}$$

As $\frac{\beta}{\beta^2-1} \to \frac{m_b}{2\sqrt{-t}}$ for the HCL, we find that the expression (3.16) already contains the required classical correction when the leading term of the integrand, around $\beta = 1$, is extracted. We can straightforwardly evaluate the 3pt amplitudes at $\beta = 1$, giving finite contributions. This simplification will indeed prove extremely useful for the $S > 0$ cases in

[5] Also the choice $y = 0$ is permitted for the contour, i.e. $\Gamma_{\rm LS} = S^1_0$. This choice does not matter in the HCL since the leading piece in (3.16) is invariant under the inversion of the contour [67].

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section 4. On the other hand, for the 4pt amplitude the limit $\beta \to 1$ is needed to obtain a finite answer, since it contains a pole in the t channel.

Explicitly, at $\beta = 1$ the internal momenta are given by

$$\begin{aligned} k_3^0(y) &= \underbrace{\frac{1}{2}|\lambda](1+y)}_{|k_3]} \underbrace{\frac{-1}{2y}\langle\lambda|(1+y)}_{\langle k_3|}\,, \\ k_4^0(y) &= \underbrace{\frac{1}{2}|\lambda](1-y)}_{|k_4]} \underbrace{\frac{1}{2y}\langle\lambda|(1-y)}_{\langle k_4|}\,. \end{aligned} \tag{3.18}$$

We thus note that in the HCL both internal momenta are collinear and aligned with the momentum transfer K. For the standard non-relativistic limit defined in the COM frame the condition $\beta \to 1$ certainly implies the soft limit $K \to 0$ and in general leads to vanishing momenta for the gravitons and vanishing 3pt amplitudes at $\beta = 1$.

Now, using the expression (3.1) for the momenta P_3 and (outgoing) P_4, we readily find

$$x_3 = x_4 = -y\,, \tag{3.19}$$

such that the 3pt amplitudes are given (at $\beta = 1$) by

$$\begin{aligned} &A_3(P_3, -L(y), -k_3^{+|h|}(y))A_3(-P_4, L(y), -k_4^{-|h|}(y))\Big|_{\beta=1} = \alpha^2 m_b^2 \\ &A_3(P_3, -L(y), -k_3^{+|h|}(y))A_3(-P_4, L(y), -k_4^{+|h|}(y))\Big|_{\beta=1} = \alpha^2 m_b^2 (y^2)^{|h|}\,. \end{aligned} \tag{3.20}$$

We note that for $h_3 = -h_4$ the contribution from the 3pt amplitudes is invariant under conjugation. In fact, as can be already checked from (3.18) the conjugation is induced by $y \to -y$. Even though the full contribution from the triangle leading singularity requires to sum over internal helicities, in the HCL $\beta \to 1$ the conjugate configuration $h_3 = -h_4 = -|h|$ yields the same residue, while the configurations $h_3 = h_4$ yield none as we explain below. This means that the full result can be obtained by evaluating the case $h_3 = -h_4 = +|h|$ and inserting a factor of 2. Returning to the computation, (3.16) now reads

$$M^{(1,b)}_{(0,0,|h|)} = \frac{\alpha^2}{16}\left(\frac{m_b}{\sqrt{-t}}\right)\int_\infty \frac{dy}{y} A^{(-+)}_{(4,|h|)}(\beta \to 1)\,, \tag{3.21}$$

where $A^{(-+)}_{(4,|h|)}(\beta \to 1)$ is the leading order of the 4pt. Compton-like amplitude, given by

$$A^{(-+)}_{(4,|h|)} = \alpha^2 \begin{cases} \dfrac{\langle k_3|P_1|k_4]^2}{\langle k_3|P_1|k_3]\langle k_3|P_2|k_3]} & |h| = 1 \\ \frac{1}{t} \times \dfrac{\langle k_3|P_1|k_4]^4}{\langle k_3|P_1|k_3]\langle k_3|P_2|k_3]} & |h| = 2 \end{cases} \tag{3.22}$$

We note that the stripped Compton amplitudes (3.22) exhibit the double-copy factorization $A_{(4,2)} = 4\frac{(k_3\cdot P_1)(k_3\cdot P_2)}{t}(A_{(4,1)})^2$ as explained in [60]. We will come back at this point in

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section 4. By considering the definitions (3.12), and using (3.17) together with momentum conservation constraints, we find the HCL expansions

$$\begin{aligned}\langle k_3|P_1|k_4] &= (\beta-1)\left(u\frac{1-y}{2}+v\frac{1+y}{2}+\frac{(v-u)(1-y^2)}{4y}\right)+O(\beta-1)^2\,,\\ \langle k_3|P_1|k_3] &= \langle k_3|P_2|k_3]+O(\beta-1)^2=(\beta-1)\frac{(v-u)(1-y^2)}{4y}+O(\beta-1)^2\,.\end{aligned} \tag{3.23}$$

where it is understood that u, v are evaluated at $\beta = 1$. We note that the conjugation $y \to -y$ is equivalent to change $u \leftrightarrow v$, as expected. Also, we can now argue why the Compton amplitude gives a finite answer in the limit $\beta \to 1$. Consider for instance the gravitational case. By unitarity, such limit induces a t channel factorization into a 3-graviton amplitude and a scalar-scalar-graviton amplitude A_3. Because of the collinear configuration (3.18) at $\beta = 1$, the 3-graviton amplitude vanishes at the same rate as the t channel propagator $\sim (\beta-1)^2$, yielding a finite result. Note that, for this factorization, regular terms in t will contribute to the result and hence these 3pt factors are not enough to compute the HCL answer.

At this stage we exhibit for completeness the expressions for the Compton amplitude in the case of same helicities. It is given by

$$A^{(4,|h|)}_{(++)} = \alpha^2 \begin{cases} \dfrac{m_a^2\,[k_3k_4]^2}{\langle k_3|P_1|k_3\rangle\langle k_3|P_2|k_3]} & |h|=1 \\ \frac{1}{t}\times\dfrac{m_a^4\,[k_3k_4]^4}{\langle k_3|P_1|k_3\rangle\langle k_3|P_2|k_3]} & |h|=2 \end{cases} \tag{3.24}$$

By expanding $[k_3k_4]$ in an analogous form to (3.23) and, together with (3.20), inserting it back into (3.16) we easily find that this gives indeed vanishing residue. In fact, this can also be checked to any order in $(\beta-1)$, i.e. with no expansion at all [67]. As anticipated, the configurations $h_3 = h_4$ simply do not lead to a classical potential.

Finally, by inserting (3.23) into (3.21) we find that the residue is trivial ($\text{Res}_\infty = 1$) for $|h| = 1$, while for $|h| = 2$ we have

$$M^{(1,b)}_{(0,0,2)} = \frac{3\alpha^4 m_b}{2^7\sqrt{-t}}(5u^2+6uv+5v^2)\,. \tag{3.25}$$

The expression is indeed symmetric in u, v, as expected by parity invariance. By using (3.5) we can write (3.26) in an analogous form to its tree level counterpart (3.8)

$$M^{(1,b)}_{(0,0,2)} = G^2\pi^2\frac{3m_b}{2\sqrt{-t}}\left(5(s-m_a^2-m_b^2)^2-4m_a^2m_b^2\right)\,. \tag{3.26}$$

The contribution $M^{(1,a)}_{(0,0,2)}$ is obtained by exchanging $m_a \leftrightarrow m_b$. After implementing the Born approximation as explained in [24, 40], this indeed recovers the 1PN form of the effective potential including the corrections in $\vec{p}^2$ [26, 40, 59, 67, 72].

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3.3 Massless probe particle

Here we show that the massless case $m_a = 0$ can be regarded as a smooth limit defined in the variables u, v. In this case such limit is natural to define since both massless and massive scalar fields contain the same number of degrees of freedom. In appendix A.1 we show, however, how to extend this construction to representations with nonzero spin. In the following we focus for simplicity in the gravitational case, the electromagnetic analog being straightforward. Moreover, the gravitational case is motivated by the study of light bending phenomena within the framework of EFT, see [61, 66].

In order to discuss the massless limit, it is convenient to absorb the mass into the definition of $x, \bar{x}$ given in (3.1), i.e. these quantities now carry units of energy. Then, the massless condition $P_3\bar{P}_3 = 0$ is equivalent to $x_3\bar{x}_3 = 0$, thus one of the helicity configurations in (3.2) must vanish at $\beta = 1$. This choice corresponds to selecting one of the graviton polarizations to give vanishing contribution, that is either $u = 0$ or $v = 0$. Due to parity invariance the election is not relevant, hence we put $v = 0$ and find from (3.5)

$$u = s - m_b^2 \,, \tag{3.27}$$

which in turn yields

$$\begin{aligned} M^{(0)}_{m_a=0} &= \alpha^2 \frac{u^2}{t} \\ &= \alpha^2 \frac{(s-m_b^2)^2}{t} \,. \end{aligned} \tag{3.28}$$

Analogously, for the 1-loop correction (3.25) we find

$$\begin{aligned} M^{(1,b)}_{m_a=0} &= \frac{3\alpha^4 m_b (5u^2)}{2^7\sqrt{-t}} \\ &= \frac{15\alpha^4}{2^7} \times \frac{m_b (s-m_b^2)^2}{\sqrt{-t}} \,. \end{aligned} \tag{3.29}$$

After including the normalization factor $(4E_aE_b)^{-1} \approx (4E_am_b)^{-1}$ we find that this recovers the 1PN correction of the effective potential for a massless probe particle [61, 65]. It is important to note that in this result only the $b-$topology LS contributes, i.e. no symmetrization is needed. This is because the triangle LS scales with the mass, i.e. for the $a-$topology is proportional to $\frac{m_a}{\sqrt{-t}}$ and thus vanishes in this case. In fact, classical effects require at least one massive propagator entering the loop diagram [69], see also discussion. We will again resort to this fact in section 4.3, where we construct the massless limit for spinning particles.

4 HCL for spinning particles

In this section we proceed to consider the case of particles with nonzero spin. That is, we extend the HCL evaluation of the triangle leading singularity presented in section 3 but this time for external particles with masses m_a, m_b and spins S_a, S_b respectively. By

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using the Born approximation, the LS leads to the 1-loop effective potential arising in gravitational or electromagnetic scattering of spinning objects, already computed in [40] for $S_a, S_b \in \{0, \frac{1}{2}, 1\}$. Here we provide an explicit expression for the tree level LS and a contour integral representation for the 1-loop correction, both valid for any spin. We explicitly expand the contour integral for $S_a \leq 1$, S_b arbitrary. In appendix B we explain how to recover the results of [40] by projecting our corresponding expression in the HCL to the standard EFT operators.

We start by explaining a novel spinor helicity representation for the little group of a massive particle of spin S, first introduced by Arkani-Hamed et al. [68]. The space is spanned by $2S+1$ polarization states, corresponding to the spin S representation of SU(2). Following the lines of section 3 we will focus on the 3pt. amplitudes $A_3(P_3, P_4, K)$ as operators acting on in this space, which will then serve as building blocks for the leading singularities. In our case, it will be natural to take advantage of the parametrization of the previous section,

$$\begin{aligned} P_3 &= |\eta]\langle\lambda| + |\lambda]\langle\eta|\,, \\ P_4 &= \beta|\eta]\langle\lambda| + \frac{1}{\beta}|\lambda]\langle\eta| + |\lambda]\langle\lambda|\,, \end{aligned} \tag{4.1}$$

to construct the little group representation for momenta P_3 and P_4 (carrying the same spin S) in a simultaneous fashion. We will denote the respective $2S+1$ dimensional Hilbert spaces by V_3^S and $\bar{V}_4^S$. In appendix A we explicitly construct $V_3^{\frac{1}{2}}$ and $\bar{V}_4^{\frac{1}{2}}$ starting from the well known Dirac spinor representation. For general spin, a basis for these spaces is given by the $2S$-th rank tensors[6]

$$\begin{aligned} |m\rangle &= \frac{1}{[\lambda\eta]^S} \underbrace{|\lambda] \odot \ldots \odot |\lambda]}_{m} \odot \underbrace{|\eta] \odot \ldots \odot |\eta]}_{2S-m} \in V_3^S\,, \\ \langle n| &= \frac{1}{[\lambda\eta]^S} \underbrace{[\lambda| \odot \ldots \odot [\lambda|}_{n} \odot \underbrace{[\eta| \odot \ldots \odot [\eta|}_{2S-n} \in \bar{V}_4^S\,, \end{aligned} \tag{4.2}$$

with $m, n = 0, \ldots, 2S$. Here the symbol $\odot$ denotes the symmetrized tensor product. The normalization is chosen for latter convenience, i.e.

$$\begin{aligned} \eta_{\dot\alpha} \odot \lambda_{\dot\beta} &= \frac{\eta_{\dot\alpha}\lambda_{\dot\beta} + \eta_{\dot\beta}\lambda_{\dot\alpha}}{\sqrt{2}}\,, \\ \eta_{\dot\alpha} \odot \lambda_{\dot\beta} \odot \lambda_{\dot\gamma} &= \frac{\eta_{\dot\alpha}\lambda_{\dot\beta}\lambda_{\dot\gamma} + \eta_{\dot\beta}\lambda_{\dot\alpha}\lambda_{\dot\gamma} + \eta_{\dot\gamma}\lambda_{\dot\alpha}\lambda_{\dot\beta}}{\sqrt{3}}\,, \end{aligned} \tag{4.3}$$

etc. As we explicitly show below, in this framework we regard the 3pt amplitudes as operators $A_S : \bar{V}_4^S \otimes V_3^S \to \mathbb{C}$, that is, they are to be contracted with the states given in (4.2) for both particles. The representation is symmetric and anti-chiral in the sense

[6] The notation $|m\rangle$ for the states may seem unfortunate since it is similar to the one for angle (chiral) spinors. However, as we will be mostly using the anti-chiral representation for spinors, the risk of confusion is low.

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that it is spanned by symmetrizations of the anti-chiral spinors $|\lambda]$, $|\eta]$. Further details on the choice of basis and the chirality are given in appendix A (see also [68]).

Consider then the 3pt amplitudes for two particles of momenta P_3, P_4 and spin S interacting with a massless particle of momenta $K = P_4 - P_3$ and helicity $\pm h$. From (4.1) we see that the on-shell condition $K^2 = 0$ sets $\beta = 1$, i.e. $K = |\lambda]\langle\lambda|$. For the massless particle, we choose the standard representation in terms of the spinors $\langle k| = \frac{\langle\lambda|}{\sqrt{x}}$ and $[k| = \sqrt{x}[\lambda|$, where x carries helicity weight $+1$ and agrees with the definition (3.1) for our parametrization. Note that $[\lambda|$ and $\langle\lambda|$ remain fixed under little group transformations. With these conventions the minimally coupled 3pt amplitudes are given by the operators

$$\begin{aligned} A_S^{(+h)} &= \alpha(mx)^h \left(1 - \frac{|\lambda][\lambda|}{m}\right)^{\otimes 2S} = \alpha(mx)^h \left(1 - \frac{|\lambda][\lambda|}{m}\right) \otimes \ldots \otimes \left(1 - \frac{|\lambda][\lambda|}{m}\right) , \\ A_S^{(-h)} &= \alpha(m\bar{x})^h = \alpha \left(\frac{m}{x}\right)^h . \end{aligned} \tag{4.4}$$

These expressions represent extensions of the ones given in (3.2). Note that we have omitted trivial tensor structures (i.e. the identity operator) in (4.4). For example, in the second line the explicit index structure is

$$\left(A_S^{(-h)}\right)^{\dot\alpha_1 \ldots \dot\alpha_{2S}}_{\dot\beta_1 \ldots \dot\beta_{2S}} = \alpha \left(\frac{m}{x}\right)^h \left(\mathbb{I}^{\otimes 2S}\right)^{\dot\alpha_1 \ldots \dot\alpha_{2S}}_{\dot\beta_1 \ldots \dot\beta_{2S}} = \alpha \left(\frac{m}{x}\right)^h \delta^{\dot\alpha_1}_{\dot\beta_1} \ldots \delta^{\dot\alpha_{2S}}_{\dot\beta_{2S}} . \tag{4.5}$$

The value for the amplitude is now obtained as the matrix element $\langle n|A_S^{(\pm h)}|m\rangle$. This contraction is naturally induced by the bilinear product $[\,,\,]$ of spinors. For instance, consider the matrix element associated to the transition of particle of momenta P_3 and polarization $|m\rangle$ to momenta P_4 and polarization $|n\rangle$, while absorbing a graviton:

$$A^{m+(-h)\to n} = \langle n|A_S^{(-h)}|m\rangle = \alpha \left(\frac{m_b}{x}\right)^h \langle n|m\rangle , \tag{4.6}$$

where the contraction

$$\langle n|m\rangle = (-1)^m \delta_{m+n,2S} \tag{4.7}$$

is induced by (4.2). The relation of this contraction with the inner product, and the corresponding normalizations, are discussed in appendix A. We note further that for helicity $-h$ the only non trivial amplitudes are of the form $\langle n|A_S^{(-h)}|2S-n\rangle$ and correspond to the scalar amplitude. This is a consequence of choosing the anti-chiral basis. For $+h$ helicity this is not the case, but the fact that $A_S^{(+h)}$ is to be contracted with totally symmetric states of V_3^S and $\bar{V}_4^S$ allows us to commute any two factors in the tensor product of (4.4). That is, we can expand without ambiguity

$$\begin{aligned} A_S^{(+h)} &= \alpha(mx)^h \left(1 - \frac{|\lambda][\lambda|}{m}\right)^{\otimes 2S} \\ &= \alpha(mx)^h \left(1 - 2S\frac{|\lambda][\lambda|}{m} + \binom{2S}{2}\frac{|\lambda][\lambda| \otimes |\lambda][\lambda|}{m^2} + \ldots\right) , \end{aligned} \tag{4.8}$$

where we again omitted the trivial operators in the tensor product. As we explain in appendix A, $|\lambda][\lambda|$ is proportional to the spin vector, hence we call it *spin operator* hereafter (see also [68]). Here we can see that in general the contraction $\langle 0|A_S|2S\rangle$ projects out the spin operator, again recovering the scalar amplitude.

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4.1 Tree amplitudes

We follow the lines of section 3 and evaluate the $2 \to 2$ t channel residue. This time we assign spins S_a, S_b to the particles of mass m_a, m_b, respectively. However, in order to construct the corresponding SU(2) representation (4.2) for the momenta P_1, P_2, we need to repeat the parametrization for P_3 and P_4 given in (4.1). This time we have

$$\begin{aligned} P_1 &= |\hat{\eta}]\langle\hat{\lambda}| + |\hat{\lambda}]\langle\hat{\eta}|\,, \\ P_2 &= \beta|\hat{\eta}]\langle\hat{\lambda}| + \frac{1}{\beta}|\hat{\lambda}]\langle\hat{\eta}| + |\hat{\lambda}]\langle\hat{\lambda}|\,, \end{aligned} \tag{4.9}$$

together with the normalization $[\hat{\lambda}\hat{\eta}] = m_a$. Both parametrizations can be matched in the HCL, effectively reducing the apparent degrees of freedom. In fact, $\beta \to 1$ yields $|\lambda]\langle\lambda| \to -|\hat{\lambda}]\langle\hat{\lambda}|$. Recall that at $\beta = 1$ the tree level process of figure 2 consists of a photon/graviton exchange, with corresponding momentum $K = |\lambda]\langle\lambda|$. For this internal particle we choose the spinors

$$|K] = |\hat{\lambda}] = \frac{|\lambda]}{\gamma}\,,\ |K\rangle = |\hat{\lambda}\rangle = -\gamma|\lambda\rangle\,, \tag{4.10}$$

for some $\gamma \in \mathbb{C}$. By using the definitions (3.1) for both P_1 and P_3 we find $x_1 = 1\,,\ \bar{x}_3 = -\gamma^2\,,$ Using (3.4) we can then solve for γ, completely determining $|\hat{\lambda}]$ and $\langle\hat{\lambda}|$:

$$\gamma^2 = -\frac{u}{m_a m_b} = -\frac{m_a m_b}{v}. \tag{4.11}$$

After this detour, we are ready to compute the tree level residue. The $2 \to 2$ amplitude is here regarded as the operator $M^{(0)}_{(S_a,S_b,|h|)} : V_1^{S_a} \otimes \bar{V}_2^{S_a} \otimes V_3^{S_b} \otimes \bar{V}_4^{S_b} \to \mathbb{C}$, where $V_1^{S_a}, \bar{V}_2^{S_a}$ are constructed in analogous manner to (4.2). Using the expansion (4.8) we find our first main result

$$\begin{aligned} M^{(0)}_{(S_a,S_b,|h|)} &= \alpha^2 \frac{(m_a m_b)^h}{t}\left((x_1\bar{x}_3)^h\left(1 - \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a}\right)^{2S_a} + (\bar{x}_1 x_3)^h\left(1 - \frac{|\lambda][\lambda|}{m_b}\right)^{2S_b}\right) \\ &= \frac{\alpha^2}{t}\left(u^h\left(1 - \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a}\right)^{2S_a} + v^h\left(1 - \frac{|\lambda][\lambda|}{m_b}\right)^{2S_b}\right) \\ &= \frac{\alpha^2}{t}\left(u^h - 2u^h S_a \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a} \otimes \mathbb{I}_b + S_a(2S_a - 1)\frac{|\tilde{\lambda}][\hat{\lambda}|[\hat{\lambda}|[\tilde{\lambda}|}{m_a^2} \otimes \mathbb{I}_b \right. \\ &\quad \left. + v^h - 2v^h S_b\, \mathbb{I}_a \otimes \frac{|\lambda][\lambda|}{m_b} + \ldots\right)\,, \end{aligned} \tag{4.12}$$

where $h = 1$ for Electromagnetism and $h = 2$ for Gravity. In the third and fourth line we exhibited explicitly the identity operators for both representations to emphasize that the spin operators act on different spaces and hence cannot be summed. In appendix A it is argued, by examining the $S = \frac{1}{2}$ and $S = 1$ case, that the binomial expansion is in direct correspondence with the expansion in multipoles moments and hence to the PN expansion for the gravitational case. That is to say we can match the operators $|\hat{\lambda}][\hat{\lambda}|^{\otimes 2n}$, $|\hat{\lambda}][\hat{\lambda}|^{\otimes 2n-1}$

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to the spin operators (2.7) in the HCL and compute the respective coefficients in the EFT expression, as we demonstrate in appendix B for the cases in the literature, i.e. $S \leq 1$. Note further that we can easily identify universal multipole interactions as predicted by [40, 60] for the minimal coupling, the leading one corresponding to scalar (orbital) interaction. Here we emphasize again that all these multipole interactions can be easily seen at $\beta = 1$, in contrast with the COM frame limit.

Finally, note that the parametrization that we introduced did not seem relevant in order to obtain (4.12). However, it is indeed implicit in the choice of basis of states needed to project the operator $M^{(0)}_{(S_a,S_b,h)}$ into a particular matrix element. Next we compute the 1-loop correction for this process, which requires extensive use of the parametrization.

4.2 1-loop amplitude

We now compute the triangle LS (3.13) for the case in which the external particles carry spin. We explicitly expand the contour integral in the HCL for the case $S_a \leq 1$ (EM) and $S_a \leq 2$ (gravity), leaving S_b arbitrary. The limitation for S_a simply comes from the fact that for $S_a \leq 1, S_a \leq 2$ the four point Compton amplitude has a well known compact form [60, 68] both for EM and gravity. Let us remark that the expression for higher spins is also known in terms of the new spinor helicity formulation [68], but we will leave the explicit treatment for future work. Additionally, the case $S_a \leq 1$ is enough to recover all the 1-loop results for the scattering amplitude in the literature [25, 40], and suffices here to demonstrate the effectiveness of the method (see appendix B). Note that the final result is obtained by considering the two triangle topologies for the leading singularity, which can be obtained by symmetrization as we explain below.

In the following we regard the 3pt and 4pt amplitudes entering the integrand (3.13) as 2×2 operators equipped with the natural multiplication. Analogous to the scalar case, only the opposite helicities contribute to the residue and both configurations give the same contribution, hence we focus only on $(+-)$. Furthermore, the 3pt amplitudes can also be readily obtained at $\beta = 1$, by using (3.18) into (4.4). They give

$$\begin{aligned} A_3(P_3, -L(y), k_3^{+i}(y)) A_3(-P_4, L(y), k_4^{-i}(y))\Big|_{\beta=1} &= \alpha^2 m_b^2 \left(1 - \frac{|k_3][k_3|}{y m_b}\right)^{2S_b} , \\ &= \alpha^2 m_b^2 \left(1 - \frac{(1+y)^2}{4y} \frac{|\lambda][\lambda|}{m_b}\right)^{2S_b} . \end{aligned} \tag{4.13}$$

This time note that the y variable carries helicity weight $+1$, as can be seen from plugging k_3 and P_3 in (3.1). This means that we needed to restore the helicity factor y in the first line in order to account for little group transformations of k_3. As in the tree level case, eq. (4.13) corresponds to an expansion in terms of spin structures that "survive" the limit $\beta = 1$.

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We now proceed to compute the 4pt Compton amplitude in the limit $\beta \to 1$. For this, consider

$$A^{(S_a)}_{(4,|h|)} = \alpha^2 \begin{cases} \Gamma^{\otimes 2S_a} \dfrac{\langle k_3|P_1|k_4]^{2-2S_a}}{\langle k_3|P_1|k_3]\langle k_3|P_2|k_3]} & |h| = 1, \\ \Gamma^{\otimes 2S_a} \frac{1}{t} \times \dfrac{\langle k_3|P_1|k_4]^{4-2S_a}}{\langle k_3|P_1|k_3]\langle k_3|P_2|k_3]} & |h| = 2, \end{cases} \tag{4.14}$$

for $S_a \in \{0, \frac{1}{2}, 1\}$ for EM and $S_a \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ for gravity. For higher spins it is easy to see that the proposed expression ceases to be local and hence a more complicated form is in turn needed [68]. Here we have defined the 2×2 matrix

$$\Gamma = |k_4]\langle k_3|P_1 + P_2|k_3\rangle[k_4| \,. \tag{4.15}$$

As anticipated, the 4pt. amplitude takes a compact form for $S_a \leq 1$, and exhibits remarkable factorizations relating EM and gravity [60]. Furthermore, we have already computed the expansions (3.23), hence we only need to compute the leading term in Γ! Using the parametrizations (3.10), (4.9), (4.10) together with (3.17), we find

$$\Gamma = (\beta - 1)\left(\hat{u}\frac{(1-y)}{2} + v\frac{(1+y)}{2} + (v - \hat{u})\frac{1-y^2}{4y}\right) + O(\beta - 1)^2 \,, \tag{4.16}$$

where

$$\hat{u} = u\left(1 - \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a}\right) . \tag{4.17}$$

We see that the expansion effectively attaches a "spin factor" $\left(1 - \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a}\right)$ to u in the expression (3.23). This is expected since the $A^{(S_a)}_{(4,i)}$ is built from the 3pt amplitudes (4.4), which can be obtained from the scalar case by promoting $x_1^h \to x_1^h \left(1 - \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a}\right)^{S_a}$ while $\bar{x}_1$ remains the same. Consequently, the expression (4.16) precisely reduces to its scalar counterpart once the spin operator is projected out: comparing both expansions we find

$$\mathrm{Tr}(\Gamma) = 2\langle k_3|P_1|k_4] \,, \tag{4.18}$$

as required by (4.15). The conjugation $y \to -y$ in Γ effectively swaps $\tilde{u} \leftrightarrow v$. This time this transformation also modifies the contribution from the 3pt amplitudes (4.13), but once the residue is computed the leading singularity is still invariant (in the HCL).

Finally, considering the contribution $h_3 = -h_4 = -2$ in eq. (3.16):

$$\begin{aligned} M^{(1,b)}_{(S_a,S_b,2)} = \frac{\beta}{8(\beta^2-1)m_b^2} \int_{\Gamma_{\mathrm{LS}}} \frac{dy}{y} A_4(P_1, -P_2, k_3^{-2}(y), k_4^{+2}(y)) \\ \otimes A_3(P_3, -L(y), -k_3^{+2}(y)) A_3(-P_4, L(y), -k_4^{-2}(y)) \,, \end{aligned} \tag{4.19}$$

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and inserting (4.16), (3.23), (4.14) together with (4.13), we find our second main result for the classical piece associated to spinning particles

$$M^{(1,b)}_{(S_a,S_b,2)} = \frac{\alpha^4}{16}\frac{m_b}{\sqrt{-t}(v-u)^2}\int_\infty \frac{dy}{y^3(1-y^2)^2}\left(\hat{u}y(1-y)+vy(1+y)+(v-\hat{u})\frac{1-y^2}{2}\right)^{\otimes 2S_a} \times\left(uy(1-y)+vy(1+y)+\frac{(v-u)(1-y^2)}{2}\right)^{4-2S_a}\otimes\left(1-\frac{(1+y)^2}{4y}\frac{|\lambda][\lambda|}{m_b}\right)^{\otimes 2S_b} \tag{4.20}$$

together with the analogous expression for $|h|=1$. The residue can then be computed and expanded as a polynomial in spin operators. Evidently, the factor $\Gamma^{\otimes 2S_a}$ is responsible for these higher multipole interactions, together with the spin operators coming from the 3pt amplitudes (4.13). Finally, symmetrization is needed in order to derive the classical potential. This means that we need to consider the triangle topology obtained by exchanging particles m_a and m_b. For instance, for the gravitational potential this can be easily done since our expressions are general as long as S_a, $S_b \leq 2$. In appendix B we explicitly show how to construct the full answer for $S_a = S_b = \frac{1}{2}$ in terms of the standard EFT operators, and find full agreement with the results in [40]. This time it can be readily checked that the Electromagnetic case also leads to analogous spin corrections, which coincide with those given in [25].

4.3 Light bending for arbitrary spin

We will now implement the construction of appendix A.1 to obtain the massless limit in a similar fashion as we did for the scalar case in section 3.3. We will again focus on the gravitational case since it is of interest for studying light bending phenomena, addressed in detail in [63, 64] for particles with non trivial helicity.

Let us then proceed to take the massless limit of the parametrization (4.9) (at $\beta=1$) corresponding to $\tau|\hat{\eta}]\to 0$. This yields $x_1\to 0$, which is in turn equivalent to $u\to 0$. We get from (4.12), using (3.5)

$$\begin{aligned} M^{(0)}_{(h_a,S_b,2)} &= \alpha^2\frac{v^2}{t}\left(1-\frac{|\lambda][\lambda|}{m_b}\right)^{2S_b} \\ &= \alpha^2\frac{(s-m_b^2)^2}{t}\left(1-\frac{|\lambda][\lambda|}{m_b}\right)^{2S_b}, \end{aligned} \tag{4.21}$$

where $S_a = h_a$ now corresponds to the helicity of particle a. This operator is to be contracted with the states $|0\rangle$, $|2h_a\rangle$ associated to momenta P_3 and the corresponding ones for P_4, which carry the information of the polarizations. It is however trivial in the sense that it is proportional to the identity for such states, in particular being independent of h_a. In the non-relativistic limit we find $s-m_b^2\to 2m_bE$, with $E\ll m_b$ corresponding to the energy of the massless particle. This shows how the low energy effective potential obtained from (4.21) is independent of the type of massless particle, as long as it is minimally coupled to gravity. This is the universality of the light bending phenomena previously proposed in [63]. It may seem that this claim depends on the choice $u=0$ or $v=0$ for defining the

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massless limit, since for $v=0$ the operator $\left(1-\frac{[\hat{\lambda}][\hat{\lambda}|}{[\hat{\lambda}\hat{\eta}]}\right)^{2h_a}$ would certainly show up in the result. However, as argued in the appendix A.1, the choice $v=0$ is supplemented by the choice of a different basis of states for the massless representation, such that this operator is again proportional to the identity and hence independent of h_a.

To argue for the universality at the 1-loop level, we consider the massless limit of (4.16), given by

$$\Gamma \to (\beta-1)\left(v(1+y)+v\frac{1-y^2}{2y}\right), \tag{4.22}$$

which is precisely the massless limit of $\langle k_3|P_1|k_4]$, i.e. the corresponding factor for the scalar case. The conclusion is that the behavior of $A^{(S_a)}_{(4,2)}$ is again independent of $S_a=h_a$, hence showing the universality. The LS for gravity now reads

$$M^{(1,b)}_{(h_a,S_b,2)}=\left(\frac{\alpha^4}{2^8}\right)\frac{(s-m_b)^2m_b}{\sqrt{-t}}\int_\infty \frac{dy\,(1+y)^6}{y^3(1-y)^2}\left(1-\frac{(1+y)^2}{4y}\frac{|\lambda][\lambda|}{m_b}\right)^{2S_b}. \tag{4.23}$$

This leading singularity is all what is needed to compute the classical potential for the massless case, since as explained in subsection 3.3 the $a-$topology has vanishing LS. Thus, we note that because there is no need to symmetrize there is no restriction on S_a at all. This means that, up to 1-loop, we have access to all orders of spin corrections for a massless particle interacting with a rotating point-like source. The expression can be used in principle to recover the multipole expansion of the Kerr black hole solution up to order G^2, see discussion.

5 Discussion

In this work we have proposed the implementation of a new set of techniques in order to extract in a direct manner the classical behavior of a variety of scattering amplitudes, including arbitrarily high order spin effects. This classical piece can then be used to construct an effective field theory for long range gravitational or electromagnetic interactions. It was shown in [67] that for the gravitational case the 1-loop correction to such interaction is completely encoded into the triangle leading singularity. In this work we have reproduced this result and extended the argument to the electromagnetic case in a trivial fashion. The reason this is possible is because the triangle LS captures the precise non-analytic dependence of the form $t^{-\frac{1}{2}}$, which carries the subleading contribution to the potential. As explained in [69], this structure arises from the interplay between massive and massless propagators entering the loop diagrams. This is the case whenever massive particles exchange multiple massless particles which mediate long range forces, such as photons or gravitons.

We have also included the tree level residues for both cases, which correspond to the leading Newtonian and Coulombian potentials. In this case, both computations were completely analogous and the gravitational contribution could be derived by "squaring" the electromagnetic one. This is reminiscent of the double copy construction, which has been shown to be realized even for the case where massive particles are involved [60, 62].

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At 1-loop level, such construction is most explicitly realized in the factorization properties of the Compton amplitude. In the overall picture, this set of relations between gravity and EM amplitudes renders the computations completely equivalent. Even though the latter carries phenomenological interest by itself, it can also be regarded as a model for understanding long range effects arising in higher PN corrections, including higher loop and spin orders.

The HCL was designed as a suitable limit to extract the relevant orders in t from the complete classical leading singularities introduced in [67]. When embedded in this framework, the computation of the triangle LS proves not only simpler but also leads directly to $t^{-\frac{1}{2}}$ contribution including all the spin interactions. As explained in section 2.1 and explicitly shown in appendix B, the covariant form of these interactions allows us to discriminate them from the purely quantum higher powers of t, which appear merged in the COM frame. In order to distinguish them we resorted to the following criteria: for a given power of G, a subleading order in $|\vec{q}|$ can be classical if it appears multiplied by the appropriate power of the spin vector $|\vec{S}|$. In the HCL framework this is easily implemented since the combination $|\vec{q}||\vec{S}|$ will always emerge from a covariant form which does not vanish for $t \to 0$. For instance, for $S = \frac{1}{2}$, the spin-orbit interaction only arises from $\epsilon_{\alpha\beta\gamma\delta} P_1^\alpha P_3^\beta K^\gamma S^\delta$ and can be tracked directly at leading order.

In striking contrast with previous approaches, the evaluation of spin effects does not involve increased difficulty with respect to the scalar case and can be put on equal footing. This is a direct consequence of implementing the massive representation with spinor helicity variables, which certainly bypasses all the technical difficulties associated to the manipulation of polarization tensors. As an important outcome we have proved that the forms of the higher multipole interactions are independent of the spin we assign to the scattered particles. This is a consequence of the equivalence principle, which we have implemented by assuming minimally coupled amplitudes. The expressions have been explicitly shown to agree with the previous results in the literature for the lowest spin orders, corresponding to $S = 1$ and $S = \frac{1}{2}$, yielding spin-orbit, quadrupole and spin-spin interactions. We emphasize that the proposed expressions correspond to a relativistic extension of these results, in the sense that they contain the full $\vec{p}^2$ expansion. However, as the PN expansion demands, a completion of these results would also require arbitrary orders in both spins: this means one would need a closed form of the Compton amplitude for $S > 2$ [68] in order to evaluate the LS.

At this point one could argue that the former difficulty of the diagrammatic computations has been transferred here to the difficulty in performing the matching to the EFT operators. In fact, in order to obtain the effective potential (in terms of vector fields) it is certainly necessary to translate the spinor helicity operators to their standard forms, as was done in appendix A for $S = \frac{1}{2}$ and $S = 1$. We do not think that this should be regarded as a complication. First, as a consequence of the universality we have found, it is clear that we only need to perform the translation once and for a particle of a given spin S, as high as the order of multipole corrections we require. Second and more important, we think that this work along with e.g. [31, 59–61, 66, 67] will serve as a further motivation towards a reformulation of the EFT framework which naturally integrates recent

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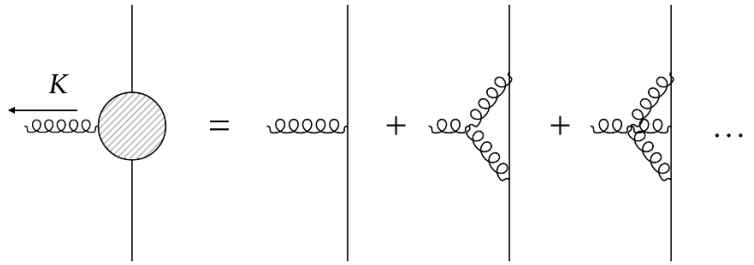

Figure 4. The matrix element of the stress-energy tensor $\langle T_{\mu\nu}(K)\rangle$ corresponds to the 3 point function associated to a pair of massive particles and an external off-shell graviton. The coupling to internal gravitons emanating from the massive source yields, in the long-range behavior, higher corrections in G.

developments in scattering amplitudes. For instance, one could aim for a reformulation of the effective potential, or even better, its replacement by a gauge invariant observable, solely in terms of spinor helicity variables so that no translation is needed to address the dynamics of astrophysical objects.

Next we give some proposals for future work along these lines.

The most pressing future direction is the extension of these techniques in the context of classical corrections at higher loops. By consistency with the PN expansion such higher orders would require to include spin multipole corrections. In fact, in [67] Cachazo and the author have provided several examples of higher-loop Leading Singularities. In particular, at two-loops, the computation of the gravitational double-triangle LS presented there may be easily translated into our methods, since it only uses 3pt. amplitudes which are trivial in the HCL for any spin. In general Leading Singularities extract non-analytic pieces of the full amplitude. It remains to see whether this or other LS can be used to match the non-analytic piece of a 3PN potential, either by directly comparing to classical results or by providing a dispersion relation argument as the one given in [67] for 1-loop.

In that direction, a first consistency check could be to reproduce known solutions to Einstein equations. In the spirit of [20, 59] and the more modern implementations [73, 74] we could argue that this work indeed represent progress towards the derivation of classical spacetimes from scattering amplitudes. As argued by Donoghue [21, 75] a way to obtain the spacetime metric is to compute the long-range behavior of the off-shell expectation value $\langle T_{\mu\nu}(K)\rangle$ illustrated in figure 4, which yields the Schwarzschild/Kerr solutions through Einstein equations. At first glance it would seem that is not possible to compute this matrix element using the on-shell methods here exposed. However, this is simply analogous to the fact that we require an off-shell two-body potential for the PN problem. The solution is, of course, to attach another external particle to turn figure 4 into the scattering process of figure 1. In this way we can get information about off-shell subprocesses by examining the $2 \to 2$ amplitude.

A simple way to proceed in that direction is to incorporate probe particles whose backreaction can be neglected. In fact, the massless case explored in subsections 3.3 and 4.3

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can be regarded as a probe particle choice. The lack of backreaction is realized in the fact that only one triangle topology is needed for obtaining the classical piece of the amplitude, which in turn can be thought of containing the process of figure 4. Furthermore, this piece has no restriction in the spin S of the massive particle, i.e. we can compute both the tree level and 1-loop potential to arbitrarily high multipole terms. By extracting the matrix element $\langle T_{\mu\nu}(K)\rangle$ we could recover both leading and subleading orders in G to arbitrary order in angular momentum of the Kerr solution, see also [26]. In fact, it was recently proposed [70] that by examining a probe particle in the Kerr background the generic form of the multipole terms entering the 2-body Hamiltonian can be extracted at leading order in G and arbitrary order in spin.

Of course, it is also tempting to explore the opposite direction, outside the probe particle limit. One could try to obtain an expression for the effective (i.e. long-range) vertex of figure 4, including higher couplings with spin, expressed in terms of spinor variables. Then an effective potential could be constructed in terms of several copies of these vertices, for instance to address the n-body problem in GR [76–81].

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A Spinor helicity variables for massive kinematics

Here we construct the SU(2) states (4.2) and their respective operators written in terms of anti-chiral spinors, first proposed in [68] as a presentation of the massive little group. In (4.2) we considered two massive particles (with same mass m_b and spin S) and constructed the spaces V_3^S, $\bar{V}_4^S$ associated to their respective states. We also introduced the contraction between these states which will naturally occur in the matrix element of the scattering processes:

$$\langle n|m\rangle = (-1)^m \delta_{m+n,2S}\,.$$

This follows from the normalization explained in (4.3). It is also natural to define an inner product for each space if we identify $\bar{V}_4^S = \left(V_3^S\right)^*$, i.e. as providing a dual basis for V_3^S.[7]

[7] The contraction $\langle n|m\rangle$, as defined, is antisymmetric for fermions. This is reminiscent of the spin-statistics theorem, as such form is proportional to the minimally coupled 3pt amplitude. On the other hand, in order to interpret this contraction as an inner product it is necessary to introduce the dual map $\zeta : V^S \to \left(V^S\right)^*$. For instance, defining $\zeta : |n\rangle \mapsto (-1)^{2s}\langle n|$ leads to a symmetric expression.

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With these conventions, we can expand any operator $O \in \left(V_3^S\right)^* \otimes \left(\bar{V}_4^S\right)^*$ as

$$O = \sum_{n,m \leq 2S+1} (-1)^{n+m-2S} |\bar{n}\rangle\langle\bar{m}| \, \langle n|O|m\rangle \,, \tag{A.1}$$

where $\bar{m} = 2S - m, \bar{n} = 2S - n$. Of course, this expansion is general for any choice of basis as long as $|\bar{n}\rangle, \langle\bar{m}|$ are defined as the duals. It is even possible to use different states for V_3^S and V_4^S, spanned by different spinors $\{|\lambda], |\eta]\}$ and $\{|\bar{\lambda}], |\bar{\eta}]\}$. However, it is natural to use the basis (4.2) as it coincides for both massive particles entering the 3pt amplitude, and also coincides with the dual basis up to relabelling. Next we explicitly illustrate the natural map between the states (4.2) and the well known Dirac spinor representation for $S = \frac{1}{2}$. We also show how to construct the chiral presentation in terms of angle spinors, in which the basis for both particles turn out to be different.

First, consider the parametrization (3.10). The basis of solutions for the (momentum space) Dirac equation are given in terms of the spinors

$$\begin{aligned} u_3^+ &= \begin{pmatrix} \langle\lambda| \\ [\lambda| \end{pmatrix} , & u_3^- &= \begin{pmatrix} -\langle\eta| \\ [\eta| \end{pmatrix} , \\ \bar{u}_4^+ &= \begin{pmatrix} -\beta|\lambda\rangle & |\lambda] \end{pmatrix} , & \bar{u}_4^- &= \begin{pmatrix} \dfrac{|\eta\rangle}{\beta} + |\lambda\rangle & |\eta] \end{pmatrix} . \end{aligned} \tag{A.2}$$

(For $\beta = 1$, note that (3.1) follows from the Dirac equation with $x = 1$). Thus it is now natural to use $|\eta]$ and $|\lambda]$ to construct the $S = \frac{1}{2}$ representation for the (outgoing) particle P_4, and similarly for P_3. This yields an anti-chiral representation of SU(2). From the definition (4.2) we find (slightly abusing the notation)

$$|+\rangle = \frac{|\lambda]}{\sqrt{m_b}} , \quad |-\rangle = \frac{|\eta]}{\sqrt{m_b}} \quad \in V_3^{\frac{1}{2}} . \tag{A.3}$$

and analogously for $\langle\pm| \in \bar{V}_4^{\frac{1}{2}}$. The expansion (A.1) leads to the 2×2 operator

$$O = \frac{1}{m_b} \left(-|\lambda][\lambda| \, O_{(--)} + |\lambda][\eta| \, O_{(-+)} + |\eta][\lambda| \, O_{(+-)} - |\eta][\eta| \, O_{(++)} \right) . \tag{A.4}$$

Had we used the chiral part, we would have selected a different basis for each of the massive particles. In fact, the chiral part is obtained by acting with P_3, P_4 on the anti-chiral states, respectively. This means that the change of basis (for $S = \frac{1}{2}$) is given by

$$\bar{O} = \frac{\bar{P}_3 O P_4}{m^2} , \tag{A.5}$$

where we have used matrix multiplication, with the extension to higher values of spin being straightforward.

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For completeness we present here some useful expressions obtained at $\beta = 1$, even though they can easily be computed in general

$$\begin{aligned} m^2\bar{u}_4\gamma_\mu u_3 \to m^2\gamma_\mu &= (P_4)_\mu|\eta][\lambda| - (P_3)_\mu|\lambda][\eta| - v_\mu|\lambda][\lambda| \\ &= m(P_3)_\mu + K_\mu|\eta][\lambda| - v_\mu|\lambda][\lambda|\,, \\ \bar{u}_4 u_3 \to \mathbb{I}_{2\times 2} &= \frac{(P_3)^\mu}{m}\gamma_\mu = 1 - \frac{|\lambda][\lambda|}{2\,m}\,, \\ \frac{m^2}{2}\bar{u}_4\gamma_5\gamma_\mu u_3 \to m^2 S_\mu &= K_\mu|\eta][\eta| - \left(R_\mu + \frac{1}{2}v_\mu\right)|\lambda][\lambda| \\ &\quad + (u_\mu - v_\mu + K_\mu)|\eta][\lambda| + (u_\mu - v_\mu)|\lambda][\eta|\,, \end{aligned} \tag{A.6}$$

where

$$\begin{aligned} 2v_\mu &= [\eta|\sigma_\mu|\lambda\rangle\,, & 2u_\mu &= [\lambda|\sigma_\mu|\eta\rangle\,, \\ v_\mu + u_\mu &= (P_3)_\mu\,, & 2R_\mu &= [\eta|\sigma_\mu|\eta\rangle\,. \end{aligned} \tag{A.7}$$

Here $\mathbb{I}_{2\times 2}$ is the identity operator for Dirac spinors, projected into the two-dimensional subspaces spanned by the wavefunctions $u^\pm$. In the rules (A.6) we have included an overall factor of $\frac{1}{2}$ to account for the normalization of the wavefunctions in (A.2), i.e. $\bar{u}_3^+ u_3^- = 2\,m$. On the other hand, in the second line we used the identity

$$1 = \frac{|\eta][\lambda| - |\lambda][\eta|}{[\lambda\eta]}\,. \tag{A.8}$$

From the fourth line of (A.6), using $2R\cdot K = -m^2$ we find in the HCL

$$S_\mu K^\mu = \frac{1}{2}|\lambda][\lambda|\,. \tag{A.9}$$

This is the reason we call $|\lambda][\lambda|$ a spin operator. One may wonder why the spin operator appears in the expansion of $\mathbb{I}_{2\times 2}$, which contains the scalar contribution. Even though $\mathbb{I}$ and $\gamma_5\gamma_\mu$ are orthogonal as Dirac matrices, this does not hold once they are projected into the 2D subspace of physical states. This is consistent with the non-relativistic expansions of [40], where the form $\bar{u}_4 u_3$ indeed contributes to the spin interaction. In fact, this is also true for higher spin generalizations as we now show.

Motivated by the manifest universality found in section 4, i.e. expression (4.12), we consider the following extensions for arbitrary spin S_b (not to be confused with the spin vector S_μ)

$$\begin{aligned} S_\mu K^\mu &= S_b\,|\lambda][\lambda|\,, \\ \mathbb{I}_{(2S_b+1)} &= \left(1 - S_b\frac{|\lambda][\lambda|}{m}\right)\,, \end{aligned} \tag{A.10}$$

As explained in the discussion after eq. (4.4), we omit the trivial part of the operators on the r.h.s. . This allows to keep the expressions compact and makes the universality manifest. Let us briefly perform a nontrivial check of equations (A.10) for higher spins. We do so by examining the representation for $S_b = 1$, which in the standard framework is given by polarization vectors satisfying $\epsilon^{(i)}\cdot P = 0$, $i = 1, 2, 3$, for a given momentum

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$P^2 = m_b^2$. In terms of spinor helicity variables the polarization vectors ϵ_3 and ϵ_4 are represented as operators acting on V_3^1 and $\bar{V}_4^1$ respectively. Explicitly, we can choose the normalized wavefunctions[8]

$$\begin{aligned}\frac{m_b^2(\epsilon_3)_\mu}{\sqrt{2}} &\to [\lambda|[\lambda|R_\mu - [\lambda|[\eta|(u-v+K)_\mu - [\eta|[\eta|K_\mu\,,\\ \frac{m_b^2(\epsilon_4)_\mu}{\sqrt{2}} &\to |\lambda]|\lambda]\left(R+\frac{1}{2}P_3\right)_\mu - |\lambda]|\eta](u-v-K)_\mu - |\eta]|\eta]K_\mu\,,\end{aligned}$$

Using this expression it is easy to check the validity of (A.10) for $S_b = 1$, with [40]

$$\begin{aligned}\epsilon_3\cdot\epsilon_4 &\to \mathbb{I}_3\,,\\ \frac{1}{2m_b}\epsilon_{\mu\alpha\beta\gamma}\epsilon_4^\alpha\epsilon_3^\beta(P_3+P_4)^\gamma &\to S_\mu\,.\end{aligned} \tag{A.11}$$

Also, we can now derive the form of the quadrupole interaction. It is given by

$$(\epsilon_4\cdot K)(\epsilon_3\cdot K) = \frac{1}{2}|\lambda]|\lambda]\otimes[\lambda|[\lambda|\,. \tag{A.12}$$

We will use this expression in appendix B to translate the leading singularity into standard EFT operators.

For illustration purposes, let us close this section by constructing the representation of the 3pt amplitudes for $S=\frac{1}{2}$ massive fields with a graviton. Let the polarization of the massless particle be described by $|\bar{\lambda}] = \sqrt{x}|\lambda]$, $\langle\bar{\lambda}| = \frac{\langle\lambda|}{\sqrt{x}}$, where x carries helicity 1 (recall $|\lambda]$ is fixed) and agrees with (3.1). The 3pt amplitude is given by [26]

$$\begin{aligned}A_{\frac{1}{2}}^{(+2)} &= \frac{\alpha m}{2}\gamma_\mu\frac{[\bar{\lambda}|\sigma^\mu|\eta\rangle[\bar{\lambda}|P_3|\eta\rangle}{\langle\eta\bar{\lambda}\rangle^2}\,,\\ A_{\frac{1}{2}}^{(-2)} &= \frac{\alpha m}{2}\gamma_\mu\frac{[\eta|\sigma^\mu|\bar{\lambda}\rangle[\eta|P_3|\bar{\lambda}\rangle}{[\eta\bar{\lambda}]^2}\,.\end{aligned} \tag{A.13}$$

Here we have fixed the reference spinor entering in the 3pt. amplitudes to be η. Using (A.6) together with (3.10) we find

$$\begin{aligned}A_{\frac{1}{2}}^{(+2)} &= \alpha(mx)^2\left(1-\frac{|\lambda][\lambda|}{m}\right)\,,\\ A_{\frac{1}{2}}^{(-2)} &= \alpha\left(\frac{m}{x}\right)^2\,,\end{aligned} \tag{A.14}$$

precisely agreeing with (4.4) for $|h|=2$. Furthermore, in the chiral representation we find, using (A.5)

$$\begin{aligned}\bar{A}_{\frac{1}{2}}^{(+2)} &= \alpha\left(\frac{m}{\bar{x}}\right)^2\,,\\ \bar{A}_{\frac{1}{2}}^{(-2)} &= \alpha(m\bar{x})^2\left(1-\frac{|\lambda\rangle\langle\lambda|}{m}\right)\,.\end{aligned} \tag{A.15}$$

where $\bar{x}$ is defined in (3.1).

[8]Here we use the notation $[\lambda|[\eta|$ to account for the standard tensor product, i.e. not symmetrized. Of course, we can replace $[\lambda|[\eta| \to \frac{1}{\sqrt{2}}[\lambda|\odot[\eta|$, where $\odot$ involves the normalization (4.3).

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A.1 Massless representation

We can extend the treatment described in section 3.3 in order to construct the states for massless particles. The idea is to use the two highest weight states $|0\rangle$, $|2S\rangle$ of the massive representation as the physical polarizations of the massless one, after the limit is taken. The massless case can be formally defined by introducing a variable τ in the parametrization (3.10), i.e. by putting either $|\eta] \mapsto \tau|\eta]$ or $|\eta\rangle \mapsto \tau|\eta\rangle$ and then proceed to take the limit $\tau \to 0$. This parametrizes the mass of both $P_3(\tau)$ and $P_4(\tau)$ as $m^2(\tau) = \tau m^2$. Next we proceed to sketch the procedure leading to the massless 3pt. amplitudes[9] and study both choices $\tau|\eta] \to 0$ and $\tau|\eta\rangle \to 0$. As these amplitudes correspond to the building blocks for both the tree level residue and the triangle LS in section 4, showing that they can be recovered from our expressions (4.4) proves the equivalence with the standard spinor helicity approach for massless particles. This approach was recently implemented in [63].

In the following we will consider $\beta = 1$. Indeed, the massless deformation of the momenta is only consistent in the HCL since $t = \tau\frac{(\beta-1)^2}{\beta}m_b^2 \to 0$ as $\tau \to 0$. This is enough for our purposes in section 4 since we evaluate both the tree level residue and triangle LS by neglecting subleading contributions in t. For the choice $|\eta] \mapsto \tau|\eta]$ we thus have

$$\begin{aligned} P_3 &= \tau|\eta]\langle\lambda| + |\lambda]\langle\eta| \;\longrightarrow |\lambda]\langle\eta|\,, \\ P_4 &= \tau|\eta]\langle\lambda| + |\lambda](\langle\eta| + \langle\lambda|) \;\longrightarrow |\lambda](\langle\eta| + \langle\lambda|)\,, \\ K &= |\lambda]\langle\lambda|\,. \end{aligned} \tag{A.16}$$

In the following we choose $|\lambda]$,$\langle\lambda|$ to represent the polarizations of the particle K. As explained in section 3.3, it is convenient to reabsorb the mass into the definition of x (3.1), thus we have

$$x = \tau[\lambda\eta] = \tau m\,, \quad \bar{x} = \langle\lambda\eta\rangle = m\,. \tag{A.17}$$

This means $\tau|\eta] \to 0$ is equivalent to the limit $x \to 0$, keeping $\bar{x}$ fixed. As the reference spinor $|\eta]$ is also fixed, we can assume that the neither the basis (4.2) nor the operators (4.4) depend on τ in any other way that is not through x. With these considerations, we find for the massless limit

$$A_S^{(h)} = 0\,, \quad A_S^{(-h)} = \alpha\bar{x}^h\,, \tag{A.18}$$

where at this stage $\bar{x} = \langle\lambda\eta\rangle$ is not restricted since the original mass m is not relevant after the limit is taken. We then note that all the positive helicity amplitudes vanish. In fact, these amplitudes can be described in terms of square brackets, thus it is expected that they vanish for the $\tau = 0$ limit in (A.16). Now, the negative helicity amplitudes in the standard spinor helicity notation read [71]

$$\begin{aligned} A(3^{+S}, 4^{-S}, K^{-h}) &= \alpha\frac{\langle K3\rangle^{h-2S}\langle K4\rangle^{h+2S}}{\langle 43\rangle^h} \\ &= \alpha\bar{x}^h\,. \end{aligned} \tag{A.19}$$

[9] At this level we keep the discussion general for S and h. Of course, (interacting) massless higher spin particles are known to be inconsistent by very fundamental principles, thus effectively restricting our choices to $S, h \leq 2$.

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Note that this derivation is also valid for $A(3^{-S}, 4^{+S}, K^{-h})$ up to a possible sign. Also, the configuration $A(3^{+S}, 4^{+S}, K^{-h})$ together with its conjugate do not correspond to the minimal coupling and thus vanish. In order to interpret these amplitudes as matrix elements of (A.18), we need to specify the basis of states for the massless particles. It turns out that just the highest weight states in (4.2) are enough for this purpose. That is, we find

$$\begin{aligned} A(3^{+S}, 4^{-S}, K^{-h}) &= \langle 2S|A_S|0\rangle, & A(3^{-S}, 4^{+S}, K^{-h}) &= \langle 0|A_S|2S\rangle\,, \\ A(3^{+S}, 4^{+S}, K^{-h}) &= \langle 2S|A_S|2S\rangle, & A(3^{+S}, 4^{+S}, K^{-h}) &= \langle 2S|A_S|2S\rangle\,, \end{aligned} \tag{A.20}$$

therefore showing the equivalence of both approaches for massless particles. Here we note that a somehow more straightforward approach is to define the massless limit directly in the expectation values (A.20), following [68]. Instead, we have opted for constructing the corresponding operators (A.18), since our integral expressions in section 4 are given in terms of them. These operators are defined for the basis built from the fixed spinors $|\lambda]$ and $|\eta]$, which are reminiscent of the massive representation in (A.16).

The choice $|\eta\rangle \mapsto \tau|\eta\rangle$ is completely analogous and yields

$$A_S^{(h)} = \alpha x^h \left(1 - \frac{|\lambda][\lambda|}{[\lambda\eta]}\right)^S\,, \quad A_S^{(-h)} = 0\,, \tag{A.21}$$

i.e. vanishing negative helicity amplitudes. This is expected since we have

$$\begin{aligned} P_3 &= |\eta]\langle\lambda| + \tau|\lambda]\langle\eta| \;\longrightarrow |\eta]\langle\lambda|\,, \\ P_4 &= |\eta]\langle\lambda| + \tau|\lambda]\langle\eta| + |\lambda]\langle\lambda| \;\longrightarrow (|\lambda] + |\eta])\langle\lambda|\,. \end{aligned} \tag{A.22}$$

However, this time we note that the natural basis of spinors for P_4 is given by $|\bar{\eta}] := |\lambda] + |\eta]$ and $|\lambda]$. When expressed in terms of this basis, the expression (A.21) takes a form analogous to (A.18). Hence we construct the states $\langle 0|,\langle 2S|$ in $\bar{V}_4^S$ in terms of these spinors.

B Matching the spin operators

Here we explain how to recover the standard form of the potential in terms of generic spin operators (2.7), starting from the results of section 4. As usual throughout this work, we focus on the gravitational case since it presents greater difficulty in the standard approaches. We give two examples which illustrate how the procedure works. First, we present the tree level result for the case $S_a = 0$, $S_b = 1$, which yields both a spin-orbit and a quadrupole interaction. Second, we discuss the matching at 1-loop level for the case $S_a = S_b = \frac{1}{2}$. Both computations were done in [40] using standard Feynman diagrammatic techniques, which lead to notably increased difficulty with respect to the scalar case. Here we find that the computations are straightforward using the techniques introduced throughout this work. In fact, all the computations in [40] can be redone in a direct way and we leave them as an exercise for the reader. The same can be done for the EM case in order to recover the results previously presented in [25].

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The starting point for both cases are the explicit expressions for the variables u, v that we used to construct the amplitudes. We can easily solve them from eq. (3.5). We find

$$\begin{aligned} 2u &= s - m_a^2 - m_b^2 + \sqrt{\left(s - m_a^2 - m_b^2\right)^2 - 4m_a^2 m_b^2}\,, \\ 2v &= s - m_a^2 - m_b^2 - \sqrt{\left(s - m_a^2 - m_b^2\right)^2 - 4m_a^2 m_b^2}\,, \end{aligned} \tag{B.1}$$

where the square root corresponds to the parity odd piece. From the definition (3.4) it is clear that under the exchange $P_1 \leftrightarrow P_3$ (which we perform below), u and v must also be exchanged. Now, to keep the notation compact, let us write

$$P_1 \cdot P_3 = r m_a m_b\,, \quad r > 1.$$

Note that in the non-relativistic regime we have $r \to 1$. Now we can write (B.1) as

$$u = m_a m_b \left(r + \sqrt{r^2 - 1}\right)\,, \quad v = m_a m_b \left(r - \sqrt{r^2 - 1}\right)\,. \tag{B.2}$$

Consider now the case $S_a = 0$, $S_b = 1$. Let us construct a linear combination of the EFT operators associated to scalar, spin-orbit, and quadrupole interaction, that is [26, 40]

$$\bar{M}^{(1)}_{(0,1,2)} = \alpha^2 \frac{(m_a m_b)^2}{t} \left(c_1(r) \epsilon_3 \cdot \epsilon_4 + c_2(r) \frac{\epsilon_{\mu\alpha\beta\gamma} K^\mu P_1^\alpha P_3^\beta S^\gamma}{m_a m_b^2} + c_3(r) \frac{(\epsilon_4 \cdot K)(\epsilon_3 \cdot K)}{m_b^2} \right)\,. \tag{B.3}$$

The reason we call $\epsilon_3 \cdot \epsilon_4$ a scalar interaction is because, as will be evident in a moment, it is the only piece surviving the contraction $\langle 0|\bar{M}^{(1)}_{(0,1,2)}|2\rangle$, which we identified as the scalar amplitude (see discussion below eq. (4.8)).

Note that we have not assumed the non-relativistic limit in the u, v variables, only the HCL $t = 0$ which selects the classical contribution. The operators can now be expanded using (A.10), (A.12). For this, it is enough to note that in the HCL the spin-orbit piece takes the form

$$\epsilon_{\mu\alpha\beta\gamma} K^\mu P_1^\alpha P_3^\beta S^\gamma = -\frac{K \cdot S}{2} \sqrt{\left(s - m_a^2 - m_b^2\right)^2 - 4m_a^2 m_b^2} = m_a m_b (K \cdot S) \sqrt{r^2 - 1}\,. \tag{B.4}$$

We then find

$$\bar{M}^{(0)}_{(0,1,2)} = \alpha^2 \frac{(m_a m_b)^2}{t} \left(c_1 - \frac{|\lambda][\lambda|}{m_b}(c_1 - c_2\sqrt{r^2 - 1}) + c_3 \frac{|\lambda]|\lambda] \otimes [\lambda|[\lambda|}{2\, m_b^2} \right)\,.$$

Comparing now with the expression (4.12), which in this case reads

$$\begin{aligned} M^{(0)}_{(0,1,2)} &= \frac{\alpha^2}{t} \left(u^2 + v^2 \left(1 - \frac{|\lambda][\lambda|}{m_b}\right)^2 \right) \\ &= \frac{\alpha^2}{t} \left(u^2 + v^2 - 2v^2 \frac{|\lambda][\lambda|}{m_b} + v^2 \frac{|\lambda]|\lambda] \otimes [\lambda|[\lambda|}{m_b^2} \right) \\ &= \alpha^2 \frac{(m_a m_b)^2}{t} \left((4r^2 - 2) - 2\left(2r^2 - 1 - 2r\sqrt{r^2 - 1}\right) \frac{|\lambda][\lambda|}{m_b} \right. \\ &\quad \left. + \left(2r^2 - 1 - 2r\sqrt{r^2 - 1}\right) \frac{|\lambda]|\lambda] \otimes [\lambda|[\lambda|}{m_b^2} \right)\,, \end{aligned}$$

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we find

$$\begin{aligned} c_1 &= 4r^2 - 2\,, \\ c_2 &= 4r\,, \\ c_3 &= 4r^2 - 2 - 4r\sqrt{r^2-1}\,. \end{aligned}$$

The result in [40] can then be recovered by imposing the non-relativistic limit $s \to s_0$, which in this case reads $r \to 1$.[10] Even though we computed the residue in (B.3) at $t = 0$, it is evident that this expression can be analytically extended to the region $t \neq 0$ in which the COM frame can be imposed, as described in (2.1). This is precisely done in [40] where the effective potential is obtained from this expression after implementing the Born approximation.

The 1-loop result for $S_a = 0$, $S_b = 1$ can be computed in the same fashion, by using the expressions provided in section 4.2. Expectedly, the EFT operators are exactly the same that appeared at tree level, but the behavior of the coefficients c_1, c_2 and c_3 as functions of r differs in the sense that it can involve poles at $r = 1$. We now illustrate all this by considering the more complex case also addressed in [40], namely $S_a = S_b = \frac{1}{2}$.

For $S = \frac{1}{2}$ the multipole operators are restricted to the scalar and spin-orbit interaction. They read [40]

$$\mathcal{U} = \bar{u}_4 u_3 \ , \ \mathcal{E} = \epsilon_{\alpha\beta\gamma\delta} P_1^\alpha P_3^\beta K^\gamma S^\delta \,.$$

In our case we will consider two copies of these operators, one for each particle. That is to say we propose the following form for the 1-loop leading singularity

$$\begin{aligned} \bar{M}^{(1)}_{(\frac{1}{2},\frac{1}{2},2)} &= \left(\frac{\alpha^4}{16}\right) \frac{(m_a m_b)^2}{\sqrt{-t}} \left(c_{11}\mathcal{U}_a\mathcal{U}_b + c_{12}\frac{\mathcal{U}_a\mathcal{E}_b}{m_b^2 m_a} + c_{21}\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2 m_b} + c_{22}\frac{\mathcal{E}_a\mathcal{E}_b}{m_b^3 m_a^3} \right) \\ &= \alpha^4 \frac{(m_a m_b)^2}{16\sqrt{-t}} \left(c_{11} - \frac{c_{11} - c_{12}\sqrt{r^2-1}}{2} \left(\frac{|\hat{\lambda}][\hat{\lambda}|}{m_a}\right) - \frac{c_{11} + c_{21}\sqrt{r^2-1}}{2} \left(\frac{|\lambda][\lambda|}{m_b}\right) \right. \\ &\quad \left. + \frac{\left(c_{11} - (c_{12}-c_{21})\sqrt{r^2-1} - c_{22}(r^2-1)\right)}{4} \frac{|\hat{\lambda}][\hat{\lambda}|}{m_a} \otimes \frac{|\lambda][\lambda|}{m_b} \right) . \end{aligned} \tag{B.5}$$

Here we have used (B.4), (A.9) and (A.6). A minus sign was introduced when implementing (A.9) for particle m_a, which arises from the mismatch between both parametrizations in the HCL, i.e. $K = |\lambda]\langle\lambda| = -|\hat{\lambda}][\hat{\lambda}|$. We proceed to compare this with the sum of the two triangle leading singularities given by (3.16), using the results of section 4.2. The result can be written

$$M^{(1,\text{full})}_{(\frac{1}{2},\frac{1}{2},2)} = M^{(1,b)}_{(\frac{1}{2},\frac{1}{2},2)} + M^{(1,a)}_{(\frac{1}{2},\frac{1}{2},2)} \,,$$

[10] There are, however, some discrepancies in conventions which may be fixed by replacing $-\epsilon_f^{b*} \to \epsilon_4$, $iS_b \to S_b$ in [40]. We find our conventions more appropriated since the sign in the scalar interaction is the same for any spin.

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where $M^{(1,a)}_{(\frac{1}{2},\frac{1}{2},2)}$ is obtained by exchanging $m_a \leftrightarrow m_b$, $|\hat{\lambda}][\hat{\lambda}| \leftrightarrow |\lambda][\lambda|$ and $u \leftrightarrow v$ in

$$M^{(1,b)}_{(\frac{1}{2},\frac{1}{2},2)} = \left(\frac{\alpha^4}{16}\right)\frac{m_b}{\sqrt{-t}(v-u)^2}\int_\infty \frac{dy}{y^3(1-y^2)^2}\left(\hat{u}y(1-y)+vy(1+y)+(v-\hat{u})\,\frac{1-y^2}{2}\right)$$
$$\times\left(uy(1-y)+vy(1+y)+\frac{(v-u)(1-y^2)}{2}\right)^3 \otimes\left(1-\frac{(1+y)^2}{4y}\frac{|\lambda][\lambda|}{m_b}\right), \quad \text{(B.6)}$$

with $\hat{u} = u\left(1-\frac{|\hat{\lambda}][\hat{\lambda}|}{m_a}\right)$. After computing the contour integral, we can easily solve for the coefficients c_{ij}, $i,j \in \{1,2\}$. In order to compare with the results in the literature, we first perform the non-relativistic expansion

$$\begin{aligned} c_{11} &= 6(m_a+m_b)+\dots \\ c_{12} &= \frac{4m_a+3m_b}{2(r-1)}+11\left(m_a+\frac{3}{4}m_b\right)+\dots \\ c_{21} &= \frac{3m_a+4m_b}{2(r-1)}+11\left(\frac{3}{4}m_a+m_b\right)+\dots \\ c_{22} &= \frac{m_a+m_b}{4(r-1)^2}+\frac{9(m_a+m_b)}{2(r-1)}+\dots \end{aligned} \quad \text{(B.7)}$$

Note that even though the coefficients present poles, they are parity invariant in the sense that they do not contain square roots. To put the result in the same form as [40], we need to further extract the standard spin-spin interaction term from our operator $\mathcal{E}_a\mathcal{E}_b$. This accounts for extracting the classical piece, which can be obtained by returning to the physical region $t = K^2 \neq 0$. Using (B.4) we find

$$\mathcal{E}_a\mathcal{E}_b = m_am_b(r^2-1)\left((S_a\cdot K)(S_b\cdot K)-K^2(S_a\cdot S_b)\right)+rK^2(P_1\cdot S_b)(P_3\cdot S_a)+O(K^3),$$

where $O(K^3) = O(|\vec{q}|^3)$ denotes quantum contributions, i.e. higher orders in $|\vec{q}|$ for a fixed power of spin $|\vec{S}|$. However, we note the presence of the couplings $P_i\cdot S \sim \vec{v}\cdot\vec{S}$ which certainly do not appear in the effective potential [26, 39, 40]. In fact, in the standard EFT framework these couplings are dropped by the so-called Frenkel-Pirani conditions or Tulczyjew conditions [82].[11] In our case they have emerged due to our bad choice of ansatz (B.5). In fact, the right choice is now clearly obtained by replacing

$$\mathcal{E}_a\mathcal{E}_b \to m_am_b(r^2-1)\left((S_a\cdot K)(S_b\cdot K)-K^2(S_a\cdot S_b)\right),$$

corresponding to the correct spin-spin interaction term [29], which is already visible at tree level [25, 26, 40]. Note, however, that this does not modify the HCL of this operator, which comes solely from the first term. Consequently, our results (B.7) are still valid and indeed they agree with the ones in the literature [40]. They can be regarded as a relativistic extension leading to the effective potential up to order G^2.

[11] They can arise, however, when including non-minimal couplings corresponding to higher dimensional operators, see e.g. [50].

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