

# Thermodynamic and classical instability of AdS black holes in fourth-order gravity

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**ABSTRACT:** We study thermodynamic and classical instability of AdS black holes in fourth-order gravity. These include the BTZ black hole in new massive gravity, Schwarzschild-AdS black hole, and higher-dimensional AdS black holes in fourth-order gravity. All thermodynamic quantities which are computed using the Abbot-Deser-Tekin method are used to study thermodynamic instability of AdS black holes. On the other hand, we investigate the *s*-mode Gregory-Laflamme instability of the massive graviton propagating around the AdS black holes. We establish the connection between the thermodynamic instability and the GL instability of AdS black holes in fourth-order gravity. This shows that the Gubser-Mitra conjecture holds for AdS black holes found from fourth-order gravity.

**KEYWORDS:** Black Holes, Classical Theories of Gravity

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**Contents**

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>BTZ black hole in new massive gravity</b>	<b>3</b>
<b>3</b>	<b>Thermodynamics of AdS black holes in fourth-order gravity</b>	<b>6</b>
<b>4</b>	<b>SAdS black hole in Einstein-Weyl gravity</b>	<b>8</b>
4.1	Thermodynamic instability for small black holes	8
4.2	GL instability for small black holes	10
<b>5</b>	<b>Higher-dimensional AdS black holes in fourth-order gravity</b>	<b>12</b>
5.1	Thermodynamic instability for small black holes	12
5.2	GL instability	13
<b>6</b>	<b>Discussions</b>	<b>16</b>
<b>A</b>	<b>Coefficients of the perturbation equation for <math>H_{tr}</math></b>	<b>16</b>

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**1 Introduction**

Concerning the thermodynamic analysis of a black hole, the Schwarzschild black hole in Einstein gravity is in an unstable equilibrium with the heat reservoir of the temperature  $T$  [1]. Its fate under small fluctuations will be either to decay to hot flat space by emitting Hawking radiation or to grow without limit by absorbing thermal radiations in the infinite heat reservoir [2]. This means that an isolated black hole is never in thermal equilibrium in asymptotically flat spacetimes because of its negative heat capacity. Thus, one has to find a way of getting a stable black hole which might be in an equilibrium with a finite heat reservoir. A black hole could be rendered thermodynamically stable by placing it in four-dimensional anti-de Sitter ( $AdS_4$ ) spacetimes because  $AdS_4$  spacetimes play the role of a confining box. An important point to understand is to know how a stable black hole with positive heat capacity could emerge from thermal radiation through a phase transition. The Hawking-Page (HP) phase transition occurs between thermal AdS spacetimes (TAdS) and Schwarzschild-AdS (SAdS) black hole [3, 4], which is known to be one typical example of the first-order phase transition in the gravitational system. Its higher dimensional extension and the AdS/CFT correspondence of confinement-deconfinement phase transition were studied in [5].

To study the HP phase transition in Einstein gravity explicitly, we are necessary to know the Arnowitt-Deser-Misner (ADM) mass [6], the Hawking temperature, and the Bekenstein-Hawking (BH) entropy. These are combined to give the on-shell Helmholtz

free energy in canonical ensemble which determines the global thermodynamic stability. The other important quantity is the heat capacity which determines the local thermodynamic stability. If one uses the Euclidean action approach, one also finds these quantities consistently [7].

However, the black hole thermodynamics was not completely known in fourth-order gravity because one has encountered some difficulty to compute their conserved quantities in asymptotically AdS spacetimes exactly. Recently, there was some progress on computation scheme of mass and related thermodynamic quantities by using the Abbot-Deser-Tekin (ADT) method [8, 9]. The ADM method is suitable for computing conserved quantities of a black hole in asymptotically flat spacetimes, while the ADT method is useful to compute conserved quantities of a black hole in asymptotically AdS spacetimes found from fourth-order gravity [10]. After computing all ADT thermodynamic quantities depending on a mass parameter  $m_d^2 (= 1/\beta)$ , one is ready to study thermodynamics and phase transition between TAdS and AdS black hole in fourth-order gravity. For  $m_d^2 > m_c^2$  with critical mass parameter  $m_c$  giving  $\mathcal{M}_d^2(m_c^2) = 0$ , all thermodynamic properties are dominantly determined by Einstein gravity, while for  $m_d^2 < m_c^2$ , all thermodynamic properties are dominantly by Weyl-squared term. The former is completely understood, but the latter becomes a new area of black hole thermodynamics appeared when one studies the black hole by using the ADT thermodynamic quantities.

On the other hand, there was a connection between thermodynamic instability and classical [Gregory-Laflamme (GL) [11]] instability for the black strings/branes. This Gubser-Mitra proposal [12] was referred to as the the correlated stability conjecture (CSC) [13] which states that the classical instability of a black string/brane with translational symmetry and infinite extent sets in precisely when the corresponding thermodynamic system becomes (locally) thermodynamically unstable (Hessian matrix<sup>1</sup>  $< 0$  or heat capacity  $< 0$ ). Here the additional assumption of translational symmetry and infinite extent has been added to ensure that finite size effects do not spoil the thermodynamic nature of the argument and to exclude a well-known case of the Schwarzschild black hole which is classically stable, but thermodynamically unstable because of its negative heat capacity.

Interestingly, it is very important to mention that the stability of the Schwarzschild black hole in four-dimensional massive gravity is determined by using the GL instability of a five-dimensional black string. Although the Schwarzschild black hole stability has been performed in Einstein gravity forty years ago [14–16], the stability analysis of the Schwarzschild black hole in massive gravity theory were very recently announced. The massless spin-2 graviton has 2 degrees of freedom (DOF) in Einstein gravity, while the massive graviton has 5 DOF in massive gravity theory. Even a massive spin-2 graviton has 5 DOF, one has a single physical DOF when one considers the  $s(l = 0)$ -mode of massive graviton. Also, it was proved that the  $s$ -wave perturbation gives unstable modes

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<sup>1</sup>There are two representations when defining the Hessian matrix:  $H_M^S$  and  $H_S^M$ . The matrix  $H_M^S$  ( $H_S^M$ ) can be expressed in terms of the second-order derivatives of the entropy (mass) with respect to the mass (entropy) and the conserved charges. Here, Hessian matrix  $< 0$  denotes a negative eigenvalue of the matrix  $H_S^M$ .

only in the higher dimensional black string perturbation [17]. It turned out that the small Schwarzschild black holes in the dRGT massive gravity [18, 19] and fourth-order gravity [20, 21] are unstable against the metric and Ricci tensor perturbations, respectively. This implies that the massiveness of  $m^2 \neq 0$  gives rise to unstable modes propagating around the Schwarzschild black hole. If one may find thermodynamic instability from the ADT thermodynamic quantities of AdS black hole in fourth-order gravity, then it could be compared with the GL-instability found from the linearized Einstein equation [22]. If one finds a connection between them, it might imply that the Gubser-Mitra conjecture holds even for a compact object of the SAdS black hole found in fourth-order gravity. This is our main motivation of why we study fourth-order gravity here.

In this work, we investigate thermodynamic and classical instability of AdS black holes in fourth-order gravity. These include the BTZ black hole in new massive gravity, Schwarzschild-AdS black hole and higher-dimensional AdS black holes in fourth-order gravity. All thermodynamic quantities are computed using the ADT method. Here we use the ADT conserved quantities, since they respect the first-law of thermodynamics and the ADT mass and entropy are reliable to use a thermodynamic study of the AdS black holes in fourth-order gravity. Finally, we establish a connection between the thermodynamic instability of AdS black holes and the GL instability of AdS black holes in the linearized fourth-order gravity.

## 2 BTZ black hole in new massive gravity

As a prototype, we consider the BTZ black hole in new massive gravity (NMG) which is known to be a three-dimensional version of fourth-order gravity. The NMG action [23] composed of the Einstein-Hilbert action with a cosmological constant  $\lambda$  and fourth-order curvature terms is given by

$$S_{\text{NMG}} = S_{\text{EH}} + S_{\text{FOT}}, \tag{2.1}$$

$$S_{\text{EH}} = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g} (R - 2\lambda), \tag{2.2}$$

$$S_{\text{FOT}} = -\frac{1}{16\pi G_3 m^2} \int d^3x \sqrt{-g} \left( R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right), \tag{2.3}$$

where  $G_3$  is a three-dimensional Newton constant and  $m^2$  a positive mass parameter with mass dimension 2 [ $m^2 \in (0, \infty)$ ]. In the limit of  $m^2 \rightarrow \infty$ ,  $S_{\text{NMG}}$  recovers the Einstein gravity  $S_{\text{EH}}$ , while it reduces to purely fourth-order term  $S_{\text{FOT}}$  in the limit of  $m^2 \rightarrow 0$ . The field equation is given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} - \frac{1}{2m^2} K_{\mu\nu} = 0, \tag{2.4}$$

where

$$\begin{aligned} K_{\mu\nu} = & 2\Box R_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu R - \frac{1}{2}\Box R g_{\mu\nu} \\ & + 4R_{\mu\rho\nu\sigma}R^{\rho\sigma} - \frac{3}{2}RR_{\mu\nu} - R_{\rho\sigma}R^{\rho\sigma}g_{\mu\nu} + \frac{3}{8}R^2g_{\mu\nu}. \end{aligned} \tag{2.5}$$

The non-rotating BTZ black hole solution to eq. (2.4) is given by [24, 25]

$$ds_{\text{BTZ}}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2, \quad f(r) = -M + \frac{r^2}{\ell^2} \quad (2.6)$$

under the condition of  $1/\ell^2 + \lambda + 1/(4m^2\ell^4) = 0$ . Here  $M$  is an integration constant related to the the ADM mass of black hole. The horizon radius  $r_+$  is determined by the condition of  $f(r) = 0$  and  $\ell$  denotes the curvature radius of AdS<sub>3</sub> spacetimes.

Its Hawking temperature is found to be

$$T_{\text{H}} = \frac{f'(r_+)}{4\pi} = \frac{r_+}{2\pi\ell^2}. \quad (2.7)$$

Using the ADT method, one can derive all thermodynamic quantities of its mass [26], heat capacity ( $C = \frac{dM_{\text{ADT}}}{dT_{\text{H}}}$ ), entropy [27], and on-shell (Helmholtz) free energy

$$\begin{aligned} M_{\text{ADT}}(m^2, r_+) &= \left(1 - \frac{1}{2m^2\ell^2}\right)M(r_+), & C_{\text{ADT}}(m^2, r_+) &= \left(1 - \frac{1}{2m^2\ell^2}\right)C(r_+), \\ S_{\text{ADT}}(m^2, r_+) &= \left(1 - \frac{1}{2m^2\ell^2}\right)S_{\text{BH}}(r_+), & F_{\text{ADT}}^{\text{on}}(m^2, r_+) &= \left(1 - \frac{1}{2m^2\ell^2}\right)F^{\text{on}}(r_+), \end{aligned} \quad (2.8)$$

whose thermodynamic quantities in Einstein gravity have already given by [28–30]

$$M(r_+) = \frac{r_+^2}{8G_3\ell^2}, \quad C(r_+) = \frac{\pi r_+}{2G_3}, \quad S_{\text{BH}}(r_+) = \frac{\pi r_+}{2G_3}, \quad F^{\text{on}}(r_+) = M - T_{\text{H}}S_{\text{BH}} = -\frac{r_+^2}{8G_3\ell^2}. \quad (2.9)$$

These all are positive regardless of the horizon size  $r_+$  except that the free energy is always negative. This means that the BTZ black hole is thermodynamically stable in Einstein gravity. Here we check that the first-law of thermodynamics is satisfied as

$$dM_{\text{ADT}} = T_{\text{H}}dS_{\text{ADT}} \quad (2.10)$$

as the first-law is satisfied in Einstein gravity

$$dM = T_{\text{H}}dS_{\text{BH}} \quad (2.11)$$

where ‘ $d$ ’ denotes the differentiation with respect to the horizon size  $r_+$  only. In this work, we treat  $m^2$  differently from the black hole charge  $Q$  and angular momentum  $J$  to achieve the first-law (2.10). Here we observe that in the limit of  $m^2 \rightarrow \infty$  one recovers thermodynamics of the BTZ black hole in Einstein gravity, while in the limit of  $m^2 \rightarrow 0$  we recover the black hole thermodynamics in purely fourth-order gravity which is similar to recovering the conformal Chern-Simons gravity from the topologically massive gravity (TMG) [31].

On the other hand, the linearized equation to (2.4) upon choosing the transverse-traceless (TT) gauge of  $\bar{\nabla}^\mu h_{\mu\nu} = 0$  and  $h^\mu{}_\mu = 0$  leads to the fourth-order equation for the metric perturbation  $h_{\mu\nu}$  [32, 33]

$$\left(\bar{\nabla}^2 - 2\Lambda\right)\left[\bar{\nabla}^2 - 2\Lambda - \mathcal{M}^2(m^2)\right]h_{\mu\nu} = 0, \quad \Lambda = -\frac{1}{\ell^2} \quad (2.12)$$

which might imply the two second-order linearized equations

$$\left(\bar{\nabla}^2 - 2\Lambda\right)h_{\mu\nu} = 0, \tag{2.13}$$

$$\left[\bar{\nabla}^2 - 2\Lambda - \mathcal{M}^2(m^2)\right]h_{\mu\nu} = 0. \tag{2.14}$$

Here the mass squared of a massive spin-2 graviton is given by

$$\mathcal{M}^2(m^2) = m^2 - \frac{1}{2\ell^2}. \tag{2.15}$$

eq. (2.14) describes a massive graviton with 2 DOF propagating around the BTZ black hole under the TT gauge.

Expressing all thermodynamic quantities in (2.8) in terms of the mass squared leads to

$$M_{\text{ADT}} = \frac{\mathcal{M}^2}{m^2}M, \quad C_{\text{ADT}} = \frac{\mathcal{M}^2}{m^2}C, \quad S_{\text{ADT}} = \frac{\mathcal{M}^2}{m^2}S_{\text{BH}}, \quad F_{\text{ADT}}^{\text{on}} = \frac{\mathcal{M}^2}{m^2}F^{\text{on}} \tag{2.16}$$

which shows clearly that all thermodynamical quantities depend on the sign of  $\mathcal{M}^2$ . The local thermodynamic stability is determined by the positive heat capacity ( $C_{\text{ADT}} > 0$ ) and the global stability is determined by the negative free energy ( $F_{\text{ADT}} < 0$ ). Hence, it implies that the thermodynamic stability is determined by the sign of the heat capacity, while the phase transition is mainly determined by the sign of the free energy.

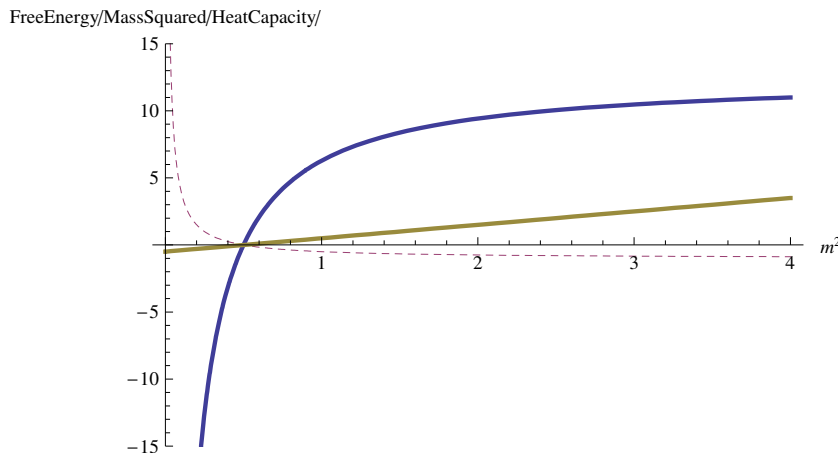
For  $\mathcal{M}^2 > 0$  ( $m^2 > 1/2\ell^2$ ), all thermodynamic quantities have the same property as those for Einstein gravity (2.2), whereas for  $\mathcal{M}^2 < 0$  ( $m^2 < 1/2\ell^2$ ), all thermodynamic quantities have the same property as those for fourth-order term (2.3). We observe from figure 1 that for  $\mathcal{M}^2 > 0$ , the BTZ black hole is thermodynamically stable regardless of the horizon size  $r_+$  because of  $C_{\text{ADT}} > 0$ .

On the other hand, the classical (in) stability condition of the BTZ black hole was recently determined by the condition of  $\mathcal{M}^2 > 0$  ( $< 0$ ) regardless of the horizon size  $r_+$  [34] (see also [35] for stability issue in NMG). The case of  $\mathcal{M}^2 = 0$  corresponds to the critical gravity where all thermodynamical quantities are zero and logarithmic modes appear. For  $\mathcal{M}^2 < 0$ , the BTZ black hole is thermodynamically unstable because of  $C_{\text{ADT}} < 0$  as well as it is classically unstable against the metric perturbations. *Hence, it shows a clear connection between thermodynamic and classical instability for the BTZ black hole regardless of the horizon size in new massive gravity.*

Finally, let us turn to the issue related to a phase transition from the thermal AdS<sub>3</sub> (TAdS) to BTZ black hole. For this purpose, we first consider thermodynamic quantities for the TAdS [29]

$$M_{\text{TAdS}} = -\frac{1}{8G_3}, \quad F_{\text{TAdS}} = -\frac{1}{8G_3}. \tag{2.17}$$

It turns out that for  $\mathcal{M}^2 < 0$  ( $m^2 < 1/2\ell^2$ ) [fourth-order term (2.3) contributes dominantly to black hole thermodynamics], the TAdS is always favored than the BTZ black hole because of  $F_{\text{TAdS}} < F_{\text{ADT}}^{\text{on}}$ , where  $F_{\text{ADT}}^{\text{on}}$  is given by (2.16). In this case, we might not define a possible phase transition because the ground state is the TAdS. Alternatively, this implies



**Figure 1.** Heat capacity (solid curve)  $C_{\text{ADT}}(m^2, r_+ = 1)$ , mass squared (line)  $\mathcal{M}^2(m^2)$ , and free energy (dotted curve)  $F_{\text{ADT}}^{\text{on}}(m^2, r_+ = 1)$  with  $G_3 = 1/8$  and  $\ell = 1$ . For  $m^2 < 0.5$ , one has  $C_{\text{ADT}} < 0$  ( $F_{\text{ADT}}^{\text{on}} > 0$ ) and  $\mathcal{M}^2 < 0$ , while for  $m^2 > 0.5$ , one has  $C_{\text{ADT}} > 0$  ( $F_{\text{ADT}}^{\text{on}} < 0$ ) and  $\mathcal{M}^2 > 0$ . At the critical point of  $m^2 = 0.5$ , we have  $F_{\text{ADT}}^{\text{on}} = 0$  and  $\mathcal{M}^2 = 0$ . In the limit of  $m^2 \rightarrow 0/\infty$ , fourth-order term/Einstein gravity are recovered for free energy and heat capacity.

that a gap between  $F_{\text{ADT}}^{\text{on}}$  and  $F_{\text{TAdS}}$  might not allow a continuous phase transition.<sup>2</sup> It is noted that a phase transition from the TAdS to the BTZ black hole is possible to occur in new massive gravity for  $\mathcal{M}^2 > 0$  ( $m^2 = 1 > 1/2\ell^2$ ) (see arXiv:1311.6985v1 for details), in which the Einstein gravity (2.2) contributes dominantly to black hole thermodynamics. The corresponding phase transition in this case is similar to that obtained in the literature [36] for  $z = 1$ .

### 3 Thermodynamics of AdS black holes in fourth-order gravity

Let us start with the  $d(\geq 4)$ -dimensional fourth-order gravity action [37]

$$S_{\text{dFO}} = \frac{1}{16\pi G_d} \int d^d x \sqrt{-g} \left[ R - (d-2)\Lambda_0 + \alpha R_{\mu\nu} R^{\mu\nu} + \beta R^2 \right] \quad (3.1)$$

with two parameters  $\alpha$  and  $\beta$ . Here we do not include the Gauss-Bonnet term [38] because (3.1) admits solutions of the higher-dimensional Einstein gravity including the higher dimensional AdS black holes. From (3.1), the Einstein equation is derived to be

$$G_{\mu\nu} + E_{\mu\nu} = 0, \quad (3.2)$$

where the Einstein tensor is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \frac{d-2}{2} \Lambda_0 g_{\mu\nu} \quad (3.3)$$

<sup>2</sup>Concerning this issue, a phase transition between hot flat space and flat space cosmological spacetimes was recently studied in TMG by using on-shell free energies [31].

and  $E_{\mu\nu}$  takes the form

$$E_{\mu\nu} = 2\alpha \left( R_{\mu\rho\nu\sigma} R^{\rho\sigma} - \frac{1}{4} R^{\rho\sigma} R_{\rho\sigma} g_{\mu\nu} \right) + 2\beta R \left( R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right) + \alpha \left( \nabla^2 R_{\mu\nu} + \frac{1}{2} \nabla^2 R g_{\mu\nu} - \nabla_\mu \nabla_\nu R \right) + 2\beta \left( g_{\mu\nu} \nabla^2 R - \nabla_\mu \nabla_\nu R \right). \quad (3.4)$$

For the Einstein space of  $R_{\mu\nu} = \Lambda g_{\mu\nu}$  and  $R = d\Lambda$  together with  $\Lambda_0 = \Lambda + (d-4)(\alpha + d\beta)\Lambda^2/(d-2)$ , eq. (3.2) allows a  $d$ -dimensional AdS black hole solution

$$ds_{\text{ST}}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -V(r) dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_{d-2}^2 \quad (3.5)$$

with the metric function

$$V(r) = 1 - \left( \frac{r_0}{r} \right)^{d-3} - \frac{\Lambda}{d-1} r^2, \quad \Lambda = -\frac{d-1}{\ell^2}. \quad (3.6)$$

The horizon is located at  $r = r_+$  ( $V(r_+) = 0$ ) which means that  $r_0$  differs from  $r_+$ . Hereafter we denote the background quantities with the ‘‘overbar’’. In this case, the black hole background spacetimes is given by

$$\bar{R}_{\mu\nu} = \Lambda \bar{g}_{\mu\nu}, \quad \bar{R} = 4\Lambda. \quad (3.7)$$

The Hawking temperature is derived as

$$T_{\text{H}}^d = \frac{V'(r_+)}{4\pi} = \frac{1}{4\pi r_+} \left[ (d-3) + \frac{d-1}{\ell^2} r_+^2 \right]. \quad (3.8)$$

Using the ADT method [8, 9], all thermodynamic quantities of its mass [37], heat capacity, entropy [37], and on-shell free energy are given by

$$\begin{aligned} M_{\text{dADT}} &= \frac{4(d-1)}{dm_d^2} \mathcal{M}_d^2 M_d(r_+), & C_{\text{dADT}}(m^2, r_+) &= \frac{4(d-1)}{dm_d^2} \mathcal{M}_d^2 C_d(r_+), \\ S_{\text{dADT}} &= \frac{4(d-1)}{dm_d^2} \mathcal{M}_d^2 S_{\text{BH}}(r_+), & F_{\text{dADT}}^{\text{on}} &= \frac{4(d-1)}{dm_d^2} \mathcal{M}_d^2 F_d^{\text{on}}(r_+), \end{aligned} \quad (3.9)$$

where

$$m_d^2 = \frac{1}{\beta}, \quad \alpha = -\frac{4(d-1)}{d} \beta, \quad \mathcal{M}_d^2(m_d^2) = \frac{d}{4} \left[ \frac{m_d^2}{d-1} - \frac{2(d-2)^2}{d\ell^2} \right]. \quad (3.10)$$

At this stage, we note that even though all thermodynamic quantities are obtained for arbitrary  $\alpha$  and  $\beta$ , we require a condition of  $\alpha = -4(d-1)\beta/d$  because the classical stability could be achieved only under this condition. This means that we have a single mass parameter  $m_d^2 = 1/\beta$  by eliminating a massive spin-0 graviton in fourth-order gravity, reducing to the Einstein-Weyl gravity. All thermodynamic quantities in  $d(\geq 4)$ -dimensional



Einstein gravity were known to be [39]

$$M_d(r_+) = \frac{\Omega_{d-2}(d-2)}{16\pi G_d} r_+^{d-3} \left[ 1 + \frac{r_+^2}{\ell^2} \right], \quad (3.11)$$

$$C_d(r_+) = \frac{dM_d}{dT_H^d} = \frac{\Omega_{d-2}(d-2)r_+^{d-2}}{4G_d} \left[ \frac{(d-1)r_+^2 + (d-3)\ell^2}{(d-1)r_+^2 - (d-3)\ell^2} \right], \quad (3.12)$$

$$S_{BH}(r_+) = \frac{\Omega_{d-2}}{4G_d} r_+^{d-2}, \quad (3.13)$$

$$F_d^{\text{on}}(r_+) = M_d - T_H S_{BH} = \frac{\Omega_{d-2}}{16\pi G_d} r_+^{d-3} \left[ 1 - \frac{r_+^2}{\ell^2} \right] \quad (3.14)$$

with the area of  $S^{d-2}$

$$\Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}. \quad (3.15)$$

We observe from (3.9) that for  $\mathcal{M}_d^2 > 0$ , thermodynamic stability of AdS black hole is determined by the higher-dimensional Einstein gravity. On the other hand, for  $\mathcal{M}_d^2 < 0$ , thermodynamic stability of black hole is determined by Weyl-squared term (conformal gravity).

We check that the first-law of thermodynamics is satisfied as

$$dM_{\text{dADT}} = T_H^d dS_{\text{dADT}} \quad (3.16)$$

as the first-law is satisfied in  $d$ -dimensional Einstein gravity

$$dM_d = T_H^d dS_{\text{BH}}, \quad (3.17)$$

where ‘ $d$ ’ denotes the differentiation with respect to the horizon size  $r_+$  only. In this work, we treat  $m_d^2$  differently from the black hole charge  $Q$  and angular momentum  $J$  to obtain the first-law (3.16). Here we observe that in the limit of  $m_d^2 \rightarrow \infty$  we recover thermodynamics of the AdS black hole in  $d$ -dimensional Einstein gravity, while in the limit of  $m_d^2 \rightarrow 0$  we recover that in Weyl-squared term (conformal gravity).

## 4 SAdS black hole in Einstein-Weyl gravity

### 4.1 Thermodynamic instability for small black holes

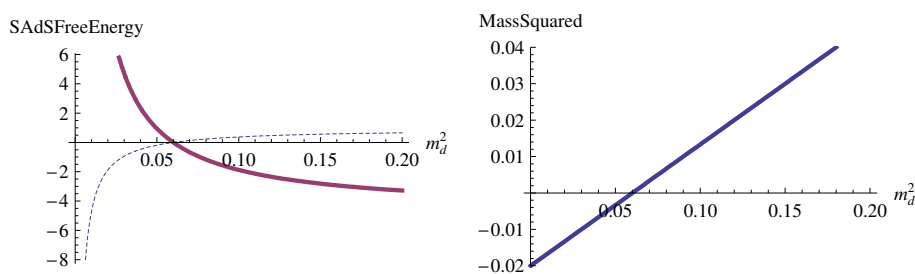
For a definite description of black hole thermodynamics, we choose  $d = 4$  which provides a SAdS black hole. Its thermodynamic quantities of mass [40], heat capacity, entropy [40], and on-shell free energy are given by

$$M_{\text{dADT}}(m_d^2, r_+) = \left( 1 - \frac{6}{m_d^2 \ell^2} \right) M_{\text{SAdS}}, \quad (4.1)$$

$$C_{\text{dADT}}(m_d^2, r_+) = \left( 1 - \frac{6}{m_d^2 \ell^2} \right) C_{\text{SAdS}}, \quad (4.2)$$

$$S_{\text{dADT}}(m_d^2, r_+) = \left( 1 - \frac{6}{m_d^2 \ell^2} \right) S_{\text{BH}}, \quad (4.3)$$

$$F_{\text{dADT}}^{\text{on}}(m_d^2, r_+) = \left( 1 - \frac{6}{m_d^2 \ell^2} \right) F_{\text{SAdS}}^{\text{on}}, \quad (4.4)$$



**Figure 2.** Left: free energy (solid)  $F_{\text{dADT}}^{\text{on}}(m_d^2, r_+ = 15)$  for large black hole and free energy (dotted)  $F_{\text{dADT}}^{\text{on}}(m_d^2, r_+ = 5)$  for small black hole with  $G_4 = 1/2$  and  $\ell = 10$ . Right: mass squared  $\mathcal{M}_d^2(m_d^2)$ . At the critical point of  $m_d^2 = 0.06$ , we have  $F_{\text{dADT}}^{\text{on}} = 0$  and  $\mathcal{M}_d^2 = 0$ . In the limit of  $m_d^2 \rightarrow 0/\infty$ , Weyl-squared term (conformal gravity)/Einstein gravity are recovered for free energy.

where all thermodynamic quantities of SAdS black hole are shown in the eqs. (3.11)–(3.14) for  $d = 4$ . The mass squared takes the form [40]

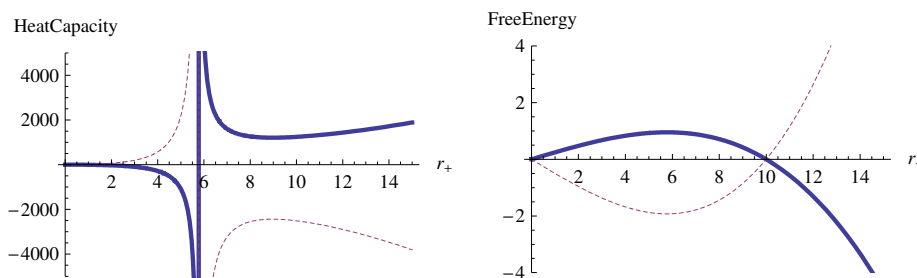
$$\mathcal{M}_d^2 = \frac{m_d^2}{3} - \frac{2}{\ell^2} \tag{4.5}$$

which is negative/postive for  $m_d^2 \leq 6/\ell^2$ .

First of all, we depict the free energy as a function of  $m_d^2$  in figure 2. Since the sign of free energy depends on the horizon size  $r_+$  critically, we plot the two free energies as function of  $m_d^2$  for large ( $r_+ = 15 > \ell = 10$ ) and small black hole ( $r_+ = 5 < \ell$ ), respectively. For  $m_d^2 < 0.06$ , one has  $F_{\text{dADT}}^{\text{on}}(m_d^2, r_+ = 15) > 0$  ( $F_{\text{dADT}}^{\text{on}}(m_d^2, r_+ = 5) < 0$ ) and  $\mathcal{M}_d^2 < 0$ , while for  $m_d^2 > 0.06$ , one has  $F_{\text{dADT}}^{\text{on}}(m_d^2, r_+ = 15) < 0$  ( $F_{\text{dADT}}^{\text{on}}(m_d^2, r_+ = 5) > 0$ ) and  $\mathcal{M}_d^2 > 0$ .

We consider first the case of  $\mathcal{M}_d^2 > 0$  ( $m_d^2 > 0.06$ ) which is dominantly described by the Einstein gravity. Since the heat capacity of  $C_{\text{SAdS}}$  blows up at  $r_+ = r_* = \ell/\sqrt{3} = 0.577\ell$ , we divide the black hole into the small black hole with  $r_+ < r_*$  and the large black hole with  $r_+ > r_*$ . As is shown in the solid (familiar) curves in figure 3, we have the small black hole with  $r_+ < r_*$  which is thermodynamically unstable because  $C_{\text{dADT}} < 0$ , while the large black hole with  $r_+ > r_*$  is thermodynamically stable because  $C_{\text{dADT}} > 0$ . Especially for  $\mathcal{M}_d^2 = 1.98 > 0$  ( $m_d^2 = 6$ ), one has  $M_{\text{dADT}} = 0.99M_{\text{SAdS}}$ ,  $C_{\text{dADT}} = 0.99C_{\text{SAdS}}$ ,  $S_{\text{dADT}} = 0.99S_{\text{BH}}$ , and  $F_{\text{dADT}} = 0.99F_{\text{SAdS}}$ . Hence we could describe the Hawking-Page phase transition well as for the SAdS black hole in Einstein gravity [3]. We wish to comment that the free energy has the maximum value at  $r_+ = r_*$ . Since the free energy becomes negative (positive) for  $r_+ < \ell$  ( $r_+ > \ell$ ), we did not choose  $r_+ = \ell$  as a boundary point to divide the black hole into small and large black holes.

On the other hand, for  $\mathcal{M}_d^2 = -0.013 < 0$  ( $m_d^2 = 0.02 < 6/\ell^2$ ) which is dominantly described by conformal gravity [41], the small black hole ( $r_+ < r_*$ ) is thermodynamically stable because  $C_{\text{dADT}} > 0$ , while the large black hole ( $r_+ > r_*$ ) is thermodynamically unstable because  $C_{\text{dADT}} < 0$ . See the dotted (unfamiliar) curves in figure 3 for observation. It seems that there is no known phase transition from thermal AdS to the SAdS black hole in conformal gravity.



**Figure 3.** Left: familiar heat capacity (solid)  $C_{\text{dADT}}(m_d^2 = 6, r_+)$  for  $\mathcal{M}_d^2 = 1.98 > 0$  and unfamiliar heat capacity (dotted)  $C_{\text{dADT}}(m_d^2 = 0.02, r_+)$  for  $\mathcal{M}_d^2 = -0.013 < 0$  with  $G_4 = 1/2$  and  $\ell = 10$ . Right: familiar free energy (solid)  $F_{\text{dADT}}^{\text{on}}(m_d^2 = 6, r_+)$  for  $\mathcal{M}_d^2 = 1.98$  and unfamiliar free energy (dotted)  $F_{\text{dADT}}^{\text{on}}(m_d^2 = 0.02, r_+)$  for  $\mathcal{M}_d^2 = -0.013$ .

### 4.2 GL instability for small black holes

We briefly review the Gregory-Laflamme  $s$ -mode instability for a massive spin-2 graviton with mass  $\mathcal{M}_d \geq 0$  propagating on the SAdS black hole spacetimes in Einstein-Weyl gravity. Choosing the TT gauge, its linearized equation to (3.2) takes the form

$$\bar{\nabla}^2 h_{\mu\nu} + 2\bar{R}_{\alpha\mu\beta\nu} h^{\alpha\beta} - \mathcal{M}_d^2 h_{\mu\nu} = 0. \quad (4.6)$$

which describes 5 DOF of a massive spin-2 graviton propagating on the SAdS black hole spacetimes. We note that choosing the condition of  $\alpha = -3\beta$  eliminates a massive spin-0 graviton with 1 DOF.

Before we proceed, we wish to mention that the stability of the Schwarzschild black hole in four-dimensional massive gravity is determined by using the Gregory-Laflamme instability of a five-dimensional black string. It turned out that the small Schwarzschild black holes in the dRGT massive gravity [18, 19] and fourth-order gravity [20] are unstable against the metric and Ricci tensor perturbations because the inequality is satisfied as

$$\mathcal{M}_d \leq \frac{\mathcal{O}(1)}{r_0}, \quad r_0 = 2M_S. \quad (4.7)$$

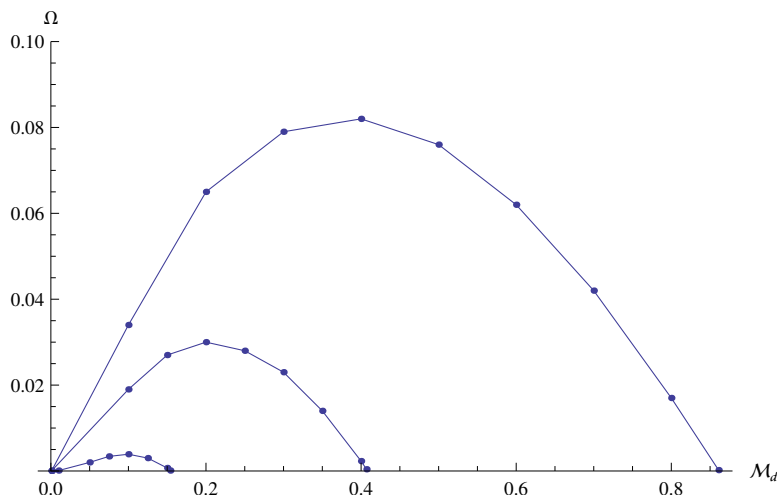
For the massless case of  $\mathcal{M}_d = 0$ , eq. (4.6) leads to the linearized equation around the Schwarzschild black hole with the TT gauge which is known to be stable in the Einstein gravity.

Choosing the  $s$ -mode ansatz whose form is given by  $H_{tt}$ ,  $H_{tr}$ ,  $H_{rr}$ , and  $K$  as

$$h_{\mu\nu}^s = e^{\Omega t} \begin{pmatrix} H_{tt}(r) & H_{tr}(r) & 0 & 0 \\ H_{tr}(r) & H_{rr}(r) & 0 & 0 \\ 0 & 0 & K(r) & 0 \\ 0 & 0 & 0 & \sin^2 \theta K(r) \end{pmatrix}, \quad (4.8)$$

a relevant equation for  $H_{tr}$  takes the same form (see appendix for explicit forms of  $A$ ,  $B$ ,  $C$ )

$$A(r; r_0, \ell, \Omega^2, \mathcal{M}_d^2) \frac{d^2}{dr^2} H_{tr} + B \frac{d}{dr} H_{tr} + C H_{tr} = 0, \quad (4.9)$$



**Figure 4.** Plots of unstable modes on three curves with  $r_+ = 1, 2, 4$  and  $\ell = 10$ . These belong to small black holes because of  $r_+ < r_* = 5.77$ . The  $y(x)$ -axis denote  $\Omega(\mathcal{M}_d)$ . Also we check that for  $r_+ = 6 > r_* = 5.77$ , the maximum value of  $\Omega$  is less than  $10^{-4}$ , which implies that there is no instability for large black hole. In this figure, the smallest curve represents  $r_+ = 4$ , the medium denotes  $r_+ = 2$ , and the largest one shows  $r_+ = 1$ .

which shows the same unstable modes for

$$0 < \mathcal{M}_d < \frac{\mathcal{O}(1)}{r_0} \tag{4.10}$$

with the mass

$$\mathcal{M}_d = \sqrt{\frac{m_d^2}{3} - \frac{2}{l^2}}. \tag{4.11}$$

The condition of (4.10) could be read off from figure 4 when one notes the difference between  $r_+$  and  $r_0$ :  $r_+ = 1, 2, 4$  and  $r_0 = 1.01, 2.08, 4.64$ . On the other hand, the stable condition of the SAdS black hole in Einstein-Weyl gravity is given by

$$\mathcal{M}_d > \frac{\mathcal{O}(1)}{r_0}. \tag{4.12}$$

At this stage, we would like to mention the classical stability of  $\mathcal{M}_d = 0$  case. In this case, its linearized equation reduces to

$$\bar{\nabla}^2 h_{\mu\nu} + 2\bar{R}_{\alpha\mu\beta\nu} h^{\alpha\beta} = 0, \tag{4.13}$$

which is exactly the linearized equation around the SAdS black hole in Einstein gravity. From the observation<sup>3</sup> of figure 4, the GL instability disappears at  $\mathcal{M}_d = 0$ , which may imply that the SAdS black hole is stable against the  $s$ -mode metric perturbation. The SAdS black hole was known to be stable against the metric perturbation even though a negative potential appeared near the event horizon in odd-parity sector [42]. Later

<sup>3</sup>In the next section 5.2, we introduce the corresponding numerical analysis to observe this GL instability.

on, one could achieve the positivity of gravitational potentials by using the  $S$ -deformed technique [43], proving the stability of SAdS black hole exactly [44]. This implies that there is no connection between classical stability and thermodynamic instability ( $C_{\text{SAdS}} < 0$ ) for small SAdS black hole. This situation is similar to the Schwarzschild black hole which shows a violation of the CSC between thermodynamic instability ( $C_S < 0$ ) and the classical stability [14–16]. Therefore, we could not apply the Gubser-Mitra conjecture to the SAdS black hole in Einstein gravity.

Let us see how things are improved in Einstein-Weyl gravity. We note that at the critical point of  $\mathcal{M}_d^2 = 0$ , all thermodynamic quantities vanish exactly. For  $\mathcal{M}_d^2 > 0$  and small black hole with  $r_+ < r_*$ , the heat capacity takes the form

$$C_{\text{dADT}} = \frac{3\mathcal{M}_d^2}{m_d^2} C_{\text{SAdS}} < 0, \tag{4.14}$$

which shows thermodynamic instability like that of a small SAdS black hole in Einstein gravity. From the condition of (4.10), however, we find that a small black hole is unstable against the  $s$ -mode massive graviton perturbation. This implies that the CSC holds for the SAdS black hole in Einstein-Weyl gravity.

Also the stability condition of (4.12) is consistent with thermodynamic stability condition for large black hole with  $r_+ > r_*$  in Einstein gravity

$$C_{\text{dADT}} = \frac{3\mathcal{M}_d^2}{m_d^2} C_{\text{SAdS}} > 0. \tag{4.15}$$

As was previously emphasized, there is no connection between thermodynamic instability and classical stability for small SAdS black hole in Einstein gravity. However, the GL instability condition picks up the small SAdS black hole which is thermodynamically unstable in Einstein-Weyl gravity. Hence, we conclude that there is a connection between the GL instability and thermodynamic instability for small black hole in fourth-order (Einstein-Weyl) gravity.

## 5 Higher-dimensional AdS black holes in fourth-order gravity

### 5.1 Thermodynamic instability for small black holes

In this section, we comment briefly on the thermodynamic (in)stability for higher-dimensional AdS black hole. To this end, we first recall the thermodynamic quantities (3.9), obtained in the  $d$ -dimensional fourth order gravity. Among them, taking into account the heat capacity together with (3.12), the small and large black holes can be divided by choosing the blow-up heat capacity at

$$r_+ = r_*^{(d)} = \sqrt{\frac{d-3}{d-1}} \ell. \tag{5.1}$$

For  $\mathcal{M}_d^2 > 0$  [ $m_d^2 > 2(d-1)(d-2)^2/d\ell^2$ ], we have the small black hole for  $r_+ < r_*^{(d)}$  which is thermodynamically unstable because  $C_{\text{dADT}} < 0$  in (3.9), while we have the large

black hole for  $r_+ > r_*^{(d)}$  which is thermodynamically stable because  $C_{\text{dADT}} > 0$ . This is dominantly described by the higher-dimensional Einstein gravity. We would like to mention that for  $\mathcal{M}_d^2 > 0$  we will establish the connection between the GL instability and thermodynamic instability of the small black hole.

On the other hand, for  $\mathcal{M}_d^2 < 0$  [ $m_d^2 < 2(d-1)(d-2)^2/d\ell^2$ ] which is dominantly described by Weyl-squared term, the small black hole is thermodynamically stable because  $C_{\text{dADT}} > 0$ , whereas the large black hole is thermodynamically unstable because of  $C_{\text{dADT}} < 0$ . This case requires a newly black hole thermodynamics.

## 5.2 GL instability

In order to investigate the classical instability for higher-dimensional AdS black hole, we first consider two coupled first order differential equations<sup>4</sup>

$$H' = \left[ \frac{3-d-(d-1)r^2/\ell^2}{rV} - \frac{1}{r} \right] H + \frac{\Omega}{2V}(H_+ + H_-) \quad (5.2)$$

$$H'_- = \frac{\mathcal{M}_d^2}{\Omega} H + \frac{d-2}{2r} H_+ + \left[ \frac{d-3+(d-1)r^2/\ell^2}{2rV} - \frac{2d-3}{2r} \right] H_- \quad (5.3)$$

with the constraint equation

$$\begin{aligned} & r^2\Omega \left[ 4r\Omega^2 - rV'^2 + (d-2)VV' + 2rV\mathcal{M}_d^2 + 2rVV'' \right] H_- \\ & - \Omega r^2V \left[ 2\mathcal{M}_d^2r + (d-2)V' \right] H_+ - 2r^2V \left[ 2(d-2)\Omega^2 - 2\mathcal{M}_d^2V + r\mathcal{M}_d^2V' \right] H = 0, \end{aligned} \quad (5.4)$$

where

$$H \equiv H_{tr}, \quad H_{\pm} \equiv \frac{H_{tt}}{V(r)} \pm V(r)H_{rr} \quad \text{with} \quad V(r) = 1 - \left( \frac{r_0}{r} \right)^{d-3} + \frac{r^2}{\ell^2}. \quad (5.5)$$

At infinity of  $r \rightarrow \infty$ , asymptotic solutions to eqs. (5.2) and (5.3) are

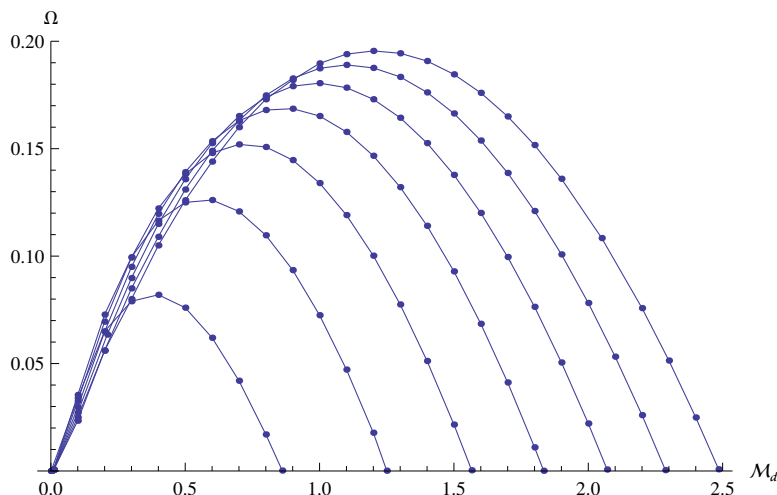
$$\begin{aligned} H^{(\infty)} &= C_1^{(\infty)} r^{-(d+1)/2 + \sqrt{\mathcal{M}_d^2\ell^2 + (d-1)^2/4}} + C_2^{(\infty)} r^{-(d+1)/2 - \sqrt{\mathcal{M}_d^2\ell^2 + (d-1)^2/4}}, \\ H_-^{(\infty)} &= \tilde{C}_1^{(\infty)} r^{-(d-1)/2 + \sqrt{\mathcal{M}_d^2\ell^2 + (d-1)^2/4}} + \tilde{C}_2^{(\infty)} r^{-(d-1)/2 - \sqrt{\mathcal{M}_d^2\ell^2 + (d-1)^2/4}}, \end{aligned} \quad (5.6)$$

where  $\tilde{C}_{1,2}^{(\infty)}$  are

$$\begin{aligned} \tilde{C}_1^{(\infty)} &= \frac{\mathcal{M}_d^2}{(1-d)/2 + \sqrt{\mathcal{M}_d^2\ell^2 + (d-1)^2/4}} C_1^{(\infty)}, \\ \tilde{C}_2^{(\infty)} &= \frac{\mathcal{M}_d^2}{(1-d)/2 - \sqrt{\mathcal{M}_d^2\ell^2 + (d-1)^2/4}} C_2^{(\infty)}. \end{aligned} \quad (5.7)$$

---

<sup>4</sup>We note that these first order differential and constraint equations can be obtained from using the perturbation equation (4.6) and TT gauge condition. Finally we have checked, after some manipulations, that these equations are consistent with the second order equation (4.9) and for  $V = 1 - (r_0/r)^{d-3}$  [in the  $\ell^2 \rightarrow \infty$ -limit], they reduce to those found in the original literature [11].



**Figure 5.**  $\Omega$  graphs as function of  $\mathcal{M}_d$  for a small black hole with  $r_+ = 1$ ,  $\ell = 10$  and  $d = 4, 5, \dots, 10$  from left to right curve. The most left curve in this figure corresponds to the largest one in the figure 5.

At the horizon  $r_+$ , their asymptotic solutions are given by

$$\begin{aligned}
 H^{(r_+)} &= C_1^{(r_+)}(r^{d-3} - r_+^{d-3})^{-1+\Omega/V'(r_+)} + C_2^{(r_+)}(r^{d-3} - r_+^{d-3})^{-1-\Omega/V'(r_+)}, \\
 H_-^{(r_+)} &= \tilde{C}_1^{(r_+)}(r^{d-3} - r_+^{d-3})^{\Omega/V'(r_+)} + \tilde{C}_2^{(r_+)}(r^{d-3} - r_+^{d-3})^{-\Omega/V'(r_+)}, \quad (5.8)
 \end{aligned}$$

where  $\tilde{C}_{1,2}^{(r_+)}$  are

$$\begin{aligned}
 \tilde{C}_1^{(r_+)} &= \frac{(d-3)r_+^{d-3}\Omega(2\Omega - V'(r_+))}{2V'(r_+)(\mathcal{M}_d^2 r_+ + (d-2)\Omega)} C_1^{(r_+)}, \\
 \tilde{C}_2^{(r_+)} &= -\frac{(d-3)r_+^{d-3}\Omega(2\Omega + V'(r_+))}{2V'(r_+)(\mathcal{M}_d^2 r_+ - (d-2)\Omega)} C_2^{(r_+)}.
 \end{aligned}$$

We note that two boundary conditions of the regular solutions correspond to  $C_1^{(\infty)} = 0$  and  $C_2^{(r_+)} = 0$  at infinity and horizon, respectively.

Eliminating  $H_+$  in eqs. (5.2) and (5.3) with the help of the constraint (5.4), one can find the coupled equations with  $H$ ,  $H_-$  only. For given dimensions  $d = 4, 5, \dots, 10$ , fixed  $\mathcal{M}_d$ , and various values of  $\Omega$ , we solve these coupled equations numerically, which yields possible values of  $\Omega$  as a function of  $\mathcal{M}_d$  given by

$$\mathcal{M}_d = \sqrt{\frac{dm_d^2}{4(d-1)} - \frac{(d-2)^2}{2\ell^2}}. \quad (5.9)$$

Figure 5 shows that the curve of possible values of  $\Omega$  and  $\mathcal{M}_d$  intersects the  $\mathcal{M}_d$ -axis at two places:  $\mathcal{M}_d = 0$  and  $\mathcal{M}_d = \mathcal{M}_d^c$  where  $\mathcal{M}_d^c$  is a critical non-zero mass. The fact that

the curve does not intersect the  $\mathcal{M}_d$ -axis at  $\mathcal{M}_d < 0$  follows from the stability of the AdS black hole in higher-dimensional Einstein gravity. Explicitly, for  $\mathcal{M} < \mathcal{M}_d^c$  ( $\mathcal{M} > \mathcal{M}_d^c$ ), the AdS black hole is unstable (stable) against the metric perturbations. From the observation of figure 5, we read off the critical mass  $\mathcal{M}_d^c$  depending on the dimension  $d$  as

$$\left( \begin{array}{c|cccccccc} d & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline \mathcal{M}_d^c & 0.86 & 1.26 & 1.57 & 1.83 & 2.07 & 2.29 & 2.49 \end{array} \right). \quad (5.10)$$

For higher-dimensional black strings with  $V_{\text{bs}}(r) = 1 - (r_0/r)^{d-3}$ , the critical wave number marks the lower bound of possible wavelengths for which there is an unstable mode. Especially for  $e^{\frac{\Omega}{r_0}t} e^{i\frac{k}{r_0}z}$  setting, there exists a critical wave number  $k_d^c$  where for  $k < k_d^c$  ( $k > k_d^c$ ), the black string is unstable (stable) against the metric perturbations. There is an unstable (stable) mode for any wavelength larger (smaller) than the critical wavelength  $\lambda_{\text{GL}} = 2\pi r_0/k_d^c$ :  $\lambda > \lambda_{\text{GL}}$  ( $\lambda < \lambda_{\text{GL}}$ ). The critical wave number  $k_d^c$  depends on the dimension  $d$  as [13]

$$\left( \begin{array}{c|cccccccc} d & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline k_d^c & 0.88 & 1.24 & 1.60 & 1.86 & 2.08 & 2.30 & 2.50 \end{array} \right). \quad (5.11)$$

From  $V(r)$  in (3.5), one has the relation between  $r_+$  and  $r_0$  as

$$r_0 = \left[ \frac{r_+^{d-1}}{\ell^2} + r_+^{d-3} \right]^{\frac{1}{d-3}}. \quad (5.12)$$

For  $r_+ = 1$  and  $\ell = 10$ , it takes the form

$$r_0 = \left( \frac{101}{100} \right)^{\frac{1}{d-3}} \quad (5.13)$$

which implies  $r_0 = \{1.01, 1.005, 1.003, 1.002, 1.002, 1.002, 1.001\}$ . Here, the corresponding  $\tilde{k}_d^c = r_0 \mathcal{M}_d^c$  is given by

$$\left( \begin{array}{c|cccccccc} d & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline \tilde{k}_d^c & 0.87 & 1.27 & 1.57 & 1.83 & 2.07 & 2.29 & 2.49 \end{array} \right). \quad (5.14)$$

We observe that  $\tilde{k}_d^c = k_d^c - 0.01$  for  $d = 4, 8, 9, 10$  whereas  $\tilde{k}_d^c = k_d^c \pm 0.03$  for  $d = 5, 6, 7$ .

Now let us derive the GL instability condition from (5.14). We propose the bound for unstable modes approximately as

$$0 < \mathcal{M}_d < \frac{\tilde{k}_d^c}{r_0}. \quad (5.15)$$

We note that there is no connection between thermodynamic instability and classical stability for small AdS black hole in higher-dimensional Einstein gravity. However, the GL instability condition (massiveness) picks up the small AdS black hole with  $r_+ < r_*^{(d)}$  which is thermodynamically unstable in fourth-order gravity. Hence, we conclude that there is a connection between the GL instability and thermodynamic instability for small AdS black hole in fourth-order gravity.



## 6 Discussions

First of all, we have studied the thermodynamics and phase transitions of AdS black holes using the ADT thermodynamic quantities in fourth-order gravity. For  $m_d^2 > m_c^2$ , all thermodynamic properties are dominantly determined by Einstein gravity, while for  $m_d^2 < m_c^2$ , all thermodynamic properties are dominated by Weyl-squared term (conformal gravity). The former is completely understood, but the latter has a new feature when one studies the black hole thermodynamics by using the ADT thermodynamic quantities. A further study is necessary to understand the latter completely.

We have confirmed a close connection between thermodynamic and classical instability for the BTZ black hole in new massive gravity. Also, there is a connection between the classical (GL) and thermodynamic instability for small AdS black holes in fourth-order gravity. This implies that the Gubser-Mitra conjecture (CSC) holds for the AdS black holes found from fourth-order gravity theory, which corresponds to our main result.

Finally, we wish to comment on the linearized equation (4.6) for the metric perturbation, which is obtained by splitting the linearized fourth-order equation. One confronts with ghost states with negative kinetic term when one uses the second-order equation (4.6) in fourth-order gravity. In order to avoid this problem, one may express the linearized equation (the linearized fourth-order equation for  $h_{\mu\nu}$ ) in terms of the linearized Einstein tensor as

$$\bar{\nabla}^2 \delta G_{\mu\nu} + 2\bar{R}_{\alpha\mu\beta\nu} \delta G^{\alpha\beta} - \mathcal{M}_d^2 \delta G_{\mu\nu} = 0 \tag{6.1}$$

which is surely a second-order differential equation. This equation describes 5 DOF of a massive spin-2 graviton propagating on the AdS black hole spacetimes when one imposes the tracelessness of  $\delta G^\mu{}_\mu = -\delta R = 0$  and the transversality of  $\bar{\nabla}^\mu \delta G_{\mu\nu} = 0$  from the contracted Bianchi identity. Actually, eq. (6.1) is a boosted-up version of eq. (4.6) which indicates the GL instability for small AdS black holes. However, the former is a ghost free equation, while the latter has the ghost problem in fourth-order gravity.

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## A Coefficients of the perturbation equation for $H_{tr}$

As was shown in section 4.2, the master equation for  $H_{tr}$  is given by

$$A(r; r_0, \ell, \Omega^2, \mathcal{M}_d^2) \frac{d^2}{dr^2} H_{tr} + B \frac{d}{dr} H_{tr} + C H_{tr} = 0, \tag{A.1}$$

where  $A$ ,  $B$ , and  $C$  are

$$\begin{aligned}
 A &= -\mathcal{M}_d^2 V - \Omega^2 + \frac{V'^2}{4} - \frac{VV''}{2} - \frac{(d-2)VV'}{2r}, \\
 B &= -2\mathcal{M}_d^2 V' - \frac{3V'V''}{2} - \frac{3\Omega^2 V'}{V} + \frac{3V'^3}{4V} + \frac{(d-2)\mathcal{M}_d^2 V}{r} + \frac{(d-2)\Omega^2}{r} + \frac{3(d-2)V'^2}{4r} \\
 &\quad + \frac{(d-2)VV''}{2r} - \frac{(d-2)^2 VV'}{2r^2}, \\
 C &= \mathcal{M}_d^4 + \frac{\Omega^4}{V^2} + \frac{2\mathcal{M}_d^2 \Omega^2}{V} - \frac{5\Omega^2 V'^2}{4V^2} + \frac{\mathcal{M}_d^2 V'^2}{4V} + \frac{V'^4}{4V^2} - \frac{\mathcal{M}_d^2 V''}{2} - \frac{\Omega^2 V''}{2V} - \frac{V'^2 V''}{4V} - \frac{V''^2}{2} \\
 &\quad - \frac{d\mathcal{M}_d^2 V'}{2r} - \frac{(d-2)\Omega^2 V'}{2rV} + \frac{(d-2)V'^3}{2rV} - \frac{3(d-2)V'V''}{2r} + \frac{(d-2)\Omega^2}{r^2} + \frac{(d-2)\mathcal{M}_d^2 V}{r^2} \\
 &\quad - \frac{(d-2)(2d-3)V'^2}{4r^2} + \frac{(d-2)VV''}{2r^2} + \frac{(d-2)^2 VV'}{2r^3}.
 \end{aligned}$$

One can easily check that for  $V = 1 - (r_0/r)^{d-3}$  or  $V = 1 - r_0/r + r^2/\ell^2$ , eq. (A.1) reduces to the master equation for  $H_{tr}$  given in the literature [11] or [45].

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## References

- [1] D.J. Gross, M.J. Perry and L.G. Yaffe, *Instability of Flat Space at Finite Temperature*, *Phys. Rev. D* **25** (1982) 330 [[INSPIRE](#)].
- [2] J.W. York, *Black hole thermodynamics and the Euclidean Einstein action*, *Phys. Rev. D* **33** (1986) 2092 [[INSPIRE](#)].
- [3] S.W. Hawking and D.N. Page, *Thermodynamics of Black Holes in anti-de Sitter Space*, *Commun. Math. Phys.* **87** (1983) 577 [[INSPIRE](#)].
- [4] J.D. Brown, J. Creighton and R.B. Mann, *Temperature, energy and heat capacity of asymptotically anti-de Sitter black holes*, *Phys. Rev. D* **50** (1994) 6394 [[gr-qc/9405007](#)] [[INSPIRE](#)].
- [5] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [[hep-th/9803131](#)] [[INSPIRE](#)].
- [6] R.L. Arnowitt, S. Deser and C.W. Misner, *The Dynamics of general relativity*, *Gen. Rel. Grav.* **40** (2008) 1997 [[gr-qc/0405109](#)] [[INSPIRE](#)].
- [7] C.V. Johnson, *D-Branes*, Cambridge University Press, Cambridge (2003).
- [8] L.F. Abbott and S. Deser, *Stability of Gravity with a Cosmological Constant*, *Nucl. Phys. B* **195** (1982) 76 [[INSPIRE](#)].
- [9] S. Deser and B. Tekin, *Energy in generic higher curvature gravity theories*, *Phys. Rev. D* **67** (2003) 084009 [[hep-th/0212292](#)] [[INSPIRE](#)].
- [10] W. Kim, S. Kulkarni and S.-H. Yi, *Quasilocal Conserved Charges in a Covariant Theory of Gravity*, *Phys. Rev. Lett.* **111** (2013) 081101 [[arXiv:1306.2138](#)] [[INSPIRE](#)].

- [11] R. Gregory and R. Laflamme, *Black strings and p-branes are unstable*, *Phys. Rev. Lett.* **70** (1993) 2837 [[hep-th/9301052](#)] [[INSPIRE](#)].
- [12] S.S. Gubser and I. Mitra, *Instability of charged black holes in Anti-de Sitter space*, [hep-th/0009126](#) [[INSPIRE](#)].
- [13] T. Harmark, V. Niarchos and N.A. Obers, *Instabilities of black strings and branes*, *Class. Quant. Grav.* **24** (2007) R1 [[hep-th/0701022](#)] [[INSPIRE](#)].
- [14] T. Regge and J.A. Wheeler, *Stability of a Schwarzschild singularity*, *Phys. Rev.* **108** (1957) 1063 [[INSPIRE](#)].
- [15] F.J. Zerilli, *Effective potential for even parity Regge-Wheeler gravitational perturbation equations*, *Phys. Rev. Lett.* **24** (1970) 737 [[INSPIRE](#)].
- [16] C.V. Vishveshwara, *Stability of the Schwarzschild metric*, *Phys. Rev. D* **1** (1970) 2870 [[INSPIRE](#)].
- [17] H. Kudoh, *Origin of black string instability*, *Phys. Rev. D* **73** (2006) 104034 [[hep-th/0602001](#)] [[INSPIRE](#)].
- [18] E. Babichev and A. Fabbri, *Instability of black holes in massive gravity*, *Class. Quant. Grav.* **30** (2013) 152001 [[arXiv:1304.5992](#)] [[INSPIRE](#)].
- [19] R. Brito, V. Cardoso and P. Pani, *Massive spin-2 fields on black hole spacetimes: Instability of the Schwarzschild and Kerr solutions and bounds on the graviton mass*, *Phys. Rev. D* **88** (2013) 023514 [[arXiv:1304.6725](#)] [[INSPIRE](#)].
- [20] Y.S. Myung, *Stability of Schwarzschild black holes in fourth-order gravity revisited*, *Phys. Rev. D* **88** (2013) 024039 [[arXiv:1306.3725](#)] [[INSPIRE](#)].
- [21] Y.S. Myung, *Unstable Schwarzschild-Tangherlini black holes in fourth-order gravity*, *Phys. Rev. D* **88** (2013) 084006 [[arXiv:1308.3907](#)] [[INSPIRE](#)].
- [22] Y.S. Myung, *Instability of Schwarzschild-AdS black hole in Einstein-Weyl gravity*, *Phys. Lett. B* **728** (2014) 422 [[arXiv:1308.1455](#)] [[INSPIRE](#)].
- [23] E.A. Bergshoeff, O. Hohm and P.K. Townsend, *Massive Gravity in Three Dimensions*, *Phys. Rev. Lett.* **102** (2009) 201301 [[arXiv:0901.1766](#)] [[INSPIRE](#)].
- [24] M. Bañados, C. Teitelboim and J. Zanelli, *The Black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [[hep-th/9204099](#)] [[INSPIRE](#)].
- [25] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, *Phys. Rev. D* **48** (1993) 1506 [[gr-qc/9302012](#)] [[INSPIRE](#)].
- [26] G. Clement, *Warped AdS<sub>3</sub> black holes in new massive gravity*, *Class. Quant. Grav.* **26** (2009) 105015 [[arXiv:0902.4634](#)] [[INSPIRE](#)].
- [27] W. Kim, S. Kulkarni and S.-H. Yi, *Conserved quantities and Virasoro algebra in new massive gravity*, *JHEP* **05** (2013) 041 [[arXiv:1303.3691](#)] [[INSPIRE](#)].
- [28] Y.S. Myung, *No Hawking-Page phase transition in three dimensions*, *Phys. Lett. B* **624** (2005) 297 [[hep-th/0506096](#)] [[INSPIRE](#)].
- [29] Y.S. Myung, *Phase transition between the BTZ black hole and AdS space*, *Phys. Lett. B* **638** (2006) 515 [[gr-qc/0603051](#)] [[INSPIRE](#)].
- [30] M. Eune, W. Kim and S.-H. Yi, *Hawking-Page phase transition in BTZ black hole revisited*, *JHEP* **03** (2013) 020 [[arXiv:1301.0395](#)] [[INSPIRE](#)].

- [31] A. Bagchi, S. Detournay, D. Grumiller and J. Simon, *Cosmic evolution from phase transition of 3-dimensional flat space*, *Phys. Rev. Lett.* **111** (2013) 181301 [[arXiv:1305.2919](#)] [[INSPIRE](#)].
- [32] Y.S. Myung, Y.-W. Kim, T. Moon and Y.-J. Park, *Classical stability of BTZ black hole in new massive gravity*, *Phys. Rev. D* **84** (2011) 024044 [[arXiv:1105.4205](#)] [[INSPIRE](#)].
- [33] T. Moon and Y.S. Myung, *Stability of the massive graviton around a BTZ black hole in three dimensions*, *Gen. Rel. Grav.* **45** (2013) 2493 [[arXiv:1303.5893](#)] [[INSPIRE](#)].
- [34] T. Moon and Y.S. Myung, *Gregory-Laflamme instability of BTZ black hole in new massive gravity*, *Phys. Rev. D* **88** (2013) 124014 [[arXiv:1310.3024](#)] [[INSPIRE](#)].
- [35] A. Ghodsi and D.M. Yekta, *Stability of vacua in New Massive Gravity in different gauges*, *JHEP* **08** (2013) 095 [[arXiv:1212.6876](#)] [[INSPIRE](#)].
- [36] Y.S. Myung, *Phase transitions for the Lifshitz black holes*, *Eur. Phys. J. C* **72** (2012) 2116 [[arXiv:1203.1367](#)] [[INSPIRE](#)].
- [37] H. Liu, H. Lü and M. Luo, *On Black Hole Stability in Critical Gravities*, *Int. J. Mod. Phys. D* **21** (2012) 1250020 [[arXiv:1104.2623](#)] [[INSPIRE](#)].
- [38] S. Deser, H. Liu, H. Lü, C.N. Pope, T.C. Sisman et al., *Critical Points of D-Dimensional Extended Gravities*, *Phys. Rev. D* **83** (2011) 061502 [[arXiv:1101.4009](#)] [[INSPIRE](#)].
- [39] Y.S. Myung, *Phase transitions for the topological de Sitter spaces and Schwarzschild-de Sitter black hole*, *Phys. Lett. B* **645** (2007) 369 [[hep-th/0603200](#)] [[INSPIRE](#)].
- [40] H. Lü and C.N. Pope, *Critical Gravity in Four Dimensions*, *Phys. Rev. Lett.* **106** (2011) 181302 [[arXiv:1101.1971](#)] [[INSPIRE](#)].
- [41] H. Lü, Y. Pang, C.N. Pope and J.F. Vazquez-Poritz, *AdS and Lifshitz Black Holes in Conformal and Einstein-Weyl Gravities*, *Phys. Rev. D* **86** (2012) 044011 [[arXiv:1204.1062](#)] [[INSPIRE](#)].
- [42] V. Cardoso and J.P.S. Lemos, *Quasinormal modes of Schwarzschild anti-de Sitter black holes: Electromagnetic and gravitational perturbations*, *Phys. Rev. D* **64** (2001) 084017 [[gr-qc/0105103](#)] [[INSPIRE](#)].
- [43] A. Ishibashi and H. Kodama, *Stability of higher dimensional Schwarzschild black holes*, *Prog. Theor. Phys.* **110** (2003) 901 [[hep-th/0305185](#)] [[INSPIRE](#)].
- [44] T. Moon, Y.S. Myung and E.J. Son, *Stability analysis of f(R)-AdS black holes*, *Eur. Phys. J. C* **71** (2011) 1777 [[arXiv:1104.1908](#)] [[INSPIRE](#)].
- [45] T. Hirayama and G. Kang, *Stable black strings in anti-de Sitter space*, *Phys. Rev. D* **64** (2001) 064010 [[hep-th/0104213](#)] [[INSPIRE](#)].