

Chiral four-dimensional F-theory compactifications with SU(5) and multiple U(1)-factors

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ABSTRACT: We develop geometric techniques to determine the spectrum and the chiral indices of matter multiplets for four-dimensional F-theory compactifications on elliptic Calabi-Yau fourfolds with rank two Mordell-Weil group. The general elliptic fiber is the Calabi-Yau onefold in dP_2 . We classify its resolved elliptic fibrations over a general base B . The study of singularities of these fibrations leads to explicit matter representations, that we determine both for $U(1) \times U(1)$ and $SU(5) \times U(1) \times U(1)$ constructions. We determine for the first time certain matter curves and surfaces using techniques involving prime ideals. The vertical cohomology ring of these fourfolds is calculated for both cases and general formulas for the Euler numbers are derived. Explicit calculations are presented for a specific base $B = \mathbb{P}^3$. We determine the general G_4 -flux that belongs to $H_V^{(2,2)}$ of the resolved Calabi-Yau fourfolds. As a by-product, we derive for the first time all conditions on G_4 -flux in general F-theory compactifications with a non-holomorphic zero section. These conditions have to be formulated after a circle reduction in terms of Chern-Simons terms on the 3D Coulomb branch and invoke M-theory/F-theory duality. New Chern-Simons terms are generated by Kaluza-Klein states of the circle compactification. We explicitly perform the relevant field theory computations, that yield non-vanishing results precisely for fourfolds with a non-holomorphic zero section. Taking into account the new Chern-Simons terms, all 4D matter chiralities are determined via 3D M-theory/F-theory duality. We independently check these chiralities using the subset of matter surfaces we determined. The presented techniques are general and do not rely on toric data.

KEYWORDS: Flux compactifications, F-Theory

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1 Introduction and summary of results

Four-dimensional F-theory compactifications provide a broad domain of the string theory landscape with the potential to derive promising particle physics consequences of string theory. It has a number of advantages, such as encoding geometric structures of string theory compactifications at a finite string couplings. The main focus in recent years has been on studies of F-theory GUT models with $SU(5)$ as well as $SO(10)$ gauge groups, initiated in [1–4] within local model building. Techniques for constructions of global models were developed in [5–9], and efforts to embed local models into global ones have been pursued, see e.g. [10–12] for reviews.

The origin of non-Abelian gauge symmetries in four-dimensional F-theory compactifications is well understood since the origins of F-theory [13–15], and it is due to the full classification of codimension one singularities of elliptically fibered Calabi-Yau fourfolds with a section in the Weierstrass or Tate model [16–18]. Matter multiplets appear at codimension two singularities and were studied originally in [13, 14, 19] and more recently in [9–12].¹ Constructions with chiral matter in four-dimensions require the addition of

¹Recent efforts clarified and filled the gaps in classifications singularities at higher co-dimensions [20–22]. For most recent complementary advances, employing deformations of singularities, see [23].

G_4 -flux, which belongs to the middle (vertical) cohomology on the resolved Calabi-Yau fourfold. Specific fluxes of this type were constructed in [21, 24].

On the other hand Abelian gauge symmetries in F-theory are less understood and studied. This is primarily due to the fact that the classification of Abelian gauge symmetries and their matter spectrum depends crucially on the global geometry of the elliptic fibration. Nevertheless, some aspects of Abelian gauge symmetries have been studied in local F-theory models employing spectral cover methods [6, 25–31]. The study of higher rank Abelian sectors in four-dimensional F-theory compactifications is also of phenomenological interest, since it can play an important rôle in model building beyond the Standard Model model.

Abelian gauge theory sectors appear in compactifications of F-theory on elliptic Calabi-Yau varieties with a general fiber being an elliptic curve with rational points. Elliptic curves with a so-called non-trivial Mordell-Weil group of rational points are a classical subject in mathematics [32–36]. These rational points lift to rational sections of the elliptically fibered Calabi-Yau manifold, which contribute new harmonic two-forms to the cohomology that support Abelian gauge fields in the F-theory effective action. The number of Abelian gauge fields is set by the rank of the Mordell-Weil group of the elliptic curve and its torsion subgroup gives rise to non-simply connected groups [15, 37, 38]. Rank one Mordell-Weil groups in compact elliptic fibrations have been studied recently in the F-theory literature in a variety of contexts [39–44, 44–47]. Elliptic fibrations with elliptic fiber of D_5 -type have been constructed in the context of counting BPS states in [48]. The resolutions of their most general fibrations, which have a Mordell-Weil group of rank three, as well as the complete induced F-theory matter spectrum have been analyzed in [49].

While a classification of possible Abelian gauge sectors in F-theory, analogous to the well-studied non-Abelian sector, is lacking (see, however [50] for a systematic study of rational sections on toric K3-surfaces), significant progress has been made recently for the systematic study of $U(1) \times U(1)$ gauge symmetry on elliptically fibered Calabi-Yau manifolds with a rank two Mordell-Weil group [46, 51].² The analysis found that the natural presentation of an elliptic curve with two rational points and a zero point is the generic Calabi-Yau one-fold in dP_2 and the birational map to its Tate and Weierstrass form was derived [51]. While the discussion of its resolved elliptic fibrations was done for a Calabi-Yau variety over a general base B , their classification was first performed for the base $B = \mathbb{P}^2$ and later for any two-dimensional base B in [54]. One key finding of this classification, which we further elaborate on also in this work, was the identification of all topological degrees of freedom in the construction of a fibrations of a fixed elliptic curve

²The elliptic curves in $\mathbb{P}^2(1, 1, 2)$ and \mathbb{P}^2 have also appeared in [52] and independently in [53], cf. also [48], as F-theory duals of heterotic backgrounds with $U(1)$ -Wilson lines. In the former, elliptic threefolds with these elliptic fibers over \mathbb{F}_n are constructed and the non-Abelian gauge groups are matched between the dual theories in dependence on n as in [15]. In the latter, the Kaehler classes of these elliptic threefolds, which are dual to the Wilson lines breaking $E_8 \rightarrow E_{8-k} \times U(1)^k$, $k = 0, 1, 2$, in the heterotic string, are identified by matching certain Gromov-Witten invariants with BPS-states of non-critical strings from E_{8-k} small instantons computed on the heterotic side. However, the detailed analysis and classification of the Abelian sectors on the F-theory side, by studying the singularities of these elliptic fibrations, has first been carried out in [51] and in this work using the resolved dP_2 -elliptic fibrations.

over a fixed base B . These are encoded in the choice of two divisor classes $\mathcal{S}_7, \mathcal{S}_9$ [51]. Allowing for all their possible values yields elliptic fibrations with novel properties, most notably with a non-holomorphic zero section, that were not constructed before because the existence of a holomorphic zero section was enforced. Next, a thorough analysis of the generic codimension two singularities of this most general family of elliptic Calabi-Yau threefolds was given. This determines geometrically all the matter representations under $U(1) \times U(1)$ and their multiplicities, that were shown to be consistent with anomaly cancellations in the six-dimensional compactified theory. Explicit toric examples were constructed, both with $U(1) \times U(1)$ and $SU(5) \times U(1) \times U(1)$ gauge symmetries.

It is the purpose of this paper to develop explicit techniques for the calculation of the matter spectrum and their chiralities for four-dimensional F-theory compactifications with two $U(1)$ -factors. This involves now F-theory compactifications on elliptically fibered Calabi-Yau fourfolds with rank two Mordell-Weil groups. Furthermore the appearance of the chiral matter requires an explicit construction of G_4 -flux. We would like to emphasize that our calculations are performed for general globally defined elliptic Calabi-Yau fourfolds and are *not* restricted to geometries described by toric reflexive polytopes. In particular, we find closed formulas for the basis of the vertical cohomology groups, the consistent G_4 -flux and the chiral indices for entire discrete families of Calabi-Yau fourfolds which depend explicitly on the aforementioned degrees of freedom $\mathcal{S}_7, \mathcal{S}_9$ of these families. We underline that even if each member of this family could be realized torically, it would be hard to obtain this dependence on $\mathcal{S}_7, \mathcal{S}_9$ by considering reflexive polytopes. In recent works [45, 47, 55] a classification of toric Calabi-Yau fourfolds with $SU(5)$ and allowed $U(1)$ factors was given (see also [46]). Note however that techniques employed have, at least to the knowledge of the authors, not culminated in the determination of the matter representations and their 4D chiralities yet which are the main results of this paper.

We advance the program in several important ways:

- The general elliptic fiber is the generic Calabi-Yau one-fold in dP_2 with two rational points and a zero point (as analyzed in detail in [51]). The chosen fiber determines the representations of the matter multiplets under $U(1) \times U(1)$ over any base B . The spectrum for $U(1) \times U(1)$ and $SU(5) \times U(1) \times U(1)$ (for a specific $SU(5)$) is derived and summarized in table 1. However, the analysis is performed now over a general three-dimensional base and the matter appears over codimension two Riemann surfaces in the base, denoted as matter curves. We employ explicit algebraic geometry techniques using prime ideals³ to represent these matter curves and classify the singularities of the elliptic fibration over them using the Calabi-Yau fourfold with the resolved elliptic fibration (see sections 2, 3.1 and 6.1). At present these techniques allow us, however, to determine only a subset of matter surfaces (three out of six in the $U(1) \times U(1)$ and six out of twelve in the $SU(5) \times U(1) \times U(1)$ case).
- As a preparation for the construction of G_4 -flux on these Calabi-Yau fourfolds, we find for the first time consistent conditions on the G_4 -flux in F-theory compactifications

³Note that in the case of Calabi-Yau threefolds these geometric techniques were sufficient to determine multiplicities of all the matter multiplets. [51].

with a non-holomorphic zero section. These conditions have to be formulated in three dimensions after the compactification on a circle and require the use of M-theory/F-theory duality. We show the connections between holomorphicity of the zero section and Kaluza Klein-states via Chern-Simons (CS) terms on the 3D Coulomb branch. In particular, there are new CS-terms for a non-holomorphic zero section.

- We develop techniques to calculate explicitly the most general G_4 -flux that belongs to the vertical cohomology $H_V^{(2,2)}$ resolved Calabi-Yau fourfolds (see section 3.2 and 4). To this end we algebraically calculate the full vertical cohomology ring of these Calabi-Yau fourfolds. These cohomology calculations allow us to compute the general expression for the Euler number and the Chern classes of these fourfolds for an arbitrary base B , both in the $U(1) \times U(1)$ and $SU(5) \times U(1) \times U(1)$ cases. As an application of these techniques we derive an explicit basis of the cohomology group for all elliptically fibered Calabi-Yau fourfolds with fiber in dP_2 and base $B = \mathbb{P}^3$.⁴ Again these techniques are general and not restricted to toric examples. In particular the dependence on the divisors $\mathcal{S}_7, \mathcal{S}_9$ is manifest. When the G_4 -flux is integrated over matter surfaces (determined via the geometric techniques mentioned above) we obtain the chiralities of three matter representations (second set in table 1). Chiralities of the remaining matter representations are determined by a subset of the 3D CS-terms of the dual M-theory invoking M-/F-theory duality. 4D anomalies are found to be cancelled. We note that the rest of the 3D CS-terms, in particular those for the Kaluza-Klein vector, provide an independent check for chiralities of matter multiplets obtained via geometric techniques. It is important to note that, given the list of representations that are realized, all CS-terms taken together are sufficient to determine chiralities of all the matter multiplets. Our geometric techniques allow us to have an independent determination for a subset of them (see section 5). We also perform an independent check that with the obtained spectrum the four-dimensional anomalies are cancelled. Explicit results are presented for the most general G_4 -flux for all generic resolved elliptic Calabi-Yau fourfolds over the base $B = \mathbb{P}^3$, both for $U(1) \times U(1)$ and $SU(5) \times U(1) \times U(1)$ (for a specific embedding of $SU(5)$).

We note that the fibrations over the chosen base $B = \mathbb{P}^3$ are generally non-flat at a single codimension three locus. This can be circumvented in two ways. Either, one can forbid the existence of the non-flat fiber geometrically, or restrict the allowed G_4 -flux by requiring a vanishing integral of it over the non-flat fiber. Both approaches yield an anomaly-free 4D spectrum with no chiral excess of additional light states.

The paper is organized in the following way: in section 2 we summarize the geometry of the general elliptic curve with rank two Mordell Weil group in dP_2 and classify its fibrations. Section 3.1 is devoted to the analysis of the codimension two and three singularities of the fibrations specifying, respectively, the matter content, the matter curves and surfaces as well as the Yukawa couplings. In section 3.2 we determine the cohomology ring and Chern classes for resolved elliptic Calabi-Yau fourfolds with rank two Mordell-Weil group.

⁴These techniques have been used in the context of mirror symmetry on Calabi-Yau fourfolds in [56–58].

$U(1) \times U(1)$	$SU(5) \times U(1) \times U(1)$
$(1, 0) (0, 1) (1, 1)$	$(\mathbf{5}, -\frac{2}{5}, 0) (\mathbf{5}, \frac{3}{5}, 0) (\mathbf{5}, -\frac{2}{5}, -1)$
$(-1, 1) (0, 2) (-1, -2)$	$(\mathbf{5}, -\frac{2}{5}, 1) (\mathbf{5}, \frac{3}{5}, 1) (\overline{\mathbf{10}}, -\frac{1}{5}, 0)$

Table 1. Matter representation for F-theory compactifications with rank- two Mordell Weil group. While $U(1) \times U(1)$ charges for $SU(5)$ singlets are general, $U(1) \times U(1)$ charges for the additional non-singlet matter representations of $SU(5)$ depend on a specific realization of the $SU(5)$ gauge symmetry.

Section 4 is devoted to a detailed discussion of G_4 -flux and conditions imposed on them in general F-theory compactifications with a non-holomorphic zero section. This study is based on relations between 3D Chern-Simons terms under F-theory/M-theory duality. A special emphasis is on quantum loop corrections due to Kaluza-Klein states, that are computed explicitly. In section 5 the general G_4 -flux is explicitly calculated (section 5.1), and matter chiralities both geometrically and via 3D Chern-Simons terms of dual the M-theory are derived (section 5.2). In section 5.3 anomaly cancellation of the four- dimensional field theory is checked and in section 5.4 an explicit toric example is presented. A generalization to F-theory compactifications with an additional $SU(5)$ non-Abelian gauge symmetry is spelled out in sections 6 and 7. In section 6.1 the singularities at codimension one, two (matter representations) and three (Yukawa points) are presented. A non-flat fiber at codimension three is discovered. The cohomology ring and the general Euler number are calculated in 6.2. In section 7 the G_4 -flux and the 4D chiralities are computed and anomaly cancellation is checked in detail for all elliptic fibrations with dP_2 -fiber over \mathbb{P}^3 . Section 7.1 demonstrates the general construction of G_4 -flux in F-theory compactifications with rank two Mordell-Weil group and a resolved $SU(5)$ -singularity with $B = \mathbb{P}^3$. In section 7.2 the 4D chiralities are evaluated and anomaly cancellation is checked. We finish the section with one concrete toric example in section 7.3. Conclusions and future directions can be found in section 8.

This work has seven appendices. Appendix A contains formulas for all Chern classes of dP_2 -elliptic fibrations over an arbitrary base B along with the Chern classes and Euler numbers for their resolved Calabi-Yau two-, three- and fourfolds. In appendix B we present the cohomology ring of the generic fourfolds with $B = \mathbb{P}^3$ and dP_2 -elliptic fiber. Appendix C contains the Chern classes of dP_2 -fibrations with resolved $SU(5)$ -singularities, in appendices D and E the intersections and vertical cohomology ring of fourfolds with $B = \mathbb{P}^3$ are computed. Appendix F contains all 3D CS-terms and chiralities. Appendix G concludes by describing how to systematically construct the toric polytopes for the example of Calabi-Yau fourfolds with base $B = \mathbb{P}^3$.

2 The elliptic curve in dP_2 and its fibrations

In this section we review the construction of the elliptic curve \mathcal{E} in dP_2 and its Calabi-Yau elliptic fibrations over a general B . These Calabi-Yau manifolds have a rank two Mordell-Weil group, that gives rise to $U(1) \times U(1)$ gauge symmetry in F-theory.

In section 2.1 we review the geometry of \mathcal{E} as the generic Calabi-Yau onefold in the del Pezzo surface dP_2 following the conventions and notations of [51], to which we also refer for more details. The reader familiar with the geometry of this elliptic curve \mathcal{E} can safely skip the first part of the section. Then, in section 2.2 we construct resolved elliptically fibered Calabi-Yau manifolds $\hat{\pi} : \hat{X} \rightarrow X$ over an arbitrary base B with this elliptic curve \mathcal{E} as the general fiber. The singular Calabi-Yau manifold is denoted by X . We show that these Calabi-Yau manifolds \hat{X} are classified by the choice of two divisors $\mathcal{S}_7, \mathcal{S}_9$ in the base B . In particular, we work out all the line bundles that are relevant to formulate the Calabi-Yau constraint of \hat{X} , which is the analog of the Tate model for elliptic fibrations with dP_2 -elliptic fiber.

The content of section 2.2 is a direct extension of the discussion in [51], where the possibility of a full classification of all Calabi-Yau elliptic fibrations with general fiber \mathcal{E} was pointed out, but demonstrated explicitly only for $B = \mathbb{P}^2$.

2.1 The elliptic curve with rank two Mordell-Weil group

The hypersurface description of the elliptic curve \mathcal{E} with a zero point P and two rational points Q and R has been derived in [51] from the existence of a degree three ample line bundle $M = \mathcal{O}(P + Q + R)$ on \mathcal{E} . The result of this analysis is that such an elliptic curve is naturally represented as the generic Calabi-Yau hypersurface in the del Pezzo dP_2 . The Calabi-Yau constraint in dP_2 takes the form

$$p = u(s_1 u^2 e_1^2 e_2^2 + s_2 u v e_1 e_2^2 + s_3 v^2 e_2^2 + s_5 u w e_1^2 e_2 + s_6 v w e_1 e_2 + s_8 w^2 e_1^2) + s_7 v^2 w e_2 + s_9 v w^2 e_1, \tag{2.1}$$

where we introduced the homogeneous coordinates $[u : v : w : e_1 : e_2]$ on dP_2 .⁵ One readily checks that p is the most general section of the anti-canonical bundle $K_{dP_2}^{-1} = \mathcal{O}(3H - E_1 - E_2)$, by noting the following divisor classes of the homogeneous coordinates,

	divisor class	\mathbb{C}^* -actions
u	$D_u = H - E_1 - E_2$	1 1 1
v	$D_v = H - E_2$	1 0 1
w	$D_w = H - E_1$	1 1 0
e_1	E_1	0 -1 0
e_2	E_2	0 0 -1

(2.2)

Here we also introduced the divisors D_u, D_v and D_w obtained by setting u, v, w to zero, respectively. The group of divisors is generated by H, E_1 and E_2 whose geometric

⁵We deviate here from the conventions of [51] by denoting coordinates on dP_2 by $[u : v : w : e_1 : e_2]$ and those on \mathbb{P}^2 by $[u' : v' : w']$.

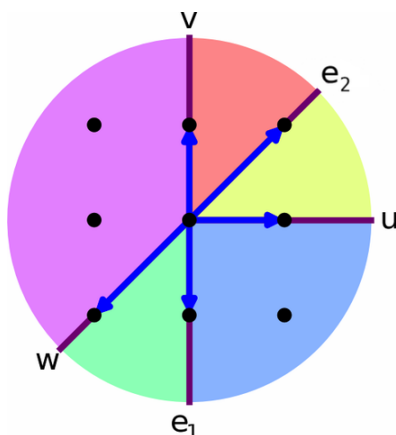


Figure 1. Fan of dP_2 . The coordinates corresponding to its rays are indicated.

interpretation we give momentarily. We note that dP_2 is a toric variety with the $(\mathbb{C}^*)^3$ -action on the homogeneous coordinates⁶ specified by the last three columns of (2.2). Its reflexive two-dimensional polytope is given in figure 1.

It is useful for our purposes to recall that a general del Pezzo surface dP_n is obtained from \mathbb{P}^2 by a blow-up at n generic points. Thus, the del Pezzo surface dP_2 is the blow-up of \mathbb{P}^2 at two generic points. In terms of the homogeneous coordinates $[u' : v' : w']$ on \mathbb{P}^2 , this blow-up is performed in our case at $u' = w' = 0$ and $u' = v' = 0$, so that the blow-down map takes the form

$$u' = ue_1e_2, \quad v' = ve_2, \quad w' = we_1, \quad (2.3)$$

with the two sections e_i associated to the exceptional divisors E_i . We note that one can use this map to represent the elliptic curve \mathcal{E} in (2.1) as a *non-generic* Calabi-Yau onefold in \mathbb{P}^2 . As discussed in [51] this presentation of \mathcal{E} suffices to obtain its Weierstrass model and discriminant. However, for the understanding of elliptic fibrations the completely resolved curve \mathcal{E} in dP_2 with is inevitable.

The divisors classes in (2.2) are the classes of the e_i , i.e. the exceptional divisors E_i of the blow-up (2.3), and the divisor class H , that is the pullback of the hyperplane class on \mathbb{P}^2 . We note the intersections on dP_2 are given as

$$H^2 = 1, \quad H \cdot E_i = 0, \quad E_i \cdot E_j = -\delta_{ij}. \quad (2.4)$$

These intersections immediately follow on the one hand from the rule that two divisors corresponding to rays of the same two-dimensional cone have intersection number one and on the other hand from the exceptional set, the Stanley-Reissner ideal SR . The latter encodes all rays which do not share a two dimensional cone and have intersections number zero. From figure 1 one readily obtains

$$SR = \{uv, uw, e_1e_2, e_1v, e_2w\}. \quad (2.5)$$

⁶Denoting one column vector in the last three columns of (2.2) by ℓ and the homogeneous coordinates collectively as x_i , then the corresponding \mathbb{C}^* -action is defined as $x_i \mapsto \lambda^{\ell_i} x_i$ with $\lambda \in \mathbb{C}^*$.

The rational points Q and R as well as the zero point P on \mathcal{E} are simply given by the intersections of the three independent divisors in (2.2) with the elliptic curve (2.1). We have chosen the three different points on \mathcal{E} that are obtained by intersecting the three divisors D_u , E_1 and E_2 on dP_2 with $p = 0$ in (2.1). Upon setting all coordinates to one that can not vanish simultaneously with the divisor under consideration due to the exceptional set (2.5) we obtain

$$\begin{aligned} P : E_2 \cap p &= [-s_9 : s_8 : 1 : 1 : 0], & Q : E_1 \cap p &= [-s_7 : 1 : s_3 : 0 : 1], \\ R : D_u \cap p &= [0 : 1 : 1 : -s_7 : s_9]. \end{aligned} \tag{2.6}$$

The point P is considered as the zero point on \mathcal{E} and the points Q , R are the rational points. They are the two generators of the rank two Mordell-Weil group of rational points, with the group law given by the addition of points on \mathcal{E} . We note that these three points are generically distinct. P is always distinct from Q , however, we observe $P = R$ for $s_9 = 0$ and $Q = R$ for $s_7 = 0$.

When considering elliptically fibered Calabi-Yau fourfolds \hat{X} with the curve (2.1) as the general elliptic fiber, the points P , Q and R lift to rational sections of the fibration. We note that the coefficients s_i in (2.1) are then base-dependent functions, that can vanish on B . In particular, we see from (2.6) that the points P , Q , and R , respectively, are ill-defined when $s_8 = s_9 = 0$, $s_3 = s_8 = 0$ and $s_7 = s_9 = 0$, respectively. This behavior is typical for rational sections and further discussed in section 3.1. In F-theory compactifications on such a fourfold \hat{X} each of the rational sections gives rise to an Abelian gauge symmetry. Thus, Calabi-Yau fourfolds with general elliptic fiber in dP_2 generically have a rank two Abelian gauge group, i.e. an $U(1) \times U(1)$ gauge symmetry.

2.2 General Calabi-Yau fibrations with dP_2 -elliptic fiber

In this section we discuss the construction of resolved elliptically fibered Calabi-Yau manifolds \hat{X} with general elliptic fiber in dP_2 . The following results hold for general complex dimension of \hat{X} , in particular for Calabi-Yau three- and fourfolds. We end this section with the concrete example of $B = \mathbb{P}^3$.

Classifying dP_2 -fibrations and their Calabi-Yau hypersurfaces \hat{X}

In general an elliptically fibered Calabi-Yau manifold $\mathcal{E} \rightarrow \hat{X} \xrightarrow{\pi} B$ with π denoting the projection to the base B is constructed by first considering the defining equation for the desired elliptic curve \mathcal{E} alone and then by lifting the coefficients in this equation to sections over the base B . In the case at hand, the elliptic curve is described by (2.1). Thus, all we have to do to obtain an elliptic fibration is to promote the coefficients s_i to sections of line bundles on the base B . Finally, the Calabi-Yau condition for (2.1) fixes the respective line bundles for the sections s_i .

The procedure of lifting the s_i to sections of B is described as follows. First, we have to define the ambient space in which the elliptically fibered manifold $\hat{X} \rightarrow B$ is embedded. Since the constraint (2.1) merely cuts the elliptic curve \mathcal{E} out of dP_2 , the ambient space is

simply a dP_2 -fibration over the base B of \hat{X} . It takes the form

$$\begin{array}{ccc}
 dP_2 & \longrightarrow & dP_2^B(\mathcal{S}_7, \mathcal{S}_9) \\
 & & \downarrow \\
 & & B
 \end{array} \tag{2.7}$$

which can be viewed as a generalization of a projective bundle. Here \mathcal{S}_7 and \mathcal{S}_9 are two divisors on B associated to the vanishing loci of the sections s_7 and s_9 in (2.1). The total space is denoted $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ since it is uniquely determined by these divisors \mathcal{S}_7 and \mathcal{S}_9 if we demand that the constraint (2.1) defines a Calabi-Yau manifold \hat{X} . In fact, we first note that any the dP_2 -fibration is specified by only two divisors on B . This can be seen by noting that in a general such fibrations the homogeneous coordinates $[u : v : w : e_1 : e_2]$ on dP_2 are sections of five different line bundles on the base B , respectively. However, we can always use the three \mathbb{C}^* -actions in (2.2) to eliminate three of these line bundles, so that only two of the five coordinates on dP_2 take values in non-trivial line bundles. We make the following assignment of line bundles on B to the coordinates,

$$u \in \mathcal{O}_B(\mathcal{S}_9 + [K_B]), \quad v \in \mathcal{O}_B(\mathcal{S}_9 - \mathcal{S}_7), \tag{2.8}$$

where K_B denotes the canonical bundle on B and $[K_B]$ the associated divisor. All other coordinates on dP_2 transform as the trivial bundle on B . We note that this parametrization of the two line bundles for u and v is completely general, because \mathcal{S}_7 and \mathcal{S}_9 are completely general divisors on B at the moment.

Next, we use these results to readily calculate the total Chern class of $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ from adjunction, see (A.2) in appendix A, from which we obtain its anti-canonical bundle

$$K_{dP_2^B}^{-1} = \mathcal{O}(3H - E_1 - E_2 + 2\mathcal{S}_9 - \mathcal{S}_7), \tag{2.9}$$

where we suppressed the dependence on $\mathcal{S}_7, \mathcal{S}_9$ for brevity of our notation. Then the Calabi-Yau condition implies that the constraint (2.1) has to be a section of $K_{dP_2^B}^{-1}$. This immediately fixes the line bundles of all the sections s_i on B . We summarize the sections defining the elliptically fibered Calabi-Yau manifold \hat{X} as follows

section	bundle	section	bundle
u	$\mathcal{O}(H - E_1 - E_2 + \mathcal{S}_9 + [K_B])$	s_1	$\mathcal{O}(3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$
v	$\mathcal{O}(H - E_2 + \mathcal{S}_9 - \mathcal{S}_7)$	s_2	$\mathcal{O}(2[K_B^{-1}] - \mathcal{S}_9)$
w	$\mathcal{O}(H - E_1)$	s_3	$\mathcal{O}([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)$
e_1	$\mathcal{O}(E_1)$	s_5	$\mathcal{O}(2[K_B^{-1}] - \mathcal{S}_7)$
e_2	$\mathcal{O}(E_2)$	s_6	$\mathcal{O}([K_B^{-1}])$
		s_7	$\mathcal{O}(\mathcal{S}_7)$
		s_8	$\mathcal{O}([K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7)$
		s_9	$\mathcal{O}(\mathcal{S}_9)$

(2.10)

In particular we see that with the parametrization (2.8) the divisors \mathcal{S}_7 and \mathcal{S}_9 are indeed associated to s_7 and s_9 as claimed at the beginning.

Basic geometry of Calabi-Yau manifolds with dP_2 -elliptic fiber

Having constructed the general elliptically fibered Calabi-Yau manifolds \hat{X} over B , we discuss next the group of divisors on \hat{X} . By construction, the basis of divisors on a generic⁷ \hat{X} is induced by a basis of divisors on the ambient space $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$, which consists of divisors of the base B and the fiber dP_2 . The divisors induced from a basis of divisors D_α^b of the base B are the vertical divisors $D_\alpha = \pi^*(D_\alpha^b)$ of the elliptic fibration $\pi : \hat{X} \rightarrow B$. Similarly, the classes H, E_1, E_2 of the fiber dP_2 in (2.2) become divisors on \hat{X} . Then, the points P, Q and R in (2.6) lift to, in general, *rational sections* of the fibration of $\pi : \hat{X} \rightarrow B$, denoted \hat{s}_P, \hat{s}_Q and \hat{s}_R , with \hat{s}_P the zero section. We denote the homology classes of the associated divisors by capital letters,

$$S_P = E_2, \quad S_Q = E_1, \quad S_R = H - E_1 - E_2 + \mathcal{S}_9 + [K_B]. \quad (2.11)$$

In general, a rational section is a non-holomorphic map of the base B into \hat{X} , such as $\hat{s}_P : B \rightarrow \hat{X}$ for example. A rational section $B \rightarrow \hat{X}$ is ill-defined over codimension two loci to the effect that it wraps entire fiber components over these loci. From a given rational section, one can easily obtain a *holomorphic section*, i.e. a holomorphic map $\hat{B} \rightarrow \hat{X}$, by a birational transformation, namely a blow-up $\hat{B} \rightarrow B$ at those codimension two loci of B . Usually the zero section \hat{s}_P has been assumed to be holomorphic in F-theory. Only lately, the possibility of a non-holomorphic zero section \hat{s}_P in F-theory has been studied [45, 51, 55]. The group of sections excluding the zero section \hat{s}_P is the *Mordell-Weil group* of rational sections on \hat{X} , which in the case at hand is rank two and generated by \hat{s}_Q, \hat{s}_R . For brevity of our notation, we will occasionally denote the generators of the Mordell-Weil group and their divisor classes collectively as

$$\hat{s}_m = (\hat{s}_Q, \hat{s}_R), \quad S_m = (S_Q, S_R). \quad (2.12)$$

There are some characteristic intersections involving the divisors S_P, S_Q and S_R in (2.11) that immediately follow from the defining properties of a section. We list them in the following and refer to [43, 44, 51, 59] for a more thorough discussion. We also give a simple criterion to distinguish between rational and holomorphic sections. A more detailed account on intersections in the presence of a rational zero section can be found in [55]. Here, we content ourselves with noting that \hat{s}_P is holomorphic if $\mathcal{S}_9 = 0$ or $\mathcal{S}_8 = 0$, \hat{s}_Q is holomorphic if $\mathcal{S}_3 = 0$ or $\mathcal{S}_7 = 0$ and \hat{s}_R is holomorphic if $\mathcal{S}_7 = 0$ or $\mathcal{S}_9 = 0$, cf. the paragraph following (2.6) and [51].

⁷By generic we mean the absence of Cartan divisors D_i from resolutions of codimension one singularities of the fibration of \hat{X} . We will briefly discuss the geometry of \hat{X} in the presence of D_i at the end of this section. We refer to section 6 for more details.

The following intersections and definitions will be crucial in the rest of this work:

Universal intersection:	$S_P \cdot F = S_m \cdot F = 1$ with general fiber $F \cong \mathcal{E}$, (2.13)
Rational sections:	$\pi(S_P^2 + [K_B^{-1}] \cdot S_P) = \pi(S_m^2 + [K_B^{-1}] \cdot S_m) = 0$, (2.14)
	$\mathcal{S}_7 = \pi(S_P \cdot S_R)$, $\mathcal{S}_9 = \pi(S_Q \cdot S_R)$, (2.15)
Holomorphic sections:	$S_P^2 + [K_B^{-1}] \cdot S_P = S_m^2 + [K_B^{-1}] \cdot S_m = 0$, (2.16)
Shioda maps:	$\sigma(\hat{s}_Q) = S_Q - S_P - [K_B^{-1}]$, (2.17)
	$\sigma(\hat{s}_R) = S_R - S_P - [K_B^{-1}] - \mathcal{S}_9$,
Height pairing:	$\pi(\sigma(\hat{s}_m) \cdot \sigma(\hat{s}_n)) = \begin{pmatrix} 2[K_B] & [K_B] - \mathcal{S}_7 + \mathcal{S}_9 \\ [K_B] + \mathcal{S}_7 - \mathcal{S}_9 & 2[K_B] - 2\mathcal{S}_9 \end{pmatrix}_{mn}$ (2.18)

Let us briefly comment on these intersections in the order of their appearance. The intersection (2.13) is an immediate consequence of the definition of a section: its divisor class intersects the general class of the fiber $F \cong \mathcal{E}$ at a point. The relation (2.14) can be shown by an adjunction argument, see section 3.2 for direct cohomology computations. Here we have defined the a projection onto the homology $H^4(B)$ of the base as

$$\pi(\mathcal{C}) = (\mathcal{C} \cdot \Sigma^\alpha) D_\alpha^b, \quad \Sigma_b^\alpha \cdot D_\beta^b = \delta_\beta^\alpha \tag{2.19}$$

for every complex surface \mathcal{C} in \hat{X} . The intersection pairings on \hat{X} , respectively, B are denoted \cdot and the $\Sigma^\alpha = \pi^*(\Sigma_b^\alpha)$ arise from a basis of curves Σ_b^α dual to the divisors D_α^b on B as indicated in the last equation in (2.19). We emphasize that in the case of a holomorphic section, the relations (2.14) hold in the full homology of \hat{X} as indicated in (2.16). The divisors $\mathcal{S}_7, \mathcal{S}_9$ are the codimension one loci where the sections collide in the fiber \mathcal{E} , as discussed below (2.6). They are encoded in the intersections (2.15). Next, we introduce the divisors $\sigma(\hat{s}_Q), \sigma(\hat{s}_R)$ in (2.17). The map σ is the Shioda map that takes here the form

$$\sigma(\hat{s}_m) := S_m - \tilde{S}_P - \pi(S_m \cdot \tilde{S}_P), \tag{2.20}$$

where we introduced the combination [60, 61]

$$\tilde{S}_P = S_P + \frac{1}{2}[K_B^{-1}]. \tag{2.21}$$

We refer to [33, 43, 44, 51, 59] for more details on the Shioda map and to section 6 for the inclusion of an $SU(5)$ -sector. We note that the divisors (2.17) support $U(1)$ -gauge fields in F-theory due to their vanishing intersections with vertical divisors D_α and the zero-section, as well as potential Cartan divisors D_i of non-Abelian groups. Finally, we have calculated the intersection matrix of the Shioda map of \hat{s}_Q, \hat{s}_R in (2.18).

We finish this section by some concluding definitions and remarks on the general structure of the fibrations (2.7) and \hat{X} . First, we summarize the basis of divisors on \hat{X} as

$$D_A = (\tilde{S}_P, D_\alpha, D_i, \sigma(\hat{s}_m)), \quad A = 0, 1, \dots, h^{(1,1)}(\hat{B}) + \text{rk}(G) + 3, \quad (2.22)$$

where we have collectively denoted the basis (2.17) as $\sigma(\hat{s}_m)$. We have also introduced one set of Cartan divisors D_i with $i = 1, \dots, \text{rk}(G)$ in order to prepare for the presence of a non-Abelian group G , as in section 6 with $G = \text{SU}(5)$. These divisors D_i are present for non-generic \hat{X} with a resolved singularity of type G of the elliptic fibration over codimension one in B . The D_i admit a fibration

$$\begin{array}{ccc} c_{-\alpha_i} & \longrightarrow & D_i \\ & & \downarrow \\ & & \mathcal{S}_G^b \end{array} \quad (2.23)$$

where the general fiber is a rational curve $c_{-\alpha_i} \cong \mathbb{P}^1$ that corresponds to the simple root $-\alpha_i$ of G . The divisor \mathcal{S}_G^b in B physically supports 7-branes that give rise to the non-Abelian gauge symmetry G in F-theory [14, 15, 18].

Next, we expand the canonical bundle K_B of the base B in terms of the vertical divisors D_α as

$$[K_B] = K^\alpha D_\alpha \quad (2.24)$$

with coefficients K^α . Similarly, we expand the divisors

$$\mathcal{S}_7 = n_7^\alpha D_\alpha^b, \quad \mathcal{S}_9 = n_9^\alpha D_\alpha^b, \quad (2.25)$$

with general positive integral coefficients $n_7^\alpha, n_9^\alpha, \alpha = 1, \dots, h^{(1,1)}(B)$. It is important to emphasize that the coefficients n_7^α, n_9^α are in general further bounded from above by the requirement that all sections s_i in (2.10) are generic, i.e. that the line bundle of s_i admits sufficiently many holomorphic sections. If this is not the case we expect additional singularities in \hat{X} , potentially corresponding to a minimal (non-Abelian) gauge symmetry in F-theory. For this reason, we will in the rest of this work assume that \hat{X} can be constructed with generic s_i .

Despite these restrictions on the integers n_7^α and n_9^α we would like to point out that the constructions of the fibration (2.7) and of \hat{X} hold in general for an arbitrary base B and arbitrary complex dimension. In particular this analysis applies to an arbitrary choice of divisors \mathcal{S}_7 and \mathcal{S}_9 within these bounds. In particular the general construction here reproduce immediately the classification in [51] with $B = \mathbb{P}^2$ as a special case.

dP_2 -fibrations over $B = \mathbb{P}^3$ with generic Calabi-Yau hypersurfaces \hat{X}

We conclude with the discussion of the special case $B = \mathbb{P}^3$, which will be considered in later sections of this work. In this case, there is only one divisor in the base, the hyperplane H_B , so that the dP_2 -fibration (2.7) is specified only by two integers $n_7 \equiv n_7^1, n_9 \equiv n_9^1$. In

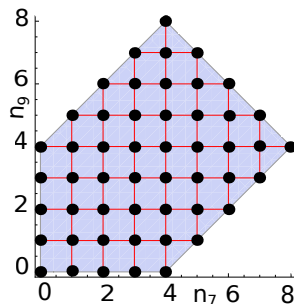


Figure 2. Each dot corresponds to a dP_2 -fibration over \mathbb{P}^3 with generic Calabi-Yau \hat{X} .

this case we use the notation

$$\begin{array}{ccc}
 dP_2 & \longrightarrow & dP_2(n_7, n_9) \\
 & & \downarrow \\
 & & \mathbb{P}^3
 \end{array} \tag{2.26}$$

where we suppress the base $B = \mathbb{P}^3$ when denoting the total space (2.7) of the fibration if the context is clear.

We note that $K_{\mathbb{P}^3}^{-1} = \mathcal{O}_{\mathbb{P}^3}(4)$. In this case all sections s_i exist iff all bundles in the second table in (2.10) have non-negative degree. This puts the following conditions on the integers n_7, n_9 ,

$$0 \leq n_7, n_9 \leq 8, \quad n_7 + n_9 \leq 12, \quad 0 \leq 4 + n_7 - n_9, \quad 0 \leq 4 + n_9 - n_7. \tag{2.27}$$

The domain of allowed values for n_7 and n_9 are displayed in figure 2. As we will see in section 5.4 we can torically construct the Calabi-Yau fourfolds \hat{X} for a wide range of values of n_7, n_9 . The general strategy to build the corresponding reflexive polytopes is outlined in appendix G. It is satisfying, that in the toric context the conditions (2.27) are enforced by reflexivity of the toric polytope, i.e. for values n_7, n_9 exceeding the bounds (2.27) the toric polytope is no longer reflexive.

3 Calabi-Yau fourfolds with rank two Mordell-Weil

In this section we analyze F-theory compactifications to four dimensions on a generic elliptically fibered Calabi-Yau fourfold \hat{X} over a base B with general fiber \mathcal{E} in dP_2 . These compactifications have a gauge theory with $U(1) \times U(1)$ gauge group and a number of chiral matter fields in representations $\mathbf{1}_{(q_1, q_2)}$. The possible $U(1) \times U(1)$ -charges (q_1, q_2) have been determined recently in [46, 51] and the full 6D anomaly-free spectrum including matter multiplicities has been derived for $B = \mathbb{P}^2$ in [51].

Here we extend this geometric analysis to fourfolds. The main difference to the 6D case is that matter is not localized anymore at points in B , but on in general rather complicated matter curves. The determination of these matter curves and some of their associated matter surfaces, along with the Yukawa points, is presented in section 3.1. Then, in section 3.2 we present a method to determine the cohomology ring of the fourfold \hat{X} . We

use these techniques to derive general expressions for the Euler number of \hat{X} and its second Chern class. For the example of $B = \mathbb{P}^3$ we finally compute the full vertical cohomology group. These calculations serve as a preparation for the computation of 4D chiralities in section 5, which requires both the knowledge of matter surfaces and the construction of G_4 -flux.

3.1 Singularities of the fibration: matter Surfaces & Yukawa points

We organize this section into a detailed discussion of codimension two singularities in section 3.1.1 and a very brief account on codimension three singularities in section 3.1.2.

3.1.1 Matter: codimension two

In general, the determination of the matter sector in F-theory vacua with general gauge group requires a detailed analysis of singularities of the elliptic fibration of the Calabi-Yau fourfold at codimension two in the base B , where the elliptic fiber \mathcal{E} becomes reducible. Then one has to identify the isolated rational curve $c_{\mathbf{w}}$ in the fiber over these loci, since these correspond in F-theory to matter in a representation \mathbf{R} from wrapped M2-brane states. These curves are in one-to-one correspondence to the weights \mathbf{w} of the representations \mathbf{R} and accordingly labeled. In the case of elliptically fibered Calabi-Yau fourfolds, the codimension two matter loci are Riemann surfaces of genus g , the so-called matter curves $\Sigma_{\mathbf{R}}$ in B conveniently labeled by the corresponding matter representation \mathbf{R} . In addition, for the determination of four-dimensional chirality, compare section 5, we have to know the homology classes of the associated matter surfaces

$$\begin{array}{ccc}
 c_{\mathbf{w}} & \longrightarrow & C_{\mathbf{R}}^{\mathbf{w}} \\
 & & \downarrow \\
 & & \Sigma_{\mathbf{R}}
 \end{array}
 \tag{3.1}$$

which are constructed as the fibration of the rational curve $c_{\mathbf{w}}$ corresponding to a given weight \mathbf{w} of the representation \mathbf{R} fibered over $\Sigma_{\mathbf{R}}$.

In this section we determine the matter curves $\Sigma_{\mathbf{R}}$ and the matter surfaces $C_{\mathbf{R}}^{\mathbf{w}}$ for the six representations occurring in the Calabi-Yau fourfold \hat{X} . As we demonstrate, their determination is complicated by the fact that three of the six the codimension two loci in the base B where the elliptic fiber \mathcal{E} becomes reducible are themselves reducible curves. Their irreducible components are multiple different matter curves $\Sigma_{\mathbf{R}}$. Some of these matter curves, denoted $\Sigma_{\mathbf{R}'}$, fail to be complete intersection and can only be described in terms of their prime ideals. These prime ideals are straightforwardly constructed from the two equations of the original reducible codimension two locus. However, the isolation of rational curves $c_{\mathbf{w}}$ over those matter curves $\Sigma_{\mathbf{R}'}$ is very involved. Thus, in these cases we can not determine the corresponding matter surfaces (3.1) explicitly. Fortunately, we can obtain the other three matter surfaces straightforwardly, and are still able to determine the full F-theory matter spectrum for the fourfold \hat{X} , as outlined in section 5. It would be desirable, however, to reproduce the results obtained there invoking M-/F-theory duality

by direct geometric computation based on a better understanding of the matter surfaces $\mathcal{C}_{\mathbf{R}'}$ in general.

In any case, we can qualitatively describe all the matter surfaces $\mathcal{C}_{\mathbf{R}}^{\mathbf{w}}$ by recalling the construction of the resolved fourfold \hat{X} . The smooth fourfold \hat{X} is formed by two consecutive blow-ups of a singular Weierstrass model X . We depict this schematically as

$$\begin{array}{ccc}
 \hat{X} \subset dP_2^B(\mathcal{S}_7, \mathcal{S}_9) & \xrightarrow{\hat{\pi}} & X \subset (\mathbb{P}^2(1, 2, 3) \rightarrow B) \\
 \text{generic CY} & & \text{non-generic WSF} \\
 \searrow \pi_2 & & \nearrow \pi_1 \\
 & \tilde{X} \subset (\mathbb{P}^2 \rightarrow B) & \\
 & \text{non-generic cubic} &
 \end{array} \tag{3.2}$$

where the full blow-down map $\hat{\pi} : \hat{X} \rightarrow X$ is consequently a composition $\hat{\pi} = \pi_1 \circ \pi_2$. On the left we have the smooth geometry with elliptic fiber constructed in section 2. It can be understood as a toric blow-up $\pi_2 : \hat{X} \rightarrow \tilde{X}$ from a non-generic cubic in \mathbb{P}^2 , with corresponding fourfold denoted by \tilde{X} . A final blow-down π_1 yields the singular Weierstrass form (WSF) X with $\mathbb{P}^2(1, 2, 3)$ -fiber. The birational map π_1 is derived in detail in [51], see its defining equations eqs. (3.18) and (3.20) therein.

Having the diagram (3.2) in mind, the three matter surfaces $\mathcal{C}_{\mathbf{R}}$ which have a simple description are those generated in the blow-up π_2 . There are three simple codimension two singularities in \tilde{X} , which are precisely the three simple matter curves $\Sigma_{\mathbf{R}}$. Their pull-backs under π_2 are precisely the matter surfaces $\mathcal{C}_{\mathbf{R}} = \pi_2^*(\Sigma_{\mathbf{R}})$. Because of the simplicity of both $\Sigma_{\mathbf{R}}$ and the blow-up π_2 , these surfaces have a description as a simple complete intersection in the ambient space $dP_2(\mathcal{S}_7, \mathcal{S}_9)$. In contrast, the other three matter curves $\Sigma_{\mathbf{R}'}$ are the loci of codimension two singularities in the WSF X , which are resolved by the map π_1 . However, these curves $\Sigma_{\mathbf{R}'}$ have a description only in terms of prime ideals and the map π_1 is not a simple toric blow-up but a fully-fledged birational map [51]. These two complications make an explicit determination of the surfaces $\mathcal{C}_{\Sigma_{\mathbf{R}'}}^{\mathbf{w}}$ hard. Nevertheless, the matter surfaces are again abstractly given by $\mathcal{C}_{\mathbf{R}'} = \pi_1^*(\Sigma_{\mathbf{R}'})$, which are ruled surfaces over $\Sigma_{\mathbf{R}'}$. Thus, the determination of the exceptional loci of the map π_1 might be a first step towards an understanding of these matter surfaces.

Summary of matter representations & their matter curves

Before going into technical calculations of matter curves and surfaces, let us briefly summarize the matter content as it has been determined in [46, 51].

There are six different matter representations $\mathbf{R} = \mathbf{1}_{(q_1, q_2)}$ in the F-theory compactification on the fourfold \hat{X} . The list of realized $U(1) \times U(1)$ -charges, together with the

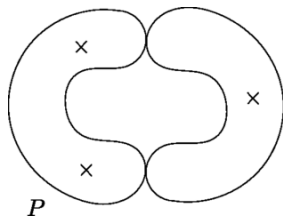


Figure 3. I_2 -fiber from resolving a codimension two singularity of the fibration of \hat{X} .

cohomology class of the corresponding matter curves $\Sigma_{\mathbf{R}}$ determined below, reads

Matter	Homology class of $\Sigma_{\mathbf{R}}$ in B	
$\mathbf{1}_{(1,0)}$	$6[K_B^{-1}]^2 + 4[K_B^{-1}] \cdot \mathcal{S}_7 - 5[K_B^{-1}] \cdot \mathcal{S}_9 + \mathcal{S}_9^2 + \mathcal{S}_7 \cdot \mathcal{S}_9 - 2\mathcal{S}_7^2$	
$\mathbf{1}_{(0,1)}$	$6[K_B^{-1}]^2 + 4[K_B^{-1}] \cdot (\mathcal{S}_7 + \mathcal{S}_9) - 2\mathcal{S}_7^2 - 2\mathcal{S}_9^2$	
$\mathbf{1}_{(1,1)}$	$6[K_B^{-1}]^2 + 4[K_B^{-1}] \cdot \mathcal{S}_9 - 5[K_B^{-1}] \cdot \mathcal{S}_7 + \mathcal{S}_7^2 + \mathcal{S}_7 \cdot \mathcal{S}_9 - 2\mathcal{S}_9^2$	(3.3)
$\mathbf{1}_{(-1,1)}$	$([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9) \cdot \mathcal{S}_7$	
$\mathbf{1}_{(0,2)}$	$\mathcal{S}_7 \cdot \mathcal{S}_9$	
$\mathbf{1}_{(-1,-2)}$	$\mathcal{S}_9 \cdot ([K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7)$	

Here we used as before the notation $[K_B^{-1}]$ for the anti-canonical divisor of the base and denoted the intersection on B as ‘ \cdot ’. These representations of matter fields are model-independent and in particular do not depend on the choice of base B . The last three matter representations arise from rational curves created in the blow-up π_2^{-1} in (3.2). Their matter curves are simply described by $s_3 = s_7 = 0$, $s_7 = s_9 = 0$ and $s_8 = s_9 = 0$ in the order of their appearance in (3.3). The first three representations arise from rational curves from the blow-up π_1^{-1} in (3.2). The determination of their matter curves is more involved and presented below.

All the matter representations in (3.2) arise from M2-branes on rational curves $c_{\mathbf{w}}$ with weight $\mathbf{w} = (q_1, q_2)$. These charges are calculated by the intersection of the curve $c_{\mathbf{w}}$ with the Shioda maps $\sigma(\hat{s}_Q)$, $\sigma(\hat{s}_R)$ defined in (2.17) as

$$q_m \equiv \sigma(\hat{s}_m) \cdot c_{\mathbf{w}} = (S_m \cdot c_{\mathbf{w}}) - (S_P \cdot c_{\mathbf{w}}), \tag{3.4}$$

All curves $c_{\mathbf{w}}$ are part of an I_2 -fiber. Along the matter surfaces in (3.3) the general elliptic fiber \mathcal{E} splits into two rational curves $c_1, c_2 \cong \mathbb{P}^1$ intersecting in two points with one curve, say c_1 , the original singular fiber and the other curve $c_2 \equiv c_{\mathbf{w}}$. We write this as

$$I_2\text{-fiber} : \quad \mathcal{E} = c_1 + c_2, \quad c_1 \cdot c_2 = 2. \tag{3.5}$$

A cartoon of such a reducible fiber together with possible locations of the points P, Q and R is depicted in figure 3. In terms of the Calabi-Yau constraint (2.1) the split of \mathcal{E} into an I_2 -fiber is visible as a factorization at a point $pt \in \Sigma_{\mathbf{R}}$ as

$$p|_{pt} = p_1 \cdot p_2. \tag{3.6}$$

Here the two rational curves in (3.5) are described by one of these two factors, for example $c_1 = \{p_1 = 0, pt \in \Sigma_{\mathbf{R}}\}$ and $c_2 = \{p_2 = 0, pt \in \Sigma_{\mathbf{R}}\}$.

Matter surfaces $\mathcal{C}_{(-1,1)}$, $\mathcal{C}_{(0,2)}$, $\mathcal{C}_{(-1,-2)}$ and their homology classes

Next, we determine the matter surfaces $\mathcal{C}_{\mathbf{R}}^{\mathbf{w}}$ for the last three representations in (3.3). As we will see, this is straightforward since the matter curves $\Sigma_{\mathbf{R}}$ in these cases are irreducible varieties of the simple form $s_i = s_j = 0$ for appropriate i, j . This implies that the factorization of the elliptic fiber $\mathcal{E} = c_1 + c_2$ is manifest over the entire matter curve $\Sigma_{\mathbf{R}}$ and the matter surfaces $\mathcal{C}_{\mathbf{R}}^{\mathbf{w}}$ can be described by a complete intersection of three constraints in the ambient space $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$. Then its homology class is given simply by the product of the divisor classes of each of these constraints.

The resulting homology classes of matter surfaces read

Matter surface	Homology class	
$\mathcal{C}_{\mathbf{1}_{(-1,1)}}$	$([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9) \cdot \mathcal{S}_7 \cdot E_1$	(3.7)
$\mathcal{C}_{\mathbf{1}_{(0,2)}}$	$\mathcal{S}_7 \cdot \mathcal{S}_9 \cdot ([K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7 + 2H)$	
$\mathcal{C}_{\mathbf{1}_{(-1,-2)}}$	$([K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7) \cdot \mathcal{S}_9 \cdot (3H - E_1 - 2E_2 + 2\mathcal{S}_9 - \mathcal{S}_7)$	

Here we suppressed the weight \mathbf{w} since it is identical to the charges (q_1, q_2) . We obtain the homology class of the first matter surface $\mathcal{C}_{\mathbf{1}_{(-1,1)}}$ by noting its description as the complete intersection

$$\mathcal{C}_{\mathbf{1}_{(-1,1)}} = \{s_3 = s_7 = 0, e_1 = 0\} \tag{3.8}$$

in the ambient space $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$. Here the first two equations describe the matter curve $\Sigma_{\mathbf{1}_{(-1,1)}} = \{s_3 = s_7 = 0\}$ over which the Calabi-Yau constraint (2.1) factorizes as $p = p_1 p_2$ with one factor given by e_1 , cf. section 4.3 of [51]. Thus, the rational isolated curve is described as $c_{(-1,1)} = \{e_1 = 0, pt \in \Sigma_{(-1,1)}\}$ over all the points of $\Sigma_{(-1,1)}$. The homology class in the first line of (3.7) for $\mathcal{C}_{\mathbf{1}_{(-1,1)}}$ in the ambient space $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ is then easily obtained from (3.8) employing the assignments (2.10) of line bundles to s_3, s_7 and e_1 .

Similarly we obtain the homology classes of the matter surfaces $\mathcal{C}_{\mathbf{1}_{(0,2)}}$ and $\mathcal{C}_{\mathbf{1}_{(-1,-2)}}$. In the former case the matter curve is $\Sigma_{\mathbf{1}_{(0,2)}} = \{s_7 = s_9 = 0\}$ over which the Calabi-Yau constraint (2.1) factorizes globally with the isolated rational curve given by [51]

$$c_{(0,2)} = \{s_1 e_1^2 e_2^2 u^2 + s_2 e_1 e_2^2 uv + s_3 e_2^2 v^2 + s_5 e_1^2 e_2 u w + s_6 e_1 e_2 v w + s_8 e_1^2 w^2 = 0\}. \tag{3.9}$$

Here it is understood that the sections s_i are evaluated on $\Sigma_{\mathbf{1}_{(0,2)}}$. The homology class of this complete intersection is the product of the class of $c_{(0,2)}$ and of $\Sigma_{\mathbf{1}_{(0,2)}}$ and we immediately reproduce the second line in (3.7) using (2.10). Finally, the matter curve for $\mathbf{1}_{(-1,-2)}$ is given by $\Sigma_{\mathbf{1}_{(-1,-2)}} = \{s_8 = s_9 = 0\}$ and the isolated rational curve is [51]

$$c_{(-1,-2)} = \{s_1 e_1^2 e_2 u^3 + s_2 e_1 e_2 u^2 v + s_3 e_2 u v^2 + s_5 e_1^2 u^2 w + s_6 e_1 u v w + s_7 v^2 w = 0\}, \tag{3.10}$$

where as before the s_i are evaluated on $\Sigma_{\mathbf{1}_{(-1,-2)}}$. Then, the matter surface is again a complete intersection in the ambient space (2.7) and its homology class, employing the line bundles (2.10), is indeed given by the third line in (3.7).

Matter curves $\Sigma_{(1,0)}$, $\Sigma_{(0,1)}$, $\Sigma_{(1,1)}$ and their prime ideals

As mentioned before, the three remaining matter curves $\Sigma_{\mathbf{R}'}$ are themselves no simple complete intersections, but contained in a reducible codimension two subvarieties in B that are complete intersections. To isolate the component $\Sigma_{\mathbf{R}'}$ of interest we have to determine its prime ideal. This prime ideal is generated by more than two constraints, but still describes a codimension two variety in B . In addition, the factorization (3.6) describing the split $\mathcal{E} = c_1 + c_2$ of the elliptic curve does not occur globally over the matter curves $\Sigma_{\mathbf{R}'}$, but is manifest only at generic points of $\Sigma_{\mathbf{R}'}$. These combined effects render the determination of the homology class of the matter surfaces $\mathcal{C}_{\mathbf{R}'}$ unfeasible. However, we can obtain the homology class of the matter curves $\Sigma_{\mathbf{R}'}$ as shown next. For completeness we will also present the prime ideal for one illustrative example. In general the prime ideals are needed for a thorough analysis of codimension three singularities presented in section 3.1.2.

We begin with the determination of the homology classes of the matter curves $\Sigma_{\mathbf{R}'}$. They can be obtained by first determining the homology class of the two equations for the reducible codimension two locus in B and by then subtracting the classes of those components $\Sigma_{\mathbf{R}}$ we are not interested in. As in the six-dimensional case [51] we have to subtract the components $\Sigma_{\mathbf{R}}$ with the right multiplicity, which is computed by the resultant⁸ of the two equations at the root corresponding to $\Sigma_{\mathbf{R}}$. The resulting homology classes of this computations give the first three lines of (3.3). We work out these homology classes in detail in the remainder of this section.

First we present the equations for the reducible codimension two loci in B that contain the three matter curves $\Sigma_{\mathbf{R}}$ that we are interested in as irreducible components. These codimension two loci read [51]

$$\begin{aligned} \text{loc}_1 &= \{s_7 s_8^2 + s_9(s_5 s_9 - s_6 s_8) = s_3 s_8^2 - s_2 s_8 s_9 + s_1 s_9^2 = 0\}, \\ \text{loc}_2 &= \{s_3 s_6 s_8 - s_2 s_7 s_8 - s_3 s_5 s_9 + s_1 s_7 s_9 = s_3^2 s_8^2 + s_7(s_1 s_7 s_8 + s_2 s_5 s_9 - s_2 s_6 s_8) \\ &\quad + s_3(s_6^2 s_8 - s_5 s_7 s_8 - s_5 s_6 s_9 - s_2 s_8 s_9 + s_1 s_9^2) = 0\}, \\ \text{loc}_3 &= \{2s_7^3 s_8^3 + s_3 s_9^3(s_5 s_9 - s_6 s_8) + s_7^2 s_8 s_9(2s_5 s_9 - 3s_6 s_8) - s_7 s_9^2(s_5 s_6 s_9 + s_2 s_8 s_9 - s_6^2 s_8 \\ &\quad - 2s_3 s_8^2 - s_1 s_9^2) = s_7^4 s_8^4 + 2s_7^3 s_8^2 s_9(s_5 s_9 - s_6 s_8) + s_7 s_9^3(2s_3 s_8 - s_2 s_9)(s_5 s_9 - s_6 s_8) \\ &\quad - s_7^2 s_9^2(2s_5 s_6 s_8 s_9 + s_2 s_8^2 s_9 - s_6^2 s_8^2 - 2s_3 s_8^3 - s_5^2 s_9^2) + s_3 s_9^4(s_3 s_8^2 + s_9(s_1 s_9 - s_2 s_8)) = 0\} \end{aligned} \tag{3.11}$$

By calculating the associated prime ideals⁹ of loc_1 we see that it has two irreducible components. One is obviously the matter curve $\Sigma_{\mathbf{1}_{(-1,-2)}} = \{s_8 = s_9 = 0\}$ and the other one is the matter curve $\Sigma_{\mathbf{1}_{(1,1)}}$. Then we determine the homology class of the reducible variety loc_1 . We further recall from [51] that the order of the root $(s_8, s_9) = (0, 0)$ of the two polynomials in loc_1 is 4. Thus, decompose the class of loc_1 as

$$\begin{aligned} [\text{loc}_1] &= (2[K_B^{-1}] + 2\mathcal{S}_9 - \mathcal{S}_7) \cdot (3[K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7) \cong \Sigma_{(1,1)} + 4\Sigma_{(-1,-2)} \\ &\Rightarrow \Sigma_{(1,1)} \cong 6[K_B^{-1}]^2 + 4[K_B^{-1}] \cdot \mathcal{S}_9 - 5[K_B^{-1}] \cdot \mathcal{S}_7 + \mathcal{S}_7^2 + \mathcal{S}_7 \cdot \mathcal{S}_9 - 2\mathcal{S}_9^2, \end{aligned} \tag{3.12}$$

⁸In general, the resultant gives the order of a root of two polynomials in two variables.

⁹The ideal generated by loc_1 is the intersection of its associated primary ideals. An ideal I is primary ideal if $ab \in I$ implies $a \in I$ or $b^n \in I$ for some $n > 0$. If $n = 1$, the ideal I is a prime ideal.

where we used (2.10) and denote the equivalence relation in homology as ‘ \cong ’. Here we also used that the homology class of $\Sigma_{\mathbf{1}_{(-1,-2)}}$ as given in the third line of (3.7) by the first two factors. Thus, we have obtained the homology class of $\Sigma_{\mathbf{1}_{(1,1)}}$ as in (3.3).

Similarly, we obtain the homology class of $\Sigma_{\mathbf{1}_{(1,0)}}$. We calculate four associated prime ideals of loc_2 in (3.11) that correspond to four different irreducible components. These components are the curves $\Sigma_{\mathbf{1}_{(-1,1)}} = \{s_3 = s_7 = 0\}$, $\Sigma_{\mathbf{1}_{(-1,-2)}} = \{s_8 = s_9 = 0\}$, $\Sigma_{\mathbf{1}_{(1,1)}}$ and finally $\Sigma_{\mathbf{1}_{(1,0)}}$. By calculating the resultants of loc_2 at the relevant roots, we obtain multiplicities one for all irreducible components. Thus, we obtain the homology class of $\Sigma_{(1,0)}$ from decomposition of the class of loc_2 as

$$\begin{aligned} [\text{loc}_2] &= 12[K_B^{-1}]^2 = \Sigma_{(-1,1)} + \Sigma_{(-1,-2)} + \Sigma_{(1,1)} + \Sigma_{(1,0)} \\ &\Rightarrow \Sigma_{(1,0)} \cong 6[K_B^{-1}]^2 + 4[K_B^{-1}] \cdot \mathcal{S}_7 - 5[K_B^{-1}] \cdot \mathcal{S}_9 + \mathcal{S}_9^2 + \mathcal{S}_7 \cdot \mathcal{S}_9 - 2\mathcal{S}_7^2, \end{aligned} \quad (3.13)$$

where we have used (2.10), the homology classes of matter curves in (3.7) as well as in (3.12). This is the result in (3.3).

Finally, we determine the homology class of the matter curve $\Sigma_{\mathbf{1}_{(0,1)}}$. The ideal loc_3 in (3.11) has five prime ideals corresponding to the matter curves $\Sigma_{\mathbf{1}_{(-1,1)}}$, $\Sigma_{\mathbf{1}_{(0,2)}}$, $\Sigma_{\mathbf{1}_{(-1,-2)}}$, $\Sigma_{\mathbf{1}_{(1,1)}}$ and the matter curve $\Sigma_{\mathbf{1}_{(0,1)}}$ we are interested in. The multiplicities of the irreducible components we are not interested are calculated as one, 16, 16, one, respectively, by the corresponding resultants of loc_3 . Thus, we calculate the homology class of the curve $\Sigma_{\mathbf{1}_{(0,1)}}$ from the homology class of loc_3 as

$$\begin{aligned} \text{loc}_3 &= 12[K_B^{-1}]^2 = \Sigma_{(-1,1)} + 16 \cdot \Sigma_{(0,2)} + 16 \cdot \Sigma_{(-1,-2)} + \Sigma_{(1,1)} + \Sigma_{(0,1)} \\ &\Rightarrow \Sigma_{(0,1)} = 6[K_B^{-1}]^2 + 4[K_B^{-1}] \cdot (\mathcal{S}_7 + \mathcal{S}_9) - 2\mathcal{S}_7^2 - 2\mathcal{S}_9^2, \end{aligned} \quad (3.14)$$

where we used the homology class of the matter curves in (3.7) and (3.12). This is the homology class in (3.3).

We conclude this discussion by presenting the associated prime ideal of selected matter surfaces as an instructive preparation of section 3.1.2. The prime ideal of $\Sigma_{\mathbf{1}_{(1,1)}}$ reads

$$\begin{aligned} \mathcal{P} &= \left\{ s_3^2 s_5^2 + s_7 (s_2^2 s_5 - s_1 s_2 s_6 + s_1^2 s_7) + s_3 (-s_2 s_5 s_6 + s_1 (s_6^2 - 2s_5 s_7)), \right. \\ &\quad s_3 s_5^2 s_9 + s_2 s_5 (s_7 s_8 - s_6 s_9) + s_1 (-s_6 s_7 s_8 + s_6^2 s_9 - s_5 s_7 s_9), \\ &\quad s_3 s_5 s_8 - s_1 s_7 s_8 - s_2 s_5 s_9 + s_1 s_6 s_9, s_3 s_6 s_8 - s_2 s_7 s_8 - s_3 s_5 s_9 + s_1 s_7 s_9, \\ &\quad \left. s_3 s_8^2 + s_9 (-s_2 s_8 + s_1 s_9), s_7 s_8^2 + s_9 (-s_6 s_8 + s_5 s_9) \right\}. \end{aligned} \quad (3.15)$$

The dimension of \mathcal{P} is calculated to be six in the ring generated by the s_i which confirms that the irreducible variety described by it is codimension two in B as expected. It is evident from (3.15) that the two irreducible components $\Sigma_{\mathbf{1}_{(1,1)}}$ and $\Sigma_{\mathbf{1}_{(-1,-2)}} = \{s_8 = s_9 = 0\}$ of loc_1 intersect at points in B . These points may correspond to Yukawa points in F-theory since the fiber in the resolved space \hat{X} splits further into three components, as can be seen by a prime ideal analysis. We also determine the prime ideal of the matter curve $\Sigma_{\mathbf{1}_{(1,0)}}$ as

$$\begin{aligned} \mathcal{P} &= \left\{ -s_2^2 s_8^2 + s_2 (s_5 s_6 s_8 - s_5^2 s_9 + 2s_1 s_8 s_9) - s_1 (s_6^2 s_8 - s_5 s_6 s_9 + s_1 s_9^2), \right. \\ &\quad s_2 s_5 s_7 - s_1 s_6 s_7 - s_2 s_3 s_8 + s_1 s_3 s_9, s_3 s_5 s_7 - s_1 s_7^2 - s_3^2 s_8, \\ &\quad \left. s_3 s_6 s_7 - s_2 s_7^2 - s_3^2 s_9, s_3 s_6 s_8 - s_2 s_7 s_8 - s_3 s_5 s_9 + s_1 s_7 s_9 \right\}. \end{aligned} \quad (3.16)$$

From this ideal we see that $\Sigma_{\mathbf{1}_{(1,0)}}$ intersects the matter curve $\Sigma_{\mathbf{1}_{(-1,1)}}$, where again there is an additional split of the fiber.

3.1.2 Yukawa couplings: codimension three

At codimension three the singularities of the fibration enhance further signaling the presence of a Yukawa point. In the case at hand we find an enhancement to an I_3 -singularity, which is resolved into three intersecting \mathbb{P}^1 's in \hat{X} . The loci of I_3 -fibers are determined by looking at zeros of higher order of the discriminant and by checking whether the fiber in the resolution \hat{X} splits further. We find the following loci,

Loci	Yukawa	
$s_8 = s_9 = s_7 = 0$	$\mathbf{1}_{(-1,-2)} \times \mathbf{1}_{(0,2)} \times \mathbf{1}_{(1,0)}$	(3.17)
$s_3 = s_7 = s_9 = 0$	$\mathbf{1}_{(0,2)} \times \bar{\mathbf{1}}_{(1,-1)} \times \bar{\mathbf{1}}_{(-1,-1)}$	
$\Sigma_{\mathbf{1}_{(1,0)}} \cap \Sigma_{\mathbf{1}_{(1,0)}} \cap \Sigma_{\mathbf{1}_{(1,0)}}$	$\bar{\mathbf{1}}_{(-1,-1)} \times \mathbf{1}_{(1,0)} \times \mathbf{1}_{(0,1)}$	
$\Sigma_{\mathbf{1}_{(1,0)}} \cap \{s_3 = s_7 = 0\}$	$\bar{\mathbf{1}}_{(1,-1)} \times \bar{\mathbf{1}}_{(-1,0)} \times \mathbf{1}_{(0,1)}$	
$\Sigma_{\mathbf{1}_{(1,1)}} \cap \{s_8 = s_9 = 0\}$	$\mathbf{1}_{(1,1)} \times \mathbf{1}_{(-1,-2)} \times \mathbf{1}_{(0,1)}$	

We note that the first three agree with earlier results, see [46]. The last two loci also produce reducible fibers with three irreducible components that can be described in terms of the prime ideals. The study of these new Yukawa points, along with a more thorough discussion of the use of prime ideals, will be postponed to future work.

3.2 The cohomology ring and the Chern classes of \hat{X}

In this section we abstractly calculate the cohomology ring of the fourfold \hat{X} . The central result of these computations is the basis of surfaces or dual $(2, 2)$ -forms in $H^{(2,2)}(\hat{X})$, which is relevant for the construction of G_4 -flux, see section 5.1. Furthermore, for the calculation of the D3-brane tadpole and the quantization of the G_4 -flux, we use these techniques to calculate the general expression for the Euler number and the second Chern class of \hat{X} for a general base B . In addition, we derive the full cohomology ring explicitly for $B = \mathbb{P}^3$, leaving the straightforward generalization to other bases for future works.

We note that the presentation of the cohomology of \hat{X} used here has been employed in the context of toric mirror symmetry for a long time and is in this sense not new. For an F-theory context see e.g. [58, 62] and references therein. We refer also to [21] for cohomology calculations in the same spirit. However, we emphasize that, except for the language that we borrow from toric geometry, the following discussion is based only on reasonable assumptions on the intersections of \hat{X} . Thus, we expect the following procedure to work also in the non-toric case. In particular, not all fourfolds considered here have, to our knowledge, a description in terms of a reflexive polytope, which does, however, not keep us from using them for F-theory and computing their full chiral 4D spectrum.

The basic idea to calculate the cohomology ring $H_V^{(*,*)}(\hat{X})$ of a general elliptically fibered Calabi-Yau fourfold \hat{X} over a base B with general fiber the elliptic curve in dP_2 is

to exploit the Stanley-Reissner (SR) ideal¹⁰ SR of the ambient space $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ together with the linear equivalences of divisors. After dividing out the linear equivalences, the cohomology ring $H_V^{(*,*)}(\hat{X})$ can be represented as the quotient ring R of the form¹¹

$$H_V^{(*,*)}(\hat{X}) \cong \frac{\mathbb{C}[D_\alpha, S_P, S_Q, S_R] \cdot [\hat{X}]}{SR}, \quad (3.18)$$

where the basis (2.11) together with the vertical divisors D_α are the variables of the free polynomial ring $\mathbb{C}[D_\alpha, S_P, S_Q, S_R]$ and SR is considered as an ideal in this ring. For this purpose, the ideal SR has to be translated into intersection relations of those divisors. Note that we have to multiply by the homology class of \hat{X} in $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ in (2.9) to restrict the intersections on the ambient space $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ to \hat{X} .¹² By the Calabi-Yau condition this class is precisely given as $\mathcal{O}(\hat{X}) = K_{dP_2^B}^{-1}$.

The quotient ring (3.18) is graded with each graded piece being finitely generated by monomials in the divisors D_α, S_P and S_m of appropriate degree. We denote this ring by R . The k -th graded piece is then identified with

$$H_V^{(k,k)}(\hat{X}) = R^{(k)}, \quad (3.19)$$

after restriction to \hat{X} , i.e. after dropping the overall factor $K_{dP_2^B}^{-1}$ in (3.18). More precisely, at grade zero we obtain $H^{(0,0)}(\hat{X}) = \langle 1 \rangle$, at grade one we have $H^{(1,1)}(\hat{X}) = \langle D_\alpha, S_P, S_Q, S_R \rangle$. At higher grade we obtain naively as many generators as homogeneous monomials of appropriate degree in the divisors in (3.18). However, due to equivalence relation in R the number of independent monomials is in general smaller. In fact, by Poincaré duality the rings $R^{(3)}$, and $R^{(4)}$ are fixed, i.e. the corresponding Hodge numbers are related as $h^{(3,3)}(\hat{X}) \stackrel{!}{=} h^{(1,1)}(\hat{X}) = 3 + h^{(1,1)}(B)$ and $h^{(4,4)}(\hat{X}) \stackrel{!}{=} h^{(0,0)}(\hat{X}) = 1$. At degree $k \geq 5$ we trivially have $R^{(k)} = \{0\}$ by reasons of dimensionality. In this sense the only non-trivial piece is $R^{(2)} \cong H_V^{(2,2)}(\hat{X})$, and the corresponding Hodge-number $h_V^{(2,2)}(\hat{X})$. Furthermore, it is precisely the elements $H_V^{(2,2)}(\hat{X})$ of independent surfaces on \hat{X} into which the general G_4 -flux on \hat{X} has to be expanded, as discussed in section 5.1.

The main advantage of the representation (3.18) compared to concrete toric models is that it allows us to determine the cohomology ring for all fibrations $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ in (2.7) over a given base B with general divisors $\mathcal{S}_7, \mathcal{S}_9$. Of course the relevant computations depend on the geometry of the base B since the ideal SR in (3.18) in general is generated by the SR-ideal (2.5) of the fiber dP_2 and of the base B , which differs from case to case. Nevertheless, as we demonstrate next, it is possible to calculate the total Chern class $c(\hat{X})$ and Euler number $\chi(\hat{X})$ of \hat{X} for any base B using minimal assumptions.

¹⁰We merely borrow this term from toric geometry. In general, SR can be any ideal containing all vanishing intersections of divisors on $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$, which not necessarily has to be a toric variety.

¹¹We note that this polynomial ring is only the primary vertical cohomology $H_V^{(*,*)}(\hat{X})$ [63]. This is the subspace of $H^{(*,*)}(\hat{X})$ relevant for G_4 -flux inducing chirality in F-theory. Its complement in $H^{(2,2)}(\hat{X})$ is the horizontal cohomology $H_H^{(2,2)}(\hat{X})$ that encodes complex structure moduli of \hat{X} . See e.g. [58] for an analysis of G_4 -flux in $H_H^{(2,2)}(\hat{X})$ in F-theory.

¹²Generally, not all divisors on \hat{X} arise as restrictions of divisors on the ambient space. However, for generic \hat{X} with elliptic fiber in dP_2 , only divisors in B can potentially miss this assumption. We exclude those B in the following. Note that non-generic \hat{X} can have additional divisors, see the footnote 13.

3.2.1 Second Chern class and Euler number of \hat{X} : general formulas

For the purpose of finding the general expression for $c(\hat{X})$ and, thus, the Euler number $\chi(\hat{X})$, it suffices to know, that the intersections of more than three vertical divisors D_α in both $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ and \hat{X} are zero. The latter is true because of the properties of fibrations. Thus, we are working in the following with the ideal of vanishing intersections on $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ generated by the ideal (2.2) of the fiber dP_2 supplemented by the vanishing of quartic intersection of vertical divisors,

$$SR' = \{S_R \cdot (S_R + S_Q - \mathcal{S}_7 - [K_B]), S_R \cdot (S_R + S_P - \mathcal{S}_9 - [K_B]), S_Q \cdot S_P, \quad (3.20)$$

$$S_Q \cdot (S_R + S_Q - \mathcal{S}_7 - [K_B]), S_P \cdot (S_R + S_P - \mathcal{S}_9 - [K_B]), D_\alpha \cdot D_\beta \cdot D_\gamma \cdot D_\delta\}.$$

Here we have employed (2.10) in combination with (2.11) to translate (2.5) into intersection relations. The prime in SR' reminds us that we are not working with the full SR-ideal of the base B , but just assume vanishing quartic intersections. As before ‘ \cdot ’ denotes the intersections product in $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$.

Next we can perform the calculation of the total Chern class of \hat{X} . For this purpose we first compute the formal expression of the Chern class $c(\hat{X})$ by adjunction,

$$c(\hat{X}) = \frac{c(dP_2^B)}{1 + c_1(\mathcal{O}(\hat{X}))}. \quad (3.21)$$

The numerator denotes the total Chern class of $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ and the denominator is the Chern class of its anti-canonical bundle (2.9), which is the class of \hat{X} as mentioned above.

Then, we reduce this expression in the quotient ring (3.18) with SR replaced by the reduced ideal SR' in (3.20). We refer to appendix A for the detailed calculations leading to the following results, as well as for the general expression of the total Chern classes $c(dP_2^B)$ and $c(\hat{X})$ for Calabi-Yau two-, three- and fourfolds. We obtain for the second Chern class $c_2(\hat{X})$ of \hat{X} the expression

$$c_2(\hat{X}) = 3c_1^2 + c_2 - 2S_Q^2 - 3S_P^2 + c_1(2S_Q + S_P + 4S_R - 2(\mathcal{S}_7 + \mathcal{S}_9))$$

$$+ 2\mathcal{S}_7(S_Q - S_P) + \mathcal{S}_9(3S_P - 2S_Q - S_R + \mathcal{S}_7), \quad (3.22)$$

where we have expressed all cohomology classes in terms of the basis of divisors (2.11) on the fiber dP_2 and the first and second Chern classes $c_1 \equiv c_1(B)$, respectively, $c_2 \equiv c_2(B)$ of the base B . By abuse of notation we denote a divisor and its Poincaré dual (1,1)-form by the same symbol.

As a first sanity check we note that (3.22) is consistent with the formula for the second Chern class of a fourfold with a generic E_6 -elliptic fiber, i.e. with the elliptic curve in \mathbb{P}^2 . In fact, in the limit $\mathcal{S}_7 = \mathcal{S}_9 = 0$, the total space $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ formally turns into $\mathbb{P}(\mathcal{O}_B \oplus K_B^{-1} \oplus K_B^{-1})$, the sections S_P, S_Q and S_R become indistinguishable and fuse into a single holomorphic three-section σ , following conventions in the literature. Then the second line in (3.22) vanishes and we use the relation (2.16) for σ^2 to rewrite the first line as

$$c_2(\hat{X}) \rightarrow c_2(X_{E_6}) = 3c_1^2 + c_2 + 12\sigma c_1, \quad (3.23)$$

where we denote by X_{E_6} the fourfold with E_6 -elliptic fiber. This expression is in line with the results obtained in [53].

Similarly, the Euler number of \hat{X} is calculated from the integration of the fourth Chern class $c_4(\hat{X})$ as

$$\chi(\hat{X}) = 3 \int_B [24c_1^3 + 4c_1c_2 - 16c_1^2(\mathcal{S}_7 + \mathcal{S}_9) + c_1(8\mathcal{S}_7^2 + \mathcal{S}_7\mathcal{S}_9 + 8\mathcal{S}_9^2) - \mathcal{S}_7\mathcal{S}_9(\mathcal{S}_7 + \mathcal{S}_9)] . \tag{3.24}$$

Here the integral over \hat{X} has been reduced to an integral over the base B by first consecutive application of the relation (2.14) and then by employing (2.13), which can be rewritten for Calabi-Yau fourfolds as the intersection relation

$$S_P \cdot D_\alpha \cdot D_\beta \cdot D_\gamma = S_m \cdot D_\alpha \cdot D_\beta \cdot D_\gamma = (D_\alpha \cdot D_\beta \cdot D_\gamma)|_B . \tag{3.25}$$

Here S_m collectively denotes the divisors S_Q, S_R of the sections \hat{s}_Q, \hat{s}_R and $D_\alpha, D_\beta, D_\gamma$ are general vertical divisors. We emphasize that our expression of the Euler number (3.24) reproduces the Euler number of [53] as the special case $\mathcal{S}_7 = \mathcal{S}_9 = 0$. As before S_P, S_Q and S_R become homologous and we obtain

$$\chi(\hat{X}_{E_6}) = 72 \int_B c_1^3 + 12 \int_B c_1c_2 . \tag{3.26}$$

As another consistency check, and also for the sake of the discussion of general flux quantization and the D3-brane tadpole in section 5.1, we calculate the arithmetic genus $\chi_0(\hat{X})$ on \hat{X} . It is calculated from the Todd class $\text{Td}_4(\hat{X})$ by the Hirzebruch-Riemann-Roch index theorem. Since \hat{X} is a simply-connected Calabi-Yau fourfold, its arithmetic genus has to be two,

$$\chi_0(\hat{X}) := \sum_p (-1)^p h^{(p,0)}(\hat{X}) \stackrel{!}{=} 2 . \tag{3.27}$$

This immediately follows from $h^{(0,0)}(\hat{X}) = h^{(4,0)}(\hat{X}) = 1$ and $h^{(p,0)}(\hat{X}) = 0$ otherwise. From index theory, however, we obtain

$$\chi_0(\hat{X}) = \int_{\hat{X}} \text{Td}_4(\hat{X}) = \frac{1}{720} \int_{\hat{X}} (3c_2(\hat{X})^2 - c_4(\hat{X})) = \frac{1}{720} \left(3 \int_{\hat{X}} c_2(\hat{X})^2 - \chi(\hat{X}) \right) . \tag{3.28}$$

Evaluating this integral using our expressions (3.22), (3.24) for the second Chern class and Euler number on \hat{X} , cf. appendix A for details, we obtain

$$\chi_0(\hat{X}) = \frac{1}{12} \int_B c_1c_2 = 2\chi_0(B) \stackrel{!}{=} 2 . \tag{3.29}$$

Here the first equality is due to a remarkable cancellation of all terms containing the divisors $\mathcal{S}_7, \mathcal{S}_9$ in the second Chern class and Euler number of \hat{X} . In the second equality we used the index theorem for the arithmetic genus of the base

$$\chi_0(B) = \frac{1}{24} \int_B c_1c_2 , \tag{3.30}$$

and the last equality follows from the constraint (3.27). Thus we see, that the arithmetic genus $\chi_0(\hat{X}) = 2$ precisely iff $\chi_0(B) = 1$. In general, one demands the stronger conditions $h^{(1,0)}(B) = h^{(2,0)}(B) = h^{(3,0)}(B) = 0$ since non-trivial $(p, 0)$ -forms of the base B would pull back to $(p, 0)$ -forms on \hat{X} under the projection $\pi : \hat{X} \rightarrow B$, which we excluded by assumption.

We note that our result (3.29) for the arithmetic genus is in line with the computations in [57, 64], whose analysis we followed. We also refer to [21] for an application of these techniques to F-theory with $SU(5)$ gauge group.

3.2.2 The full cohomology ring of \hat{X} : base $B = \mathbb{P}^3$

As we demonstrate next, the representation (3.18) for a concrete base B allows us to calculate the full cohomology ring for a general Calabi-Yau fourfold \hat{X} in $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ with general divisors $\mathcal{S}_7, \mathcal{S}_9$. We exemplify this in the following for the base $B = \mathbb{P}^3$, but note that this analysis can be generalized to other bases. For all details of the intersection calculations as well as the quartic intersections, we refer the reader to appendix B.

In the case $B = \mathbb{P}^3$ the cohomology $H^{(1,1)}(\hat{X})$ is generated according to (2.22) by the divisors D_A . We choose the following basis,

$$H^{(1,1)}(\hat{X}) = \langle H_B, S_P, S_Q, S_R \rangle, \tag{3.31}$$

where H_B is the only vertical divisor of the fibration, which is pullback of the hyperplane of \mathbb{P}^3 to \hat{X} . The three other divisors are related to the in general rational sections \hat{s}_P, \hat{s}_Q and \hat{s}_R . Employing (2.10) and (2.11), their associated divisor classes are

$$S_P = E_2, \quad S_Q = E_1, \quad S_R = H - E_1 - E_2 + (n_9 - 4)H_B, \tag{3.32}$$

where we have used $c_1(K_{\mathbb{P}^3}) = -c_1(\mathbb{P}^3) = -4H_B$. We recall that the divisors $\mathcal{S}_7, \mathcal{S}_9$ on \mathbb{P}^3 are specified by integers n_7, n_9 in the region in figure 2 specifying the total space (2.26) of the dP_2 -fibration over \mathbb{P}^3 .

We set up the construction of the cohomology ring of $\pi : \hat{X} \rightarrow \mathbb{P}^3$ via (3.18) by specifying the ideal SR . We note that the SR -ideal in the case of $B = \mathbb{P}^3$ is generated by the Stanley-Reissner ideal (2.5) of the fiber dP_2 and the base, which is just $H_B^4 = 0$. Thus, using the divisor classes (3.31), the resulting ideal is identical to (3.20) with all vertical divisors equal to H_B and with $K_{\mathbb{P}^3}^{-1} = \mathcal{O}_{\mathbb{P}^3}(4)$. Then, we need the anti-canonical bundle $K_{dP_2(n_7, n_9)}^{-1}$ of $dP_2(n_7, n_9)$. It is given in general in (2.9) and easily specialized to $B = \mathbb{P}^3$ using $\mathcal{S}_7 = n_7 H_B$ and $\mathcal{S}_9 = n_9 H_B$ as well as expressed in the basis (3.32). Now we are equipped with all the necessary quantities to construct the quotient ring representation (3.18) of the cohomology ring $H_V^{(*,*)}(\hat{X})$.

We begin by summarizing the Hodge numbers of the vertical cohomology of \hat{X} as

$$h^{(0,0)}(\hat{X}) = h^{(4,4)}(\hat{X}) = 1, \quad h^{(1,1)}(\hat{X}) = h^{(3,3)}(\hat{X}) = 4, \quad h_V^{(2,2)}(\hat{X}) = 5 \quad (4), \tag{3.33}$$

where the subscript V indicates that we are considering the vertical subspace, and the number in the bracket denotes the non-generic case with (n_7, n_9) on the boundary¹³ of the allowed region in figure 2. These are the lines $n_7 = 0$, $n_9 = 0$, $n_9 = 4 + n_7$ for $n_7 \leq 4$, $n_9 = n_7 - 4$ and $n_9 = 12 - n_7$, both of the latter two for $4 < n_7$.

Indeed, we obtain these Hodge numbers as follows from the ring (3.18). At degree zero, which is $H^{(0,0)}(\hat{X})$, the only generator is the trivial element 1. The graded piece $R^{(1)} \cong H^{(1,1)}(\hat{X})$ is generated by the four divisors D_A . At degree two, i.e. $H_V^{(2,2)}(\hat{X})$, there are ten different combinations $D_A \cdot_{\hat{X}} D_B$, of which, however, only five are generically inequivalent. As outlined in appendix B a choice of basis, denoted in general by \mathcal{C}_r with $r = 1, \dots, h_V^{(2,2)}(\hat{X})$, for $H_V^{(2,2)}(\hat{X})$ is given by

$$H_V^{(2,2)}(\hat{X}) = \langle H_B^2, H_B \cdot S_P, H_B \cdot \sigma(\hat{s}_Q), H_B \cdot \sigma(\hat{s}_R), S_P^2 \rangle. \quad (3.34)$$

Here $\sigma(\hat{s}_Q)$, respectively, $\sigma(\hat{s}_R)$ are the Shioda maps (2.17) of the sections \hat{s}_Q , \hat{s}_R . In the case at hand these take the form

$$\sigma(\hat{s}_Q) = S_Q - S_P - 4H_B, \quad \sigma(\hat{s}_R) = S_R - S_P - (4 + n_9)H_B \quad (3.35)$$

We can evaluate the 5×5 -intersection matrix $\eta^{(2)}$ in the basis (3.34) using the quartic intersections in (B.4) as

$$\eta^{(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 & -4 \\ 1 & -4 & 0 & 0 & 16 + (n_7 - n_9 - 4)n_9 \\ 0 & 0 & -8 & n_7 - n_9 - 4 & n_9(4 - n_7 + n_9) \\ 0 & 0 & \eta_{34}^{(2)} & -2(4 + n_9) & 2n_9(4 - n_7 + n_9) \\ -4 & \eta_{25}^{(2)} & \eta_{35}^{(2)} & \eta_{45}^{(2)} & -64 - (8 + n_7 - 2n_9)(n_7 - n_9 - 4)n_9 \end{pmatrix}. \quad (3.36)$$

Here entries $\eta_{rs}^{(2)}$ that are determined by symmetry are omitted and denoted by $\eta_{sr}^{(2)}$. We note that for values of (n_7, n_9) on the boundary of figure 2, there are only four inequivalent such surfaces. A quick way to see this is by calculating the rank of the matrix (3.36) which is generically five, but decreases to four in these cases. In all these cases we can drop the basis element S_P^2 in (3.34) since it becomes homologous to the other four basis elements, cf. (B.8), (B.10), (B.11) and (B.12) in appendix B. The corresponding intersection matrix $\eta^{(2)}$ is then obtained from (3.36) by deleting the last row and column. We note that both the knowledge of the basis (3.34) as well as of the intersections (3.36) is essential for the construction of G_4 -flux in section 5.1.

At degree three, there are 20 combinations of three divisors D_A , however, there are only four inequivalent ones, which is expected by duality of $H^{(3,3)}(\hat{X})$ and $H^{(1,1)}(\hat{X})$. Finally, at degree four, which is $H^{(4,4)}(\hat{X})$, there are 35 different quartic monomials in the D_A ,

¹³We note that for the two special values $(n_7, n_9) = (4, 8)$, $(8, 4)$ there is one additional divisor on \hat{X} that is not induced from the ambient space. For these special values we see from (2.10) that \hat{X} is not generic since s_1, s_2, s_3 , respectively, s_1, s_5, s_8 are constants. Then, we can perform a variable transformation on the fiber coordinates to achieve $s_1 = 0$, i.e. the elliptic curve will have an additional section at $u = 1$, $v = w = 0$. The elliptic fiber can then be embedded into dP_3 with all sections toric. We thank Jan Keitel for pointing out the existence of a non-toric divisor for non-generic $\hat{X} \rightarrow \mathbb{P}^3$.

of which there is only one inequivalent combination. This combination is precisely the quartic intersections on \hat{X} . The higher graded pieces of $H^{(k,k)}(\hat{X})$, $k > 4$ vanish, which is intuitively clear since there are at most quartic intersections on a Calabi-Yau fourfold.

We conclude by summarizing some key intersections on \hat{X} which are discussed in section 2.2 as general properties of the fibrations, that can, however, be proven explicitly using the representation (3.18). Of the complete quartic intersections summarized in B.4 of appendix B we highlight the following intersections,

$$S_P \cdot S_R \cdot H_B^2 = n_9 S_* \cdot H_B^3, \quad S_Q \cdot S_R \cdot H_B^2 = n_7 S_* \cdot H_B^3, \quad S_* \cdot H_B^3 = 1, \quad S_*^2 \cdot H_B^2 = -4, \tag{3.37}$$

where S_* collectively denotes all the divisor classes S_P , S_Q and S_R of the sections. Here the first two relations are the versions of (2.15), respectively, on $B = \mathbb{P}^3$. The third relation implies that a section of the elliptic fibration of \hat{X} intersects the generic fiber $F = \pi^*(pt)$ for a generic point pt in B precisely at one point, cf. (2.13). Finally, the last relation is the analog of (2.14) on $B = \mathbb{P}^3$. We note that for a holomorphic section, this relation holds without intersection with H_B^2 . Indeed, this is confirmed by the concrete cohomology calculation in appendix B for the zero-section S_P in (B.12).

4 G_4 -flux conditions in F-theory from CS-terms: Kaluza-Klein states on the 3D Coulomb branch

In this section we discuss the construction of G_4 -flux in F-theory compactifications on general elliptically fibered Calabi-Yau fourfolds \hat{X} with a non-trivial Mordell-Weil group and a non-holomorphic zero section.

We define G_4 -flux in F-theory through the *M-theory* compactification on the resolved fourfold \hat{X} , that is dual to F-theory reduced on a circle to 3D. The general constraints on G_4 -flux in M-theory compactifications are reviewed in section 4.1. In addition to these conditions, G_4 -flux that is admissible for an *F-theory* compactification has to obey additional constraints. The form of G_4 -flux that yields a consistent F-theory has been derived first in [65] by requiring a Lorentz-invariant uplift to four dimensions. Here we discuss a different logic to obtain constraints on the G_4 -flux. As we point out in section 4.2, these constraints are appropriately formulated as the requirement of the vanishing of certain Chern-Simons (CS) terms on the Coulomb branch of the effective three-dimensional theory.¹⁴ In particular, consistent conditions on the G_4 -flux are obtained only if one-loop corrections of both massive states on the 3D Coulomb branch as well as *Kaluza-Klein* (KK) states are taken into account. Most importantly, the presence of a non-holomorphic zero section is linked to the existence of new CS-terms for the KK-vector, that are generated by KK-states, whereas other CS-terms receive additional shifts.

We present for the first time a consistent set of conditions on G_4 -flux in F-theory compactifications with a non-holomorphic zero section. We also evaluate explicitly the

¹⁴See also [66, 67] for recent related studies of connections between CS-terms and contact terms in 3D effective field theories with background fields in the context of F-maximization.

corrections of massive states to 3D CS-levels for the F-theory/M-theory compactification on the fourfold \hat{X} with dP_2 -elliptic fiber.

The following discussion is an extension of [44], where KK-states have first been discussed in the context of 4D anomaly cancellation, and inspired by the analogous six-dimensional analysis in [55],¹⁵ see also [68] for the relevance of KK-states in the description of self-dual two-forms in 6D/5D.

4.1 A brief portrait of G_4 -flux in M-theory

Let \hat{X} denote an arbitrary smooth Calabi-Yau fourfold. In general, G_4 -flux in M-theory can only be defined on such a smooth manifold, that in the context of F-theory typically arises from resolutions of both codimension one singularities from non-Abelian gauge groups or, as in the case considered here, from higher codimension singularities in the presence of a non-trivial Mordell-Weil group.

Then, G_4 -flux in F-theory is defined as G_4 -flux in M-theory with as set of additional F-theoretic restrictions discussed in the next section 4.2. G_4 -flux in M-theory is consistent if it obeys two basic conditions. First, G_4 has to be quantized as [69]

$$G_4 + \frac{c_2(\hat{X})}{2} \in H^4(\hat{X}, \mathbb{Z}), \tag{4.1}$$

which depends on the second Chern class $c_2(\hat{X})$ of \hat{X} . In addition, the M2-brane tadpole has to be cancelled [64, 70],¹⁶

$$\frac{\chi(\hat{X})}{24} = n_3 + \frac{1}{2} \int_{\hat{X}} G_4 \wedge G_4, \tag{4.2}$$

where $\chi(\hat{X})$ is the Euler characteristic of \hat{X} and n_3 the number of spacetime-filling M2-branes. This tadpole lifts to the D3-brane tadpole in F-theory with n_3 denoting the number of D3-branes.

In addition, one can distinguish G_4 -flux further by decomposing $H^4(\hat{X})$ into its primary vertical and horizontal subspaces $H^4(\hat{X}) = H_V^4(\hat{X}) \oplus H_H^4(\hat{X})$ [63]. As mentioned before, only G_4 -flux in the vertical homology induces 4D chirality as well as gaugings of axions in F-theory and is considered here. A general G_4 -flux in the vertical cohomology $H_V^{(2,2)}(\hat{X})$ than has an expansion as

$$G_4 = m^r \mathcal{C}_r, \tag{4.3}$$

where \mathcal{C}_r with $r = 1, \dots, h_V^{(2,2)}(\hat{X})$ denotes an integral basis of $H_V^{(2,2)}(\hat{X})$ and m^r are the flux-quanta with integrality fixed by the quantization condition (4.1).¹⁷ Such a basis can be constructed explicitly, as demonstrated in sections 3.2 and 6.2, for concrete examples.

¹⁵We are grateful to Thomas W. Grimm for explanations and comments on the importance of Θ_{00} .

¹⁶We are working here in the carefully checked conventions of [71], where also comparison with other, inconsistent sign choices in the literature can be found.

¹⁷In general, $c_2(\hat{X})$ has to be decomposed into the integral basis of $H_V^{(2,2)}(\hat{X})$. However, the determination of this basis is very involved and requires more sophisticated techniques that would exceed the scope of this work. We refer to [58] for the application of mirror symmetry to fix the integral basis. See also [72] for a discussion of potential conflicts between the split of $H^4(\hat{X})$ into vertical and horizontal subspace and the choice of an integral basis.

4.2 Deriving conditions on G_4 -flux in F-theory

The additional constraints on the G_4 -flux in F-theory compactifications are most conveniently formulated in the three-dimensional theory obtained after compactification of the 4D $\mathcal{N} = 1$ effective action of F-theory on S^1 . Then we can use the basic duality between three-dimensional F-theory on $\hat{X} \times S^1$ and M-theory on \hat{X} . Consideration of the resolved fourfold \hat{X} means in terms of the 3D $\mathcal{N} = 2$ effective theory obtained in the circle reduction to go to the 3D Coulomb. The corresponding fields acquiring a VEV are the adjoint valued scalars ζ^A , $A = 0, \dots, h^{(1,1)}(B) + \text{rk}(G) + n_{U(1)}$, along the Cartan directions of the 4D gauge group $G \times U(1)^{n_{U(1)}}$. The zeroth component denotes the scalar in the multiplet of the KK-vector. Then the 3D gauge group is broken in an ordinary Higgs effect to the maximal torus $U(1)^{h^{(1,1)}(B) + \text{rk}(G) + n_{U(1)} + 1}$ and in addition the charged fermions, in particular those from the 4D massless matter multiplets, obtain a mass (shift) as $m = q_A \cdot \zeta^A$, where q_A denotes the full 3D charge vector.

This Coulomb branch then describes the IR dynamics of the dual M-theory compactification on the fourfold \hat{X} in the supergravity approximation. It is the key point of the following analysis that the matching of the two dual descriptions works only on the level of the quantum effective action after massive degrees of freedom have been integrated out on the F-theory side. As we emphasize in the following, also corrections due to KK-states have to be considered. In fact, consistent conditions on the G_4 -flux in the presence of a non-holomorphic zero section are only obtained if new CS-terms for the KK-vector are taken into account.

The general approach of matching F- and M-theory in 3D initiated in [73] has been exploited recently in [42, 44, 45, 74] to study various aspects of the F-theory effective action. We refer to these references for the background of the following discussion.

4.2.1 F-theory conditions from KK-states corrected CS-terms

First we recall that G_4 -flux in M-theory induces CS-terms for the $U(1)$ -gauge fields A^A on the 3D Coulomb branch that read

$$S_{CS}^{(3)} = -\frac{1}{2} \int \Theta_{AB}^M A^A \wedge F^B \quad \Theta_{AB}^M = \frac{1}{2} \int_{\hat{X}} G_4 \wedge \omega_A \wedge \omega_B. \quad (4.4)$$

This can be shown by reducing the M-theory three-form C_3 along $(1, 1)$ -forms ω_A on the fourfold \hat{X} that are dual to the basis of divisors D_A , $A = 0, 1, \dots, h^{(1,1)}(\hat{X}) - 1$. We recall, cf. (2.22) for the case $n_{U(1)} = 2$, that this basis is given by $h^{(1,1)}(B)$ vertical divisors D_α , the divisor $\tilde{S}_P = S_P + \frac{1}{2}[K_B^{-1}]$, see (2.21), associated to the zero section, the Shioda maps $\sigma(\hat{s}_m)$ for a rank $n_{U(1)}$ Mordell-Weil group and $\text{rk}(G)$ Cartan divisors D_i in the presence of a four-dimensional non-Abelian gauge group G .

Now the strategy to formulate conditions on the G_4 -flux for a valid F-theory compactification is as follows. We require that the CS-levels on the F-theory side, denoted Θ_{AB}^F , have to agree with the M-theory CS-levels, denoted Θ_{AB}^M and given by the flux integral in (4.4),

$$\Theta_{AB}^F \stackrel{!}{=} \Theta_{AB}^M. \quad (4.5)$$

This means, whenever a non-vanishing CS-level Θ_{AB}^F is there on the F-theory side, as a classical CS-level or generated by loops of massive matter on the 3D Coulomb branch, then the same CS-level has to be there on the M-theory side, i.e. it has to be generated by the G_4 -flux. In contrast, when a CS-term is not there on the F-theory effective field theory side, the corresponding flux integral in (4.4) for the same CS-term in M-theory has to vanish. However, the critical point is to allow in the corresponding F-theory loop-computation also for loops with an infinite tower of KK-states. If KK-states are not included, the G_4 -flux is in general over-constrained, in particular in the presence of a non-holomorphic zero section. In addition, certain CS-levels get shifted by KK-states and a consistent match of CS-terms in F- and M-theory is only possible if these corrections are included.

The general form for the correction to the classical CS-level on the F-theory side, denoted by $\Theta_{cl,AB}^F$, has been worked out in [75–77]. The correction is one-loop exact with all 3D massive fermions contributing in the loop. Assuming a fermion with charge vector q_A , the loop corrected CS-term takes the simple form

$$\Theta_{AB}^F = \Theta_{cl,AB}^F + \frac{1}{2} \sum_{\underline{q}} n(\underline{q}) q_A q_B \text{sign}(q_A \zeta^A), \tag{4.6}$$

where $n(\underline{q})$ is the number of fermions with charge vector \underline{q} and the sum runs over all these charge vectors. We note that since real masses can be negative in 3D, the sign-function is non-trivial. We can rewrite this expression further by noting the general form of the charge vector

$$q_A = (n, q_\alpha, q_i, q_m), \tag{4.7}$$

where we recall the 3D gauge group $U(1)^{h^{(1,1)}(B)+\text{rk}(G)+n_{U(1)}+1}$ on the Coulomb branch and that the charge under the KK-vector A^0 is just the KK-label of KK-states, $q_0 \equiv q_{KK} = n$.

Next, we assume that in a theory obtained by circle reduction from 4D, there are no states with charge under the A^α , since these do not correspond to a 4D gauge symmetry, i.e. we assume $q_\alpha = 0$.¹⁸ We consider in the following the massive charged states obtained from the reduction of four-dimensional massless chiral matter to 3D, along with their KK-states. Denoting their charge vectors as $q_A = (n, 0, q_i, q_m)$ with q_i the Dynkin labels of their non-Abelian representations \mathbf{R} under G , we write the loop-correction (4.6) as

$$\begin{aligned} \Theta_{00}^F &= \frac{1}{2} \sum_{\mathbf{R}_{q_m}} \sum_{\underline{q} \in \mathbf{R}_{q_m}} \chi(\mathbf{R}_{q_m}) \sum_{n=-\infty}^{\infty} n^2 \text{sign}(m_{CB} + n \cdot m_{KK}), \\ \Theta_{0\Lambda}^F &= \frac{1}{2} \sum_{\mathbf{R}_{q_m}} \sum_{\underline{q} \in \mathbf{R}_{q_m}} q_\Lambda \chi(\mathbf{R}_{q_m}) \sum_{n=-\infty}^{\infty} n \text{sign}(m_{CB} + n \cdot m_{KK}), \\ \Theta_{\Sigma\Lambda}^F &= \frac{1}{2} \sum_{\mathbf{R}_{q_m}} \sum_{\underline{q} \in \mathbf{R}_{q_m}} q_\Sigma q_\Lambda \chi(\mathbf{R}_{q_m}) \sum_{n=-\infty}^{\infty} \text{sign}(m_{CB} + n \cdot m_{KK}), \end{aligned} \tag{4.8}$$

where we invoked the absence of classical CS-terms. Here we have suppressed a labeling of the KK-charges n by the weights \mathbf{w} of the representation \mathbf{R}_{q_m} and unified the labels i

¹⁸One might wonder whether these states correspond, on the M-theory side, to M2-branes wrapping curves in the base B , e.g. generated from resolved conifolds in B .

and m as $\Lambda = (i, m)$ and $\chi(\mathbf{R}_{q_m})$ denote the 4D chiralities of chiral matter fields. We have also defined the Coulomb branch mass $m_{CB} = q_\Sigma \zeta^\Sigma$ and the KK-mass $m_{KK} = \frac{1}{R_{KK}}$ with R_{KK} the radius of the S^1 . The terms (4.8) are, bearing our assumptions on the spectrum of massive fermions in mind, the only CS-terms receiving loop-corrections via (4.6). Other CS-terms are classically generated, either in 4D or in the 3D reduction.

We summarize all CS-terms, the presence of classical terms, the potential correction via loop-effects, their physical interpretation and a related reference in the following,

Θ_{AB}^F	$\Theta_{\text{cl}, AB}^F$	loop-corr.	G_4 -condition	interpretation
Θ_{00}^F	-	yes	-	4D chiralities
$\Theta_{0\alpha}^F$	yes	-	$\stackrel{!}{=} 0$	S^1 -circle fluxes [60]
Θ_{0i}^F	-	yes	-	4D chiralities
Θ_{0m}^F	-	yes	-	4D anomaly cancellation [44] (for holomorphic \hat{s}_P)
$\Theta_{\alpha\beta}^F$	- (?)	-	$\stackrel{!}{=} 0$	non-geometric flux?
$\Theta_{\alpha i}^F$	yes	-	$\stackrel{!}{=} 0$	4D gaugings by GUT Cartans
$\Theta_{\alpha m}^F$	yes	-	-	4D gaugings by $U(1)_m$ (4D GS-mechanism)
$\Theta_{\Sigma\Lambda}^F$	-	yes	-	4D chiralities

(4.9)

Here we also mention in one column whether the corresponding CS-term is used to impose a condition on the G_4 -flux. Indeed, the effect of CS-terms $\Theta_{0\alpha}^F$ is the induction of circle fluxes along the S^1 in compactification from 4D and, thus, not a physical effect in 4D. Therefore, we impose these CS-terms to vanish. Then, the CS-terms $\Theta_{\alpha\beta}^F$ obstruct the lift back to 4D [73], potentially by non-geometric effects, and are required to vanish. Finally, the CS-terms $\Theta_{\alpha i}$ correspond to 4D gaugings of axions via the maximal torus of the GUT and are thus set to zero. Thus, using the M-/F-theory duality relation (4.5) we formulate the F-theory conditions on the G_4 -flux in terms of 3D CS-levels Θ_{AB} as

G_4 -flux conditions: $\Theta_{0\alpha} = \Theta_{\alpha\beta} = \Theta_{i\alpha} = 0$

(4.10)

We note that these conditions on the G_4 -flux look much weaker than the ones considered in the literature before. We claim that these conditions are the appropriate ones, in particular in cases with a non-holomorphic the zero section \hat{s}_P . In contrast, however, in compactifications with a holomorphic zero section, the vanishing of the CS-levels in (4.10) implies the vanishing of other, dependent CS-levels. We discuss this in the following and contrast it to the situation with a non-holomorphic \hat{s}_P .

4.2.2 KK-corrected 3D CS-terms: field theory computations

In certain cases, the loop-corrections (4.8) can vanish, leading to additional vanishing CS-terms. In particular, for a holomorphic zero-section, the CS-term Θ_{00}^F in field theory

vanishes, and consistently also the geometric CS-term Θ_{00}^M in M-theory. The latter can be seen easily using the conditions (4.10) in the relation

$$\Theta_{00} = \frac{1}{4} K^\alpha K^\beta \Theta_{\alpha\beta}, \tag{4.11}$$

which is derived employing the definition (2.21), the intersection property (2.16), and K^α introduced in (2.24). In order to see the same from the field theory side and, in general, to compute the loop-corrections to the CS-terms, we first have to evaluate the sign-function in (4.8).

We calculate the sign-function geometrically, recalling from the discussion of section 3.1 that to every weight \mathbf{w} of a representation \mathbf{R}_{q_m} realized in F-theory there is a corresponding curve $c_{\mathbf{w}}$, cf. (3.1). The sign-function in (4.8) is then determined by testing whether the curve $c_{\mathbf{w}}$ associated to a given weight \mathbf{w} with Dynkin labels q_Λ and KK-charge n is in the Mori cone $M(\hat{X})$ of effective curves on \hat{X} . We define

$$\text{sign}(q_A \zeta^A) = \begin{cases} 1, & c_{\mathbf{w}} \in M(\hat{X}), \\ -1, & \text{otherwise.} \end{cases} \tag{4.12}$$

Here the KK-charge $q_0 = n$ of a curve $c_{\mathbf{w}}$ is obtained geometrically as

$$n = c_{\mathbf{w}} \cdot \tilde{S}_P. \tag{4.13}$$

In general, a curve is in the Mori cone if it is described by holomorphic equations and an analysis of the geometry allows in general to find all holomorphic curves corresponding to matter, cf. sections 3.1 and 6.1 as well as [51].¹⁹ In the toric context, the relevant parts of the Mori cone can be constructed systematically as recently demonstrated in [55].

Now, in the presence of a holomorphic zero section the sign-function is centered around 0 because no curve $c_{\mathbf{w}}$ has KK-charge. This follows from the simple geometric fact that by definition any rational curve $c_{\mathbf{w}}$ does not intersect S_P , which always goes through the original singular curve, cf. figure 3. This implies that the loop-corrections in (4.8) that are odd in the KK-level n vanish since KK-states with charge $-|n|$ cancel those with charge $|n|$. In particular, $\Theta_{00}^F = 0$, confirming the geometric result (4.11). In addition, the sum over KK-states in $\Theta_{\Lambda\Sigma}^F$ reduces to

$$\Theta_{\Lambda\Sigma}^F = \frac{1}{2} \sum_{\mathbf{R}_{q_m}} \sum_{q \in \mathbf{R}_{q_m}} q_\Sigma q_\Lambda \chi(\mathbf{R}_{q_m}) \text{sign}(m_{CB}), \tag{4.14}$$

which has been used in [42, 44].

In contrast, the CS-levels $\Theta_{0\Lambda}$ receive an infinite loop-correction that has to be regularized by zeta-function regularization. In [44] this zeta function regularization has been

¹⁹In general, the values of the sign-function on a given representation \mathbf{R}_{q_m} depend on the phase of \hat{X} , respectively, of the 3D gauge theory. See [78] for a detailed discussion of phases structure of Calabi-Yau fourfolds and 3D $SU(5)$ gauge theories. However, it can be shown by a similar argument as in [79] that 4D observables like the chiralities $\chi(\mathbf{R}_{q_m})$ are not expected to depend on the phase.

performed and it was shown that the field theory result for Θ_{0m} agrees with the 4D mixed Abelian-gravitational anomaly,

$$\Theta_{0m}^F = -\frac{1}{12} \sum_{\underline{q}} n(\underline{q}) q_m, \tag{4.15}$$

with $n(\underline{q})$ denoting the number of fermions in 4D with U(1)-charge vector \underline{q} . In particular, anomaly cancellation follows then using the M-/F-theory relation (4.5) and the geometric result

$$\Theta_{0\Lambda}^M = \frac{1}{2} K^\alpha \Theta_{\alpha\Lambda}^M, \tag{4.16}$$

where the right side is immediately identified with the 4D Green-Schwarz term for $\Lambda = m$, cf. section 5.3. Here we used the general result $S_P \cdot D_\Lambda = 0$ for a holomorphic zero section. We also infer from (4.16) that the CS-terms Θ_{0i} are set to zero recalling (4.10).

In the presence of a non-holomorphic section \hat{s}_P the only thing that changes on the field theory side is a shift of the sign function in (4.12). It is no longer centered symmetrically around the origin of the sum over KK-labels n . This is geometrically clear because there are now rational curves $c_{\mathbf{w}}$ in the Mori cone that have non-zero intersection with the rational zero section \hat{s}_P , i.e. that have non-zero KK-charge (4.13). These curves have to be located precisely at the loci where the rational zero section is ill-defined and wraps a whole \mathbb{P}^1 in the fiber, which is the original singular curve. Everywhere else in the base B the zero section is only a point on the original singular fiber and does not intersect curves $c_{\mathbf{w}}$.

The effect of the shift of the sign-function is dramatic because now KK-states with positive and negative KK-label $|n|$, respectively, $-|n|$ do no longer cancel in corrections (4.8) that are odd in n . The infinite parts of the sums over KK-states still cancel, but with a non-zero remainder. Thus, the CS-term Θ_{00}^F that was zero for a holomorphic zero section is now generated by a contribution of a finite number of KK-states. For example, assuming, with k denoting an integer, a sign-function of the form

$$\text{sign}(m_{CB} + n \cdot m_{KK}) = \begin{cases} 1, & \text{for } n \geq -k, \\ -1, & \text{for } n < -k, \end{cases} \tag{4.17}$$

for only one weight of a single matter multiplet \mathbf{R}_{q_m} the loop induced CS-term in (4.8) reads

$$\Theta_{00}^F = \frac{k(k+1)(2k+1)}{6} \chi(\mathbf{R}_{q_m}). \tag{4.18}$$

See e.g. [55] for a formal derivation or the following for an intuitive understanding,

Integers	$-k-1$	$-k$	$-k+1$	\dots	0	1	$\sum_n n^2 \text{sign}$	
$n^2 \text{sign}(n)$	$-(k+1)^2$	$-k^2$	$-(k-1)^2$	\dots	0	1	0	(4.19)
$n^2 \text{sign}(n+k)$	$-(k+1)^2$	k^2	$(k-1)^2$	\dots	0	1	$\frac{k(k+1)(2k+1)}{3}$	

Here the unshifted sum was normalized to zero, then the shifted sum differs only by the amount obtained as the finite sum over the differences between the first and second row in (4.19) for each column.

Thus we see that Θ_{00}^F in (4.18) is directly proportional to one chirality. Consequently, imposing $\Theta_{00}^M = 0$, as done in the literature with holomorphic zero sections, also in the non-holomorphic case would unnecessarily set this chirality to zero. We will see that the loop-correction to Θ_{00}^F precisely takes the form (4.18) for dP_2 -elliptic fibrations, both without and with an additional $SU(5)$ gauge group, cf. sections 5.2 and 6.1, since only one singlet has a non-trivial KK-charge $n = 2$ and, thus, a shifted sign-function (4.17) with $k = -2$.

We conclude by mentioning that the CS-terms $\Theta_{\Lambda\Sigma}$ are shifted in a similar way, where as in (4.19) the shift originates from a finite number of KK-states. We note that this shift has to be taken into account when we determine certain 4D chiralities via 3D CS-terms in sections 5.2 and 7.2. In addition, also the relation (4.15) is modified by a finite correction due to KK-states. Assuming again that only one weight in the representation \mathbf{R}_{q_m} has a non-trivial KK-charge with shifted sign-function (4.17), we obtain the corrected expression

$$\Theta_{0m}^F = -\frac{1}{12} \sum_q n(q)q_m - \frac{k(k+1)}{2} \chi(\mathbf{R}_{q_m})q_m. \tag{4.20}$$

Thus, we see that these CS-levels are no longer directly related to the 4D mixed Abelian-gravitational anomaly as in (4.15) with a holomorphic zero section. Geometrically this is also clear since in general $S_P \cdot \sigma(\hat{s}_m) \neq 0$ in compactifications with rational zero sections. Thus, (4.16) does not hold and the cancellation of 4D mixed Abelian-gravitational anomaly is not geometrically implied. However, after subtracting the contribution of KK-states in (4.20), 4D anomaly cancellation requires the remaining piece to again equal $\frac{1}{2}K^\alpha\Theta_{\alpha m}$. In other words, we can formulate the relation

$$\frac{1}{4} \int_{\hat{X}} S_P \cdot \sigma(\hat{s}_m) \cdot G_4 = \frac{k(k+1)}{2} q_m \chi(\mathbf{R}_{q_m}) \tag{4.21}$$

on the fourfold \hat{X} as a necessary and sufficient condition for 4D anomaly cancellation. It would be nice to prove this relation purely geometrically. Finally, we note that also the CS-terms Θ_{0i}^F need no longer vanish, since geometrically $\Theta_{0i}^M \neq \frac{1}{2}K^\alpha\Theta_{\alpha i}^M$. However, in the geometry \hat{X} at hand, cf. section 6.1, this relation still holds since the singularity of the zero section happens away from the resolved $SU(5)$ singularity and, thus, no $SU(5)$ -matter has non-trivial intersections with S_P . Moreover, we obtain $\tilde{S}_P \cdot D_i = 0$ for all $SU(5)$ -Cartan divisors D_i in section 6.2.

5 G_4 -flux & chiralities on fourfolds with two $U(1)$ s

In this section we analyze chirality-inducing G_4 -flux in F-theory on the fourfolds \hat{X} with dP_2 -elliptic fiber and a non-holomorphic zero section. In section 5.1 we first construct the general G_4 -flux for the fourfold $\pi : \hat{X} \rightarrow \mathbb{P}^3$, where we also comment on the general D3-brane tadpole. Then in section 5.2 we outline first our general strategy to obtain chiralities on Calabi-Yau fourfolds with higher rank Mordell-Weil group and a rational zero section, before applying it again to the fourfold \hat{X} with $B = \mathbb{P}^3$. Then in section 5.3 we first review anomaly cancellation conditions in general 4D effective field theories. Then we show that

the general spectrum obtained for the Calabi-Yau fourfold \hat{X} is anomaly-free for the entire allowed region of figure 2. We conclude in section 5.4 with a concrete toric example to exemplify the general findings.

The case of an un-Higgsed $SU(5)$ GUT-group is considered separately in section 6.

5.1 G_4 -flux on fourfolds with two rational sections

In this section we first show that on a general fourfold \hat{X} with dP_2 -elliptic fiber the D3-brane tadpole can always be solved by adding a sufficient amount of n_{D3} of integral D3-brane charge. Then we construct viable G_4 -flux for F-theory on a general fourfold \hat{X} with base $B = \mathbb{P}^3$. We show that for different elliptic fibrations, i.e. for different values of the integers n_7, n_9 , the number of independent G_4 -flux quanta changes.

Integral D3-tadpole on general fourfolds with $U(1) \times U(1)$

In the following we prove the necessary condition for D3-tadpole cancellation on \hat{X} , namely that the induced D3-brane charge from the combination of the Euler number and the quantized G_4 -flux in the tadpole equation (4.2) is always integral. The following discussion is an application of the arguments in [57, 64, 69], that immediately carry over to elliptic fibrations \hat{X} with dP_2 -elliptic curve.

To this end, we use the relation (3.28) between the arithmetic genus $\chi_0(\hat{X})$ and the Euler number $\chi(\hat{X})$ to rewrite the tadpole (4.2) as

$$n_{D3} = \frac{\chi(\hat{X})}{24} - \frac{1}{2} \int_{\hat{X}} G_4^2 = -60 + \frac{1}{2} \int_{\hat{X}} \left(\frac{1}{4} c_2(\hat{X})^2 - G_4^2 \right). \quad (5.1)$$

Here we have also employed that $\chi_0(\hat{X}) = 2$, cf. (3.29). Using the flux quantization condition (4.1) this can be written as

$$n_{D3} = -60 - \frac{1}{2} \int_{\hat{X}} \left(x^2 - x \wedge c_2(\hat{X}) \right), \quad (5.2)$$

where we used $x = G_4 + \frac{1}{2} c_2(\hat{X})$. By flux quantization (4.1) we know that x is integral, i.e. an element in $H^4(\hat{X}, \mathbb{Z})$. This implies by Wu's theorem that $x^2 \cong c_2(\hat{X}) \wedge x \pmod{2}$ [69], so that the integrand in (5.2) is divisible by two. Thus, the number n_{D3} of D3-branes is integral for every elliptically Calabi-Yau fourfold \hat{X} with general elliptic fiber in dP_2 .

The G_4 -flux on \hat{X} with $B = \mathbb{P}^3$

Next we explicitly determine the G_4 -flux on \hat{X} for a general elliptic fibration over the base $B = \mathbb{P}^3$, i.e. for all integers n_7, n_9 in the allowed region in figure 2.

We begin by expanding the G_4 -flux according to (4.3) into the basis of $H_V^{(2,2)}(\hat{X})$ determined in (3.34),

$$G_4 = a_1 H_B^2 + a_2 H_B \cdot S_P + a_3 H_B \cdot \sigma(\hat{s}_Q) + a_4 H_B \cdot \sigma(\hat{s}_R) + a_5 S_P^2, \quad (5.3)$$

for general coefficients a_i , where as before the application of Poincaré duality is understood. Then we calculate the CS-levels (4.4) employing the intersection ring (B.4) in the basis of

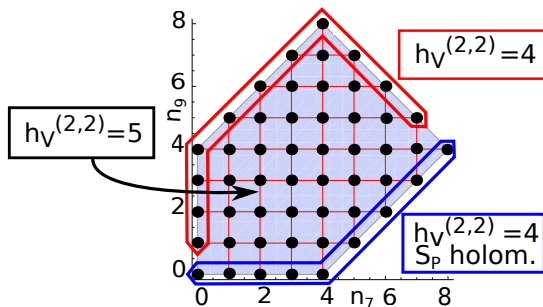


Figure 4. The region of allowed values for (n_7, n_9) from figure 2. On the entire region, there are two conditions on the flux. In the interior of this region, (5.4) holds. On the red and the blue boundary, there are only four independent $(2, 2)$ -forms in the expansion (5.3). On the blue boundary, \hat{s}_P is holomorphic.

divisors (3.31), but with $\tilde{S}_P = S_P + 2H_B$ as defined in (2.21) replacing the zero section S_P . The generic solution is a three-parameter family of G_4 -flux given by

$$G_4 = a_5 n_9 (4 - n_7 + n_9) H_B^2 + 4a_5 H_B \cdot S_P + a_3 H_B \cdot \sigma(\hat{s}_Q) + a_4 H_B \cdot \sigma(\hat{s}_R) + a_5 S_P^2, \quad (5.4)$$

which is valid for all values of n_7 and n_9 in the allowed region figure 2.

This generic three-parameter solution for the G_4 -flux is expected since there are generically five different surfaces in (5.3) and two independent conditions (4.10), namely $\Theta_{0\alpha} = \Theta_{\alpha\beta} = 0$ with $\alpha, \beta = 1$. However, the situation becomes more interesting at special values for (n_7, n_9) . First, we recall that for (n_7, n_9) on the boundary of the region in figure 2, the dimensionality of $H_V^{(2,2)}(\hat{X})$ decreases to 4. The surface S_P^2 becomes linearly dependent in homology on the four other surfaces, as noted below (3.36). At the same time, the number of independent conditions on the G_4 -flux remains two. Thus, we find two independent G_4 -fluxes on the boundary. We have depicted this situation in figure 3

In all these cases we can obtain the expression for the G_4 -flux by specializing (5.4). This ensures that all quantities, in particular the chiralities of 4D charged matter, that are calculated from the most general G_4 -flux specialize correctly for non-generic values of (n_7, n_9) . We discuss this specialization at the end of this subsection, but note that the reader may want to skip these details on a first read and proceed with the chirality formulas in section 5.2.

Before delving into the details of this analysis, we evaluate the D3-brane tadpole (4.2) for the Calabi-Yau fourfold $\hat{X} \rightarrow \mathbb{P}^3$ and the G_4 -flux (5.4). First, we calculate the individual terms in the D3-brane tadpole. We obtain the Euler number for \hat{X} with $B = \mathbb{P}^3$ from the general formula (3.24) as

$$\chi(\hat{X}) = 4896 + 3 [-256(n_7 + n_9) + 4(8n_7^2 + n_7n_9 + 8n_9^2) - n_7n_9(n_7 + n_9)], \quad (5.5)$$

where we employed $c_1(\mathbb{P}^3) = 4H_B$, $c_2(\mathbb{P}^3) = 6H_B^2$. We note that this expression is manifestly positive due to the bounds on n_7 and n_9 in (2.27), respectively, figure 4. Then we

calculate the contribution of the G_4 -flux (5.4) to the D3-tadpole as

$$\int_{\hat{X}} G_4^2 = -8a_3^2 + 2a_3a_4(n_7 - n_9 - 4) + a_5^2n_9(2n_9 - n_7)(n_7 - n_9 - 4) - 2a_4^2(4 + n_9) + 2a_5(a_3 + 2a_4)n_9(4 - n_7 + n_9). \quad (5.6)$$

Finally, for the purpose of G_4 -flux quantization via (4.1) we note the following expression for the second Chern-class of \hat{X} ,

$$c_2(\hat{X}) = (182 + 3n_9(n_7 - n_9 - 4))H_B^2 + 28H_B \cdot S_P + 2(8 - n_9)H_B \cdot \sigma(\hat{s}_Q) + (16 + 2n_7 - 3n_9)H_B \cdot \sigma(\hat{s}_R) - 5S_P^2, \quad (5.7)$$

where we expanded the general expression (3.22) for a better comparison with the G_4 -flux (5.4) in the basis (3.34) of the vertical cohomology. Also for completeness we calculate the square of the second Chern-class from the general formula (A.15) as

$$\int_{\hat{X}} c_2(\hat{X})^2 = 2112 - 256(n_7 + n_9) + 4(8n_7^2 + n_7n_9 + 8n_9^2) - n_7n_9(n_7 + n_9). \quad (5.8)$$

Comparing this with the Euler number (5.5) we reconfirm the general relation (3.28) between the arithmetic genus and the Euler number of a Calabi-Yau fourfold.

Finally, we conclude by a discussion of (5.4) for non-generic values of (n_7, n_9) . Along each component of the boundary we have to use the homology relations between S_P^2 and the four remaining basis elements (3.34) of $H_V^{(2,2)}(\hat{X})$. The relevant relations are (B.8), (B.10), (B.11) and (B.12) worked out in the appendix. In all these cases the general formula (5.4) for the G_4 -flux can be shown to reduce to the two-parameter

$$G_4 = \tilde{a}_3H_B \cdot \sigma(\hat{s}_Q) + \tilde{a}_4H_B \cdot \sigma(\hat{s}_R). \quad (5.9)$$

Here the coefficients \tilde{a}_3 and \tilde{a}_4 are the following linear combination of a_3 , a_4 and a_5 on the three boundary components,

$$\begin{aligned} \{n_7 = 0\} : \tilde{a}_3 = a_3, \tilde{a}_4 = a_4 - n_9a_5, \quad \{n_9 = n_7 + 4\} : \tilde{a}_3 = a_3 - n_7a_5, \tilde{a}_4 = a_4 - 4a_5, \\ \{n_9 = 12 - n_7\} : \tilde{a}_3 = a_3 + (n_7 - 8)a_5, \tilde{a}_4 = a_4 + (n_7 - 8)a_5, \\ \{n_9 = n_7 - 4\} \cup \{n_9 = 0\} : \tilde{a}_3 = a_3, \tilde{a}_4 = a_4. \end{aligned} \quad (5.10)$$

It is satisfying to see that the parameters \tilde{a}_3 , \tilde{a}_4 are continuous at the intersection points of boundary components. We note that the blue boundaries in figure 4, $n_9 = 0$ and $n_9 = n_7 - 4$, are special because \hat{s}_P is holomorphic.

We conclude with one remark on the form of the G_4 -flux obtained here. In some cases, Type IIB seven-brane gauge fluxes $F^{(1)}$, $F^{(2)}$ of two single D7-branes can be lifted into F-theory by considering a G_4 -flux of the type

$$G_4 = F^{(1)} \cdot \sigma(\hat{s}_Q) + F^{(2)} \cdot \sigma(\hat{s}_R). \quad (5.11)$$

We note that the flux (5.9) is precisely of this form. In contrast, the interpretation of the general G_4 -flux in (5.4) in terms of Type IIB quantities, if possible, is less clear and would be very interesting to investigate.

5.2 4D chiralities from matter surfaces & 3D CS-terms

Finally we are prepared to calculate the chirality of matter in four-dimensional F-theory compactifications on \hat{X} with $U(1) \times U(1)$ gauge group. We demonstrate that the chirality of all six matter representations discussed in section 3 can be determined uniquely. Half of the chiralities are determined by integration of the G_4 -flux over matter surfaces $\mathcal{C}_{\mathbf{R}}^{\mathbf{w}}$, the other half from the 3D CS-terms $\Theta_{\Sigma\Lambda}$. As we see explicitly, the 3D CS-terms $\Theta_{\Lambda\Sigma}$ are not sufficient to fix all chiralities, in contrast to earlier works with a holomorphic zero section. However, supplemented by new CS-terms present only for non-holomorphic zero sections, see section 4.2, the chiralities can be obtained also exclusively via 3D CS-terms. We emphasize that this analysis requires the inclusion of KK-charges for all curves $c_{\mathbf{w}}$. In the fourfold \hat{X} only the representation $\mathbf{1}_{(-1,-2)}$ has non-trivial KK-charge $q_{KK} = 2$.

We outline the general strategy to obtain 4D chiralities in section 5.2.1 before we determine them explicitly in section 5.2.2. The concrete calculations are performed for the Calabi-Yau fourfold \hat{X} with base $B = \mathbb{P}^3$ for the entire allowed region in figure 2.

5.2.1 General strategy to determine 4D chiralities

We begin our discussion by recalling how to extract the chiral index of 4D matter in a representation \mathbf{R} under a gauge group G in a global F-theory compactification on an arbitrary resolved elliptically fibered Calabi-Yau fourfold \hat{X} .

As explained in section 3, each matter representation \mathbf{R} possesses an associated ruled surface, the matter surface $\mathcal{C}_{\mathbf{R}}^{\mathbf{w}}$, where \mathbf{w} denotes a weight of \mathbf{R} . Then, the chiral index of charged matter in this representation \mathbf{R} is given by the flux integral²⁰ [1, 21, 40, 80]

$$\chi(\mathbf{R}) = -\frac{1}{4} \int_{\mathcal{C}_{\mathbf{R}}^{\mathbf{w}}} G_4, \tag{5.12}$$

where the G_4 -flux is a quantized M-theory flux subject to the conditions (4.10). It is important to note that these conditions, more precisely the conditions $\Theta_{i\beta} = 0$, imply that the integral (5.12) is independent on the choice of a particular weight \mathbf{w} , since different weights \mathbf{w}, \mathbf{w}' of the same representation \mathbf{R} are related by a root α_i , $\mathbf{w} - \mathbf{w}' = \alpha_i$.

In cases where all the matter surfaces $\mathcal{C}_{\mathbf{R}}^{\mathbf{w}}$ are known, the integral (5.12) is the most direct and geometric way of calculating the chirality $\chi(\mathbf{R})$. However, for the fourfold \hat{X} at hand, but also for the fourfold $\hat{X}_{SU(5)}$ with an additional resolved $SU(5)$ singularity constructed in section 6, the homology class of all matter surfaces $\mathcal{C}_{\mathbf{R}}^{\mathbf{w}}$ is not known. Fortunately, since we know the expected matter spectrum from the geometric analysis there is to determine the missing chiralities $\chi(\mathbf{R})$ indirectly via 3D CS-terms.

For this purpose we recall the three-dimensional M-/F-duality explained in section 4.2. There we have seen that the 3D CS-terms (4.4) for the $U(1)$ -vector fields $A^\Lambda = (A^i, A^m)$ on the Coulomb branch can be calculated in two independent ways. On the one hand, after solving the G_4 -flux condition (4.10), one can calculate all CS-terms $\Theta_{\Lambda\Sigma}^M$ on the M-theory side by evaluating the classical flux integrals in (4.4). On the other hand, the CS-levels

²⁰The factor $-\frac{1}{4}$ has been introduced to be consistency with the conventions of [44]. It can be reabsorbed into the G_4 -flux.

Θ_{AB}^F on the F-theory side are generated and corrected at one-loop from integrating out charged matter and, thus, contain, as shown in section 4.2, cf. (4.8), the 4D chiral indices $\chi(\mathbf{R}_{q_m})$ of all 4D matter. Since we know which representations \mathbf{R}_{q_m} are there from our geometric analysis in section 3.1 and the later section 6.1 for the SU(5) case, we can solve the matching condition (4.5) for the chiralities $\chi(\mathbf{R}_{q_m})$.

We note that in both fourfolds \hat{X} and $\hat{X}_{\text{SU}(5)}$ we know the homology classes for a number of matter surfaces $\mathcal{C}_{\mathbb{R}}^w$, but not for all. For these matter surfaces, we can evaluate the index (5.12) directly. The remaining chiralities can be fixed, as demonstrated in sections 5.2 and 7.2 by the matching of the CS-terms $\Theta_{\Lambda\Sigma}$, that are given on the F-theory side by the loop-expression given in (4.8). However, we can also obtain *all* 4D chiralities by taking into account the CS-terms Θ_{00} and Θ_{0m} in (4.8), respectively, in (4.18) and (4.20). As we will see concretely, the obtained results agree with the direct computations via (5.12), confirming the validity of application of the M-/F-theory duality (4.5) to the determination of 4D chiralities.

We conclude by noting that the matching (4.5) of CS-terms $\Theta_{\Sigma\Lambda}$ has been used in [42, 44, 45] to calculate successfully the chiralities of F-theory compactifications with SU(5) and SU(5)×U(1) gauge symmetry. In this work, however, we encounter the novel situation that the conditions arising from $\Theta_{\Sigma\Lambda}$ are not sufficient to determine the chiralities of the full spectrum with both U(1)×U(1) as well as SU(5)×U(1)×U(1) gauge symmetry as outlined in the following and in section 6. The reason for this is precisely the existence of a non-holomorphic zero section. However, either supplemented by the chiralities that can be directly determined by the flux integrals (5.12) or by the CS-terms Θ_{00} and Θ_{0m} , we obtain all chiralities. For an application of the latter resolution, see also the analogous analysis of [55] in 6D.

One might wonder whether the CS-terms always provide enough conditions to solve for the chiralities of a known F-theory spectrum. By a simple counting argument assuming a rank $n_{\text{U}(1)}$ Mordell-Weil group, we enumerate the number of conditions arising from the matching of these CS-terms as

$$\#(\text{CS-terms}) = \frac{(n_{\text{U}(1)} + 2)(n_{\text{U}(1)} + 1)}{2}. \tag{5.13}$$

If all these conditions remain independent, the CS-terms might indeed be sufficient to determine the chiralities in F-theory compactifications with more U(1)-symmetries. This can be seen by a similar estimate on the growth of the number of different representations as a function of the rank $n_{\text{U}(1)}$ of the Mordell-Weil group.

5.2.2 Chiralities on \hat{X} with $B = \mathbb{P}^3$: matter surfaces & CS-terms

In the following we calculate for the first time 4D chiralities $\chi(\mathbf{R})$ of an F-theory compactification on a general elliptically fibered Calabi-Yau fourfold \hat{X} with rank two Mordell-Weil group and with the full matter spectrum analyzed in section 3. The following is a direct extension of the six-dimensional analysis in [51] to a 4D chiral theory.

As mentioned before, the 4D chirality of a given representation \mathbf{R} is computed by the flux integrals (5.12) given the G_4 -flux and the corresponding matter surfaces $\mathcal{C}_{\mathbf{R}}^w$. The

matter surfaces for the matter fields $\mathbf{1}_{(q_1, q_2)}$ with the three different $U(1)^2$ -charges $(q_1, q_2) = (-1, 1), (0, 2), (-1, -2)$ have been determined in (3.7). Using the general G_4 -flux (5.4) on $\hat{X} \rightarrow \mathbb{P}^3$ we obtain the following chiralities

$$\begin{aligned} \chi(\mathbf{1}_{(-1,1)}) &= -\frac{1}{4} \int_{\mathcal{C}_{(-1,1)}} G_4 = \frac{1}{4} (a_3 - a_4) n_7 (4 + n_7 - n_9) , \\ \chi(\mathbf{1}_{(0,2)}) &= -\frac{1}{4} \int_{\mathcal{C}_{(0,2)}} G_4 = \frac{1}{4} n_7 n_9 (-2a_4 + a_5 (4 - n_7 + n_9)) , \\ \chi(\mathbf{1}_{(-1,-2)}) &= -\frac{1}{4} \int_{\mathcal{C}_{(-1,-2)}} G_4 = \frac{1}{4} n_9 (4 - n_7 + n_9) (a_3 + 2a_4 + a_5 (n_7 - 2n_9)) . \end{aligned} \quad (5.14)$$

In order to evaluate the involved intersections we have made use of the topological metric $\eta^{(2)}$ on the cohomology $H_V^{(2,2)}(\hat{X})$ calculated in (3.36).

In order to obtain the chiralities for the matter fields $\mathbf{1}_{(q_1, q_2)}$ with charges $(q_1, q_2) = (1, 0), (0, 1), (1, 1)$, we have to use the 3D CS-levels and the matching condition (4.5). For this purpose we have to determine the KK-charges of all six matter representations on \hat{X} . As mentioned before, a non-trivial KK-charge is calculate by the intersection (4.13) of the curve c_w and the zero section \tilde{S}_P . Such a non-trivial intersection can only occur at loci, where the zero section is ill-defined and wraps fiber components. This is precisely the case at the loci $s_8 = s_9 = 0$, where $\mathbf{1}_{(-1,-2)}$ is supported. Since the fiber is an I_2 -fiber over all matter loci, we obtain a KK-charge

$$q_{KK}(\mathbf{1}_{(-1,-2)}) = c_{(-1,-2)} \cdot \tilde{S}_P = 2, \quad (5.15)$$

and zero for all other matter representations. We note that (5.15) implies that the sign-function (4.12) is given by the shifted sign-function (4.17) with $k = -2$ whereas the other matter fields retain a point-symmetric sign-function.

We can immediately cross-check this result using the field theory computations of sections 4.2.2. We first calculate the CS-level Θ_{00}^F for the 3D KK-vector on the field theory side. Using the general expression (4.18) for $k = -2$ we obtain

$$\Theta_{00}^F = -\chi(\mathbf{1}_{(-1,-2)}), \quad (5.16)$$

with the chirality $\chi(\mathbf{1}_{(-1,-2)})$ determined in (5.14). We readily calculate the corresponding flux integral Θ_{00}^M via (4.4) and immediately reproduce (5.16). Next we check the relation (4.20) for the CS-level Θ_{0m} , respectively, (4.21). Again we start with the field theory result for the right hand side of (4.21) which requires

$$\frac{1}{4} \int_{\hat{X}} S_P \cdot \sigma(\hat{s}_Q) \cdot G_4 \stackrel{!}{=} -\chi(\mathbf{1}_{(-1,-2)}), \quad \frac{1}{4} \int_{\hat{X}} S_P \cdot \sigma(\hat{s}_R) \cdot G_4 \stackrel{!}{=} -2\chi(\mathbf{1}_{(-1,-2)}). \quad (5.17)$$

We confirm this relation easily by calculating the intersections on the left hand side directly from the G_4 -flux (5.4). Thus, as we have just demonstrated the results for the chiralities in (5.14) obtained from the matter surface integrals can be employed as an independently check of the field theory expressions for the CS-levels in 4.2.2 and the M-/F-theory duality relation (4.5).

Next we proceed with the computation of the other CS-levels Θ_{mn} , $m, n = 1, 2$, for the two $U(1)$ gauge fields A^m corresponding to the divisors $\sigma(\hat{s}_Q)$, respectively, $\sigma(\hat{s}_R)$. Beginning with the M-theory expressions, we obtain using the Shioda maps (3.35), (4.4) and the general G_4 -flux (5.4),

$$\begin{aligned}\Theta_{11}^M &= \frac{1}{2} [a_3(96 - n_7(4 + n_7) + n_9(4 + n_9)) + a_5(4 - n_7 + n_9)n_9(n_7 - 12 - 2n_9) \\ &\quad + a_4(n_7^2 + 2(4 + n_9)(6 + n_9) - n_7(8 + 3n_9))] , \\ \Theta_{12}^M &= \frac{1}{2} [a_3(n_7^2 + 2(4 + n_9)(6 + n_9) - n_7(8 + 3n_9)) + (4 - n_7 + n_9)(a_5(-12 + 3n_7 - 5n_9)n_9 \\ &\quad + a_4(12 + 5n_9))] , \\ \Theta_{22}^M &= \frac{1}{2} [96a_4 + a_3(4 - n_7 + n_9)(12 + 5n_9) + 2a_4n_9(32 - 4n_7 + 5n_9) \\ &\quad - 2a_5n_9(4 - n_7 + n_9)(12 - 2n_7 + 5n_9)] ,\end{aligned}\tag{5.18}$$

with $\Theta_{21}^M = \Theta_{12}^M$. Here we have used the quartic intersections (B.4) in appendix B.

Then we compute the one-loop CS-terms Θ_{mn}^F in (4.8) on the F-theory side. For the matter spectrum at hand, cf. (3.3), we obtain

$$\begin{aligned}\Theta_{11}^F &= \frac{1}{2} (\chi(\mathbf{1}_{(1,0)}) + \chi(\mathbf{1}_{(1,1)}) + \chi(\mathbf{1}_{(-1,1)}) - 3\chi(\mathbf{1}_{(-1,-2)})) , \\ \Theta_{12}^F &= \frac{1}{2} (\chi(\mathbf{1}_{(1,1)}) - \chi(\mathbf{1}_{(-1,1)}) - 6\chi(\mathbf{1}_{(-1,-2)})) , \\ \Theta_{22} &= \frac{1}{2} (\chi(\mathbf{1}_{(0,1)}) + \chi(\mathbf{1}_{(1,1)}) + \chi(\mathbf{1}_{(-1,1)}) + 4(\chi(\mathbf{1}_{(0,2)}) - 3\chi(\mathbf{1}_{(-1,-2)}))) .\end{aligned}\tag{5.19}$$

We note that the factor of -3 in front of $\chi(\mathbf{1}_{(-1,-2)})$ occurs due to the shifted sign-function (4.17) with $k = -2$. Indeed, the relevant sum over KK-states in this case yields

$$\sum_n \text{sign}(n + k) = -3.\tag{5.20}$$

We note that the matching of 3D CS-terms (4.5) only using the Θ_{mn} yields three conditions for the six a priori unknown chiralities in (5.19). Thus, it is impossible to determine the full matter spectrum from these CS-terms alone, which is in contrast to earlier studies in the literature with holomorphic zero sections. We emphasize that the same complication arises in the case of $SU(5) \times U(1)^2$ gauge group considered in section 6, since the representations in the $SU(5)$ -singlet sector will be identical to the six representations considered here. However, we can either use the results (5.14) from the integral of the G_4 -flux over the matter surfaces or have to incorporate the CS-terms Θ_{00} and Θ_{0m} to obtain three further conditions and to fix all six chiralities.

Consequently, taking into account the results (5.14), we apply the matching condition (4.5) for the dual CS-terms (5.18) and (5.19) we obtain the remaining three chiralities as

$$\begin{aligned}\chi(\mathbf{1}_{(1,0)}) &= \frac{1}{4} [a_5n_7n_9(4 - n_7 + n_9) + a_3(2n_7^2 - (12 - n_9)(8 - n_9) - n_7(16 + n_9))] , \\ \chi(\mathbf{1}_{(0,1)}) &= \frac{1}{2} [a_5n_9(4 - n_7 + n_9)(12 - n_9) - a_4(n_7(8 - n_7) + (12 - n_9)(4 + n_9))] , \\ \chi(\mathbf{1}_{(1,1)}) &= \frac{1}{4} [2a_5n_9(4 - n_7 + n_9)(12 - n_9) - (a_3 + a_4)(n_7^2 + n_7(n_9 - 20) + 2(12 - n_9)(4 + n_9))] .\end{aligned}\tag{5.21}$$

We conclude by noting that the chiralities we obtain include factors of $\frac{1}{2}$ and $\frac{1}{4}$. These factors should disappear once the G_4 -flux has been quantized appropriately according to (4.1). The precise quantization will, however, depend on the values of n_7, n_9 and has to be done in a case by case analysis. In addition, for concrete n_7, n_9 the factors in the numerator (5.14) and (5.21) have different divisibility properties and can cancel the denominators. Therefore, in order to not obscure these cancellation effects, we keep the normalization in (5.14), (5.21) and the mild fractions of $\frac{1}{2}$ and $\frac{1}{4}$. In concrete toric examples, they can be cancelled appropriately.

5.3 4D anomaly cancellation: F-theory with multiple U(1)s

Finally, after having calculated the matter spectrum of an F-theory compactification, we check consistency of the obtained low-energy effective physics. One check is anomaly cancellation. In the following we introduce the necessary quantities to analyze anomalies and refer to [44] for more details on anomalies in general and, in particular, in F-theory. Then we use these techniques to show that the general spectrum found in section 5.2 for F-theory compactifications on $\hat{X} \rightarrow \mathbf{P}^3$ with gauge group $U(1)^2$ is anomaly-free.

5.3.1 4D anomaly cancellation: general discussion

In the following we assume a 4D gauge group G with a single non-Abelian factors, see for example [44] for the general case, and with a number $n_{U(1)}$ of Abelian factors. Charges are summarized by a charge vector $\underline{q} = (q_m)$ and the number of left Weyl fermions in a matter representation $\mathbf{R}_{\underline{q}}$ are denoted in general by $n(\mathbf{R}_{\underline{q}})$. The number of fermions in a representation \mathbf{R} irrespective of their U(1)-charges is denoted by $n(\mathbf{R})$, whereas $n(\underline{q})$ indicates the number of fermions with charges \underline{q} regardless of their non-Abelian representation. All these numbers can be expressed in terms of the chiralities $\chi(\mathbf{R}_{\underline{q}})$.

The conditions for 4D anomaly cancellation via the generalized Green-Schwarz (GS) mechanism [81, 82] yield a system of linear equations involving the spectrum of the theory as well as parameters encoding the GS-counter terms and the gaugings of axions. These conditions for cancellation of 4D purely non-Abelian, purely Abelian, mixed Abelian-non-Abelian and mixed Abelian-gravitational anomalies read, in the same order,

$$\begin{aligned}
 \text{purely non-Abelian anomaly} &: \sum_{\mathbf{R}} n(\mathbf{R})V(\mathbf{R}) = 0, \\
 U(1)_k \times U(1)_l \times U(1)_m\text{-anomaly} &: \frac{1}{6} \sum_{\underline{q}} n(\underline{q})q_{(m}q_nq_{k)} = \frac{1}{4}b_{(mn}^\alpha \Theta_{k)\alpha}, \\
 U(1)_m\text{-non-Abelian anomaly} &: \frac{1}{2} \sum_{\mathbf{R}} \sum_{\underline{q}} n(\mathbf{R}_{\underline{q}})U(\mathbf{R})q_m = \frac{1}{4\lambda}b^\alpha \Theta_{\alpha m}, \\
 U(1)_m\text{-gravitational anomaly} &: \frac{1}{48} \sum_{\underline{q}} n(\underline{q})q_m = -\frac{1}{16}a^\alpha \Theta_{m\alpha}. \tag{5.22}
 \end{aligned}$$

Here $V(\mathbf{R})$ and $U(\mathbf{R})$ denote group theoretical constants that arise when rewriting traces in the representation \mathbf{R} as traces in the fundamental representation \mathbf{f} . Letting F denote

the non-Abelian field strength, we set

$$\mathrm{tr}_{\mathbf{R}} F^3 = V(\mathbf{R}) \mathrm{tr}_{\mathbf{f}} F^3, \quad \mathrm{tr}_{\mathbf{R}} F^2 = U(\mathbf{R}) \mathrm{tr}_{\mathbf{f}} F^2. \quad (5.23)$$

Similarly, we note that $\lambda = 2c_G/V(\mathbf{adj})$ with c_G the dual Coxeter number of G and \mathbf{adj} its adjoint representation. Note that $\lambda = 1$ for $G = \mathrm{SU}(N)$, the case of interest here.

There are some remarks in order. The left hand sides of (5.22) are the actual one-loop triangle anomalies of the field theory. These are determined entirely by the spectrum. The right hand sided of (5.22) are the contribution of the anomalous tree-level GS-counter terms. In an F-theory compactification on a Calabi-Yau fourfold \hat{X} over a base B they are identified as follows [44, 83]

$$\Theta_{m\alpha} = \Theta_{m\alpha}^M, \quad b_{mn}^\alpha = -\pi(\sigma(\hat{s}_m) \cdot \sigma(\hat{s}_n)) \cdot \Sigma_b^\alpha, \quad b^\alpha = S_G^b \cdot \Sigma_b^\alpha, \quad a^\alpha = K^\alpha. \quad (5.24)$$

Here the CS-terms Θ_{AB}^M are defined in (4.4), the Néron-Tate height pairing has been introduced in (2.18) for two sections and can be straightforwardly generalized to an arbitrary number of sections, Σ_b^α denotes a basis of curves defined in (2.19), S_G^b is the divisor in the base B supporting the gauge symmetry G , cf. 2.23, and K^α is the coefficient (2.24) in the expansion of K_B .

5.3.2 Anomaly cancellation in 4D F-theory with a $\mathrm{U}(1)^2$ -sector: $B = \mathbb{P}^3$

The spectrum of the F-theory compactification to four dimensions on the fourfold $\hat{X} \rightarrow \mathbb{P}^3$ has been calculated in (5.14) and (5.21). The various anomalies for this spectrum read

$$\begin{aligned} A_{111}^{\mathrm{U}(1)} &: 2[a_3(n_9 - n_7 - 12) + a_5 n_7 (4 - n_7 + n_9)], \\ A_{222}^{\mathrm{U}(1)} &: n_7 (a_3 + a_5 (4 - n_9)) (4 + n_9), \\ A_{112}^{\mathrm{U}(1)} &: \frac{1}{6} [a_5 n_7 (48 + n_7^2 + n_9^2 - 2n_7 (4 + n_9)) + a_3 (n_7^2 - 2n_7 (n_9 - 8) + (n_9 - 12) (4 + n_9))], \\ A_{122}^{\mathrm{U}(1)} &: \frac{1}{6} (a_5 n_7 (n_7 - n_9 - 4) (n_9 - 12) + a_3 (-2n_7^2 + n_7 (4 + n_9) + (n_9 - 12) (4 + n_9))), \\ A_1^{\mathrm{U}(1)\text{-grav}} &: a_3 (n_9 - n_7 - 12) + a_5 n_7 (4 - n_7 + n_9), \\ A_2^{\mathrm{U}(1)\text{-grav}} &: 2n_7 (a_3 + a_5 (4 - n_9)), \end{aligned} \quad (5.25)$$

where we brought all numerical factors in (5.22) to the left hand sides. We denoted by $A_{klm}^{\mathrm{U}(1)}$ the $\mathrm{U}(1)_k \times \mathrm{U}(1)_l \times \mathrm{U}(1)_m$ - and by $A_k^{\mathrm{U}(1)\text{-grav}}$ the $\mathrm{U}(1)$ -gravitational anomalies. Clearly, there are no non-Abelian anomalies due to the absence of a non-Abelian group G , however, see section 6 for the inclusion of $G = \mathrm{SU}(5)$. Thus, since the anomalies (5.25) are all non-vanishing, a non-trivial GS-mechanism is required for consistency of the theory.

Therefore, all that is left to check cancellation to prove anomaly cancellation of these F-theory compactifications is to calculate the quantities on the right of (5.24) that encode the GS-mechanism. First, we obtain

$$\begin{aligned} \Theta_{\alpha m} &= \left(\frac{1}{4} [(-12 - n_7 + n_9)a_3 + n_7(4 - n_7 + n_9)a_5], \frac{1}{2} n_7 [a_3 + (4 - n_9)a_5] \right)_m, \\ b_{mn}^\alpha &= \begin{pmatrix} 8 & 4 - n_7 + n_9 \\ 4 - n_7 + n_9 & 8 + 2n_9 \end{pmatrix}, \quad a^\alpha = -4. \end{aligned} \quad (5.26)$$

where we evaluated the CS-terms (4.4) for the flux (5.4), computed the height pairing (2.18) and (2.24) for $K_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-4)$. We note that the index $\alpha = 1$ since the only vertical divisor is the hyperplane H_B and $m = 1, 2$ for the two rational sections \hat{s}_Q, \hat{s}_R . Equipped with the coefficients in (5.26) we finally calculate the GS-terms on the right side of (5.22), which precisely yield (5.25). Thus, we see that all anomalies are cancelled by the GS-mechanism.

We conclude note that the spectrum calculated in section 5.2 is the uniquely determined anomaly-free spectrum if only one chirality $\chi(\mathbf{1}_{(q_1, q_2)})$ is calculated independently. Anomaly cancellation is not sufficient to fix the spectrum completely, since the $U(1)_1^3$ -anomaly is proportional to the $U(1)_1$ -gravitational anomaly as is evident from (5.25).

5.4 A toric example

We conclude the analysis of F-theory compactifications with $U(1) \times U(1)$ gauge group with explicit toric constructions of the fourfolds \hat{X} with dP_2 -elliptic curve \mathcal{E} . We focus on elliptically fibered Calabi-Yau fourfolds $\hat{X} \rightarrow \mathbb{P}^3$ over \mathbb{P}^3 .

First we note that we can construct using the algorithm of appendix G toric fourfolds realizing a wide variety of values for n_7 and n_9 inside the allowed region in figure 2 of dP_2 -fibrations over \mathbb{P}^3 with generic Calabi-Yau hypersurface \hat{X} . In the following, we present the case $n_7 = 4$ and $n_9 = 5$ to illustrate key ideas of the toric construction.

The toric reflexive polytope defining the toric variety $dP_2(4, 5)$ takes the form

variable	vertices	\mathbb{C}^* -action	divisor class
z_0	1 1 1 -1 -1	1 0 0 0	H_B
z_1	-1 0 0 0 0	1 0 0 0	H_B
z_3	0 -1 0 0 0	1 0 0 0	H_B
z_2	0 0 -1 0 0	1 0 0 0	H_B
u	0 0 0 1 0	0 1 1 -1	$H - E_1 - E_2 + H_B$
v	0 0 0 0 1	0 1 0 0	$H - E_2 + H_B$
w	0 0 0 -1 -1	-1 0 1 0	$H - E_1$
e_1	0 0 0 0 -1	0 0 -1 1	E_1
e_2	0 0 0 1 1	0 -1 0 1	E_2

(5.27)

Here the first column denoted the projective coordinates on the toric variety $dP_2(4, 5)$, the next five columns are the vertices of the polytope, that are given as five-dimensional row vectors. Next we displayed the four \mathbb{C}^* -actions of $dP_2(4, 5)$ along with the divisor classes in the last column.

Comparing the divisor classes in (5.27) with the general assignments in (2.10) we immediately confirm that the above polytope indeed describes a fibration with divisors $\mathcal{S}_7 = 4H_B$ and $\mathcal{S}_9 = 5H_B$. This can also be checked explicitly by torically calculating the intersections (2.15) in the toric variety associated to the single star triangulation of the polytope specified in (5.27). The Euler number, as well as the Hodge numbers, of the fourfold \hat{X} in $dP_2(4, 5)$ read

$$\chi(\hat{X}) = 1620, \quad h^{(1,1)}(\hat{X}) = 4, (0), \quad h_V^{(2,2)}(\hat{X}) = 5, \quad h^{(3,1)}(\hat{X}) = 258, (0), \quad (5.28)$$

where the zeros in parenthesis indicate the absence of any non-toric divisors, respectively, complex structure moduli on \hat{X} as expected by construction. The Euler number $\chi(\hat{X})$ as well as the number $h_V^{(2,2)}(\hat{X})$ of independent surfaces in (5.28) agree with the general formula in (5.5), respectively, the cohomology calculations leading to (3.34). The topological metric $\eta^{(2)}$ of these five classes is read off from (3.36) with $n_7 = 4$, $n_9 = 5$.

The codimension two singularities of this concrete Calabi-Yau fourfold \hat{X} yield the full spectrum (3.3) of six different singlet representations $\mathbf{1}_{(q_1, q_2)}$. The cohomology classes of the three sections are given by the toric divisors $S_P = E_2$, $S_Q = E_1$ and $S_R = H - E_1 - E_2 + H_B$ in (5.27) and agree with the general expression (3.32). We obtain the (1, 1)-forms inducing the $U(1) \times U(1)$ -gauge fields by evaluating the Shioda map of the sections \hat{s}_Q , \hat{s}_R following (3.35) as

$$\sigma(\hat{s}_Q) = S_Q - S_P - 4H_B, \quad \sigma(\hat{s}_R) = S_R - S_P - 9H_B. \quad (5.29)$$

The Nero-Tate height pairing b_{mn}^α of these two classes follows from (5.26) with $n_7 = 4$, $n_9 = 5$.

Finally, we calculate the general G_4 -flux on \hat{X} employing the procedure of section 5.1. We precisely obtain the G_4 -flux in (5.4) in the case $n_7 = 4$, $n_9 = 5$. With this result, we readily calculate the chiralities for the matter representations, along with the CS-terms (4.4), again obtaining a perfect match with the results (5.14), (5.21) and (5.18) evaluated for $n_7 = 4$, $n_9 = 5$. Then we also calculate the GS-terms (5.26), most prominently the gaugings $\Theta_{m\alpha}$, and show, following section 5.3, that all anomalies are cancelled.

6 Calabi-Yau fourfolds with $SU(5) \times U(1) \times U(1)$

Building on the results of previous sections, the aim of this section is to develop tools for the construction of phenomenologically appealing F-theory compactifications with additional non-Abelian gauge symmetries. For concreteness we consider in this section a Calabi-Yau fourfold $\hat{X}_{SU(5)}$ with a resolved $SU(5)$ -singularity, while maintaining the rank two Mordell-Weil group. Compactifications of F-theory on $\hat{X}_{SU(5)}$ give rise to a four-dimensional GUT with $SU(5) \times U(1) \times U(1)$ gauge group.

We consider here one particular geometric way of adding and resolving the $SU(5)$ -singularity. Once this non-Abelian sector is added, it is rather straightforward to implement the techniques developed in previous sections for the rank-two Abelian symmetry. As the first step in section 6.1 we describe the resolution to $\hat{X}_{SU(5)}$, determine the matter representations as well as Yukawa points and the matter surfaces for a subset of matter multiplets. We note, that the spectrum that we obtain in this analysis has not been found via the toric classification of $SU(5)$ -TOPs in [46]. As the next step we calculate the vertical cohomology ring $H_V^{(*,*)}(\hat{X}_{SU(5)})$ in section 6.2. For a general base B we derive expressions for the Chern classes and Euler number of $\hat{X}_{SU(5)}$, before we calculate the full vertical cohomology for the fourfold with $B = \mathbb{P}^3$ explicitly.

We emphasize one finding of our analysis. With the addition of an $SU(5)$ -singularity we encounter at one codimension three locus a non-flat fiber (NFF), i.e. at this particular codimension three locus the fiber of $\hat{X}_{SU(5)}$ is no longer a complex one-dimensional elliptic

curve, but a complex two-dimensional surface \mathcal{C}_{NFF} . We note that this was also observed in [46]. The physics of non-flat fibers has been studied in [84], to which we refer for more details. In summary, the complex surface in the fibration can be wrapped by an M5-brane, that gives rise, in the blow-down of the resolved fibration of $\hat{X}_{SU(5)}$, to a charged tensionless string with an infinite tower of massless excitations. In addition, further light states are contributed by M2-branes wrapping holomorphic curves in the surface of the non-flat fiber.

Compactifications of F-theory with such a spectrum of additional light states are problematic for phenomenology. One obvious way out is to geometrically avoid the presence of the Yukawa couplings supporting the dimension two fiber. This can be achieved by considering a base B where the dangerous Yukawa coupling does not exist but all the other physical features of the compactification on $\hat{X}_{SU(5)}$ are retained. See also [45] for a toric analysis. Such bases are easily constructed and realize phenomenologically interesting F-theory compactifications [85].

Another way to deal with the non-flat fiber is to forbid at least a chiral excess of the additional light states induced by \mathcal{C}_{NFF} . This requires the G_4 -flux on $\hat{X}_{SU(5)}$ to integrate to zero over \mathcal{C}_{NFF} ,

Non-flat fiber condition: $\int_{\mathcal{C}_{NFF}} G_4 = 0,$

(6.1)

Following this approach later in section 7, we show explicitly that we can obtain a physical and consistent low-energy spectrum with all 4D field theory anomalies cancelled even in the presence of a non-flat fiber at codimension three. It would be interesting to investigate the microscopic meaning of (6.1) in more detail.

6.1 Singularities of the fibration: gauge group, matter & Yukawa Couplings

In this section we analyze the codimension one, two and three singularities and their resolutions in $\hat{X}_{SU(5)}$. We thoroughly determine all matter representations and determine some of the associated matter surfaces.

Before going into the details of this geometric analysis of $\hat{X}_{SU(5)}$ it is instructive to introduce some terminology and the following simple geometric picture. We construct $\hat{X}_{SU(5)}$ by first considering a non-generic Calabi-Yau fourfold \hat{X} with an $SU(5)$ -singularity over codimension one in B . Then we resolve all singularities and obtain the smooth $\hat{X}_{SU(5)}$. Thus, the smooth Calabi-Yau fourfold $\hat{X}_{SU(5)}$ is schematically obtained from a singular Weierstrass model X as,

$$\hat{X}_{SU(5)} \xrightarrow{\pi_{SU(5)}} \hat{X} \xrightarrow{\hat{\pi}} X. \tag{6.2}$$

Here $\pi_{SU(5)}$ blows down the four Cartan divisors of $SU(5)$ and $\hat{\pi} = \pi_1 \circ \pi_2$ is the two-step blow-up in (3.2). Thus the fourfold $\hat{X}_{SU(5)}$ is constructed via a combination of a total of six resolutions starting from the singular Weierstrass model X .

6.1.1 Explicit resolution of a codimension one $SU(5)$ -singularity

In the following we first engineer and then resolve an $SU(5)$ -singularity over codimension one in B . We introduce the section $z \in \mathcal{O}(\mathcal{S}_{SU(5)})$ that vanishes along the divisor $\mathcal{S}_{SU(5)}^b$

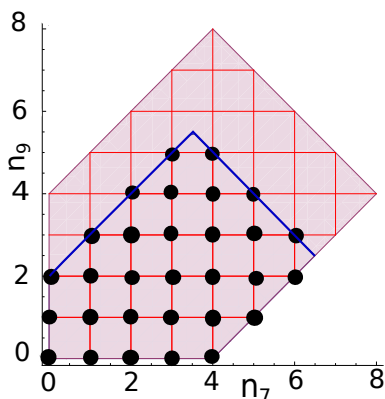


Figure 5. Each dot represents a dP_2 -fibration over \mathbb{P}^3 with generic Calabi-Yau $\hat{X}_{\text{SU}(5)}$. The red region is the set of dP_2 -fibration with generic \hat{X} without $\text{SU}(5)$, cf. figure 2.

in the base B of $\text{SU}(5)$ -singularities of the elliptic fibration of \hat{X} . Here we focus on the interplay between the non-Abelian and Abelian sector and choose one particular $\text{SU}(5)$ -singularity first constructed in [51] that has been missed in the literature before.

The $\text{SU}(5)$ -singularity of interest is obtained by considering non-generic coefficients s_i in the Calabi-Yau equation (2.1) of the form

$$s_1 = z^3 s'_1, \quad s_2 = z^2 s'_2, \quad s_3 = z^2 s'_3, \quad s_5 = z s'_5. \tag{6.3}$$

Since the s_i are still required to be sections of the bundles (2.10) by the Calabi-Yau condition, this implies that the s'_i have to be sections of the following line bundles:

section	bundle
s'_1	$\mathcal{O}(3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9 - 3\mathcal{S}_{\text{SU}(5)})$
s'_2	$\mathcal{O}(2[K_B^{-1}] - \mathcal{S}_9 - 2\mathcal{S}_{\text{SU}(5)})$
s'_3	$\mathcal{O}([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9 - 2\mathcal{S}_{\text{SU}(5)})$
s'_5	$\mathcal{O}(2[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_{\text{SU}(5)})$

We note that we assume in the following that we obtain a fourfold $\hat{X}_{\text{SU}(5)}$ that is a generic besides the resolved $\text{SU}(5)$ -singularity, i.e. that all the s_i respectively s'_i in (2.10) and (6.9) exist. As before in the $\text{U}(1)^2$ -case, this poses upper and lower bounds on the coefficients n_7^α , n_9^α in the expansion (2.25) of \mathcal{S}_7 , \mathcal{S}_9 , that depend on the base B . For the case $B = \mathbb{P}^3$ that is of most interest in this work, these bounds replace (2.27) and read, using $K_{\mathbb{P}^3}^{-1} = \mathcal{O}_{\mathbb{P}^3}(4)$ and $\mathcal{S}_{\text{SU}(5)} = H_B$,

$$0 \leq n_7 \leq 7, \quad 0 \leq n_9 \leq 6, \quad n_7 + n_9 \leq 9, \quad 0 \leq 2 + n_7 - n_9, \quad 0 \leq 4 + n_9 - n_7. \tag{6.5}$$

The allowed region is displayed in figure 5.

Next we confirm that the specialization 6.3 of the coefficients indeed gives rise to an $\text{SU}(5)$ -singularity. The discriminant takes the form

$$\Delta = -z^5 (\beta_5^4 P + z\beta_5^2 P_2 R + \mathcal{O}(z^2)) \tag{6.6}$$

where we defined

$$\beta_5 = s_6, \quad P := P_1 P_2 P_3 P_4 P_5 = (s'_2 s'_5 - s'_1 s_6) s_7 (s'_2 s_7 - s'_3 s_6) s_8 (s_6 s_9 - s_7 s_8), \quad (6.7)$$

and R is a polynomial in s_i with no common factors.

We note that the factor z^5 in (6.6) signals an A_4 -singularity over the divisor $z = 0$, as desired. In addition, we see that the singularity gets further enhanced at $\beta_5 = 0$ to D_5 and at $P = 0$ to A_5 , indicating appearance of matter multiplets at codimension two singularities. Yukawa points are located at the codimension three loci.

In order to resolve all singularities, we perform four blow-ups of the ambient space (2.7) that induce two blow-ups and two small resolutions on the Calabi-Yau fourfold. The blow-down map $\pi_{\text{SU}(5)} : \hat{X}_{\text{SU}(5)} \rightarrow \hat{X}$ of this resolution then reads [51],

$$\pi_{\text{SU}(5)} : \quad w = d_1 d_2^2 d_3^3 d_4^2 \tilde{w}, \quad v = d_1 d_2 d_3 d_4 \tilde{v}, \quad u = d_3 d_4 \tilde{u}, \quad z = d_1 d_2 d_3 d_4 \tilde{z}, \quad (6.8)$$

where we have introduced new coordinates \tilde{w} , \tilde{v} , \tilde{u} and \tilde{z} as well as the d_i which are sections of the line bundles $\mathcal{O}(D_i)$ associated with the exceptional divisors D_i for $i = 1 \dots 4$. The latter are the Cartan divisors of $\hat{X}_{\text{SU}(5)}$ and admit the fibration structure (2.23) with $\mathcal{S}_{\text{SU}(5)} = \pi(\mathcal{S}_G^b)$. In the following it will be convenient to denote the new class of \tilde{z} as D_0 and supplement the Cartan divisors as $D_I = (D_0, D_i)_I$ with $I = 0, 1, \dots, 4$. The divisor classes of all homogeneous coordinates in (6.8) are then given by

section	bundle	
\tilde{u}	$\mathcal{O}(H - E_1 - E_2 + \mathcal{S}_9 + [K_B] - D_3 - D_4)$	
\tilde{v}	$\mathcal{O}(H - E_2 + \mathcal{S}_9 - \mathcal{S}_7 - D_1 - D_2 - D_3 - D_4)$	(6.9)
\tilde{w}	$\mathcal{O}(H - E_1 - D_1 - 2D_2 - 3D_3 - 2D_4)$	
\tilde{z}	$\mathcal{O}(D_0) \equiv \mathcal{O}(\mathcal{S}_{\text{SU}(5)} - D_1 - D_2 - D_3 - D_4)$	

The total transform of the Calabi-Yau hypersurface (2.1) then reads

$$p = s_6(e_1 e_2) u v w + s_7(d_1 d_2) e_2 v^2 w + s_8(d_2 d_3^2 d_4) e_1^2 u w^2 + s_9(d_1 d_4 d_3^2 d_2^2) e_1 v w^2 + z_2 s_5(d_4 d_3) e_1^2 e_2 u^2 w + z_2^2 s_2(d_1 d_4) e_1 e_2^2 u^2 v + z_2^2 s_3(d_1^2 d_2 d_4) e_2^2 u v^2 + z_2^3 s_1(d_1 d_4^2 d_3) e_1^2 e_2^2 u^3, \quad (6.10)$$

where we dropped, by abuse of notation, all tildes of the coordinates in (6.8) and all primes of the sections in (6.9). The hypersurface (6.10) is a section of the anti-canonical bundle of the new ambient space, denoted $\widehat{dP}_2^B(\mathcal{S}_7, \mathcal{S}_9)$, after the blow-up (6.8). It reads

$$K_{\widehat{dP}_2^B}^{-1} = \pi_{\text{SU}(5)}^*(K_{dP_2^B}^{-1}) - 2D_1 - 3D_2 - 5D_3 - 4D_4, \quad (6.11)$$

where $K_{dP_2^B}^{-1}$ denotes the anti-canonical bundle (2.9) of $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ that is pulled back via the blow-down map (6.8).

There is one caveat when working with a particular resolution such as (6.8). It has been discussed in detail in [20–22], but has also been long clear in the toric context, see e.g. [41, 42, 44, 78] for recent studies, that the resolution of an $\text{SU}(5)$ -singularity over codimension one in B is not unique. The different resolutions are related by mild birational

transformations. In the context of toric geometry, this is visible as different phases of the toric ambient variety²¹, that descend to different phases on the Calabi-Yau fourfold $\hat{X}_{\text{SU}(5)}$. One way to specify these different phases is by the Stanley-Reissner ideal SR , which describes which divisors in (6.9) do not intersect. However, we can always restrict ourselves to one particular resolution in order to extract physical quantities like the matter content, Yukawa points and chiralities in F-theory. This is intuitively clear since the resolved geometry $\hat{X}_{\text{SU}(5)}$ is just a tool to analyze the geometry and not physical in F-theory. This can be argued more precisely in the dual M-theory compactification similar to the discussion in [79].

Thus, we focus here on one particular phase of the resolution (6.8) to extract the effective physics of F-theory on $\hat{X}_{\text{SU}(5)}$. Following the spirit of toric geometry, we specify the precise resolution (6.8) by stating a specific SR-ideal. Here we consider an ideal of the form

$$SR = \{ve_1, e_1e_2, uv, uw, we_2\} \cup \{zd_3, d_1d_3, d_1d_4\} \cup \{d_1e_1, ze_1, d_2e_1, d_4e_1, d_1u, d_2u, d_1e_2, d_2e_2, d_3e_2, d_4e_2, zw, d_1w, d_4w, d_3v, d_4v, zd_2v\} \cup \pi_{\text{SU}(5)}^*(SR_B). \quad (6.12)$$

We immediately recover the SR-ideal (2.5) of the dP_2 -fiber as the first set in (6.12), ensuring the intersections (2.4) of the fiber. The second set gives rise to the correct intersections of the Cartan divisors to obtain the intersection pattern of the A_4 Dynkin diagram. The third part encodes the intersections of the Cartan divisors with the sections \hat{s}_P, \hat{s}_m . The last part in (6.12) is the pull-back of the SR-ideal SR_B of the base B under the blow-down map $\pi_{\text{SU}(5)}$ in (6.8). We will comment on the structure of this ideal in more detail in section 6.2. For now we just state that it contains the SR-ideal of the base and additional intersections involving the D_i and the vertical divisors, that take care of the split of the divisor z into multiple components under the blow-up map (6.8).

Let us next discuss the structure of the resolution $\hat{X}_{\text{SU}(5)}$ at codimension one in B . The general fiber of $\hat{X}_{\text{SU}(5)}$ is the dP_2 -elliptic curve \mathcal{E} . However, over $\mathcal{S}_{\text{SU}(5)}^b$ the fiber becomes reducible. One way to see this is by noting the following intersections of the Cartan divisors D_I ,

$$D_I \cdot D_J \cdot D_\alpha \cdot D_\beta = -C_{IJ} \mathcal{S}_{\text{SU}(5)} \cdot S_P \cdot D_\alpha \cdot D_\beta \quad (6.13)$$

where D_α, D_β are arbitrary vertical divisors and C_{IJ} is the extended Cartan matrix of $\text{SU}(5)$ reading

$$(-C_{IJ}) = \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}. \quad (6.14)$$

²¹In general, there can also be phases of the toric variety and $\hat{X}_{\text{SU}(5)}$ that do not correspond to different resolutions but for example to different phases of B or of the general fiber \mathcal{E} . These might even change the intersections e.g. of the fiber \mathcal{E} from the ones considered in section 2.1 by going for instance into a phase corresponding to F_1 blown up at a point. We exclude these phases here and in particular in sections 5.4 and 7.3.

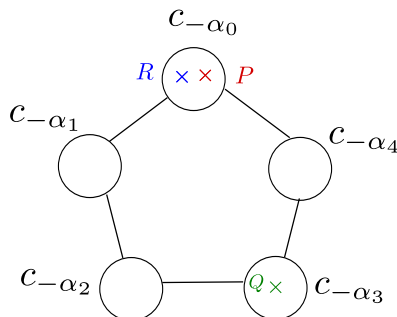


Figure 6. I_4 -fiber over $\mathcal{S}_{\text{SU}(5)}^b$ marked according to the intersections of the nodes with S_P, S_Q, S_R , corresponding to a 2 – 3-split of a spectral cover.

We note that (6.13) can be calculated via the abstract presentation (3.18) of the cohomology ring of $\hat{X}_{\text{SU}(5)}$, see section 6.2 and appendix D for details. The intersections (6.13) then immediately imply that the divisors D_I resolve a singularity of type $G = \text{SU}(5)$. Indeed, we note first that (6.13) implies a fibration structure (2.23) of the Cartan divisors $\pi : D_I \rightarrow \mathcal{S}_{\text{SU}(5)}^b$ with the rational fibers $c_{-\alpha_I}$ corresponding to the simple roots $-\alpha_i$ and the extended root $-\alpha_0 = \sum_i \alpha_i$ of $\text{SU}(5)$. The latter conclusion can be shown by representing the individual curves $c_{-\alpha_I}$ as intersections of the divisors D_i with a curve Σ_b dual to $\mathcal{S}_{\text{SU}(5)}^b$ in the base, i.e. assuming $\mathcal{S}_{\text{SU}(5)}^b \cdot \Sigma_b = 1$ we obtain

$$D_I \cdot \Sigma_b = c_{-\alpha_I}. \tag{6.15}$$

Then, we immediately confirm from (6.13) and (6.14) that the $c_{-\alpha_I}$ in fact intersect as the affine Dynkin diagram of $\text{SU}(5)$, see figure 6, justifying the assignment $c_{-\alpha_I} \leftrightarrow -\alpha_I$.

Next we turn to the behavior of the rational sections of $\hat{X}_{\text{SU}(5)}$. Generically, the sections are points in the fiber and only intersect one of its irreducible components. As we have just seen the fiber over $\mathcal{S}_{\text{SU}(5)}^b$ splits into five rational curves. The component which is pierced by the three sections \hat{s}_P, \hat{s}_Q and \hat{s}_R can be determined by calculating the intersections of their divisor classes $S_P, S_{Q,R}$ with the curves (6.15). We find that S_P and S_R intersect $c_{-\alpha_0}$, and S_Q passes through $c_{-\alpha_3}$, i.e.

$$S_P \cdot c_{-\alpha_I} = S_R \cdot c_{-\alpha_I} = (1, 0, 0, 0, 0)_I, \quad S_Q \cdot c_{-\alpha_I} = (0, 0, 0, 1, 0, 0)_I. \tag{6.16}$$

This situation is summarized in figure 6. We note that this intersection pattern of the rational sections would correspond to a 2 – 3 split in the language of spectral covers. Again, the result (6.16) can be obtained via the abstract intersection computation outlined in section 6.2 and in appendix D.

Equipped with these intersections of the sections with the nodes of the Dynkin diagram, we can calculate the Shioda maps of the sections \hat{s}_Q, \hat{s}_R . We obtain

$$\sigma(\hat{s}_Q) = S_Q - S_P - [K_B^{-1}] + \frac{1}{5}(2D_1 + 4D_2 + 6D_3 + 3D_4), \quad \sigma(\hat{s}_R) = \hat{S}_R - S_P - [K_B^{-1}] - \mathcal{S}_9. \tag{6.17}$$

Here we have used the general formula for the Shioda map in the presence of one set of Cartan divisors D_i due to a codimension one singularity of type G of the elliptic fibration,

$$\sigma(\hat{s}_m) := S_m - \tilde{S}_P - \pi(S_m \cdot \tilde{S}_P) + (\mathcal{C}^{-1})^{ij} (S_m \cdot c_{-\alpha_i}) D_j. \quad (6.18)$$

The first two terms are identical to those in the Shioda map (2.20), whereas the last term incorporates the presence of G and involves the fibral curves $c_{-\alpha_i}$ corresponding to the simple roots of G as well as the inverse of the normalized coroot matrix \mathcal{C}_{ij} of G . For the case of $SU(5)$ considered here \mathcal{C}_{ij} is identical to the Cartan matrix C_{ij} .

6.1.2 Matter: codimension two

Now we proceed to analyze the resolved codimension two singularities in $\hat{X}_{SU(5)}$. There are five loci $z = P_i = 0$, $i = 1, \dots, 5$, in (6.7) where the order of vanishing of the discriminant enhances, indicating the presence of an I_5 -singularity and of **5** representations in F-theory. In addition, there is one codimension two loci $z = \beta_5 = 0$, where the singularity enhances to type D_5 and where matter in the **10** representation is located F-theory.

We analyze in the following the splitting of the nodes of the A_4 -fiber over all these codimension two loci, determine the $U \times U(1)$ -charges (q_1, q_2) of all the non-Abelian representations along with their KK-charges, which we find to be zero. As in the $U(1) \times U(1)$ case there will be only the $SU(5)$ -singlet $\mathbf{1}_{(-1,-2)}$ with non-trivial KK-charge. We also determine which weights of the respective representations are realized as holomorphic curves and thus lie in the Mori cone of $\hat{X}_{SU(5)}$. This fixes the sign-function in (4.12) for each of these representation and allows for the field theoretic computations outlined in section 4.2, that will be used in section 7. We conclude with a determination of some matter surfaces $\mathcal{C}_{\mathbf{R}}^w$.

The analysis of the splitting of nodes at codimension two presented in this section is very similar to the ones performed earlier in the literature based on the Tate model of elliptic fibrations. Therefore, our discussion will be brief and we refer to [21, 22] for more background on the general methodology.

The **5**-representations and their $U(1) \times U(1)$ -charges

In the following we study the codimension two loci $z = P_1 = 0$ in detail. The analysis of the other loci $z = P_i = 0$ of **5**-representations is very similar. Therefore, we will only summarize the results of our computations at the end of this section.

At the loci $P_1 = (s_2 s_5 - s_1 s_6) = 0$ defined (6.7) the second node in figure of the fiber becomes reducible, as depicted figure 7. This can be observed by intersecting the locus $d_2 = P_1 = 0$ with the Calabi-Yau equation (6.10). Since we are only interested in the split of the fiber, we evaluate all coefficients s_i at generic points on B , cf. (6.15). Then, we locally obtain

$$p|_{d_2=P_1=0} = \frac{1}{s_1} (s_2 v + s_1 d_3 d_4 z) (s_5 w + s_1 d_1 d_4 z^2). \quad (6.19)$$

This means that the class of the curve $c_{-\alpha_2}$ splits into two curves c_1, c_2 when restricted to the non-generic loci $P_1 = 0$. From the perspective of the ambient space $\widehat{dP}_2^B(\mathcal{S}_7, \mathcal{S}_9)$ we can write this splitting using (6.19) as

$$c_{-\alpha_2} = c_1 + c_2, \quad [c_1] = H_B^3 \cdot [v], \quad [c_2] = H_B^3 \cdot [w], \quad (6.20)$$

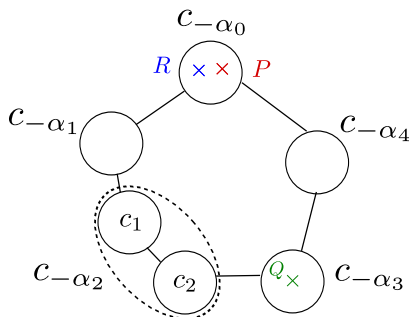


Figure 7. Splitting of the generic I_4 -fiber over $P_1 = 0$.

where we denoted the homology classes of the curve c_i as $[c_i]$ and used the shorthand notation $[v]$ for the homology class of v , cf. (6.9). In order to arrive at this result we first have to use the SR-ideal 6.12 and the fact that $H_B^4 = 0$.

The splitted curves c_1, c_2 correspond to a weight \mathbf{w} , respectively, $-\mathbf{w}$ of the $\mathbf{5}$ -representation according to the general correspondence between isolated rational curves over codimension two in B and matter in F-theory explained before (3.1). The Dynkin label²² (q_i) of a weight \mathbf{w} is calculated in general geometrically as the intersection

$$q_i = D_i \cdot c_{\mathbf{w}}, \tag{6.21}$$

which is in complete analogy with the formula (3.4) for the computation of U(1)-charges. We apply (6.21) for the curves in (6.20) to obtain the Dynkin labels of c_1 and c_2 as

$$D_i \cdot c_1 = (1, -1, 0, 0)_i, \quad D_i \cdot c_2 = (0, -1, 1, 0)_i, \tag{6.22}$$

respectively. As an immediate consistency check we recover the Dynkin label of $-\alpha_2$ as the sum of these Dynkin labels. We also note, for completeness, that in this case, none of the curves c_1, c_2 intersect the rational sections, as can be seen by calculating intersections as in (6.16). This is again expected since the original unsplit curve $c_{-\alpha_2}$ does not intersect the rational sections either, see figure 7.

The first set of Dynkin labels in (6.22) is recognized as the Dynkin label of the weight $-(\mu_5 - \alpha_1)$, where $-\mu_5$ is the highest weight with Dynkin label $\Lambda_5 = (-1, 0, 0, 0)$. Since the Cartan divisors correspond to $-\alpha_i$, however, we interpret this as $\mathbf{5}$ -representation. The second Dynkin label corresponds to the weight $(\mu_5 - \alpha_1 - \alpha_2)$ of the $\mathbf{5}$ representation. All the other weights of $\mathbf{5}$ can then be obtained by adding appropriately one of the curves $c_{1,2}$ and a number of curves $c_{-\alpha_i}$ of the simple roots $c_{-\alpha_i}$. We exemplify this here to obtain the remaining weights of the $\bar{\mathbf{5}}$,

Dynkin label	Curves	
(-1,0,0,0)	$c_1 + c_{-\alpha_1}$	(6.23)
(1,-1,0,0)	c_1	
(0,1,-1,0)	$-(c_2)$	
(0,0,1,-1)	$-(c_2 + c_{-\alpha_3})$	
(0,0,0,1)	$-(c_2 + c_{-\alpha_3} + c_{-\alpha_4})$	

²²The Dynkin label of \mathbf{w} is the vector of ‘U(1)-charges’ of \mathbf{w} under all Cartan generators.

From this table we can directly see which curves are in the Mori cone of $\hat{X}_{\text{SU}(5)}$. The two curves c_1, c_2 are effective because we explicitly know their holomorphic representation (6.19). Also by construction, the curves associated to the simple roots $c_{-\alpha_i}$ are effective and so are the sums of effective curves with positive coefficients. This implies that we can now evaluate the sign-function (4.12). It is positive for the first two representations in table 6.23, but negative for the other three curves, that are not in the Mori cone. We summarize these findings, along with the signs of the other representations in (6.27).

Next, we calculate the $U(1) \times U(1)$ -charge using the first equality in the charge formula (3.4). For the $U(1)$ associated to the S_Q section, we intersect the Shioda map $\sigma(\hat{s}_Q)$ in (6.17) with any of the curves in (6.23) of the representation. For example using the locations in figure 7 of the sections on the fiber, as well as the intersections (6.13) and (6.22) we obtain

$$q_1 = \sigma(\hat{s}_Q) \cdot (c_1 + c_{-\alpha_1}) = 0 - 0 + (-1, 0, 0, 0) \frac{1}{5} \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = -\frac{2}{5}, \quad (6.24)$$

The second $U(1)$ -charge related to the section S_R follows similarly as

$$q_2 = \sigma(\hat{s}_R) \cdot (c_1 + c_{-\alpha_1}) = 0. \quad (6.25)$$

In summary, we have found the matter representation $\mathfrak{5}_{(-\frac{2}{5}, 0)}$ is realized at the codimension two loci $P_1 = z = 0$.

A similar analysis can be performed at all other loci $z = P_i = 0, i = 2, 3, 4, 5$ of $\mathfrak{5}$ -representations. The computations are completely analogous to the ones performed above, but not very enlightening. In all cases we obtain a split of nodes similar to the one in figure 7 and obtain an I_5 -fiber. The $U(1) \times U(1)$ -charges can be straightforwardly calculated. We skip these details and just summarize all charges of $\mathfrak{5}$ representations we obtain:

Locus	$P_1 = 0$	$P_2 = 0$	$P_3 = 0$	$P_4 = 0$	$P_5 = 0$
(q_1, q_e)	$(-\frac{2}{5}, 0)$	$(-\frac{2}{5}, 1)$	$(\frac{3}{5}, 0)$	$(\frac{3}{5}, 1)$	$(-\frac{2}{5}, -1)$

(6.26)

We note that we are working in a normalization of charges which allows for fraction. We can rescale our charges as usually done in the literature in this case by a factor of 5 to obtain integral charges.

We also summarize which curves c_w of which weights are in the Mori cone of $\hat{X}_{\text{SU}(5)}$. As before we use this to determine the values of the sign-function in (4.12). We obtain the following results:

Weight	$\mathfrak{5}_{(-\frac{2}{5}, 0)}$	$\mathfrak{5}_{(-\frac{2}{5}, 1)}$	$\mathfrak{5}_{(\frac{3}{5}, 0)}$	$\mathfrak{5}_{(\frac{3}{5}, 1)}$	$\mathfrak{5}_{(-\frac{2}{5}, -1)}$
$(-1, 0, 0, 0)$	+	+	+	+	-
$(1, -1, 0, 0)$	+	+	+	+	-
$(0, 1, -1, 0)$	-	+	+	+	-
$(0, 0, 1, -1)$	-	-	-	+	-
$(0, 0, 0, 1)$	-	-	-	+	-

(6.27)

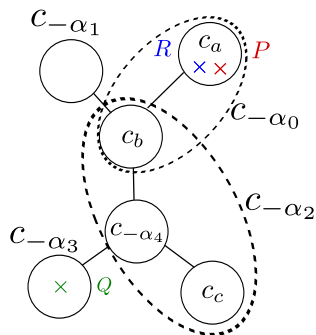


Figure 8. D_5 -curve over $z = \beta_5 = 0$.

The 10-representation and its $U(1) \times U(1)$ -charges

For completeness, we briefly discuss the fiber at the loci $z = \beta_5 = 0$. In this case more curves $c_{-\alpha_i}$ become reducible. Repeating the same procedure as above, we insert $z = \beta_5 = 0$ and $d_i = \beta_5 = 0$ into the Calabi-Yau equation (6.10), we obtain the following splitting of nodes,

Node	splits into	Cartan charges
$z = 0$	$c_a : d_4 s_8 u + d_1 d_2 d_4 s_9 v + d_1 e_2 s_7 v^2 = 0$ $c_b : d_2 = 0$	$(0, 1, 0, 0)$ $(1, -1, 0, 1)$
$d_1 = 0$	$c_{-\alpha_1} : d_2 s_8 + s_5 z_2 = 0$	$(-2, 1, 0, 0)$
$d_2 = 0$	$c_b : z_2 = 0$ $c_{-\alpha_4} : d_4 = 0$ $c_c : d_1 d_3 d_4 s_1 z_2^2 + d_1 s_2 v z_2 + d_3 s_5 w = 0$	$(1, -1, 0, -1)$ $(0, 0, 1, -2)$ $(0, -1, 0, 1)$
$d_3 = 0$	$c_{-\alpha_3} : d_2 d_4 s_3 u + d_4 e_1 s_2 u^2 + d_2 s_7 w = 0$	$(0, 1, -2, 1)$
$d_4 = 0$	$c_{-\alpha_4} : d_4 = 0$	$(0, 0, 1, -2)$

We observe that both the curves $c_{-\alpha_0}$ and $c_{-\alpha_2}$ split into two, respectively, three components, denoted c_a, c_b , respectively, $c_b, c_{-\alpha_4}$ and c_c , where we note that the curve c_a is the extended node, i.e. the original singular fiber. The fact that the curves c_b and $c_{-\alpha_4}$ appear twice means that they have multiplicity two. The intersection numbers between all the curves in (6.28) can be calculated taking into account the SR-ideal of the ambient space. The intersection pattern of the curves reproduces a D_5 -curve, see figure 8.

The curves $c_{a,b,c,d}$ carry some weights of the $\mathbf{10}$ and $\overline{\mathbf{10}}$ representations. All the other weights are obtained as before by addition of curves $c_{-\alpha_i}$. The $U(1) \times U(1)$ -charges of the $\overline{\mathbf{10}}$ representation are determined by intersections with the Shioda maps $\sigma(\hat{s}_m)$ of the rational sections. We obtain the representation

$$\mathbf{10}_{(q_1, q_2)} = \mathbf{10}_{(\frac{1}{5}, 0)}. \tag{6.29}$$

Finally, we enumerate which curves c_w of this representation are in the Mori cone of $\hat{X}_{SU(5)}$.

As before we indicate this by the values of the sign-function (4.12), which reads

Weight	Sign	Weight	Sign
(0, -1, 0, 0)	+	(-1, 0, 0, 1)	+
(-1, 1, -1, 0)	+	(0, 1, 0, -1)	-
(1, 0, -1, 0)	+	(1, -1, 0, 1)	+
(-1, 0, 1, -1)	+	(0, 1, -1, 1)	-
(1, -1, 1, -1)	+	(0, 0, 1, 0)	-

(6.30)

KK-charges

We note that at all codimension two singularities inside the divisor $\mathcal{S}_{\text{SU}(5)}$ the section S_P is holomorphic. This means for these type of matter the KK-charges vanish, $q_{KK} = 0$. The only representation with non-trivial KK-charge is, as before in section 5.2 the matter field $\mathbf{1}_{(-1,-2)}$. Its KK-charge is $q_{KK} = 2$.

Matter surfaces

We conclude the discussion of codimension two singularities by constructing the classes of certain matter surfaces $\mathcal{C}_{\mathbf{R}}^{\mathbf{w}}$. As before in section 3.1 we are able to obtain these classes only for a subset of the representations.

We first comment on the calculation of matter curves in the base. A very convenient way to calculate all matter curves $\Sigma_{\mathbf{R}}$ is by projecting the intersections $d_i^* = P_l = 0$, $l = 1, \dots, 5$ or $d_i^* = \beta_5 = 0$ to the base, where d_i^* is the coordinates associated to the Cartan divisor D_i that has split at that codimension two loci under consideration. The homology classes of the matter surfaces are then given as

$$\Sigma_{\mathbf{R}} \cong [B] \cdot [D_i^*] \cdot [P_i] = [B] \cdot [\mathcal{S}_{\text{SU}(5)}] \cdot [P_i], \quad (6.31)$$

where the divisor D_i^* has to correspond to the node that has split at the location of the matter representation \mathbf{R} . Here we have used the property $\pi(D_I) = \mathcal{S}_{\text{SU}(5)}$ in the second equality.

As in the Abelian case, we can only specify some matter surfaces completely. As before, this complication arises since we are only able to isolate the irreducible components in the elliptic fiber locally, but can not infer the total space of the fibration $\mathcal{C}_{\mathbf{R}}^{\mathbf{w}} \rightarrow \Sigma_{\mathbf{R}}$. The subset of matter surfaces, that we can explicitly determine, along with their homology classes, reads,

Rep.	Matter curve $\Sigma_{\mathbf{R}}$	Matter surface $\mathcal{C}_{\mathbf{R}}$
$\mathbf{5}_{(-\frac{2}{5}, 0)}$	$\mathcal{S}_{\text{SU}(5)} \cdot ([s_2] + [s_5])$	-
$\mathbf{5}_{(-\frac{2}{5}, 1)}$	$\mathcal{S}_{\text{SU}(5)} \cdot \mathcal{S}_7$	$([K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7 + 2H - 2D_1 - 3D_2 - 4D_3 - 3D_4) \cdot D_3 \cdot \mathcal{S}_7$
$\mathbf{5}_{(\frac{3}{5}, 0)}$	$\mathcal{S}_{\text{SU}(5)} \cdot ([s_3] + [K_B^{-1}])$	-
$\mathbf{5}_{(\frac{3}{5}, 1)}$	$\mathcal{S}_{\text{SU}(5)} \cdot [s_8]$	$(\mathcal{S}_9 + 2H - E_1 - D_1 - 2D_2 - 4D_3 - 3D_4) \cdot D_0 \cdot ([K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7)$
$\mathbf{5}_{(-\frac{2}{5}, -1)}$	$\mathcal{S}_{\text{SU}(5)} \cdot (\mathcal{S}_7 + \mathcal{S}_9)$	-
$\mathbf{10}_{(\frac{1}{5}, 0)}$	$\mathcal{S}_{\text{SU}(5)} \cdot [K_B^{-1}]$	$D_0 \cdot D_2 \cdot [K_B^{-1}]$

(6.32)

Here we made use of the SR-ideal (6.12), the assignments of sections given in (6.4), (6.9) and have abbreviated the divisors class of the s_i as $[s_i]$. In addition, we have suppressed the intersection with the class of the base B when denoting the matter curves $\Sigma_{\mathbf{R}}$ and by abuse of notation used same symbol for vertical divisors and divisor in the base.

The matter surfaces for the singlets $\mathbf{1}_{(-1,1)}$, $\mathbf{1}_{(0,2)}$ and $\mathbf{1}_{(-1,-2)}$ have been analyzed in detail in section 3.1 in the case of a $U(1) \times U(1)$ gauge group only. In the presence of an $SU(5)$ at codimension one, the matter surfaces determined in (3.7) change. Taking into account the new classes (6.4), (6.9) and the new equation of the Calabi-Yau fourfold (6.10) we obtain the matter surfaces as

Matter surface	Homology class
$\mathcal{C}_{\mathbf{1}_{(-1,1)}}$	$([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9 - 2\mathcal{S}_{SU(5)}) \cdot \mathcal{S}_7 \cdot E_1$
$\mathcal{C}_{\mathbf{1}_{(0,2)}}$	$\mathcal{S}_7 \cdot \mathcal{S}_9 \cdot ([K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7 + 2H - 2D_1 - 3D_2 - 4D_3 - 3D_4)$
$\mathcal{C}_{\mathbf{1}_{(-1,-2)}}$	$([K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7) \cdot \mathcal{S}_9 \cdot (3H - E_1 - 2E_2 + 2\mathcal{S}_9 - \mathcal{S}_7 - 2D_1 - 3D_2 - 5D_3 - 4D_4)$

(6.33)

This can be seen by specializing (6.10) to the matter curves $\Sigma_{\mathbf{1}_{(-1,1)}} = \{s'_3 = s_7 = 0\}$, $\Sigma_{\mathbf{1}_{(0,2)}} = \{s_7 = s_9 = 0\}$, respectively, $\Sigma_{\mathbf{1}_{(-1,-2)}} = \{s_8 = s_9 = 0\}$ and, then, by multiplying with the class of the isolated rational curves over each matter curves. We recall that the latter are given by $c_{(-1,1)} = \{e_1\}$ as well as the total transforms of (3.9) and (3.10).

6.1.3 Yukawa couplings: codimension three

We focus in this section on the determination of those Yukawa points that involve at least one the non-trivial representations under $SU(5)$. For an analysis of the Yukawa points of the singlets, we refer to the discussion in section 3.1.2, that applies without any changes.

Further enhancement of singularity type of the elliptic fibration can be read directly from the discriminant (6.6) of $\hat{X}_{SU(5)}$. Our strategy in determining these loci is to look for two polynomials, typically describing two matter curves, such that at their common vanishing locus the discriminant vanishes with order n greater than five, i.e. $\Delta \sim z^n$ for $n > 5$. Then we check explicitly that the fiber of the resolution $\hat{X}_{SU(5)}$ splits further. We perform this analysis in the following for all present matter representations of $\hat{X}_{SU(5)}$.

We begin with the loci of the single $\mathbf{10}$ representation. We note that here, since $\beta_5 = s_6 = 0$, the discriminant already vanishes to degree seven and takes the form

$$\Delta = -16z^7(s_5s_7)^3(s_2s_8s_7)^2 + \mathcal{O}(z^8). \tag{6.34}$$

Vanishing of any of these prefactors will enhance the zero of the discriminant to degree eight or higher. First, we analyze the vanishing $s_6 = s_5 = 0$ on the divisor $z = 0$. The polynomial P_1 vanishes but no other P_i does, so this point belongs both to the matter curve of $\mathbf{5}_{2/5,0}$ and of $\mathbf{10}$. Although the fiber does not degenerated to an E_6 -fiber the Yukawa coupling exists. The gauge invariant coupling we read off is

$$s_5 = s_6 = z = 0 : \quad \mathbf{10}_{(\frac{1}{5},0)} \times \mathbf{10}_{(\frac{1}{5},0)} \times \mathbf{5}_{(-\frac{2}{5},0)}, \tag{6.35}$$

Next, we study the vanishings of s_2 or s_8 with $s_6 = z = 0$. In this case we obtain couplings of the type $5 \times 5 \times \overline{10}$. We note that the fiber does not enhance to a D_6 -fiber, but the Yukawa is still realized geometrically.

Finally, we consider the last enhancement of the vanishing of (6.34) with $s_7 = s_6 = 0$. In this case the intersection of the Cartan divisor D_4 with the Calabi Yau hypersurface is automatically zero. Since we are now only specifying three conditions $d_4 = s_6 = s_7 = 0$ in a five-dimensional ambient space, the fiber has to be complex two-dimensional, namely the whole del Pezzo surface dP_2 . The elliptic fibration of $\hat{X}_{\text{SU}(5)}$ is non-flat, with a complex two-dimensional fiber over codimension three. We note that this effect occurs also for other $\text{SU}(5)$ -embeddings [46].

The physics of this a non-flat fibration has been discussed in [84]. As mentioned before, an M5-brane can wrap the surface in the non-flat fiber, and give rise to a string in a three-dimensional M-theory compactification. In addition, M2-branes can wrap holomorphic curves in dP_2 . In the blow-down of $\hat{X}_{\text{SU}(5)}$, and in particular in the F-theory limit, the states associated to these degrees of freedom become massless. For phenomenology these additional light states are usually undesired and can render the compactification unphysical, since we have to deal with an infinite tower of charged excitations of the string, which cannot easily be represented by a finite number of fields.

As discussed in the introduction of the section, one attempt to deal with the presence of the extra degrees of freedom from the non-flat fiber is to forbid a chiral excess of states. As we will demonstrate in section 7 this can be achieved by tuning the G_4 -flux as in (6.1), so that it integrates to zero over the non-flat fiber. Alternatively, we can simply forbid the Yukawa point with the non-flat fiber geometrically. Recalling $[s_6] = [K_B^{-1}]$, cf. (2.10), and $\pi(D_4) = \mathcal{S}_{\text{SU}(5)}$, all we have to demand is that

$$\mathcal{S}_7 \cdot [K_B^{-1}] \cdot \mathcal{S}_{\text{SU}(5)} = 0 \tag{6.36}$$

One obvious but drastic solution is $\mathcal{S}_7 = 0$ since it deprives us from some singlets and non-singlets representations. A more sophisticated solution to (6.36) is obtained as follows. We recall the basis expansion (2.25) of the divisor \mathcal{S}_7 and in addition expand

$$\mathcal{S}_{\text{SU}(5)} = n_{\text{SU}(5)}^\alpha D_\alpha. \tag{6.37}$$

Then, it is possible over an appropriate base B to tune the integers n_7^α and $n_{\text{SU}(5)}^\alpha$ in such a way to only forbid the dangerous Yukawa point (6.36) without restricting the spectrum of $\hat{X}_{\text{SU}(5)}$ and the other Yukawa points. As a concrete simple example one can choose $B = \text{Bl}\mathbb{P}^3$, the blow-up of \mathbb{P}^3 along the curve $x_i = x_j = 0$. Then (6.36) yields $n_{\text{SU}(5)}^1(4n_7^1 + 3n_7^2) + 3n_{\text{SU}(5)}^2 n_7^1 = 0$ for $D_\alpha = \{H, H - E\}$ with H the hyperplane and E the exceptional divisor and $K_{\text{Bl}\mathbb{P}^3}^{-1} = 4H - E$. Clearly, this can be solved without removing other codimension two or three singularities.

We summarize all Yukawa couplings involving the **10**-representation at $s_6 = z = 0$ in the following table,

Loci	Yukawa coupling
$s_5 = 0$	$\mathbf{5}_{(-\frac{2}{5},0)} \times \mathbf{10}_{(\frac{1}{5},0)} \times \mathbf{10}_{(\frac{1}{5},0)}$
$s_2 = 0$	$\mathbf{5}_{(-\frac{2}{5},0)} \times \mathbf{5}_{(\frac{3}{5},0)} \times \overline{\mathbf{10}}_{(-\frac{1}{5},0)}$
$s_8 = 0$	$\mathbf{5}_{(\frac{3}{5},1)} \times \mathbf{5}_{(-\frac{2}{5},-1)} \times \overline{\mathbf{10}}_{(-\frac{1}{5},0)}$
$s_7 = 0$	Non-flat fiber

(6.38)

Besides these Yukawa couplings there are additional Yukawa couplings involving singlets and the **5**-representations. All these Yukawa couplings are localized at $z = s_7 = 0$. Evaluating the discriminant at this locus we obtain the first non-vanishing terms as

$$\Delta \cong -s_3^2 s_6^5 (s_2 s_5 - s_1 s_6) s_8 s_9^2. \tag{6.39}$$

Setting each prefactor to zero we obtain Yukawa couplings, that we summarize in the following table,

Loci	Yukawa coupling
$s_3 = 0$	$\overline{\mathbf{5}}_{(\frac{2}{5},-1)} \times \mathbf{5}_{(\frac{3}{5},0)} \times \mathbf{1}_{(-1,1)}$
$(s_2 s_5 - s_1 s_6) = 0$	$\mathbf{5}_{(-\frac{2}{5},0)} \times \overline{\mathbf{5}}_{(\frac{2}{5},-1)} \times \mathbf{1}_{(0,1)}$
$s_8 = 0$	$\overline{\mathbf{5}}_{(\frac{2}{5},-1)} \times \mathbf{5}_{(\frac{3}{5},1)} \times \overline{\mathbf{1}}_{(-1,0)}$
$s_9 = 0$	$\overline{\mathbf{5}}_{(\frac{2}{5},-1)} \times \mathbf{5}_{(-\frac{2}{5},-1)} \times \mathbf{1}_{(0,2)}$

(6.40)

6.2 The cohomology ring and the Chern classes of $\hat{X}_{\text{SU}(5)}$

In this section we apply the techniques described in section 3.2 to perform computations in the vertical cohomology $H_V^{(*,*)}(\hat{X}_{\text{SU}(5)})$ of the Calabi-Yau fourfold $\hat{X}_{\text{SU}(5)}$ with a resolved $\text{SU}(5)$ -singularity over $\mathcal{S}_{\text{SU}(5)}$. The computations are completely analogous to the ones performed to obtain the cohomology of \hat{X} . All we have to do is to add the Cartan divisors D_i , $i = 1, \dots, 4$, as additional variables to the polynomial ring R in (3.18). We note that the extended node $D_0 = \mathcal{S}_{\text{SU}(5)} - \sum_i D_i$ corresponding to z is then automatically included. We also have to replace the ideal SR in (3.18) by the ideal (6.12) arising after the resolution process (6.8). Finally, we have to employ the anti-canonical bundle (6.11), which is the class of $\hat{X}_{\text{SU}(5)}$ in the blown-up ambient space $d\widehat{P}_2^B(\mathcal{S}_7, \mathcal{S}_9)$. We summarize the presentation of the vertical cohomology ring of $\hat{X}_{\text{SU}(5)}$ as

$$H_V^{(*,*)}(\hat{X}_{\text{SU}(5)}) \cong \frac{\mathbb{C}[D_\alpha, \mathcal{S}_{\text{SU}(5)}, S_P, S_Q, S_R, D_i] \cdot [\hat{X}_{\text{SU}(5)}]}{SR}, \tag{6.41}$$

with the ideal SR given in (6.12). Here we have singled out the vertical divisor $\mathcal{S}_{\text{SU}(5)}$ of the original $\text{SU}(5)$ -singularity.

We begin by using these techniques to calculate the Euler number and Chern classes of $\hat{X}_{\text{SU}(5)}$ for a general base B . Then we construct the full cohomology in the case of $B = \mathbb{P}^3$. In the following discussion we will quote the main results from this analysis and refer to

appendix D for more details on the calculation, in particular the full quartic intersections of $\hat{X}_{\text{SU}(5)}$ from which all the following is derived.

6.2.1 Second Chern classes and Euler number of $\hat{X}_{\text{SU}(5)}$: general formulas

The strategy for the computation of the Euler number is a mixture of the strategy in section 3.2 and the approach of [21] to compute Chern classes of fourfolds with a resolved SU(5)-singularity. The key point is to find a SR-ideal that contains sufficiently fine information to do concrete calculations, but that is sufficiently coarse in order to not depend on the details of B . The starting point is the SR-ideal (6.12). The only degree of freedom here is the pullback $\pi_{\text{SU}(5)}(SR_B)$ of the SR-ideal of the base. Instead of using the full ideal that depends on the details of B we again work with a simplified version denoted SR'_B that is based on some universal geometric properties of the fibration of $\hat{X}_{\text{SU}(5)}$.

As in (3.20) we assume that SR'_B contains all quartic intersections of the vertical divisors D_α . In addition we assume as in [21] that three vertical D_α never intersect the Cartan divisors D_i due to a too large codimension in the base. Thus we use the ideal

$$SR'_B = \{D_\alpha \cdot D_\beta \cdot D_\gamma \cdot D_\delta, D_\alpha \cdot D_\beta \cdot D_\gamma \cdot D_i\} \tag{6.42}$$

instead of $\pi_{\text{SU}(5)}(SR_B)$ in (6.12). As we demonstrate next this is sufficient to calculate the Chern classes and Euler number as well as to check basic intersections of $\hat{X}_{\text{SU}(5)}$, as for the fourfold \hat{X} considered in section 3.2.

Next we calculate the total Chern class of $\hat{X}_{\text{SU}(5)}$. All we need is the Chern class of the blown-up ambient space $\widehat{dP}_2^B(\mathcal{S}_7, \mathcal{S}_9)$ and to apply adjunction. The computation of the Chern class of the ambient space is straightforward, but lengthy and little illuminating. We simply use the result (C.2) and refer to appendix C for details. Then we use adjunction as in (3.21) to obtain the total Chern class of $\hat{X}_{\text{SU}(5)}$. We present here the results of the second Chern class $c_2(\hat{X}_{\text{SU}(5)})$ that reads

$$c_2(\hat{X}_{\text{SU}(5)}) = c_2(\hat{X}) - c_1(4D_1 + 6D_2 + 2D_3 + D_4) + 2D_1D_2 + D_2^2 + 2D_3D_4 + \mathcal{S}_{\text{SU}(5)}(D_1 + D_2 + D_4 - 2\mathcal{S}_Q) + \mathcal{S}_7(D_1 + 2D_2 - D_4) + \mathcal{S}_9(D_1 + D_2 + D_4), \tag{6.43}$$

where $c_2(\hat{X})$ denotes the second Chern class (3.22) on \hat{X} . As a sanity check we take the blow-down limit by formally setting $D_i \rightarrow 0$ and $\mathcal{S}_{\text{SU}(5)} \rightarrow 0$ in which we precisely recover the Chern class (3.22) before resolution.

Next we calculate the Euler number on $\hat{X}_{\text{SU}(5)}$. For this purpose we can either bootstrap ourselves and use (6.43) in combination with the relation (3.28) or calculate the fourth Chern class on $\hat{X}_{\text{SU}(5)}$ directly by expanding (C.2) to higher order in the adjunction formula expression for the Chern class for $\hat{X}_{\text{SU}(5)}$. In either case we obtain

$$\chi(\hat{X}_{\text{SU}(5)}) = \chi(\hat{X}) - 3 \int_B \mathcal{S}_{\text{SU}(5)}(52c_1^2 - c_1(41\mathcal{S}_{\text{SU}(5)} + 17\mathcal{S}_7 + 31\mathcal{S}_9) + 10\mathcal{S}_{\text{SU}(5)}^2 + \mathcal{S}_{\text{SU}(5)}(5\mathcal{S}_7 + 13\mathcal{S}_9) + 5\mathcal{S}_7^2 + 2\mathcal{S}_7\mathcal{S}_9 + 5\mathcal{S}_9^2), \tag{6.44}$$

where $\chi(\hat{X})$ denotes the Euler number (3.24) of \hat{X} before the blow-up, to which our result (6.44) specializes correctly in the limit $D_i, \mathcal{S}_{\text{SU}(5)} \rightarrow 0$.

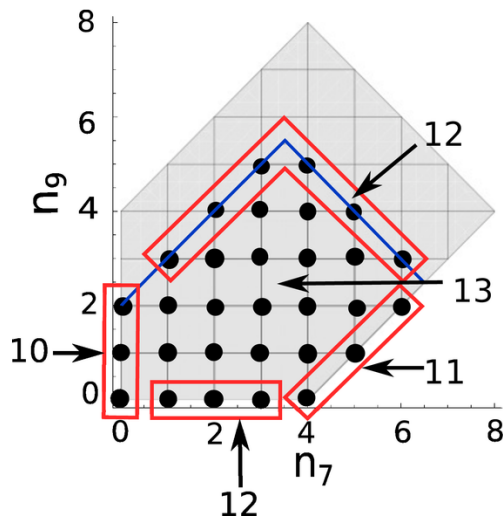


Figure 9. Number of independent surfaces $h_V^{(2,2)}(\hat{X}_{\text{SU}(5)})$ for the various values of allowed n_7, n_9 . On the blue boundary and at $n_7 = n_9 = 0$ we have a holomorphic zero section.

The full cohomology ring of $\hat{X}_{\text{SU}(5)}$: $B = \mathbb{P}^3$

Next we compute the full vertical cohomology of $\hat{X}_{\text{SU}(5)}$ with base $B = \mathbb{P}^3$. In this case, there is only one divisor in the base, the hyperplane H_B , thus, we have $D_\alpha = H_B$. Furthermore, the resolved $\text{SU}(5)$ -singularity of $\hat{X}_{\text{SU}(5)}$ is located over $\mathcal{S}_{\text{SU}(5)}^b = H_B$.

The Hodge numbers are calculated to be

$$h^{(1,1)}(\hat{X}_{\text{SU}(5)}) = h^{(3,3)}(\hat{X}_{\text{SU}(5)}) = 8, \quad h_V^{(2,2)}(\hat{X}_{\text{SU}(5)}) \in [10, 13], \quad (6.45)$$

where the number of independent surfaces as in the Abelian case depends on the values of $n_7, n_9 = 0$, see figure 9. As the basis of $H^{(1,1)}$, we use the following divisors

$$H^{(1,1)}(\hat{X}) = \langle H_B, S_P, S_Q, S_R, D_i \rangle, \quad (6.46)$$

that fixes by Poincaré duality the basis of $H_V^{(3,3)}(\hat{X}_{\text{SU}(5)})$. All quartic intersections of the divisors (6.46) can be found in (D.1).

Then we compute the grade two piece $R^{(2)}$ of the ring (6.41). We obtain 64 different monomials in two divisors, but due to equivalence relations in (6.41) at most 13 of them are independent, with jumps along the boundary of the allowed region of n_7, n_9 in figure 9. A carefully chosen basis of $H_V^{(2,2)}(\hat{X}_{\text{SU}(5)})$, thus, has to have always the correct number of surfaces for the entire allowed region. The analysis of appendix E proves that one convenient basis is given by

$$H_V^{(2,2)}(\hat{X}_{\text{SU}(5)}) = \langle H_B^2, H_B \cdot S_P, H_B \cdot \sigma(\hat{s}_Q), H_B \cdot \sigma(\hat{s}_R), S_P \cdot \sigma(\hat{s}_R), \sigma(\hat{s}_Q) \cdot \sigma(\hat{s}_R), \sigma(\hat{s}_Q)^2, H_B \cdot D_1, H_B \cdot D_2, H_B \cdot D_3, H_B \cdot D_4, D_2 \cdot D_4, D_1^2 \rangle. \quad (6.47)$$

At the boundaries of figure 9, one or more elements in $H_V^{(2,2)}(\hat{X}_{\text{SU}(5)})$ become linearly dependent. Thus, they can be expressed in terms of a smaller basis that one obtains

by dropping the linearly dependent elements from (6.47). The expansions of the linearly dependent elements in terms of this reduced basis for the various boundary components of figure 9 can be found in appendix E. The result of this analysis is that we can obtain a basis on all boundaries from the basis 6.47 by successively dropping the elements $S_P \cdot \sigma(\hat{s}_R)$, $H_B \cdot \sigma(\hat{s}_Q)$ and $D_2 \cdot D_4$. This way we obtain, as required, in steps twelve, eleven and ten independent surfaces on the relevant boundary components.

7 G_4 -flux & chiralities on fourfolds with $SU(5) \times U(1)^2$

With the cohomology ring at hand we finally calculate the general G_4 -flux, see section 7.1, and 4D chiralities in this section. As before, the lack of knowledge of all the matter surfaces forces us to employ the 3D Chern-Simons terms and M-/F-theory duality to determine the remaining chiralities, cf. section 7.2. It turns out again, that the Chern-Simons terms by themselves are sufficient to determine all the chiralities. The chiralities determined by geometric techniques for a subset of matter therefore provide an independent check. We also check that for the obtained spectrum all four-dimensional anomalies are cancelled. Full explicit results are again presented for the base $B = \mathbb{P}^3$. We also present one concrete toric examples in section 7.3.

There is one accompanying appendix F with further details on the G_4 -flux and the full expressions for the 4D chiralities.

We recall that there can be non-flat fibers at codimension three in $\hat{X}_{SU(5)}$. As mentioned before, these can be avoided for general bases B , but are generically present for $B = \mathbb{P}^3$.²³ We deal with these complications here by forbidding a chiral excess of the additional light degrees of freedom associated to the non-flat fiber. This is achieved by requiring one additional condition on the G_4 -flux, namely the vanishing of the integral (6.1). As we demonstrate here, this ensures that the 4D chiralities can successfully be determined both via geometric techniques and M-/F-theory Chern-Simons terms, and that the resulting chiral spectrum cancels all anomalies of the four-dimensional quantum field theory.

7.1 G_4 -flux on fourfolds with two rational sections & $SU(5)$

The construction of G_4 -flux presented in this section is very similar to the one of section 5. Therefore, we will keep the following discussion as brief as possible and choose, as before, the base $B = \mathbb{P}^3$ to demonstrate our techniques.

We begin by expanding the G_4 -flux in the generically 13-dimensional basis (6.47) of the cohomology group $H_V^{(2,2)}(\hat{X}_{SU(5)})$, see (4.3). We note that the dimensionality of $H_V^{(2,2)}(\hat{X}_{SU(5)})$ jumps at the boundaries of figure 9, however, our general formulas for the G_4 -flux derived in this section will remain valid. Then we enforce the generically six independent flux-conditions (4.10). Again we emphasize, that unlike in compactifications with a holomorphic zero section, the CS-terms $\Theta_{00}^M, \Theta_{0i}^M$ must not be required to vanish, nor vanish automatically.

²³As mentioned above, we can forbid these points by setting $\mathcal{S}_7 = 0$, however, at the cost of losing many representations of the general spectrum obtained from $\hat{X}_{SU(5)}$, too.

Since we are imposing generically six independent conditions 4.10 on the 13-parameter G_4 -flux, we obtain a seven-dimensional G_4 -flux,

$$G_4 = a_3 G_4^{(3)} + a_4 G_4^{(4)} + a_5 G_4^{(5)} + a_6 G_4^{(6)} + a_7 G_4^{(7)} + a_{12} G_4^{(12)} + a_{13} G_4^{(13)}, \quad (7.1)$$

where the a_i denote the free parameters of the G_4 -flux. The independent fluxes $G_4^{(i)}$ read

$$\begin{aligned} G_4^{(3)} &= H_B \cdot \sigma(\hat{s}_Q), & G_4^{(4)} &= H_B \cdot \sigma(\hat{s}_R), & G_4^{(12)} &= D_4 \cdot (D_2 - 4H_B), \\ G_4^{(5)} &= -2H_B^2 n_9 (4 - n_7 + n_9) + S_P \cdot \sigma(\hat{s}_R), \\ G_4^{(6)} &= \frac{1}{25} [4n_7 D_1 \cdot H_B + 8n_7 D_2 \cdot H_B + 12n_7 D_3 \cdot H_B + 6D_4 n_7 \cdot H_B \\ &\quad + 50 \cdot H_B^2 (4 - n_7 + n_9)(2 + n_9) + 25H_B \cdot S_P (4 - n_7 + n_9) + 25\sigma(\hat{s}_Q) \cdot \sigma(\hat{s}_R)], \\ G_4^{(7)} &= \frac{1}{125} \left[-2D_3 \cdot H_B (282 + 31n_7 - 35n_9) - 2D_1 \cdot H_B (144 + 7n_7 - 15n_9) \right. \\ &\quad - 4D_2 \cdot H_B (144 + 7n_7 - 15n_9) + D_4 \cdot H_B (-282 - 31n_7 + 35n_9) \\ &\quad \left. + 125 \cdot H_B^2 \left(\frac{136}{5} + n_9 (4 - n_7 + n_9) \right) + 850 \cdot H_B \cdot S_P + 125\sigma(\hat{s}_Q)^2 \right], \\ G_4^{(13)} &= \frac{1}{5} [5D_1^2 + 18D_1 \cdot H_B - 24D_2 \cdot H_B - 16D_3 \cdot H_B - 8D_4 \cdot H_B + 40 \cdot H_B^2 \\ &\quad + 3n_9 D_1 \cdot H_B + 6n_9 D_2 \cdot H_B + 4n_9 D_3 \cdot H_B + 2n_9 D_4 \cdot H_B + 10H_B \cdot S_P] \end{aligned} \quad (7.2)$$

The general G_4 -flux (7.1) specializes correctly at the boundaries of figure 9 and the concrete expressions for it can be obtained at every boundary component, completely analogous to the discussion of section 5.1. The necessary analysis of the behavior of the basis (6.47) along with the homology relations between linearly dependent elements on the respective boundaries can be found in appendix E.

7.2 4D chiralities from matter surfaces & 3D CS-terms

In this section we finally calculate the chiralities of the total matter content of the F-theory compactification on $\hat{X}_{\text{SU}(5)}$, that we have determined in section 6.1. We recall that there are six singlets, five different **5**-representations and one **10**, cf. section 6.1. After having determined these chiralities, we check cancellation of 4D anomalies at the end of this section. We find that all anomalies are cancelled in general.

As mentioned before, all chiralities can in principle be calculated from the G_4 -flux in (7.1) and the matter surfaces by evaluating the index (5.12). However, as in the discussion of section (5.2), we only have the explicit homology classes of the six matter surfaces (3.7) and (6.32). Again, we can obtain all chiralities by combining this information with the matching of 3D Chern-Simons terms as outline in section 5.2.1. We first calculate the chiralities from the six matter surfaces for the general G_4 -flux (7.1) and then impose the non-flat fiber condition (6.1). Only then it is possible to obtain the remaining chiralities from the matching of 3D CS-terms.

The chiralities for the six representations whose surfaces $\mathcal{C}_{\mathbf{R}}^{\mathbf{w}}$ we know, cf. (6.33) and (6.32), are calculated for the general G_4 -flux (7.1) employing (5.12) as

$$\begin{aligned}
 \chi(\mathbf{5}_{(-\frac{2}{5},1)}) &= -\frac{1}{500}n_7[-50a_3 + 125a_4 + 5(21n_7 - 5(8 + 3n_9))a_6 \\
 &\quad + (158 - 111n_7 + 115n_9)a_7 + 500a_{12} + 50(n_9 - 4)a_{13}], \\
 \chi(\mathbf{5}_{(\frac{3}{5},0)}) &= -\frac{1}{500}(4 - n_7 + n_9)[75a_3 + 125a_4 + 5(21n_7 - 65n_9 - 160)a_6 \\
 &\quad + (14n_7 - 135n_9 - 742)a_7 + 50(n_9 - 24)a_{13}], \\
 \chi(\mathbf{10}_{(\frac{1}{5},0)}) &= \frac{1}{125}[-25a_3 - 5(2n_7 - 5(4 + n_9))a_6 + (134 + 7n_7 + 20n_9)a_7 \\
 &\quad + 125n_7a_{12} + 50(-27 + 5n_7 - 2n_9)a_{13}], \\
 \chi(\mathbf{1}_{(-1,1)}) &= -\frac{1}{20}n_7(2 + n_7 - n_9)[-5a_3 + 5a_4 + (5n_7 - 4)a_6 + (58 - 10n_7 + 5n_9)a_7], \\
 \chi(\mathbf{1}_{(0,2)}) &= \frac{1}{100}n_7n_9[-50a_4 + 50(n_7 - n_9 - 4)a_5 + 5(64 - 15n_7 + 15n_9)a_6 + 162a_7 + 50a_{13}], \\
 \chi(\mathbf{1}_{(-1,-2)}) &= -\frac{1}{20}(4 - n_7 + n_9)n_9[-5a_3 - 10a_4 + (10n_7 - 20n_9)a_5 \\
 &\quad + (60 - 15n_7a_6 + 25n_9)a_6 + (54 - 5n_7 + 10n_9)a_7 + 10a_{13}], \tag{7.3}
 \end{aligned}$$

We then obtain the other chiralities by using the identification (4.5) of 3D CS-levels for $\Theta_{\Sigma\Lambda}$. Both the CS-levels $\Theta_{\Sigma\Lambda}^M$ on the M-theory side and the loop-generated CS-levels $\Theta_{\Sigma\Lambda}^F$ on the F-theory side are readily computed. All we have to know for the computation of the CS-terms $\Theta_{\Sigma\Lambda}^F$ is the KK-charge of the matter representations. We recall from section 6.1 that only the singlet $\mathbf{1}_{(-1,-2)}$ has a non-trivial KK-charge $q_{KK} = 2$ and thus a shifted sign-function (4.17) with $k = -2$. All the other matter representations have $q_{KK} = 0$ and a symmetric sign-function. Then, all the computations are straightforward but lengthy. Thus, we refer the interested reader to appendix F, where the results of these calculations are presented.

We make a short stop to show some interesting results/checks of the CS-levels and the discussion in section 4.2.2. First, we calculate the CS-term Θ_{00} for the KK-vector. On the F-theory side, we have to evaluate the loop-corrections of KK-states as in (4.18). Since only the singlet $\mathbf{1}_{(-1,-2)}$ has a non-trivial KK-charge, we again obtain the result (5.16). On the M-theory side, Θ_{00}^M is calculated directly from G_4 -flux and we obtain

$$\begin{aligned}
 \Theta_{00}^M &= -\frac{1}{20}(4 - n_7 + n_9)n_9[-5a_3 - 10a_4 + (10n_7 - 20n_9)a_5 \\
 &\quad + (60 - 15n_7a_6 + 25n_9)a_6 + (54 - 5n_7 + 10n_9)a_7 + 10a_{13}] \\
 &= \chi(\mathbf{1}_{(-1,-2)}), \tag{7.4}
 \end{aligned}$$

where we used (7.3) in the last equality. Thus, we confirm the result (5.16).

Similarly, we check the matching of the CS-levels Θ_{0m} . As before, a matching of the M- and F-theory expression for these CS-levels requires the relation (5.17) to hold,

$$\frac{1}{4} \int_{\hat{X}} S_P \cdot \sigma(\hat{s}_Q) \cdot G_4 \stackrel{!}{=} -\chi(\mathbf{1}_{(-1,-2)}), \quad \frac{1}{4} \int_{\hat{X}} S_P \cdot \sigma(\hat{s}_R) \cdot G_4 \stackrel{!}{=} -2\chi(\mathbf{1}_{(-1,-2)}). \tag{7.5}$$

Indeed, we evaluate the flux integral on the left to confirm this equality.

Finally, we comment on the CS-terms Θ_{0i} . From the direct computation of the relevant integrals of the G_4 -flux, cf. (4.4), we obtain $\Theta_{0i}^M = 0$, since $S_P \cdot D_i = 0$ in the homology of $\tilde{X}_{\text{SU}(5)}$, cf. (D.5) in appendix D. This immediately agrees with the field theory result Θ_{0i}^F in (4.8) since there the only charged matter states in 3D with non-trivial KK-charge is the singlet $\mathbf{1}_{(-1,-2)}$, which however has $q_i = 0$ for all i .

Coming back to the calculation of chiralities, we recall that for $n_7 > 0$ there is a non-flat fiber, since the codimension three point $s_6 = s_7 = d_4 = 0$ exists, cf. section 6.1.3. If we ignore this fact and try to solve for the chiralities using the matching condition 4.5 we find

$$\begin{aligned} \Theta_{i=2,j=4}^F &= \Theta_{i=2,j=4}^M, & \Theta_{i=4,m=1}^F &= \Theta_{i=4,m=1}^M, & \Theta_{i=4,m=2}^F &= \Theta_{i=4,m=2}^M, \\ 0 &= \frac{2}{5}n_7a_{12}, & 0 &= 2n_7a_{12}, & 0 &= n_7a_{12}, \end{aligned} \tag{7.6}$$

where the left hand sides are functions of chiralities. These equations are obviously inconsistent on the F-theory side. However, this is not surprising since the existence of a non-flat fiber implies the presence of additional light states, that we have not taken into account on the field theory side. Since these states are probably unwanted in setups aimed at the construction of phenomenologically appealing F-theory compactifications, we require the absence of a *chiral excess* of these light states. As we see from (7.6) the appropriate condition to impose is to demand the vanishing of the second line.

It is satisfying to see that the vanishing of this second line is precisely the non-flat fiber condition (6.1) on the G_4 -flux. Indeed, using the general G_4 -flux (7.1) and the homology class of the loci of the non-flat fiber on the left of equation (6.36), we obtain

$$\int_{X_5} G_4 D_4 \cdot \mathcal{S}_7 \cdot [s_6] = -8n_7a_{12} \stackrel{!}{=} 0, \tag{7.7}$$

which implies exactly a vanishing of the second line in (7.6). Thus, we see that we obtain a consistent set of equations in (7.6) if we impose the non-flat fiber condition (6.1), as claimed.

In summary, for $n_7 = 0$ the vanishing of (7.7) does not impose any additional conditions on the G_4 -flux, whereas for $n_7 > 0$ its vanishing implies the additional condition

$$a_{12} \stackrel{!}{=} 0. \tag{7.8}$$

Thus, the number of free parameters in the G_4 -flux in (7.1) is reduced to six. Furthermore, as we demonstrate in the following imposing the condition (7.7) will yield a consistent 4D chiral spectrum with all anomalies cancelled. For a better overview over the conditions we impose, we summarize the number of independent elements in the G_4 -flux graphically in figure 10.

Imposing now the condition (7.8) we solve the first line of (7.6). The solutions are straightforwardly obtained, but lengthy and can be found in appendix F. However, instead of parametrizing the chiralities in the parameters a_i of the G_4 -flux in (7.1), we can alternatively express the chiralities in terms of the chiralities of the **5**-representations and one singlet. This will, on the one hand, reduce the length of the expressions we obtain and, on

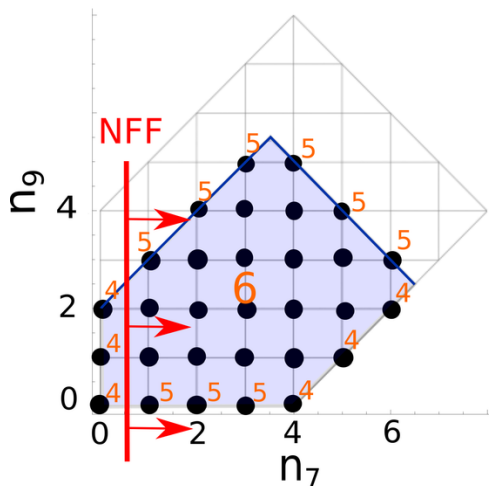


Figure 10. There are six independent constraints on the G_4 -flux from the CS-terms. In order to obtain an anomaly free spectrum, for $n_7 > 0$, we force the vanishing condition (7.7), respectively, (7.8). The small number in orange next to each point is the final number of parameters in G_4 -flux. The maximal number is six in the bulk.

the other hand, guarantee that all chiralities are parametrized in terms of integers.²⁴ We then obtain the following chiralities,

$$\begin{aligned} \chi(\mathbf{5}_{(-\frac{2}{5},0)}) &= c_1, & \chi(\mathbf{5}_{(-\frac{2}{5},1)}) &= c_2, & \chi(\mathbf{5}_{(\frac{3}{5},0)}) &= c_3, \\ \chi(\mathbf{5}_{(\frac{3}{5},1)}) &= c_4, & \chi(\mathbf{5}_{(-\frac{2}{5},-1)}) &= c_5, & \chi(\mathbf{1}_{(-1,-2)}) &= c_6, \end{aligned} \quad (7.9)$$

with integral parameters c_i in terms of which the other chiralities are expressed as

$$\begin{aligned} \chi(\mathbf{10}_{(\frac{1}{5},0)}) &= -c_1 - c_2 - c_3 - c_4 - c_5, & \chi(\mathbf{1}_{(0,2)}) &= c_6 + (c_2 + c_4 - c_5)n_9, \\ \chi(\mathbf{1}_{(1,0)}) &= -22c_2 - c_3 - 3(2(c_4 + c_5) + 3c_6) + 2(c_2 + c_4 - c_5)n_7 + 2c_1(n_9 - 6) - 2(c_4 - 2c_5)n_9, \\ \chi(\mathbf{1}_{(1,1)}) &= -14c_5 + 3c_6 + 2c_5n_7 - (c_2 + 3c_5)n_9 + c_4(5 - n_7 + n_9) + c_1(n_7 - 2(4 + n_9)), \\ \chi(\mathbf{1}_{(0,1)}) &= 14c_4 - 5c_5 - 2c_6 + c_2(21 - 2n_7) - 2c_5n_7 - 2c_4n_9 + 4c_5n_9 + c_1(8 - 2n_7 + 2n_9), \\ \chi(\mathbf{1}_{(-1,1)}) &= -2c_2 - c_6 + c_1n_7 + 2c_2n_7 + c_4n_7 - (c_2 + c_4 - c_5)n_9. \end{aligned} \quad (7.10)$$

We note that the chirality of the representation $\mathbf{10}_{(\frac{1}{5},0)}$ has the opposite sign of the c_i . For positive c_i this would yield a negative chirality of left-handed Weyl fermions in $\mathbf{10}_{(\frac{1}{5},0)}$, which has to be understood as left-handed Weyl fermions in the representation $\overline{\mathbf{10}}_{(-\frac{1}{5},0)}$. We adapt this convention in the following.

7.2.1 Anomaly cancellation

We conclude this general discussion of the spectrum by checking the cancellation of 4D anomalies. For a short review, see section 5.3, or see [44] for more details.

²⁴As mentioned before, the determination of the integral basis of $H_V^{(2,2)}(\hat{X}_{\text{SU}(5)})$ is a subtle and difficult problem, that requires the application of more advanced techniques like mirror symmetry.

Having derived the spectrum in the previous paragraph, all we have to calculate to check anomaly cancellation are the gaugings $\Theta_{\alpha m}$ for $m = 1, m = 2$, cf. (5.22). The results of this computations in terms of the parameters a_i of the G_4 -flux can be found in appendix F. As mentioned before, we can alternatively use the chiralities (7.9) as parameters. Then we obtain

$$\Theta_{\alpha, m=1} = -2(c_1 + c_2 + c_5), \quad \Theta_{\alpha, m=2} = 2(c_2 + c_4 - c_5). \quad (7.11)$$

The coefficients b_{mn}^α of the GS-counter terms change slightly compared to the $U(1) \times U(1)$ -case of section 5.3 because of the altered Shioda map (6.17), with and a^α remaining the same. We obtain

$$b_{mn}^\alpha = \begin{pmatrix} \frac{34}{5} & 4 - n_7 + n_9 \\ 4 - n_7 + n_9 & 8 + 2n_9 \end{pmatrix}, \quad a^\alpha = -4. \quad (7.12)$$

We readily check that all anomalies in (5.22) cancel beautifully. First, the purely non-Abelian anomaly vanishes trivially

$$\chi(\overline{\mathbf{10}}_{(-\frac{1}{5}, 0)}) = \sum_{(q_1, q_2)} \chi(\mathbf{5}_{(q_1, q_2)}), \quad (7.13)$$

where the sum is over all different $\mathbf{5}$ -representations, that are labeled by their charges (q_1, q_2) , and where we used the representation $\overline{\mathbf{10}}_{(-\frac{1}{5}, 0)}$ as explained below (7.10). The purely Abelian anomalies are

$$\begin{aligned} A_{1,1,1}^{U(1)} &: -\frac{17}{5}(c_1 + c_2 + c_5), \\ A_{1,1,2}^{U(1)} &: \frac{1}{15}(17c_4 - 37c_5 + 5c_5n_7 + c_2(5n_7 - 5n_9 - 3) + 5c_1(n_7 - n_9 - 4) - 5c_5n_9), \\ A_{1,2,2}^{U(1)} &: \frac{1}{3}(-8c_5 - c_2n_7 + c_5n_7 - 2c_5n_9 - c_1(4 + n_9) + c_4(4 - n_7 + n_9)), \\ A_{2,2,2}^{U(1)} &: (c_2 + c_4 - c_5)(4 + n_9), \end{aligned} \quad (7.14)$$

The mixed non-Abelian-Abelian, and Abelian-gravitational anomalies read

$$A_1^{U(1)\text{-SU}(5)} = A_1^{U(1)\text{-grav}} : \frac{1}{2}(-c_1 - c_2 - c_5), \quad A_2^{U(1)\text{-SU}(5)} = A_2^{U(1)\text{-grav}} : \frac{1}{2}(c_2 + c_4 - c_5). \quad (7.15)$$

Using the gaugings (7.11) and the coefficients (7.12) of the GS-terms, we confirm that the 4D anomaly cancellation conditions (5.22) hold.

7.3 A toric example

In this concluding subsection we construct explicitly a toric model with all characteristics described in this section, i.e. an $SU(5) \times U(1)^2$ 4D gauge group, the full spectrum computed in sections 6.1, 7.2 and with a non-flat fiber.

The concrete fourfold $\hat{X}_{SU(5)}$ we construct has $n_7 = n_9 = 4$ in figure 5 and has $S_{SU(5)} = H_B$. It is engineered following the general guideline of appendix G. The vertices

specifying the reflexive polytope, along with their associated homogeneous coordinates and divisor classes, are as follows,

variable	vertices					divisor class
z_0	1	1	1	0	0	H_B
z_1	-1	0	0	0	1	H_B
z_3	0	-1	0	-1	-1	H_B
z_2	0	0	-1	1	0	$H_B - D_1 - D_2 - D_3 - D_4$
d_1	0	0	-1	0	0	D_1
d_2	0	0	-1	-1	-1	D_2
d_3	0	0	-1	-1	-2	D_3
d_4	0	0	-1	0	-1	D_4
u	0	0	0	1	0	$-3H_B + H - D_2 - D_3 - E_1 - E_2$
v	0	0	0	0	1	$H - D_1 - D_2 - D_3 - D_4 - E_2$
w	0	0	0	-1	-1	$H - D_1 - 2D_2 - 3D_3 - 2D_4 - E_1$
e_1	0	0	0	0	-1	E_1
e_2	0	0	0	1	1	E_2

(7.16)

Here we follow the notation of section 6.1, i.e. H_B is the hyperplane of \mathbb{P}^3 , the D_i are the Cartan divisors of the $SU(5)$ and H, E_1, E_2 are the classes of the dP_2 -fiber.

The Euler number is calculated as

$$\chi(\hat{X}_{SU(5)}) = 1110 \tag{7.17}$$

which agrees perfectly with general formula (6.44), where we have used $\chi(\hat{X}) = 1632$ and $\mathcal{S}_{SU(5)} = H_B, \mathcal{S}_7 = \mathcal{S}_9 = 4H_B$. For the construction of the corresponding toric variety we have to choose the right triangulation so that the toric Stanley-Reissner ideal is of the same form as the Stanley-Reissner ideal in (6.12). The toric divisors of the sections are given by (3.32) for $n_9 = 4$ and the Shioda maps (6.17) read

$$\sigma(\hat{s}_Q) = S_Q - S_P - 4H_B + \frac{1}{5}(2D_1 + 4D_2 + 6D_3 + 3D_4) \tag{7.18}$$

$$\sigma(\hat{s}_R) = S_R - S_P - 8H_B; \tag{7.19}$$

The full basis of $H_V^{(2,2)}$ is given by the 13 elements in equation (6.47). The G_4 -flux follows from (7.1) by inserting $n_7 = n_9 = 4$. Since we have $n_7 \neq 0$, we have to impose the non-flat fiber condition (7.8). Our main interest lies in the chiralities, which are then either computed torically or follow from the general formulas in 7.2 and appendix F as

$$\begin{aligned} \chi(\mathbf{5}_{(-\frac{2}{5}, 0)}) &= \frac{1}{2}a_3 - \frac{2}{25}(40a_6 + 63a_7 + 75a_{13}), \\ \chi(\mathbf{5}_{(-\frac{2}{5}, 1)}) &= \frac{1}{125}(50a_3 - 125a_4 + 80a_6 - 174a_7), \\ \chi(\mathbf{5}_{(\frac{3}{5}, 0)}) &= \frac{1}{250}(-225a_3 + 1640a_6 + 1888a_7), \\ \chi(\mathbf{5}_{(\frac{3}{5}, 1)}) &= -\frac{2}{5}(-5a_3 + 5a_4 + 16a_6 + 38a_7), \\ \chi(\mathbf{5}_{(-\frac{2}{5}, -1)}) &= \frac{4}{5}a_3 + 2a_4 + 8a_5 - \frac{468}{25}a_6 - \frac{1608}{125}a_7 + 4a_{13}, \\ \chi(\mathbf{10}_{(\frac{1}{5}, 0)}) &= -\frac{1}{5}a_3 + \frac{32}{25}a_6 + \frac{242}{125}a_7 - 6a_{13}, \end{aligned} \tag{7.20}$$

for the non-trivial non-Abelian representations and

$$\begin{aligned}
 \chi(\mathbf{1}_{(1,0)}) &= -\frac{27}{2}a_3 - 32a_5 + \frac{596}{5}a_6 + \frac{3128}{25}a_7 + 8a_{13}, \\
 \chi(\mathbf{1}_{(1,1)}) &= -13a_3 - 13a_4 - 88a_5 + 200a_6 + \frac{4226}{25}a_7 + 16a_{13}, \\
 \chi(\mathbf{1}_{(0,1)}) &= -29a_4 - 88a_5 + \frac{836}{5}a_6 + \frac{2006}{25}a_7 + 28a_{13}, \\
 \chi(\mathbf{1}_{(-1,1)}) &= -\frac{2}{5}(-5a_3 + 5a_4 + 16a_6 + 38a_7), \\
 \chi(\mathbf{1}_{(0,2)}) &= -8a_4 - 32a_5 + \frac{256}{5}a_6 + \frac{648}{25}a_7 + 8a_{13}, \\
 \chi(\mathbf{1}_{(-1,-2)}) &= 4a_3 + 8a_4 + 32a_5 - 80a_6 - \frac{296}{5}a_7 - 8a_{13},
 \end{aligned} \tag{7.21}$$

for the singlets. As before, it is convenient to parametrize the chiralities as in the previous subsections, in terms of the c_i . We then obtain

$$\begin{aligned}
 \chi(\mathbf{5}_{(-\frac{2}{5},0)}) &= c_1, & \chi(\mathbf{5}_{(-\frac{2}{5},1)}) &= c_2, & \chi(\mathbf{5}_{(\frac{3}{5},0)}) &= c_3, \\
 \chi(\mathbf{5}_{(\frac{3}{5},1)}) &= c_4, & \chi(\mathbf{5}_{(-\frac{2}{5},-1)}) &= c_5, & \chi(\mathbf{1}_{(-1,-2)}) &= c_6, \\
 \chi(\mathbf{10}_{(\frac{1}{5},0)}) &= -c_1 - c_2 - c_3 - c_4 - c_5, & \chi(\mathbf{1}_{(1,0)}) &= -4c_1 - 14c_2 - c_3 - 6c_4 + 2c_5 - 3c_6, \\
 \chi(\mathbf{1}_{(0,1)}) &= 8c_1 + 13c_2 + 6c_4 + 3c_5 - 2c_6, & \chi(\mathbf{1}_{(-1,1)}) &= 4c_1 + 2c_2 + 4c_5 - c_6, \\
 \chi(\mathbf{1}_{(0,2)}) &= 4c_2 + 4c_4 - 4c_5 + c_6, & \chi(\mathbf{1}_{(1,1)}) &= -12c_1 - 4c_2 + 5c_4 - 18c_5 + 3c_6.
 \end{aligned} \tag{7.22}$$

It follows that all 4D anomalies are cancelled.

8 Conclusions and future directions

In this paper we have advanced the program on F-theory compactifications on elliptic Calabi-Yau manifolds with rank two Mordell-Weil group to four-dimensional chiral compactifications.

The analysis of resolved Calabi-Yau elliptic fibrations with dP_2 -elliptic fiber has now been performed for Calabi-Yau fourfolds over a general three dimensional base B , extending earlier results in six dimensions [46, 51]. We study general compactifications with $U(1) \times U(1)$ and a specific $SU(5) \times U(1) \times U(1)$ gauge symmetry. We determined the general matter representations associated with the codimension two singularities of the fibration as well as its Yukawa points. However, at present the geometric techniques, relying on the use prime ideals, that we employed allow us to determine all matter curves and only a subset of matter surfaces. As a next step, we determine explicitly the general G_4 -flux of these resolved Calabi-Yau fourfolds. Since we considered Calabi-Yau fourfolds with a non-holomorphic zero section, we encountered the novel problem of having to define G_4 -flux over these manifolds. For this purpose we had to derive the general F-theory conditions on G_4 -flux on Calabi-Yau manifolds with a non-holomorphic zero section. We have formulated these conditions in terms of Chern-Simons terms on the Coulomb branch of the effective theory obtained by compactifying on a circle. The relevance of Kaluza-Klein

states generating new Chern-Simons terms in order to ensure a consistent M-/F-theory duality mapping in 3D has been pointed out

After this interlude on G_4 -fluxes in F-theory we constructed the most general G_4 -flux. We presented explicit calculations for the three dimensional base $B = \mathbb{P}^3$. In both the $U(1) \times U(1)$ and $SU(5) \times U(1) \times U(1)$ case we could determine certain chiralities directly by evaluating the chiral index (5.12) directly while the remaining matter chiralities were determined by a subset of the 3D Chern-Simons terms. It turned out that Chern-Simons terms are sufficient to determine chiralities of all the matter multiplets if also the new, Kaluza-Klein generated Chern-Simons terms are taken into account. The geometric techniques allowed an important, independent consistency check. The complications of non-flat fibers could be successfully circumvented by imposing one additional condition on the G_4 -flux, namely the vanishing of the integral of the G_4 -flux over the non-flat fiber. We also presented concrete explicit examples using toric geometry. We emphasize that our techniques are in line, but not restricted to toric geometry. In particular, we could calculate the most general G_4 -flux and chiralities, along with the proof of anomaly cancellation, for the entire class of all elliptically fibered Calabi-Yau fourfolds, cf. figure 2 and figure 5, with dP_2 -elliptic fiber, an additional $SU(5)$ GUT-sector and a fixed base B , here chosen to be $B = \mathbb{P}^3$ for simplicity.

There paper leaves room for a number of further studies and improvements:

- The geometric techniques developed here apply to general elliptic Calabi-Yau fourfolds, and they are *not* restricted to toric examples. In particular, we could derived closed formulas for e.g. the Euler number, the G_4 -flux and the 4D chiralities that explicitly depend on the divisors \mathcal{S}_7 , \mathcal{S}_9 , introduced in section 2.2, which label the members in the family of all inequivalent Calabi-Yau fourfolds obtained by varying the topologically data specifying their elliptic fibration. It would be desirable to obtain these formula for an arbitrary base B . Our results are independent of the existence of toric realizations of the Calabi-Yau fourfolds. We emphasize that employing toric geometry techniques one can typically study only one polytope at a time, which obscures the dependence on \mathcal{S}_7 , \mathcal{S}_9 .
- The finite number of choices for the divisors \mathcal{S}_7 , \mathcal{S}_9 on the F-theory side fix the topology of the fibration of dP_2 over the base B . It would be interesting to understand these new degrees of freedom from the point of view of the heterotic string. In particular it would be important to understand how the parameters in \mathcal{S}_7 , \mathcal{S}_9 enter the heterotic vector bundle and potentially modify the spectral cover construction [85].
- Although all 4D chiralities could still be derived by a combination of geometric and field theoretic techniques, we were at this point unable to determine explicitly the matter surfaces for a subset of representations. It would be desirable to derive these missing homology classes purely geometrically by further extending the geometric techniques of section 3.1.1. For this analysis important lessons might be learned from a geometrical interpretation of the field theoretically derived G_4 -flux conditions in section 4.2.

- While explicit results were presented for a specific three-dimensional base $B = \mathbb{P}^3$, it would be important to present similar computations for systematically classified three-dimensional bases, extending a similar analysis in 6D [86, 87]. These studies are also motivated by a search for SU(5) GUT models with flat fibers.
- In this paper we have not systematically performed a classification of SU(5) GUT models. We presented models for a specific SU(5) construction with a specific base $B = \mathbb{P}^3$, only. The example does not have a flat fibration at codimension three, which may lead to an infinite tower of massless states (a so-called tensionless string spectrum). We removed chiral states in the tower by further constraining the G_4 -flux. It would be important to classify SU(5) constructions by carrying over the Tate classification to dP_2 -elliptic fibrations, and to construct a base B where the fibration can be engineered to be flat. For efforts in these directions, primarily employing toric techniques, see [46, 47]. We note that the classification of SU(n)-singularities for general n should work similarly to the SU(5) case and thus will be facilitated by techniques developed in this work.
- The techniques developed in this paper pave the way to phenomenological studies of chiral four-dimensional models with rank-two Abelian sectors. However in order to achieve these goals a number of further details have to be addressed. First, the quantization of G_4 -flux is not well understood in general. However, as pointed out in section 4 we can choose the chiralities of our matter multiplets to parameterize the spectrum which should in turn also yield integer values for the flux-quanta in the G_4 -flux. Second, the constraints imposed by the self-duality condition on the G_4 -flux and by D3-brane tadpole cancellation have to be investigated. Those are important topics for future research.

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A Chern classes of dP_2 -elliptic fibrations

In this appendix we analyze the total Chern-class of the total space of dP_2 -fibrations $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ over a general base B . The following discussions hold in any complex dimension, although we specialize to Calabi-Yau two-, three- and fourfolds in the following.

The final goal of this section is the derivation of general formulas for the Euler number of smooth Calabi-Yau manifolds \hat{X} of complex dimension two, three and four in $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$. For the reader only interested in these formulas, we first state the results of the analysis and refer to the remainder of this section for a detailed derivation. The Euler numbers for Calabi-Yau two-, three- and fourfolds are expressed as the following integrals over the one-, two- respectively, three-dimensional base B of the Calabi-Yau manifold \hat{X} ,

$$d = 2: \quad \chi(X) = 12 \int_B c_1 \tag{A.1}$$

$$d = 3: \quad \chi(X) = \int_B (-24c_1^2 + 8c_1\mathcal{S}_7 - 4\mathcal{S}_7^2 + 8c_1\mathcal{S}_9 + 2\mathcal{S}_7\mathcal{S}_9 - 4\mathcal{S}_9^2)$$

$$d = 4: \quad \chi(X) = 3 \int_B [24c_1^3 + 4c_1c_2 - 16c_1^2(\mathcal{S}_7 + \mathcal{S}_9) + c_1(8\mathcal{S}_7^2 + \mathcal{S}_7\mathcal{S}_9 + 8\mathcal{S}_9^2) - \mathcal{S}_7\mathcal{S}_9(\mathcal{S}_7 + \mathcal{S}_9)]$$

Here we denoted the Poincaré dual $(1, 1)$ -forms by abuse of notation by the same symbol as their corresponding divisors $\mathcal{S}_7, \mathcal{S}_9$ in the base B . Furthermore, wedge-products have been omitted for brevity of our notation. We emphasize that these formulas for $\chi(\hat{X})$ are a direct generalization of the formulas in [57] for fibrations by the E_6 elliptic curve, that are obtained as the special case $\mathcal{S}_7 = \mathcal{S}_9 \equiv 0$.

We prepare the derivation of the Euler numbers (A.1) by a computation of the total Chern class of the ambient space $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$. The total space of the dP_2 -fibration $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ in (2.7) has the structure of a generalized projective bundle: the two-dimensional fiber is dP_2 and it can be understood as the projectivization of a rank five vector bundle over B by three \mathbb{C}^* -actions. Consequently, the Chern class of the $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ is calculated analogous to the Chern-class of an ordinary projective bundle by adjunction. Denoting the projective coordinates of the general dP_2 fiber by $[u : v : w : e_1 : e_2]$ we follow the assignments (2.2) of line bundles over the base B from the main text. Thus we obtain for the total Chern-class of $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$,

$$c(dP_2^B) = c(B)(1+H-E_1-E_2+\mathcal{S}_9-[K_B^{-1}])(1+H-E_2+\mathcal{S}_9-\mathcal{S}_7)(1+H-E_1)(1+E_1)(1+E_2). \tag{A.2}$$

Here we suppressed the dependence on $\mathcal{S}_7, \mathcal{S}_9$ for brevity of our expression. By abuse of notation we denote the divisors E_i and H and the first Chern-classes of their associated divisor bundles by the same symbol. In addition, we used $c(B) = 1 + c_1(B) + c_2(B) + c_3(B) + \dots$ to denote the total Chern class of B . We readily expand (A.2) order by order to obtain the following Chern classes of $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$,

$$c_1(dP_2^B) = 3H - E_1 - E_2 + 2\mathcal{S}_9 - \mathcal{S}_7, \tag{A.3}$$

$$c_2(dP_2^B) = c_2 - c_1^2 - 2E_1^2 - 3E_2^2 + 2E_1\mathcal{S}_7 - 2E_2\mathcal{S}_7 - E_1\mathcal{S}_9 + 4E_2\mathcal{S}_9 - H\mathcal{S}_9 + \mathcal{S}_7\mathcal{S}_9 - \mathcal{S}_9^2 + c_1(4H - 2E_1 - 3E_2 - 2\mathcal{S}_7 + 3\mathcal{S}_9), \tag{A.4}$$

$$c_3(dP_2^B) = c_3 - c_2(E_1 + E_2 - 3H + \mathcal{S}_7 - 2\mathcal{S}_9) + c_1^2(H - E_1 - 2E_2 - \mathcal{S}_7 + \mathcal{S}_9) - c_1(c_2 + 2E_1^2 + 3E_2^2 + 2\mathcal{S}_7(E_2 - E_1) + \mathcal{S}_9(H + E_1 - 4E_2 - \mathcal{S}_7 + \mathcal{S}_9)), \tag{A.5}$$

$$c_4(dP_2^B) = c_3(3H - E_1 - E_2) + c_2(\mathcal{S}_7(2E_1 - 2E_2) + \mathcal{S}_9(4E_2 - E_1 - H) - 2E_1^2 - 3E_2^2) + c_1c_2(H - E_1 - 2E_2), \tag{A.6}$$

$$c_5(dP_2^B) = -c_3(2E_1^2 + 3E_2^2) = 5c_3|_B. \tag{A.7}$$

Here we have used the notation $c_i \equiv c_i(B)$. All divisors in the above expressions are understood as their Poincaré dual $(1, 1)$ -forms, that are given as the first Chern class of the associated divisor line bundle, e.g. $c_1(\mathcal{O}(E_1))$ for $\mathcal{O}(E_1)$. The expressions for the first to third Chern classes are completely general, whereas we have assumed $\dim_{\mathbb{C}}(B) = 3$ for the fourth and fifth Chern class to be able to drop terms containing more than three vertical divisors. In the last line we have used the toric intersections $E_1^2 = E_2^2 = -1$ in (2.4) since $[c_3] \cong (\int_B c_3)pt$ with pt denoting a point on B . All expressions in (A.3) have been reduced modulo the ideal SR' in (3.20).

Now we are in the position to calculate the total Chern class of the Calabi-Yau fourfold \hat{X} . For this purpose we apply adjunction to obtain

$$c(\hat{X}) = \frac{c(dP_2^B)}{1 + 3H - E_1 - E_2 + 2S_9 - S_7} \quad (\text{A.8})$$

where the numerator is to be computed using (A.2) and the denominator is the Chern class of the anti-canonical bundle of $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$, cf. the first line in (A.3), employing that \hat{X} is the anti-canonical divisor. Expanding out this expression for the total Chern class of the fourfold \hat{X} yields the following individual Chern classes $c_i(\hat{X})$,

$$\begin{aligned} c_1(\hat{X}) &= 0, \\ c_2(\hat{X}) &= 3c_1^2 + c_2 - 2S_Q^2 - 3S_P^2 + \mathcal{S}_7(2S_Q - 2S_P) + \mathcal{S}_9(3S_P - 2S_Q - S_R + \mathcal{S}_7) \\ &\quad + c_1(2S_Q + S_P - 2(\mathcal{S}_7 + \mathcal{S}_9 - 2S_R)), \\ c_3(\hat{X}) &= c_3 - 8c_1^3 - c_1c_2 + \mathcal{S}_7(2S_Q^2 - 2S_P^2 - 2S_Q\mathcal{S}_7 - 2S_P\mathcal{S}_7 + 2S_P\mathcal{S}_9 + \mathcal{S}_7\mathcal{S}_9 + \mathcal{S}_9^2) \\ &\quad + c_1(7S_P^2 + 2S_Q\mathcal{S}_7 + 6S_P\mathcal{S}_7 - 2\mathcal{S}_7^2 + 8S_Q\mathcal{S}_9 - 7S_P\mathcal{S}_9 - 6\mathcal{S}_7\mathcal{S}_9 - 2\mathcal{S}_9^2 + 7S_9S_R) \\ &\quad - c_1^2(8S_Q + S_P - 8(\mathcal{S}_7 + \mathcal{S}_9 - S_R)) - 2S_Q\mathcal{S}_9^2 - 2S_RS_9^2, \\ c_4(\hat{X}) &= c_1^2(8S_Q^2 - 32S_P^2 - \mathcal{S}_7(22S_Q - 18S_P) + \mathcal{S}_9(41S_P - 41S_Q - 40S_R)) - 2S_Q^2\mathcal{S}_7^2 \\ &\quad - 2S_P^2\mathcal{S}_7^2 + 2S_Q\mathcal{S}_7^3 - 2S_P\mathcal{S}_7^3 + 3S_Q^2\mathcal{S}_7\mathcal{S}_9 - S_P^2\mathcal{S}_7\mathcal{S}_9 - 5S_Q\mathcal{S}_7^2\mathcal{S}_9 + 2S_P\mathcal{S}_7^2\mathcal{S}_9 + 2S_Q^2\mathcal{S}_9^2 \\ &\quad - 6S_P^2\mathcal{S}_9^2 - 2S_Q\mathcal{S}_7\mathcal{S}_9^2 - S_P\mathcal{S}_7\mathcal{S}_9^2 - 2S_Q\mathcal{S}_9^3 + 6S_P\mathcal{S}_9^3 - 4\mathcal{S}_9^3S_R + 8c_1^3(4S_Q - S_P + 3S_R) \\ &\quad - c_2(2S_Q^2 + 3S_P^2 - 2S_Q\mathcal{S}_7 + 2S_P\mathcal{S}_7 + 2S_Q\mathcal{S}_9 - 3S_P\mathcal{S}_9 + \mathcal{S}_9S_R) \\ &\quad + c_1(\mathcal{S}_7(14S_P^2 + 12S_P\mathcal{S}_7 - 11S_P\mathcal{S}_9) + 17S_P^2\mathcal{S}_9 - 19S_P\mathcal{S}_9^2 - 3S_Q^2(2\mathcal{S}_7 + 3\mathcal{S}_9) \\ &\quad + 8S_Q(\mathcal{S}_7^2 + 2\mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2) + 23\mathcal{S}_9^2S_R + c_2(2S_Q + S_P + 4S_R)). \end{aligned} \quad (\text{A.9})$$

Here we have dropped terms quartic in divisors on the base B and have reduced all expressions in the quotient ring (3.18). For the evaluation of (A.9) it proves convenient to express all the elements in terms of the divisor classes $S_P = E_2$, $S_Q = E_1$ and $S_R = H - E_1 - E_2 + \mathcal{S}_9 + [K_B]$ of the sections of \hat{X} . Then, we can make use of the intersection relations for the sections

$$(S_m^2 + [K_B^{-1}] \cdot S_m) \cdot D_\alpha \cdot D_\beta = 0, \quad (\text{A.10})$$

where we collectively denote the divisor classes of the sections by S_m . We note that this relation is a direct consequence of the property of the S_m being the classes of sections, as noted before in (2.14), but it is satisfying that it can also be proven directly in the intersection ring (3.18).

In the following we assume that \hat{X} is either complex two-, three- and four-dimensional and for each case employ (A.10) and its lower dimensional analogs to simplify the Chern classes in (A.9) even further. We obtain for the top Chern class $c_d(\hat{X})$ on a d -dimensional Calabi-Yau manifold \hat{X} the following results,

$$\begin{aligned}
 d = 2 : \quad c_2(\hat{X}) &= 4c_1(S_Q + S_P + S_R) + 2\mathcal{S}_7(S_Q - S_P) + \mathcal{S}_9(3S_P - S_R - 2S_Q), \\
 d = 3 : \quad c_3(\hat{X}) &= -2\mathcal{S}_7^2(S_Q + S_P) + 2\mathcal{S}_7\mathcal{S}_9S_P - 2\mathcal{S}_9^2(S_Q + S_R) - 8c_1^2(S_Q + S_P + S_R) \\
 &\quad + c_1(8S_P\mathcal{S}_7 + 8S_Q\mathcal{S}_9 - 7S_P\mathcal{S}_9 + 7\mathcal{S}_9S_R), \\
 d = 4 : \quad c_4(\hat{X}) &= 2\mathcal{S}_7^3(S_Q - S_P) + \mathcal{S}_7^2\mathcal{S}_9(2S_P - 5S_Q) - \mathcal{S}_7\mathcal{S}_9^2(2S_Q + S_P) - 2\mathcal{S}_9^3(S_Q - 3S_P + 2S_R) \\
 &\quad + c_1(14S_P\mathcal{S}_7^2 - 10S_P\mathcal{S}_7\mathcal{S}_9 - 13S_P\mathcal{S}_9^2 + S_Q(10\mathcal{S}_7^2 + 13\mathcal{S}_7\mathcal{S}_9 + 14\mathcal{S}_9^2) + 23\mathcal{S}_9^2S_R) \\
 &\quad + 4c_1c_2(S_Q + S_P + S_R) - 8c_1^2(4S_P\mathcal{S}_7 - 3S_P\mathcal{S}_9 + 2S_Q(\mathcal{S}_7 + 2\mathcal{S}_9) + 5\mathcal{S}_9S_R) \\
 &\quad + 24c_1^3(S_Q + S_P + S_R) + c_2(2S_Q(\mathcal{S}_7 - \mathcal{S}_9) - 2S_P\mathcal{S}_7 + 3S_P\mathcal{S}_9 - \mathcal{S}_9S_R) \quad (\text{A.11})
 \end{aligned}$$

where we now in addition also dropped intersections of more than $d - 1$ vertical divisors.

Finally, we are in the position to calculate the Euler number for \hat{X} . We obtain the Euler number for \hat{X} being a twofold, i.e. $K3$, a Calabi-Yau threefold and fourfold by integrating the above appropriate Chern classes in (A.11) over \hat{X} . Using that $S_m \cdot D_\alpha \cdot D_\beta \cdot D_\gamma = (D_\alpha \cdot D_\beta \cdot D_\gamma)|_B$ for a fourfold \hat{X} , respectively, analogous relations for the two- and threefold case, we obtain that all integrals reduce to integrals over the base B . By some algebra we immediately reproduces the results anticipated in (A.1).

As another application of the explicit formulas (A.9) for the fourfold \hat{X} we calculate the Todd class $\text{Td}_4(\hat{X})$ of \hat{X} . Upon integrating the Todd class over \hat{X} we obtain the arithmetic genus $\chi_0(\hat{X})$ by means of an Hirzebruch-Riemann-Roch index theorem as

$$\chi_0(\hat{X}) := \sum_p (-1)^p h^{(p,0)}(\hat{X}) = \int_{\hat{X}} \text{Td}_4(\hat{X}). \quad (\text{A.12})$$

In the case of a Calabi-Yau fourfold, see cf. [57] for more details, the Todd class is given as

$$\text{Td}_4(\hat{X}) = \frac{1}{720} \left(3c_2(\hat{X})^2 - c_4(\hat{X}) \right) \quad (\text{A.13})$$

in terms of the Chern classes $c_2(\hat{X})$ and $c_4(\hat{X})$. Thus, the arithmetic genus reads

$$\chi_0(\hat{X}) = \int_{\hat{X}} \text{Td}_4(\hat{X}) = \frac{1}{12} \int_B c_1 c_2 = 2\chi_0(B). \quad (\text{A.14})$$

Here we have evaluated the second equality using the expression (A.9) for the fourth Chern class of \hat{X} and computed the square of the second Chern class $c_2(\hat{X})^2$ as

$$\begin{aligned}
 c_2(\hat{X})^2 &= S_P [4(2c_1 - \mathcal{S}_7)(c_1^2 + c_2 - c_1\mathcal{S}_7) + (6c_2 + (2c_1 - \mathcal{S}_7)(5c_1 + \mathcal{S}_7))\mathcal{S}_9 + (\mathcal{S}_7 - 3c_1)\mathcal{S}_9^2 + \mathcal{S}_9^3] \\
 &\quad + 2S_Q [4c_1^3 + 2c_2(\mathcal{S}_7 - \mathcal{S}_9) - \mathcal{S}_7\mathcal{S}_9^2 - 2c_1^2(\mathcal{S}_7 + 3\mathcal{S}_9) + 2c_1(2c_2 + \mathcal{S}_7^2 + \mathcal{S}_7\mathcal{S}_9 + \mathcal{S}_9^2)] \\
 &\quad + S_R [(4c_1 - \mathcal{S}_9)(2(c_1^2 + c_2) - 3c_1\mathcal{S}_9 + \mathcal{S}_9^2)], \quad (\text{A.15})
 \end{aligned}$$

after some algebra in the cohomology ring (3.18). In the last equality in (A.14) we employed the relation

$$\chi_0(B) = \frac{1}{24} \int_B c_1 c_2 \quad (\text{A.16})$$

for the arithmetic genus of the base B . We observe a surprising cancellation of all terms involving $\mathcal{S}_7, \mathcal{S}_9$ in the fourth Chern class in (A.9) and in $c_2(\hat{X})^2$ in (A.15) so that the final result in (A.14) only depends on the Chern classes of the base B .

We conclude that for a suitable base B of a simply-connected Calabi-Yau fourfold with $\chi_0(\hat{X}) = 2$, which follows from $h^{(0,0)}(\hat{X}) = h^{(4,0)}(\hat{X}) = 1$ and $h^{(p,0)}(\hat{X}) = 0$ otherwise, we have to require $\chi_0(B) = 1$ or, using (A.16),

$$\int_B c_1 c_2 = 24. \tag{A.17}$$

We even require the stronger conditions $h^{(1,0)}(B) = h^{(2,0)}(B) = h^{(3,0)}(B) = 0$ for B , since every non-trivial element in these Hodge cohomology groups would give rise via the pullback under $\pi : \hat{X} \rightarrow B$ to a corresponding element in the Hodge cohomology of \hat{X} . However, this is excluded by assumption of a simply-connected fourfold.

B Intersection ring on \hat{X} with $B = \mathbb{P}^3$

In this section we work out the intersection ring $H_V^{(*,*)}(\hat{X})$ of the fourfold \hat{X} over the base $B = \mathbb{P}^3$. To this end, we relate the intersections on \hat{X} to intersections on the ambient space $dP_2(n_7, n_9)$. On the ambient space $dP_2(n_7, n_9)$ the intersections are determined completely by the intersections of the fiber dP_2 , the base \mathbb{P}^3 and basic properties of the fibration. This will allow us, as demonstrated below and in the main text, to work out the basis of $H_V^{(k,k)}(\hat{X})$ for all $k \leq 4$ on the one hand, and to calculate the quartic intersections on \hat{X} .

We recall from the main text, cf. (3.31) and (3.32), that the basis D_A of divisors on $\pi : \hat{X} \rightarrow \mathbb{P}^3$ reads

$$H^{(1,1)}(\hat{X}) = \langle H_B, S_P, S_Q, S_R \rangle, \tag{B.1}$$

where S_P, S_Q, S_R denote the homology classes of the sections \hat{s}_P, \hat{s}_Q and \hat{s}_R , respectively.

Intersection relations and the quartic intersections

Next we begin by calculating the intersections of two divisors D_A , where we denote the intersection pairing on \hat{X} by a subscript. We employ the representation (3.18) for $H^{(*,*)}(\hat{X})$ to relate these to intersections on the ambient space $dP_2(n_7, n_9)$, that we denote as ‘.’. First we note that there are $\frac{5!}{2!3!} = 10$ different quadratic combinations $D_A \cdot_{\hat{X}} D_B$. We evaluate these as

$$\begin{aligned} S_P^2 &= S_P(S_P[K_B^{-1}] - (\mathcal{S}_7 - \mathcal{S}_9)(\mathcal{S}_9 - [K_B^{-1}])) , & S_P S_Q &= 0 , \\ S_P S_R &= -S_P \mathcal{S}_9 (S_P - \mathcal{S}_9 + [K_B^{-1}]) , & S_Q^2 &= S_Q(S_Q[K_B^{-1}] + (\mathcal{S}_7 - [K_B^{-1}])(\mathcal{S}_7 - \mathcal{S}_9)) , \\ S_R^2 &= S_R((\mathcal{S}_7 - [K_B^{-1}]) (\mathcal{S}_9 - [K_B^{-1}]) + S_R[K_B^{-1}]) , & S_Q S_R &= -S_Q \mathcal{S}_7 (S_Q - \mathcal{S}_7 + [K_B^{-1}]) , \\ S_P H_B &= -S_P H_B (S_P + \mathcal{S}_7 - 2\mathcal{S}_9) , & S_Q H_B &= -S_Q H_B (S_Q + \mathcal{S}_9 - 2\mathcal{S}_7) , \\ S_R H_B &= S_R H_B (\mathcal{S}_7 + \mathcal{S}_9 - [K_B^{-1}] - S_R) , \\ H_B^2 &= H_B^2 (3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9 + 2S_P + 2S_Q + 3S_R) , \end{aligned} \tag{B.2}$$

where we used $\mathcal{S}_7 = n_7 H_B, \mathcal{S}_9 = n_9 H_B$ and $[K_B^{-1}] = 4H_B$ for $B = \mathbb{P}^3$. Here we suppressed the intersection product ‘.’ for brevity of our notation. The intersections on the left side of

the equations are carried out in the Calabi-Yau fourfold \hat{X} and related to the intersections on the ambient space $dP_2(n_7, n_9)$ on the right side of the equations.

Next, we calculate the triple intersections of divisors on \hat{X} . There are $\frac{6!}{3!3!} = 20$ different cubic combinations $D_A \cdot_{\hat{X}} D_B \cdot_{\hat{X}} D_C$ with intersections

$$\begin{aligned}
H_B^3 &= H_B^3 (2S_P + 2S_Q + 3S_R), & H_B^2 S_P &= S_P H_B^2 (2S_9 - S_7 - S_P), \\
H_B^2 S_Q &= S_Q H_B^2 (2S_7 - S_9 - S_Q), & H_B^2 S_R &= S_R H_B^2 (S_7 + S_9 - [K_B^{-1}] - S_R), \\
H_B S_P S_R &= -S_P H_B S_9 (S_P + [K_B^{-1}] - S_9), & H_B S_Q S_R &= -S_Q H_B S_7 (S_Q - S_7 + [K_B^{-1}]), \\
H_B S_P^2 &= S_P H_B (S_P [K_B^{-1}] + (S_9 - S_7)(S_9 - [K_B^{-1}])), \\
H_B S_Q^2 &= S_Q H_B (S_Q [K_B^{-1}] + (S_7 - [K_B^{-1}])(S_7 - S_9)), \\
H_B S_R^2 &= S_R H_B ((S_7 - [K_B^{-1}]) (S_9 - [K_B^{-1}]) + S_R [K_B^{-1}]), \\
S_P^3 &= -S_P ([K_B^{-1}] (S_9 - cS_7)(S_9 - [K_B^{-1}]) + S_P ([K_B^{-1}]^2 + (S_7 - [K_B^{-1}])S_9 - S_9^2)), \\
S_Q^3 &= S_Q (S_Q (-[K_B^{-1}]^2 + S_7^2 - S_7(S_9 - [K_B^{-1}])) - [K_B^{-1}](S_7 - [K_B^{-1}]) (S_7 - S_9)), \\
S_R^3 &= -S_R ([K_B^{-1}] (S_7 - [K_B^{-1}]) (S_9 - [K_B^{-1}]) + ([K_B^{-1}]^2 - S_7 S_9) S_R), \\
S_P^2 S_R &= S_P (S_P - S_9 + [K_B^{-1}]) (S_7 - S_9) S_9, & S_P S_R^2 &= S_P ([K_B^{-1}] - S_7) (S_P - S_9 + [K_B^{-1}]) S_9, \\
S_Q^2 S_R &= -S_Q (S_Q - S_7 + [K_B^{-1}]) S_7 (S_7 - S_9), & S_Q S_R^2 &= -S_Q (S_Q - S_7 + [K_B^{-1}]) S_7 (S_9 - [K_B^{-1}]).
\end{aligned} \tag{B.3}$$

We note that all other intersections vanish, since $S_P \cdot S_Q = 0$ and as before we suppressed the intersection product “ \cdot ” of \hat{X} , respectively, $dP_2(n_7, n_9)$ on the left, respectively, right side of the equations.

Finally we calculate the quartic intersections as

$$\begin{aligned}
\mathcal{C}_0 &= H_B^3 S_P - 4H_B^2 S_P^2 + (16 + (n_7 - 4)n_9 - n_9^2) H_B S_P^3 + H_B^3 S_Q - 4H_B^2 S_Q^2 \\
&\quad + ((32 - 4n_7 - n_7^2)n_9 + 3n_7 n_9^2 - 2n_9^3 - 64) S_P^4 + (16 - n_7^2 + n_7(n_9 - 4)) H_B S_Q^3 \\
&\quad - (64 + 2n_7^3 - 3n_7^2 n_9 - n_7(32 - 4n_9 - n_9^2)) S_Q^4 + H_B^3 S_R + n_9 H_B^2 S_P S_R \\
&\quad + n_9(n_9 - n_7) H_B S_P^2 S_R + (n_7 - n_9)^2 n_9 S_P^3 S_R + n_7 H_B^2 S_Q S_R + n_7(n_7 - n_9) H_B S_Q^2 S_R \\
&\quad + n_7(n_7 - n_9)^2 S_Q^3 S_R - 4H_B^2 S_R^2 + (n_7 - 4)n_9 H_B S_P S_R^2 + (4 - n_7)(n_7 - n_9)n_9 S_P^2 S_R^2 \\
&\quad + n_7(n_9 - 4) H_B S_Q S_R^2 + n_7(n_7 - n_9)(n_9 - 4) S_Q^2 S_R^2 + (16 - n_7 n_9) H_B S_R^3 \\
&\quad + (n_7 - 4)^2 n_9 S_P S_R^3 + n_7(n_9 - 4)^2 S_Q S_R^3 - (64 + n_7^2 n_9 + n_7(n_9 - 12)n_9) S_R^4,
\end{aligned} \tag{B.4}$$

where the coefficient of the polynomial \mathcal{C}_0 of $D_A \cdot_{\hat{X}} D_B \cdot_{\hat{X}} D_C \cdot_{\hat{X}} D_D$ is the corresponding intersection number in \hat{X} . In this computation we have used that the intersections on dP_2 in (2.4) are embedded into the intersections of $dP_2(n_7, n_9)$ as the intersections containing $H_B^3 \cdot D_A \cdot D_B$, in particular $H_B^3 \cdot S_R^2 = H_B^3 \cdot S_P^2 = H_B^3 \cdot S_Q^2 = -1$.

Since \hat{X} is complex four-dimensional, intersections of more than four divisors on \hat{X} vanish, which is confirmed by the concrete presentation (3.18) of the intersection ring with $B = \mathbb{P}^3$.

The cohomology basis of $H_V^{(*,*)}(\hat{X})$

The basis of $H_V^{(k,k)}(\hat{X})$ for each $k = 0, \dots, 4$ is now determined as follows. First, we note that there is a canonical basis both for $k = 0, 1$. Namely, the one-dimensional space $H^{(0,0)}(\hat{X})$ is spanned by the generated by 1, $H^{(0,0)}(\hat{X}) = \langle 1 \rangle$, and four-dimensional $H^{(1,1)}(\hat{X})$ is generated by the D_A as indicated in (B.1). Using the quartic intersections (B.4)

we can now calculate a basis of $H_V^{(2,2)}(\hat{X})$. For generic n_7, n_9 this space is five-dimensional and a basis is given by

$$H_V^{(2,2)}(\hat{X}) = \langle H_B^2, H_B \cdot S_P, H_B \cdot \sigma(\hat{s}_Q), H_B \cdot \sigma(\hat{s}_R), S_P^2 \rangle. \quad (\text{B.5})$$

Here $\sigma(\hat{s}_Q), \sigma(\hat{s}_R)$ denotes the Shioda map (2.17) of the rational sections, that takes in the case at hand the form

$$\sigma(\hat{s}_Q) = S_Q - S_P - 4H_B, \quad \sigma(\hat{s}_R) = S_R - S_P - (4 + n_9)H_B. \quad (\text{B.6})$$

The intersection matrix $\eta^{(2)}$ in this basis is readily calculated from (B.4) as

$$\eta^{(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 & -4 \\ 1 & -4 & 0 & 0 & 16 + (-4 + n_7 - n_9)n_9 \\ 0 & 0 & -8 & n_7 - n_9 - 4 & n_9(4 - n_7 + n_9) \\ 0 & 0 & \eta_{34}^{(2)} & -2(4 + n_9) & 2n_9(4 - n_7 + n_9) \\ -4 & \eta_{25}^{(2)} & \eta_{35}^{(2)} & \eta_{45}^{(2)} & -64 - (8 + n_7 - 2n_9)(-4 + n_7 - n_9)n_9 \end{pmatrix}, \quad (\text{B.7})$$

where lengthy entries $\eta_{rs}^{(2)}$ that are determined by symmetry of $\eta^{(2)}$ are omitted and denoted by $\eta_{sr}^{(2)}$.

From this matrix it is apparent that if (n_7, n_9) are on the boundary of the allowed region in figure 2, the rank is reduced to four. In this case, the five surfaces in (B.5) become homologous. A choice of basis in this case is given by the first four surfaces in (B.5) with the surface S_P^2 dropped. Consequently, the intersection matrix $\eta^{(2)}$ in this case is the 4×4 -submatrix of (B.7) obtained by deleting the fifth row and column. Let us discuss these non-generic cases in more detail, to illustrate the decrease from five to four basis elements.

$n_7 = 0$: we see from (2.10) that $s_7 \neq 0$, which implies that \hat{s}_Q and \hat{s}_R are holomorphic, cf. the discussion below (2.12). we obtain the cohomology relation

$$S_P^2 \cong -n_9(4 + n_9)H_B^2 - 4H_B \cdot S_P - n_9H_B \cdot \sigma(\hat{s}_R) \quad (\text{B.8})$$

in $H_V^{(2,2)}(\hat{X})$ which follows from the relations

$$\begin{aligned} H_B S_P &= H_B(S_P([K_B^{-1}] + \mathcal{S}_9) - S_R(S_R + [K_B^{-1}] - \mathcal{S}_9)), \\ H_B S_R &= H_B S_R(\mathcal{S}_9 - [K_B^{-1}] - S_R) \\ S_P^2 &= ([K_B^{-1}] + \mathcal{S}_9)(\mathcal{S}_9 - [K_B^{-1}])S_P + [K_B^{-1}]([K_B^{-1}] - \mathcal{S}_9 + S_R)S_R. \end{aligned} \quad (\text{B.9})$$

These relations can be proven for $n_7 = 0$ using the presentation (3.18).

$n_9 = 4 + n_7$: in this situation, the section \hat{s}_Q is holomorphic. Analogously, as before, we compute the homology relation on \hat{X} reading

$$S_P^2 \cong -8(4 + n_7)H_B^2 - 4H_B \cdot S_P - n_7H_B \cdot \sigma(\hat{s}_Q) - 4H_B \cdot \sigma(\hat{s}_R). \quad (\text{B.10})$$

$n_9 = 12 - n_7$: in this case all sections are rational, but $s_1 \neq 0$ in (2.10). As before, we compute the homology relation

$$S_P^2 \cong -2(n_7 - 12)(n_7 - 8)H_B^2 - 4H_B \cdot S_P + (n_7 - 8)H_B \cdot \sigma(\hat{s}_Q) + (n_7 - 8)H_B \cdot \sigma(s_R). \quad (\text{B.11})$$

$n_9 = 0$: for this value the sections \hat{s}_P and \hat{s}_R are holomorphic. Similarly, we compute for $n_9 = 0$ the following relation in homology,

$$S_P^2 + 4H_B \cdot S_P = 0. \quad (\text{B.12})$$

This is immediately clear since for a holomorphic zero section the relation (2.16) holds in the homology of \hat{X} , which is precisely the relation (B.12). It is satisfying that this relation is reproduced by the intersection calculation (3.18) by noting the relations

$$\begin{aligned} S_P^2 &= [K_B^{-1}] ((\mathcal{S}_7 - [K_B^{-1}]) S_P + S_R ([K_B^{-1}] + S_R)) , \\ S_P H_B &= -H_B ((\mathcal{S}_7 - [K_B^{-1}]) S_P + S_R ([K_B^{-1}] + S_R)) , \end{aligned} \quad (\text{B.13})$$

that are calculated for $n_9 = 0$. Combining these two relations precisely yields (B.12).

$n_9 = n_7 - 4$: for this value the sections \hat{s}_P , but this time since $s_8 \neq 0$ as is clear from (2.10). Thus, the relation (B.13) still holds.

Next, we note that the basis of $H^{(3,3)}(\hat{X})$ is determined by Poincaré duality from $H^{(1,1)}(\hat{X})$. Thus, the group $H^{(3,3)}(\hat{X})$ is generated by four elements, independently of n_7 and n_9 , which can be checked by an explicit calculation. However, we will refrain from presenting the details, since the group $H^{(3,3)}(\hat{X})$ is of no immediate relevance for F-theory, and leave this as an exercise for the interested reader. Finally, $H^{(4,4)}(\hat{X})$ is generated by a single element which is the volume form normalized by the volume of \hat{X} . It is given, up to combinatorial factors, by \mathcal{C}_0 in (B.4).

C Total Chern class of $\widehat{dP}_2^B(\mathcal{S}_7, \mathcal{S}_9)$

In this brief appendix we outline the calculation of the total Chern class of the resolved space $\widehat{dP}_2^B(\mathcal{S}_7, \mathcal{S}_9)$, which is straightforward but tedious. We keep our discussion short, leaving some details of the calculations for the interested reader.

The basic strategy to obtain the Chern class of $\widehat{dP}_2^B(\mathcal{S}_7, \mathcal{S}_9)$ is to successively modify (A.2) taking into account all changes of divisor classes in the blow-up (6.8). First we have to use the new divisor classes (6.9) for homogeneous coordinates $[\tilde{u} : \tilde{v} : \tilde{w} : e_1 : e_2]$. This alters the second to fourth factors in (A.2) accordingly. Then we have to include the classes of the Cartan divisor, which is achieved by multiplying in (A.2) by $\prod_i (1 + D_i)$. Finally we have to modify the first factor in (A.2) to incorporate the split and shift of the class of z by the D_i as in (6.9) appropriately. All we have to do is to formally replace the divisor $\mathcal{S}_{\text{SU}(5)}$ in the expression for the Chern class of B as

$$\mathcal{S}_{\text{SU}(5)} \rightarrow \mathcal{S}_{\text{SU}(5)} - \sum_i D_i, \quad (\text{C.1})$$

expand and then re-express the result we get in terms of the Chern class of B and the intersections of $\mathcal{S}_{\text{SU}(5)}$. Taking all these changes into account we obtain the following expression

$$\begin{aligned}
c(\widehat{dP}_2^B) = & \left[1 + c_1 + c_2 - D_1^2 - D_1D_2 - D_2^2 - D_2D_3 - D_3^2 - D_2D_4 - D_3D_4 - D_4^2 \right. \\
& + (\mathcal{S}_{\text{SU}(5)} - D_0) \mathcal{S}_{\text{SU}(5)} + c_3 - c_1 (D_1^2 + D_1D_2 + D_2^2 + D_3^2 + D_3D_4 + D_4^2 + D_2D_3 + D_2D_4) \\
& + \mathcal{S}_{\text{SU}(5)} (D_1^2 + 2D_1D_2 + (D_2 + D_3 + D_4)^2) + (\mathcal{S}_{\text{SU}(5)} - D_0) \mathcal{S}_{\text{SU}(5)} (c_1 - \mathcal{S}_{\text{SU}(5)}) \\
& \left. - D_1^2D_2 - D_1D_2^2 - (D_2 + D_3)(D_2 + D_4)(D_3 + D_4) \right] \\
& \times (1 + [u])(1 + [v])(1 + [w])(1 + E_1)(1 + E_2), \tag{C.2}
\end{aligned}$$

where we used the relation $\mathcal{S}_{\text{SU}(5)} - D_0 = D_1 + D_2 + D_3 + D_4$ between the extended node D_0 and $\mathcal{S}_{\text{SU}(5)}$ as well as abbreviated the ambient space as \widehat{dP}_2^B . We also denote the classes (6.9) of the homogeneous coordinates by brackets $[\cdot]$. We emphasize that the new factors $(1 + D_i)$ from the Cartan divisors D_i have been combined with the changes in the first factor in (A.2) containing the Chern class of B .

D Intersection ring for $\hat{X}_{\text{SU}(5)}$ with $B = \mathbb{P}^3$

In this section we present the core cohomology calculations of elliptically fibered Calabi-Yau fourfolds $\pi : \hat{X} \rightarrow \mathbb{P}^3$ over \mathbb{P}^3 with general elliptic fiber \mathcal{E} in dP_2 and an additional resolved $\text{SU}(5)$ -singularity over codimension one in \mathbb{P}^3 . The geometry has been introduced in section 6, to which we refer for more details. The following discussion is very similar to the one in A and B. Thus, we will keep the exposition as short as possible and only present the key results necessary for the construction of G_4 -flux and 3D CS-terms in the main text.

The starting point of the calculation of the cohomology ring $H_V^{(*,*)}(\hat{X})$ of \hat{X} is the representation (6.41) as a quotient ring. We are working with a particular phase of the fourfold \hat{X} with Stanley-Reissner ideal (6.12). The eight-dimensional basis of $H^{(1,1)}(\hat{X})$ is given in (6.46) and the class of the anti-canonical bundle $K_{dP_2}^{-1}$ in (6.11). Omitting lengthy details, that are best performed using a computer algebra program, we immediately obtain the quartic intersections on \hat{X} as

$$\begin{aligned}
C_0 = & 12D_1^3H_B + 4D_2D_3D_4H_B - 8D_2D_4^2H_B + 12D_4^3H_B - 2D_1^2H_B^2 + D_1D_2H_B^2 - 2D_2^2H_B^2 + D_2D_3H_B^2 \\
& - 2D_3^2H_B^2 + D_3D_4H_B^2 - 2D_4^2H_B^2 + (n_7 - 7)D_2D_3^2H_B - 4(n_7 - 4)D_2^2D_3D_4 + 8(n_7 - 4)D_2^2D_4^2 \\
& - (n_7 - 4)D_2^2D_3H_B - (n_7 - 7)(n_7 - 4)D_2^2D_3^2 + (n_7 - 4)^2D_2^2D_3 - 16(n_7 - 3)D_2D_4^3 - (n_9 - 7)D_1D_2^2H_B \\
& - ((n_7 - 5)n_7 + (n_9 - 6)(n_9 - 5))D_2^4 + (4 + (n_7 - n_9)^2 - 4n_9)D_3D_4^3 + 4(n_7 - n_9 - 1)D_2D_3^2D_4 \\
& + (n_7 - n_9 - 1)D_3^2D_4H_B - (n_7 - n_9 - 1)(2 + n_7 - n_9)D_3^2D_4^2 + (n_9 - 10)D_1^2D_2H_B + 4(n_9 - 2)D_2D_3D_4^2 \\
& + (6 - 2n_7 + n_9)D_3^3H_B + (n_9 - n_7 - 2)D_3D_4^2H_B + (1 - n_7 + n_9)^2D_3^3D_4 + (n_7 + n_9 - 9)D_2^3H_B \\
& + (25 + (n_7 - 14)n_7 + 4n_9)D_2D_3^3 + (16n_7 - 12 + (n_9 - 20)n_9)D_1^3D_2 - (14 + 8n_7 + (n_9 - 17)n_9)D_1^2D_2^2 \\
& + (21 + 4n_7 + (n_9 - 14)n_9)D_1D_2^3 - 2(16n_7 - 36 + (n_9 - 14)n_9)D_1^4 - 2(28 + n_7^2 + n_9(2 + n_9) - 2n_7(7 + n_9))D_4^4 \\
& - (20 + 2n_7^2 + n_9(5 + n_9) - 2n_7(7 + n_9))D_3^4 + H_B^3S_P + (16 + (n_7 - n_9 - 4)n_9)H_B^3S_P \\
& - 4H_B^2S_P^2 + (-64 - (8 + n_7 - 2n_9)(n_7 - n_9 - 4)n_9)S_P^4 + D_3^3S_Q + D_3^2H_B^2S_Q + D_3H_B^2S_Q + H_B^3S_Q \\
& - 4D_3^2S_Q^2 - 4D_3H_B^2S_Q^2 - 4H_B^2S_Q^2 + (16 + n_7(n_9 - n_7 - 2))D_3S_Q^3 + (16 + n_7(n_9 - n_7 - 2))H_B^3S_Q^3 + H_B^3S_R
\end{aligned}$$

$$\begin{aligned}
 & -(64+n_7(2+n_7-n_9)(2n_7-n_9-10))S_Q^4-4n_7D_4^3S_R-n_7D_3^2H_B S_R+D_3D_4H_B n_7S_R-2n_7D_4^2H_B S_R \\
 & +n_7(n_7-n_9-1)D_3^2D_4S_R+n_7(n_9-n_7)D_3^3S_R+n_7(2-n_7+n_9)D_3D_4^2S_R+n_9H_B^2S_P S_R+n_7D_3H_B S_Q S_R \\
 & +n_9(n_9-n_7)H_B S_P^2 S_R+(n_7-n_9)^2n_9S_P^3 S_R+n_7D_3^2S_Q S_R+n_7H_B^2S_Q S_R+n_7(n_7-n_9-2)D_3S_Q^2 S_R \\
 & +n_7(n_7-n_9-2)H_B S_Q^2 S_R+n_7(2-n_7+n_9)^2S_Q^3 S_R-4H_B^2S_R^2-D_3H_B n_7S_R^2+(n_7-4)n_7D_3D_4S_R^2 \\
 & -2(n_7-4)n_7D_4^2S_R^2-(n_7-3)n_7D_3^2S_R^2+H_B(n_7-4)n_9S_P S_R^2-(n_7-4)(n_7-n_9)n_9S_P^2 S_R^2 \\
 & +n_7(n_9-2)D_3S_Q S_R^2+n_7(n_9-2)H_B S_Q S_R^2+(n_7-4)^2n_9S_P S_R^3+n_7(n_7-n_9-2)(n_9-2)S_Q^2 S_R^2 \\
 & -n_7(n_7+n_9-6)D_3S_R^3+(16-n_7(2+n_9))H_B S_R^3+n_7(n_9-2)^2S_Q S_R^3-(64+n_7(2+n_9)(n_7+n_9-10))S_R^4.
 \end{aligned} \tag{D.1}$$

As before, all intersections are understood to be evaluated on \hat{X} and the quartic intersections $D_A \cdot_{\hat{X}} D_B \cdot_{\hat{X}} D_C \cdot_{\hat{X}} D_D$ are read off as the coefficient of the appropriate monomial in \mathcal{C}_0 .

As one immediate application we calculate the Cartan matrix C_{IJ} of the affine Lie algebra of $SU(5)$ as

$$C_{IJ} = -D_I \cdot D_J \cdot H_B^2 = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}, \tag{D.2}$$

where we supplemented as before the Cartan divisors D_i , $i = 1, \dots, 4$, by the divisor $D_0 = H_B - D_1 - D_2 - D_3 - D_4$ corresponding to the extended node $-\alpha_0$ as $D_I = (D_0, D_i)$. In addition, we confirm the intersections (2.13), (2.14) and (2.15) explicitly employing (D.1). Finally, we calculate the intersections of the sections with the nodes of the Dynkin diagram of $SU(5)$ as

$$S_P \cdot D_I \cdot H_B^2 = S_R \cdot D_I \cdot H_B^2 = (1, 0, 0, 0, 0)_I, \quad S_Q \cdot D_I \cdot H_B^2 = (0, 0, 0, 1, 0)_I \tag{D.3}$$

reproducing the independent findings in (6.16) and, consequently, the Shioda maps (6.17). Here we exploited (6.15) to represent the curve $c_{-\alpha_I}$ corresponding to the simple root α_I as

$$c_{-\alpha_I} = D_I \cdot H_B^2, \tag{D.4}$$

employing $H_B^2 \cdot \mathcal{S}_{SU(5)}^b = H_B^3 = 1$. We note that we can even make the stronger statement

$$S_P \cdot D_i = 0 \tag{D.5}$$

in the homology of $\hat{X}_{SU(5)}$, which will be important for the discussion of section 7.2.

The cohomology group $H_V^{(2,2)}(\hat{X}_{SU(5)})$ is readily calculated. We generically obtain a 13-dimensional vector space, with certain jumps in the cohomology for non-generic values of n_7 and n_9 , see figure 5. For the application of the construction of G_4 -flux, it proves useful to make a particular choice of basis by hand. We present a thorough analysis of this basis in the next appendix.

E Basis of $H_V^{(2,2)}(\hat{X}_{\text{SU}(5)})$ with $B = \mathbb{P}^3$

The vector space $H_V^{(2,2)}$ is 13-dimensional for arbitrary values of n_7 and n_9 . A basis is (6.47), that we recall here for convenience

$$H_{(V,13)}^{(2,2)} = \langle H_B^2, H_B \cdot S_P, H_B \cdot \sigma(S_Q), H_B \cdot \sigma(S_R), S_P \cdot \sigma(S_R), \sigma(S_Q) \cdot \sigma(S_R), \sigma(S_Q)^2, H_B \cdot D_1, H_B \cdot D_2, H_B \cdot D_3, H_B \cdot D_4, D_2 \cdot D_4, D_1^2 \rangle. \quad (\text{E.1})$$

To obtain the smaller vector spaces of the boundaries we need to drop some elements of this basis. The vectors that become linearly depended at the boundary have to be re-expressed as a linear combination of the smaller basis on the boundary. In the next subsections, we choose bases for the different boundaries. We write the linearly dependent elements, that we can drop from the above 13-dimensional basis, as linear combinations of the rest of the vectors in the smaller basis.

12 dimensional

At the boundaries with a twelve-dimensional basis we can take

$$H_{(V,12)}^{(2,2)} = H_{(V,13)}^{(2,2)} \setminus \{S_P \cdot \sigma(S_R)\}, \quad (\text{E.2})$$

in which case the surface $S_P \cdot \sigma(S_R)$ is re-expressed as:

- At $n_9 = 0$

$$S_P \cdot \sigma(S_R) \cong 0, \quad (\text{E.3})$$

- At the boundary $n_9 = n_7 + 2$

$$S_P \cdot \sigma(S_R) \cong -\frac{2}{35} \left(30H_B^2(26 + 3n_7) + 25\sigma(\hat{s}_Q) \cdot \sigma(\hat{s}_R) + H_B \cdot [4D_1n_7 + 8D_2n_7 + 12D_3n_7 + 6D_4n_7 + 150S_P + 150\sigma(\hat{s}_Q) + 15n_7\sigma(\hat{s}_Q) + 130\sigma(\hat{s}_R)] \right), \quad (\text{E.4})$$

- At the boundary $n_9 = 9 - n_7$

$$S_P \cdot \sigma(S_R) \cong \frac{2}{475} \left(18D_1 \cdot H_B + 36D_2 \cdot H_B - 66D_3 \cdot H_B - 33D_4 \cdot H_B - 41100H_B^2 + 8400n_7H_B^2 - 400n_7^2H_B^2 - 4425H_B \cdot S_P + 550n_7H_B \cdot S_P - 5145H_B \cdot \sigma(\hat{s}_Q) + 450n_7H_B \cdot \sigma(\hat{s}_Q) - 125\sigma(\hat{s}_Q)^2 - 2790H_B \cdot \sigma(\hat{s}_R) + 350n_7H_B \cdot \sigma(\hat{s}_R) - 275\sigma(\hat{s}_Q) \cdot \sigma(\hat{s}_R) \right). \quad (\text{E.5})$$

11 dimensional

We obtain an eleven-dimensional basis by choosing

$$H_{(V,12)}^{(2,2)} = H_{(V,13)}^{(2,2)} \setminus \{S_P \cdot \sigma(S_R), H_B \cdot \sigma(S_Q)\}, \quad (\text{E.6})$$

- At $n_7 = n_9 - 4$ we obtain the homology relation

$$\begin{aligned} S_P \cdot \sigma(S_R) &\cong 0, \\ H_B \cdot \sigma(S_Q) &\cong \frac{1}{1220 + 150n_7} \left(25D_1^2 + 684D_3 \cdot H_B + 342D_4 \cdot H_B - 3200H_B^2 \right. \\ &\quad + (438 - 45n_7)D_1 \cdot H_B + 18(32 - 5n_7)D_2 \cdot H_B - 120n_7D_3 \cdot H_B \\ &\quad - 60n_7D_4 \cdot H_B - 800H_B \cdot S_P - 125\sigma(\hat{s}_Q)^2 \\ &\quad \left. - 490H_B \cdot \sigma(\hat{s}_R) - 275\sigma(\hat{s}_Q) \cdot \sigma(\hat{s}_R) \right). \end{aligned} \quad (\text{E.7})$$

10 dimensional

Finally a ten-dimensional basis is given by

$$H_{(V,12)}^{(2,2)} = H_{(V,13)}^{(2,2)} \setminus \{S_P \cdot \sigma(S_R), H_B \cdot \sigma(S_Q), D_2 \cdot D_4\}. \quad (\text{E.8})$$

- At $n_7 = 0$

$$\begin{aligned} S_P \cdot \sigma(S_R) &\cong 2n_9 [(4 + n_9)H_B^2 + H_B \cdot \sigma(\hat{s}_R)], \\ H_B \cdot \sigma(S_Q) &\cong -2(2 + n_9)H_B^2 - 2 \left(\frac{2 + n_9}{4 + n_9} \right) \sigma(\hat{s}_R) \cdot H_B \\ &\quad - \left(\frac{1}{4 + n_9} \right) \sigma(\hat{s}_Q) \cdot \sigma(\hat{s}_R) - H_B \cdot S_P, \\ D_2 \cdot D_4 &\cong 4D_4 \cdot H_B. \end{aligned} \quad (\text{E.9})$$

F CS-levels & chiralities for $\hat{X}_{\text{SU}(5)}$ with $B = \mathbb{P}^3$

In this appendix we present the 3D Chern-Simons terms on both the M- and F-theory side and the chiralities, all of which parametrized by the parameters a_i of the general G_4 -flux.

F.1 Chern-Simons levels

F-theory loop induced CS terms

The one loop Chern-Simons levels read as follows. For the Cartan generators, we obtain Θ_{ij}^F as:

$$\begin{aligned} \Theta_{1,3}^F &= \Theta_{1,4}^F = 0, \\ \Theta_{1,1}^F &= -\frac{1}{2} \left[6\chi(\mathbf{10}_{(\frac{1}{5},0)}) + 2\chi(\mathbf{5}_{(-\frac{2}{5},0)}) + 2\chi(\mathbf{5}_{(-\frac{2}{5},1)}) + 2\chi(\mathbf{5}_{(\frac{3}{5},0)}) + 2\chi(\mathbf{5}_{(\frac{3}{5},1)}) - 2\chi(\mathbf{5}_{(-\frac{2}{5},-1)}) \right], \\ \Theta_{1,2}^F &= -\frac{1}{2} \left[-3\chi(\mathbf{10}_{(\frac{1}{5},0)}) - \chi(\mathbf{5}_{(-\frac{2}{5},0)}) - \chi(\mathbf{5}_{(-\frac{2}{5},1)}) - \chi(\mathbf{5}_{(\frac{3}{5},0)}) - \chi(\mathbf{5}_{(\frac{3}{5},1)}) + \chi(\mathbf{5}_{(-\frac{2}{5},-1)}) \right], \\ \Theta_{2,2}^F &= -\frac{1}{2} \left[2\chi(\mathbf{10}_{(\frac{1}{5},0)}) + 2\chi(\mathbf{5}_{(-\frac{2}{5},1)}) + 2\chi(\mathbf{5}_{(\frac{3}{5},0)}) + 2\chi(\mathbf{5}_{(\frac{3}{5},1)}) - 2\chi(\mathbf{5}_{(-\frac{2}{5},-1)}) \right], \end{aligned}$$

$$\begin{aligned}
 \Theta_{2,3}^F &= -\frac{1}{2} \left[-\chi(\mathbf{10}_{(\frac{1}{5},0)}) + \chi(\mathbf{5}_{(-\frac{2}{5},0)}) - \chi(\mathbf{5}_{(-\frac{2}{5},1)}) - \chi(\mathbf{5}_{(\frac{3}{5},0)}) - \chi(\mathbf{5}_{(\frac{3}{5},1)}) + \chi(\mathbf{5}_{(-\frac{2}{5},-1)}) \right], \\
 \Theta_{3,3}^F &= -\frac{1}{2} \left[2\chi(\mathbf{10}_{(\frac{1}{5},0)}) - 2\chi(\mathbf{5}_{(-\frac{2}{5},0)}) + 2\chi(\mathbf{5}_{(\frac{3}{5},1)}) - 2\chi(\mathbf{5}_{(-\frac{2}{5},-1)}) \right], \\
 \Theta_{3,4}^F &= -\frac{1}{2} \left[-\chi(\mathbf{10}_{(\frac{1}{5},0)}) + \chi(\mathbf{5}_{(-\frac{2}{5},0)}) + \chi(\mathbf{5}_{(-\frac{2}{5},1)}) + \chi(\mathbf{5}_{(\frac{3}{5},0)}) - \chi(\mathbf{5}_{(\frac{3}{5},1)}) + \chi(\mathbf{5}_{(-\frac{2}{5},-1)}) \right], \\
 \Theta_{4,4}^F &= -\frac{1}{2} \left[2\chi(\mathbf{10}_{(\frac{1}{5},0)}) - 2\chi(\mathbf{5}_{(-\frac{2}{5},0)}) - 2\chi(\mathbf{5}_{(-\frac{2}{5},1)}) - 2\chi(\mathbf{5}_{(\frac{3}{5},0)}) + 2\chi(\mathbf{5}_{(\frac{3}{5},1)}) - 2\chi(\mathbf{5}_{(-\frac{2}{5},-1)}) \right].
 \end{aligned} \tag{F.1}$$

The mixed Abelian-non-Abelian Θ_{im}^F read

$$\begin{aligned}
 \Theta_{i=1,m=1}^F &= \Theta_{i=4,m=1}^F = \Theta_{i=1,m=2}^F = \Theta_{i=2,m=2}^F = \Theta_{i=4,m=2}^F = 0, \\
 \Theta_{i=2,m=1}^F &= -\frac{1}{2} \left(-\frac{4}{5}\chi(\mathbf{10}_{(\frac{1}{5},0)}) + \frac{4}{5}\chi(\mathbf{5}_{(-\frac{2}{5},0)}) \right), \\
 \Theta_{i=3,m=1}^F &= -\frac{1}{2} \left(\frac{4}{5}\chi(\mathbf{5}_{(-\frac{2}{5},1)}) - \frac{6}{5}\chi(\mathbf{5}_{(3/5,0)}) \right), \\
 \Theta_{i=3,m=2}^F &= \chi(\mathbf{5}_{(-\frac{2}{5},1)}).
 \end{aligned} \tag{F.2}$$

The purely Abelian Θ_{mn}^F read

$$\begin{aligned}
 \Theta_{m=1,m=1}^F &= -\frac{1}{2} \left(\frac{4}{25}\chi(\mathbf{10}_{(\frac{1}{5},0)}) - \frac{4}{25}\chi(\mathbf{5}_{(-\frac{2}{5},0)}) + \frac{4}{25}\chi(\mathbf{5}_{(-\frac{2}{5},1)}) \right. \\
 &\quad \left. + \frac{9}{25}\chi(\mathbf{5}_{(\frac{3}{5},0)}) + \frac{9}{5}\chi(\mathbf{5}_{(\frac{3}{5},1)}) - \frac{4}{5}\chi(\mathbf{5}_{(-\frac{2}{5},-1)}) + \chi(\mathbf{1}_{(1,0)}) \right. \\
 &\quad \left. + \chi(\mathbf{1}_{(1,1)}) + \chi(\mathbf{1}_{(-1,1)}) - 3\chi(\mathbf{1}_{(-1,-2)}) \right), \\
 \Theta_{m=1,m=2}^F &= -\frac{1}{2} \left(-\frac{2}{5}\chi(\mathbf{5}_{(-\frac{2}{5},1)}) + 3\chi(\mathbf{5}_{(3/5,1)}) - 2\chi(\mathbf{5}_{(-\frac{2}{5},-1)}) \right. \\
 &\quad \left. + \chi(\mathbf{1}_{(1,1)}) - \chi(\mathbf{1}_{(-1,1)}) - 6\chi(\mathbf{1}_{(-1,-2)}) \right), \\
 \Theta_{m=2,m=2}^F &= -\frac{1}{4} \left(\chi(\mathbf{5}_{(-\frac{2}{5},1)}) + 5\chi(\mathbf{5}_{(\frac{3}{5},1)}) - 5\chi(\mathbf{5}_{(-2/5,-1)}) + \chi(\mathbf{1}_{(1,1)}) \right. \\
 &\quad \left. + \chi(\mathbf{1}_{(0,1)}) + \chi(\mathbf{1}_{(-1,1)}) + 4\chi(\mathbf{1}_{(0,2)}) - 12\chi(\mathbf{1}_{(-1,-2)}) \right).
 \end{aligned} \tag{F.3}$$

M-theory classical CS terms

On the M-theory side we obtain for Θ_{ij}^M :

$$\begin{aligned}
 \Theta_{1,3}^M &= \Theta_{1,4}^M = 0, & \Theta_{2,4}^M &= 2n_7 a_{12}, \\
 \Theta_{(1,1)}^M &= \frac{1}{5}(6+n_9)a_3 + \frac{1}{2}(4+n_9)a_4 + n_9(4-n_7+n_9)a_5 \\
 &\quad + \frac{1}{25}[-10(4+n_9)(8+3n_9) + 3n_7(14+9n_9)]a_6 \\
 &\quad + \frac{1}{250}[-2968 + n_7(56+111n_9) - n_9(1148+115n_9)]a_7 \\
 &\quad - \frac{1}{5}(-184 + 40n_7 - 23n_9 + n_9^2)a_{13},
 \end{aligned}$$

$$\begin{aligned}
 \Theta_{1,2}^M = & \frac{1}{500} \left(-50(6+n_9)a_3 - 125(4+n_9)a_4 \right. \\
 & + 250(-4+n_7-n_9)n_9a_5 \\
 & + 10[10(4+n_9)(8+3n_9) - 3n_7(14+9n_9)]a_6 \\
 & + (2968 - n_7(56+111n_9) + n_9(1148+115n_9))a_7 \\
 & \left. + 50(-184+40n_7+(-23+n_9)n_9)a_{13} \right), \tag{F.4}
 \end{aligned}$$

$$\begin{aligned}
 \Theta_{2,2}^M = & \frac{1}{250} \left(25(21-n_7+n_9)a_3 + 125(4+n_9)a_4 + 250n_9(4-n_7+n_9)a_5 \right. \\
 & + 5[-2n_7^2 - 5(4+n_9)(41+11n_9) + n_7(122+57n_9)]a_6 \\
 & + [-5524 + 7n_7^2 - 3n_9(273+40n_9) + n_7(277+113n_9)]a_7 \\
 & \left. + 25[64 + n_7(-24+n_9) - (-13+n_9)n_9]a_{13} \right),
 \end{aligned}$$

$$\begin{aligned}
 \Theta_{2,3}^M = & \frac{1}{500} \left(50(-17+n_7)a_3 - 125(4+n_9)a_4 + 250(-4+n_7-n_9)n_9a_5 \right. \\
 & + 10[2n_7^2 - 6n_7(14+5n_9) + 5(4+n_9)(27+5n_9)]a_6 \\
 & + [8616 - 14n_7^2 - 5n_7(94+23n_9) + 5n_9(114+25n_9)]a_7 \\
 & \left. - 500n_7a_{12} + 50(-n_7(-4+n_9) + 2(6+n_9))a_{13} \right),
 \end{aligned}$$

$$\begin{aligned}
 \Theta_{3,3}^M = & \frac{1}{500} \left(25(50-5n_7+3n_9)a_3 - 125(n_7-2(4+n_9))a_4 \right. \\
 & + 500n_9(4-n_7+n_9)a_5 + 50(-14+n_9)n_9a_{13} \\
 & + 5[-25n_7^2 - 5(4+n_9)(90+23n_9) + 2n_7(210+73n_9)]a_6 \\
 & \left. + 4[-3030 - n_9(483+65n_9) + n_7(310+66n_9)]a_7 \right),
 \end{aligned}$$

$$\begin{aligned}
 \Theta_{3,4}^M = & \frac{1}{500} \left(25(-16+3n_7-3n_9)a_3 + 125(-4+n_7-n_9)a_4 + 250(-4+n_7-n_9)n_9a_5 \right. \\
 & + 5[720 + 21(-12+n_7)n_7 + 440n_9 - 86n_7n_9 + 65n_9^2]a_6 \\
 & + [14(-55+n_7)n_7 - 149n_7n_9 + 135n_9^2 + 6(584+227n_9)]a_7 + 500n_7a_{12} \\
 & \left. + 50[-12+n_7(-4+n_9) - (-12+n_9)n_9]a_{13} \right),
 \end{aligned}$$

$$\begin{aligned}
 \Theta_{4,4}^M = & \frac{1}{10}(16-3n_7+3n_9)a_3 + \frac{1}{2}(4-n_7+n_9)a_4 + n_9(4-n_7+n_9)a_5 \\
 & + \frac{1}{50}[-21n_7^2 - 5(4+n_9)(36+13n_9) + n_7(252+86n_9)]a_6 \\
 & + \frac{1}{250}[-14n_7^2 + n_7(770+149n_9) - 3(1168+n_9(454+45n_9))]a_7 \\
 & - 4n_7a_{12} + \frac{1}{5}[4(3+n_7) - (12+n_7)n_9 + n_9^2]a_{13}. \tag{F.5}
 \end{aligned}$$

For the mixed CS-levels Θ_{im}^M we obtain:

$$\begin{aligned}
 \Theta_{i=1,m=1}^M &= \Theta_{i=1,m=2}^M = \Theta_{i=2,m=2}^M = 0, \\
 \Theta_{i=4,m=1}^M &= \frac{2}{5}n_7a_{12}, \quad \Theta_{i=4,m=2}^M = 2n_7a_{12}, \\
 \Theta_{i=2,m=1}^M &= \frac{1}{625} \left(25(-15 + n_7 + n_9)a_3 + 5(-15 + n_7 + n_9)(2n_7 - 5(4 + n_9))a_6 \right. \\
 &\quad \left. - [-3360 + 7n_7^2 + (209 - 5n_9)n_9 + n_7(179 + 2n_9)]a_7 \right. \\
 &\quad \left. - 25(-4 + n_9)(-5 + n_7 + n_9)a_{13} \right), \\
 \Theta_{i=3,m=1}^M &= \frac{1}{2500} \left(-25(54 + 13n_7 - 9n_9)a_3 + 250n_7a_4 \right. \\
 &\quad \left. + 5[54n_7^2 - 45(-6 + n_9)(4 + n_9) + n_7(52 + 3n_9)]a_6 \right. \\
 &\quad \left. + [-639n_7^2 - 6(-6 + n_9)(426 + 5n_9) + n_7(1690 + 677n_9)]a_7 \right. \\
 &\quad \left. - 500n_7a_{12} - 50(18 + n_7 - 3n_9)(-4 + n_9)a_{13} \right), \\
 \Theta_{i=4,m=1}^M &= \frac{1}{500}n_7 \left(50a_3 - 125a_4 + 5(40 - 21n_7 + 15n_9)a_6 \right. \\
 &\quad \left. + (-158 + 111n_7 - 115n_9)a_7 - 500a_{12} - 50(-4 + n_9)a_{13} \right). \tag{F.6}
 \end{aligned}$$

The CS-levels $\Theta_{\alpha m}^M$ for $D_\alpha = H_B$ read

$$\begin{aligned}
 \Theta_{\alpha,m=1}^M &= -\frac{17}{5}a_3 + \frac{1}{2}(-4 + n_7 - n_9)a_4 + (-4 + n_7 - n_9)n_9a_5 \\
 &\quad + [n_7^2/2 + \frac{1}{5}(4 + n_9)(27 + 5n_9) - \frac{1}{50}n_7(208 + 75n_9)]a_6 \\
 &\quad + \frac{1}{250}[8616 - n_7(312 + 125n_7) + 5n_9(114 + 25n_9)]a_7 \\
 &\quad + \frac{2}{5}(6 + n_9)a_{13}, \\
 \Theta_{\alpha,m=2}^M &= \frac{1}{2}(-4 + n_7 - n_9)a_3 - (4 + n_9)a_4 + 2(-4 + n_7 - n_9)n_9a_5 \\
 &\quad + [24 + 16n_9 + \frac{5}{2}n_9^2 - \frac{1}{10}n_7(56 + 25n_9)]a_6 \\
 &\quad + [\frac{1}{2}n_7^2 + \frac{1}{5}(4 + n_9)(27 + 5n_9) - \frac{1}{50}n_7(208 + 75n_9)]a_7 \\
 &\quad + (4 + n_9)a_{13}. \tag{F.7}
 \end{aligned}$$

The purely Abelian CS-levels Θ_{mn}^M read

$$\begin{aligned}
\Theta_{m=1,m=1}^M &= \frac{1}{12500} \left(25(8616 - n_7(312 + 125n_7) + 5n_9(114 + 25n_9))a_3 \right. \\
&\quad + 125(1080 - 208n_7 + 25n_7^2 + 470n_9 - 75n_7n_9 + 50n_9^2)a_4 \\
&\quad + 1250(-54 + 5n_7 - 10n_9)(-4 + n_7 - n_9)n_9a_5 \\
&\quad + 5 \left[625n_7^3 - 5(4 + n_9)(14016 + 5n_9(854 + 125n_9)) \right. \\
&\quad \left. - 4n_7^2(1411 + 625n_9) + 8n_7(6834 + 5n_9(893 + 125n_9)) \right] a_6 \\
&\quad + 2 \left[-1090272 - 3125n_7^3 - 25n_9(6130 + n_9(1852 + 125n_9)) \right. \\
&\quad \left. + n_7^2(26777 + 3125n_9) + n_7(82008 + 5n_9(2028 + 625n_9)) \right] a_7 \\
&\quad \left. + 50 \left[-2968 + n_7(56 + 111n_9) - n_9(1148 + 115n_9) \right] a_{13} \right), \\
\Theta_{m=1,m=2}^M &= \frac{1}{100} (25n_7^2 + 10(4 + n_9)(27 + 5n_9) - n_7(208 + 75n_9))a_3 \\
&\quad + \left[12 + 8n_9 + \frac{5}{4}n_9^2 - \frac{1}{20}n_7(56 + 25n_9) \right] a_4 \\
&\quad + \frac{1}{2}(-12 + 3n_7 - 5n_9)(-4 + n_7 - n_9)n_9a_5 \\
&\quad + \left[n_7^2 \left(-\frac{454}{125} - \frac{9}{4}n_9 \right) - \frac{1}{10}(4 + n_9)(228 + n_9(157 + 30n_9)) \right. \\
&\quad \left. + \frac{3}{100}n_7(1040 + n_9(892 + 175n_9)) \right] a_6 \\
&\quad + \frac{1}{2500} \left[625n_7^3 - 5(4 + n_9)(14016 + 5n_9(854 + 125n_9)) \right. \\
&\quad \left. + 8n_7(6834 + 5n_9(893 + 125n_9)) - 4n_7^2(1411 + 625n_9) \right] a_7 \\
&\quad + \frac{1}{25}(-10(4 + n_9)(8 + 3n_9) + 3n_7(14 + 9n_9))a_{13}, \\
\Theta_{m=2,m=2}^M &= \left[12 + 8n_9 + \frac{5}{4}n_9^2 - \frac{1}{20}n_7(56 + 25n_9) \right] a_3 \\
&\quad + \frac{1}{2} \left[-n_7(1 + 4n_9) + (4 + n_9)(12 + 5n_9) \right] a_4 \\
&\quad + (-12 + 2n_7 - 5n_9)(-4 + n_7 - n_9)n_9a_5 \\
&\quad + \frac{1}{100} \left[-25(4 + n_9)(12 + 5n_9)^2 - 6n_7^2(7 + 50n_9) \right. \\
&\quad \left. + 5n_7(688 + n_9(842 + 185n_9)) \right] a_6 \\
&\quad + \frac{1}{500} \left[-50(4 + n_9)(12 + 5n_9)(27 + 5n_9) \right. \\
&\quad \left. - n_7^2(1278 + 625n_9) + n_7(12164 + 5n_9(2542 + 375n_9)) \right] a_7 \\
&\quad + \left[-\frac{1}{2}(4 + n_9)(12 + 5n_9) + \frac{1}{5}n_7(4 + 9n_9) \right] a_{13}. \tag{F.8}
\end{aligned}$$

F.2 Chiralities

Finally, we present the rest of the chiralities obtained by matching the M-/F-theory CS-terms. The G_4 -flux parameter supported along the non-flat fiber is assumed to be zero by setting $a_{12} = 0$. The chiralities in terms of the a_i in the G_4 -flux are:

$$\begin{aligned}
 \chi(\mathbf{5}_{(-\frac{2}{5},0)}) &= \frac{1}{250} \left[-25(-13 + n_7 + n_9)a_3 - 5(-13 + n_7 + n_9)(2n_7 - 5(4 + n_9))a_6 \right. \\
 &\quad + [-3092 + 7n_7^2 + (249 - 5n_9)n_9 + n_7(193 + 2n_9)]a_7 \\
 &\quad \left. + 25(-88 + (-17 + n_9)n_9 + n_7(16 + n_9))a_{13} \right], \\
 \chi(\mathbf{5}_{(\frac{3}{5},0)}) &= \frac{1}{500} \left[-75(6 + n_7 - n_9)a_3 \right. \\
 &\quad + 5[4n_7^2 + 11n_7(4 + n_9) - 15(-6 + n_9)(4 + n_9)]a_6 \\
 &\quad + [-139n_7^2 - 2(-6 + n_9)(426 + 5n_9) + n_7(458 + 149n_9)]a_7 \\
 &\quad \left. - 50(6 + n_7 - n_9)(-4 + n_9)a_{13} \right], \\
 \chi(\mathbf{5}_{(-\frac{2}{5},-1)}) &= \frac{1}{500} \left[50(4 + n_9)a_3 + 125(4 + n_9)a_4 + 250n_9(4 - n_7 + n_9)a_5 + \right. \\
 &\quad 10(-10(4 + n_9)(7 + 3n_9) + n_7(38 + 27n_9))a_6 + \\
 &\quad (3n_7(28 + 37n_9) - (4 + n_9)(608 + 115n_9))a_7 \\
 &\quad \left. - 50(-76 + 20n_7 - 15n_9 + n_9^2)a_{13} \right], \tag{F.9}
 \end{aligned}$$

For the chiralities of the singlets we obtain

$$\begin{aligned}
 \chi(\mathbf{1}_{(1,0)}) &= \frac{1}{100} \left[25[2n_7^2 - (-9 + n_9)(-6 + n_9) - n_7(15 + n_9)]a_3 \right. \\
 &\quad + 50n_7(-4 + n_7 - n_9)n_9a_5 + 200n_7a_{13} \\
 &\quad + 5[5(-9 + n_9)(-6 + n_9)(4 + n_9) \\
 &\quad - 3n_7^2(12 + 5n_9) + n_7(276 + n_9(111 + 10n_9))]a_6 \\
 &\quad + [50n_7^3 + 244(-9 + n_9)(-6 + n_9) \\
 &\quad \left. - 3n_7^2(309 + 25n_9) + n_7(2694 + n_9(883 + 25n_9))]a_7 \right], \tag{F.10}
 \end{aligned}$$

$$\begin{aligned}
 \chi(\mathbf{1}_{(1,1)}) &= -\frac{1}{4}[76 + n_7^2 + n_7(-17 + n_9) + 11n_9 - 2n_9^2](a_3 + a_4) \\
 &\quad - \frac{1}{2}(-4 + n_7 - n_9)n_9(-19 + 2n_9)a_5 + \left[12 + \frac{1}{2}(2 + n_7 - n_9)n_9 \right] a_{13} \\
 &\quad + \frac{1}{4}[608 - n_7(212 + (-25 + n_7)n_7) + 316n_9 \\
 &\quad + (-56 + n_7)n_7n_9 + (17 + 6n_7)n_9^2 - 6n_9^3]a_6 \\
 &\quad + \frac{1}{100}[n_7^2(262 - 25n_9) + n_7(-4454 + 3n_9(-91 + 25n_9)) \\
 &\quad - (4 + n_9)(-4918 + n_9(39 + 50n_9))]a_7,
 \end{aligned}$$

$$\begin{aligned}
 \chi(\mathbf{1}_{(0,1)}) &= \frac{1}{4} [n_7(-15 + 2n_7) + (4 + n_9)(-19 + 2n_9)] a_4 \\
 &\quad - \frac{1}{2} (-4 + n_7 - n_9)n_9(-19 + 2n_9) a_5 \\
 &\quad + \left[76 + 49n_9 + \frac{1}{20} ((-12 + n_7)n_7(1 + 10n_7) \right. \\
 &\quad \left. - n_7(79 + 10n_7)n_9 + 10(7 + 2n_7)n_9^2 - 20n_9^3) \right] a_6 \\
 &\quad + \frac{1}{100} [-50n_7^3 + n_7^2(403 + 50n_9) + 2n_7(541 + n_9(-488 + 25n_9)) \\
 &\quad - (4 + n_9)(-212 + n_9(-449 + 50n_9))] a_7 \\
 &\quad + \frac{1}{2} [52 - n_9(-9 + n_7 + n_9)] a_{13}.
 \end{aligned} \tag{F.11}$$

G Toric tuning of \mathcal{S}_7 and \mathcal{S}_9 for $B = \mathbb{P}^3$

In this appendix we explain how to construct the toric polytopes of $dP_2^B(\mathcal{S}_7, \mathcal{S}_9)$ with base $B = \mathbb{P}^3$ for all the points in the allowed region of figure 2. In the following we present a list of vertices, each of which realizing one possible choice of (n_7, n_9) .

In order to find this list of vertices we follow the algorithm presented in [47] to construct all reflexive convex polytopes for a given toric base and a given toric fiber. This algorithm uses the $GL(\mathbb{Z}, 5)$ symmetry to bring all the vertices of the polytope in the following form

variable	vertices				
z_0	1	1	1	p_1	p_2
z_1	-1	0	0	0	0
z_3	0	-1	0	0	0
z_2	0	0	-1	0	0
u	0	0	0	1	0
v	0	0	0	0	1
w	0	0	0	-1	-1
e_1	0	0	0	0	-1
e_2	0	0	0	1	1

(G.1)

In the first four lines we see the polytope of the base, in the last five lines the polytope of the fiber. The degrees of freedom of the fibration are parametrized by the two integers (p_1, p_2) . The coordinates (p_1, p_2) are fixed by the requirement of convexity of the polytope (G.1). We obtain the following list of points (p_1, p_2) each of which giving rise to a reflexive and convex polytope (G.1), along with the corresponding values of (n_7, n_9) :

p_1	p_2	n_7	n_9	p_1	p_2	n_7	n_9	p_1	p_2	n_7	n_9
-4	-4	8	4	0	-3	4	1	2	-1	2	1
-3	-4	7	3	0	-2	4	2	2	0	2	2
-3	-3	7	4	0	-1	4	3	2	1	2	3
-3	-2	7	5	0	0	4	4	2	2	2	4
-2	-4	6	2	0	1	4	5	2	3	2	5
-2	-3	6	3	0	2	4	6	2	4	2	6
-2	-2	6	4	0	3	4	7	3	-1	1	0
-2	-1	6	5	0	4	4	8	3	0	1	1
-2	0	6	6	1	-3	3	0	3	1	1	2
-1	-4	5	1	1	-2	3	1	3	2	1	3
-1	-3	5	2	1	-1	3	2	3	3	1	4
-1	-2	5	3	1	0	3	3	3	4	1	5
-1	-1	5	4	1	1	3	4	4	0	0	0
-1	0	5	5	1	2	3	5	4	1	0	1
-1	1	5	6	1	3	3	6	4	2	0	2
-1	2	5	7	1	4	3	7	4	3	0	3
0	-4	4	0	2	-2	2	0	4	4	0	4

(G.2)

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