Two-loop QCD helicity amplitudes for $g\,g \to Z\,g$ and $g\,g \to Z\,\gamma$

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Abstract: We compute the helicity amplitudes for the processes $gg \to Zg$ and $gg \to Z\gamma$ to two loops in massless QCD. The perturbative expansion of these processes starts only at the one-loop level, such that our results are a crucial ingredient to the NLO corrections to $Z\gamma$ and Z+jet production through gluon fusion.

Contents

1 Introduction

The production of vector bosons at hadron colliders is, to a first approximation, induced by quark-antiquark annihilation. Including corrections from higher orders in the perturbative expansion in QCD, other processes will also contribute to vector boson final states. These contributions are suppressed by higher orders in the strong coupling constant α_s, but could receive a numerical enhancement through the relevant parton-parton luminosity. In particular, in high-energy proton-proton collisions at the LHC, gluon-induced higher-order processes can become of comparable importance to quark-induced processes due to the large gluon luminosity at invariant masses relevant to vector boson production.

Vector-boson production in gluon-gluon collisions is mediated through a quark loop, which vanishes for the exclusive $gg \to V$ vertex due to Furry's theorem. The gluon-gluon-induced subprocess becomes relevant for the production of vector boson pairs (WW, ZZ, $\gamma\gamma$ and $Z\gamma$), or for the production of a neutral vector boson and a gluon. The leading-order scattering amplitudes for these processes all involve a closed quark loop. The resulting gluon-induced contributions from one-loop squared [1–4] processes (that appear only at next-to-next-to-leading order in the formal perturbative expansion of the full process) were evaluated a long time ago [5–10], and typically found to yield a contribution that amounts to 10–20% of the total cross section. Inclusion of these gluon-gluon subprocess contributions often results in an enhanced theoretical uncertainty on the prediction, since the one-loop squared process is effectively Born-level for this combination of partons. To stabilise these predictions, the computation of the next perturbative order in vector-boson pair production or vector-boson-plus-jet production in gluon fusion is required. Technically, such a calculation amounts to computing the corrections from single real radiation or single virtual exchange to the Born processes. With the Born process itself being a one-loop amplitude, one thus requires the two-loop corrections to the relevant partonic amplitudes. Up to now, these were obtained [11] only for $gg \to \gamma\gamma$, where the NLO correction to the gluon-induced process was found to be sizeable and important in the stabilisation of the theoretical prediction for photon pair production [12, 13].

In this paper, we derive in massless QCD the two-loop corrections to the helicity amplitudes relevant to the production of a Z-boson in association with either a real photon or a hadronic jet in gluon-gluon collisions: $gg \to Z\gamma$ and $gg \to Zg$. For these processes, the one-loop amplitudes involving an extra gluon in the final state can be obtained using by-now standard methods for the computation of one-loop multi-leg processes [14–19]. With the results derived here, a complete NLO calculation of $Z\gamma$ and Zj production in gluon fusion becomes thus feasible.

This paper is structured as follows: in section 2, we fix the notation and discuss the basic helicity structure of the process under consideration. The general tensor structure of the amplitude is described in section 3 and expressed through helicity amplitudes in section 4. The calculation of the two-loop amplitudes, their renormalisation and infrared properties and their simplification are described in section 5. The two-loop helicity amplitudes are obtained in a closed analytic form. We performed several non-trivial checks on the results, which are described in section 6. We conclude with an outlook in section 7. We enclose appendices with the analytical form for the one-loop and two-loop helicity amplitudes in the decay kinematics $V \to ggg$ and $V \to gg\gamma$. The helicity amplitudes continued to the regions relevant for vector-boson-plus-jet and vector-boson-plus-photon production at LHC are given in Mathematica format together with the arXiv submission of this paper.

2 Kinematics and notations

The production of a massive vector boson $V = (Z^0, \gamma^*)$ and a gluon (photon) through gluon-gluon fusion is related by crossing to the decay of a massive vector boson to three gluons (two gluons and a photon) and has the same kinematics as vector-boson-plus-jet production

$q\bar{q} \to Vg$, $qg \to Vq$ and vector-boson-plus-photon production $q\bar{q} \to V\gamma$. Technically the calculation of the two-loop QCD corrections to the $gg \to Vg$ and $gg \to V\gamma$ amplitudes is thus similar to previous calculations for $3j$-production, vector-boson-plus-photon production and $H \to 3$ partons, which have been derived to two-loop accuracy in QCD [20–22].

In the following we will focus on the decay kinematics, while the crossings relevant for V-plus-jet and V-plus-photon production at hadron colliders will be discussed in section 4.3.

The relevant partonic subprocesses are:

$$l^-(p_5) + l^+(p_6) \to V(q) \to g(p_1) + g(p_2) + g(p_3) \,,$$
$$l^-(p_5) + l^+(p_6) \to V(q) \to g(p_1) + g(p_2) + \gamma(p_3) \,, \qquad (2.1)$$

where we included the production of the vector boson V through lepton-antilepton annihilation.

In the framework of massless QCD interchanging the virtual photon with a Z boson amounts only to a proper re-weighting of the final result. Moreover, note that we always assume massless fermions in the initial or final state.

The momentum of the vector boson is given by

$$q^\mu = p_1^\mu + p_2^\mu + p_3^\mu \,. \qquad (2.2)$$

It is convenient to define the usual invariants

$$s_{12} = (p_1 + p_2)^2 \,, \qquad s_{13} = (p_1 + p_3)^2 \,, \qquad s_{23} = (p_2 + p_3)^2 \,, \qquad (2.3)$$

which fulfil

$$q^2 = (p_1 + p_2 + p_3)^2 = s_{12} + s_{13} + s_{23} \equiv s_{123} \,, \qquad (2.4)$$

as well as the dimensionless invariants

$$x = s_{12}/s_{123} \,, \qquad y = s_{13}/s_{123} \,, \qquad z = s_{23}/s_{123} \,, \qquad (2.5)$$

which satisfy $x + y + z = 1$.

In the decay kinematics $V \to ggg/gg\gamma$, as in the $3j$ case, q^2 is time-like (hence positive) and all the s_{ij} are also positive, which implies that x, y, z all lie in the interval $[0; 1]$, with the above constraint $x + y + z = 1$.

The helicity amplitudes can be expressed as a product of a partonic current S_μ and a leptonic current L_μ:

$$A(p_5, p_6; g_1, g_2, b_3) = L^\mu(p_5; p_6) S_\mu(g_1; g_2; b_3) \qquad (2.6)$$

where $g_i = g(p_i)$, and $b_3 = b(p_3)$ labels a generic massless gauge boson. In our case $b = g, \gamma$ in $V \to ggg$ and $V \to gg\gamma$ respectively.

The purely vectorial tree-level leptonic current reads:

$$L^\mu(p_5, p_6) = \bar{v}(p_6) \, \gamma^\mu \, u(p_5), \qquad (2.7)$$

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where in the case of an incoming lepton-antilepton pair $L_\mu(p_5^-, p_6^+)$ corresponds to a left-handed current, and $L_\mu(p_5^+, p_6^-)$ to a right-handed current:

$$L_L^\mu(p_5^-, p_6^+) = \bar{v}_+(p_6)\,\gamma^\mu\,u_-(p_5), \qquad L_R^\mu(p_5^+, p_6^-) = \bar{v}_-(p_6)\,\gamma^\mu\,u_+(p_5). \tag{2.8}$$

Only the partonic currents receive contributions from QCD radiative corrections, and they can be perturbatively decomposed as:

$$S_\mu(g_1; g_2; g_3) = \sqrt{4\pi\alpha_s}\,d^{a_1 a_2 a_3}\left[\left(\frac{\alpha_s}{2\pi}\right)S_\mu^{(1)}(g_1; g_2; g_3) + \left(\frac{\alpha_s}{2\pi}\right)^2 S_\mu^{(2)}(g_1; g_2; g_3) + \mathcal{O}(\alpha_s^3)\right],$$

$$S_\mu(g_1; g_2; \gamma_3) = \sqrt{4\pi\alpha}\,\delta^{a_1 a_2}\left[\left(\frac{\alpha_s}{2\pi}\right)S_\mu^{(1)}(g_1; g_2; \gamma_3) + \left(\frac{\alpha_s}{2\pi}\right)^2 S_\mu^{(2)}(g_1; g_2; \gamma_3) + \mathcal{O}(\alpha_s^3)\right],$$

where we factored out the overall colour factors $\delta^{a_1 a_2}$, $d^{a_1 a_2 a_3}$.

The general form of the gauge boson coupling to fermions is:

$$\mathcal{V}_\mu^{V, f_1 f_2} = -i\,e\,\Gamma_\mu^{V, f_1 f_2} \qquad \text{with} \qquad e = \sqrt{4\pi\alpha}, \tag{2.9}$$

whose explicit form depends on the gauge boson, on the type of fermions, and on their helicities:

$$\Gamma_\mu^{V, f_1 f_2} = L_{f_1 f_2}^V\,\gamma_\mu\left(\frac{1 - \gamma_5}{2}\right) + R_{f_1 f_2}^V\,\gamma_\mu\left(\frac{1 + \gamma_5}{2}\right). \tag{2.10}$$

The left- and right-handed couplings are identical for a pure vector interaction, and are in general different if vector and axial-vector interactions contribute. Their values for a photon are

$$L_{f_1 f_2}^{\gamma^*} = R_{f_1 f_2}^{\gamma^*} = -e_{f_1}\,\delta_{f_1 f_2}, \tag{2.11}$$

while for a Z boson

$$L_{f_1 f_2}^Z = \frac{I_3^{f_1} - \sin^2\theta_w e_{f_1}}{\sin\theta_w \cos\theta_w}\,\delta_{f_1 f_2}, \qquad\qquad R_{f_1 f_2}^Z = -\frac{\sin\theta_w e_{f_1}}{\cos\theta_w}\,\delta_{f_1 f_2}. \tag{2.12}$$

The vector boson propagator can be written as:

$$P_{\mu\nu}^V(q, \xi) = \frac{i\,\Delta_{\mu\nu}^V(q, \xi)}{D_V(q)}, \tag{2.13}$$

where $\Delta_{\mu\nu}^V(q, \xi)$ and $D_V(q)$ are, respectively, the numerator and the denominator in the R_ξ gauge:

$$\Delta_{\mu\nu}^V(q, \xi) = \left(-g_{\mu\nu} + (1 - \xi)\frac{q_\mu q_\nu}{q^2 - \xi M_V^2}\right), \tag{2.14}$$

$$D_Z(q) = \left(q^2 - M_Z^2 + i\Gamma_Z M_Z\right), \tag{2.15}$$

$$D_{\gamma^*}(q) = q^2. \tag{2.16}$$

In the narrow-width approximation we can simplify expression (2.15) to

$$D_Z(q) \approx i\Gamma_Z M_Z \qquad \text{and} \qquad q^2 = M_Z^2, \tag{2.17}$$

where M_Z is the mass of the Z boson, while Γ_Z is its decay width.

Since we do not consider any electroweak corrections, the vector boson V is always coupled to a fermion line which allows us to neglect the R_ξ dependence (or equivalently to put $\xi = 1$). A further consequence is that the total amplitude is proportional to the charge weighted sum over the quark flavours, such that all electroweak couplings can be collected into a multiplicative factor Q_V^b. With this notation we obtain for an incoming right-handed lepton-antilepton pair, for the different choices of $V = (\gamma^*, Z)$, and helicity configurations (h_1, h_2, h_3):

$$\mathcal{M}_V(p_5^+, p_6^-; g_1^{h_1}, g_2^{h_2}, b_3^{h_3}) = -i\,(4\pi\alpha)\,\frac{R_{f_5 f_6}^V Q_V^b}{D_V(p_5 + p_6)}\,A_R^{(h_1\,h_2\,h_3)}(p_5, p_6; g_1, g_2, b_3), \quad (2.18)$$

In case of $V = \gamma^*$ we find

$$Q_{\gamma^*}^g = \sum_q e_q, \tag{2.19}$$

$$Q_{\gamma^*}^\gamma = \sum_q e_q^2, \tag{2.20}$$

where the sum runs over the quark flavours in the loop.

In the case of $V = Z$ we have a contribution from the vector component of the Z boson, which is given by

$$Q_Z^g = \frac{1}{2}\sum_q (L_{qq}^Z + R_{qq}^Z), \tag{2.21}$$

$$Q_Z^\gamma = \frac{1}{2}\sum_q (L_{qq}^Z + R_{qq}^Z)e_q, \tag{2.22}$$

but also a contribution involving its axial coupling. This contribution vanishes for $Z \to gg\gamma$ due to charge conjugation invariance, already before summing over the quark flavours in the loop. On the other hand, in the case of $Z \to ggg$ it vanishes only after summing over the quark flavours.

3 The general tensor structure

In order to extract the helicity amplitudes from a generic QCD process different approaches can be attempted. One possibility is to decompose the amplitude into linearly independent tensor structures, whose number and form are entirely determined by symmetry considerations and which are completely independent on the loop order we are interested in. The entire loop-dependence is then contained in the scalar coefficients which multiply the relevant tensor structures. In order to single out these coefficients we apply projectors defined in d-continuous dimensions directly on the Feynman-diagrammatic expression for the amplitude [20–23].

Using Lorentz invariance one can show that there are 138 independent Lorentz structures which can contribute to the partonic current [23]:

$$S^{\mu\nu\rho\sigma} = a_1 g^{\mu\nu} g^{\rho\sigma} + a_2 g^{\mu\rho} g^{\nu\sigma} + a_3 g^{\mu\sigma} g^{\nu\rho}$$

$$+ \sum_{j_1,j_2=1}^{3} \left(b_{j_1 j_2}^1 \, g^{\mu\nu} \, p_{j_1}^\rho \, p_{j_2}^\sigma + b_{j_1 j_2}^2 \, g^{\mu\rho} \, p_{j_1}^\nu \, p_{j_2}^\sigma + b_{j_1 j_2}^3 \, g^{\mu\sigma} \, p_{j_1}^\nu \, p_{j_2}^\rho \right.$$

$$\left. + b_{j_1 j_2}^4 \, g^{\nu\rho} \, p_{j_1}^\mu \, p_{j_2}^\sigma + b_{j_1 j_2}^5 \, g^{\nu\sigma} \, p_{j_1}^\mu \, p_{j_2}^\rho + b_{j_1 j_2}^6 \, g^{\rho\sigma} \, p_{j_1}^\mu \, p_{j_2}^\nu \right)$$

$$+ \sum_{j_1,j_2,j_3,j_4=1}^{3} c_{j_1 j_2 j_3 j_4} p_{j_1}^\mu \, p_{j_2}^\nu \, p_{j_3}^\rho \, p_{j_4}^\sigma. \tag{3.1}$$

Not all these tensors will be relevant for our computations. Defining the physical amplitude contracted with the external polatization vectors of the three massless on-shell bosons:

$$S_\mu(g_1; g_2; b_3) = S_{\mu\nu\rho\sigma}(p_1; p_2; p_3) \, \epsilon_1^\nu(g) \, \epsilon_2^\rho(g) \, \epsilon_3^\sigma(b), \tag{3.2}$$

we see that many of the structures do not contribute because of the transversality condition:

$$\epsilon_i \cdot p_i = 0, \qquad \text{with} \quad i = 1, 2, 3 \, .$$

This reduces the number of independent tensors to 57. One way of proceeding is then to apply Ward identities for the massless bosons

$$S^{\mu\nu\rho\sigma} \, p_1^\nu \, \epsilon_2^\rho \, \epsilon_3^\sigma = S^{\mu\nu\rho\sigma} \, \epsilon_1^\nu \, p_2^\rho \, \epsilon_3^\sigma = S^{\mu\nu\rho\sigma} \, \epsilon_1^\nu \, \epsilon_2^\rho \, p_3^\sigma = 0. \tag{3.3}$$

which lowers the number of relevant structures down to 18. Applying finally current conservation for the massive boson

$$S^{\mu\nu\rho\sigma} \, \epsilon_1^\nu \, \epsilon_2^\rho \, \epsilon_3^\sigma \, p_4^\mu = 0. \tag{3.4}$$

further reduces the number of independent tensor coefficients to 14.

By requiring the amplitude to be invariant under the exchange of the three (two) gluons one can find further relations among these 14 coefficients with interchanged arguments. This allows to perform different checks on the final result (see section 6).

Once the tensor structure is known, one can compute d-dimensional projection operators that applied on $S_{\mu\nu\rho\sigma}$ extract each of the 14 coefficients. The tensors and the projectors contain a large number of individual terms. Therefore applying them to an amplitude in a Feynman-diagrammatic approach will in general result in a large number of contractions with a huge proliferation of terms.

Moreover, it must be noted that the basis of tensors is not unique, namely that any set of 14 tensors, obtained as independent linear combinations of those found above, can be chosen. Choosing suitable linear combinations of the above tensors can simplify their structure substantially.

For all these reasons we decided to follow a simplified approch, which nevertheless allows us to retain the full information on the process. It is well known that when performing

a computation with a large number of external bosons a specific gauge choice can highly simplify the intermediate steps of the calculation, while gauge invariance ensures that the final result for the amplitude must be independent on the choice made. Following this idea, instead of imposing gauge invariance on the tensor structures, we chose to fix the gauge of the external particles in order to symplify the tensor structures as much as possible.

Naively one would expect the loss of gauge invariance on the tensors, together with the loss of part of the symmetry due to the gauge choice performed, to be a drawback of this approach. However, one can show that once these 14 coefficients are known, the full gauge-invariant tensor can be reconstructed. In particular one can find linear relations among the 14 coefficients obtained imposing the gauge fixing and the 14 coefficients of the gauge invariant tensor, as outlined in the following section.

3.1 The gauge-fixed tensor structure

Following the above reasoning, we replace the condition (3.3) with a gauge choice on the external on-shell bosons:

$$\epsilon_1 \cdot p_2 = \epsilon_2 \cdot p_3 = \epsilon_3 \cdot p_1 = 0. \tag{3.5}$$

This choice is arbitrary and could be substituted by any other set of gauge conditions. The advantage of this particular choice is to produce extremely compact tensor structures.

Fixing the gauge of the external bosons reduces the number of independent tensors to 18. Also in this case we impose current conservation (3.4) on the Z^0 and end up again with 14 tensor structures. As expected, the number of independent tensor structures obtained in this way is the same as for the gauge-independent tensor.

We decompose the parton current as

$$S^\mu(g_1, g_2, b_3) = \sum_{i=1}^{14} A_i^{(b)} T_i^\mu, \tag{3.6}$$

where the coefficients are functions of the mandelstam variables $A_i^{(b)} = A_i^{(b)}(s_{12}, s_{13}, s_{23})$ and their explicit values differ in general if b is a gluon or a photon.

Finally, the gauge-fixed tensors read:

$$T_1^\mu = \epsilon_1 \cdot p_3 \, \epsilon_3 \cdot p_2 \, \epsilon_2^\mu - \epsilon_2 \cdot p_1 \, \epsilon_3 \cdot p_2 \, \epsilon_1^\mu, \quad T_2^\mu = \epsilon_1 \cdot p_3 \, \epsilon_2 \cdot p_1 \, \epsilon_3^\mu - \epsilon_2 \cdot p_1 \, \epsilon_3 \cdot p_2 \, \epsilon_1^\mu, \tag{3.7}$$

$$T_3^\mu = \epsilon_1 \cdot \epsilon_2 \left[\epsilon_3 \cdot p_2 \, p_1^\mu - \frac{(s_{12} + s_{13})}{2} \, \epsilon_3^\mu \right], \tag{3.8}$$

$$T_4^\mu = \epsilon_1 \cdot \epsilon_2 \left[\epsilon_3 \cdot p_2 \, p_2^\mu - \frac{(s_{12} + s_{23})}{2} \, \epsilon_3^\mu \right], \tag{3.9}$$

$$T_5^\mu = \epsilon_1 \cdot \epsilon_2 \left[\epsilon_3 \cdot p_2 \, p_3^\mu - \frac{(s_{13} + s_{23})}{2} \, \epsilon_3^\mu \right], \tag{3.10}$$

$$T_6^\mu = \epsilon_1 \cdot \epsilon_3 \left[\epsilon_2 \cdot p_1 \, p_1^\mu - \frac{(s_{12} + s_{13})}{2} \, \epsilon_2^\mu \right], \tag{3.11}$$

$$T_7^\mu = \epsilon_1 \cdot \epsilon_3 \left[\epsilon_2 \cdot p_1 \, p_2^\mu - \frac{(s_{12} + s_{23})}{2} \, \epsilon_2^\mu \right], \tag{3.12}$$

$$T_8^\mu = \epsilon_1 \cdot \epsilon_3 \left[\epsilon_2 \cdot p_1 \, p_3^\mu - \frac{(s_{13} + s_{23})}{2} \, \epsilon_2^\mu \right], \tag{3.13}$$

$$T_9^\mu = \epsilon_2 \cdot \epsilon_3 \left[\epsilon_1 \cdot p_3 \, p_1^\mu - \frac{(s_{12} + s_{13})}{2} \, \epsilon_1^\mu \right], \tag{3.14}$$

$$T_{10}^\mu = \epsilon_2 \cdot \epsilon_3 \left[\epsilon_1 \cdot p_3 \, p_2^\mu - \frac{(s_{12} + s_{23})}{2} \, \epsilon_1^\mu \right], \tag{3.15}$$

$$T_{11}^\mu = \epsilon_2 \cdot \epsilon_3 \left[\epsilon_1 \cdot p_3 \, p_3^\mu - \frac{(s_{13} + s_{23})}{2} \, \epsilon_1^\mu \right], \tag{3.16}$$

$$T_{12}^\mu = \epsilon_2 \cdot p_1 \, \epsilon_3 \cdot p_2 \left[\epsilon_1 \cdot p_3 \, p_1^\mu - \frac{(s_{12} + s_{13})}{2} \, \epsilon_1^\mu \right], \tag{3.17}$$

$$T_{13}^\mu = \epsilon_2 \cdot p_1 \, \epsilon_3 \cdot p_2 \left[\epsilon_1 \cdot p_3 \, p_2^\mu - \frac{(s_{12} + s_{23})}{2} \, \epsilon_1^\mu \right], \tag{3.18}$$

$$T_{14}^\mu = \epsilon_2 \cdot p_1 \, \epsilon_3 \cdot p_2 \left[\epsilon_1 \cdot p_3 \, p_3^\mu - \frac{(s_{13} + s_{23})}{2} \, \epsilon_1^\mu \right]. \tag{3.19}$$

The relations among the $A_i^{(b)}$ and the coefficients of the gauge invariant tensor can be found by performing on the latter the gauge fixing (3.5). This procedure obviously does not affect the scalar coefficients which multiply the tensor structures. One ends up then with 14 new tensor structures which can be related through linear combinations to those obtained fixing the gauge from the beginning. In this way the gauge invariant tensor can be fully reconstructed. We have verified this procedure by comparing our one-loop result with the literature [5, 6] where the results are given for an on-shell Z boson, and a different gauge choice is used (see section 6).

Once the tensor structure is known, one can obtain the coefficients $A_i^{(b)}$ by applying a set of projectors $P_\mu(A_i^{(b)})$ on the Feynman-diagrammatic expression of the amplitude defined such that

$$\sum_{\text{spin}} P^\mu(A_i^{(b)}) \, S_\mu(p_1, p_2, p_3) = A_i^{(b)}.$$

Note that the projection has to be performed in d dimensions, and that special care has to be taken in performing the polarization sums when applying the projectors on the single diagrams. In particular one has to consistently use a physical polarization sum which respects the gauge choice (3.5):

$$\sum_{\text{spin}} \epsilon_1^{*\mu}(p_1) \, \epsilon_1^\nu(p_1) = -g^{\mu\nu} + \frac{p_1^\mu p_2^\nu + p_1^\nu p_2^\mu}{p_1 \cdot p_2}, \tag{3.20}$$

$$\sum_{\text{spin}} \epsilon_2^{*\mu}(p_2) \, \epsilon_2^\nu(p_2) = -g^{\mu\nu} + \frac{p_2^\mu p_3^\nu + p_2^\nu p_3^\mu}{p_2 \cdot p_3}, \tag{3.21}$$

$$\sum_{\text{spin}} \epsilon_3^{*\mu}(p_3) \, \epsilon_3^\nu(p_3) = -g^{\mu\nu} + \frac{p_3^\mu p_1^\nu + p_3^\nu p_1^\mu}{p_3 \cdot p_1}. \tag{3.22}$$

The projectors themselves can be decomposed in the tensor basis and take the form:

$$P^\mu(A_j^{(b)}) = \sum_{j=1}^{14} X_i(A_j^{(b)}) \, T_i^{*\mu} \tag{3.23}$$

where the $X_i(A_j^{(b)})$ are functions of d and the kinematical invariants s_{ij}.

4 Helicity amplitudes

By fixing the helicities of the external massless bosons the partonic current can be cast in the usual spinor helicity notation [24]. There are two independent helicity configurations in the $gggV$-case, and three independent helicity configurations in the $gg\gamma V$-case, from which all the others can be obtained. In the following we discuss separately the two cases.

4.1 $gggV$: the amplitude in spinor helicity notation

We start off considering the $gggV$-case. We choose as two independent helicity configurations (g_1^+, g_2^-, g_3^-) and (g_1^+, g_2^+, g_3^+). In order to include the spin-correlations with the leptonic decay products we contract the partonic current with the leptonic current L_μ for fixed helicities of the initial state leptons. This also helps to further simplify the result.

Consider the production of the vector boson V through lepton-antilepton annihilation:

$$l^-(p_5) + l^+(p_6) \longrightarrow V(q).$$

The leptonic currents (2.7) are

$$L_R^\mu(p_5^+, p_6^-) = [6 |\gamma^\mu| 5\rangle, \qquad L_L^\mu(p_5^-, p_6^+) = [5 |\gamma^\mu| 6\rangle = [L_R^\mu(p_5^+, p_6^-)]^*. \qquad (4.1)$$

Performing the contraction and making use of Schouten identities and momentum conservation we end up with:

$$A_R^{(+--)}(p_5, p_6; g_1, g_2, g_3) = L_R^\mu(p_5^+, p_6^-) S_\mu(g_1^+, g_2^-, g_3^-) = \frac{1}{\sqrt{2}} \frac{\langle 2\,3\rangle}{\langle 1\,2\rangle\langle 1\,3\rangle[2\,3]}$$

$$\times \left\{ \langle 2\,5\rangle\langle 3\,5\rangle[5\,6] \, \alpha_1(x,y,z) + \langle 2\,3\rangle\langle 2\,5\rangle[2\,6] \, \alpha_2(x,y,z) + \langle 2\,3\rangle\langle 3\,5\rangle[3\,6] \, \alpha_3(x,y,z) \right\}, \quad (4.2)$$

$$A_R^{(+++)}(p_5, p_6; g_1, g_2, g_3) = L_R^\mu(p_5^+, p_6^-) S_\mu(g_1^+, g_2^+, g_3^+) = \frac{1}{\sqrt{2}}$$

$$\times \left\{ \frac{[1\,3]\langle 1\,5\rangle[1\,6]}{\langle 1\,2\rangle\langle 2\,3\rangle} \beta_1(x,y,z) + \frac{[2\,3]\langle 2\,5\rangle[2\,6]}{\langle 1\,2\rangle\langle 1\,3\rangle} \beta_2(x,y,z) + \frac{[2\,3]\langle 2\,5\rangle[1\,6]}{\langle 1\,2\rangle\langle 2\,3\rangle} \beta_3(x,y,z) \right\}, \quad (4.3)$$

where the coefficients α_i and β_i are linear combinations of the 14 tensor coefficients A_i. As an explicit example we write down the relations for the α_j:

$$\alpha_1(x,y,z) = -(s_{12} + s_{13}) \left[A_2 + A_9 + \frac{s_{12}}{2} A_{12} \right] - (s_{12} + s_{23}) \left[A_1 + A_{10} + \frac{s_{12}}{2} A_{13} \right]$$
$$- (s_{13} + s_{23}) \left[A_{11} + \frac{s_{12}}{2} A_{14} \right], \qquad (4.4)$$

$$\alpha_2(x,y,z) = -s_{12} \left[A_2 + A_9 + \frac{s_{12}}{2} A_{12} \right] - (s_{12} + s_{13} + s_{23}) \left[A_1 + A_{10} + \frac{s_{12}}{2} A_{13} \right]$$
$$- (s_{13} + s_{23}) \left[A_{11} + \frac{s_{12}}{2} A_{14} \right], \qquad (4.5)$$

$$\alpha_3(x,y,z) = s_{13} \left[A_2 + A_9 - A_{11} + \frac{s_{12}}{2} A_{12} - \frac{s_{12}}{2} A_{14} \right]. \qquad (4.6)$$

The corresponding relations for the β_j are slightly longer and we do not reproduce them here for brevity. There are in total 16 different helicity configurations. From the above expressions for $A_R^{(+--)}(p_5, p_6; g_1, g_2, g_3)$ and $A_R^{(+++)}(p_5, p_6; g_1, g_2, g_3)$, all the other helicity

amplitudes can be obtained by parity conjugation and permutations of the external legs. We find:

$$L_R^\mu(p_5^+, p_6^-)\, S_\mu(g_1^-, g_2^+, g_3^-) = A_R^{(-+-)}(p_5, p_6; g_1, g_2, g_3) = A_R^{(+--)}(p_5, p_6; g_2, g_1, g_3)\,,$$

$$L_R^\mu(p_5^+, p_6^-)\, S_\mu(g_1^-, g_2^-, g_3^+) = A_R^{(--+)}(p_5, p_6; g_1, g_2, g_3) = A_R^{(+--)}(p_5, p_6; g_3, g_2, g_1)\,,$$

$$L_R^\mu(p_5^+, p_6^-)\, S_\mu(g_1^+, g_2^+, g_3^-) = A_R^{(++-)}(p_5, p_6; g_1, g_2, g_3) = [A_R^{(+--)}(p_6, p_5; g_3, g_2, g_1)]^*\,,$$

$$L_R^\mu(p_5^+, p_6^-)\, S_\mu(g_1^+, g_2^-, g_3^+) = A_R^{(+-+)}(p_5, p_6; g_1, g_2, g_3) = [A_R^{(+--)}(p_6, p_5; g_2, g_1, g_3)]^*\,,$$

$$L_R^\mu(p_5^+, p_6^-)\, S_\mu(g_1^-, g_2^+, g_3^+) = A_R^{(-++)}(p_5, p_6; g_1, g_2, g_3) = [A_R^{(+--)}(p_6, p_5; g_1, g_2, g_3)]^*\,,$$

$$L_R^\mu(p_5^+, p_6^-)\, S_\mu(g_1^-, g_2^-, g_3^-) = A_R^{(---)}(p_5, p_6; g_1, g_2, g_3) = [A_R^{(+++)}(p_6, p_5; g_1, g_2, g_3)]^*\,. \quad (4.7)$$

The corresponding amplitudes for right-handed leptonic current can be obtained by simply interchanging $p_5 \leftrightarrow p_6$. Note that the complex conjugation operation has to be applied only on the spinor structures in (4.2) (4.3), and not on the coefficients α_j, β_j.

The unrenormalised helicity amplitude coefficients are vectors in colour space and have perturbative expansions:

$$\Omega_g^{\rm un} = \sqrt{4\pi\alpha_s}\, d^{a_1 a_2 a_3} \left[\left(\frac{\alpha_s}{2\pi}\right) \Omega_g^{(1),\rm un} + \left(\frac{\alpha_s}{2\pi}\right)^2 \Omega_g^{(2),\rm un} + \mathcal{O}(\alpha_s^3) \right]\,, \quad (4.8)$$

for $\Omega_g = \alpha_i, \beta_i$. The dependence on (x, y, z) is again implicit.

4.2 $gg\gamma V$: the amplitude in spinor helicity notation

In the $gg\gamma V$-case there are three independent helicity configurations. Two of them can be chosen identical to those in the $ggg V$-case, namely $(g_1^+, g_2^-, \gamma_3^-)$ and $(g_1^+, g_2^+, \gamma_3^+)$, the third is taken as $(g_1^+, g_2^+, \gamma_3^-)$.

Fixing the helicities and contracting with the right-handed lepton current we have:

$$A_R^{(+--)}(p_5, p_6; g_1, g_2, \gamma_3) = L_R^\mu(p_5^+, p_6^-)\, S_\mu(g_1^+, g_2^-, \gamma_3^-) = \frac{1}{\sqrt{2}}\, \frac{\langle 2\,3\rangle}{\langle 1\,2\rangle\langle 1\,3\rangle[2\,3]}$$

$$\times \left\{ \langle 2\,5\rangle\langle 3\,5\rangle[5\,6]\, \eta_1(x, y, z) + \langle 2\,3\rangle\langle 2\,5\rangle[2\,6]\, \eta_2(x, y, z) + \langle 2\,3\rangle\langle 3\,5\rangle[3\,6]\, \eta_3(x, y, z) \right\}\,, \quad (4.9)$$

$$A_R^{(+++)}(p_5, p_6; g_1, g_2, \gamma_3) = L_R^\mu(p_5^+, p_6^-)\, S_\mu(g_1^+, g_2^+, \gamma_3^+) = \frac{1}{\sqrt{2}}$$

$$\times \left\{ \frac{[1\,3]\langle 1\,5\rangle[1\,6]}{\langle 1\,2\rangle\langle 2\,3\rangle}\theta_1(x, y, z) + \frac{[2\,3]\langle 2\,5\rangle[2\,6]}{\langle 1\,2\rangle\langle 1\,3\rangle}\theta_2(x, y, z) + \frac{[2\,3]\langle 2\,5\rangle[1\,6]}{\langle 1\,2\rangle\langle 2\,3\rangle}\theta_3(x, y, z) \right\}\,, \quad (4.10)$$

$$A_R^{(++-)}(p_5, p_6; g_1, g_2, \gamma_3) = L_R^\mu(p_5^+, p_6^-)\, S_\mu(g_1^+, g_2^+, \gamma_3^-) = \frac{1}{\sqrt{2}}\, \frac{[1\,2]}{\langle 1\,2\rangle[1\,3][2\,3]}$$

$$\times \left\{ [1\,2][1\,6]\langle 1\,5\rangle\, \tau_1(x, y, z) + [1\,2][2\,6]\langle 2\,5\rangle\, \tau_2(x, y, z) + [1\,6][2\,6]\langle 6\,5\rangle\, \tau_3(x, y, z) \right\}\,. \quad (4.11)$$

From $A_R^{(+--)}(p_5, p_6; g_2, g_1, \gamma_3)$, $A_R^{(++-)}(p_5, p_6; g_1, g_2, \gamma_3)$ and $A_R^{(+++)}(p_5, p_6; g_1, g_2, \gamma_3)$

all the other helicity configurations can be obtained by parity and charge conjugation:

$$A_R^{(-+-)}(p_5, p_6; g_1, g_2, \gamma_3) = A_R^{(+--)}(p_5, p_6; g_2, g_1, \gamma_3)$$
$$A_R^{(--+)}(p_5, p_6; g_1, g_2, \gamma_3) = [A_R^{(++-)}(p_6, p_5; g_1, g_2, \gamma_3)]^*$$
$$A_R^{(+-+)}(p_5, p_6; g_1, g_2, \gamma_3) = [A_R^{(+--)}(p_6, p_5; g_2, g_1, \gamma_3)]^*$$
$$A_R^{(-++)}(p_5, p_6; g_1, g_2, \gamma_3) = [A_R^{(+--)}(p_6, p_5; g_1, g_2, \gamma_3)]^*$$
$$A_R^{(---)}(p_5, p_6; g_1, g_2, \gamma_3) = [A_R^{(+++)}(p_6, p_5; g_1, g_2, \gamma_3)]^*. \qquad (4.12)$$

As before, the left-handed helicity amplitudes can be found by the exchange $p_5 \leftrightarrow p_6$, and the complex conjugation has to be performed only on the spinor structures and not on the coefficients η_j, θ_j, τ_j.

The unrenormalised helicity amplitude coefficients are vectors in colour space and have perturbative expansions:

$$\Omega_\gamma^{\mathrm{un}} = \sqrt{4\pi\alpha}\, \delta^{a_1 a_2} \left[\left(\frac{\alpha_s}{2\pi}\right) \Omega_\gamma^{(1),\mathrm{un}} + \left(\frac{\alpha_s}{2\pi}\right)^2 \Omega_\gamma^{(2),\mathrm{un}} + \mathcal{O}(\alpha_s^3) \right], \qquad (4.13)$$

for $\Omega_\gamma = \eta_i, \theta_i, \tau_i$. The dependence on (x, y, z) is again implicit.

4.3 Analytic continuation to the scattering kinematics

In order to compute the two-loop contributions to V-plus-jet and V-plus-photon production at hadron colliders, the helicity amplitudes must be continued to the appropriate kinematical situations.

The relevant partonic subprocesses are:

$$g(p_1) + g(p_2) \to g(-p_3) + V(q) \to g(-p_3) + l^+(p_5) + l^-(p_6), \qquad (4.14)$$
$$g(p_2) + g(p_3) \to g(-p_1) + V(q) \to g(-p_1) + l^+(p_5) + l^-(p_6), \qquad (4.15)$$

where the second crossing is required to fully account for all helicity combinations, and

$$g(p_1) + g(p_2) \to \gamma(-p_3) + V(q) \to \gamma(-p_3) + l^+(p_5) + l^-(p_6). \qquad (4.16)$$

With the notation above the definitions of the helicity amplitudes in terms of momentum spinors (4.2) (4.3) and (4.11) remain unchanged under crossing. Considering in fact an outgoing leptonic current defined as:

$$V(q) \longrightarrow l^+(p_5) + l^-(p_6) \qquad (4.17)$$

with

$$L^\mu(p_5, p_6)\Big|_{\mathrm{out}} = \bar{u}(p_6)\, \gamma^\mu\, v(p_5), \qquad (4.18)$$

we find that:

$$L_R^\mu(p_5^+, p_6^-)\Big|_{\mathrm{in}} = [6|\gamma^\mu|5\rangle = L_R^\mu(p_5^-, p_6^+)\Big|_{\mathrm{out}}$$
$$L_L^\mu(p_5^-, p_6^+)\Big|_{\mathrm{in}} = [5|\gamma^\mu|6\rangle = L_L^\mu(p_5^+, p_6^-)\Big|_{\mathrm{out}}.$$

This means that the expressions for the helicity amplitudes defined in the two sections above remain unchanged provided that p_5 is now considered as the label of the antilepton and p_6 the one of the lepton.

Special care has to be taken in the analytic continuation of the helicity coefficients Ω_g and Ω_γ. In the kinematical situation in (4.14) and (4.16) q^2 remains time-like, but only s_{12} becomes positive:

$$q^2 > 0 , \quad s_{12} > 0 , \quad s_{13} < 0, \quad s_{23} < 0 , \tag{4.19}$$

or, equivalently,

$$x > 0 , \quad y < 0 , \quad z < 0 . \tag{4.20}$$

As shown in [25] (where this region is denoted as $(2a)_+$) and used for example in [21], this kinematical situation can be expressed by introducing new dimensionless variables

$$u_1 = -\frac{s_{13}}{s_{12}} = -\frac{y}{x} , \qquad v_1 = \frac{q^2}{s_{12}} = \frac{1}{x} , \tag{4.21}$$

which fulfil

$$0 \leq u_1 \leq v_1 , \qquad 0 \leq v_1 \leq 1 .$$

To account for all helicity combinations in the case of $gg \to gV$, also the kinematical situation (4.15) must be considered. In this case we have

$$q^2 > 0 , \quad s_{12} < 0 , \quad s_{13} < 0, \quad s_{23} > 0 , \tag{4.22}$$

This can be treated with the following choice of variables [25] (this region is denoted as $(4a)_+$):

$$u_2 = -\frac{s_{13}}{s_{23}} = -\frac{y}{z} , \qquad v_2 = \frac{q^2}{s_{23}} = \frac{1}{z} , \tag{4.23}$$

which fulfil again

$$0 \leq u_2 \leq v_2 , \qquad 0 \leq v_2 \leq 1 .$$

Note that the two kinematical regions (4.14) and (4.15) are turned each other by the permutation $p_1 \leftrightarrow p_3$, in particular one has:

$$u_1(p_1 \leftrightarrow p_3) = u_2$$
$$v_1(p_1 \leftrightarrow p_3) = v_2.$$

As shown in (4.7), in the $gggV$-case, in order to obtain all the different helicity configurations, we also need to exploit the Bose symmetry of the external gluons. It is now clear that whenever the permutation $p_1 \leftrightarrow p_3$ is performed, this only amounts to switching from region (4.14) to region (4.15).

We provide the one-loop and two-loop coefficients in all relevant regions in Mathematica format together with the arXiv-submission of this paper.

5 Outline of the calculation

The two-loop corrections to the coefficients Ω_b can be evaluated through a calculation of the relevant Feynman diagrams. The calculation proceeds as follows. The diagrams contributing to the process are produced using QGRAF [26]. In the $gggV$-case there are 12 diagrams at one loop and 264 at two loops, while in the $gg\gamma V$-case there are 8 diagrams at one and 138 at two loops. The tensor coefficients are evaluated analytically diagram by diagram applying the projectors defined above. As a result, one obtains the tensor coefficients in terms of thousands of planar and non-planar two-loop scalar integrals, which can be classified in two auxiliary topologies, one planar and the other non-planar [27]. In order to do so, one needs to perform both shifts in the integration variables and permutations on the external legs. All the routines needed for this purpose have been coded in FORM [28, 29] and checked against the new automated shift-finder implemented in Reduze2 [30]. Through the usual IBP identities [31, 32] one can reduce independently all the integrals belonging to these two auxiliary topologies to a small set of master integrals. This reduction is performed using the Laporta algorithm [33] implemented in the Reduze code [30, 34]. All the masters for the topologies above are known as series in the parameter $\epsilon = (4-d)/2$ through a systematic approach based on the differential equation method [35–37]. The masters are expressed as Laurent expansion in ϵ, with coefficients containing harmonic polylogarithms (HPLs, [38]) and two-dimensional harmonic polylogarithms (2dHPLs, [35, 36]). Numerical implementations of these functions are available [39–44]. For all the intermediate algebraic manipulations we have made extensive use of FORM [28, 29] and Mathematica [45]. The two-loop unrenormalised helicity coefficients $\Omega_b^{(2),\mathrm{un}}$ can then be evaluated as linear combination of the tensor coefficients. The whole computation is performed in the euclidean non-physical region, where the amplitude is real. The final result is then analytically continued to the physical regions relevant for $Z + jet/\gamma$ production at LHC, as thoroughly discussed in [25] and in section 4.3.

5.1 UV renormalisation and IR subtraction

Renormalisation of ultraviolet divergences is performed in the $\overline{\mathrm{MS}}$ scheme by replacing the bare coupling α_0 with the renormalised coupling $\alpha_s \equiv \alpha_s(\mu^2)$, evaluated at the renormalisation scale μ^2. Since there is no tree level contribution to the amplitude, we only need the one loop relation between the bare and renormalised couplings:

$$\alpha_0 \mu_0^{2\epsilon} S_\epsilon = \alpha_s \mu^{2\epsilon} \left[1 - \frac{\beta_0}{\epsilon} \left(\frac{\alpha_s}{2\pi} \right) + \mathcal{O}(\alpha_s^2) \right] , \tag{5.1}$$

where

$$S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma} \qquad \text{with Euler constant } \gamma = 0.5772\ldots$$

and μ_0^2 is the mass parameter introduced in dimensional regularisation to maintain a dimensionless coupling in the bare QCD Lagrangian density. β_0 is the first coefficient of the QCD β-function:

$$\beta_0 = \frac{11C_A - 4T_R N_F}{6}, \tag{5.2}$$

with the QCD colour factors

$$C_A = N, \qquad C_F = \frac{N^2 - 1}{2N}, \qquad T_R = \frac{1}{2} \ . \tag{5.3}$$

The renormalisation is performed at fixed scale $\mu^2 = q^2$. The renormalised helicity coefficients read:

$$\Omega_g^{(1)} = S_\epsilon^{-1} \Omega_g^{(1),\text{un}},$$
$$\Omega_g^{(2)} = S_\epsilon^{-2} \Omega_g^{(2),\text{un}} - \frac{3\beta_0}{2\epsilon} S_\epsilon^{-1} \Omega_g^{(1),\text{un}} \ . \tag{5.4}$$
$$\Omega_\gamma^{(1)} = S_\epsilon^{-1} \Omega_\gamma^{(1),\text{un}},$$
$$\Omega_\gamma^{(2)} = S_\epsilon^{-2} \Omega_\gamma^{(2),\text{un}} - \frac{\beta_0}{\epsilon} S_\epsilon^{-1} \Omega_\gamma^{(1),\text{un}} \ . \tag{5.5}$$

After performing ultraviolet renormalisation, the amplitudes still contain singularities, which are of infrared origin and will be analytically cancelled by those occurring in radiative processes of the same order. Catani [46] has shown how to organise the infrared pole structure of the one- and two-loop contributions renormalised in the $\overline{\text{MS}}$-scheme in terms of the tree and renormalised one-loop amplitudes. The same procedure applies to the tensor coefficients. Since there is no tree level process contributing, their pole structure can be separated off as follows:

$$\Omega_b^{(1)} = \Omega_b^{(1),\text{finite}},$$
$$\Omega_b^{(2)} = \boldsymbol{I}_b^{(1)}(\epsilon) \Omega_b^{(1)} + \Omega_b^{(2),\text{finite}}, \tag{5.6}$$

where again $b = g, \gamma$.

In the two cases the operator $\boldsymbol{I}_b^{(1)}(\epsilon)$ is given by

$$\boldsymbol{I}_g^{(1)}(\epsilon) = -N \frac{e^{\epsilon\gamma}}{2\Gamma(1 - \epsilon)} \left[\left(\frac{1}{\epsilon^2} + \frac{\beta_0}{N\,\epsilon} \right) (\mathsf{S}_{12} + \mathsf{S}_{13} + \mathsf{S}_{23}) \right], \tag{5.7}$$

$$\boldsymbol{I}_\gamma^{(1)}(\epsilon) = -N \frac{e^{\epsilon\gamma}}{\Gamma(1 - \epsilon)} \left[\left(\frac{1}{\epsilon^2} + \frac{\beta_0}{N\,\epsilon} \right) \mathsf{S}_{12} \right], \tag{5.8}$$

where, since we have set $\mu^2 = s_{123}$:

$$\mathsf{S}_{ij} = \left(-\frac{s_{123}}{s_{ij}} \right)^\epsilon \tag{5.9}$$

Note that on expanding S_{ij}, imaginary parts are generated, depending on which kinematical configuration we are working in. In the decay kinematics $Z \to ggg\,/\,gg\gamma$ for example we have that all the s_{ij} become positive, so that all three terms will generate imaginary parts whose sign is fixed by the small imaginary part $+i0$ of s_{ij}. On the other hand if we are interested in the scattering kinematics $gg \to Zg\,/\,Z\gamma$ only s_{12} or s_{13} become positive, with the usual $s_{ij} + i0$ prescription.

For the infrared factorisation of the two-loop results, the renormalised one-loop helicity amplitude coefficients are needed through to $\mathcal{O}(\epsilon^2)$. Their decomposition in colour

structures is straightforward, namely the whole colour dependence is in the overall factors $d^{a_1 a_2 a_3}$ and $\delta^{a_1 a_2}$ for $gggV$ and $gg\gamma V$ respectively.

$$\Omega_b^{(1),\text{finite}}(x, y, z) = a_{\Omega_b}(x, y, z) \,. \tag{5.10}$$

The expansion of the coefficients through to ϵ^2 yields HPLs and 2dHPLs up to weight 4. The explicit expressions are of considerable size, such that we only quote the ϵ^0-terms in the appendix. To this order, the coefficients had been derived previously [5, 6] in terms of logarithms and dilogarithms. The expressions through to $\mathcal{O}(\epsilon^2)$ in Mathematica format are appended to the arXiv submission of this article.

The finite two-loop remainder is obtained by subtracting the predicted infrared structure (expanded through to $\mathcal{O}(\epsilon^0)$) from the renormalised helicity coefficient. We further decompose it according to the colour structures as follows:

$$\Omega_b^{(2),\text{finite}}(x, y, z) = N\, A_{\Omega_b} + \frac{1}{N}\, B_{\Omega_b} + N_f\, C_{\Omega_b}\,., \tag{5.11}$$

The helicity coefficients contain HPLs and 2dHPLs up to weight 4. The size of each helicity coefficient is comparable to the size of the helicity-averaged tree times two-loop matrix element for $3j$ production quoted in [27], and we decided not to include them here explicitly. The complete set of coefficients in Mathematica format is attached to the arXiv submission of this article.

5.2 Simplification using the Symbol formalism

After the computation of the amplitudes and subtraction of UV- and IR-divergences we used an in-house implementation of the algorithm described in [47] to express the result as far as possible in logarithms and polylogarithms of functions of the kinematic invariants. The GiNaC libary was used to evaluate the 2dHPLs [48] and the implementation of the PSLQ algorithm contained in the arprec library [49] to find the parts mapped to zero by the symbol map.

It is well known that up to transcendental weight three all two-dimensional harmonic polylogarithms can be expressed this way. However, for weight four this is not always the case. In [50] it was conjectured that a combination of 2dHPLs can be expressed in logarithms and polylogarithms if and only if its symbol fulfills a certain symmetry condition. In the present case, we found this condition in general not to be fulfilled and were also not able to express our result in logarithms and polylogarithms only. Nevertheless we reduced the number of required functions in all kinematic regions as far as possible, having to resort to 17 2dHPLs of weight four.

In the past, surprising relations between certain QCD and N=4 SYM amplitudes have been found, for example in the case of $H \to ggg$ at two loops in the heavy-top-limit [22, 51, 52]. In the leading color part of the finite two-loop amplitude, the weight four contribution without a rational factor was found [51] to be helicity-independent and equal to the three-point form factor remainder function in planar N=4 SYM. In the present cases, however, no such relation could be observed. This feature can be understood from the fact that, in contrast to the Higgs amplitudes, no purely gluonic contribution is present here, due to the internal quark loop coupling to the vector boson.

6 Checks on the result

Several non-trivial checks were applied to validate our results.

1. As a first check we computed all 14 tensor coefficients in (3.6) at one-loop order for the $gggV$-case, and we verified that we can reproduce the results in [5, 6] up to order $O(\epsilon^0)$. Performing this check was not entirely trivial. In [5, 6] the results for the one-loop helicity amplitudes are given in the case of an on-shell Z with a fixed polarization. Moreover, the amplitudes for different helicity configurations are given choosing an explicit representation for the polarization vectors of the external particles. This representation does not respect the gauge choice performed in (3.5), so that we cannot naively start from our tensor structure and fix the polarization vectors in the same way to reproduce their result. Nevertheless, as explained in section 3, the full gauge-invariant tensor can be fully reconstructed taking suitable linear combinations of the tensor coefficients of the gauge-fixed tensor. Once the gauge-invariant tensor is known, one can then use the explicit representation of the polarization vectors given in [5, 6] and demonstrate the analytic agreement of the expressions.

2. We computed all the 14 tensor coefficients both at one-loop and at two-loop order, in the $gggV$- and in the $gg\gamma V$-case. Following the procedure outlined in section 3, we obtained the 14 coefficients of the gauge invariant tensor for both processes, and we verified that they respect the expected symmetry relations under permutation of the external gluons.

3. The IR singularity structure of our results agrees with the prediction of Catani formula [46], see section 5.1.

4. We compared the helicity amplitudes $\Omega_b^{(1)}$ for the $gggV$- and the $gg\gamma V$-case. We verified the following identities for the one-loop amplitude coefficients:

$$2\,a_{\alpha_j}(x,y,z) = a_{\eta_j}(x,y,z),$$
$$2\,a_{\beta_j}(x,y,z) = a_{\theta_j}(x,y,z). \qquad j = 1,2,3\,. \tag{6.1}$$

5. Finally, we performed the same comparison at two-loop order, finding:

$$2\,B_{\alpha_j}(x,y,z) = B_{\eta_j}(x,y,z),$$
$$2\,B_{\beta_j}(x,y,z) = B_{\theta_j}(x,y,z), \qquad j = 1,2,3\,, \tag{6.2}$$

which follow from the structure of the underlying two-loop diagrams. The subleading colour coefficients B are unaffected by renormalisation and infrared subtraction. No relation of this type can be found for the coefficients C_{Ω_b}, which are determined purely from renormalisation counterterms and IR subtraction, which differ in the cases $b = g, \gamma$.

7 Conclusions and outlook

In this paper we presented the two-loop corrections to the helicity amplitudes for the processes $gg \to Vg$ and $gg \to V\gamma$. We performed the calculation in dimensional regularisation by applying d-dimensional projection operators to the most general tensor structure of the amplitude. We showed how an explicit gauge choice can reduce considerably the complexity of the basic tensor structures appearing while retaining the full information on the gauge-invariant amplitudes. We expressed our results in terms of dimensionless helicity coefficients, which multiply four-dimensional spinor structures. We extracted the infrared singularities by means of an infrared factorisation formula and provide compact analytic expressions for the finite part of the two-loop helicity coefficients in all relevant kinematical regions.

The matrix elements derived here contribute to the NLO corrections to the gluon-induced production of $Z\gamma$ and $Z + j$ final states at the LHC. Viewed in an expansion in the strong coupling constant, these contributions are formally N^{3}LO as far as the reactions $pp \to V\gamma + X$, $pp \to Vj + X$ are concerned. However, due to the large gluon-gluon luminosity at the LHC, these contributions could be comparable in size with the NNLO corrections to $q\bar{q} \to Vg$, $qg \to Vq$ and $q\bar{q} \to V\gamma$. Their inclusion will also help to stabilise the substantial scale dependence of the gluon-induced subprocesses, which were known only at Born-level up to now.

Acknowledgments

A One-loop helicity amplitudes

A.1 $V \to ggg$ at one loop

We reproduce the leading order $O(\epsilon^2)$ for $V \to ggg$. The complete expressions up to order $O(\epsilon^2)$ can be found in the attachments of the arXiv version of this paper.

$$
\begin{aligned}
a_{\alpha_1}(x,y,z) = {} & 2x\left(\frac{1}{1-x} - \frac{2}{z}\right)\log(x) + 2y\left(\frac{1}{1-y} - \frac{2}{z}\right)\log(y) \\
& - 2\left(\frac{(1-x)x + (1-y)y}{z^2}\right)\left[\frac{\pi^2}{6} + \log(x)\log(y) - (\log(1-x)\log(x) + \mathrm{Li}_2(x)) \right. \\
& \left. \qquad\qquad - (\log(1-y)\log(y) + \mathrm{Li}_2(y))\right],
\end{aligned}
\tag{A.1}
$$

$$a_{\alpha_2}(x,y,z) = 2y\left(\frac{1}{1-y} - \frac{1}{z}\right) - \frac{2x(2y+z)}{z^2}\log(x) - \frac{2xy(z+(1-y)(2y+z))}{(1-y)^2 z^2}\log(y)$$

$$+ 2\left(-\frac{x\left(2y^2 + 2yz + z^2\right)}{z^3}\right)\left[\frac{\pi^2}{6} + \log(x)\log(y) - (\log(1-x)\log(x) + \mathrm{Li}_2(x))\right.$$

$$\left. - (\log(1-y)\log(y) + \mathrm{Li}_2(y))\right], \tag{A.2}$$

$$a_{\alpha_3}(x,y,z) = -a_{\alpha_2}(y,x,z), \tag{A.3}$$

$$a_{\beta_1}(x,y,z) = -2\left(1 - \frac{1}{y}\right), \quad a_{\beta_2}(x,y,z) = -2\left(1 - \frac{1}{z}\right), \tag{}$$

$$a_{\beta_3}(x,y,z) = -4. \tag{A.4}$$

A.2 $V \to gg\gamma$ at one loop

At one loop, the two amplitudes are related to each other as follows:

$$a_{\eta_i}(x,y,z) = 2\,a_{\alpha_i}(x,y,z) \qquad \text{for } i = 1,2,3,$$

$$a_{\theta_i}(x,y,z) = 2\,a_{\beta_i}(x,y,z) \qquad \text{for } i = 1,2,3, \tag{A.5}$$

$$a_{\tau_1}(x,y,z) = 2\,a_{\alpha_3}(z,y,x), \quad a_{\tau_2}(x,y,z) = 2\,a_{\alpha_2}(z,y,x),$$

$$a_{\tau_3}(x,y,z) = 2\,a_{\alpha_1}(z,y,x) \tag{A.6}$$

B Two-loop amplitudes: all-plus helicity coefficients

Due to the length of the resulting expressions we chose to reproduce only the all-plus $(g_1^+, g_2^+, g_3^+/\gamma_3^+)$ helicity amplitudes of both processes in the decay region, which are considerably shorter than the other helicity combinations and contain only functions up to transcendental weight two. The full result can be found in the attachments to the arXiv submission of this paper in Mathematica format.

B.1 $V \to ggg$ at two loops

The coefficients for the (g_1^+, g_2^+, g_3^+) helicity configuration are:

$$A_{\beta_1}(x,y,z) = -\frac{1}{27}\left(27\left(3 - \frac{1}{1-x} - \frac{1}{y} - \frac{1}{1-z}\right) - \frac{4z}{x} - \frac{4z^2}{x^2}\right.$$

$$+x^2\left(-\frac{4}{z^2} - \frac{4z}{y^3}\right) + \frac{4x}{z}\left(-1 - \frac{z^3}{y^3}\right)\right) - \frac{11}{2}\left(1 - \frac{1}{y}\right)i\pi$$

$$-\frac{1}{12}\left(\frac{3(1-y)}{xy} - \frac{2(-1+2y)}{y^2} + \frac{3(1-y)}{yz} + \frac{14(1-y)z}{y^3} - \frac{14z^2}{y^3}\right)\pi^2$$

$$+\frac{1}{6}\left(11 + \frac{6}{(1-x)^2} - \frac{6}{1-x} - \frac{42x+11y}{y^2}\right)\log(x) + \frac{1}{6}\left(11 - \frac{47}{y}\right)\log(y)$$

$$+\frac{1}{6}\left(11 + \frac{-42+42x+31y}{y^2} + \frac{6}{(1-z)^2} - \frac{6}{1-z}\right)\log(z)$$

$$-\frac{1}{2}\left(\frac{2}{y} + \frac{3x}{yz}\right)\log(x)\log(y) - \frac{1}{2}\left(\frac{2}{y} + \frac{3z}{xy}\right)\log(y)\log(z)$$

$$- \left(\frac{7(1-x)x}{y^3} + \frac{1}{y^2} - \frac{7x}{y^2} - \frac{1}{y} \right) \log(x) \log(z)$$

$$+ \frac{1}{2} \left(\frac{2(1-7z)}{y^2} - \frac{3}{z} + \frac{3(1-z)}{yz} + \frac{14(1-z)z}{y^3} \right)$$

$$\times \left(\log(1-x) \log(x) + \mathrm{Li}_2(x) \right)$$

$$+ \frac{1}{2} \left(\frac{4}{y} + \frac{3x}{yz} + \frac{3z}{xy} \right) \left(\log(1-y) \log(y) + \mathrm{Li}_2(y) \right)$$

$$- \frac{1}{2} \left(\frac{14x^2}{y^3} - \frac{14x(1-y)}{y^3} - \frac{3(1-y)}{xy} + \frac{-2+3y}{y^2} \right)$$

$$\times \left(\log(1-z) \log(z) + \mathrm{Li}_2(z) \right) , \tag{B.1}$$

$$A_{\beta_2}(x,y,z) = - \frac{1}{27} \left(\frac{27}{1-x} + \frac{27}{1-y} - \frac{4y}{x} - \frac{4y^2}{x^2} + x^2 \left(-\frac{4}{y^2} - \frac{4y}{z^3} \right) \right.$$

$$\left. + \frac{4x}{y} \left(-1 - \frac{y^3}{z^3} \right) - \frac{54}{z} \right) - \frac{11}{2} \left(1 - \frac{1}{z} \right) i\pi$$

$$- \frac{1}{12} \left(\frac{x^2(14 + y(-58 + 45y))}{y^2 z^2} + \frac{14x^3(-2+3y)}{y^2 z^2} - \frac{2(1-y)y(-8+21y)}{xz^2} \right.$$

$$\left. - \frac{4x(1-y)(-4+5y)}{yz^2} + \frac{3(2 + 3y(-4+5y))}{z^2} + \frac{14(1-y)^2 y^2}{x^2 z^2} + \frac{14x^4}{y^2 z^2} \right) \pi^2$$

$$- \frac{1}{6} \left(\frac{53x}{z} + \frac{42x^2}{yz} - \frac{(-47 - 53(-2+x)x)y}{(1-x)^2 z} \right) \log(x)$$

$$+ \frac{1}{6} \left(-\frac{53y}{z} - \frac{42y^2}{xz} - \frac{x(47 + 53(-2+y)y)}{(1-y)^2 z} \right) \log(y)$$

$$- \frac{1}{6} \left(31 + \frac{42x}{y} + \frac{42y}{x} + \frac{11}{z} \right) \log(z)$$

$$- \frac{1}{2} \left(\frac{x^2}{z^2} + \frac{x(2 - 12y)}{z^2} + \frac{y(2+y)}{z^2} \right) \log(x) \log(y)$$

$$- \frac{1}{2} \left(1 + \frac{14(1-z)^2}{y^2} - \frac{2(1-z)(-1+7z)}{yz} \right) \log(x) \log(z)$$

$$- \frac{1}{2} \left(1 + \frac{14y^2}{x^2} + \frac{-2 + 14y + \frac{2}{z}}{x} \right) \log(y) \log(z)$$

$$+ \frac{1}{2} \left(\frac{14(1-z)^2}{y^2} + \frac{14y^2}{z^2} - \frac{14y(1-z)}{z^2} - \frac{2(1-z)(-1+7z)}{yz} \right.$$

$$\left. + \frac{3 + 2(-2+z)z}{z^2} \right) \left(\log(1-x) \log(x) + \mathrm{Li}_2(x) \right)$$

$$+ \frac{1}{2} \left(\frac{14(1-z)^2}{x^2} + \frac{14x^2}{z^2} - \frac{14x(1-z)}{z^2} - \frac{2(1-z)(-1+7z)}{xz} \right.$$

$$\left. + \frac{3 + 2(-2+z)z}{z^2} \right) \left(\log(1-y) \log(y) + \mathrm{Li}_2(y) \right)$$

$$- \left(\frac{1-x}{x} - \frac{7x^2}{y^2} + \frac{x(-8+7x)}{(1-x)y} - \frac{7y}{x} - \frac{7y^2}{x^2} - \frac{1}{(1-x)xz} \right)$$

$$\times \left(\log(1-z) \log(z) + \mathrm{Li}_2(z) \right) , \tag{B.2}$$

$$A_{\beta_3}(x,y,z) = -\frac{1}{27}\left(81 - \frac{8y^2}{z^2} - \frac{8z}{y} + \frac{8(1-z)z}{y^2} + \frac{8y}{z^2}\left(1 - z + \frac{z^3}{x^2}\right)\right) - 11i\pi$$

$$- \frac{1}{12}\left(3 - \frac{5}{x} - \frac{14(1-x)x}{y^2} + \frac{-5+14x}{y} - \frac{14(1-x)y}{x^2}\right.$$

$$\left.+ \frac{14y^2}{x^2} - \frac{14(1-x)x}{z^2} - \frac{5-14x}{z}\right)\pi^2$$

$$+ \frac{1}{3}\left(32 - \frac{3}{1-x} + \frac{21(1-x)x}{yz}\right)\log(x)$$

$$+ \frac{1}{3}\left(32 - \frac{3}{1-y} + \frac{21(1-y)y}{xz}\right)\log(y)$$

$$+ \frac{1}{3}\left(32 - \frac{3}{1-z} + \frac{21(1-z)z}{xy}\right)\log(z)$$

$$- \frac{1}{2}\left(\frac{14x^2}{z^2} - \frac{14x(1-z)}{z^2} + \frac{-5+z}{z}\right)\log(x)\log(y)$$

$$- \frac{1}{2}\left(1 - \frac{14(1-z)z}{y^2} + \frac{-5+14z}{y}\right)\log(x)\log(z)$$

$$- \frac{1}{2}\left(1 - \frac{14(1-y)y}{x^2} + \frac{-5+14y}{x}\right)\log(y)\log(z)$$

$$+ \frac{1}{2}\left(2 - \frac{(1-x)}{y^2z^2}\left(14(1-x)^2x - (1-x)(-5+42x)y + (-5+42x)y^2\right)\right)$$

$$\times \left(\log(1-x)\log(x) + \mathrm{Li}_2(x)\right)$$

$$+ \frac{1}{2}\left(2 - \frac{(1-y)}{x^2z^2}\left(14(1-y)^2y - (1-y)(-5+42y)z + (-5+42y)z^2\right)\right)$$

$$\times \left(\log(1-y)\log(y) + \mathrm{Li}_2(y)\right)$$

$$+ \frac{1}{2}\left(\frac{14x^2}{y^2} - \frac{14x(1-y)}{y^2} - \frac{14(1-y)y}{x^2} + \frac{-5+2y}{y} + \frac{-5+14y}{x}\right)$$

$$\times \left(\log(1-z)\log(z) + \mathrm{Li}_2(z)\right), \tag{B.3}$$

$$B_{\beta_1}(x,y,z) = \frac{1}{1-x} + \frac{1}{y} + \frac{1}{1-z} - 3$$

$$+ \frac{1}{12}\left(\frac{2}{y} - \frac{1-y}{xy} - \frac{1-y}{yz} - \frac{2(1-y)z}{y^3} + \frac{2z^2}{y^3}\right)\pi^2$$

$$+ \left(\frac{x}{(1-x)^2} - \frac{x}{y^2}\right)\log(x) - \left(\frac{z}{y^2} - \frac{z}{(1-z)^2}\right)\log(z)$$

$$- \frac{1}{2}\left(\frac{x}{yz}\right)\log(x)\log(y) - \left(\frac{xz}{y^3}\right)\log(x)\log(z)$$

$$- \frac{1}{2}\left(\frac{z}{xy}\right)\log(y)\log(z)$$

$$+ \frac{1}{2}\left(\frac{x}{yz} + \frac{2xz}{y^3}\right)\left(\log(1-x)\log(x) + \mathrm{Li}_2(x)\right)$$

$$+ \frac{1}{2}\left(\frac{x}{yz} + \frac{z}{xy}\right)\left(\log(1-y)\log(y) + \mathrm{Li}_2(y)\right)$$

$$- \frac{1}{2}\left(-\frac{2xz}{y^3} - \frac{z}{xy}\right)\left(\log(1-z)\log(z) + \mathrm{Li}_2(z)\right), \tag{B.4}$$

$$B_{\beta_2}(x,y,z) = -\frac{1}{1-x} - \frac{1}{1-y} + \frac{2}{z}$$

$$- \frac{1}{12}\left(3 + \frac{2x^2}{y^2} + \frac{2x}{y} + \frac{2y}{x} + \frac{2y^2}{x^2} + \frac{1-2(1-x)x}{z^2} - \frac{2(1-x)}{z}\right)\pi^2$$

$$- \left(\frac{2x}{(1-x)^2} - \frac{x}{z} + \frac{x^2 z}{(1-x)^2 y}\right)\log(x) - \left(1 + \frac{x}{y} + \frac{y}{x}\right)\log(z)$$

$$+ \left(1 - \frac{1}{(1-y)^2} + y\left(-\frac{y}{x(1-y)} + \frac{1}{z}\right)\right)\log(y)$$

$$- \frac{1}{2}\left(\frac{x^2}{z^2} + \frac{y^2}{z^2}\right)\log(x)\log(y) - \frac{1}{2}\left(1 + \frac{2x^2}{y^2} + \frac{2x}{y}\right)\log(x)\log(z)$$

$$- \frac{1}{2}\left(1 + \frac{2y}{x} + \frac{2y^2}{x^2}\right)\log(y)\log(z)$$

$$+ \frac{1}{2}\left(2 + \frac{2x(1-z)}{y^2} + \frac{1-2(1-x)x}{z^2} - \frac{2(1-x)}{z}\right)$$

$$\times \left(\log(1-x)\log(x) + \mathrm{Li}_2(x)\right)$$

$$+ \frac{1}{2}\left(2 + \frac{2y}{x} + \frac{2y^2}{x^2} + \frac{1-2(1-x)x}{z^2} - \frac{2(1-x)}{z}\right)$$

$$\times \left(\log(1-y)\log(y) + \mathrm{Li}_2(y)\right)$$

$$+ \left(1 + \frac{x^4 + x^3 y + x y^3 + y^4}{x^2 y^2}\right)\left(\log(1-z)\log(z) + \mathrm{Li}_2(z)\right), \tag{B.5}$$

$$B_{\beta_3}(x,y,z) = -3 - \frac{1}{12}\left(3 - \frac{1}{x} - \frac{2(1-x)x}{y^2} + \frac{-1+2x}{y} - \frac{2(1-x)y}{x^2}\right.$$

$$\left. + \frac{2y^2}{x^2} - \frac{2(1-x)x}{z^2} - \frac{1-2x}{z}\right)\pi^2$$

$$- \left(\frac{x}{1-x} - \frac{x}{y} - \frac{x}{z}\right)\log(x) + \left(\frac{y}{x} + \frac{xy}{(1-y)z}\right)\log(y)$$

$$+ \left(\frac{1-x}{y} - \frac{1}{1-z} + \frac{z}{x}\right)\log(z)$$

$$+ \frac{1}{2}\left(\frac{(1-x)x}{z^2} + \frac{(1-y)y}{z^2}\right)\log(x)\log(y)$$

$$- \frac{1}{2}\left(1 - \frac{2(1-z)z}{y^2} + \frac{-1+2z}{y}\right)\log(x)\log(z)$$

$$- \frac{1}{2}\left(1 - \frac{2(1-y)y}{x^2} + \frac{-1+2y}{x}\right)\log(y)\log(z)$$

$$+ \frac{1}{2}\left(2 - \frac{1}{y^2 z^2}\left((1-x)\left(2(1-x)^2 x + y + x(-7+6x)y + (-1+6x)y^2\right)\right)\right)$$

$$\times \left(\log(1-x)\log(x) + \mathrm{Li}_2(x)\right)$$

$$+ \frac{1}{2}\left(2 - \frac{1}{x^2 z^2}\left((1-y)\left(2(1-y)^2 y + z + y(-7+6y)z + (-1+6y)z^2\right)\right)\right)$$
$$\times\left(\log(1-y)\log(y) + \mathrm{Li}_2(y)\right)$$
$$+ \frac{1}{2}\left(\frac{2x^2}{y^2} - \frac{2x(1-y)}{y^2} - \frac{2(1-y)y}{x^2} + \frac{-1+2y}{x} + \frac{-1+2y}{y}\right)$$
$$\times\left(\log(1-z)\log(z) + \mathrm{Li}_2(z)\right), \tag{B.6}$$

$$C_{\beta_1}(x,y,z) = \frac{1}{3}\left(1 - \frac{1}{y}\right)\left(3i\pi - \log(x) - \log(y) - \log(z)\right), \tag{B.7}$$

$$C_{\beta_2}(x,y,z) = \frac{1}{3}\left(1 - \frac{1}{z}\right)\left(3i\pi - \log(x) - \log(y) - \log(z)\right), \tag{B.8}$$

$$C_{\beta_3}(x,y,z) = \frac{2}{3}\left(3i\pi - \log(x) - \log(y) - \log(z)\right). \tag{B.9}$$

B.2 $V \to gg\gamma$ at two loops

The coefficients for the $(g_1^+, g_2^+, \gamma_3^+)$ helicity configuration are as follows:

$$A_{\theta_1}(x,y,z) = -\frac{2}{81}\left(81 - \frac{81}{1-x} + \frac{8(1-y)}{z^2}\left(-x - \frac{z^3}{x^2} + \frac{z^3}{y^2} - \frac{(1-z)z^3}{y^3}\right)\right)$$
$$- \frac{22}{3}\left(1 - \frac{1}{y}\right)i\pi + \frac{1}{6}\left(-\frac{2(1-y)}{xy} - \frac{2(1-y)}{yz} + \frac{(-9+5y)z}{y^3} + \frac{9z^2}{y^3}\right)\pi^2$$
$$+ \frac{1}{3}\left(22 + \frac{6}{(1-x)^2} - \frac{6}{1-x} - \frac{27x+22y}{y^2}\right)\log(x)$$
$$- 9\frac{1}{y}\log(y) - 9\frac{z}{y^2}\log(z) - 2\left(\frac{z}{xy}\right)\log(y)\log(z)$$
$$- 2\left(\frac{2}{y} + \frac{x}{yz}\right)\log(x)\log(y) - \left(\frac{9xz}{y^3} + \frac{4z}{y^2}\right)\log(x)\log(z)$$
$$- \left(\frac{2}{z} + \frac{5z}{y^2} - \frac{9(1-z)z}{y^3} - \frac{2(1+z)}{yz}\right)\left(\log(1-x)\log(x) + \mathrm{Li}_2(x)\right)$$
$$+ 2\left(\frac{(1-y)^2}{xyz}\right)\left(\log(1-y)\log(y) + \mathrm{Li}_2(y)\right)$$
$$- \left(-\frac{9xz}{y^3} - \frac{4z}{y^2} - \frac{2z}{xy}\right)\left(\log(1-z)\log(z) + \mathrm{Li}_2(z)\right), \tag{B.10}$$

$$A_{\theta_2}(x,y,z) = -\frac{2}{81}\left(-\frac{8y}{x} - \frac{8y^2}{x^2} + x^2\left(-\frac{8}{y^2} - \frac{8y}{z^3}\right) + \frac{8x\left(-1-\frac{y^3}{z^3}\right)}{y} - \frac{81y}{(1-x)z}\right)$$
$$- \frac{1}{6}\left(4 + \frac{9x^2}{y^2} + \frac{12x}{y} + \frac{8y}{x} + \frac{8y^2}{x^2} - \frac{-1+(8-9x)x}{z^2} + \frac{2(-1+6x)}{z}\right)\pi^2$$
$$- \frac{1}{3}\left(\frac{46x}{z} + \frac{27x^2}{yz} + \frac{2(20+23(-2+x)x)y}{(1-x)^2 z}\right)\log(x) - \frac{22}{3}\left(1 - \frac{1}{z}\right)i\pi$$
$$+ \left(-\frac{9x}{z} - \frac{8y}{z} - \frac{8y^2}{xz}\right)\log(y) - \left(8 + \frac{9x}{y} + \frac{8y}{x}\right)\log(z)$$

$$+ \left(\frac{2(-2+x)x}{z^2} + \frac{10xy}{z^2} - \frac{y^2}{z^2} \right) \log(x) \log(y)$$

$$- \left(1 - \frac{3x(-4+x+4z)}{y^2} \right) \log(x) \log(z)$$

$$- 2 \left(\frac{(x+2y)^2}{x^2} \right) \log(y) \log(z)$$

$$+ \left(2 - \frac{-1+(8-9x)x}{z^2} - \frac{2-12x}{z} - \frac{3x(-4+x+4z)}{y^2} \right)$$

$$\times \left(\log(1-x) \log(x) + \mathrm{Li}_2(x) \right)$$

$$+ \left(\frac{8y(1-z)}{x^2} + \frac{2+y(-10+9y)}{z^2} + \frac{6y}{z} \right) \left(\log(1-y) \log(y) + \mathrm{Li}_2(y) \right)$$

$$+ \left(\frac{9x}{y^2} + \frac{3}{y} + \frac{8y}{x^2} - \frac{9xz}{y^2} - \frac{3z}{y} - \frac{8yz}{x^2} \right) \left(\log(1-z) \log(z) + \mathrm{Li}_2(z) \right), \quad (\text{B.11})$$

$$A_{\theta_3}(x,y,z) = -\frac{2}{81} \left(81 + \frac{16 \left(-y^4 - yz^3 + (1-z)z^3 + y^3 \left(1 - z + \frac{z^3}{x^2} \right) \right)}{y^2 z^2} \right) - \frac{44}{3} i\pi$$

$$+ \frac{1}{6} \left(2 + \frac{2}{y} + \frac{8(1-y)y}{x^2} - \frac{-2+8y}{x} + \frac{9(1-y)y}{z^2} + \frac{2-6y}{z} \right.$$

$$\left. + \frac{(9-6y)z}{y^2} - \frac{9z^2}{y^2} \right) \pi^2 + \frac{1}{3} \left(68 - \frac{6}{1-x} + \frac{27(1-x)x}{yz} \right) \log(x)$$

$$+ \left(9 + \frac{8y}{x} + \frac{9y}{z} \right) \log(y) + \left(\frac{9(1-x)}{y} + \frac{8z}{x} \right) \log(z)$$

$$+ \left(-\frac{9x^2}{z^2} + \frac{x(9-12z)}{z^2} - \frac{-5+z}{z} \right) \log(x) \log(y)$$

$$+ \left(-\frac{9x^2}{y^2} + \frac{x(9-12y)}{y^2} - \frac{-5+y}{y} \right) \log(x) \log(z)$$

$$- 2 \left(1 - \frac{4(1-y)y}{x^2} + \frac{-1+4y}{x} \right) \log(y) \log(z)$$

$$+ \left(2 + \frac{(1-x) \left(-9(1-x)^2 x + 5(1-x)(-1+6x)y + 5(1-6x)y^2 \right)}{y^2 z^2} \right)$$

$$\times \left(\log(1-x) \log(x) + \mathrm{Li}_2(x) \right)$$

$$- \left(\frac{2-3x}{x} + \frac{9(1-x)x}{z^2} + \frac{5-12x}{z} + \frac{8(1-x)z}{x^2} - \frac{8z^2}{x^2} \right)$$

$$\times \left(\log(1-y) \log(y) + \mathrm{Li}_2(y) \right)$$

$$- \left(\frac{2-3x}{x} + \frac{9(1-x)x}{y^2} + \frac{5-12x}{y} + \frac{8(1-x)y}{x^2} - \frac{8y^2}{x^2} \right)$$

$$\times \left(\log(1-z) \log(z) + \mathrm{Li}_2(z) \right), \quad (\text{B.12})$$

$$B_{\theta_i}(x,y,z) = 2 B_{\beta_i}(x,y,z) \quad \text{for } i = 1,2,3, \quad (\text{B.13})$$

$$C_{\theta_1}(x,y,z) = \frac{4}{3} \left(1 - \frac{1}{y} \right) \left(i\pi - \log(x) \right), \quad (\text{B.14})$$

$$C_{\theta_2}(x, y, z) = \frac{4}{3} \left(1 - \frac{1}{z} \right) (i\pi - \log(x)) , \tag{B.15}$$

$$C_{\theta_3}(x, y, z) = \frac{8}{3} (i\pi - \log(x)) . \tag{B.16}$$

The complete results, including the other helicity configurations, can again be found in the attachments of the arXiv version of this paper.

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