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Conformal algebra: R-matrix and star-triangle relation

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Abstract: The main purpose of this paper is the construction of the R-operator which acts in the tensor product of two infinite-dimensional representations of the conformal algebra and solves Yang-Baxter equation. We build the R-operator as a product of more elementary operators $\mathrm{S}_1, \mathrm{S}_2$ and S_3. Operators S_1 and S_3 are identified with intertwining operators of two irreducible representations of the conformal algebra and the operator S_2 is obtained from the intertwining operators S_1 and S_3 by a certain duality transformation. There are star-triangle relations for the basic building blocks $\mathrm{S}_1, \mathrm{S}_2$ and S_3 which produce all other relations for the general R-operators. In the case of the conformal algebra of n-dimensional Euclidean space we construct the R-operator for the scalar (spin part is equal to zero) representations and prove that the star-triangle relation is a well known star-triangle relation for propagators of scalar fields. In the special case of the conformal algebra of the 4-dimensional Euclidean space, the R-operator is obtained for more general class of infinite-dimensional (differential) representations with nontrivial spin parts. As a result, for the case of the 4-dimensional Euclidean space, we generalize the scalar star-triangle relation to the most general star-triangle relation for the propagators of particles with arbitrary spins.

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Contents

1 Introduction

Recently, the quantum integrable spin chains with higher rank symmetry algebras have attracted much attention [33–37]. However, the most part of the methods developed so far enable one to deal only with $s\ell(N)$-symmetric models for which finite-dimensional as well as infinite-dimensional representations in the quantum space have been analyzed thoroughly. The fundamental equations which underlie integrability are the universal Yang-Baxter RRR-relation and its particular cases: the RLL-relation with the general R-operator acting in two quantum spaces and the RLL-relation with R-matrix acting in two finite-dimensional auxiliary spaces (e.g., in spaces of the defining representations of $s\ell(N)$). The last case of the RLL-relation is also obtained by means of the evaluation homomorphism of the $s\ell(N)$-type Yangian: $Y(s\ell(N)) \to \mathcal{U}(s\ell(N))$. For $s\ell(N)$-symmetric models the general R-operator acting in two infinite-dimensional quantum spaces is known and it serves as a local building block in the construction of the Baxter Q-operators [21–23]. Much less is known about integrable quantum lattice models and spin chains with $so(N)$ symmetry (see, however, [24–26]). Substantially, it is related to the fact that one of the basic algebraic object — the Yangian $Y(so(N))$ [27] which can be defined by the RLL-relation with $so(N)$-type R-matrix acting in two finite-dimensional auxiliary spaces (the spaces of $so(N)$ defining representations) does not possess the evaluation homomorphism $Y(so(N)) \to \mathcal{U}(so(N))$. One can think that in this case the Yangian $Y(so(N))$ could be substituted by a more sophisticated Olshanskii twisted Yangian or by the standard reflection equation algebra for which the evaluation homomorphism exists (see [28] and [29], respectively). However these type of algebras are used only for the formulation of integrable open spin chain models with nontrivial boundary conditions.

In this paper our aim is to adapt some methods developed for $s\ell(N)$-symmetric spin chains to spin chains with $so(p+1,q+1)$ symmetry which is interpreted as the conformal symmetry in $\mathbb{R}^{p,q}$. Here we make the first step in this direction.

The plan of the paper is the following.

In section 2 we recall basic facts about conformal algebra $\mathsf{conf}(\mathbb{R}^{p,q}) = so(p+1,q+1)$ and its representations [2–8]. We construct representation of the conformal algebra $so(p+1,q+1)$ in the space of tensor fields by the method of induced representations. This material is more or less standard [2–8]. Our approach is slightly different and appropriate for our own purposes and we include it for the completeness. There is also alternative approach — the so called embedding formalism [9–14].

In section 3 we collect some basic facts about L-operators. The L-operator for $s\ell(N)$-symmetric quantum spin chain can be constructed from the Yangian $Y(s\ell(N))$ by means of the evaluation homomorphism $Y(s\ell(N)) \to \mathcal{U}(s\ell(N))$ and is expressed as a polarized Casimir operator for $g\ell(N)$:

$$\mathrm{L}(u) = u \cdot 1 + T(E_{ij}) \otimes T'(E_{ij})\,,$$

where u is a parameter, E_{ij} are generators of $g\ell(N)$ and in the first space (auxiliary space) of the tensor product the fundamental (defining) representation T is taken and in the second space (quantum space) one can choose an arbitrary representation T'. Such L-operators were considered in [15–19]. If we fix T' as a differential (induced) representation [47], then the L-operator exhibits a remarkable factorization property [21–23] and respects RLL-relation with Yang's R-matrix.

Henceforth, we define the L-*operator* as an operator L acting in the tensor product of some finite-dimensional auxiliary space and arbitrary quantum space (generically infinite-dimensional) and furthermore L respects RLL-relation with a certain numerical R-matrix acting in the auxiliary spaces. For the conformal algebra $so(p+1,q+1)$ of the pseudo-Euclidean space $\mathbb{R}^{p+1,q+1}$, we consider the operator L which is constructed from the $so(p+1,q+1)$ polarized Casimir operator acting in the tensor product of two spaces: the first one (the auxiliary space) is the space of a spinor representation (instead of fundamental one) and the second quantum space is the space of the differential representation of the conformal algebra $so(p+1,q+1)$. It happens that in general this operator respects RLL-relation only if we choose special (scalar) differential representation of the conformal algebra in the quantum space when spin part $S_{\mu\nu}$ of the Lorentz generators is equal to zero. Corresponding numerical R-matrix (acting in the spaces of spinor representation) is rather nontrivial. For the first time it appeared in [38] (see also [24, 25]), where the RLL-relation for the $so(N)$-invariant L-operator with fundamental (defining) representation in the quantum space was established. Thus, we generalize this result. Namely we prove that in the L-operator one can replace (in the quantum space) the defining representation to the infinite-dimensional scalar representation parameterized by the conformal dimension Δ. This new conformal L-operator can be factorized as well (similar to the $s\ell(N)$ case) and as we will see it corresponds to a certain integrable systems [42, 43, 46].

In section 4 we specify formulae of the previous section to the case of the conformal algebra $so(2,4)$ in 4-dimensional Minkowski space $\mathbb{R}^{1,3}$ (actually these formulas can be easily generalized to the case of any conformal algebra $so(p+1, 5-p)$ in 4-dimensional space $\mathbb{R}^{p,4-p}$, where $p = 0, 1, 2$). This case is a special one since $so(2,4)$ is isomorphic to $su(2,2)$ (for complexifications we have $so(6,\mathbb{C}) = s\ell(4,\mathbb{C})$) and consequently we can establish connection with the known construction [21–23] developed for $s\ell(N,\mathbb{C})$. Indeed, as we show the L-operators for these two algebras are related by an appropriate change of variables. The numerical R-matrix for both L-operators is the Yang's one (for the $so(2,4)$ case it is shown in subsection 3.2) and in the quantum space, for the conformal L-operator, we obtain the general differential representation of the conformal algebra with nontrivial spin part $S_{\mu\nu}$, i.e. we deal with representation $\rho_{\Delta,\ell,\dot{\ell}}$ of $so(2,4)$ parameterized by conformal dimension Δ and two spin variables $\ell, \dot{\ell}$.

In subsection 4.2, following the approach [21–23] which was developed for the $s\ell(N,\mathbb{C})$ case, we reproduce intertwining operators for the product of two $so(6,\mathbb{C})$-type L-operators. These operators are building blocks in construction of R-operator which acts in the tensor product of two infinite-dimensional representations. As we indicate at the end of subsection 4.2 the form of these intertwining operators is not manifestly Lorentz covariant. Moreover these operators are not properly defined and can be treated only formally. That is why, in the next section 5, the same intertwining operators and R operator are constructed directly without using the isomorphism $so(6,\mathbb{C}) = s\ell(4,\mathbb{C})$.

The main purpose of section 5 is the construction of the general R-operator which acts in the tensor product $\rho_1 \otimes \rho_2$ of two infinite-dimensional representations of the conformal algebra $so(n+1,1) = \mathsf{conf}(\mathbb{R}^n)$ and solves RLL-relation with conformal L-operators. For simplicity we restrict ourselves to the case of Euclidean space $\mathbb{R}^n$ because in this situation all integral operators are well defined in generic situation. In the case of the conformal algebra $so(n+1,1) = \mathsf{conf}(\mathbb{R}^n)$ we construct R-operator for the scalar ($S_{\mu\nu} = 0$) representations $\rho_{\Delta_1} \otimes \rho_{\Delta_2}$ and in the special case of $so(5,1)$, i.e. the conformal algebra of the 4-dimensional Euclidean space, the R-operator is constructed for a rather general class of representations $\rho_{\Delta_1,\ell_1,\dot{\ell}_1} \otimes \rho_{\Delta_2,\ell_2,\dot{\ell}_2}$ with nontrivial spin parts.

We build the general R-operator as a product of simpler operators $\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3$ which respect relations of RLL-type: $\mathrm{SLL}' = \mathrm{L}''\mathrm{L}'''\mathrm{S}$. Each L-operator depends on the set of four parameters $(u, \Delta, \ell, \dot{\ell})$ and the RLL-relation implies that the R-operator, intertwining the product $(\mathrm{L}_1 \cdot \mathrm{L}_2)$ of two L-operators, interchanges the sets of their parameters: $(u, \Delta_1, \ell_1, \dot{\ell}_1) \leftrightarrow (v, \Delta_2, \ell_2, \dot{\ell}_2)$. Consequently it is reasonable to implement this transposition in several steps and to consider operators $\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3$ which intertwine two L-operators and transpose (or change) only a part of their parameters. The operators S_1 and S_3 separately change the parameters in the first and second factor of the product $\mathrm{L}_1 \cdot \mathrm{L}_2$. Actually the operators S_1 and S_3 can be identified with the intertwining operators of two irreducible

representations of the conformal algebra [5, 7, 8] so that our construction has a transparent representation theory meaning. The operator S_2 interchanges parameters between the factors L_1 and L_2 and the form of S_2 is obtained from the intertwining operators S_1 and S_3 by some kind of the duality transformation. This duality transformation is very similar to the one obtained in the spin chain model [54] (see also [55]). It also resembles the dual conformal transformation for the Feynman diagrams [56, 57] and for the scattering amplitudes in maximally supersymmetric $N = 4$ super Yang-Mills theory [58, 59].
Finally, the general R-operator that implements a special transposition in the set of parameters can be factorized in a product of operators $\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3$ which represent basic elementary transpositions. There are certain relations for the basic building blocks which produce all other relations for R-operators. Indeed, the Coxeter (braid) three-term relations in the symmetric group are represented as follows $\mathrm{S}_1\mathrm{S}_2\mathrm{S}_1 = \mathrm{S}_2\mathrm{S}_1\mathrm{S}_2$ and $\mathrm{S}_3\mathrm{S}_2\mathrm{S}_3 = \mathrm{S}_2\mathrm{S}_3\mathrm{S}_2$. These Coxeter three-term relations can be interpreted as star-triangle relations and play the important role in the construction of the general R-operator. In the n-dimensional scalar case these relations are well known star-triangle relations [51, 52] for propagators of scalar fields. In the case of the conformal algebra $so(5, 1)$ of 4-dimensional Euclidean space we prove a new star-triangle relation for generic representations of the type $\rho_{\Delta,\ell,\dot{\ell}}$ included spin degrees of freedom, i.e. we generalize the scalar star-triangle relation to the star-triangle relation for the propagators of the spin particles.

Recall that the star-triangle relations happen to be a corner stone in the integrability of many lattice models of statistical mechanics [30] (see also papers [31, 32] and references therein). At the end of subsection 5.1 we show that the scalar star-triangle relations can be used for the formulation of the n-dimensional variant of the integrable lattice model proposed by Lipatov [42, 43] (see also [46] where another integrable lattice models were constructed and investigated by means of the scalar star-triangle relations). To our knowledge integrable models related to the new spinorial star-triangle relation of $so(5, 1)$ type are still unknown.

In appendix we prove this new star-triangle identity directly evaluating corresponding integrals.

2 Conformal algebra in $\mathbb{R}^{p,q}$

In this section we summarize some facts about conformal Lie algebras; these facts are needed in subsequent sections. Let $\mathbb{R}^{p,q}$ be a pseudoeuclidean space with the metric

$$g_{\mu\nu} = \mathrm{diag}(\underbrace{1, \dots, 1}_{p}, \underbrace{-1, \dots, -1}_{q}) .$$

Denote by $\mathsf{conf}(\mathbb{R}^{p,q})$ a Lie algebra of the conformal group in $\mathbb{R}^{p,q}$ with basis elements $\{L_{\mu\nu}, P_\mu, K_\mu, D\}$ $(\mu, \nu = 0, 1, \dots, p + q - 1)$ and commutation relations:

$$\begin{aligned}
&[D, P_\mu] = iP_\mu \ , \quad [D, K_\mu] = -iK_\mu \ , \quad [L_{\mu\nu}, L_{\rho\sigma}] = i(g_{\nu\rho}L_{\mu\sigma}+g_{\mu\sigma}L_{\nu\rho}-g_{\mu\rho}L_{\nu\sigma}-g_{\nu\sigma}L_{\mu\rho}) \\
&[K_\rho, L_{\mu\nu}] = i(g_{\rho\mu}K_\nu - g_{\rho\nu}K_\mu), \qquad [P_\rho, L_{\mu\nu}] = i(g_{\rho\mu}P_\nu - g_{\rho\nu}P_\mu), \\
&[K_\mu, P_\nu] = 2i(g_{\mu\nu}D - L_{\mu\nu}), [P_\mu, P_\nu] = 0, \quad [K_\mu, K_\nu] = 0, [L_{\mu\nu}, D] = 0.
\end{aligned} \tag{2.1}$$

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Note that elements $L_{\mu\nu}$ generate the Lie algebra $so(p,q)$ of the rotation group $\mathrm{SO}(p,q)$ in $\mathbb{R}^{p,q}$.

It is known [1] that conformal Lie algebra (2.1) is isomorphic to the algebra $so(p+1,q+1)$ with generators M_{ab} $(a,b=0,1,\dots,p+q+1)$ subject relations

$$[M_{ab},M_{dc}] = i(g_{bd}M_{ac}+g_{ac}M_{bd}-g_{ad}M_{bc}-g_{bc}M_{ad})\,, \tag{2.2}$$

where g_{ab} is the metric for $\mathbb{R}^{p+1,q+1}$:

$$g_{ab} = \mathrm{diag}(\underbrace{1,\dots,1}_{p},\underbrace{-1,\dots,-1}_{q},1,-1)\,. \tag{2.3}$$

The isomorphism of Lie algebras $so(p+1,q+1)$ and $\mathsf{conf}(\mathbb{R}^{p,q})$ is established by the relations (see, e.g., [2]):

$$\begin{aligned} L_{\mu\nu} &= M_{\mu\nu}\,, & K_\mu &= M_{n,\mu}-M_{n+1,\mu}\,, \\ P_\mu &= M_{n,\mu}+M_{n+1,\mu}\,, & D &= -M_{n,n+1}\,, \quad (n=p+q)\,. \end{aligned} \tag{2.4}$$

Using these formulas one can write relations (2.1) in the concise form (2.2). Define a quadratic Casimir operator

$$C_2 = \frac{1}{2}M_{ab}\,M^{ab}\,, \tag{2.5}$$

which is the center element in the enveloping algebra of $so(p+1,q+1)=\mathsf{conf}(\mathbb{R}^{p,q})$. In terms of generators of $\mathsf{conf}(\mathbb{R}^{p,q})$ the operator C_2 (2.5) is written as

$$C_2 = \frac{1}{2}\left(L_{\mu\nu}L^{\mu\nu}+P_\mu K^\mu+K_\mu P^\mu\right)-D^2\,, \tag{2.6}$$

where the identification (2.4) has been used.

Now we describe matrix representations for the conformal algebra $\mathsf{conf}(\mathbb{R}^{p,q})=so(p+1,q+1)$ which we call *spinor representations*. We consider only the case of even dimensions $n=p+q$ (the generalization to the odd dimensional case requires a separate investigation). Let γ_μ $(\mu=0,\dots,n-1)$ be $2^{\frac{n}{2}}$-dimensional gamma-matrices in $\mathbb{R}^{p,q}$:

$$\gamma_\mu\,\gamma_\nu+\gamma_\nu\,\gamma_\mu = 2\,g_{\mu\nu}\,I\,, \tag{2.7}$$

$$\gamma_{n+1}\equiv\alpha\,\gamma_0\cdot\gamma_1\cdots\gamma_{n-1}\,,\quad \alpha^2=(-1)^{q+n(n-1)/2}\,, \tag{2.8}$$

where I is the unit matrix and the constant α is such that $\gamma_{n+1}^2=I$. Using gamma-matrices (2.7) one can construct gamma-matrices Γ_a in the space $\mathbb{R}^{p+1,q+1}$ with the metric g_{ab} (2.3):

$$\begin{aligned} &\Gamma_\mu = \sigma_2\otimes\gamma_\mu = \begin{pmatrix} O & -i\gamma_\mu \\ i\gamma_\mu & O \end{pmatrix} \quad (\mu=0,\dots,n-1)\,, \\ &\Gamma_n = \sigma_1\otimes I = \begin{pmatrix} O & I \\ I & O \end{pmatrix}\,,\quad \Gamma_{n+1} = i\,\sigma_2\otimes\gamma_{n+1} = \begin{pmatrix} O & \gamma_{n+1} \\ -\gamma_{n+1} & O \end{pmatrix}\,, \\ &\Gamma_{n+3} = -\alpha\,\Gamma_0\cdot\Gamma_1\cdots\Gamma_{n+1} = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}\,, \end{aligned} \tag{2.9}$$

where O is the $2^{\frac{n}{2}}$-dimensional zero matrix. Here and below we use the standard Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.10}$$

Matrices (2.9), as it follows from (2.7), indeed satisfy Clifford relations for gamma-matrices in $\mathbb{R}^{p+1,q+1}$:

$$\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = 2\, g_{ab}\, \mathbf{I}\,, \quad \mathbf{I} \equiv I_2 \otimes I\,, \tag{2.11}$$

where I_2 is the 2×2 unit matrix. Now the standard spinor representation T for the generators M_{ab} of the Lie algebra $so(p+1, q+1)$ (2.2) is

$$T(M_{ab}) = \frac{i}{4}\,(\Gamma_a \Gamma_b - \Gamma_b \Gamma_a)\,. \tag{2.12}$$

Substitution of (2.9) into (2.12) and using (2.4) gives the spinor representation for $\mathsf{conf}(\mathbb{R}^{p,q})$

$$\begin{aligned} T(L_{\mu\nu}) &= \frac{i}{4}\, I_2 \otimes [\gamma_\mu, \gamma_\nu]\,, & T(K_\mu) &= \frac{1}{2}(I_2 \otimes \gamma_{n+1}\, \gamma_\mu - \sigma_3 \otimes \gamma_\mu)\,, \\ T(P_\mu) &= -\frac{1}{2}(I_2 \otimes \gamma_{n+1}\, \gamma_\mu + \sigma_3 \otimes \gamma_\mu)\,, & T(D) &= \frac{i}{2}\, \sigma_3 \otimes \gamma_{n+1}\,, \\ \mu, \nu &= 0, 1, \ldots, n-1\,, & n &= p+q\,. \end{aligned} \tag{2.13}$$

This representation is reducible since all matrices (2.13) have the block diagonal form

$$\begin{aligned} T(L_{\mu\nu}) &= \begin{pmatrix} \frac{i}{4}\, [\gamma_\mu, \gamma_\nu] & \mathbf{0} \\ \mathbf{0} & \frac{i}{4}\, [\gamma_\mu, \gamma_\nu] \end{pmatrix}, \quad T(K_\mu) = \begin{pmatrix} -\frac{1}{2}(1 - \gamma_{n+1})\, \gamma_\mu & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}(1 + \gamma_{n+1})\, \gamma_\mu \end{pmatrix}, \\ T(P_\mu) &= \begin{pmatrix} -\frac{1}{2}(1 + \gamma_{n+1})\, \gamma_\mu & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}(1 - \gamma_{n+1})\, \gamma_\mu \end{pmatrix}, \quad T(D) = \begin{pmatrix} \frac{i}{2}\, \gamma_{n+1} & \mathbf{0} \\ \mathbf{0} & -\frac{i}{2}\, \gamma_{n+1} \end{pmatrix}. \end{aligned} \tag{2.14}$$

Thus, the representation (2.13), (2.14) can be decomposed into the sum of two $2^{\frac{n}{2}}$-dimensional representations of $\mathsf{conf}(\mathbb{R}^{p,q})$. In fact these two representations are related to each other by the obvious automorphism of the conformal algebra (2.1):

$$L_{\mu\nu} \to L_{\mu\nu}\,, \quad P_\mu \to -K_\mu\,, \quad K_\mu \to -P_\mu\,, \quad D \to -D\,.$$

One of these representations, after applying the commutation relations for gamma-matrices, can be written in the form

$$\begin{aligned} T_1(L_{\mu\nu}) &= \frac{i}{4}\, [\gamma_\mu, \gamma_\nu] \equiv \ell_{\mu\nu}\,, & T_1(K_\mu) &= \gamma_\mu\, \frac{(1 - \gamma_{n+1})}{2} \equiv k_\mu\,, \\ T_1(P_\mu) &= \gamma_\mu\, \frac{(1 + \gamma_{n+1})}{2} \equiv p_\mu\,, & T_1(D) &= -\frac{i}{2}\, \gamma_{n+1} \equiv d\,, \end{aligned} \tag{2.15}$$

and it is not hard to check directly that the operators (2.15) possess needed commutation relations (2.1).

Further we use the common representation (see, e.g., recurrence (2.9)) for the gamma-matrices (2.7):

$$\gamma_\mu = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}_\mu \\ \bar{\boldsymbol{\sigma}}_\mu & \mathbf{0} \end{pmatrix}, \quad \gamma_{n+1} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \implies \frac{I + \gamma_{n+1}}{2} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \frac{I - \gamma_{n+1}}{2} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \tag{2.16}$$

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where $\mathbf{1}$, $\boldsymbol{\sigma}_\mu = ||(\boldsymbol{\sigma}_\mu)_{\alpha\dot{\alpha}}||$ and $\overline{\boldsymbol{\sigma}}_\mu = ||(\overline{\boldsymbol{\sigma}}_\mu)^{\dot{\alpha}\alpha}||$ are $2^{\frac{n}{2}-1}$ - dimensional matrices; $\mathbf{1}$ is unit matrix and $\boldsymbol{\sigma}_\mu$, $\overline{\boldsymbol{\sigma}}_\mu$ satisfy

$$\boldsymbol{\sigma}_\mu \overline{\boldsymbol{\sigma}}_\nu + \boldsymbol{\sigma}_\nu \overline{\boldsymbol{\sigma}}_\mu = 2\, g_{\mu\nu} \mathbf{1}\,, \;\; \overline{\boldsymbol{\sigma}}_\mu \boldsymbol{\sigma}_\nu + \overline{\boldsymbol{\sigma}}_\nu \boldsymbol{\sigma}_\mu = 2\, g_{\mu\nu} \mathbf{1}\,, \tag{2.17}$$

Equations (2.17) follow from identities (2.7) and representation (2.16) for γ_μ. In terms of gamma-matrices (2.16) conformal generators (2.15) can be represented as

$$\ell_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] = \begin{pmatrix} \frac{i}{4}(\boldsymbol{\sigma}_\mu \overline{\boldsymbol{\sigma}}_\nu - \boldsymbol{\sigma}_\nu \overline{\boldsymbol{\sigma}}_\mu) & \mathbf{0} \\ \mathbf{0} & \frac{i}{4}(\overline{\boldsymbol{\sigma}}_\mu \boldsymbol{\sigma}_\nu - \overline{\boldsymbol{\sigma}}_\nu \boldsymbol{\sigma}_\mu) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}_{\mu\nu} & \mathbf{0} \\ \mathbf{0} & \overline{\boldsymbol{\sigma}}_{\mu\nu} \end{pmatrix}, \tag{2.18}$$

$$p^\mu = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \overline{\boldsymbol{\sigma}}^\mu & \mathbf{0} \end{pmatrix}, \;\; k^\mu = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}^\mu \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \;\; d = -\frac{i}{2}\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}.$$

Recall that the matrices $\ell_{\mu\nu}$ as well as their diagonal blocks

$$\boldsymbol{\sigma}_{\mu\nu} = \frac{i}{4}(\boldsymbol{\sigma}_\mu \overline{\boldsymbol{\sigma}}_\nu - \boldsymbol{\sigma}_\nu \overline{\boldsymbol{\sigma}}_\mu) = ||(\boldsymbol{\sigma}_{\mu\nu})_\alpha^{\ \beta}||\,, \;\; \overline{\boldsymbol{\sigma}}_{\mu\nu} = \frac{i}{4}(\overline{\boldsymbol{\sigma}}_\mu \boldsymbol{\sigma}_\nu - \overline{\boldsymbol{\sigma}}_\nu \boldsymbol{\sigma}_\mu) = ||(\overline{\boldsymbol{\sigma}}_{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}}||\,, \tag{2.19}$$

are different spinor representations of the basis elements $L_{\mu\nu}$ of the algebra $so(p,q)$.

Remark 1. It is well known that any two $2^{n/2}$-dimensional representations of the Clifford algebra (2.7) are equivalent. Since the both sets of matrices $\{\gamma_\mu\}$ and $\{\gamma_\mu^\dagger\}$ represent the same Clifford algebra (2.7) we have

$$\gamma_\mu^\dagger = \mathbf{C} \cdot \gamma_\mu \cdot \mathbf{C}^{-1}\,, \quad \mu = 0, \ldots, n-1\,, \tag{2.20}$$

where matrix $\mathbf{C} \in \mathsf{Mat}(2^{n/2})$ can be fixed such that $\mathbf{C}^\dagger = \mathbf{C}$. For matrices $\ell_{\mu\nu}$ (2.18) and corresponding group elements

$$\mathbf{U} = \exp(i\omega^{\mu\nu}\ell_{\mu\nu}) = \begin{pmatrix} ||\boldsymbol{\Lambda}_\alpha^{\ \beta}|| & \mathbf{0} \\ \mathbf{0} & ||\overline{\boldsymbol{\Lambda}}^{\dot{\alpha}}_{\ \dot{\beta}}|| \end{pmatrix}, \quad \det(\mathbf{U}) = 1\,, \tag{2.21}$$

where $\omega^{\mu\nu} \in \mathbb{R}$, relations (2.20) give

$$\ell_{\mu\nu}^\dagger = \mathbf{C}\ell_{\mu\nu}\mathbf{C}^{-1} \;\;\Rightarrow\;\; \mathbf{U}^\dagger \mathbf{C} \mathbf{U} = \mathbf{C}\,. \tag{2.22}$$

The last equation means that $\mathbf{U}$ are pseudo-unitary matrices and their upper-diagonal blocks $\boldsymbol{\Lambda}$ (as well as their low-diagonal blocks $\overline{\boldsymbol{\Lambda}}$) generate matrix Lie group which is denoted as $\mathsf{Spin}(p,q)$. Definition (2.8) of γ_{n+1} and relations (2.20) yield

$$\gamma_{n+1}^\dagger = \alpha^*\, \mathbf{C} \cdot \gamma_{n-1} \cdots \gamma_0 \cdot \mathbf{C}^{-1} = (-1)^q\, \mathbf{C} \cdot \gamma_{n+1} \cdot \mathbf{C}^{-1}\,. \tag{2.23}$$

Note that there is a freedom in the definition of γ-matrices (2.16) and matrices $\boldsymbol{\sigma}_\mu$, $\overline{\boldsymbol{\sigma}}_\mu$ (2.17):

$$\gamma_\mu \to \begin{pmatrix} x & \mathbf{0} \\ \mathbf{0} & y \end{pmatrix} \gamma_\mu \begin{pmatrix} x^{-1} & \mathbf{0} \\ \mathbf{0} & y^{-1} \end{pmatrix} \;\;\Rightarrow\;\; \boldsymbol{\sigma}_\mu \to x \cdot \boldsymbol{\sigma}_\mu \cdot y^{-1}\,, \;\;\; \overline{\boldsymbol{\sigma}}_\mu \to y \cdot \overline{\boldsymbol{\sigma}}_\mu \cdot x^{-1}\,, \tag{2.24}$$

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where $x, y \in \mathsf{Mat}(2^{n/2-1})$. Then, applying this freedom[1] and using relations (2.20), (2.23) and explicit form (2.16) of matrix $\gamma_{n+1} = \gamma_{n+1}^\dagger$, we partially fix the matrix $\mathbf{C}$ according to the cases:

$$\begin{array}{ll} 1.)\ q - \text{even} \ \Rightarrow \ \mathbf{C} = \begin{pmatrix} \mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{c} \end{pmatrix}, & \mathbf{c}^\dagger = \mathbf{c} \in \mathsf{Mat}(2^{n/2-1})\,; \\ 2.)\ q - \text{odd} \ \Rightarrow \ \mathbf{C} = \begin{pmatrix} \mathbf{0} & \mathbf{g} \\ \mathbf{g} & \mathbf{0} \end{pmatrix}, & \mathbf{g}^\dagger = \mathbf{g} \in \mathsf{Mat}(2^{n/2-1})\,. \end{array} \tag{2.25}$$

Finally, from (2.25) we deduce the following relations for diagonal blocks $\mathbf{\Lambda}$ and $\overline{\mathbf{\Lambda}}$ of the matrices $\mathbf{U}$ (2.21), (2.22):

$$\begin{array}{l} 1.)\ q - \text{even} \ \Rightarrow \ \mathbf{\Lambda}^\dagger \cdot \mathbf{c} \cdot \mathbf{\Lambda} = \mathbf{c}\,, \ \overline{\mathbf{\Lambda}}^\dagger \cdot \mathbf{c} \cdot \overline{\mathbf{\Lambda}} = \mathbf{c}\,; \\ 2.)\ q - \text{odd} \ \Rightarrow \ \overline{\mathbf{\Lambda}} = \mathbf{g}^{-1} \cdot (\mathbf{\Lambda}^{-1})^\dagger \cdot \mathbf{g}\,. \end{array} \tag{2.26}$$

Remark 2. For the complexification of the group $\mathsf{Spin}(p,q)$ when parameters $\omega^{\mu\nu}$ in (2.21) are complex numbers the second relation in (2.22) is not valid. Here we present another conditions for blocks $\mathbf{\Lambda}, \overline{\mathbf{\Lambda}} \in \mathsf{Spin}(p,q)$ which are correct even for the complex case and which will be used below. Again the sets of matrices $\{\gamma_\mu\}$ and $\{\gamma_\mu^T\}$ represent the same Clifford algebra (2.7) and therefore we have

$$\gamma_\mu^T = \mathrm{C} \cdot \gamma_\mu \cdot \mathrm{C}^{-1}\,, \quad \mu = 0, \ldots, n-1\,. \tag{2.27}$$

For matrices $\ell_{\mu\nu}$ (2.18) and corresponding group elements (2.21) relations (2.27) give

$$\ell_{\mu\nu}^T = -\mathrm{C} \cdot \ell_{\mu\nu} \cdot \mathrm{C}^{-1} \ \Rightarrow \ \mathbf{U}^T \cdot \mathrm{C} \cdot \mathbf{U} = \mathrm{C}\,. \tag{2.28}$$

Definition (2.8) of γ_{n+1} and relations (2.27) yield

$$\gamma_{n+1}^T = \alpha\,(-1)^n\, \mathrm{C} \cdot \gamma_{n-1} \cdots \gamma_0 \cdot \mathrm{C}^{-1} = (-1)^{n(n-1)/2}\, \mathrm{C} \cdot \gamma_{n+1} \cdot \mathrm{C}^{-1}\,, \tag{2.29}$$

where we have used that n is the even number. Then, applying the freedom (2.24) and using relations (2.27), (2.29) and explicit form (2.16) of the matrix $\gamma_{n+1} = \gamma_{n+1}^T$, we fix operator C in (2.27) according to the cases:

$$\begin{array}{ll} 1.)\ \frac{n(n-1)}{2} - \text{even} \ \Rightarrow \ \mathrm{C} = \begin{pmatrix} \mathrm{c} & \mathbf{0} \\ \mathbf{0} & \mathrm{c} \end{pmatrix}, & \mathrm{c}^T = \mathrm{c} \in \mathsf{Mat}(2^{n/2-1})\,; \\ 2.)\ \frac{n(n-1)}{2} - \text{odd} \ \Rightarrow \ \mathrm{C} = \begin{pmatrix} \mathbf{0} & \mathrm{g} \\ \mathrm{g} & \mathbf{0} \end{pmatrix}, & \mathrm{g}^T = \mathrm{g} \in \mathsf{Mat}(2^{n/2-1})\,. \end{array} \tag{2.30}$$

Finally, from (2.30) we deduce the following relations for diagonal blocks $\mathbf{\Lambda}$ and $\overline{\mathbf{\Lambda}}$ of the matrices $\mathbf{U}$ (2.21), (2.28):

$$\begin{array}{l} 1.)\ \frac{n(n-1)}{2} - \text{even} \ \Rightarrow \ \mathbf{\Lambda}^T \cdot \mathrm{c} \cdot \mathbf{\Lambda} = \mathrm{c}\,, \ \overline{\mathbf{\Lambda}}^T \cdot \mathrm{c} \cdot \overline{\mathbf{\Lambda}} = \mathrm{c}\,; \\ 2.)\ \frac{n(n-1)}{2} - \text{odd} \ \Rightarrow \ \mathbf{\Lambda}^T \cdot \mathrm{g} \cdot \overline{\mathbf{\Lambda}} = \mathrm{g}\,. \end{array} \tag{2.31}$$

[1] The matrix γ_{n+1} does not changed under the transformations (2.24) and one can bring one of the matrices γ_μ (for μ such that $g_{\mu\mu} = +1$), say γ_0, to the standard form $\gamma_0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$.

2.1 Differential realization for conformal algebra and induced representations

The standard differential representation ρ of elements $\{L_{\mu\nu}, P_\mu, K_\mu, D\}$ of the algebra (2.1) is [2]:

$$
\begin{aligned}
&\rho(P_\mu) = -i\partial_{x_\mu} \equiv \hat{p}_\mu \,, \qquad \rho(D) = x^\mu \hat{p}_\mu - i\Delta \,, \\
&\rho(L_{\mu\nu}) = \hat{\ell}_{\mu\nu} + S_{\mu\nu} \,, \qquad \rho(K_\mu) = 2\, x^\nu \left(\hat{\ell}_{\nu\mu} + S_{\nu\mu}\right) + (x^\nu x_\nu)\hat{p}_\mu - 2i\Delta x_\mu \,,
\end{aligned} \tag{2.32}
$$

$$
\hat{\ell}_{\mu\nu} \equiv (x_\nu \hat{p}_\mu - x_\mu \hat{p}_\nu) \,,
$$

where x_μ are coordinates in $\mathbb{R}^{p,q}$, $\Delta \in \mathbb{R}$ — conformal parameter, $S_{\mu\nu} = -S_{\nu\mu}$ are spin generators with the same commutation relations as for generators $L_{\mu\nu}$ (see (2.1)):

$$
[S_{\mu\nu}, S_{\rho\sigma}] = i(g_{\nu\rho} S_{\mu\sigma} + g_{\mu\sigma} S_{\nu\rho} - g_{\mu\rho} S_{\nu\sigma} - g_{\nu\sigma} S_{\mu\rho}) \,, \tag{2.33}
$$

and $[S_{\mu\nu}, x_\rho] = 0 = [S_{\mu\nu}, \hat{p}_\rho]$. Note that in the differential representation (2.32) the quadratic Casimir operator (2.6) acquires the form:

$$
\rho(C_2) = \frac{1}{2}\left(S_{\mu\nu}\, S^{\mu\nu} - \hat{\ell}_{\mu\nu}\, \hat{\ell}^{\mu\nu}\right) + \Delta(\Delta - n) \,. \tag{2.34}
$$

In this subsection we obtain the differential realization (2.32) of the conformal algebra by means of the method of induced representations. Our method is slightly different from the method which was used in [2].

First we pack generators (2.32) into $2^{n/2}$-dimensional matrix

$$
\begin{aligned}
\frac{1}{2} T_1(M^{ab}) \cdot \rho(M_{ab}) &= \frac{1}{2}[\ell^{\mu\nu} \cdot \rho(L_{\mu\nu}) + p^\mu \cdot \rho(K_\mu) + k^\mu \cdot \rho(P_\mu)] - d \cdot \rho(D) = \\
&= \begin{pmatrix} \mathbf{L} + \mathbf{S} + \frac{i}{2}\rho(D)\, \mathbf{1} & \mathbf{p} \\ \overline{\mathbf{K}} & \overline{\mathbf{L}} + \overline{\mathbf{S}} - \frac{i}{2}\rho(D)\, \mathbf{1} \end{pmatrix} ,
\end{aligned} \tag{2.35}
$$

where the representations T_1 and ρ were defined in (2.15) and (2.32), respectively. In eq. (2.35) and below we use notations

$$
\begin{aligned}
&\mathbf{L} = \frac{1}{2}\boldsymbol{\sigma}^{\mu\nu}\, \hat{\ell}_{\mu\nu} \,, \quad \overline{\mathbf{L}} = \frac{1}{2}\overline{\boldsymbol{\sigma}}^{\mu\nu}\, \hat{\ell}_{\mu\nu} \,, \quad \mathbf{p} = \frac{1}{2}\boldsymbol{\sigma}^\mu\, \hat{p}_\mu = -\frac{i}{2}\boldsymbol{\sigma}^\mu\, \partial_{x_\mu} \,, \\
&\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}^{\mu\nu}\, S_{\mu\nu} \,, \quad \overline{\mathbf{S}} = \frac{1}{2}\overline{\boldsymbol{\sigma}}^{\mu\nu}\, S_{\mu\nu} \,, \quad \overline{\mathbf{K}} = \frac{1}{2}\overline{\boldsymbol{\sigma}}^\mu\, \rho(K_\mu) \,, \qquad \mathbf{x} = -i\,\overline{\boldsymbol{\sigma}}^\mu\, x_\mu \,.
\end{aligned} \tag{2.36}
$$

Then we need the following technical result. Namely, operators (2.36) satisfy identities

$$
\mathbf{L} = -\mathbf{p}\cdot\mathbf{x} - \frac{i}{2}(\hat{p}_\mu x^\mu)\, \mathbf{1} \,, \qquad \overline{\mathbf{L}} = \mathbf{x}\cdot\mathbf{p} + \frac{i}{2}(x^\mu \hat{p}_\mu)\, \mathbf{1} \,, \tag{2.37}
$$

$$
\overline{\mathbf{K}} = (\mathbf{x}\cdot\mathbf{S} - \overline{\mathbf{S}}\cdot\mathbf{x}) - \mathbf{x}\cdot\mathbf{p}\cdot\mathbf{x} + \left(\Delta - \frac{n}{2}\right)\, \mathbf{x} \,. \tag{2.38}
$$

Indeed, the first identity in (2.37) can be deduced as following

$$
\mathbf{L} = \frac{i}{4}\boldsymbol{\sigma}^\mu \overline{\boldsymbol{\sigma}}^\nu (\hat{p}_\mu x_\nu - \hat{p}_\nu x_\mu) = -\frac{1}{2}\,\mathbf{p}\cdot\mathbf{x} - \frac{i}{4}\,(2g^{\mu\nu}\mathbf{1} - \boldsymbol{\sigma}^\nu \overline{\boldsymbol{\sigma}}^\mu)\hat{p}_\nu x_\mu = -\mathbf{p}\cdot\mathbf{x} - \frac{i}{2}\,(\hat{p}^\mu x_\mu)\, \mathbf{1} \,,
$$

where we have applied (2.17). Second identity in (2.37) is obtained analogously. To prove identity (2.38) we note that

$$\begin{aligned} \mathbf{x}\cdot\mathbf{S}-\overline{\mathbf{S}}\cdot\mathbf{x} &= \overline{\boldsymbol{\sigma}}^{\nu}\, x^{\mu}\, S_{\mu\nu}\,, \\ \mathbf{x}\cdot\mathbf{p}\cdot\mathbf{x} &= -(\overline{\boldsymbol{\sigma}}^{\mu}x_{\mu})\,(x^{\nu}\,\hat{p}_{\nu}) + \frac{1}{2}\,x^2\,(\overline{\boldsymbol{\sigma}}^{\mu}\hat{p}_{\mu}) + \frac{n}{2}\,i\,(\overline{\boldsymbol{\sigma}}^{\mu}x_{\mu})\,, \end{aligned} \tag{2.39}$$

where we again applied (2.17). Then (2.38) follows from (2.32) and (2.39). Now we substitute (2.37), (2.38) into (2.35). As a result the matrix (2.35) can be written in the form

$$\frac{1}{2}T_1(M^{ab})\cdot\rho(M_{ab}) = \begin{pmatrix} \frac{\Delta-n}{2}\cdot\mathbf{1}+\mathbf{S}-\mathbf{p}\cdot\mathbf{x}\,, & \mathbf{p} \\ \mathbf{x}\cdot\mathbf{S}-\overline{\mathbf{S}}\cdot\mathbf{x}-\mathbf{x}\cdot\mathbf{p}\cdot\mathbf{x}+(\Delta-\frac{n}{2})\cdot\mathbf{x}\,, & -\frac{\Delta}{2}\cdot\mathbf{1}+\overline{\mathbf{S}}+\mathbf{x}\cdot\mathbf{p} \end{pmatrix}, \tag{2.40}$$

and this form of (2.35) will be extensively used below.

Now we consider the set of matrices

$$A = i(\omega^{\mu\nu}\,\ell_{\mu\nu} + a^{\mu}\,p_{\mu} + b^{\mu}\,k_{\mu} + \beta\,d)\,, \quad (\omega^{\mu\nu},\, a^{\mu},\, b^{\mu},\, \beta \in \mathbb{R})\,, \tag{2.41}$$

which are the linear combinations of the generators (2.18). These matrices form a matrix Lie algebra. The corresponding matrix Lie group G is isomorphic to the group $\mathsf{Spin}(p+1,q+1)$. The elements $g\in G$ (at least that which are closed to unity) can be represented in the exponential form

$$g = \exp(i\omega^{\mu\nu}\,\ell_{\mu\nu} + ia^{\mu}\,p_{\mu} + ib^{\mu}\,k_{\mu} + i\beta\,d)\,.$$

We stress that elements $g\in\mathsf{Spin}(p{+}1,q{+}1)$ satisfy one of the equations in (2.26) depending on the case of $(q+1)$ is even or odd. The group $G\simeq\mathsf{Spin}(p+1,q+1)$ has a subgroup $H\subset G$ which is generated by elements $\{\ell_{\mu\nu},\,k_{\mu},\,d\}$:

$$h = \exp(i\omega^{\mu\nu}\,\ell_{\mu\nu} + ib^{\mu}\,k_{\mu} + i\beta\,d)\,\in\,H\,. \tag{2.42}$$

This fact immediately follows from the commutation relations (2.1). In the representation (2.18) the elements (2.42) can be written in the matrix form

$$h = \begin{pmatrix} e^{\frac{\beta}{2}}\cdot\exp(i\omega^{\mu\nu}\boldsymbol{\sigma}_{\mu\nu}) & e^{\frac{\beta}{2}}\boldsymbol{\Lambda}_0 \\ 0 & e^{-\frac{\beta}{2}}\cdot\exp(i\omega^{\mu\nu}\overline{\boldsymbol{\sigma}}_{\mu\nu}) \end{pmatrix} = \begin{pmatrix} \delta\cdot\mathbf{1} & 0 \\ 0 & \overline{\delta}\cdot\mathbf{1} \end{pmatrix}\cdot\begin{pmatrix} \boldsymbol{\Lambda} & \boldsymbol{\Lambda}_0 \\ 0 & \overline{\boldsymbol{\Lambda}} \end{pmatrix}, \tag{2.43}$$

where we denote $\delta = \overline{\delta}^{\,-1} = e^{\frac{\beta}{2}}$. We recall that matrices $\boldsymbol{\Lambda}$, $\overline{\boldsymbol{\Lambda}}$ were defined in (2.21), (2.26) and they satisfy $\det(\boldsymbol{\Lambda}) = \det(\overline{\boldsymbol{\Lambda}}) = 1$. The coset space G/H can be parameterized by the special elements of $\mathsf{Spin}(p+1,q+1)$

$$Z = \exp(-ix^{\mu}\,p_{\mu}) = \begin{pmatrix} \mathbf{1} & 0 \\ -ix^{\mu}\overline{\boldsymbol{\sigma}}_{\mu} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ \mathbf{x} & \mathbf{1} \end{pmatrix},$$

and any element $g\in G$ is uniquely represented as a product $g = Z\cdot h$, where $Z\in G/H$ and $h\in H$. The group $G\simeq\mathsf{Spin}(p+1,q+1)$ acts on the coset space G/H as following

$$g^{-1}\cdot Z = Z'\cdot h^{-1}\,, \quad \forall g\in G\,,\ \forall Z\in G/H\,, \tag{2.44}$$

where $h \in H$ and $Z' \in G/H$ depends on g and Z. We take g^{-1} and Z' in the block form

$$g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \, , \quad Z' = \begin{pmatrix} \mathbf{1} & 0 \\ \mathbf{x}' & \mathbf{1} \end{pmatrix} \in G/H \, . \tag{2.45}$$

and from (2.44) we deduce expressions:

$$\mathbf{x}' = (C + D\,\mathbf{x})(A + B\,\mathbf{x})^{-1} \, , \tag{2.46}$$

$$h^{-1} = \begin{pmatrix} A + B\,\mathbf{x} & B \\ 0 & D - \mathbf{x}'\,B \end{pmatrix} \, . \tag{2.47}$$

For the subgroup H consisting of elements h (2.43) we define the representation T which acts in the space of tensors $\Phi^{\dot\alpha_1 \dots \dot\alpha_{2\dot\ell}}_{\alpha_1 \dots \alpha_{2\ell}}$ of the type $(\ell, \dot\ell)$:

$$[T(h) \cdot \Phi]^{\underline{\dot\alpha}}_{\underline{\alpha}} = \delta^{\Delta}\, \overline{\delta}^{\Delta}\, t(\mathbf{\Lambda})_{\underline{\alpha}}^{\;\underline{\beta}} \cdot \overline{t}(\overline{\mathbf{\Lambda}})^{\underline{\dot\alpha}}_{\;\underline{\dot\beta}} \cdot \Phi^{\underline{\dot\beta}}_{\underline{\beta}} \, . \tag{2.48}$$

Here we assume that parameters δ, $\overline{\delta}$ of h are independent and $\underline{\alpha}$ and $\underline{\dot\alpha}$ denote collections of indexes $(\alpha_1 \dots \alpha_{2\ell})$ and $(\dot\alpha_1 \dots \dot\alpha_{2\dot\ell})$, respectively. Matrices t and $\overline{t}$ are two inequivalent representations of the subgroup $\mathsf{Spin}(p,q) \subset \mathsf{Spin}(p+1,q+1)$. Matrix t corresponds to the representation of the type $(\ell, 0)$ with undotted spinor indices while $\overline{t}$ corresponds to the representation of the type $(0, \dot\ell)$ with dotted spinor indices. In particular for $(1/2, 0)$ and $(0, 1/2)$ type representations we have (see (2.21)) $t(\mathbf{\Lambda})^{\alpha}_{\;\beta} = \mathbf{\Lambda}^{\alpha}_{\;\beta}$ and $\overline{t}(\overline{\mathbf{\Lambda}})_{\dot\alpha}^{\;\dot\beta} = \overline{\mathbf{\Lambda}}_{\dot\alpha}^{\;\dot\beta}$, respectively.

Then we induce representation (2.48) of the subgroup H to the representation ρ of the whole group G. The representation ρ acts in the space of tensor fields $\Phi^{\underline{\dot\alpha}}_{\underline{\alpha}}(\mathbf{x})$ according to the rule

$$\rho(g) \cdot \Phi(\mathbf{x}) = [T(h) \cdot \Phi](\mathbf{x}') \; ; \; h \in H \; ; \; g \in G \, , \tag{2.49}$$

where elements g, h and parameters $\mathbf{x}$, $\mathbf{x}'$ are related by the formula (2.44).

Our aim is to find the infinitesimal form of (2.49). To do this we first take the element g^{-1} (2.45) in the infinitesimal form

$$g^{-1} = \begin{pmatrix} \mathbf{1} - \varepsilon_{11} & -\varepsilon_{12} \\ -\varepsilon_{21} & \mathbf{1} - \varepsilon_{22} \end{pmatrix} = I - ||\varepsilon_{ij}|| \, , \tag{2.50}$$

where the 2×2 block matrix $||\varepsilon_{ij}||$ can be represented as linear combination (2.41) of $\mathsf{Spin}(p+1,q+1)$ generators and in particular we have $\mathrm{tr}(\varepsilon_{11}) = -\,\mathrm{tr}(\varepsilon_{22}) \in \mathbb{R}$. It is easy to find from (2.46) that

$$\mathbf{x}' = \mathbf{x} + (-\varepsilon_{21} - \varepsilon_{22} \cdot \mathbf{x} + \mathbf{x} \cdot \varepsilon_{11} + \mathbf{x} \cdot \varepsilon_{12} \cdot \mathbf{x}) \, ,$$

and for the parameters δ, $\overline{\delta}$ and diagonal blocks of matrix h (2.43) we have:

$$\begin{gathered} \delta = 1 + \mathrm{tr}[\varepsilon_{11} + \varepsilon_{12} \cdot \mathbf{x}] \, , \qquad \overline{\delta} = 1 + \mathrm{tr}[\varepsilon_{22} - \varepsilon_{12} \cdot \mathbf{x}] \, , \\ \mathbf{\Lambda} = \mathbf{1} + (\varepsilon_{11} + \varepsilon_{12} \cdot \mathbf{x} - \mathrm{tr}[\varepsilon_{11} + \varepsilon_{12} \cdot \mathbf{x}] \cdot \mathbf{1}) \equiv \mathbf{1} + \varepsilon(\mathbf{x}) \, , \\ \overline{\mathbf{\Lambda}} = \mathbf{1} + (\varepsilon_{22} - \mathbf{x} \cdot \varepsilon_{12} - \mathrm{tr}[\varepsilon_{22} - \varepsilon_{12} \cdot \mathbf{x}] \cdot \mathbf{1}) \equiv \mathbf{1} + \overline{\varepsilon}(\mathbf{x}) \, , \end{gathered}$$

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where to simplify formulas we normalize the trace such that $\mathrm{tr}(\mathbf{1}) = 1$. In particular this normalization yields

$$\mathrm{tr}[\boldsymbol{\sigma}_\mu \overline{\boldsymbol{\sigma}}_\nu] = g_{\mu\nu}\,, \quad \mathrm{tr}[\boldsymbol{\sigma}_\mu \overline{\boldsymbol{\sigma}}_\nu \boldsymbol{\sigma}_\lambda \overline{\boldsymbol{\sigma}}_\rho] = 2\left(g_{\mu\nu} g_{\lambda\rho} - g_{\mu\lambda} g_{\nu\rho} + g_{\mu\rho} g_{\nu\lambda}\right)\,, \quad \ldots\,.$$

Further we assume that generators $S_{\mu\nu}$ (2.33) of infinitesimal Lorentz transformations are related to matrix representations t and $\bar{t}$ (2.48) of the Lorentz subgroup by means of the formulae

$$t_{\underline{\alpha}}^{\;\underline{\beta}}(\mathbf{1}+\varepsilon(\mathbf{x})) = \delta_{\underline{\alpha}}^{\underline{\beta}} + 2\,\mathrm{tr}[\varepsilon(\mathbf{x})\,\mathbf{S}_{\underline{\alpha}}^{\;\underline{\beta}}]\,, \qquad \bar{t}^{\underline{\dot\alpha}}_{\;\underline{\dot\beta}}(\mathbf{1}+\bar\varepsilon(\mathbf{x})) = \delta^{\underline{\dot\alpha}}_{\underline{\dot\beta}} + 2\,\mathrm{tr}[\bar\varepsilon(\mathbf{x})\cdot \overline{\mathbf{S}}^{\underline{\dot\alpha}}_{\;\underline{\dot\beta}}]\,,$$

where $\mathbf{S}_{\underline{\alpha}}^{\;\underline{\beta}} = \frac{1}{2}(S^{\mu\nu})_{\underline{\alpha}}^{\;\underline{\beta}}\cdot\boldsymbol{\sigma}_{\mu\nu}$ and $\overline{\mathbf{S}}^{\underline{\dot\alpha}}_{\;\underline{\dot\beta}} = \frac{1}{2}(S^{\mu\nu})^{\underline{\dot\alpha}}_{\;\underline{\dot\beta}}\cdot\overline{\boldsymbol{\sigma}}_{\mu\nu}$ (see (2.36)). Operators $(S^{\mu\nu})_{\underline{\alpha}}^{\;\underline{\beta}}$ and $(S^{\mu\nu})^{\underline{\dot\alpha}}_{\;\underline{\dot\beta}}$ define the action of generators $S_{\mu\nu}$ on the tensor fields of $(\ell,0)$ and $(0,\dot\ell)$ types

$$\begin{aligned} (S^{\mu\nu})_{\underline{\alpha}}^{\;\underline{\beta}}\Phi_{\underline{\beta}} &= (\boldsymbol{\sigma}_{\mu\nu})_{\alpha_1}^{\;\;\alpha}\,\Phi_{\alpha\alpha_2\cdots\alpha_{2\ell}} + \cdots + (\boldsymbol{\sigma}_{\mu\nu})_{\alpha_{2\ell}}^{\;\;\alpha}\,\Phi_{\alpha_1\cdots\alpha_{2\ell-1}\alpha}\,,\\ (S^{\mu\nu})^{\underline{\dot\alpha}}_{\;\underline{\dot\beta}}\Phi^{\underline{\dot\beta}} &= (\bar{\boldsymbol{\sigma}}_{\mu\nu})^{\dot\alpha_1}_{\;\;\dot\alpha}\,\Phi^{\dot\alpha\dot\alpha_2\cdots\dot\alpha_{2\dot\ell}} + \cdots + (\bar{\boldsymbol{\sigma}}_{\mu\nu})^{\dot\alpha_{2\ell}}_{\;\;\dot\alpha}\,\Phi^{\dot\alpha_1\cdots\dot\alpha_{2\ell-1}\dot\alpha}\,. \end{aligned} \tag{2.51}$$

Thus, for (2.49) we have

$$\begin{aligned} \rho(g)\,\Phi(\mathbf{x}) = \Big(1 + \big(&\Delta\,\mathrm{tr}[\varepsilon_{11}+\varepsilon_{12}\,\mathbf{x}] - \Delta\,\mathrm{tr}[\varepsilon_{22}-\varepsilon_{12}\,\mathbf{x}] + 2\,\mathrm{tr}\left[(\varepsilon_{11}+\varepsilon_{12}\,\mathbf{x})\,\mathbf{S}\right] \\ &+ 2\,\mathrm{tr}\left[(\varepsilon_{22}-\mathbf{x}\,\varepsilon_{12})\,\overline{\mathbf{S}}\right]\,\big)\Big)\cdot \\ &\cdot\,\Phi\left(\mathbf{x} + (-\varepsilon_{21} - \varepsilon_{22}\,\mathbf{x} + \mathbf{x}\,\varepsilon_{11} + \mathbf{x}\,\varepsilon_{12}\,\mathbf{x})\right)\,. \end{aligned} \tag{2.52}$$

According to (2.50) we denote the infinitesimal part of $\rho(g)$ as $\rho(||\varepsilon_{ij}||)$ and write the l.h.s. of (2.52) in the form $\rho(g)\,\Phi(\mathbf{x}) = \Phi(\mathbf{x}) + \rho(||\varepsilon_{ij}||)\cdot\Phi(\mathbf{x})$. Next we transform infinitesimal part of r.h.s. of (2.52) in the form of the trace by using expansion

$$\Phi\left(\mathbf{x} - \epsilon(\mathbf{x})\right) = \left(1 + 2\,\mathrm{tr}\left[\epsilon(\mathbf{x})\cdot\mathbf{p}\right]\right)\Phi(\mathbf{x})\,, \quad \mathbf{p} = -\frac{i}{2}\,\boldsymbol{\sigma}^\mu\,\partial_{x^\mu}\,, \tag{2.53}$$

and cyclicity of the trace taking into account the noncommutativity of operators $\mathbf{x}$ and $\mathbf{p}$, e.g., $\mathrm{tr}\left[\mathbf{x}\cdot\varepsilon_{11}\cdot\mathbf{p}\right] = \mathrm{tr}\left[\varepsilon_{11}(\mathbf{p}\cdot\mathbf{x}+\frac{n}{2})\right]$, etc. Note that the operator $\mathbf{p}$ is the same as in (2.36). As a result we write (2.52) in the form

$$\begin{aligned} \rho(||\varepsilon_{ij}||)\,\Phi(\mathbf{x}) &= 2\,\mathrm{tr}\Big[\varepsilon_{11}\cdot\Big(\frac{\Delta-n}{2} + \mathbf{S} - \mathbf{p}\,\mathbf{x}\Big) + \varepsilon_{12}\cdot\Big((\Delta-\frac{n}{2})\,\mathbf{x} + \mathbf{x}\,\mathbf{S} - \overline{\mathbf{S}}\,\mathbf{x} - \mathbf{x}\,\mathbf{p}\,\mathbf{x}\Big) + \\ &\qquad + \varepsilon_{21}\cdot\mathbf{p} + \varepsilon_{22}\cdot\Big(-\frac{\Delta}{2} + \overline{\mathbf{S}} + \mathbf{x}\,\mathbf{p}\Big)\Big]\,\Phi(\mathbf{x}) \\ &= \mathrm{Tr}\left[\begin{pmatrix}\varepsilon_{11} & \varepsilon_{12}\\ \varepsilon_{21} & \varepsilon_{22}\end{pmatrix}\,\Big(T_1(M^{ab})\otimes\rho(M_{ab})\Big)\right]\Phi(\mathbf{x})\,. \end{aligned}$$

From this formula we immediately recover generators (2.32) of the conformal algebra collected in the blocks as they appear in the matrix (2.40).

At the end of this section we list global forms (2.46) of four basic conformal group transformations and give corresponding elements $h \in H$, $g \in G$ which have been used in (2.49).

- Translations

$$g = e^{ia^\mu p_\mu} = \begin{pmatrix} \mathbf{1} & 0 \\ \mathbf{a} & \mathbf{1} \end{pmatrix}, \quad \mathbf{x}' = \mathbf{x} - \mathbf{a}, \quad h = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \quad \mathbf{a} := ia^\mu \overline{\boldsymbol{\sigma}}_\mu . \tag{2.54}$$

- Lorentz rotations

$$g = e^{i\omega^{\mu\nu}\ell_{\mu\nu}} = \begin{pmatrix} \boldsymbol{\Lambda} & 0 \\ 0 & \overline{\boldsymbol{\Lambda}} \end{pmatrix}, \quad \mathbf{x}' = \overline{\boldsymbol{\Lambda}}^{-1} \cdot \mathbf{x} \cdot \boldsymbol{\Lambda}, \quad h = \begin{pmatrix} \boldsymbol{\Lambda} & 0 \\ 0 & \overline{\boldsymbol{\Lambda}} \end{pmatrix}. \tag{2.55}$$

- Dilatation

$$g = e^{i\beta d} = \begin{pmatrix} e^{\frac{\beta}{2}} & 0 \\ 0 & e^{-\frac{\beta}{2}} \end{pmatrix}, \quad \mathbf{x}' = e^{\beta}\,\mathbf{x}, \quad h = \begin{pmatrix} e^{\frac{\beta}{2}} & 0 \\ 0 & e^{-\frac{\beta}{2}} \end{pmatrix}. \tag{2.56}$$

- Special conformal transformations

$$g = e^{ib^\mu k_\mu} = \begin{pmatrix} \mathbf{1} & \boldsymbol{b} \\ 0 & \mathbf{1} \end{pmatrix}, \quad \mathbf{x}' = \mathbf{x} \cdot (\mathbf{1} - \boldsymbol{b} \cdot \mathbf{x})^{-1}, \tag{2.57}$$

$$h = \begin{pmatrix} (\mathbf{1} - \boldsymbol{b} \cdot \mathbf{x})^{-1}, & (\mathbf{1} - \boldsymbol{b} \cdot \mathbf{x})^{-1} \cdot \boldsymbol{b} \cdot (1 + \mathbf{x}' \cdot \mathbf{b})^{-1} \\ 0, & (1 + \mathbf{x}' \cdot \mathbf{b})^{-1} \end{pmatrix}, \quad \boldsymbol{b} := ib^\mu \boldsymbol{\sigma}_\mu .$$

At the end of this subsection we stress that all formulas (2.54) – (2.57) are written for the case when dimension $n = p + q$ is even.

In this section the Lie group $\mathrm{SO}(p+1, q+1)$ has been used to derive straightforwardly the formula (2.40) for polarized Casimir operator by means of induced representation method. Below we will work only with conformal algebra $so(p+1, q+1)$.

2.2 Spin operators S and $\overline{\text{S}}$

According to previous subsection we consider the representation of the conformal algebra in the space of tensor fields $\Phi^{\dot\alpha_1 \cdots \dot\alpha_{2\dot\ell}}_{\alpha_1 \cdots \alpha_{2\ell}}(x)$ (see (2.48) and (2.49)) of the type $(\ell, \dot\ell)$. Here and below in the capacity of the argument of fields Φ we use the point $x \in \mathbb{R}^{p,q}$ with coordinates x_μ instead of corresponding matrix $\mathbf{x}$ (2.36). The generators $S_{\mu\nu}$ act on the tensor field of the type $(\ell, \dot\ell)$ according to the formulas (2.51):

$$\begin{aligned} [S_{\mu\nu}\Phi]^{\dot\alpha_1 \cdots \dot\alpha_{2\dot\ell}}_{\alpha_1 \cdots \alpha_{2\ell}} = (\boldsymbol{\sigma}_{\mu\nu})_{\alpha_1}^{\ \ \alpha}\, \Phi^{\dot\alpha_1 \cdots \dot\alpha_{2\dot\ell}}_{\alpha\alpha_2 \cdots \alpha_{2\ell}} + \cdots + (\boldsymbol{\sigma}_{\mu\nu})_{\alpha_{2\ell}}^{\ \ \alpha}\, \Phi^{\dot\alpha_1 \cdots \dot\alpha_{2\dot\ell}}_{\alpha_1 \cdots \alpha_{2\ell-1}\alpha} + \\ + (\bar{\boldsymbol{\sigma}}_{\mu\nu})^{\dot\alpha_1}_{\ \ \dot\alpha}\, \Phi^{\dot\alpha\dot\alpha_2 \cdots \dot\alpha_{2\dot\ell}}_{\alpha_1 \cdots \alpha_{2\ell}} + \cdots + (\bar{\boldsymbol{\sigma}}_{\mu\nu})^{\dot\alpha_{2\dot\ell}}_{\ \ \dot\alpha}\, \Phi^{\dot\alpha_1 \cdots \dot\alpha_{2\dot\ell-1}\dot\alpha}_{\alpha_1 \cdots \alpha_{2\ell}} . \end{aligned} \tag{2.58}$$

First we discuss the very special case of representations of $so(p+1, q+1)$ when tensor fields $\Phi^{\dot\alpha_1 \cdots \dot\alpha_{2\dot\ell}}_{\alpha_1 \cdots \alpha_{2\ell}}(x)$ are such that dotted and undotted indexes compose symmetric sets separately. In this situation it is convenient to work with the generating functions

$$\Phi(x, \lambda, \tilde\lambda) = \Phi^{\dot\alpha_1 \cdots \dot\alpha_{2\dot\ell}}_{\alpha_1 \cdots \alpha_{2\ell}}(x)\, \lambda^{\alpha_1} \cdots \lambda^{\alpha_{2\ell}}\, \tilde\lambda_{\dot\alpha_1} \cdots \tilde\lambda_{\dot\alpha_{2\dot\ell}} , \tag{2.59}$$

where λ and $\tilde{\lambda}$ are auxiliary spinors. Using these generating functions the action (2.58) of generators $S_{\mu\nu}$ can be written in a compact form

$$[S_{\mu\nu}\Phi]\,(x,\lambda,\tilde{\lambda}) = \left[\lambda\,\boldsymbol{\sigma}_{\mu\nu}\partial_\lambda + \tilde{\lambda}\,\bar{\boldsymbol{\sigma}}_{\mu\nu}\partial_{\tilde{\lambda}}\right]\Phi(x,\lambda,\tilde{\lambda})\,, \tag{2.60}$$

where

$$\lambda\,\boldsymbol{\sigma}_{\mu\nu}\partial_\lambda = \lambda_\alpha\,(\boldsymbol{\sigma}_{\mu\nu})^\alpha{}_\beta\,\partial_{\lambda_\beta} \quad ; \quad \tilde{\lambda}\,\bar{\boldsymbol{\sigma}}_{\mu\nu}\partial_{\tilde{\lambda}} = \tilde{\lambda}^{\dot{\alpha}}\,(\bar{\boldsymbol{\sigma}}_{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}}\,\partial_{\tilde{\lambda}^{\dot{\beta}}}\,.$$

In accordance with (2.60) we obtain the realization of the spin generators $S_{\mu\nu}$ as differential operators over spinor variables

$$S_{\mu\nu} = \lambda\,\boldsymbol{\sigma}_{\mu\nu}\partial_\lambda + \tilde{\lambda}\,\overline{\boldsymbol{\sigma}}_{\mu\nu}\partial_{\tilde{\lambda}}\,. \tag{2.61}$$

One can easily show that operators $S_{\mu\nu}$ defined in (2.61) respect commutation relations (2.33) for the algebra $so(p,q)$.

Now we consider the 4-dimensional Minkowski case $n=4$, i.e. $\mathbb{R}^{p,q} = \mathbb{R}^{1,3}$. In this case the dimension of the spinor spaces is equal to $2^{n/2} = 2$ and tensor fields $\Phi^{\dot{\alpha}_1\cdots\dot{\alpha}_{2\dot{\ell}}}_{\alpha_1\cdots\alpha_{2\ell}}(x)$ are automatically symmetric under permutations of dotted and undotted indexes separately. For the Minkowski space $\mathbb{R}^{1,3}$, in the expressions for gamma-matrices (2.16), we choose

$$\boldsymbol{\sigma}_\mu = (\sigma_0,\sigma_1,\sigma_2,\sigma_3)\,, \qquad \overline{\boldsymbol{\sigma}}_\mu = (\sigma_0,-\sigma_1,-\sigma_2,-\sigma_3)\,, \tag{2.62}$$

where $\sigma_0 = \mathrm{I}_2$ and $\sigma_1,\sigma_2,\sigma_3$ are standard Pauli matrices (2.10). One can check that $\boldsymbol{\sigma}_\mu$, $\overline{\boldsymbol{\sigma}}_\mu$ satisfy identities (2.17) with $||g_{\mu\nu}|| = \mathrm{diag}(+1,-1,-1,-1)$. To proceed further we note that

$$\boldsymbol{\sigma}_{\mu\nu}\otimes\boldsymbol{\sigma}^{\mu\nu} = \sigma_i\otimes\sigma_i \ \ , \ \ \overline{\boldsymbol{\sigma}}_{\mu\nu}\otimes\overline{\boldsymbol{\sigma}}^{\mu\nu} = \sigma_i\otimes\sigma_i \ \ , \ \ \boldsymbol{\sigma}_{\mu\nu}\otimes\overline{\boldsymbol{\sigma}}^{\mu\nu} = 0\,,$$

(sum over $i=1,2,3$ is implied) consequently by using (2.61) we get for the self-dual components of $S_{\mu\nu}$

$$\mathbf{S} = \frac{1}{2}\,\boldsymbol{\sigma}^{\mu\nu}\,S_{\mu\nu} = \frac{1}{2}\,\sigma_i\cdot(\lambda\,\sigma_i\,\partial_\lambda) = \begin{pmatrix} \frac{1}{2}\,\lambda_1\partial_{\lambda_1} - \frac{1}{2}\,\lambda_2\partial_{\lambda_2} & \lambda_2\partial_{\lambda_1} \\ \lambda_1\partial_{\lambda_2} & -\frac{1}{2}\,\lambda_1\partial_{\lambda_1} + \frac{1}{2}\,\lambda_2\partial_{\lambda_2} \end{pmatrix} \tag{2.63}$$

and for anti-self-dual components of $S_{\mu\nu}$

$$\overline{\mathbf{S}} = \frac{1}{2}\,\overline{\boldsymbol{\sigma}}^{\mu\nu}\,S_{\mu\nu} = \frac{1}{2}\,\sigma_i\cdot\left(\tilde{\lambda}\,\sigma_i\,\partial_{\tilde{\lambda}}\right) = \begin{pmatrix} \frac{1}{2}\,\tilde{\lambda}^{\dot{1}}\partial_{\tilde{\lambda}^{\dot{1}}} - \frac{1}{2}\,\tilde{\lambda}^{\dot{2}}\partial_{\tilde{\lambda}^{\dot{2}}} & \tilde{\lambda}^{\dot{2}}\partial_{\tilde{\lambda}^{\dot{1}}} \\ \tilde{\lambda}^{\dot{1}}\partial_{\tilde{\lambda}^{\dot{2}}} & -\frac{1}{2}\,\tilde{\lambda}^{\dot{1}}\partial_{\tilde{\lambda}^{\dot{1}}} + \frac{1}{2}\,\tilde{\lambda}^{\dot{2}}\partial_{\tilde{\lambda}^{\dot{2}}} \end{pmatrix} \tag{2.64}$$

In fact the operator $\mathbf{S}$ is restricted to the space of homogeneous polynomials in components of the spinor λ of degree 2ℓ (see (2.58) and (2.59)) so that one can choose new variables $\chi_1 = -\frac{\lambda_1}{\lambda_2}$, $t = -\lambda_2$ and obtain that $\mathbf{S}$ coincides with the following matrix $\mathbf{S}^{(\ell)}$ which contains parameter ℓ (the eigenvalue of the operator $\frac{1}{2}t\partial_t$):

$$\mathbf{S}^{(\ell)} = \begin{pmatrix} \chi_1\partial_{\chi_1} - \ell\,, & -\partial_{\chi_1} \\ \chi_1^2\,\partial_{\chi_1} - 2\ell\,\chi_1\,, & -\chi_1\,\partial_{\chi_1} + \ell \end{pmatrix} \equiv \begin{pmatrix} S_3\,, & S_- \\ S_+\,, & -S_3 \end{pmatrix}\,, \tag{2.65}$$

Similarly the operator $\bar{\mathbf{S}}$ is restricted to the space of homogeneous polynomials in components of the spinor $\tilde{\lambda}$ of degree $2\dot{\ell}$ so that for the the choice $\chi_2 = -\frac{\tilde{\lambda}^{\dot{1}}}{\tilde{\lambda}^{\dot{2}}}$ one obtains $\overline{\mathbf{S}} = \overline{\mathbf{S}}^{(\dot{\ell})}$, where

$$\overline{\mathbf{S}}^{(\dot{\ell})} = \begin{pmatrix} \chi_2 \partial_{\chi_2} - \dot{\ell}\,, & -\partial_{\chi_2} \\ \chi_2^2\, \partial_{\chi_2} - 2\dot{\ell}\, \chi_2\,, & -\chi_2\, \partial_{\chi_2} + \dot{\ell} \end{pmatrix} \equiv \begin{pmatrix} \overline{S}_3\,, & \overline{S}_- \\ \overline{S}_+\,, & -\overline{S}_3 \end{pmatrix} . \tag{2.66}$$

For constructing general R-operators in section 5 we will need Euclidean analogues of the previous formulas. For 4-dimaensional Euclidean space $\mathbb{R}^4$ we choose gamma-matrices (2.16) such that

$$\boldsymbol{\sigma}_\mu = (\sigma_0, i\sigma_1, i\sigma_2, i\sigma_3)\,, \qquad \overline{\boldsymbol{\sigma}}_\mu = (\sigma_0, -i\sigma_1, -i\sigma_2, -i\sigma_3)\,,$$

and $\boldsymbol{\sigma}_\mu$, $\overline{\boldsymbol{\sigma}}_\mu$ satisfy relations (2.17) with $||g_{\mu\nu}|| = \mathrm{diag}(+1, +1, +1, +1)$. Let us mention that explicit experessions for $\mathbf{S}^{(\ell)}$, $\overline{\mathbf{S}}^{(\dot{\ell})}$ (2.65), (2.66) remains valid.

3 L-operators

Let V be a vector space and I is the identity operator in V. Consider an operator $\mathrm{R}(u) \in \mathrm{End}(V \otimes V)$ which is a function of spectral parameter u and satisfies Yang-Baxter equation in the braid form

$$\mathrm{R}_{12}(u - v)\, \mathrm{R}_{23}(u)\, \mathrm{R}_{12}(v) = \mathrm{R}_{23}(v)\, \mathrm{R}_{12}(u)\, \mathrm{R}_{23}(u - v) \;\in\; \mathrm{End}(V \otimes V \otimes V)\,. \tag{3.1}$$

Here we use standard matrix notations of [15, 16, 19], i.e. we denote by $\mathrm{R}_{23}(u)$ the operator $\mathrm{R}(u)$ which acts nontrivially in the second and third factors in $V \otimes V \otimes V$ and as identity I on the first factor, then $\mathrm{R}_{12}(u) = \mathrm{R}(u) \otimes I$, etc. Let V' be another vector space and I' is the identity operator in V'. We call operator $\mathrm{L}(u) \in \mathrm{End}(V \otimes V')$ as L-*operator* in the spaces V and V' if it obeys intertwining relation

$$\mathrm{R}_{12}(u - v)\, \mathrm{L}_{13}(u)\, \mathrm{L}_{23}(v) = \mathrm{L}_{13}(v)\, \mathrm{L}_{23}(u)\, \mathrm{R}_{12}(u - v) \;\in\; \mathrm{End}(V \otimes V \otimes V')\,. \tag{3.2}$$

Here again indices $1, 2, 3$ indicate in which factors of the space $V \otimes V \otimes V'$ the corresponding operators act nontrivially, e.g., $\mathrm{L}_{23}(v) = I \otimes \mathrm{L}(v)$, $\mathrm{R}_{12}(u) = \mathrm{R}(u) \otimes I'$, etc.

In this section we consider a special form of L-operators which is related to simple Lie algebras $\mathcal{A}$ and their representations. Let X_a $(a = 1, \dots, \mathrm{dim}\mathcal{A})$ be generators of $\mathcal{A}$ and $||\mathrm{g}_{ab}||$ — matrix of the Killing form for $\mathcal{A}$ in the basis $\{X_a\}$. Introduce a polarized (or split) Casimir operator for $\mathcal{A}$

$$\mathbf{r} = \mathrm{g}^{ab}\, X_a \otimes X_b \;\in\; \mathcal{A} \otimes \mathcal{A}\,, \tag{3.3}$$

where g^{ab} is the inverse matrix of Killing form. Recall that quadratic Casimir operator $C_2 = \mathrm{g}^{ab}\, X_a \cdot X_b$ is the element of the enveloping algebra $\mathcal{U}(\mathcal{A})$. The operator $\mathbf{r}$ satisfies identity

$$[\mathbf{r}_{12} + \mathbf{r}_{13},\, \mathbf{r}_{23}] = 0\,, \tag{3.4}$$

where again we have used standard notations

$$\mathbf{r}_{13} = \mathrm{g}^{ab}\, X_a \otimes \mathbf{1} \otimes X_b\,, \;\; \mathbf{r}_{12} = \mathrm{g}^{ab}\, X_a \otimes X_b \otimes \mathbf{1}\,, \;\; \mathbf{r}_{23} = \mathrm{g}^{ab}\, \mathbf{1} \otimes X_a \otimes X_b\,,$$

and $\mathbf{1}$ is the unit element in $\mathcal{U}(\mathcal{A})$.

Let T and T' be representations of $\mathcal{A}$ in vector spaces V and V', respectively. Further we investigate special solutions of equation (3.2) which can be represented in the form:

$$\mathrm{L}(u) = (T \otimes T')(u\,\mathbf{1} \otimes \mathbf{1} + \mathbf{r}) = u\,(I \otimes I') + \mathrm{g}^{ab}\,(T_a \otimes T'_b) \;\in\; \mathrm{End}(V \otimes V')\,, \tag{3.5}$$

where $T_a := T(X_a)$ and $T'_b := T'(X_b)$. The matrix (3.5) is constructed by means of polarized Casimir operator (3.3) for the algebra $\mathcal{A}$ and as we show in next sections this matrix is a solution of equation (3.2) only for the special choice of the algebra $\mathcal{A}$ and representations T and T'.

3.1 The case of the algebra $\mathcal{A} = s\ell(N, \mathbb{C})$

Consider Lie algebra $g\ell(N, \mathbb{C})$ with generators E_{ij} $(i, j = 1, \ldots, N)$ which obey commutation relations

$$[E_{ij},\, E_{k\ell}] = \delta_{jk} E_{i\ell} - \delta_{i\ell} E_{kj}\,. \tag{3.6}$$

One can embed Lie algebra $s\ell(N, \mathbb{C})$ into the algebra $g\ell(N, \mathbb{C})$ by choosing generators X_a of $s\ell(N, \mathbb{C})$ as E_{ij} $(i \neq j$ and $i, j = 1, \ldots, N)$ and $H_k = E_{kk} - \frac{1}{N} \sum_m E_{mm}$, where only $(N-1)$ elements H_k are independent in view of the equation $\sum_k H_k = 0$. These generators satisfy commutation relations

$$\begin{array}{lll} [E_{ij},\, E_{k\ell}] = \delta_{jk} E_{i\ell} - \delta_{i\ell} E_{kj}\,, & i \neq \ell \,\mathrm{or}\, j \neq k\,, & [E_{ij},\, E_{ji}] = H_i - H_j\,, \\ [H_k,\, E_{ij}] = (\delta_{ki} - \delta_{kj}) E_{ij}\,, & & [H_k,\, H_j] = 0\,. \end{array} \tag{3.7}$$

In the defining representation T of $s\ell(N, \mathbb{C})$ the elements E_{ij} and H_k are $(N \times N)$ traceless matrices

$$T(E_{ij}) = e_{ij}\,, \;\; T(H_k) = e_{kk} - \frac{1}{N} I_N \equiv h_k\,, \tag{3.8}$$

where e_{ij} are matrix units and $I_N = \sum_j e_{jj}$. Matrices (3.8) act in the N-dimensional vector space $V_N = \mathbb{C}^N$. Introduce permutation matrix P_{12} which acts in the space $V_N \otimes V_N$ as following

$$P_{12}\, w_1 \otimes w_2 = w_2 \otimes w_1\,, \quad \forall\, w_1, w_2 \in V_N\,. \tag{3.9}$$

Proposition 1. [17] *The operator (cf. (3.5))*

$$\mathrm{L}(u) = u\, I_N \otimes \mathbf{1} + \sum_i h_i \otimes H_i + \sum_{i \neq j} e_{ij} \otimes E_{ji}\,, \tag{3.10}$$

is the universal L*-operator for the Lie algebra $s\ell(N, \mathbb{C})$. In other words the operator (3.10) satisfies intertwining relations (3.2) with Yangian* R*-matrix*

$$\mathrm{R}_{12}(u) := u\, P_{12} + I_N \otimes I_N\,, \tag{3.11}$$

and the universality means that the second factors in (3.10) can be taken in an arbitrary representation T' of $s\ell(N, \mathbb{C})$ (cf. (3.5)).

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Proof. First we write operator (3.10) in terms of $g\ell(N,\mathbb{C})$ generators

$$\mathrm{L}(u) = (u - 1/N)\, I_N \otimes \mathbf{1} + e_{ij} \otimes E_{ji}\,, \tag{3.12}$$

where the sum over all indices i and j is implied. Note that the split Casimir operator $\mathbf{r} = E_{ij} \otimes E_{ji}$ satisfies equations (3.4) and we have

$$e_{ij} \otimes E_{ji} = (T \otimes 1)\, \mathbf{r}\,, \quad (T \otimes T)\, \mathbf{r} = e_{ij} \otimes e_{ji} = P_{12}\,. \tag{3.13}$$

Substitution of (3.12) into (3.2) and using (3.13) gives relation

$$(u - v)\, P_{12}\, (T \otimes T \otimes 1)\, ([\mathbf{r}_{13},\, \mathbf{r}_{23}] + [\mathbf{r}_{12},\, \mathbf{r}_{23}]) = 0\,,$$

which is identity in view of (3.4). Thus, $\mathrm{L}(u)$ (3.10) satisfies intertwining relation (3.2) and it means that $\mathrm{L}(u)$ is the L-operator. •

Now we take the second factors of universal L-operator (3.10) in the differential representation ρ of $s\ell(N,\mathbb{C})$ (see [21–23] for details) and make the redefinition:

$$(1 \otimes \rho)\mathrm{L}(u + 1/N) \;\; \to \;\; \mathrm{L}(u) = u\, I_N \otimes \rho(\mathbf{1}) + e_{ij} \otimes \rho(E_{ji})\,. \tag{3.14}$$

The representation ρ is characterized by parameters $(\rho_1, \ldots, \rho_N)$ subject condition $\sum\limits_{k=1}^{N} \rho_k = N(N-1)/2$ and we stress that spectral parameter u and parameters ρ_k are collected in combinations

$$u_k = u - \rho_k\,. \tag{3.15}$$

The L-operator (3.14) can be written in the factorized form [21–23]

$$\mathrm{L}(u_1, \ldots, u_N) = \mathrm{Z} \cdot \mathrm{D}(u_1, \ldots, u_N) \cdot \mathrm{Z}^{-1}\,, \tag{3.16}$$

where low-triangular and upper-triangular $(N \times N)$ matrices Z and D are

$$\mathrm{Z} = I_N + \sum_{k>m} z_{km}\, e_{km}\,, \quad \mathrm{D}(u_1, \ldots, u_N) = \sum_{k=1}^{N} u_k\, e_{kk} + \sum_{i<j} D_{ij}\, e_{ij}\,. \tag{3.17}$$

Here we use notation

$$D_{ij} = -\partial_{ji} - \sum_{k=j+1}^{N} z_{kj}\, \partial_{ki}\,, \quad \partial_{ji} \equiv \tfrac{\partial}{\partial z_{ji}}\,, \quad (i < j)\,. \tag{3.18}$$

In other words, the low-triangular matrix Z is parameterized by z_{km} and according to eqs. (3.14), (3.16) the generators $\rho(E_{ji})$ of $s\ell(n, C)$ are realized as differential operators which act in the space of functions of variables z_{km}.

We stress that all elements of matrices Z and D have to be interpreted as operators acting in the space of functions $f(\mathrm{Z})$. The important fact is that there exist operators $\mathcal{T}_k$ $(k = 1, \ldots, N-1)$ which permute parameters u_k and u_{k+1} in L-operator:

$$\mathcal{T}_k \cdot \mathrm{L}(u_1, \ldots, u_k, u_{k+1}, \ldots, u_n) = \mathrm{L}(u_1, \ldots, u_{k+1}, u_k, \ldots, u_n) \cdot \mathcal{T}_k\,. \tag{3.19}$$

One can find that

$$\mathcal{T}_k = (D_{k,k+1})^{u_{k+1}-u_k} \,, \tag{3.20}$$

where $D_{k,k+1}$ are the elements of the matrix D.

The operators $\mathcal{T}_k$ have clear group theoretical meaning as intertwining operators [47–49] for equivalent representations which differ by the permutation of parameters ρ_k and ρ_{k+1}. These intertwining operators corresponds to the elementary transpositions s_k in the Weyl group. In the case under consideration the Weyl group of $s\ell(N,\mathbb{C})$ is the group of permutation of parameters $(\rho_1,\dots,\rho_n)$:

$$s_k: \;(\rho_1,\dots,\rho_k,\rho_{k+1},\dots,\rho_n) \;\to\; (\rho_1,\dots,\rho_{k+1},\rho_k,\dots,\rho_n)\,. \tag{3.21}$$

As an illustration we present L-operator (3.16) for the simplest $s\ell(2,\mathbb{C})$ case ($N=2$). In this case $\rho_1+\rho_2=1$ and it is convenient to use standard spin parameters ℓ instead of parameters ρ_1 and ρ_2:

$$\rho_1 = \ell+1 \;,\quad \rho_2 = -\ell\,,\quad u_1 = u-\ell-1\,,\quad u_2 = u+\ell\,. \tag{3.22}$$

Then we write operator $\mathrm{L}(u_1,u_2)$ (3.16) for $N=2$ in the form

$$\begin{aligned}\mathrm{L}(u_1,u_2) &= \begin{pmatrix} 1 & 0 \\ z & 1\end{pmatrix}\begin{pmatrix} u_1 & -\partial_z \\ 0 & u_2\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -z & 1\end{pmatrix} = \\ &= u\, I_2 + \begin{pmatrix} z\,\partial_z-\ell\,, & -\partial_z \\ z^2\,\partial_z - 2\ell\, z\,, & -z\,\partial_z+\ell\end{pmatrix} = u\, I_2 + \mathbf{S}^{(\ell)}\,,\end{aligned} \tag{3.23}$$

where $z=z_{21}$ and elements of matrix $\mathbf{S}^{(\ell)}$ are generators of $s\ell(2,\mathbb{C})$ in the standard differential realization. One can directly check the identity

$$\partial_z^{2\ell+1}\cdot \mathbf{S}^{(\ell)} = \mathbf{S}^{(-\ell-1)}\cdot\partial_z^{2\ell+1}\,,$$

which corresponds to the permutation $\rho_1 \leftrightarrow \rho_2$ and justifies (3.19) and (3.20). In section 4.1 we investigate L-operator (3.16) for the $s\ell(4,\mathbb{C})$ case.

3.2 The case of the Lie algebra $\mathcal{A}=so(p+1,q+1)$. Spinorial R-matrix

Let Γ_a $(a=0,\dots,n+1)$ be $2^{\frac{n}{2}+1}$-dimensional gamma-matrices in $\mathbb{R}^{p+1,q+1}$ (2.9), where $n=p+q$. Operators Γ_a are generators of the Clifford algebra (2.11) which, as a vector space, has dimension 2^{n+2}. The standard basis in this space is formed by antisymmetrized products of the Γ-matrices

$$\Gamma_{a_1\dots a_k} = \frac{1}{k!}\sum_s (-1)^{\mathrm{p}(s)}\Gamma_{s(a_1)}\cdots\Gamma_{s(a_k)} \equiv \Gamma_{A_k} \quad (\forall k\le n+2)\,,\quad \Gamma_{A_k}=0 \quad (\forall k>n+2)\,, \tag{3.24}$$

here the summation is taken over all permutations s of k indices $\{a_1,\dots,a_k\}\to\{s(a_1),\dots,s(a_k)\}$ and $\mathrm{p}(s)$ denote the parity of the permutation s. We start from the general $\mathrm{SO}(p+1,q+1)$-invariant expression for the R-matrix which acts in the tensor product of two vector spaces V of $2^{\frac{n}{2}+1}$-dimensional spinor representations of $\mathrm{SO}(p+1,q+1)$

$$\mathrm{R}(u) = \sum_{k=0}^{\infty}\frac{\mathrm{R}_k(u)}{k!}\cdot\Gamma_{a_1\dots a_k}\otimes\Gamma^{a_1\dots a_k} \;\in\; \mathrm{End}(V\otimes V)\,. \tag{3.25}$$

Note that in the r.h.s. of (3.25) the summation over k does not run up to infinity since it is automatically truncated by the condition $k \le n+2$ (see (3.24)). We claim (see also [24, 38–40]) that the R-matrix (3.25) satisfies Yang-Baxter equation (3.1) if coefficient functions $\mathrm{R}_k(u)$ obey the recurrent relation

$$\mathrm{R}_{k+2}(u) = -\frac{u+k}{u+n-k}\,\mathrm{R}_k(u)\,. \tag{3.26}$$

Recall now that the Lie algebra $so(p+1,q+1)$ is generated by elements M_{ab} subject commutation relations (2.2). The operator $\mathrm{L}(u)$ (3.5) of $so(p+1,q+1)$-type can be written in the form

$$\mathrm{L}(u) = u\,\mathbf{I}\otimes 1\, + \frac{1}{2}\,T(M_{ab})\otimes M^{ab}\,, \tag{3.27}$$

where T denote the spinor representation (2.12) of $so(p+1,q+1)$ which acts in the space V. The generators M_{ab} which are in the second factors of $\mathrm{L}(u)$ can be thought as taken in arbitrary representation T'. Now we investigate the cases when operator $\mathrm{L}(u)$ defined in (3.27) satisfies intertwining equation (3.2). After substitution of $\mathrm{L}(u)$ (3.27) into (3.2) (with R-matrix (3.25)) and some calculations equation (3.2) acquires the form

$$\sum_{k=0}^{\infty}\frac{1}{k!}\Big[(u{+}n{-}k)\,\mathrm{R}_{k+2}(u){+}(u{+}k)\,\mathrm{R}_k(u)\Big]\,T'(M_{ab})\,\Big[\Gamma^{abc_1\dots c_k}\otimes\Gamma_{c_1\dots c_k}{-}\Gamma_{c_1\dots c_k}\otimes\Gamma^{abc_1\dots c_k}\Big]{+}$$
$$-\frac{i}{2}\sum_{k=0}^{\infty}\frac{1}{k!}\Big[\mathrm{R}_{k+3}(u){+}\mathrm{R}_{k+1}(u)\Big]T'\left(\{M^{ab}\,,M^{c}{}_{d}\}\right)\Big[\Gamma_{abcc_1\dots c_k}\otimes\Gamma^{dc_1\dots c_k}{+}\Gamma^{dc_1\dots c_k}\otimes\Gamma_{abcc_1\dots c_k}\Big] = 0\,, \tag{3.28}$$

where $\{A\,,B\} = A\cdot B + B\cdot A$. The first term in the previous equation turns to zero due to recurrence relation (3.26). The second term could be equal to zero for special choice of the representation T' of generators M_{ab} and for special projections in spinor spaces V, e.g., Weyl projections $V \to V_\pm = \frac{\mathbf{I}\pm\Gamma_{n+3}}{2}\,V$ or choice of the Majorana representation for gamma-matrices. We consider more restrictive condition which is

$$T'\left(\{M_{[ab}\,,M_{c]d}\}\right) = 0\,. \tag{3.29}$$

Here square brackets denote antisymmetrization. Below we itemize some cases when the condition (3.29) is fulfilled

- The differential representation T':

$$M_{ab} \to T'(M_{ab}) = i(y_a\,\partial_b - y_b\,\partial_a)\,, \tag{3.30}$$

  where $\partial_a = \frac{\partial}{\partial y^a}$ and y_a are coordinates in the space $\mathbb{R}^{p+1,q+1}$.

- Fundamental (defining) $(n+2)$-dimensional representation T':

$$M_{ab} \to T'(M_{ab}) = ig(e_{ab} - e_{ba})\,, \tag{3.31}$$

  where e_{ab} are matrix units and $g = ||g_{ab}||$. Corresponding L-operator (3.5) can be written as

$$\mathrm{L}(u) = u\,I_{n+2}\otimes\mathbf{1}+\frac{1}{2}\cdot T(M^{ab})\otimes T'(M_{ab}) = u\,I_{n+2}\otimes\mathbf{1}-\frac{1}{4}\cdot\Gamma^{ab}\otimes g(e_{ab}-e_{ba})\,, \tag{3.32}$$

Then by direct calculation one can prove that operator $\mathrm{L}(u)$ (3.32) satisfies intertwining relation

$$\mathrm{R}'_{23}(u-v)\,\mathrm{L}_{12}(u)\,\mathrm{L}_{13}(v) = \mathrm{L}_{12}(v)\,\mathrm{L}_{13}(u)\,\mathrm{R}'_{23}(u-v) \in \mathrm{End}(V\otimes V'\otimes V')$$

with $so(p+1,q+1)$-type Yangian R-matrix [44, 45]

$$\mathrm{R}'_{23}(u) = u\,\mathrm{P}_{23} + \mathrm{I}_{23} - \frac{u}{u+\frac{n}{2}}\mathrm{K}_{23}\,,$$

where matrices I_{23}, P_{23} were described in (3.9) and operator K is

$$\mathrm{K}(\vec{e}_a\otimes\vec{e}_b) = \left(\vec{e}_c\otimes\vec{e}_d\,g^{cd}\right)\cdot g_{ab}\,,$$

where $\vec{e}_a$ are basis vectors in the space V' of the defining representation T'. In [38] it was also shown that there exists a spinorial Yang-Baxter R-matrix (3.25) $\mathrm{R}(u)\in\mathrm{End}(V\otimes V)$ which intertwines L-operators (3.32) in spinorial spaces (3.2).

- The differential representation $T'=\rho$ (2.32) for the scalar case $S_{\mu\nu}=0$ and arbitrary Δ:

$$M_{ab}\to T'(M_{ab}) = \rho(M_{ab})\,,\quad S_{\mu\nu}=0\,. \tag{3.33}$$

Using relations (2.4) the conditions (3.29) are written as

$$\rho\left(\{L_{[\mu\nu}\,,L_{\lambda]\sigma}\}\right)=0\,;\quad \rho\left(\{L_{[\mu\nu}\,,P_{\lambda]}\}\right)=0\,;\quad \rho\left(\{L_{[\mu\nu}\,,K_{\lambda]}\}\right)=0\,;$$
$$\rho\left(\{\,L_{\mu\nu}\,,\,D\,\}\right)+\frac{1}{2}\rho\left(\{\,K_\mu\,,\,P_\nu\,\}\right)-\frac{1}{2}\rho\left(\{\,K_\nu\,,\,P_\mu\,\}\right)=0\,.$$

One can directly check that these conditions are identities. One can also check that the representations (3.31) and (3.33) can be extracted from the differential representation (3.30).

Detailed derivation of (3.28) as well as the direct check of Yang-Baxter equation (3.1) with spinorial R-matrix (3.25) is carried out in [41]. Let us note that the first calculation concerning RLL-relation can be implemented in a rather straightforward albeit lengthy way using standard formula for the product of a Γ-matrix with two antysimmetrized indices and a Γ-matrix with k antisymmetrized indices (3.24). However the check of Yang-Baxter happens to be much more complicated task to get through using textbook formulae for the products of Γ-matrices with arbitry number of antisymmetrized indices. In order to perform such intricate calculation we address to generating function technique [65] which is exteremely efficient tool to deal with the algebra of Γ-matrices.

In the following section it will be important that in particular case of conformal algebra $so(2,4)$ of 4-dimensional Minkowski space ($n=4$) the RLL-relation (3.2) with R-matrix (3.25) and L-operator (3.27) can be satisfied for any representation T' of the generators $\{M_{ab}\}$ so that the condition (3.29) is dispensable. Indeed all gamma-matrix structures in (3.25) have block-diagonal representation that can be seen from (2.9). Therefore it is reasonable to consider projections of (3.25) on corresponding irreducible subspaces. As before we introduce subspaces V_+ and V_- obtained by Weyl projections: $V_\pm = \frac{1\pm\Gamma_7}{2}V$. Then at a special choice of solutions $\mathrm{R}_k(u)$ of recurrent relations (3.26), namely with vanishing odd coefficients: $\mathrm{R}_{2k+1}(u) = 0$, in the case $n = 4$ we have

$$\mathrm{R}(u)|_{V_+ \otimes V_-} = \mathrm{R}(u)|_{V_- \otimes V_+} = 0 \tag{3.34}$$

and the other two projections give rise to Yang R-matrix

$$\mathrm{R}(u)|_{V_- \otimes V_-} = \mathrm{I} \otimes \mathrm{I} + u \cdot \mathrm{P}_- \,, \tag{3.35a}$$

$$\mathrm{R}(u)|_{V_+ \otimes V_+} = \mathrm{I} \otimes \mathrm{I} + u \cdot \mathrm{P}_+ \,, \tag{3.35b}$$

where P_+ and P_- are permutation operators in the spaces $V_+ \otimes V_+$ and $V_- \otimes V_-$ correspondingly.

The L-operator (3.27) is also reducible since $T(M_{ab})$ consists of two irreducible blocks (2.14). Therefore its projection to the subspace V_-

$$\mathrm{L}(u) = u\,\mathrm{I} \otimes 1 \, + \, \frac{1}{2}\, T_1(M_{ab}) \otimes M^{ab} \,, \tag{3.36}$$

in the case of 4-dimensional Minkowski space ($n = 4$), fulfils RLL-relation with Yang R-matrix. The second projection of the L-operator (3.27) (on the subspace V_+) fulfils Yangian relation as well.

In [41] we show that at $n = 4$ (3.34), (3.35) are satisfied and demonstrate that in this case due to simplification of gamma-matrix structure in (3.28) the condition (3.29) happens to be superfluous.

At the end of this section we consider operator $\mathrm{L}(u)$ (3.5) for the algebra $so(p+1, q+1)$ for the special choice of the representations $T = T_1$ (cf. eq. (3.36)) and $T' = \rho$, where T_1 is the spinor representation (2.15) and ρ is the differential representation (2.32). This operator $\mathrm{L}(u)$ is written in the form:

$$\begin{aligned} \mathrm{L}^{(\rho)}(u) &= u\,\mathrm{I} + \frac{1}{2} T_1(M^{ab}) \otimes \rho(M_{ab}) \\ &= \begin{pmatrix} u_+ \cdot \mathbf{1} + \mathbf{S} - \mathbf{p} \cdot \mathbf{x} \,, & \mathbf{p} \\ \mathbf{x} \cdot \mathbf{S} - \overline{\mathbf{S}} \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{p} \cdot \mathbf{x} + (\Delta - \frac{n}{2}) \cdot \mathbf{x} \,, & u_- \cdot \mathbf{1} + \overline{\mathbf{S}} + \mathbf{x} \cdot \mathbf{p} \end{pmatrix} , \end{aligned} \tag{3.37}$$

where

$$u_+ = u + \frac{\Delta - n}{2} \,, \quad u_- = u - \frac{\Delta}{2} \,, \quad n = p + q \,, \tag{3.38}$$

and we have used the expression (2.40) for the matrix $\frac{1}{2} T_1(M^{ab}) \otimes \rho(M_{ab})$ which was introduced in (2.35). The important fact is:

Proposition 2. *The operator (3.37) is expressed in the following factorized form*[2]

$$\mathrm{L}^{(\rho)}(u) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{x} & \mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} u_+ \cdot \mathbf{1} + \mathbf{S} & \mathbf{p} \\ \mathbf{0} & u_- \cdot \mathbf{1} + \overline{\mathbf{S}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{x} & \mathbf{1} \end{pmatrix} . \tag{3.39}$$

[2]The factorized form (3.39) of the so-type L-operator for the scalar representation ($\mathbf{S} = \overline{\mathbf{S}} = 0$) was obtained independently by G.Korchemsky and V.Pasquier (private communication).

Proof. One can show that (3.39) is equivalent to (3.37) by direct calculation. •

Remark 1. The formula (3.39) for the $so(p+1, q+1)$-type operator $\mathrm{L}^{(\rho)}(u)$ can be considered as the recurrent formula if we interpret operators $(u_+ \cdot \mathbf{1} + \mathbf{S})$ and $(u_- \cdot \mathbf{1} + \overline{\mathbf{S}})$ as two smaller $so(p,q)$-type operators $\mathrm{L}^{(\rho)}(u)$.

Remark 2. Consider $so(p+1, q+1)$-type L-operator (3.27) which satisfies RLL-relations (3.2) with spinorial R-matrix (3.25), (3.26):

$$\mathrm{L}(u) = u\,\mathbf{I} - \frac{1}{8}\,(\Gamma_a\,\Gamma_b - \Gamma_b\,\Gamma_a)(y^a\partial^b - y^b\partial^a)\,, \tag{3.40}$$

where for the generators M^{ab} we have used the representation T' given in (3.30). The L-operator (3.40) satisfies crossing symmetry relation

$$\mathrm{L}^T(u)\cdot \mathrm{C}\cdot \mathrm{L}(u') = \left(u\,u' - \frac{1}{4}\,T'(C_2)\right)\,\mathrm{C}\,, \tag{3.41}$$

where $u' = u + \frac{n}{2}$, $n = (p+q)$, C is the $2^{\frac{n}{2}+1}$-dimensional analog of matrix C introduced in (2.27) and C_2 is the Cazimir operator (2.5). Since the representation (2.12) is reducible (see (2.14)) the operator $\mathrm{L}(u)$ (3.40) has the block diagonal form

$$\mathrm{L}(u) = \begin{pmatrix} \mathrm{L}_+(u) & \mathbf{0} \\ \mathbf{0} & \mathrm{L}_-(u) \end{pmatrix}\,, \tag{3.42}$$

and in view of relations (2.30) one can rewrite relation (3.41) for blocks $\mathrm{L}_\pm(u)$ as following

$$\begin{array}{lll} 1.) \ \frac{(n+2)(n+1)}{2} \ - \text{ even} & \Rightarrow & \mathrm{L}^T_\pm(u)\cdot \mathrm{c}\cdot \mathrm{L}_\pm(u') = z(u)\cdot \mathrm{c} \\ 2.) \ \frac{(n+2)(n+1)}{2} \ - \text{ odd} & \Rightarrow & \mathrm{L}^T_+(u)\cdot \mathrm{g}\cdot \mathrm{L}_-(u') = z(u)\cdot \mathrm{g}\,. \end{array} \tag{3.43}$$

where $z(u) = \big(u\,u' - \frac{1}{16}\,T'(C_2)\big)$. It is clear that the irreducible parts $\mathrm{L}_\pm(u)$ of the operator (3.40) satisfy RLL-relations (3.2):

$$\mathrm{R}^{(\pm)}_{12}(u-v)\,\mathrm{L}_{\pm 1}(u)\,\mathrm{L}_{\pm 2}(v) = \mathrm{L}_{\pm 1}(v)\,\mathrm{L}_{\pm 2}(u)\,\mathrm{R}^{(\pm)}_{12}(u-v)\,, \tag{3.44}$$

where $\mathrm{R}^{(\pm)}(u) = \mathrm{R}(u)|_{V_\pm\otimes V_\pm}$ and the matrix $\mathrm{R}(u)$ is given in (3.25).

Consider instead of operators $\mathrm{L}_\pm(u)$ defined in (3.40), (3.42) more general operators

$$\mathrm{L}_\pm(u) = \mathrm{I} + \sum_{k=1}^{\infty} \frac{1}{u^k}\,L^{(k)}_\pm\,. \tag{3.45}$$

Then relations (3.44) will define the infinite-dimensional quadratic algebra generated by the set of elements $\{(L^{(0)}_\pm)_{\alpha\beta}, (L^{(1)}_\pm)_{\alpha\beta}, \dots\}$, $(\alpha,\beta = 1,2,\dots,2^{\frac{n}{2}})$. We denote this algebra as $\mathrm{Y}(\mathsf{spin}(n+2,\mathbb{C}))$. For the generators of $\mathrm{Y}(\mathsf{spin}(n+2,\mathbb{C}))$ it is also necessary to add additional constraints (3.43), where the function $z(u)$ have to be considered as a central element of $\mathrm{Y}(\mathsf{spin}(n+2,\mathbb{C}))$. The results of this subsection show that the algebra $\mathrm{Y}(\mathsf{spin}(n+2,\mathbb{C}))$ possesses evaluation representations when $(L^{(k)}_\pm)_{\alpha\beta} \to 0$ for $k > 1$ and $(L^{(1)}_\pm) \to \frac{1}{2}T_1(M_{ab})T'(M^{ab})$. Here M_{ab} are generators of $\mathsf{spin}(n+2,\mathbb{C})$ and special representations T' are itemized in (3.31) – (3.33). For the special case $n=4$ the matrix $\mathrm{R}(u)|_{V_-\otimes V_-}$ is the Yang R-matrix (3.35), all 4×4 matrices $(\Gamma^{a_1\dots a_{2k}})|_{V_-}$ form the basis for $s\ell(4)$ and we see that the algebra $\mathrm{Y}(\mathsf{spin}(6))$ is isomorphic to the Yangian $\mathrm{Y}(s\ell(4))$.

4 L-operator for the conformal algebra in four dimensions

Now we restrict our consideration to the case of 4-dimensional Minkowski space $\mathbb{R}^{1,3}$, i.e., $p = 1$, $q = 3$ and $n = 4$. In this case generators (2.15) are

$$\ell_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] \,, \qquad p_\mu = \gamma_\mu \frac{1+\gamma_5}{2} \,, \qquad k_\mu = \gamma_\mu \frac{1-\gamma_5}{2} \,, \qquad d = -\frac{i}{2}\gamma_5 \,, \tag{4.1}$$

where $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$ and we choose common representation (2.16)

$$\gamma_\mu = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}_\mu \\ \overline{\boldsymbol{\sigma}}_\mu & \mathbf{0} \end{pmatrix} , \; \gamma_5 = \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_2 \end{pmatrix} , \; \frac{1+\gamma_5}{2} = \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} , \; \frac{1-\gamma_5}{2} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix} , \tag{4.2}$$

constructed by means of 2×2-matrices $\boldsymbol{\sigma}_\mu$ and $\overline{\boldsymbol{\sigma}}_\mu$ (2.62). Note that in the representation (4.2) we have identities

$$\gamma_\mu^\dagger = \gamma_0 \, \gamma_\mu \, \gamma_0 \,, \quad \gamma_5^\dagger = \gamma_5 = -\gamma_0 \, \gamma_5 \, \gamma_0 \,, \tag{4.3}$$

which are analogs of (2.20) and (2.23) and correspond to the case 2.) stated in (2.25) and (2.26) for $\mathbf{C} = \gamma_0$ and $\mathbf{g} = I_2$.

It is known that fifteen matrices (4.1) form the basis in the space $\mathsf{Mat}(4)$ of all traceless 4×4 matrices. Then one can check by using (4.3) that any 4×4 matrix (2.41) which belongs to a linear span of (4.1) satisfies equation $A^\dagger \, \gamma_0 + \gamma_0 \, A = 0$ which defines Lie algebra $su(2,2)$. This equation means that 4×4 matrices A (2.41) not only represent all elements of the conformal algebra $so(2,4) = spin(2,4)$ but also generate the Lie algebra $su(2,2)$. In other words we have established the well known isomorphism $so(2,4) = su(2,2)$. For complexifications of these algebras we have $so(6,\mathbb{C}) = s\ell(4,\mathbb{C})$. Below we use this isomorphism to relate operators $\mathrm{L}(u)$ (3.14), (3.37) for the special choices of algebras $s\ell(4,\mathbb{C})$ and $so(6,\mathbb{C})$. Then one can investigate the $so(6,\mathbb{C})$-type L-operator by applying known facts [21–23] about $s\ell$-type L-operators.

To proceed further we present explicitly the L-operator for the conformal algebra $so(2,4)$. This L-operator is given by formulas (3.37) and (3.39) (for $n = p + q = 4$):

$$\mathrm{L}^{(\rho)}(u) = \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{x} & \mathrm{I}_2 \end{pmatrix} \cdot \begin{pmatrix} u_+ \cdot \mathrm{I}_2 + \mathbf{S} \,, & \mathbf{p} \\ \mathbf{0} \,, & u_- \cdot \mathrm{I}_2 + \overline{\mathbf{S}} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ -\mathbf{x} & \mathrm{I}_2 \end{pmatrix} = \tag{4.4}$$

$$= \begin{pmatrix} u_+ \cdot \mathrm{I}_2 + \mathbf{S} - \mathbf{p} \cdot \mathbf{x} \,, & \mathbf{p} \\ \mathbf{x} \cdot (u_+ \cdot \mathrm{I}_2 + \mathbf{S}) - (u_- \cdot \mathrm{I}_2 + \overline{\mathbf{S}}) \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{p} \cdot \mathbf{x} \,, & u_- \cdot \mathrm{I}_2 + \overline{\mathbf{S}} + \mathbf{x} \cdot \mathbf{p} \end{pmatrix} , \tag{4.5}$$

where $u_+ = u + \frac{\Delta - 4}{2}$, $u_- = u - \frac{\Delta}{2}$ and 2×2 matrices $\mathbf{p}$, $\mathbf{x}$, $\mathbf{S}$, $\overline{\mathbf{S}}$ were defined in (2.36), (2.63), (2.64). We stress that 2×2 matrices $u_+ \cdot \mathrm{I}_2 + \mathbf{S}$ and $u_- \cdot \mathrm{I}_2 + \overline{\mathbf{S}}$ are two L-operators (see (2.63) – (2.66), (3.23)) for the case of $s\ell(2,\mathbb{C}) = so(1,3)$. We note that the basis of the algebra $so(2,4)$ (which is the real form of $so(6,\mathbb{C})$) is the basis of the algebra $so(6,\mathbb{C})$ and therefore the operator (3.37), (4.4) can be considered (after a complexification when all coordinates x_μ are complex numbers) as the L-operator of the algebra $so(6,\mathbb{C})$ as well.

4.1 L-operators for $s\ell(4,\mathbb{C})$ and $so(6,\mathbb{C})$

Now using the construction of L-operator for $s\ell(N,\mathbb{C})$ (see section 3) and the isomorphism $so(6,\mathbb{C}) = s\ell(4,\mathbb{C})$ we investigate relations of L-operator for $s\ell(4,\mathbb{C})$ (which satisfies (3.2)) and L-operator (4.4) for the algebra $so(2,4)$. The complexification of the last L-operator is also given by (3.14), (3.16) but with the special choice of the basis $\{\rho(E_{ij})\} \to \{L_{\mu\nu}, P_\mu, K_\nu, D\}$ and $\{e_{ij}\} \to \{\ell_{\mu\nu}, p_\mu, k_\nu, d\}$ in the representations ρ (2.32) and T_1 (4.1).

Consider L-operator (3.16), (3.17) for $s\ell(4,\mathbb{C})$ case, where weights $\rho_1,\dots,\rho_4$ are related by condition $\rho_1+\dots+\rho_4 = 6$. The factorized form of this L-operator in the right hand side of (3.16) contains (4×4) matrices Z and D:

$$\mathrm{Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ z_{21} & 1 & 0 & 0 \\ z_{31} & z_{32} & 1 & 0 \\ z_{41} & z_{42} & z_{43} & 1 \end{pmatrix}, \quad \mathrm{D} = \begin{pmatrix} u_1 & D_{12} & D_{13} & D_{14} \\ 0 & u_2 & D_{23} & D_{24} \\ 0 & 0 & u_3 & D_{34} \\ 0 & 0 & 0 & u_4 \end{pmatrix}, \tag{4.6}$$

where elements D_{ij} are differential operators defined in (3.18). Note that we have $D_{k4} = -\partial_{4k}$ $(k = 1,2,3)$. In view of the isomorphism $s\ell(4,\mathbb{C}) = so(6,\mathbb{C})$ one can expect that factorized form (3.16) for $N = 4$ is transformed into the factorized form of the L-operator (4.4) for conformal algebra $so(2,4)$. To obtain explicitly this transformation we write 4×4 matrices (4.6) and then (3.16) in a 2×2 block-matrix form with blocks

$$\mathrm{z}_1 = \begin{pmatrix} 1 & 0 \\ z_{21} & 1 \end{pmatrix}, \qquad \mathrm{z}_2 = \begin{pmatrix} 1 & 0 \\ z_{43} & 1 \end{pmatrix}, \qquad \mathrm{z} = \begin{pmatrix} z_{31} & z_{32} \\ z_{41} & z_{42} \end{pmatrix}, \tag{4.7}$$

$$\mathrm{d}_1 = \begin{pmatrix} u_1 & D_{12} \\ 0 & u_2 \end{pmatrix}, \qquad \mathrm{d}_2 = \begin{pmatrix} u_3 & -\partial_{43} \\ 0 & u_4 \end{pmatrix}, \qquad \mathrm{d} = -\begin{pmatrix} \partial_{31} & \partial_{41} \\ \partial_{32} & \partial_{42} \end{pmatrix}. \tag{4.8}$$

Indeed, using these blocks we first deduce factorized expressions for Z and D:

$$\mathrm{Z} = \begin{pmatrix} \mathrm{z}_1 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix} \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathrm{z} & \mathrm{I}_2 \end{pmatrix} \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{z}_2 \end{pmatrix}, \quad \mathrm{D} = \begin{pmatrix} \mathrm{d}_1\,, & \mathrm{d}\cdot\mathrm{z}_2 \\ \mathbf{0}\,, & \mathrm{d}_2 \end{pmatrix} = \begin{pmatrix} \mathrm{d}_1\,, & \mathrm{d} \\ \mathbf{0}\,, & \mathrm{d}_2\cdot\mathrm{z}_2^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{z}_2 \end{pmatrix}, \tag{4.9}$$

and then $s\ell(4,\mathbb{C})$-type L-operator (3.16) is also written, after multiplication of the matrices in the middle, in the factorized form

$$\mathrm{L}(u) = \mathrm{Z}\cdot\mathrm{D}\cdot\mathrm{Z}^{-1} = \begin{pmatrix} \mathrm{z}_1 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix} \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathrm{z} & \mathrm{I}_2 \end{pmatrix} \begin{pmatrix} \mathrm{d}_1\,, & \mathrm{d} \\ \mathbf{0}\,, & \mathrm{z}_2\cdot\mathrm{d}_2\cdot\mathrm{z}_2^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ -\mathrm{z} & \mathrm{I}_2 \end{pmatrix} \begin{pmatrix} \mathrm{z}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix}. \tag{4.10}$$

We note that here matrix $\mathrm{z}_2\cdot\mathrm{d}_2\cdot\mathrm{z}_2^{-1}$ is just the usual L-operator (3.23) for $s\ell(2,\mathbb{C})$ case

$$\mathrm{z}_2\,\mathrm{d}_2\,\mathrm{z}_2^{-1} = \begin{pmatrix} 1 & 0 \\ z_{43} & 1 \end{pmatrix} \begin{pmatrix} u_3 & -\partial_{43} \\ 0 & u_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z_{43} & 1 \end{pmatrix}, \tag{4.11}$$

and the whole dependence on z_{43} in $\mathrm{L}(u)$ (4.10) is absorbed only in this operator (4.11).

Multiplication of all matrices in (4.10) gives

$$\mathrm{L}(u)=\begin{pmatrix} \mathrm{z}_1\cdot(\mathrm{d}_1-\mathrm{d}\cdot\mathrm{z})\cdot\mathrm{z}_1^{-1}\,, & \mathrm{z}_1\cdot\mathrm{d} \\ \mathrm{z}\cdot(\mathrm{d}_1-\mathrm{d}\cdot\mathrm{z})\cdot\mathrm{z}_1^{-1}-(\mathrm{z}_2\cdot\mathrm{d}_2\cdot\mathrm{z}_2^{-1})\cdot\mathrm{z}\cdot\mathrm{z}_1^{-1}\,, & \mathrm{z}\cdot\mathrm{d}+(\mathrm{z}_2\cdot\mathrm{d}_2\cdot\mathrm{z}_2^{-1}) \end{pmatrix}\,, \tag{4.12}$$

and comparing of this expression with $so(2,4)$-type L-operator (4.5) suggests the natural change of variables

$$\mathbf{x}=\mathrm{z}\cdot\mathrm{z}_1^{-1}\,,\ \ \mathbf{p}=\mathrm{z}_1\cdot\mathrm{d}\,,\ \ \boldsymbol{\chi}_1=\mathrm{z}_1\,,\ \ \boldsymbol{\chi}_2=\mathrm{z}_2\,, \tag{4.13}$$

where in view of the explicit form of matrices z_1 and z_2 (4.7) we have to fix

$$\boldsymbol{\chi}_1=\begin{pmatrix}1&0\\ \chi_1&1\end{pmatrix}\,,\ \ \boldsymbol{\chi}_2=\begin{pmatrix}1&0\\ \chi_2&1\end{pmatrix}\ \ \Rightarrow\ \chi_1=z_{21}\,,\ \ \chi_2=z_{43}\,.$$

The inverse transformations with respect to (4.13) are:

$$\mathrm{z}=\mathbf{x}\cdot\boldsymbol{\chi}_1\,,\ \ \mathrm{d}=\boldsymbol{\chi}_1^{-1}\cdot\mathbf{p}\,,\ \ \mathrm{z}_1=\boldsymbol{\chi}_1\,,\ \ \mathrm{z}_2=\boldsymbol{\chi}_2\,. \tag{4.14}$$

Now we fix

$$\begin{aligned} u_1=u_+-\ell-1\,,\ \ u_2=u_++\ell\,,\\ u_3=u_--\dot{\ell}-1\,,\ \ u_4=u_-+\dot{\ell}\,. \end{aligned} \tag{4.15}$$

In terms of new variables $\mathbf{x}$, $\boldsymbol{\chi}_1$, $\boldsymbol{\chi}_2$ and $\mathbf{p}$ (4.13) the L-operator (4.10), (4.12) acquires the form(cf. (4.5)):

$$\mathrm{L}(u)=\begin{pmatrix} u_+\cdot\mathrm{I}_2+\mathbf{S}^{(\ell)}-\mathbf{p}\cdot\mathbf{x}\,, & \mathbf{p} \\ \mathbf{x}\cdot\left(u_+\cdot\mathrm{I}_2+\mathbf{S}^{(\ell)}\right)-\left(u_-\cdot\mathrm{I}_2+\overline{\mathbf{S}}^{(\dot{\ell})}\right)\cdot\mathbf{x}-\mathbf{x}\cdot\mathbf{p}\cdot\mathbf{x}\,, & u_-\cdot\mathrm{I}_2+\overline{\mathbf{S}}^{(\dot{\ell})}+\mathbf{x}\cdot\mathbf{p} \end{pmatrix}\,, \tag{4.16}$$

where we have introduced two $s\ell(2,\mathbb{C})$-type L-operators

$$\begin{aligned}\mathrm{z}_1\cdot\mathrm{d}_1\cdot\mathrm{z}_1&=\begin{pmatrix}1&0\\ z_{21}&1\end{pmatrix}\begin{pmatrix}u_1&D_{12}\\ 0&u_2\end{pmatrix}\begin{pmatrix}1&0\\ -z_{21}&1\end{pmatrix}=\begin{pmatrix}1&0\\ \chi_1&1\end{pmatrix}\begin{pmatrix}u_1&-\partial_{\chi_1}\\ 0&u_2\end{pmatrix}\begin{pmatrix}1&0\\ -\chi_1&1\end{pmatrix}\\ &=u_+\mathrm{I}_2+\mathbf{S}^{(\ell)}\,,\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\mathrm{z}_2\,\mathrm{d}_2\,\mathrm{z}_2^{-1}&=\begin{pmatrix}1&0\\ z_{43}&1\end{pmatrix}\begin{pmatrix}u_3&-\partial_{43}\\ 0&u_4\end{pmatrix}\begin{pmatrix}1&0\\ -z_{43}&1\end{pmatrix}=\begin{pmatrix}1&0\\ \chi_2&1\end{pmatrix}\begin{pmatrix}u_3&-\partial_{\chi_2}\\ 0&u_4\end{pmatrix}\begin{pmatrix}1&0\\ -\chi_2&1\end{pmatrix}\\ &=u_-\mathrm{I}_2+\mathbf{S}^{(\dot{\ell})}\,.\end{aligned} \tag{4.18}$$

Here we have used notations $\mathbf{S}^{(\ell)}$ (2.65) and $\mathbf{S}^{(\dot{\ell})}$ (2.66) for the matrices of $s\ell(2,\mathbb{C})$ generators (3.23) and interpret $\mathbf{S}^{(\ell)}$ and $\mathbf{S}^{(\dot{\ell})}$ as matrices $\mathbf{S}$ and $\overline{\mathbf{S}}$ of spin operators $S_{\mu\nu}$ (see (2.36)) appeared in the differential representation (2.32) of conformal algebra $so(2,4)$. The generators of two $s\ell(2,\mathbb{C})$ algebras which were packed into the matrices (2.65) and (2.66) are differential operators over variables χ_1 and χ_2 and act in spaces of functions of χ_1 and χ_2. It is natural to call χ_1 and χ_2 as harmonic variables.

Let us summarize connection between variables in the first and second approaches. From (2.36) and (4.13) we have:

$$\begin{aligned}
&\chi_1 = z_{21}, \\
&(\mathbf{x})_{11} = -i(x_0 + x_3) = z_{31} - z_{32}z_{21}\,, \qquad (\mathbf{x})_{12} = -i(x_1 - ix_2) = z_{32} \\
&(\mathbf{x})_{21} = -i(x_1 + ix_2) = z_{41} - z_{42}z_{21}\,, \qquad (\mathbf{x})_{22} = -i(x_0 - x_3) = z_{42}\,, \qquad \chi_2 = z_{43}
\end{aligned}$$

In the next subsection we also use light-cone coordinates

$$x_{\pm} = -i(x_0 \pm x_3)\,,\;\; x = -i(x_1 - ix_2)\,,\;\; \bar{x} = -i(x_1 + ix_2)\,, \tag{4.19}$$

so that 2×2 blocks (4.13) inside L-operator (4.4) have the form

$$\mathbf{x} = \begin{pmatrix} x_+ & x \\ \bar{x} & x_- \end{pmatrix} \qquad \mathbf{p} = \begin{pmatrix} -\partial_{x_+} & -\partial_{\bar{x}} \\ -\partial_x & -\partial_{x_-} \end{pmatrix} \equiv \mathbf{p}_x\,. \tag{4.20}$$

A solution of equations (3.38), (3.15) (for $n = 4$) and (4.15) gives the connection between parameters ρ_k and $\Delta, \ell, \dot{\ell}$

$$\rho_1 = -\frac{\Delta}{2} + \ell + 3 \quad \rho_2 = -\frac{\Delta}{2} - \ell + 2 \quad \rho_3 = \frac{\Delta}{2} + \dot{\ell} + 1 \quad \rho_4 = \frac{\Delta}{2} - \dot{\ell} \tag{4.21}$$

Thus, we have the bridge between two formulations of L-operator in $s\ell(4,\mathbb{C})$ and $so(6,\mathbb{C})$ (or $su(2,2)$ and $so(2,4)$) cases. In the next subsection we shall reproduce all constructions [21–23] of intertwining operators for the $s\ell(4,\mathbb{C})$ case and apply them for $so(6,\mathbb{C})$ case.

4.2 Intertwining operators and star-triangle relation. The $so(6,\mathbb{C}) = s\ell(4,\mathbb{C})$ case

In section 3.1 we have introduced operators $\mathcal{T}_k$ which intertwine two $s\ell(N,\mathbb{C})$-type L-operators (3.16) and permute their spectral parameters as it is shown in (3.19). In this subsection we consider intertwining operators for a product of two $s\ell(4,\mathbb{C})$-type L-operators (3.16):

$$\mathrm{L}_1(u_1,u_2,u_3,u_4)\,\mathrm{L}_2(v_1,v_2,v_3,v_4) \in \mathrm{End}(\mathbb{C}^4 \otimes V_{\Delta_1,\ell_1,\dot{\ell}_1} \otimes V_{\Delta_2,\ell_2,\dot{\ell}_2})\,. \tag{4.22}$$

Here operators L_1 and L_2 act in different quantum spaces $V_{\Delta_1,\ell_1,\dot{\ell}_1}$ and $V_{\Delta_2,\ell_2,\dot{\ell}_2}$ (the spaces of the differential representations ρ) and indices 1 and 2 indicate these spaces, respectively. Recall the definition of the spectral parameters in operators L_1 and L_2 (see (4.21)):

$$\begin{aligned}
(u_1,u_2,u_3,u_4) &= \left(u + \frac{\Delta_1}{2} - \ell_1 - 3, u + \frac{\Delta_1}{2} + \ell_1 - 2, u - \frac{\Delta_1}{2} - \dot{\ell}_1 - 1, u - \frac{\Delta_1}{2} + \dot{\ell}_1\right), \\
(v_1,v_2,v_3,v_4) &= \left(v + \frac{\Delta_2}{2} - \ell_2 - 3, v + \frac{\Delta_2}{2} + \ell_2 - 2, v - \frac{\Delta_2}{2} - \dot{\ell}_2 - 1, v - \frac{\Delta_2}{2} + \dot{\ell}_2\right),
\end{aligned} \tag{4.23}$$

where $\Delta_1\,, \Delta_2$ are the scaling dimensions and $(\ell_1,\, \dot{\ell}_1)$, $(\ell_2,\, \dot{\ell}_2)$ are the spin values. For the general case of $s\ell(N,\mathbb{C})$-type L-operators the intertwiners S such that

$$\begin{aligned}
\mathrm{S} \cdot \mathrm{L}_1(u_1,\dots,u_N)\,\mathrm{L}_2(v_1,\dots,v_N) &= \mathrm{L}_1(u'_1,\dots,u'_N)\,\mathrm{L}_2(v'_1,\dots,v'_N) \cdot \mathrm{S}\,, \\
(v'_1,\dots,v'_N,u'_1,\dots,u'_N) &= s(v_1,\dots,v_N,u_1,\dots,u_N)\,,
\end{aligned} \tag{4.24}$$

were constructed in [21–23]. In equation (4.24) we denote by s any permutation of $2N$ spectral parameters $(v_1,\dots,v_N,u_1,\dots,u_N)$. In this subsection we briefly discuss the intertwining operators S for the product $\mathrm{L}_1(u_1,\cdots,u_4)\,\mathrm{L}_2(v_1,\cdots,v_4)$ of two $s\ell(4,\mathbb{C})$-type L-operators which permute parameters inside the set $\mathbf{u}=(v_1,\cdots,v_4,u_1,\cdots,u_4)$. First, we choose the following variables for the operator L_1: light-cone coordinates $\vec{x}_1=(y_+,y_-,y,\bar{y})$ for space-time vector (see (4.19)) and χ_1 and χ_2 for harmonic variables. For the operator L_2 we choose $\vec{x}_2=(z_+,z_-,z,\bar{z})$ for space-time vector and η_1 and η_2 for harmonic variables. In terms of these variables the differential operators $D_{k,k+1}$ (3.18) for L_1 and L_2 have the following representations

$$\begin{array}{l}\mathrm{L}_1:\;\; \mathrm{D}_{12}\to\partial_{\chi_1}\,,\;\; \mathrm{D}_{23}\to\mathrm{D}_y=\partial_y+\chi_2\partial_{y_-}-\chi_1\partial_{y_+}-\chi_1\chi_2\partial_{\bar{y}}\,,\;\; \mathrm{D}_{34}\to\partial_{\chi_2}\,,\\ \mathrm{L}_2:\;\; \mathrm{D}_{12}\to\partial_{\eta_1}\,,\;\; \mathrm{D}_{23}\to\mathrm{D}_z=\partial_z+\eta_2\partial_{z_-}-\eta_1\partial_{z_+}-\eta_1\eta_2\partial_{\bar{z}}\,,\;\; \mathrm{D}_{34}\to\partial_{\eta_2}\,.\end{array} \tag{4.25}$$

Then, according to the results of [21–23] (see also section 3.1), the intertwining operators $\mathcal{T}_k$ (3.19), (3.20) which separately permute the spectral parameters $(v_1,\cdots,v_4)$ in L_2 and $(u_1,\cdots,u_4)$ in L_1 are

$$\mathrm{L}_2:\;\; \mathcal{T}_1(\mathbf{u})=\partial_{\eta_1}^{v_2-v_1}\qquad \mathcal{T}_2(\mathbf{u})=\mathrm{D}_z^{v_3-v_2}\qquad \mathcal{T}_3(\mathbf{u})=\partial_{\eta_2}^{v_4-v_3}\,, \tag{4.26}$$

$$\mathrm{L}_1:\;\; \mathcal{T}_5(\mathbf{u})=\partial_{\chi_1}^{u_2-u_1}\qquad \mathcal{T}_6(\mathbf{u})=\mathrm{D}_y^{u_3-u_2}\qquad \mathcal{T}_7(\mathbf{u})=\partial_{\chi_2}^{u_4-u_3}\,. \tag{4.27}$$

The middle intertwining operator which correspond to the permutation $u_1\leftrightarrow v_4$ in the product of two L-operators (4.22) is [21–23]

$$\mathcal{T}_4(\mathbf{u})=\mathrm{S}(\vec{x}_1-\vec{x}_2)^{u_1-v_4}\,,$$

where

$$\mathrm{S}(\vec{x}_1-\vec{x}_2)=(\bar{y}-\bar{z})+\chi_1(y_--z_-)+\eta_2(z_+-y_+)+\chi_1\eta_2(z-y)\,. \tag{4.28}$$

Next we construct the composite intertwining operators S_1 and S_2. The first operator S_1 interchanges pairs (v_1,v_2) and (v_3,v_4): $(\underline{v_1,v_2},v_3,v_4)\to(v_3,v_4,\underline{v_1,v_2})$. In terms of physical parameters this permutation is written as $(\Delta,\ell_2,\dot{\ell}_2)\to(4-\Delta,\dot{\ell}_2,\ell_2)$. We explain the choice of this intertwining operator at the end of this subsection (see Remark 2). According to (4.26) the explicit form of S_1 is

$$\mathrm{S}_1=\mathcal{T}_2(s_1s_3s_2\mathbf{u})\,\mathcal{T}_1(s_3s_2\mathbf{u})\,\mathcal{T}_3(s_2\mathbf{u})\,\mathcal{T}_2(\mathbf{u})=\mathrm{D}_z^{v_4-v_1}\,\partial_{\eta_2}^{v_4-v_2}\,\partial_{\eta_1}^{v_3-v_1}\,\mathrm{D}_z^{v_3-v_2}\,. \tag{4.29}$$

We stress that in (4.29) for each $\mathcal{T}_k$ the previous permutations s_m (3.21) of the spectral parameters should be taken into account.

The second intertwining operator S_2 interchanges pairs (v_3,v_4) and (u_1,u_2):

$$(v_1,v_2,\underline{v_3,v_4},\underline{\underline{u_1,u_2}},u_3,u_4)\to(v_1,v_2,\underline{\underline{u_1,u_2}},\underline{v_3,v_4},u_1,u_2)\,,$$

and explicit form of S_2 is

$$\mathrm{S}_2=\mathcal{T}_4(s_5s_3s_4\mathbf{u})\mathcal{T}_5(s_3s_4\mathbf{u})\mathcal{T}_3(s_4\mathbf{u})\mathcal{T}_4(\mathbf{u})=\mathrm{S}(\vec{x}_1-\vec{x}_2)^{u_2-v_3}\,\partial_{\chi_1}^{u_2-v_4}\,\partial_{\eta_2}^{u_1-v_3}\,\mathrm{S}(\vec{x}_1-\vec{x}_2)^{u_1-v_4}\,. \tag{4.30}$$

The remarkable fact [21–23] is that the operators S_1 and S_2 satisfy the braid relation

$$\mathrm{S}_1\,\mathrm{S}_2\,\mathrm{S}_1=\mathrm{S}_2\,\mathrm{S}_1\,\mathrm{S}_2\,. \tag{4.31}$$

In next section 5 we interpret the identity (4.31) as the star-triangle relation for propagators of spin particles in certain conformal field theory.

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Remark 1. One can try to write operators D_y, D_z in (4.25) and $S(\vec{x}_1 - \vec{x}_2)$ in (4.28) in covariant form (under the transformations of the subgroup $\mathrm{SO}(4,\mathbb{C}) \subset \mathrm{SO}(6,\mathbb{C})$ with generators $\rho(L_{\mu\nu})$ (2.32)) by means of introducing new homogeneous variables $\lambda_\alpha, \tilde{\lambda}^{\dot{\alpha}}, \mu_\alpha, \tilde{\mu}^{\dot{\alpha}}$ (see subsection 2.2):

$$\chi_1 = \frac{\lambda_2}{\lambda_1}\,, \qquad \chi_2 = \frac{\tilde{\lambda}^{\dot{2}}}{\tilde{\lambda}^{\dot{1}}}\,, \qquad \eta_1 = \frac{\mu_2}{\mu_1}\,, \qquad \eta_2 = \frac{\tilde{\mu}^{\dot{2}}}{\tilde{\mu}^{\dot{1}}}\,,$$
$$\partial_{\chi_1} = \lambda_1\,\partial_{\lambda_2}\,, \qquad \partial_{\eta_1} = \mu_1\,\partial_{\mu_2}\,, \qquad \partial_{\eta_2} = \tilde{\mu}^{\dot{1}}\,\partial_{\tilde{\mu}^{\dot{2}}}\,,$$

In terms of these new variables we have

$$D_y = \frac{1}{(\lambda_1\tilde{\lambda}^{\dot{1}})}\lambda^\alpha\,(\mathbf{p}_y)_{\alpha\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}\,, \qquad D_z = \frac{1}{(\mu_1\tilde{\mu}^{\dot{1}})}\mu^\alpha\,(\mathbf{p}_z)_{\alpha\dot{\alpha}}\tilde{\mu}^{\dot{\alpha}}\,, \tag{4.32}$$
$$S(\vec{x}_1 - \vec{x}_2) = \frac{1}{(\lambda_1\tilde{\mu}^{\dot{1}})}\tilde{\mu}_{\dot{\alpha}}(\mathbf{y}-\mathbf{z})^{\dot{\alpha}\alpha}\,\lambda_\alpha\,,$$

where $\lambda^\alpha = \lambda_\beta\varepsilon^{\beta\alpha}$, $\mu^\alpha = \mu_\beta\varepsilon^{\beta\alpha}$, $\tilde{\mu}_{\dot{\alpha}} = \tilde{\mu}^{\dot{\beta}}\varepsilon_{\dot{\beta}\dot{\alpha}}$ ($\varepsilon^{\beta\alpha}$ and $\varepsilon_{\dot{\beta}\dot{\alpha}}$ — antisymmetric tensors) and $\mathbf{p}_y, \mathbf{p}_z, \mathbf{y}, \mathbf{z}$ were defined in (4.20). Then the operators (4.29), (4.30) are represented in the form

$$\mathrm{S}_1 = (\mu\,\mathbf{p}_z\,\tilde{\mu})^{v_4-v_1}\left(\frac{1}{\mu_1}\partial_{\tilde{\mu}^{\dot{2}}}\right)^{v_4-v_2}\left(\frac{1}{\tilde{\mu}^{\dot{1}}}\partial_{\mu_2}\right)^{v_3-v_1}(\mu\,\mathbf{p}_z\,\tilde{\mu})^{v_3-v_2}\,, \tag{4.33}$$

$$\mathrm{S}_2 = (\tilde{\mu}(\mathbf{y}-\mathbf{z})\,\lambda)^{u_2-v_3}\left(\frac{1}{\tilde{\mu}^{\dot{1}}}\partial_{\lambda_2}\right)^{u_2-v_4}\left(\frac{1}{\lambda_1}\partial_{\tilde{\mu}^{\dot{2}}}\right)^{u_1-v_3}(\tilde{\mu}(\mathbf{y}-\mathbf{z})\,\lambda)^{u_1-v_4}\,. \tag{4.34}$$

The covariance of the operators S_1 (4.33) and S_2 (4.34) under $\mathrm{SO}(4,\mathbb{C})$ transformations (or the Lorentz covariance for real forms of S_1 and S_2) is broken in view of the presence of noncovariant operators $\frac{1}{\mu_1}\partial_{\tilde{\mu}^{\dot{2}}}$, $\frac{1}{\tilde{\mu}^{\dot{1}}}\partial_{\lambda_2}$ etc. in (4.33) and (4.34). In next section 5, using slightly different approach, we derive another operators S_1, S_2 and S_3 which are represented in the Lorentz covariant form and therefore their physical interpretation as propagators of spining particles will be clarified.

Remark 2. The irreducible representation of the algebra $so(6,\mathbb{C})$ (complexification of the conformal algebra $so(2,4)$) in the differential realization (2.32) is characterized by the conformal dimension Δ and spin parameters $(\ell,\,\dot{\ell})$ which are labels of the representations of the subalgebra $so(4,\mathbb{C}) = s\ell(2,\mathbb{C}) + s\ell(2,\mathbb{C})$.[3] If all Casimir operators for two such representations of $so(6,\mathbb{C})$ coincide then these representations are equivalent (or partially equivalent) and the intertwining operator between these representations should exist. For the algebra $so(6,\mathbb{C})$ (2.32) there are three Casimir operators: the first one is $\rho(C_2)$ (2.34) and two others are

$$\rho(C_3) = \epsilon^{abcdef}\,\rho(M_{ab}\,M_{cd}\,M_{ef})\,, \quad \rho(C_4) = \rho(M_{ab}\,M^{bc}\,M_{cd}\,M^{da})\,.$$

In view of the isomorphism $so(6,\mathbb{C}) = s\ell(4,\mathbb{C})$, the eigenvalues of these Casimir operators are elementary symmetric polynomials in four variables $(\rho_1, \rho_2, \rho_3, \rho_4)$ (4.21) and therefore

[3]There is also parameter which is eigenvalue of the operator $\hat{\ell}_{\mu\nu}\hat{\ell}^{\mu\nu}$ (see (2.34)) but this additional parameter is not important for our consideration.

any permutations of these variables lead to the equivalent representations. Consider the spectral parameters (4.15), (4.23):

$$(u_1, u_2, u_3, u_4) = (u-\rho_1,\, u-\rho_2,\, u-\rho_3,\, u-\rho_4) = \left(u_+ - \ell - 1,\, u_+ + \ell,\, u_- - \dot{\ell} - 1,\, u_- + \dot{\ell}\right),$$

instead of parameters (4.21). Note that the permutation $u_1 \leftrightarrow u_2$ is equivalent to the transformation $\ell \to -1 - \ell$ while permutation $u_3 \leftrightarrow u_4$ is equivalent to $\dot{\ell} \to -1 - \dot{\ell}$. Both permutation are not appropriate for us since we would like to work with the finite dimensional representations of spin algebras (2.65) and (2.66) when parameters 2ℓ and $2\dot{\ell}$ are nonnegative integers. The other permutations of (u_1, u_2, u_3, u_4) include the interchanging $u_+ \leftrightarrow u_-$. In this case we have two possibilities $\ell \to -1 - \dot{\ell}$ or $\ell \to \dot{\ell}$. Again the first possibility is not appropriate for us since it is not compatible with the finite dimensional representations of spin algebras. As the final result we have only one variant of intertwining operator which permutes $u_+ \leftrightarrow u_-$, $\ell \to \dot{\ell}$ and therefore corresponds to the permutation of pairs of the spectral parameters (u_1, u_2) and (u_3, u_4). Precisely this intertwining operator was constructed in (4.29) and will be investigated in the next section.

Remark 3. In the paper [21–23] the complex group $\mathrm{SL}(N, C)$ were considered and there we have $\frac{N(N-1)}{2}$ complex variables z_{ik} and $\frac{N(N-1)}{2}$ complex conjugate variables $\bar{z}_{ik}$. In the case of $\mathrm{SL}(4, C)$ we have 6 complex variables and 6 complex conjugate variables. In subsection 4.1 all operators are well defined because we work with the differential operators and one can restrict everything to complex variables and forget about complex conjugated variables — the holomorphic and antiholomorphic sectors can be separated. In this subsection the situation is different because operators like ∂_z^α (i.e., the operators $D_z^\alpha \sim (\mu \mathbf{p}_z \tilde{\mu})^\alpha$ in (4.29), (4.33)) for noninteger α needs antiholomorphic part $\partial_{\bar{z}}^{\bar{\alpha}}$ so that only the product $\partial_z^\alpha \cdot \partial_{\bar{z}}^{\bar{\alpha}}$ can be defined as usual integral operator acting on the functions $f(z, \bar{z})$ defined on $\mathbb{R}^2$. We omit the antiholomorphic part everywhere in this subsection so that intertwining operators are not properly defined and can be treated only as formal operators. This is another reason why in next section we develop slightly different approach.

5 General R-operator

In this section we are going to construct R-operator as solution of the defining RLL-equation [17, 20]

$$\mathrm{R}_{12}(u - v)\, \mathrm{L}_1(u)\, \mathrm{L}_2(v) = \mathrm{L}_1(v)\, \mathrm{L}_2(u)\, \mathrm{R}_{12}(u - v)$$

with conformal L-operator (3.39). Here indices $1, 2$ correspond to two infinite-dimensional spaces of differential representation ρ of the conformal algebra $\mathsf{conf}(\mathbb{R}^n)$ (2.32) and we consider two cases:

- Dimension n of the Euclidean space $\mathbb{R}^n$ is arbitrary and representations of the conformal algebra are restricted to the case of scalars: $\mathbf{S} = 0$ and $\bar{\mathbf{S}} = 0$.
- Dimension n of the space $\mathbb{R}^n$ is fixed by $n = 4$ and representations of the conformal algebra are generic: $\mathbf{S} \neq 0$ and $\bar{\mathbf{S}} \neq 0$.

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5.1 n-dimensional scalar case

In this case the defining RLL-equation has the form

$$\mathrm{R}_{12}(u-v)\,\mathrm{L}_1(u_+,u_-)\,\mathrm{L}_2(v_+,v_-)=\mathrm{L}_1(v_+,v_-)\,\mathrm{L}_2(u_+,u_-)\,\mathrm{R}_{12}(u-v)\,, \tag{5.1}$$

where

$$\mathrm{L}_1(u_+,u_-)=\begin{pmatrix}\mathbf{1}&\mathbf{0}\\ \mathbf{x}_1&\mathbf{1}\end{pmatrix}\cdot\begin{pmatrix}u_+\cdot\mathbf{1}&\mathbf{p}_1\\ \mathbf{0}&u_-\cdot\mathbf{1}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{1}&\mathbf{0}\\ -\mathbf{x}_1&\mathbf{1}\end{pmatrix},$$

$$\mathrm{L}_2(v_+,v_-)=\begin{pmatrix}\mathbf{1}&\mathbf{0}\\ \mathbf{x}_2&\mathbf{1}\end{pmatrix}\cdot\begin{pmatrix}v_+\cdot\mathbf{1}&\mathbf{p}_2\\ \mathbf{0}&v_-\cdot\mathbf{1}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{1}&\mathbf{0}\\ -\mathbf{x}_2&\mathbf{1}\end{pmatrix},$$

and $u_+=u+\frac{\Delta_1-n}{2}$, $u_-=u-\frac{\Delta_1}{2}$, $v_+=v+\frac{\Delta_2-n}{2}$, $v_-=v-\frac{\Delta_2}{2}$ (3.38).

The R-operator in (5.1) interchanges a pair of parameters (u_+,u_-) in the first L-operator with a pair (v_+,v_-) from the second L-operator. It seems to be reasonable to consider also operators which perform the other interchanges of four parameters. In order to carry out it systematically we joint them in the set $\mathbf{u}=(v_+,v_-,u_+,u_-)$. Then R-operator represents the permutation s such that

$$s\mapsto \mathrm{R}(u-v)\ ;\ s\,\mathbf{u}=(\underline{u_+,u_-},\underline{v_+,v_-}). \tag{5.2}$$

An arbitrary permutation can be builded from elementary transpositions s_1, s_2 and s_3

$$s_1\mathbf{u}=(\underline{v_-},\underline{v_+},u_+,u_-)\ ;\ s_2\mathbf{u}=(v_+,\underline{u_+},\underline{v_-},u_-)\ ;\ s_3\mathbf{u}=(v_+,v_-,\underline{u_-},\underline{u_+})\,.$$

In particular: $s=s_2s_1s_3s_2$. Thus we reduce the problem to construction of operators $\mathrm{S}_i(\mathbf{u})$ $(i=1,2,3)$ which represent elementary transpositions

$$(\underline{v_+\,,\ v_-},u_+,u_-)\ :\ \mathrm{S}_1(\mathbf{u})\,\mathrm{L}_2(v_+,v_-)=\mathrm{L}_2(v_-,v_+)\,\mathrm{S}_1(\mathbf{u}) \tag{5.3}$$

$$(v_+,\underline{v_-\,,\ u_+},u_-)\ :\ \mathrm{S}_2(\mathbf{u})\,\mathrm{L}_1(u_+,u_-)\,\mathrm{L}_2(v_+,v_-)=\mathrm{L}_1(v_-,u_-)\,\mathrm{L}_2(v_+,u_+)\,\mathrm{S}_2(\mathbf{u}) \tag{5.4}$$

$$(v_+,v_-,\underline{u_+\,,\ u_-})\ :\ \mathrm{S}_3(\mathbf{u})\,\mathrm{L}_1(u_+,u_-)=\mathrm{L}_1(u_-,u_+)\,\mathrm{S}_3(\mathbf{u}) \tag{5.5}$$

We have the correspondence

$$s_i\mapsto \mathrm{S}_i(\mathbf{u})\ \ ;\ \ s_is_j\mapsto \mathrm{S}_i(s_j\mathbf{u})\,\mathrm{S}_j(\mathbf{u})\ \ ;\ \ s_is_js_k\mapsto \mathrm{S}_i(s_js_k\mathbf{u})\,\mathrm{S}_j(s_k\mathbf{u})\,\mathrm{S}_k(\mathbf{u})\ \ ;\ \ \cdots \tag{5.6}$$

and for the proof that it is indeed the representation of the permutation group of four parameters we have to check the corresponding defining (Coxeter) relations

$$s_is_i=\mathbb{1}\longrightarrow \mathrm{S}_i(s_i\mathbf{u})\,\mathrm{S}_i(\mathbf{u})=\mathbb{1}\ ;\ s_1s_3=s_3s_1\longrightarrow \mathrm{S}_1(s_3\mathbf{u})\,\mathrm{S}_3(\mathbf{u})=\mathrm{S}_3(s_1\mathbf{u})\,\mathrm{S}_1(\mathbf{u}) \tag{5.7}$$

$$s_1s_2s_1=s_2s_1s_2\longrightarrow \mathrm{S}_1(s_2s_1\mathbf{u})\,\mathrm{S}_2(s_1\mathbf{u})\,\mathrm{S}_1(\mathbf{u})=\mathrm{S}_2(s_1s_2\mathbf{u})\,\mathrm{S}_1(s_2\mathbf{u})\,\mathrm{S}_2(\mathbf{u}) \tag{5.8}$$

$$s_2s_3s_2=s_3s_2s_3\longrightarrow \mathrm{S}_2(s_3s_2\mathbf{u})\,\mathrm{S}_3(s_2\mathbf{u})\,\mathrm{S}_2(\mathbf{u})=\mathrm{S}_3(s_2s_3\mathbf{u})\,\mathrm{S}_2(s_3\mathbf{u})\,\mathrm{S}_3(\mathbf{u}) \tag{5.9}$$

Then R-operator can be constructed form these building blocks:

$$\mathrm{R}(\mathbf{u})=\mathrm{S}_2(s_1s_3s_2\mathbf{u})\,\mathrm{S}_1(s_3s_2\mathbf{u})\,\mathrm{S}_3(s_2\mathbf{u})\,\mathrm{S}_2(\mathbf{u}) \tag{5.10}$$

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We will see that operators S_i depend on their parameters in a special way

$$\mathrm{S}_1(\mathbf{u}) = \mathrm{S}_1(v_- - v_+)\ ;\ \mathrm{S}_2(\mathbf{u}) = \mathrm{S}_2(u_+ - v_-)\ ;\ \mathrm{S}_3(\mathbf{u}) = \mathrm{S}_3(u_- - u_+), \tag{5.11}$$

so that the operator $\mathrm{R}(\mathbf{u})$ depends on the difference of spectral parameters $u - v$ as it should

$$\mathrm{R}(\mathbf{u}) = \mathrm{S}_2(u_- - v_+)\,\mathrm{S}_1(u_+ - v_+)\,\mathrm{S}_3(u_- - v_-)\,\mathrm{S}_2(u_+ - v_-)\,. \tag{5.12}$$

The Yang-Baxter relation for this R-operator is the direct consequence of the Coxeter relations for the building blocks $\mathrm{S}_i(\mathbf{u})$. In explicit notations relations (5.8) and (5.9) have the form

$$\mathrm{S}_1(u_+ - v_-)\,\mathrm{S}_2(u_+ - v_+)\,\mathrm{S}_1(v_- - v_+) = \mathrm{S}_2(v_- - v_+)\,\mathrm{S}_1(u_+ - v_+)\,\mathrm{S}_2(u_+ - v_-)\,,$$

$$\mathrm{S}_2(u_- - u_+)\,\mathrm{S}_3(u_- - v_-)\,\mathrm{S}_2(u_+ - v_-) = \mathrm{S}_3(u_+ - v_-)\,\mathrm{S}_2(u_- - v_-)\,\mathrm{S}_3(u_- - u_+)\,,$$

and are particular examples of the general relations

$$\mathrm{S}_1(a)\,\mathrm{S}_2(a+b)\,\mathrm{S}_1(b) = \mathrm{S}_2(b)\,\mathrm{S}_1(a+b)\,\mathrm{S}_2(a)\ \ ;\ \ \mathrm{S}_2(a)\,\mathrm{S}_3(a+b)\,\mathrm{S}_2(b) = \mathrm{S}_3(b)\,\mathrm{S}_2(a+b)\,\mathrm{S}_3(a)\,. \tag{5.13}$$

We are going to construct operators $\mathrm{S}_i(\mathbf{u})$ and at the first stage we consider operators S_1 and S_3 which are examples of the operator S being defined by the equation

$$\hat{\mathrm{S}}\cdot \mathrm{L}(u_+, u_-) = \mathrm{L}(u_-, u_+)\cdot \hat{\mathrm{S}} \tag{5.14}$$

As soon as here we deal with a scalar case differential representation of the conformal algebra is parameterized by one parameter — conformal dimension Δ. We denote it by ρ^{Δ}. Taking in mind the definition of the parameters u_+ and u_- we see that their transposition corresponds to $\Delta \to n - \Delta$. Since L-operator is linear on spectral parameter and in view of equation (5.14) we conclude that S intertwines two representations of the conformal algebra: ρ^{Δ} and $\rho^{n-\Delta}$. Note that such a change of conformal dimension do preserve the Casimir operator (2.34).

Let us represent the intertwining operator as an integral operator acting on fields $\Phi(x)$ where $x \in \mathbb{R}^{p,q}$

$$[\mathrm{S}\,\Phi]\,(x) = \int \mathrm{d}^n y\,\mathrm{S}\,(x,y)\ \Phi(y)\,,$$

then defining equation for S (5.14) is equivalent to the set of equations

$$\int \mathrm{d}^n y\,\mathrm{S}(x,y)\,G_y^{\Delta}\,\Phi(y) = \int \mathrm{d}^n y\,G_x^{n-\Delta}\,\mathrm{S}(x,y)\,\Phi(y)\,,$$

which can be rewritten as the set of differential equations for the kernel $\mathrm{S}\,(x,y)$

$$\left(G_y^{\Delta}\right)^T \mathrm{S}\,(x,y) = G_x^{n-\Delta}\,\mathrm{S}\,(x,y)\ . \tag{5.15}$$

Here G_x^{Δ} denotes generators of conformal group in scalar ($S_{\mu\nu} = 0$) differential representation (2.32) and T stands for transposition arising after integration by parts

$$\int \mathrm{d}^n y\,\mathrm{S}(x,y)\,G_y\,\Phi(y) = \int \mathrm{d}^n y\ \left[G_y^T\,\mathrm{S}(x,y)\right]\ \Phi(y)\,.$$

We obtain the following equations:

- translation

$$\left(\frac{\partial}{\partial x_\mu}+\frac{\partial}{\partial y_\mu}\right)\mathrm{S}\,(x,y)=0\,, \tag{5.16}$$

- Lorentz rotation

$$\left(y_\nu\frac{\partial}{\partial y_\mu}-y_\mu\frac{\partial}{\partial y_\nu}\right)\mathrm{S}(x,y)=\left(x_\mu\frac{\partial}{\partial x_\nu}-x_\nu\frac{\partial}{\partial x_\mu}\right)\mathrm{S}(x,y)\,, \tag{5.17}$$

- dilatation

$$\left(x_\mu\frac{\partial}{\partial x_\mu}+y_\mu\frac{\partial}{\partial y_\mu}\right)\mathrm{S}\,(x,y)=-2\,(n-\Delta)\;\mathrm{S}\,(x,y)\;, \tag{5.18}$$

- conformal boost

$$\left(x^2\frac{\partial}{\partial x_\mu}-2x_\mu x_\nu\frac{\partial}{\partial x_\nu}+y^2\frac{\partial}{\partial y_\mu}-2y_\mu y_\nu\frac{\partial}{\partial y_\nu}\right)\mathrm{S}(x,y)=2(n-\Delta)\,(x_\mu+y_\mu)\,\mathrm{S}(x,y)\,. \tag{5.19}$$

Note that in the scalar case $S_{\mu\nu}=0$ the conformal boost equation is dispensable since it can be derived from (5.16) – (5.18).

The set of equations for the kernel of $\hat{\mathrm{S}}$ coincides with the set of equations for the Green function of the two scalar fields with equal scaling dimensions in conformal field theory [50]. The solution is well known

$$\mathrm{S}(x,y)=\frac{c}{(x-y)^{2(n-\Delta)}}\,,$$

and is fixed up to an arbitrary multiplicative constant. The action of the integral operator with the kernel $\mathrm{S}(x,y)$ on the function $\Phi(x)$ can be represented in different forms

$$[\hat{\mathrm{S}}\,\Phi](x)=c\int\frac{\mathrm{d}^ny}{(x-y)^{2(n-\Delta)}}\cdot\Phi(y)=c\int\frac{\mathrm{d}^ny}{y^{2(n-\Delta)}}\cdot\Phi(x-y)=c\int\frac{\mathrm{d}^ny\;e^{iy\hat{p}}}{y^{2(n-\Delta)}}\cdot\Phi(x)\,, \tag{5.20}$$

where $\hat{p}_\nu=-i\partial_{x^\nu}$. There exists useful expression for this operator

$$\hat{\mathrm{S}}(u_--u_+)=\hat{p}^{\,2(u_--u_+)}=\hat{p}^{\,2\left(\frac{n}{2}-\Delta\right)}\,. \tag{5.21}$$

Indeed, using the well-known formula for the Fourier transformation

$$\int\mathrm{d}^ny\,\frac{\mathrm{e}^{-i\,yp}}{y^{2\left(\frac{n}{2}-\alpha\right)}}=\frac{a(\alpha)}{p^{2\alpha}}\quad;\quad a(\alpha)\equiv\pi^{\frac{n}{2}}\,4^\alpha\,\frac{\Gamma\,(\alpha)}{\Gamma\left(\frac{n}{2}-\alpha\right)}\,,$$

it is possible to present the integral operator of considered type as

$$\int\frac{\mathrm{d}^ny}{(x-y)^{2\left(\frac{n}{2}-\alpha\right)}}\,\Phi(y)=a(\alpha)\,\hat{p}^{\,-2\alpha}\,\Phi(x)\,.$$

In our case $\alpha=\Delta-\frac{n}{2}$, so that it remains to choose the normalization constant c in (5.20) in a special way

$$c=\frac{1}{a\left(\Delta-\frac{n}{2}\right)}=4^{\frac{n}{2}-\Delta}\pi^{\frac{n}{2}}\,\frac{\Gamma(n-\Delta)}{\Gamma(\Delta-\frac{n}{2})}\,,$$

to fix operator $\hat{\mathrm{S}}$ in the form (5.21). Thus we have constructed operators S_1 and S_3 using solely their representation theory meaning. Explicit expressions are the following

$$\mathrm{S}_1(v_- - v_+) = \hat{p}_2^{2(v_- - v_+)} \quad ; \quad \mathrm{S}_3(u_- - u_+) = \hat{p}_1^{2(u_- - u_+)} \, .$$

We stress that operators $\hat{p}_1^2$ and $\hat{p}_2^2$ have got zero modes in the case of the pseudo-euclidean space. It leads to the problem of correct definition of corresponding Green's functions. We postpone this problem to further investigations.

Remark. Solution (5.21) can be obtained directly if write equations (5.16) – (5.19) in the operator form (cf. (5.14)):

- translation : $$[\hat{p}_\mu \, , \, \hat{\mathrm{S}}] = 0 \, , \tag{5.22}$$
- Lorentz rotation : $$[x_\nu \, \hat{p}_\mu - x_\mu \, \hat{p}_\nu \, , \, \hat{\mathrm{S}}] = 0 \, , \tag{5.23}$$
- dilatation : $$(x^\mu \, \hat{p}_\mu - i(n-\Delta)) \, \hat{\mathrm{S}} = \hat{\mathrm{S}} \, (x^\mu \, \hat{p}_\mu - i\Delta) \, , \tag{5.24}$$
- conformal boost : $$\left(x_\mu (x^\nu \, \hat{p}_\nu - 2i(n-\Delta)) - x^2 \hat{p}_\mu\right) \hat{\mathrm{S}} = \hat{\mathrm{S}} \, \left(x_\mu (x^\nu \, \hat{p}_\nu - 2i\Delta) - x^2 \hat{p}_\mu\right) \, . \tag{5.25}$$

Then equation (5.22) gives that $\hat{\mathrm{S}}$ depends only on $\hat{p}_\mu$, from (5.23) we obtain that $\hat{\mathrm{S}}$ depends on the Lorentz invariant combination $\hat{p}^2$ and (5.24) leads to the solution (5.21) up to a normalization constant. Equation (5.25) is optional since operator (5.21) satisfies (5.25) automatically.

It remains to construct the last building block for R-operator — operator S_2. It happens that it can be produced directly from the operator S obtained above using some kind of duality transformation

$$p \to x_2 - x_1 \equiv x_{21} \; ; \; u_+ \to v_- \; ; \; u_- \to u_+ \, ,$$

so that S_2 is the operator of multiplication by the function

$$\mathrm{S}_2(u_+ - v_-) = x_{12}^{2(u_+ - v_-)} \, .$$

To explain the origin of these duality we start from the defining equation (5.14) for S

$$\mathrm{S} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{x} & \mathbf{1} \end{pmatrix} \begin{pmatrix} u_+ \cdot \mathbf{1} & \mathbf{p} \\ \mathbf{0} & u_- \cdot \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{x} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{x} & \mathbf{1} \end{pmatrix} \begin{pmatrix} u_- \cdot \mathbf{1} & \mathbf{p} \\ \mathbf{0} & u_+ \cdot \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{x} & \mathbf{1} \end{pmatrix} \mathrm{S} \, , \tag{5.26}$$

and show that the defining equation for S_2 can be obtained from considered ones by the duality transformation. For this purpose we rewrite the defining equation (5.4) for the operator S_2 in an explicit form using factorization of the L-operator ($\mathbf{x}_{21} \equiv \mathbf{x}_2 - \mathbf{x}_1$)

$$\mathrm{S}_2 \underline{\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{x}_1 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & u_- \cdot \mathbf{1} \end{pmatrix}} \begin{pmatrix} u_+ \cdot \mathbf{1} & \mathbf{p}_1 \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{x}_{21} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{p}_2 \\ \mathbf{0} & v_- \cdot \mathbf{1} \end{pmatrix} \underline{\begin{pmatrix} v_+ \cdot \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{x}_2 & \mathbf{1} \end{pmatrix}} =$$

$$= \underline{\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{x}_1 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & u_- \cdot \mathbf{1} \end{pmatrix}} \begin{pmatrix} v_- \cdot \mathbf{1} & \mathbf{p}_1 \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{x}_{21} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{p}_2 \\ \mathbf{0} & u_+ \cdot \mathbf{1} \end{pmatrix} \underline{\begin{pmatrix} v_+ \cdot \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{x}_2 & \mathbf{1} \end{pmatrix}} \mathrm{S}_2$$

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By the condition $[\mathrm{S}_2, \mathbf{x}_1] = [\mathrm{S}_2, \mathbf{x}_2] = 0$ it is possible to cancel the underlined factors so that equation is transformed to the much more simple form

$$\mathrm{S}_2 \begin{pmatrix} \mathbf{1} & \mathbf{p}_1 \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} u_+ \cdot \mathbf{1} & \mathbf{0} \\ \mathbf{x}_{21} & v_- \cdot \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{p}_2 \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{p}_1 \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} v_- \cdot \mathbf{1} & \mathbf{0} \\ \mathbf{x}_{21} & u_+ \cdot \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{p}_2 \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \mathrm{S}_2$$

One more requirement we impose is the translation invariance: $[\mathrm{S}_2, \mathbf{p}_1 + \mathbf{p}_2] = 0$, so that we obtain

$$\mathrm{S}_2 \begin{pmatrix} \mathbf{1} & \mathbf{p}_1 \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} u_+ \cdot \mathbf{1} & \mathbf{0} \\ \mathbf{x}_{21} & v_- \cdot \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & -\mathbf{p}_1 \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{p}_1 \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} v_- \cdot \mathbf{1} & \mathbf{0} \\ \mathbf{x}_{21} & u_+ \cdot \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & -\mathbf{p}_1 \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \mathrm{S}_2 \,. \tag{5.27}$$

Next we perform similarity transformation of the previous equation by means of the matrix $\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$:

$$\mathrm{S}_2 \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{p}_1 & \mathbf{1} \end{pmatrix} \begin{pmatrix} v_- \cdot \mathbf{1} & \mathbf{x}_{21} \\ \mathbf{0} & u_+ \cdot \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{p}_1 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{p}_1 & \mathbf{1} \end{pmatrix} \begin{pmatrix} u_+ \cdot \mathbf{1} & \mathbf{x}_{21} \\ \mathbf{0} & v_- \cdot \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{p}_1 & \mathbf{1} \end{pmatrix} \mathrm{S}_2 \,. \tag{5.28}$$

It remains to compare this equation with the defining equation (5.26) for the intertwining operator S which suggests that the change

$$\mathbf{x} \to \mathbf{p}_1 \;;\; \mathbf{p} \to \mathbf{x}_{21} \;;\; u_+ \to v_- \;;\; u_- \to u_+$$

transforms (5.26) into (5.28). Thus we have that S_2 is the operator of multiplication by the function

$$\mathrm{S}_2(u_+ - v_-) = x_{12}^{2(u_+ - v_-)} \,.$$

Coxeter relations (5.7) are evident and Coxeter relations (5.13) have the following explicit forms

$$\hat{p}_2^{2a}\, x_{12}^{2(a+b)}\, \hat{p}_2^{2b} = x_{12}^{2b}\, \hat{p}_2^{2(a+b)}\, x_{12}^{2b} \;\;;\;\; \hat{p}_1^{2a}\, x_{12}^{2(a+b)}\, \hat{p}_1^{2b} = x_{12}^{2b}\, \hat{p}_1^{2(a+b)}\, x_{12}^{2a} \,, \tag{5.29}$$

and are both equivalent to the operator identity [62, 63]:

$$\hat{p}^{\,2a}\, x^{2(a+b)}\, \hat{p}^{\,2b} = x^{2b}\, \hat{p}^{\,2(a+b)}\, x^{2a} \,, \tag{5.30}$$

which can be rewritten in the standard integral form

$$\int \mathrm{d}^n w \frac{1}{(x-w)^{2\alpha}(y-w)^{2\beta}(z-w)^{2\gamma}} = \mathrm{V}(\alpha,\beta,\gamma) \cdot \frac{1}{(y-z)^{2\alpha'}(x-z)^{2\beta'}(x-y)^{2\gamma'}} \,, \tag{5.31}$$

where

$$\mathrm{V}(\alpha,\beta,\gamma) = \pi^{\frac{n}{2}} \frac{\Gamma(\alpha')\,\Gamma(\beta')\,\Gamma(\gamma')}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)} \;\;;\;\; \alpha' = \frac{n}{2} - \alpha \,,\; \beta' = \frac{n}{2} - \beta \,,\; \gamma' = \frac{n}{2} - \gamma$$

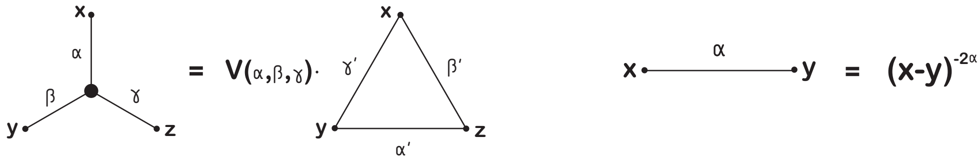

Figure 1. Scalar star-triangle relation.

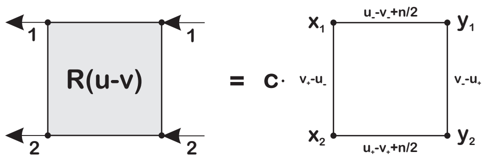

Figure 2. Integral kernel of the R-operator.

and parameters respect the uniqueness condition

$$\alpha + \beta + \gamma = n\,.$$

This integral identity is a well-known star-triangle relation [51–53]. It is useful to represent the identity in figure 1 where marked vertex represents the integration over the variable w.

Finally we find explicit expression for R-operator using (5.10)

$$\mathrm{R}_{12}(u-v) = x_{12}^{2(u_- - v_+)}\,\hat{p}_2^{2(u_+ - v_+)}\,\hat{p}_1^{2(u_- - v_-)}\,x_{12}^{2(u_+ - v_-)}\,. \tag{5.32}$$

This operator can be rewritten as an integral one

$$[\mathrm{R}_{12}(u-v)\,\Phi](x_1, x_2) = c\cdot\int \frac{\mathrm{d}^n y_1 \mathrm{d}^n y_2\,\Phi(y_1, y_2)}{x_{12}^{2(v_+ - u_-)}(x_2 - y_2)^{2(u_+ - v_+ + \frac{n}{2})}(x_1 - y_1)^{2(u_- - v_- + \frac{n}{2})}y_{12}^{2(v_- - u_+)}}$$

where

$$c = 4^{u_+ + u_- - v_+ - v_-}\pi^{-n}\frac{\Gamma(u_+ - v_+ + \frac{n}{2})\,\Gamma(u_- - v_- + \frac{n}{2})}{\Gamma(v_+ - u_+)\,\Gamma(v_- - u_-)}$$

We depict its kernel (up to a constant function of spectral parameters) in figure 2 using the graphical rules outlined above.

Coxeter relations (5.29) are basic relations which enable to establish Yang-Baxter equation for R-operator (5.32). Corresponding prove is rather straightforward and in our notations it repeats literally the one presented in [60] for the case of $\mathrm{SL}(2, \mathbb{C})$. Here we illustrate the prove in figure 3. The sequence of transformations in the picture is performed by means of the star-triangle relation.

Using R-matrix (5.32) which satisfies the Yang-Baxter equation one can construct, by using the standard method, the set of commuting operators (Hamiltonians) and formulate the corresponding quantum integrable system on the chain. Here we obtain one of these Hamiltonians and describe the integrable chain model. Consider the chain model with N sites. The states of this chain are the vectors in the space $V_{\Delta_1} \otimes \cdots \otimes V_{\Delta_N}$, where

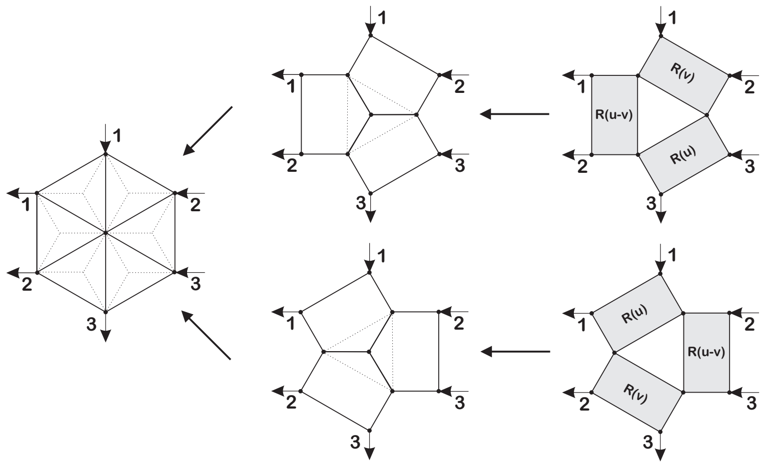

Figure 3. The proof of the Yang-Baxter equation.

each V_{Δ_a} is the vector space of the differential representation ρ of the conformal algebra $\mathsf{conf}(\mathbb{R}^n)$ (2.32). For $\Delta_a = \Delta$ $(\forall a)$ the R-matrix (5.32) is written in the form

$$\begin{aligned} \mathrm{R}_{ab}(\alpha;\xi) &:= x_{ab}^{2(\alpha+\xi)}\, \hat{p}_a^{2\alpha}\, \hat{p}_b^{2\alpha}\, x_{ab}^{2(\alpha-\xi)} = 1 + \alpha\, \mathrm{h}_{a,b}(\xi) + \alpha^2 \dots\,, \\ a, b\; &= 1, 2, \dots, N\,, \end{aligned} \tag{5.33}$$

where $\xi = \frac{n}{2} - \Delta$, parameter $\alpha = u - v$ is taken as a small one and operators

$$\begin{aligned} \mathrm{h}_{a,b}(\xi) = 2\, \ln x_{ab}^2 + x_{ab}^{2\xi}\, \ln(\hat{p}_a^2\, \hat{p}_b^2)\, x_{ab}^{-2\xi} = \\ \hat{p}_a^{2\xi} + \hat{p}_b^{-2\xi}\, \ln(x_{ab}^2)\, \hat{p}_b^{2\xi} + \ln(\hat{p}_a^2\, \hat{p}_b^2)\,, \end{aligned} \tag{5.34}$$

are interpreted for $b = a + 1$ as local Hamiltonian densities. The second expression for $\mathrm{h}_{a,b}(\xi)$ in (5.34) is deduced from the R-matrix (5.33) which is written by means of (5.29) in another form

$$\mathrm{R}_{ab}(\alpha;\xi) := \hat{p}_a^{-2\xi}\, x_{ab}^{2\alpha}\, \hat{p}_a^{2(\alpha+\xi)}\, \hat{p}_b^{2(\alpha-\xi)}\, x_{ab}^{2\alpha}\, \hat{p}_b^{2\xi}\,.$$

Then the whole Hamiltonian for the integrable chain model is given by the operator

$$H(\xi) = \sum_{a=1}^{N-1} \mathrm{h}_{a,a+1}(\xi)\,,$$

where N is the length of the chain. This operator is the high-dimensional analog of the Hamiltonian for the integrable model which was considered in [42, 43]. For $n = 1$ and special choice of $\xi = 1/2$ this operator formally reproduces (up to an additional constant) the holomorphic part of the Hamiltonian [42, 43]. The whole Hamiltonian [42, 43] is the sum of holomorphic and anti-holomorphic parts and is obtained from (5.34) for $n = 2$ and $\xi = 1$. The more general two-dimensional model was considered in [60].

Another example of the integrable lattice model based on the particular star-triangle relation (5.31) was formulated in [46].

5.2 General R-operator in the case $so(5,1)$

In the previous section we have described the general strategy for the simplest nontrivial example. Now we repeat everything step by step in more complicated situation explaining the needed modifications at each stage.

All modifications are due to the use of the more complicated representations of conformal group. The scalar representation is characterized by one parameter — scaling dimension Δ so that operator $\mathrm{L}(u)$ contains two parameters u and Δ, which are combined in a natural linear combinations $u_+ = u + \frac{\Delta - n}{2}$ and $u_- = u - \frac{\Delta}{2}$.

The tensor representation is characterized by three parameters — scaling dimension Δ and two spins ℓ, $\dot{\ell}$ and now the operator $\mathrm{L}(u)$ contains four parameters u and Δ, ℓ, $\dot{\ell}$. These parameters are combined in a pairs $\mathbf{u}_+$ and $\mathbf{u}_-$ which are analogs of u_+ and u_-

$$\mathbf{u}_+ \equiv (u_+\,,\,\ell) = \left(u + \frac{\Delta - n}{2}\,,\,\ell\right) \quad ; \quad \mathbf{u}_- \equiv (u_-\,,\,\dot{\ell}) = \left(u - \frac{\Delta}{2}\,,\,\dot{\ell}\right).$$

We have the following expression for the operator $\mathrm{L}(u)$

$$\mathrm{L}(\mathbf{u}_+\,,\,\mathbf{u}_-) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{x} & \mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} u_+ \cdot \mathbf{1} + \mathbf{S}^{(\ell)} & \mathbf{p} \\ \mathbf{0} & u_- \cdot \mathbf{1} + \overline{\mathbf{S}}^{(\dot{\ell})} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{x} & \mathbf{1} \end{pmatrix},$$

and the defining RLL-relation has the form

$$\mathrm{R}_{12}(u-v)\,\mathrm{L}_1(\mathbf{u}_+,\mathbf{u}_-)\,\mathrm{L}_2(\mathbf{v}_+,\mathbf{v}_-) = \mathrm{L}_1(\mathbf{v}_+,\mathbf{v}_-)\,\mathrm{L}_2(\mathbf{u}_+,\mathbf{u}_-)\,\mathrm{R}_{12}(u-v) \tag{5.35}$$

where (3.39)

$$\mathrm{L}_1(u_+,u_-) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{x}_1 & \mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} u_+ \cdot \mathbf{1} + \mathbf{S}_1^{(\ell_1)} & \mathbf{p}_1 \\ \mathbf{0} & u_- \cdot \mathbf{1} + \overline{\mathbf{S}}_1^{(\dot{\ell}_1)} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{x}_1 & \mathbf{1} \end{pmatrix},$$

$$\mathrm{L}_2(v_+,v_-) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{x}_2 & \mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} v_+ \cdot \mathbf{1} + \mathbf{S}_2^{(\ell_2)} & \mathbf{p}_2 \\ \mathbf{0} & v_- \cdot \mathbf{1} + \overline{\mathbf{S}}_2^{(\dot{\ell}_2)} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{x}_2 & \mathbf{1} \end{pmatrix}$$

To avoid misunderstanding we collect all parameters

$$\begin{array}{llll} u_+ = u + \dfrac{\Delta_1 - n}{2}\,, & u_- = u - \dfrac{\Delta_1}{2}\,, & v_+ = v + \dfrac{\Delta_2 - n}{2}\,, & v_- = v - \dfrac{\Delta_2}{2} \\ \mathbf{u}_+ \equiv (u_+\,,\,\ell_1)\,, & \mathbf{u}_- \equiv (u_-\,,\,\dot{\ell}_1)\,, & \mathbf{v}_+ \equiv (v_+\,,\,\ell_2)\,, & \mathbf{v}_- \equiv (v_-\,,\,\dot{\ell}_2)\,. \end{array}$$

We construct R-operator from basic building blocks $\mathrm{S}_1(\mathbf{u}), \mathrm{S}_2(\mathbf{u})$ and $\mathrm{S}_3(\mathbf{u})$ which satisfy more simple relations like (5.3), (5.4), (5.5) with substitution $(v_+, v_-, u_+, u_-) \to (\mathbf{v}_+, \mathbf{v}_-, \mathbf{u}_+, \mathbf{u}_-)$ and represent elementary transpositions in the set of four pairs of parameters $\mathbf{u} = (\mathbf{v}_+, \mathbf{v}_-, \mathbf{u}_+, \mathbf{u}_-)$.

Let us start with operators S_1 and S_3 which are two copies of the operator S defined by the equation

$$\mathrm{S}\,\mathrm{L}(\mathbf{u}_+,\mathbf{u}_-) = \mathrm{L}(\mathbf{u}_-,\mathbf{u}_+)\,\mathrm{S} \tag{5.36}$$

The exchange $\mathbf{u}_+ \leftrightarrow \mathbf{u}_-$ is equivalent to $u_+ \leftrightarrow u_-$ and $\ell \leftrightarrow \dot{\ell}$, i.e. $\Delta \leftrightarrow 4 - \Delta$ and $\ell \leftrightarrow \dot{\ell}$. Differential representation of the conformal algebra $\mathsf{conf}(\mathbb{R}^4)$ is parameterized by three numbers $\Delta,\ \ell,\ \dot{\ell}$ and we denote it by $\rho^{\Delta,\ell,\dot{\ell}}$. Thus operator S intertwines two representations[4] $\rho^{\Delta,\ell,\dot{\ell}} \sim \rho^{4-\Delta,\dot{\ell},\ell}$. As in the previous section operator S has transparent representation theory meaning.

We consider representation of the conformal algebra on tensor fields $\Phi^{\dot{\alpha}_1\cdots\dot{\alpha}_{2\dot{\ell}}}_{\alpha_1\cdots\alpha_{2\ell}}(x)$ of the type $(\ell,\dot{\ell})$ where dotted and undotted indexes are symmetric separately and where $x \in \mathbb{R}^4$. In this situation it is convenient to work with the generating functions

$$\Phi(x,\lambda,\tilde{\lambda}) = \Phi^{\dot{\alpha}_1\cdots\dot{\alpha}_{2\dot{\ell}}}_{\alpha_1\cdots\alpha_{2\ell}}(x)\,\lambda^{\alpha_1}\cdots\lambda^{\alpha_{2\ell}}\,\tilde{\lambda}_{\dot{\alpha}_1}\cdots\tilde{\lambda}_{\dot{\alpha}_{2\dot{\ell}}}\,,$$

where λ and $\tilde{\lambda}$ are auxiliary spinors. Let us introduce the convolution

$$\mathrm{F}(\lambda,\tilde{\lambda}) * \mathrm{G}(\lambda,\tilde{\lambda}) = \mathrm{F}(\partial_\lambda,\partial_{\tilde{\lambda}})\,\mathrm{G}(\lambda,\tilde{\lambda})\Big|_{\lambda=0,\tilde{\lambda}=0}$$

and use it to represent the intertwining operator as an integral operator acting on generating functions

$$[\mathrm{S}\,\Phi]\,(X) = \int \mathrm{d}^4y\,\mathrm{S}(X,Y) * \Phi(Y)$$

where we combine space-time coordinates and two spinors in one compact notation $X = (x,\lambda,\tilde{\lambda})$, $Y = (y,\eta,\tilde{\eta})$ and denote generating function by $\Phi(X)$.

The defining equation for S is equivalent to the set of differential equations for its kernel $\mathrm{S}\,(X,Y)$

$$\left(G_Y^{\Delta,\ell,\dot{\ell}}\right)^T \mathrm{S}\,(X,Y) = G_X^{4-\Delta,\dot{\ell},\ell}\,\mathrm{S}\,(X,Y)\ . \tag{5.37}$$

Here $G_X^{4-\Delta,\dot{\ell},\ell}$ denotes generators of conformal group in representation $\rho^{4-\Delta,\dot{\ell},\ell}$, $G_Y^{\Delta,\ell,\dot{\ell}}$ — generators in representation $\rho^{\Delta,\ell,\dot{\ell}}$. The generators $S_{\mu\nu}$ are taken in the form (2.61)

$$S_{\mu\nu} = \lambda\,\boldsymbol{\sigma}_{\mu\nu}\,\partial_\lambda + \tilde{\lambda}\,\overline{\boldsymbol{\sigma}}_{\mu\nu}\,\partial_{\tilde{\lambda}}\,.$$

T stands for transposition

$$\int \mathrm{d}^4y\,\mathrm{S}(X,Y) * G_Y^{\Delta,\ell,\dot{\ell}}\,\Phi(Y) = \int \mathrm{d}^4y\,\left[\left(G_Y^{\Delta,\ell,\dot{\ell}}\right)^T\,\mathrm{S}(X,Y)\right] *\,\Phi(Y)\,.$$

arising after integration by parts and using some evident properties of convolution like

$$\mathrm{F}(\lambda) * \partial_\lambda \mathrm{G}(\lambda) = \lambda\,\mathrm{F}(\lambda) * \mathrm{G}(\lambda)$$
$$\mathrm{F}(\lambda) * \lambda\,\mathrm{G}(\lambda) = \partial_\lambda \mathrm{F}(\lambda) * \mathrm{G}(\lambda)$$

After substitution of explicit expression for generators one obtains the following set of equations

- Translation

$$\left(\frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu}\right)\mathrm{S}\,(X,Y) = 0$$

[4]It is easy to see that values of the Casimir operator (2.34) coincides for these two representations.

- Lorentz rotations

$$\left[i\left(y_\nu\frac{\partial}{\partial y_\mu}-y_\mu\frac{\partial}{\partial y_\nu}\right)+\eta\,\boldsymbol{\sigma}^T_{\mu\nu}\,\partial_\eta+\tilde\eta\,\overline{\boldsymbol{\sigma}}^T_{\mu\nu}\,\partial_{\tilde\eta}\right]\mathrm{S}(X,Y)=$$

$$=\left[i\left(x_\mu\frac{\partial}{\partial x_\nu}-x_\nu\frac{\partial}{\partial x_\mu}\right)+\lambda\,\boldsymbol{\sigma}_{\mu\nu}\,\partial_\lambda+\tilde\lambda\,\overline{\boldsymbol{\sigma}}_{\mu\nu}\,\partial_{\tilde\lambda}\right]\mathrm{S}(X,Y)$$

- Dilatation

$$\left(x_\mu\frac{\partial}{\partial x_\mu}+y_\mu\frac{\partial}{\partial y_\mu}\right)\mathrm{S}\,(X,Y)=-2\,(4-\Delta)\;\mathrm{S}\,(X,Y)$$

- Conformal boosts

$$\left(-iy^2\frac{\partial}{\partial y_\mu}+2iy_\mu y_\nu\frac{\partial}{\partial y_\nu}+2y^\nu(\eta\,\boldsymbol{\sigma}^T_{\nu\mu}\,\partial_\eta+\tilde\eta\,\overline{\boldsymbol{\sigma}}^T_{\nu\mu}\,\partial_{\tilde\eta})+2i(4-\Delta)\,y_\mu\right)\mathrm{S}(X,Y)=$$

$$=\left(ix^2\frac{\partial}{\partial x_\mu}-2ix_\mu x_\nu\frac{\partial}{\partial x_\nu}+2x^\nu(\lambda\,\boldsymbol{\sigma}_{\nu\mu}\,\partial_\lambda+\tilde\lambda\,\overline{\boldsymbol{\sigma}}_{\nu\mu}\,\partial_{\tilde\lambda})-2i(4-\Delta)\,x_\mu\right)\mathrm{S}(X,Y)$$

This set of equations for the kernel of S coincides with the set of equations for a Green function for two fields of the types $(\ell,\dot\ell)$ and $(\dot\ell,\ell)$ in conformal field theory and the solution is well known [50]

$$\mathrm{S}(X,Y)=\frac{1}{(2\ell)!}\frac{1}{(2\dot\ell)!}\frac{\left(\tilde\lambda\,(\overline{\mathbf{x}-\mathbf{y}})\,\eta\right)^{2\ell}\left(\lambda\,(\mathbf{x}-\mathbf{y})\,\tilde\eta\right)^{2\dot\ell}}{(x-y)^{2(4-\Delta)}}\,.$$

In this section for simplicity we shall use compact notation

$$\mathbf{x}=\sigma_\mu\frac{x^\mu}{|x|}\;\;;\;\;\overline{\mathbf{x}}=\overline{\sigma}_\mu\frac{x^\mu}{|x|} \tag{5.38}$$

where $\frac{x^\mu}{|x|}$ is the unit vector in the direction x^μ. Formula for the kernel $\mathrm{S}(X,Y)$ leads to the following explicit expression for the action of operator S on the generating function (shadow transformation [12])

$$[\mathrm{S}\,\Phi](X)=\int\frac{\mathrm{d}^4y\;\Phi\left(y\,,\,\tilde\lambda\,\overline{(\mathbf{x}-\mathbf{y})}\,,\,\lambda\,(\mathbf{x}-\mathbf{y})\right)}{(x-y)^{2(4-\Delta)}}$$

which can be represented in more transparent form (remember that $u_- - u_+ = 2-\Delta$)

$$[\mathrm{S}(u_- - u_+)\,\Phi](X)=\int\frac{\mathrm{d}^4y}{y^{2(u_- - u_+ +2)}}\cdot\Phi(x-y\,,\,\tilde\lambda\,\overline{\mathbf{y}}\,,\,\lambda\,\mathbf{y}\,)=\int\frac{\mathrm{d}^4y\;e^{iy\hat p}}{y^{2(u_- - u_+ +2)}}\cdot\Phi\big(x\,,\,\tilde\lambda\,\overline{\mathbf{y}}\,,\,\lambda\,\mathbf{y}\,\big) \tag{5.39}$$

where $\hat p = i\partial_x$. These formulae clearly show the analogy and difference in comparison to the considered scalar case. The last formula with operator $\hat p$ is very similar to (5.20) but there exists additional action of operator S on the spinor variables λ and $\tilde\lambda$. These spinors are transformed by the matrices $\mathbf{y}$ and $\overline{\mathbf{y}}$ and after transformations are interchanged: $\lambda\to\tilde\lambda\,\overline{\mathbf{y}}\,;\,\tilde\lambda\to\lambda\,\mathbf{y}$.

The operators S_1 and S_3 act on the function $\Phi(X_1;X_2)$ in a similar manner

$$[\mathrm{S}_1(v_- - v_+)\,\Phi](X_1;X_2) = \int \frac{\mathrm{d}^4y\; e^{iy\hat{p}_2}}{y^{2(v_- - v_+ + 2)}}\;\Phi(X_1;x_2\,,\,\tilde{\lambda}_2\,\bar{\mathbf{y}}\,,\,\lambda_2\,\mathbf{y})$$

$$[\mathrm{S}_3(u_- - u_+)\,\Phi](X_1;X_2) = \int \frac{\mathrm{d}^4y\; e^{iy\hat{p}_1}}{y^{2(u_- - u_+ + 2)}}\;\Phi(x_1\,,\,\tilde{\lambda}_1\,\bar{\mathbf{y}}\,,\,\lambda_1\,\mathbf{y}\,;X_2) \tag{5.40}$$

In order to construct operator S_2 we take into account the same observation as in a scalar case: it can be produced directly from the operator S using duality transformation

$$y \to p\;;\;p \to x_2 - x_1 \equiv x_{21}\;;\;\mathbf{u}_+ \to \mathbf{v}_-\;;\;\mathbf{u}_- \to \mathbf{u}_+\,,$$

The change $\mathbf{u}_+ \to \mathbf{v}_-\;;\;\mathbf{u}_- \to \mathbf{u}_+$ implies the corresponding change of spinors so that the expression for the action of operator S_2 on the generating function $\Phi\big(X_1\,;\,X_2\big)$ is

$$[\mathrm{S}_2(u_+ - v_-)\,\Phi]\big(X_1\,;\,X_2\big) = \int \frac{\mathrm{d}^4p\; e^{ip\,x_{21}}}{p^{2(u_+ - v_- + 2)}}\,\Phi\big(x_1\,,\,\tilde{\lambda}_2\,\bar{\mathbf{p}}\,,\,\tilde{\lambda}_1\,;\,x_2\,,\,\lambda_2\,,\,\lambda_1\,\mathbf{p}\big)\,. \tag{5.41}$$

In the case of scalars there is no dependence on λ_1,λ_2 and $\tilde{\lambda}_1,\tilde{\lambda}_2$ so that integral over p can be calculated explicitly and operator S_2 reduces to the operator of the multiplication to the function $x_{12}^{v_- - u_+}$.

The proof of the duality rules is almost the same as in a scalar case. We rewrite the defining equation (5.36) for operator S in the factorized form (3.39)

$$\mathrm{S}\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{x} & \mathbf{1}\end{pmatrix}\begin{pmatrix}u_+\cdot\mathbf{1}+\mathbf{S}^{(\ell)} & \mathbf{p}\\ \mathbf{0} & u_-\cdot\mathbf{1}+\bar{\mathbf{S}}^{(\dot{\ell})}\end{pmatrix}\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ -\mathbf{x} & \mathbf{1}\end{pmatrix} = \tag{5.42}$$

$$= \begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{x} & \mathbf{1}\end{pmatrix}\begin{pmatrix}u_-\cdot\mathbf{1}+\mathbf{S}^{(\dot{\ell})} & \mathbf{p}\\ \mathbf{0} & u_+\cdot\mathbf{1}+\bar{\mathbf{S}}^{(\ell)}\end{pmatrix}\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ -\mathbf{x} & \mathbf{1}\end{pmatrix}\mathrm{S}\,.$$

Using the same argumentation as in the previous section one can easily see that the defining equation for operator S_2

$$\mathrm{S}_2\,\mathrm{L}_1(\underline{\mathbf{u}_+},\mathbf{u}_-)\,\mathrm{L}_2(\mathbf{v}_+,\underline{\mathbf{v}_-}) = \mathrm{L}_1(\underline{\mathbf{v}_-},\mathbf{u}_-)\,\mathrm{L}_2(\mathbf{v}_+,\underline{\mathbf{u}_+})\,\mathrm{S}_2$$

can be transformed to the form

$$\mathrm{S}_2\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{p}_1 & \mathbf{1}\end{pmatrix}\begin{pmatrix}v_-\cdot\mathbf{1}+\mathbf{S}_2^{(\dot{\ell}_2)} & \mathbf{x}_{21}\\ \mathbf{0} & u_+\cdot\mathbf{1}+\bar{\mathbf{S}}_1^{(\ell_1)}\end{pmatrix}\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ -\mathbf{p}_1 & \mathbf{1}\end{pmatrix} = \tag{5.43}$$

$$= \begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{p}_1 & \mathbf{1}\end{pmatrix}\begin{pmatrix}u_+\cdot\mathbf{1}+\mathbf{S}_2^{(\ell_1)} & \mathbf{x}_{21}\\ \mathbf{0} & v_-\cdot\mathbf{1}+\bar{\mathbf{S}}_1^{(\dot{\ell}_2)}\end{pmatrix}\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ -\mathbf{p}_1 & \mathbf{1}\end{pmatrix}\mathrm{S}_2\,,$$

if we require that

$$[\mathrm{S}_2,\mathbf{x}_1] = [\mathrm{S}_2,\mathbf{x}_2] = [\mathrm{S}_2,\mathbf{u}_-] = [\mathrm{S}_2,\mathbf{v}_+] = 0\,.$$

Comparing equations (5.42) and (5.43) we conclude that the change

$$\mathbf{x} \to \mathbf{p}_1\;;\;\mathbf{p} \to \mathbf{x}_{21}\;;\;\mathbf{u}_+ \to \mathbf{v}_-\;;\;\mathbf{u}_- \to \mathbf{u}_+ \tag{5.44}$$

transforms (5.42) into (5.43). Using the formula (5.39) with momentum $\hat{p}$ and duality rules (5.44) we obtain the expression (5.41) for the action of operator $\mathrm{S}_2(\mathbf{u})$ on the generating function $\Phi\big(X_1\,;\,X_2\big)$.

Thereby we have constructed operator representation of elementary transpositions s_1, s_2, s_3:

$$s_1\mathbf{u} = (\underline{\mathbf{v}_-}, \underline{\mathbf{v}_+}, \mathbf{u}_+, \mathbf{u}_-)\ ;\ s_2\mathbf{u} = (\mathbf{v}_+, \underline{\mathbf{u}_+}, \underline{\mathbf{v}_-}, \mathbf{u}_-)\ ;\ s_3\mathbf{u} = (\mathbf{v}_+, \mathbf{v}_-, \underline{\mathbf{u}_-}, \underline{\mathbf{u}_+})$$

following the line of the previous section. The corresponding Coxeter relations have the more complicated form in comparison to the scalar case. The first triple relation

$$\mathrm{S}_1(a)\,\mathrm{S}_2(a+b)\,\mathrm{S}_1(b) = \mathrm{S}_2(b)\,\mathrm{S}_1(a+b)\,\mathrm{S}_2(a)$$

in explicit form looks as follows

$$\int \frac{\mathrm{d}^4z\,\mathrm{d}^4k\,\mathrm{d}^4y\ e^{i\,z\,\hat{p}_2}e^{i\,k\,x_{21}}e^{i\,y\,\hat{p}_2}}{z^{2(a+2)}k^{2(a+b+2)}y^{2(b+2)}}\cdot\Phi(x_1\,,\,\lambda_2\,\mathbf{z}\,\overline{\mathbf{k}}\,,\,\tilde{\lambda}_1\,;\,x_2\,,\,\lambda_1\,\mathbf{k}\,\overline{\mathbf{y}}\,,\,\tilde{\lambda}_2\,\overline{\mathbf{z}}\,\mathbf{y}\,) =$$

$$= \int \frac{\mathrm{d}^4q\,\mathrm{d}^4y\,\mathrm{d}^4k\ e^{i\,q\,x_{21}}e^{i\,y\,\hat{p}_2}e^{i\,k\,x_{21}}}{q^{2(b+2)}y^{2(a+b+2)}k^{2(a+2)}}\cdot\Phi(x_1\,,\,\lambda_2\,\mathbf{y}\,\overline{\mathbf{k}}\,,\,\tilde{\lambda}_1\,;\,x_2\,,\,\lambda_1\,\mathbf{q}\,\overline{\mathbf{y}}\,,\,\tilde{\lambda}_2\overline{\mathbf{q}}\,\mathbf{k}\,)\,, \qquad (5.45)$$

and the second triple relation

$$\mathrm{S}_3(a)\,\mathrm{S}_2(a+b)\,\mathrm{S}_3(b) = \mathrm{S}_2(b)\,\mathrm{S}_3(a+b)\,\mathrm{S}_2(a)$$

is equivalent to the similar integral relation

$$\int \frac{\mathrm{d}^4z\,\mathrm{d}^4k\,\mathrm{d}^4y\,e^{i\,z\,\hat{p}_1}e^{i\,k\,x_{21}}e^{i\,y\,\hat{p}_1}}{z^{2(a+2)}k^{2(a+b+2)}y^{2(b+2)}}\,\Phi(x_1\,,\,\lambda_1\,\mathbf{z}\,\overline{\mathbf{y}}\,,\,\tilde{\lambda}_2\,\overline{\mathbf{k}}\,\mathbf{y}\,;\,x_2\,,\,\lambda_2\,,\,\tilde{\lambda}_1\,\overline{\mathbf{z}}\,\mathbf{k}\,) =$$

$$= \int \frac{\mathrm{d}^4q\,\mathrm{d}^4y\,\mathrm{d}^4k\,e^{i\,q\,x_{21}}e^{i\,y\,\hat{p}_1}e^{i\,k\,x_{21}}}{q^{2(b+2)}y^{2(a+b+2)}k^{2(a+2)}}\,\Phi(x_1\,,\,\lambda_1\,\mathbf{q}\,\overline{\mathbf{k}}\,,\,\tilde{\lambda}_2\,\overline{\mathbf{q}}\,\mathbf{y}\,;\,x_2\,,\,\lambda_2\,,\,\tilde{\lambda}_1\,\overline{\mathbf{y}}\,\mathbf{k}\,). \qquad (5.46)$$

These relations are equivalent to the following generalization of the scalar star-triangle relation (5.30)

$$\frac{\hat{p}^{\mu_1}\cdots\hat{p}^{\mu_m}}{\hat{p}^{\,2(a+m)}}\,\frac{\mathrm{A}_{\mu_1\nu_1}\cdots\mathrm{A}_{\mu_m\nu_m}}{x^{2(a+b+m)}}\,\frac{\hat{p}^{\nu_1}\cdots\hat{p}^{\nu_m}}{\hat{p}^{\,2(b+m)}} = \frac{x^{\mu_1}\cdots x^{\mu_m}}{x^{\,2(b+m)}}\,\frac{\mathrm{A}_{\mu_1\nu_1}\cdots\mathrm{A}_{\mu_m\nu_m}}{\hat{p}^{2(a+b+m)}}\,\frac{x^{\nu_1}\cdots x^{\nu_m}}{x^{\,2(a+m)}} \qquad (5.47)$$

where $m = 0,1,2,\cdots$ and the matrix A respects properties $\mathrm{A}_{\mu\nu}\,\mathrm{A}^{\mu}_{\ \lambda} = 0\ ;\ \mathrm{A}_{\nu\mu}\,\mathrm{A}_{\lambda}^{\ \mu} = 0$. The equivalence of relations (5.45), (5.46) and (5.47) and the validity of relation (5.47) are proven in appendix.

Coxeter relations (5.45) and (5.46) are basic relations which enable to prove Yang-Baxter equation for R-operator constructed from the basic building blocks

$$[\mathrm{R}_{12}\,\Phi](X_1;X_2) = \int \frac{\mathrm{d}^4q\,\mathrm{d}^4k\,\mathrm{d}^4y\,\mathrm{d}^4z\ \ e^{i\,(q+k)\,x_{21}}\ e^{i\,k\,(y-z)}}{q^{2(u_--v_++2)}z^{2(u_+-v_++2)}y^{2(u_--v_-+2)}k^{2(u_+-v_-+2)}}\cdot \\ \cdot\,\Phi(x_1-y\,,\,\lambda_2\,\mathbf{z}\,\overline{\mathbf{k}}\,,\,\tilde{\lambda}_2\,\overline{\mathbf{q}}\,\mathbf{y}\,;\,x_2-z\,,\,\lambda_1\,\mathbf{q}\,\overline{\mathbf{z}}\,,\,\tilde{\lambda}_1\,\overline{\mathbf{y}}\,\mathbf{k}\,)\,. \qquad (5.48)$$

To conclude this subsection we would like to stress that in the case of the conformal algebra $so(5,1)$ of 4-dimensional Euclidean space we proved the new star-triangle relations (5.45) and (5.46) for generic representations of the type $\rho_{\Delta,\ell,\dot{\ell}}$ included spin degrees of freedom, i.e. we generalize the scalar star-triangle relation to the star-triangle relation for the propagators of particles with any spin.[5] It seems that the integrable models of the type [42, 43, 60] or [46] related to the spinorial R-matrix (5.48) and spinorial star-triangle relations (5.45) and (5.46) are not known.

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A Direct proof of the star-triangle relation

In this appendix we prove identities (5.45) and (5.46) which are the corner stone of our construction. The first relation (5.45) can be rewritten as an operator identity

$$\int \frac{\mathrm{d}^4 z\, \mathrm{d}^4 k\, \mathrm{d}^4 y\ (\lambda_2\, \mathbf{z}\, \overline{\mathbf{k}}\, \eta_1)^{2\ell_1} (\lambda_1\, \mathbf{k}\, \overline{\mathbf{y}}\, \eta_2)^{2\ell_2} (\tilde{\lambda}_2\, \overline{\mathbf{z}}\, \mathbf{y}\, \tilde{\eta}_2)^{2\dot{\ell}_2}}{z^{2(a+2)} k^{2(a+b+2)} y^{2(b+2)}}\, e^{i\, z\, \hat{p}_2} e^{i\, k\, x_{21}} e^{i\, y\, \hat{p}_2} =$$

$$= \int \frac{\mathrm{d}^4 q\, \mathrm{d}^4 y\, \mathrm{d}^4 k\ (\lambda_2\, \mathbf{y}\, \overline{\mathbf{k}}\, \eta_1)^{2\ell_1} (\lambda_1\, \mathbf{q}\, \overline{\mathbf{y}}\, \eta_2)^{2\ell_2} (\tilde{\lambda}_2\, \overline{\mathbf{q}}\, \mathbf{k}\, \tilde{\eta}_2)^{2\dot{\ell}_2}}{q^{2(b+2)} y^{2(a+b+2)} k^{2(a+2)}}\, e^{i\, q\, x_{21}} e^{i\, y\, \hat{p}_2} e^{i\, k\, x_{21}} \qquad \text{(A.1)}$$

and the second one (5.46) as

$$\int \frac{\mathrm{d}^4 z\, \mathrm{d}^4 k\, \mathrm{d}^4 y\, (\lambda_1\, \mathbf{z}\, \overline{\mathbf{y}}\, \eta_1)^{2\ell_1} (\tilde{\lambda}_2\, \overline{\mathbf{k}}\, \mathbf{y}\, \tilde{\eta}_1)^{2\dot{\ell}_1} (\tilde{\lambda}_1\, \overline{\mathbf{z}}\, \mathbf{k}\, \tilde{\eta}_2)^{2\dot{\ell}_2}}{z^{2(a+2)} k^{2(a+b+2)} y^{2(b+2)}}\, e^{i\, z\, \hat{p}_1} e^{i\, k\, x_{21}} e^{i\, y\, \hat{p}_1} =$$

$$= \int \frac{\mathrm{d}^4 q\, \mathrm{d}^4 y\, \mathrm{d}^4 k\, (\lambda_1\, \mathbf{q}\, \overline{\mathbf{k}}\, \eta_1)^{2\ell_1} (\tilde{\lambda}_2\, \overline{\mathbf{q}}\, \mathbf{y}\, \tilde{\eta}_1)^{2\dot{\ell}_1} (\tilde{\lambda}_1\, \overline{\mathbf{y}}\, \mathbf{k}\, \tilde{\eta}_2)^{2\dot{\ell}_2}}{q^{2(b+2)} y^{2(a+b+2)} k^{2(a+2)}}\, e^{i\, q\, x_{21}} e^{i\, y\, \hat{p}_1} e^{i\, k\, x_{21}} \qquad \text{(A.2)}$$

where we use compact notations (5.38). In a particular case $\ell_1 = \ell_2 = \dot{\ell}_1 = \dot{\ell}_2 = 0$ (i.e. scalar one) spinor variables disappear from (A.1) and (A.2) and corresponding integrals can be easily evaluated. Therefore these identities reduce to (5.29).

[5]Other generalizations of the scalar star-triangle relation and special star-triangle identities which include γ-matrices and propagators of spin particles were also considered in [51, 62] (see eq. (27)) and [64].

Both previous relations are equivalent to the following generating integral identity

$$\int \mathrm{d}^4 z\, \mathrm{d}^4 k\, \frac{\langle (x-z)\,\mathrm{A}\,(y-z)\rangle^m\, \mathrm{e}^{-i\,\langle k\left[z-\alpha\,\mathrm{B}(y-z)-\beta\,\mathrm{C}(x-z)\right]\rangle}}{(x-z)^{2a}\, k^{2b}\,(y-z)^{2c}} =$$
$$= \frac{1}{(x-y)^{2b}} \int \mathrm{d}^4 k\, \mathrm{d}^4 p\, \frac{\langle p\,\mathrm{A} k\rangle^m\, \mathrm{e}^{i\,\langle p\left[x+\alpha\,\mathrm{B}(x-y)\right]\rangle}\, \mathrm{e}^{-i\,\langle k\left[y+\beta\,\mathrm{C}(y-x)\right]\rangle}}{k^{2a}\, p^{2c}} \tag{A.3}$$

provided that parameters respect uniqueness condition

$$a + c - b = 2 + m\,,\ m = 0, 1, 2, \cdots \tag{A.4}$$

Here α, β are numerical parameters, matrices A , B , C fulfil the following properties

$$\begin{array}{ccc} A_{\mu\nu}A^{\mu}{}_{\lambda} = A_{\nu\mu}A_{\lambda}{}^{\mu} = 0\,; & B_{\mu\nu}B^{\mu}{}_{\lambda} = B_{\nu\mu}B_{\lambda}{}^{\mu} = 0\,; & C_{\mu\nu}C^{\mu}{}_{\lambda} = C_{\nu\mu}C_{\lambda}{}^{\mu} = 0\,; \\ A_{\mu\nu} + A_{\nu\mu} = 2\,g_{\mu\nu}\,\mathrm{tr}\,A\,; & B_{\mu\nu}+B_{\nu\mu} = 2\,g_{\mu\nu}\,\mathrm{tr}\,B\,; & C_{\mu\nu}+C_{\nu\mu} = 2\,g_{\mu\nu}\,\mathrm{tr}\,C\,; \\ \mathrm{B}_{\mu\nu}\mathrm{C}_{\lambda}{}^{\nu}+\mathrm{B}_{\lambda\nu}\mathrm{C}_{\mu}{}^{\nu} = 2\,g_{\mu\lambda}\,\mathrm{tr}\,BC\,; & \mathrm{A}_{\mu\nu}\mathrm{B}_{\lambda}{}^{\nu} = \mathrm{B}_{\nu\mu}\mathrm{A}^{\nu}{}_{\lambda}\,; & \mathrm{A}_{\mu\nu}\mathrm{C}_{\lambda}{}^{\nu} = \mathrm{C}_{\mu\nu}\mathrm{A}_{\lambda}{}^{\nu} \end{array} \tag{A.5}$$

where we normalize the trace such that $\mathrm{tr}(\mathbb{1}) = 1$. We also use shortcut notations $\langle x\,\mathrm{M}\,y\rangle = x_\mu\, y_\nu\, \mathrm{M}_{\mu\nu}$.

In order to obtain (A.1) we have to take in (A.3)

$$\left(\tilde\lambda_2\,\overline{\boldsymbol\sigma}_\mu\,\boldsymbol\sigma_\nu\,\tilde\eta_2\right) = \mathrm{A}_{\mu\nu}\,;\ (\lambda_1\,\boldsymbol\sigma_\mu\,\overline{\boldsymbol\sigma}_\nu\,\eta_2) = \mathrm{B}_{\mu\nu}\,;\ (\lambda_2\,\boldsymbol\sigma_\nu\,\overline{\boldsymbol\sigma}_\mu\,\eta_1) = \mathrm{C}_{\mu\nu}\,;\ m = 2\dot\ell_2$$

and apply $\partial_\alpha^{2\ell_2}\partial_\beta^{2\ell_1}|_{\alpha=\beta=0}$. To obtain (A.2) we take in (A.3)

$$(\lambda_1\,\boldsymbol\sigma_\mu\,\overline{\boldsymbol\sigma}_\nu\,\eta_1) = \mathrm{A}_{\mu\nu}\,;\ \left(\tilde\lambda_2\,\overline{\boldsymbol\sigma}_\mu\,\boldsymbol\sigma_\nu\,\tilde\eta_1\right) = \mathrm{B}_{\mu\nu}\,;\ \left(\tilde\lambda_1\,\overline{\boldsymbol\sigma}_\nu\,\boldsymbol\sigma_\mu\,\tilde\eta_2\right) = \mathrm{C}_{\mu\nu}\,;\ m = 2\ell_1$$

and apply $\partial_\alpha^{2\dot\ell_1}\partial_\beta^{2\dot\ell_2}|_{\alpha=\beta=0}$. Using (2.17) and Fierz identity $\sigma_\mu\otimes\overline{\sigma}^\mu = 2\,\mathrm{P}$ it is easy to check that previous expressions for matrices A , B , C fulfil relations (A.5).

Thus our aim is to prove (A.3) that we will perform in two steps. On the first step we implement ceratin change of variables which enables to remove matrices B and C from (A.3) obtaining an integral relation equivalent to the operator identity (5.47). On the second step we prove (5.47).

Let us consider the integral in the right hand side of (A.3)

$$\int \mathrm{d}^4 k\, \mathrm{d}^4 p\, \frac{\langle p\,\mathrm{A} k\rangle^m\, \mathrm{e}^{i\,\langle p\left[x+\alpha\,\mathrm{B}(x-y)\right]\rangle}\, \mathrm{e}^{-i\,\langle k\left[y+\beta\,\mathrm{C}(y-x)\right]\rangle}}{k^{2a}\, p^{2c}} =$$
$$= \langle \partial_{w_1}\mathrm{A}\,\partial_{w_2}\rangle^m \int \mathrm{d}^4 k\, \mathrm{d}^4 p\, \left.\frac{\mathrm{e}^{i\,\langle p\left[x+w_1+\alpha\,\mathrm{B}(x-y)\right]\rangle}\, \mathrm{e}^{-i\,\langle k\left[y+w_2+\beta\,\mathrm{C}(y-x)\right]\rangle}}{k^{2a}\, p^{2c}}\right|_{w_1=w_2=0} =$$

then we implement Fourier transform (here $a' \equiv 2-a$, $b' \equiv 2-b$, $c' \equiv 2-c$)

$$= 4^{a'+c'}\pi^4 \frac{\Gamma(a')\Gamma(c')}{\Gamma(a)\Gamma(c)} \langle\partial_{w_1}\mathrm{A}\partial_{w_2}\rangle^m \left.\frac{1}{\left[x+w_1+\alpha\mathrm{B}(x-y)\right]^{2c'}\left[y+w_2+\beta\mathrm{C}(y-x)\right]^{2a'}}\right|_{w_1=w_2=0} =$$

and perform differentiation

$$= 4^{a'+c'+m}\pi^4 \frac{\Gamma(a'+m)\,\Gamma(c'+m)}{\Gamma(a)\,\Gamma(c)} \frac{\langle [x+\alpha\,\mathrm{B}(x-y)]\,\mathrm{A}\,[y+\beta\,\mathrm{C}(y-x)]\rangle^m}{\big[x+\alpha\,\mathrm{B}(x-y)\big]^{2(c'+m)}\big[y+\beta\,\mathrm{C}(y-x)\big]^{2(a'+m)}}\,.$$

Further we introduce new variables which absorb matrices B and C

$$X \equiv x+\alpha\,\mathrm{B}(x-y)\;;\; Y \equiv y+\beta\,\mathrm{C}(y-x)\,. \tag{A.6}$$

Then $X-Y = \mathrm{S}\cdot(x-y)$ where

$$\mathrm{S} \equiv \mathbb{1}+\alpha\,\mathrm{B}+\beta\,\mathrm{C}\,. \tag{A.7}$$

Using properties (A.5) one can easily obtain that

$$\mathrm{S}\cdot\mathrm{S}^T = \lambda\cdot\mathbb{1}\,;\; \lambda \equiv 1+2\alpha\,\mathrm{tr}\,B+2\beta\,\mathrm{tr}\,C+2\alpha\beta\,\mathrm{tr}\,BC \tag{A.8}$$

therefore

$$(X-Y)^2 = \lambda\cdot(x-y)^2$$

and the right hand side of (A.3) takes the form

$$4^{b'}\pi^4\lambda^b\,\frac{\Gamma(a'+m)\,\Gamma(c'+m)}{\Gamma(a)\,\Gamma(c)}\,\frac{\langle XAY\rangle^m}{X^{2(c'+m)}(X-Y)^{2b}Y^{2(a'+m)}}\,. \tag{A.9}$$

Then we consider the left hand side of (A.3) where we shift the integration variable

$$\int \mathrm{d}^4z\,\mathrm{d}^4k\,\frac{\langle (z-x+y)\,\mathrm{A}\,z\rangle^m\,\mathrm{e}^{-i\,\langle k\,\left[z+y+\alpha\,\mathrm{B}z+\beta\,\mathrm{C}(z-x+y)\right]\rangle}}{(x-y-z)^{2a}\,k^{2b}\,z^{2c}} =$$

and perform Fourier transform

$$= 4^{b'}\pi^2\frac{\Gamma(b')}{\Gamma(b)}\int \mathrm{d}^4z\,\frac{\langle (z-x+y)\,\mathrm{A}\,z\rangle^m}{(x-y-z)^{2a}\,\big[z+y+\alpha\,\mathrm{B}z+\beta\,\mathrm{C}(z-x+y)\big]^{2b'}\,z^{2c}}\,.$$

Further we change the integration variables $Z = \mathrm{S}\cdot z$ (A.7) in the previous integral and introduce variables (A.6) instead of x and y. Let us note that $X-Y+Z = \mathrm{S}\cdot(x-y+z)$, consequently due to (A.8)

$$Z^2 = \lambda\cdot z^2\,;\,(X-Y-Z)^2 = \lambda\cdot(x-y-z)^2\,.$$

From (A.8) it follows that the Jacobian of the linear change is equal to $|\det\mathrm{S}| = \lambda^2$. Using (A.5) it is possible to deduce that $\mathrm{S}\cdot\mathrm{A}\cdot\mathrm{S}^T = \lambda\cdot\mathrm{A}$, thus

$$(z-x+y)\,\mathrm{A}\,z = (Z-X+Y)\,\mathrm{S}^{-1T}\cdot\mathrm{A}\cdot\mathrm{S}^{-1}\,Z = \frac{1}{\lambda}\cdot(Z-X+Y)\,\mathrm{A}\,Z\,.$$

Therefore the left-hand side of (A.3) takes the form

$$4^{b'}\pi^2\frac{\Gamma(b')}{\Gamma(b)}\,\lambda^{a+c-2-m}\int \mathrm{d}^4Z\,\frac{\langle (Z-X+Y)\,\mathrm{A}\,Z\rangle^m}{(X-Y-Z)^{2a}(Z+Y)^{2b'}Z^{2c}} \tag{A.10}$$

Finally equating (A.9) with (A.10), performing a shift of integration variable in the later and taking into account uniqueness condition (A.4) we obtain that (A.3) is equivalent to

$$\int \mathrm{d}^4 Z \frac{\langle (Z-X)\, \mathrm{A}\, (Z-Y)\rangle^m}{(X-Z)^{2a} Z^{2b'} (Z-Y)^{2c}} = \pi^2 \frac{\Gamma(a'+m)\, \Gamma(b)\, \Gamma(c'+m)}{\Gamma(a)\, \Gamma(b')\, \Gamma(c)} \frac{\langle XAY\rangle^m}{X^{2(c'+m)} (X-Y)^{2b} Y^{2(a'+m)}} \tag{A.11}$$

where

$$a + c - b = 2 + m\,.$$

(A.11) is an integral form of the operator identity (5.47) in the same way as (5.31) is an integral form of the scalar star-triangle identity (5.30).

Now we are going to prove (A.11). At first let us evaluate the integral

$$I(x,y) \equiv \int \mathrm{d}^4 z \, \frac{\langle (z-x)\, \mathrm{A}\, z\rangle^m}{(x-z)^{2a} (z-y)^{2b'}\, z^{2c}}\,.$$

by means of inversion transform

$$x \to \tfrac{x}{x^2}\ ;\ y \to \tfrac{y}{y^2}\ ;\ z \to \tfrac{z}{z^2}\ ;\ \mathrm{d}^4 z \to \tfrac{\mathrm{d}^4 z}{z^8}\ ;\ (x-z)^2 \to \tfrac{(x-z)^2}{x^2\, z^2}\ ;\ (z-y)^2 \to \tfrac{(z-y)^2}{z^2\, y^2}$$
$$\langle (z-x)\, \mathrm{A}\, z\rangle \to \langle \left(\tfrac{z}{z^2} - \tfrac{x}{x^2}\right)\, \mathrm{A}\, \tfrac{z}{z^2}\rangle = \tfrac{\langle x\, \mathrm{A}\, (x-z)\rangle}{x^2 z^2}$$

In the previous transformation we take into account (A.5). Then due to uniqueness condition (A.4)

$$I\left(\frac{x}{x^2}, \frac{y}{y^2}\right) = x^{2a} y^{2b'} \int \mathrm{d}^4 z \, \frac{\langle \frac{x}{x^2}\, \mathrm{A}\, (x-z)\rangle^m}{(x-z)^{2a} (z-y)^{2b'}}\,.$$

In order to evaluate the previous integral we take into account a well-known formula for convolution of two "propagators"

$$\int \mathrm{d}^4 z \, \frac{1}{(x-w-z)^{2(a-m)} (z-y)^{2b'}} = \pi^2 \frac{\Gamma(a'+m)\, \Gamma(b)\, \Gamma(c')}{\Gamma(a-m)\, \Gamma(b')\, \Gamma(c)} \frac{1}{(x-y-w)^{2c'}}$$

and apply to it $\langle \frac{x}{x^2}\, \mathrm{A}\, \partial_w\rangle^m\big|_{w=0}$. Thus we have

$$I\left(\frac{x}{x^2}, \frac{y}{y^2}\right) = \pi^2 \frac{\Gamma(a'+m)\, \Gamma(b)\, \Gamma(c'+m)}{\Gamma(a)\, \Gamma(b')\, \Gamma(c)}\, x^{2a} y^{2b'} \frac{\langle \frac{x}{x^2}\, \mathrm{A}\, (x-y)\rangle^m}{(x-y)^{2(c'+m)}} \tag{A.12}$$

To obtain $I(x,y)$ we perform inverse transform

$$x \to \frac{x}{x^2}\ ;\ y \to \frac{y}{y^2}\ ;\ (x-y)^2 \to \frac{(x-y)^2}{x^2\, y^2}\ ;\ \langle \frac{x}{x^2}\, \mathrm{A}\, (x-y)\rangle \to \langle (y-x)\, \mathrm{A}\, \frac{y}{y^2}\rangle$$

in (A.12)

$$I(x,y) = \pi^2 \frac{\Gamma(a'+m)\, \Gamma(b)\, \Gamma(c'+m)}{\Gamma(a)\, \Gamma(b')\, \Gamma(c)} \frac{\langle (y-x)\, \mathrm{A}\, y\rangle^m}{(y-x)^{2(c'+m)}\, x^{2b}\, y^{2(a'+m)}}\,. \tag{A.13}$$

Finally we note that (A.11) coincides with (A.13) at $x \to X - Y$, $y \to -Y$.

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