Anomaly/transport in an Ideal Weyl gas

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ABSTRACT: We study some of the transport processes which are specific to an ideal gas of relativistic Weyl fermions and relate the corresponding transport coefficients to various anomaly coefficients of the system. We propose that these transport processes can be thought of as arising from the continuous injection of chiral states and their subsequent adiabatic flow driven by vorticity. This in turn leads to an elegant expression relating the anomaly induced transport coefficients to the anomaly polynomial of the Ideal Weyl gas.

Contents

1   Introduction

Anomalies are arguably among the most interesting phenomena to come out of studies of quantum matter. Their importance lies in their robustness across various length/energy scales — as one passes from one description of matter into another, anomaly matching ala 't Hooft ensures that the underlying anomalies of a theory survive in various disguises.

While this statement is relatively better understood within the realm of effective theories (as exemplified by the phenomenology of WZW term in particle physics and solid-state physics), we have only a limited understanding of the role of anomalies in various finite temperature/finite chemical potential setups. Any progress in the phenomenology of anomalies is welcome especially given the important role of quantum anomalies and their associated transport phenomena in fields ranging from solid-state physics to cosmology.

To be more precise, we are concerned with the following situation: consider a quantum system with a continuous symmetry[1] which via Noether theorem corresponds to conserved Noether currents. Consider gauging this symmetry by introducing a set of external non-dynamical gauge fields. We will say the quantum system has an anomaly if in the presence of such non-dynamical gauge fields the covariant Noether currents are no more conserved. One can now consider instead turning on temperature/chemical potential for the Noether currents and ask what novel processes are characteristic of a quantum system with underlying anomalies.

By now, such transport processes have been studied from various points of view — they are known to be constrained by thermodynamics/adiabaticity in arbitrary dimensions [1–4], the corresponding transport coefficients can be derived via a Kubo-like formula in $4d$ [5]. They are an established feature in various holographic fluid phases in CFTs dual to AdS_3 [6] and AdS_5 [7–11] where the CFT anomalies are in one to one correspondence with the Chern-Simons terms in the AdS bulk. The effect of Chern-Simons term for a U(1) gauge field in arbitrary AdS_{2n+1}/CFT_{2n} was worked out in [3] — no analogous results are known for gravitational Chern-Simons terms in higher AdS spacetimes.[2]

In 2d field theories, the relation between U(1) anomaly and transport is one the foundations of modern theories of Hall effects and there is an analogous relation between thermal transport and gravitational anomaly (see for example [14]). There are by now various ways in which such transport processes in 4d free fermion theories have been derived — some of them quite old [15–20] and others more recent [21–23]. Further, since 4d Weyl fermions are systems with Berry phases, these transport processes have close links to the general theory of Berry phases and transport [24, 25] and in particular transport in Weyl semi-metals [26, 27]. The transport processes linked to anomalies have also made their appearance in discussions about classification of topological insulators [28, 29].

At present, the most general set of results were derived via thermodynamic arguments employing adiabaticity [3, 4]. These results can be summarized as follows:[3] In a fluid the energy, charge and entropy transport are given by

$$
\begin{aligned}
T^{\mu\nu} &\equiv \varepsilon u^\mu u^\nu + pP^{\mu\nu} + q_{\text{anom}}^\mu u^\nu + u^\mu q_{\text{anom}}^\nu + T_{\text{diss}}^{\mu\nu} \\
J^\mu &\equiv nu^\mu + J_{\text{anom}}^\mu + J_{\text{diss}}^\mu \\
J_S^\mu &\equiv su^\mu + J_{S,\text{anom}}^\mu + J_{S,\text{diss}}^\mu
\end{aligned}
\tag{1.1}
$$

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[1]To simplify our discussion, we will assume this global symmetry is not spontaneously broken - in other words, we are interested in transport processes in normal fluid not superfluids. Though various ideas that we discuss in the article have their counterparts in superfluids, we believe the phenomenology of anomaly-induced transport in superfluids is sufficiently different to merit a separate discussion. There is by now a vast literature on such transport phenomena which are beset with their own subtleties. Since it would be too tedious/distracting to compare and contrast the effect of anomalies in the two situations, we will choose entirely focus on normal fluids in what follows.

[2]It would be interesting to construct and study rotating solutions in AdS_{2n+1} with pure/mixed gravitational Chern-Simons terms and link the proposed modification of Wald entropy [12, 13] against the anomaly-induced entropy transport in the dual CFT.

[3]We will use the notations of [4] in the following. See section 2 and the appendix D for a discussion of the basic setup and notations.

where u^μ is the velocity of the fluid under consideration which obeys $u^\mu u_\mu = -1$ when contracted using the spacetime metric $g_{\mu\nu}$. Further, $P^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$, pressure of the fluid is p and $\{\varepsilon, n, s\}$ are the energy,charge and the entropy densities respectively. We have denoted by $\{q^\mu_{\text{anom}}, J^\mu_{\text{anom}}, J^\mu_{S,\text{anom}}\}$ the anomalous heat/charge/entropy currents and by $\{T^{\mu\nu}_{\text{diss}}, J^\mu_{\text{diss}}, J^\mu_{S,\text{diss}}\}$ the dissipative currents.

We are primarily interested in the anomalous currents in what follows. It is convenient to work with forms — let $\{\bar{q}_{\text{anom}}, \bar{J}_{\text{anom}}, \bar{J}_{S,\text{anom}}\}$ be the Hodge duals of the corresponding currents.[4] Using adiabaticity the following statements can be made in flat spacetime [4]

1. All these currents are derivable from a single Gibbs current $\bar{\mathcal{G}}_{\text{anom}}$ which describes the transport of Gibbs free energy $G \equiv E - TS - \mu Q$ in the fluid. We have the following thermodynamic relations

$$\bar{J}_{\text{anom}} = -\frac{\partial \bar{\mathcal{G}}_{\text{anom}}}{\partial \mu}$$

$$\bar{J}_{S,\text{anom}} = -\frac{\partial \bar{\mathcal{G}}_{\text{anom}}}{\partial T} \tag{1.2}$$

$$\bar{q}_{\text{anom}} = \bar{\mathcal{G}}_{\text{anom}} + T\bar{J}_{S,\text{anom}} + \mu \bar{J}_{\text{anom}}$$

2. $\bar{\mathcal{G}}_{\text{anom}}$ is determined in terms of the fluid vorticity 2-form ω , the rest-frame magnetic field 2-form B and a $(n+1)^{th}$ degree polynomial $\mathfrak{F}^\omega_{\text{anom}}[T, \mu]$ in temperature T and chemical potential μ (where $d = 2n$ is the number of spacetime dimensions). The explicit expression for $\bar{\mathcal{G}}_{\text{anom}}$ is given by[5]

$$\bar{\mathcal{G}}_{\text{anom}} = \frac{1}{(2\omega)^2}\Big\{ \mathfrak{F}^\omega_{\text{anom}}[T(2\omega), B + \mu(2\omega)] - \Big[\mathfrak{F}^\omega_{\text{anom}}[T(2\omega), B + \mu(2\omega)]\Big]_{\omega=0}$$
$$-\omega\Big[\frac{\delta}{\delta\omega}\mathfrak{F}^\omega_{\text{anom}}[T(2\omega), B + \mu(2\omega)]\Big]_{\omega=0} \Big\} \wedge u \tag{1.3}$$

3. The polynomial $\mathfrak{F}^\omega_{\text{anom}}[T, \mu]$ obeys two constraints — first, it has no term linear in T. Second, its value at zero temperature is completely determined by the U(1) anomaly in the system.

4. To these statements, we can add the following statement which does not follow from thermodynamic arguments in [4] but nevertheless seems to be true across various systems — the T^2 coefficient in $2d$ and $4d$ seem to be related to gravitational anomalies (see for example [11, 23] for results in $4d$.

This summary encompasses everything that is known till now about these transport processes just from flat spacetime thermodynamics alone. But, this is unsatisfactory for

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[4]Throughout this article, we will use overbars to denote Hodge duals.

[5]This is just a convenient rephrasing of the formulae/results presented in the appendix A of [4]. In particular, we have the following relation relating the functions appearing there to the functions appearing here

$$\mathfrak{F}^\omega_{\text{anom}}[T(2\omega), B + \mu(2\omega)] = \mathfrak{f}[B + \mu(2\omega)] + \frac{1}{2}T^2(2\omega)^2\mathfrak{g}[B + \mu(2\omega), T\omega].$$

various reasons — first of all, as we had mentioned various gravitational anomalies of the system show up even in the flat spacetime transport and this is mysterious even from the point of view of flat spacetime thermodynamics. Second, we do not know how these relations to gravitational anomalies generalize to higher dimensions. Further, a more microscopic understanding of these transport processes would clearly be useful for various reasons — for example, we would like to study these transport processes away from equilibrium/in the presence of disorder/ in the lattice analogues of continuum Weyl fermions. Our aim in this article is to begin addressing these questions in the simplest system exhibiting such transport — free fermion theory in even spacetime dimensions $d = (2n-1)+1$ with a collection of fermions with different chiralities $\chi_{d=2n}$ and charges q under some specific U(1) global symmetry of the free theory.

After establishing the basic setup/notations in the section 2, we proceed to study in detail the simplest case of $1 + 1$d chiral fermions in flat spacetime in section 3. We then translate the thermodynamic arguments of adiabaticity in flat spacetime into a more microscopic set of equations which one can take as an alternate starting point for the $2d$ analysis and which easily generalizes to higher dimensions. We propose the following intuitive picture for how the transport phenomena linked to anomaly arise: first of all, we propose that these transport processes can be thought of as arising from a certain *chiral* density of states and their *spectral flow*. In a continuum description, we capture this by a *chiral spectral current* $\mathcal{J}_q^\mu(x, E_p)$ whose time component is the chiral density of 1-particle states with charge q and the rest-frame energy E_p and whose spatial components tell us about the flow of such states as the fluid flows. In section 4, we argue that adiabaticity can be simply seen as a conservation type equation for this chiral spectral current.

This is of course a very well-known aspect of anomalies whereby the basic process driving the anomaly is the continuous injection of chiral zero modes by the external magnetic field (which we will assume to be small and slowly varying) into the system. We make this precise in section 5 by simply matching the microscopic discussion of adiabaticity to the thermodynamic discussion of adiabaticity. This matching gives a boundary condition for the conservation equation for the chiral spectral current by giving the rate at which chiral modes are injected into the fluid.

$$\bar{\mathcal{J}}_q|_{E_p=0} = \frac{\chi_{d=2n}}{2\pi} \left(\frac{qB}{2\pi} \right)^{n-1} \wedge \frac{u}{(n-1)!} \tag{1.4}$$

These modes, once injected by the magnetic field are then convected along with the fluid flow and the adiabaticity is just the statement that there is no more creation/annihilation of states at finite energies.

Armed with this intuitive picture, we proceed in section 6 to solve the conservation equation in flat spacetime. This results in a simple expression for the chiral spectral current

$$\bar{\mathcal{J}}_q = \frac{\chi_{d=2n}}{2\pi} \left(\frac{qB + 2\omega E_p}{2\pi} \right)^{n-1} \wedge \frac{u}{(n-1)!} \tag{1.5}$$

where $\chi_{d=2n}$ is the chirality of the 1-particle state. This solution tells us how the states of different energies and charges flow hence solving once for all the spectral flow problem in

an arbitrary fluid flow (in flat spacetime). We now notice a remarkable result — if we take the external field strengths to zero $B \to 0$, there is no more an injection of new zero-energy states into the fluid, but the chiral spectral flow is still non-zero even if the anomaly is turned off ! In this case, the vorticity is sufficient to drive the chiral spectral current and this is the basic reason why rotational response encodes information about the anomalies in the system — both gravitational and non-gravitational.

In the next section 7, we add up the Gibbs-free energy contribution of each 1-particle state to get the anomaly-induced free-energy current $\bar{\mathcal{G}}_{\mathrm{anom}}$. We find that $\bar{\mathcal{G}}_{\mathrm{anom}}$ is of the form given in eq. (1.3) which was derived in [4] by thermodynamic considerations. While this is not surprising, we find the the polynomial $\mathfrak{F}^{\omega}_{\mathrm{anom}}$ is derived by a very simple formula from the anomaly polynomial of the system. We get

$$\mathfrak{F}^{\omega}_{\mathrm{anom}} = \mathcal{P}_{\mathrm{anom}} \left(F \mapsto \mu, \ p_1(\mathfrak{R}) \mapsto -T^2, \ p_{k>1}(\mathfrak{R}) \mapsto 0 \right) \tag{1.6}$$

where $\mathcal{P}_{\mathrm{anom}}$ is the anomaly polynomial of the system written in terms of the U(1) field strength F and the kth Pontryagin forms[6] of the spacetime curvature $p_k(\mathfrak{R})$. The above formula gives a simple replacement rule by which one can go from the anomaly polynomial to the polynomial $\mathfrak{F}^{\omega}_{\mathrm{anom}}$.

Note that this a generalization of the observation made in [23] that T^2 coefficient in $4d$ free theories seem to be related to gravitational anomalies. Since the $4d$ relation continues to hold in strongly coupled holographic phases too [11], it is tempting to conjecture that the above replacement rule would continue to hold even beyond free theories. We emphasize that this is quite a non-trivial conjecture and what we have is a preliminary evidence that it might be true. We discuss this along with other further directions in our discussion section 8. We collect various useful results in our appendices. In the next section, we begin by explaining our basic setup and defining our notation - most of it being quite standard and elementary, the reader should feel free to skim through the section just noting various remarks on notation.

2 The basic setup

The main system we will be concerned about throughout this article is a system of free relativistic fermions at finite temperature and chemical potential in a flat spacetime with the spacetime dimension $d = 2n = (2n - 1) + 1$ being even. Every particle state occurs along with its anti-particle state and we will call such a particle/anti-particle pair as a *species*. Hence, we will denote by $\sum_{\mathrm{species}}$ the summation where each particle/anti-particle pair contributes a single term to the sum. This should be distinguished from $\sum_F$ which denotes the summation where each particle (or anti-particle) state contributes separately to the sum.

Among the large symmetry that the free theory enjoys, we will choose a specific U(1) symmetry for which we will turn on the chemical potential μ and put the system at a finite temperature T (we will use the letter $\beta \equiv 1/T$ to denote the inverse temperature).

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[6]See appendices B for definitions of various quantities related to anomaly polynomials.

The thermodynamic T and μ are of course defined on a local rest frame defined via a unit time-like vector u^μ. The thermodynamic potential appropriate to such a grand-canonical description is the Gibbs free-energy density $\mathcal{G} \equiv \varepsilon - Ts - \mu n$ where ε, s, n are the rest-frame energy density, entropy density and the charge density respectively. The first law takes the form $d\mathcal{G} = -sdT - nd\mu$.

For a free theory, the Gibbs free-energy is obtained by simply multiparticling contribution from the 1-particle sector. We will denote the Gibbs free-energy contribution of a fermionic state with a charge q and rest frame energy E_p by g_q. Hence

$$g_q \equiv -\frac{1}{\beta} \ln \left[1 + e^{-\beta(E_p - q\mu)} \right]$$

The occupation of a given 1-particle fermionic state is given by the Fermi-Dirac distribution denoted by f_q

$$f_q \equiv \frac{1}{e^{\beta(E_p - q\mu)} + 1}$$

The contribution of a 1-particle state to the entropy is given by the *negative* of the Boltzmann's $\mathcal{H}$-function which for fermionic states takes the form

$$\mathcal{H}_q \equiv f_q \ln f_q + (1 - f_q) \ln (1 - f_q)$$

Using the standard terminology , we will call a state with slowly varying T, μ, u^μ with the local density matrix being close to the thermal density matrix as an *Ideal gas*. Since the constituents are Weyl fermions, we will call this an Ideal Weyl gas. In this article, we will be concerned about a specific subset of transport processes which are linked to various anomalies in the ideal Weyl gas. Before proceeding let us dispose of a specific technicality: as is well known, in the strictly non-interacting limit, all the dissipation length/time scales diverge which in turn means that dissipative transport coefficients like viscosity also diverge in this limit. While this is true, this shall not worry us too much since the effects that we are looking for are non-dissipative and are not plagued by such 'free-theory' infinities. While we expect addition of interactions/dissipation would not modify our results (this expectation is partially justified by various existent calculations in strongly coupled holographic phases), it would be nice to explicitly prove this statement.

Having addressed that technicality, let us continue: we want to study such an ideal Weyl gas in presence of non-dynamical background electric and magnetic fields. Let F be the field-strength 2-form, we define the rest frame electric 1-form via $E_\mu \equiv F_{\mu\nu} u^\nu$. We can then do an electric-magnetic decomposition

$$F_{\mu\nu} - [u_\mu E_\nu - E_\mu u_\nu] \equiv B_{\mu\nu} \tag{2.1}$$

or in the language of forms

$$F = B + u \wedge E \tag{2.2}$$

where B is the rest-frame magnetic 2-form completely transverse to u^μ, i.e., $B_{\mu\nu} u^\nu = 0$. We will also use the standard decomposition of the velocity gradients

$$D_\mu u_\nu = \sigma_{\mu\nu} + \omega_{\mu\nu} - u_\mu a_\nu + \frac{\theta}{d-1} P_{\mu\nu} \tag{2.3}$$

in terms of the shear strain rate $\sigma_{\mu\nu}$, the vorticity $\omega_{\mu\nu}$, the acceleration a_μ and the expansion rate θ of the fluid. Further $P_{\mu\nu} \equiv g_{\mu\nu} u_\mu u_\nu$ as before. This in particular means the exterior derivative of the velocity 1-form has the decomposition

$$Du = 2\omega - u \wedge a \tag{2.4}$$

where ω is the vorticity 2-form.

Before we enter the main argument of the paper, we will make some comments regarding our conventions for chirality in higher dimensions. Consider the 1-particle states which are given by the solutions of the Weyl equation of appropriate chirality (which is just the massless Dirac equation with the opposite chirality projected out). We will define our conventions for chirality now by essentially equating it to the helicity. To do this, let us divide the $2n - 1$ spatial directions into a direction x_1 and $n - 1$ planes where the kth plane is the $(x_{i_{2k}}, x_{i_{2k+1}})$ plane where $k = 1, \ldots, n-1$.

Consider first the positive frequency solutions of the Weyl equation. Further we will consider only the solutions with only $p^0 > 0, p^1 \neq 0$ — all other components of momentum being zero and the spin along k^{th} plane being $S_{i_{2k} i_{2k+1}} \equiv \frac{1}{2}\sigma_k$ with $\sigma_k = \pm 1$. Any other solution can be obtained by rotating this solution or by linear combinations thereof. For these solutions, we will assign the chirality via their helicity

$$\chi_{d=2n} \equiv \text{sign}(p_1) \prod_{k=1}^{n-1} \sigma_k$$

For a Weyl equation of a particular chirality $\chi_{d=2n}$ only the σ_k obeying the above equation are allowed. When $n = 1$ (when $d = 1 + 1d$) this gives a single state with the sign of p_1 fixed. For $n \geq 2$, we get 2^{n-2} states with p_1 being arbitrary.

We now turn to the negative frequency solutions which are complex conjugates of positive frequency anti-particle solutions. The complex conjugate of a Weyl spinor of chirality $\chi_{d=2n}$ is another Weyl spinor of chirality $(-1)^{n-1}\chi_{d=2n}$. Hence, for corresponding to every particle state above we get an anti-particle state with chirality $(-1)^{n-1}\chi_{d=2n}$. We will call the Weyl fermions with positive chirality as left fermions and those with negative chirality as right fermions. This summarizes the basic definitions needed for the rest of the paper. The reader can find a table of notation in the appendix D for ready reference.

3 Anomaly and transport in 2d Weyl gas

Before trying to tackle the case of a Weyl gas in higher dimensions, it is instructive to work out the simplest case of $1 + 1$ dimensions. This is a very well-studied system and in some sense we will not have anything new[7] to add except for a way of looking at the standard results which will prepare us for the more subtle effects in higher dimensions. With this objective in mind, we will focus on this simple case in some detail.

We will begin by considering a single species of a free left Weyl fermion in $1 + 1$ dimensions with charge q. We will assign this fermion a chirality $\chi_{d=2} = +1$. The anti-particle of this fermion is again a left Weyl fermion with charge $-q$. This follows from the

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[7]See for example [30] for a different take on anomaly/transport in 2d fluids.

general rule that in $(2n - 1) + 1$d the chirality of the anti-particle is $(-1)^{n+1}$ times the chirality of the particle.

We are interested in a gas of such Weyl fermions at a finite temperature T and chemical potential μ. These quantities are of course defined in a center of mass frame of the ideal Weyl gas — let this frame be defined by a $1 + 1$-d unit time-like vector u^μ which we will take it to be constant. It is this u^μ which in the hydrodynamic description will describe the fluid velocity.

The question we want to address is this — what is the hydrodynamic description of such an ideal gas? We will first give a naive answer to this question which will later correct.

The conventional intuition is that this system behaves like an ideal fluid with the following naive constitutive relations for energy/charge/entropy currents

$$
\begin{aligned}
T^{\mu\nu}_{\text{naive}} &= \varepsilon u^\mu u^\nu + p \left(g^{\mu\nu} + u^\mu u^\nu \right) \\
J^\mu_{\text{naive}} &= n u^\mu \\
J^\mu_{S,\text{naive}} &= s u^\mu
\end{aligned}
\tag{3.1}
$$

where the energy density ε, pressure p, charge density n and entropy density s can be calculated from the usual statistical mechanics of an ideal fermion gas. We will calculate this in a moment, but before that we will argue that the above form is definitely incomplete!

The reason is simple — a theory of a free Weyl fermion is a holomorphic 2d CFT and hence only the holomorphic components of the currents can be non-zero. The above relations are in clear contradiction with holomorphy — for one, the charge/entropy currents are time-like rather than null as would be predicted by holomorphy. So we are led to the surprising statement that conventional semi-classical intuition about the ideal gas is in direct contradiction with holomorphy in this simple system. Having concluded thus, let us actually calculate carefully what the actual constitutive relations should be.

A left Weyl fermion field in $2d$ is just a single component complex field ψ which obeys the Weyl equation (or the massless Dirac equation)

$$
[\partial_t + \partial_x] \psi = 0
$$

This follows from the particular choice for the Gamma matrices $\{\Gamma^t, \Gamma^x\} = \{-i\sigma_y, \sigma_x\}$. We can repeat the same exercise for the right Weyl fermion with $\chi_{d=2} = -1$ where we just flip the sign of the ∂_x term. In the following we will write our formulae in such a way that the expressions for the right Weyl fermion can be obtained by putting $\chi_{d=2} = -1$. Hence we write the Weyl equation as

$$
[\partial_t + \chi_{d=2} \partial_x] \psi = 0
$$

This equation is easily solved - solutions are just plane-waves that travel from left to right in the space

$$
\psi = \int_0^\infty \frac{dE_p}{2\pi} \frac{1}{\sqrt{2E_p}} \left[a_p e^{ip.x} + b_p^\dagger e^{-ip.x} \right]_{p^\mu = E_p \{1, \chi_{d=2}\}}
$$

where $a_p^\dagger$ and $b_p^\dagger$ are the creation operators for the particle and the anti-particle respectively and E_p is the energy in some arbitrary frame.

– 8 –

Let us work in the rest frame define by u^μ from now on — so we take $u^\mu = \{1, 0\}$. Let $\epsilon^{\mu\nu}$ be the completely antisymmetric tensor in 2d with $\epsilon^{tx} = 1$ which implies $\epsilon^{\mu\nu} u_\nu = \{0, 1\}$. The Weyl equation is

$$\left[u^\mu + \chi_{d=2} \epsilon^{\mu\nu} u_\nu \right] \partial_\mu \psi = 0$$

and the plane-wave solutions above are

$$\psi = \int_0^\infty \frac{dE_p}{2\pi} \frac{1}{\sqrt{2E_p}} \left[a_p e^{ip.x} + b_p^\dagger e^{-ip.x} \right]_{p^\mu = E_p \left[u^\mu + \chi_{d=2} \epsilon^{\mu\nu} u_\nu \right]}$$

We want to populate the states of these fermions/anti-fermions in this frame according to the Fermi-Dirac distribution and calculate the conserved currents in the thermal ensemble. This gives

$$
\begin{aligned}
T^{\mu\nu} &= \sum_{\text{species}} \int_0^\infty \frac{dE_p}{2\pi} \left(f_q + f_{-q} \right) E_p \left[u^\mu + \chi_{d=2} \epsilon^{\mu\alpha} u_\alpha \right] \left[u^\nu + \chi_{d=2} \epsilon^{\nu\lambda} u_\lambda \right] \\
&= \varepsilon u^\mu u^\nu + p \left(g^{\mu\nu} + u^\mu u^\nu \right) + q_{\text{anom}}^\mu u^\nu + q_{\text{anom}}^\nu u^\mu \\
J^\mu &= \sum_{\text{species}} \int_0^\infty \frac{dE_p}{2\pi} \left(q f_q - q f_{-q} \right) \left[u^\mu + \chi_{d=2} \epsilon^{\mu\alpha} u_\alpha \right] \\
&= n u^\mu + J_{\text{anom}}^\mu \\
J_S^\mu &= - \sum_{\text{species}} \int_0^\infty \frac{dE_p}{2\pi} \left(\mathcal{H}_q + \mathcal{H}_{-q} \right) \left[u^\mu + \chi_{d=2} \epsilon^{\mu\alpha} u_\alpha \right] \\
&= s u^\mu + J_{S,\text{anom}}^\mu
\end{aligned}
\tag{3.2}
$$

where we have used the relation $\chi_{d=2} \epsilon^{\mu\alpha} u_\alpha \, \chi_{d=2} \epsilon^{\nu\lambda} u_\lambda = g^{\mu\nu} + u^\mu u^\nu$. We have collected together the deviations from the conventional hydrodynamic expectation under the objects with the subscript *anom*. We get the conventional expressions which could have been naively guessed

$$
\begin{aligned}
\varepsilon = p &= \sum_{\text{species}} \int_0^\infty \frac{dE_p}{2\pi} \left(f_q + f_{-q} \right) E_p = \sum_F \int_0^\infty \frac{dE_p}{2\pi} f_q E_p \\
n &= \sum_{\text{species}} \int_0^\infty \frac{dE_p}{2\pi} \left(q f_q - q f_{-q} \right) = \sum_F \int_0^\infty \frac{dE_p}{2\pi} q f_q E_p \\
s &= - \sum_{\text{species}} \int_0^\infty \frac{dE_p}{2\pi} \left(\mathcal{H}_q + \mathcal{H}_{-q} \right) = - \sum_F \int_0^\infty \frac{dE_p}{2\pi} \mathcal{H}_q
\end{aligned}
\tag{3.3}
$$

where the sum is over every fermionic species with particles and antiparticles counted separately. The anomalous corrections are given by

$$
\begin{aligned}
q_{\text{anom}}^\mu &= \sum_F \chi_{d=2} \epsilon^{\mu\alpha} u_\alpha \int_0^\infty \frac{dE_p}{2\pi} f_q E_p \\
J_{\text{anom}}^\mu &= \sum_F \chi_{d=2} \epsilon^{\mu\alpha} u_\alpha \int_0^\infty \frac{dE_p}{2\pi} q f_q \\
J_{S,\text{anom}}^\mu &= - \sum_F \chi_{d=2} \epsilon^{\mu\alpha} u_\alpha \int_0^\infty \frac{dE_p}{2\pi} \mathcal{H}_q
\end{aligned}
\tag{3.4}
$$

At this point, the curious reader might wonder what happens to these relations in the case of a Dirac fermion. The massless Dirac fermion is just a left Weyl fermion of charge q along with a right Weyl fermion of charge q. In this case, it is evident from the expressions above that the conventional terms add up and the anomalous terms cancel out. Hence the naive guess turns out to be correct for a massless Dirac fermion. It is not very difficult to convince oneself by explicit computation that the naive guess works even for the massive Dirac fermion. This then is the first lesson from this exercise: *there are transport processes in hydrodynamics to which only chiral species contribute.*

Since we will be studying these anomalous contributions in more detail — let us simplify them by writing the currents above as 1-forms $\{q_{\text{anom}}, J_{\text{anom}}, J_{S,\text{anom}}\}$. Further take a Hodge-dual on both sides (which we will denote by an overbar) to remove the $\epsilon^{\mu\nu}$ to get

$$\bar{q}_{\text{anom}} = \sum_F \int_0^\infty \frac{dE_p}{2\pi} f_q E_p \ \chi_{d=2} u$$

$$\bar{J}_{\text{anom}} = \sum_F \int_0^\infty \frac{dE_p}{2\pi} q f_q \ \chi_{d=2} u \tag{3.5}$$

$$\bar{J}_{S,\text{anom}} = -\sum_F \int_0^\infty \frac{dE_p}{2\pi} \mathcal{H}_q \ \chi_{d=2} u$$

These are easily calculated from the Gibbs free-energy current

$$\bar{\mathcal{G}}_{\text{anom}} = \sum_F \int_0^\infty \frac{dE_p}{2\pi} g_q \ \chi_{d=2} u$$

$$\bar{J}_{\text{anom}} = -\frac{\partial \bar{\mathcal{G}}_{\text{anom}}}{\partial \mu} \ , \quad \bar{J}_{S,\text{anom}} = -\frac{\partial \bar{\mathcal{G}}_{\text{anom}}}{\partial T} \tag{3.6}$$

$$\text{and} \quad \bar{q}_{\text{anom}} = \bar{\mathcal{G}}_{\text{anom}} + T \bar{J}_{S,\text{anom}} + \mu \bar{J}_{\text{anom}}$$

This means we basically have to evaluate only one thermal integral. This can be done by pairing up the particle and anti-particle contribution and using the identity[8]

$$\int_0^\infty \frac{dE_p}{2\pi} \left(g_q + g_{-q}\right) = -2\pi \left[\frac{1}{2!} \left(\frac{q\mu}{2\pi}\right)^2 + \frac{T^2}{4!} \right]$$

This gives

$$\bar{\mathcal{G}}_{\text{anom}} = -2\pi \left[\frac{\mu^2}{2!(2\pi)^2} \left(\sum_{\text{species}} \chi_{d=2} q^2 \right) + \frac{T^2}{4!} \left(\sum_{\text{species}} \chi_{d=2} \right) \right] u \tag{3.7}$$

where the sum is performed over the fermionic species with a particle-antiparticle pair contributing to a single term in the sum. Now we note a crucial fact: the anomalous contribution is completely proportional to the U(1) anomaly coefficient $\sum_{\text{species}} \chi_{d=2} q^2$ and the Lorentz anomaly coefficient $\sum_{\text{species}} \chi_{d=2}$. So, we come to the second lesson of this exercise: *The anomalous transport is completely determined by the anomalies in the system.* Hence, we will call such a transport as *anomaly-induced.*

---

[8]See appendix A for a derivation of this/related identities.

Let us make this more precise — the anomaly coefficients of a system are neatly summarized by a polynomial in gauge field strength F and spacetime curvature $\mathfrak{R}$. For a collection of $2d$ Weyl fermions the anomaly polynomial is given by (see the appendix B for a review of anomaly polynomials for fermions)

$$\mathcal{P}_{\text{anom}}(F, \mathfrak{R}) \equiv -2\pi \left[\frac{F^2}{2!(2\pi)^2} \left(\sum_{\text{species}} \chi_{d=2} q^2 \right) - \frac{p_1(\mathfrak{R})}{4!} \left(\sum_{\text{species}} \chi_{d=2} \right) \right]_{2d} \tag{3.8}$$

where $p_1(\mathfrak{R})$ is the first Pontryagin class of spacetime curvature defined as

$$p_1(\mathfrak{R}) \equiv -\frac{1}{2(2\pi)^2} \mathfrak{R}_{a_1}{}^{a_2} \wedge \mathfrak{R}_{a_2}{}^{a_1} = -\frac{\mathfrak{R}_2}{(2\pi)^2}$$

where we use the notation

$$\mathfrak{R}_k \equiv \frac{1}{2} \mathfrak{R}_{a_1}{}^{a_2} \wedge \mathfrak{R}_{a_2}{}^{a_3} \dots \mathfrak{R}_{a_k}{}^{a_1}$$

Using this, we can write a simple rule to get from the anomaly polynomial to the anomaly-induced Gibbs free current

$$\bar{\mathcal{G}}_{\text{anom}} = u \, \mathcal{P}_{\text{anom}} \left(F \mapsto \mu, \; p_1(\mathfrak{R}) \mapsto -T^2 \right) \tag{3.9}$$

This elegant result tells us that one can just read off the anomaly-induced Gibbs-free current of an ideal Weyl gas in $2d$ directly from its anomaly polynomial. Let us now compare this form with eq. (1.3) which was derived in [4] by thermodynamic arguments. By inspection, it is clear that the above equation follows from (1.3) if we take

$$\mathfrak{F}_{\text{anom}}^{\omega} = \mathcal{P}_{\text{anom}} \left(F \mapsto \mu, \; p_1(\mathfrak{R}) \mapsto -T^2 \right) \tag{3.10}$$

Hence, there is a straightforward algorithm in $2d$ free fermion theories which takes us from the anomaly polynomial of the theory to the polynomial $\mathfrak{F}_{\text{anom}}^{\omega}$ in T and μ which determines the anomaly-induced transport.

One of the main aims of the rest of this article is to generalize this result to arbitrary even dimensions. However, in higher dimensions one does not have a powerful principle like holomorphy to help us and the anomaly induced transport is hence a more subtle effect to derive. So we will spend the next section to formulate a principle which will help us find the higher dimensional analogues of the above result.

4 Adiabaticity of spectral flow

One of the crucial lessons one draws from the previous section is that there is a single Gibbs free energy current from which energy/charge/entropy currents could be derived. We will make the reasonable assumption that this continues to hold true in higher dimensions. In $2d$ we derived the expression for this current in terms of a thermal integral

$$\bar{\mathcal{G}}_{\text{anom}} = \sum_F \int_0^\infty \frac{dE_p}{2\pi} g_q \, \chi_{d=2} u \tag{4.1}$$

and a natural generalization of the above expression is

$$\bar{\mathcal{G}}_{\text{anom}} = \sum_F \int_0^\infty dE_p \bar{\mathcal{J}}_q \; g_q \tag{4.2}$$

where $\bar{\mathcal{J}}_q$ should be a $2n-1$ form in $d = (2n-1)+1$ dimensions since it is proportional to the Hodge dual of the Gibbs current 1-form $\mathcal{G}_{\text{anom}}^\mu$. Taking a Hodge dual on both sides, we can remove the overbars and write

$$\mathcal{G}_{\text{anom}}^\mu = \sum_F \int_0^\infty dE_p \mathcal{J}_q^\mu \; g_q \tag{4.3}$$

Our task is to understand better the physical meaning of $\mathcal{J}_q^\mu$ and hence formulate some sort of an equation for it which then can be solved.

So what is $\mathcal{J}_q^\mu$? It is clear from the way it occurs in the expression for Gibbs-current that $\mathcal{J}_q^\mu$ is a current whose time-component is just the 1-particle chiral density of states participating in anomalous transport. Hence, we will call it the *chiral spectral current* from now on. It tells us how a subset of states related to anomaly flow as the whole fluid(the ideal Weyl gas in this case) flows. In $2d$, we have

$$\mathcal{J}_q^\mu|_{2d} = \chi_{d=2} \frac{1}{2\pi} \epsilon^{\mu\nu} u_\nu$$

Hence, two identical states which differ only in their chirality have opposite chiral spectral currents. We will assume that this is true in higher dimensions too $\mathcal{J}_q^\mu|_{d=2n} \sim \chi_{d=2n}$ hence justifying the adjective chiral.

In general, we expect the chiral spectral current to be a function of both spacetime co-ordinates x and the local rest frame energy E_p i.e., $\mathcal{J}_q^\mu = \mathcal{J}_q^\mu(x, E_p)$ which emphasizes the fact that the chiral spectral current at different energies could be different. Further, the subscript q denotes the fact that 1-particle states with different charges can have different chiral spectral currents.

With this understanding, we now turn to the question — how do we determine the chiral spectral current $\mathcal{J}_q^\mu$ for an ideal Weyl gas? Our crucial tool would be adiabaticity — as was argued by authors of [3, 4] (which is a generalization of a $d = 4$ argument in [1]) the anomaly induced transport in arbitrary dimensions is heavily constrained by adiabaticity. At the level of states, adiabaticity is basically a statement that the states responsible for anomaly induced transport do not get created or annihilated[9] as they move up or down in the energy or in case of localized states as they move around in spacetime. This assumption as stated is easily formulated in terms of a chiral spectral current — it is just the continuity equation for the chiral spectral current in the (x, E_p) space,i.e.,

$$\nabla_\mu \mathcal{J}_q^\mu + \frac{\partial}{\partial E_p} \mathcal{J}_q^E = 0 \tag{4.4}$$

where $\mathcal{J}_q^E(x, E_p)$ is the current of chiral states in the energy direction — which is the density of states times rate at which their energies are increasing.

---

[9]Except at zero energy due to the anomaly as we will see below.

For single particle states with the charge q — there are two forces which lead to a change in energy. First is the rest-frame electric force — the work done by the electric force is just the electric field times the charge current $E_\mu q \mathcal{J}_q^\mu$. Second is the pseudo force (since the rest frame of the fluid is in general accelerating) given by $-E_p a_\mu$ where a_μ is the acceleration of the fluid. The work done by it is $-a_\mu$ times the energy current i.e., $-a_\mu E_p \mathcal{J}_q^\mu$. Combining these together, we get

$$\mathcal{J}_q^E = E_\mu q \mathcal{J}_q^\mu - a_\mu E_p \mathcal{J}_q^\mu \tag{4.5}$$

Taking Hodge duals back again we get the equation that we were after

$$D\bar{\mathcal{J}}_q + \frac{\partial}{\partial E_p}\bar{\mathcal{J}}_q^E = 0 \quad \text{with} \quad \bar{\mathcal{J}}_q^E = (qE - E_p a) \wedge \bar{\mathcal{J}}_q \tag{4.6}$$

It is easily checked that the $2d$ result $\bar{\mathcal{J}}_q = \frac{1}{2\pi}\chi_{d=2}u$ solves the above equation.

5 Chiral spectral current and anomaly

A curious reader might wonder how the equation we just derived relates to the adiabaticity assumption as it appears in [1, 3, 4] where the discussion was entirely macroscopic with no reference to microscopic states. Further, we have not input anywhere the information about the anomaly in the above equations. In this section we will show that those macroscopic equations could be thought of as arising from the above microscopic equation. Further this would also clarify how anomalies are related to the chiral spectral current.

The flux of density of states is associated with the following energy/charge/entropy currents

$$\bar{q}_{\text{anom}} = \sum_F \int_0^\infty dE_p \bar{\mathcal{J}}_q E_p f_q$$

$$\bar{J}_{\text{anom}} = \sum_F \int_0^\infty dE_p \bar{\mathcal{J}}_q q f_q \tag{5.1}$$

$$\bar{J}_{S,\text{anom}} = -\sum_F \int_0^\infty dE_p \bar{\mathcal{J}}_q \mathcal{H}_q$$

For a general system with anomalies, the statement that these transport processes have to be adiabatic is equivalent to the following equation (as shown in [4])

$$D\bar{q}_{\text{anom}} + a \wedge \bar{q}_{\text{anom}} - E \wedge \bar{J}_{\text{anom}} = TD\bar{J}_{S,\text{anom}} + \mu\left[D\bar{J}_{\text{anom}} - \bar{\mathfrak{A}}[F]\right] \tag{5.2}$$

where $\bar{\mathfrak{A}}[F]$ is the anomaly $2n$-form in a given theory. This equation above assumes that one is working in flat spacetime and hence all gravitational anomalies are turned off. We will refer the reader to [4] for a proper derivation of this equation and its consequences for a general system. Here we are mainly interested in exploring what it means for the chiral spectral current. Substituting the above expressions for the various currents and using the

following relations

$$f_q \equiv \frac{1}{e^{\beta(E_p - q\mu)} + 1} = \frac{\partial g_q}{\partial E_p}$$

$$E_p f_q + T\mathcal{H}_q - q\mu f_q = g_q \tag{5.3}$$

$$E_p D f_q + T D\mathcal{H}_q - q\mu D f_q = 0$$

the adiabaticity equation of [4] assumes an especially simple form

$$\sum_F \int_0^\infty dE_p \left[g_q D\bar{\mathcal{J}}_q - \bar{\mathcal{J}}_q^E \frac{\partial g_q}{\partial E_p} \right] + \mu\bar{\mathfrak{A}} = 0 \tag{5.4}$$

where as before $\bar{\mathcal{J}}_q^E = (qE - E_p a) \wedge \bar{\mathcal{J}}_q$. In this form, the relation to the microscopic equation that we had derived before is evident. To see this we should integrate by parts — this should be done carefully since the boundary contributions do not vanish. We get

$$\sum_F \int_0^\infty dE_p \, g_q \left[D\bar{\mathcal{J}}_q + \frac{\partial}{\partial E_p} \bar{\mathcal{J}}_q^E \right] + \mu\bar{\mathfrak{A}} - \sum_F \left[g_q \bar{\mathcal{J}}_q^E \right]_{E_p=0}^{E_p=\infty} = 0 \tag{5.5}$$

We can now use the continuity equation for the chiral spectral current to set the integrand inside the integral to zero.

Now we turn to the boundary contributions: first the UV contribution — g_q falls exponentially as the rest-frame energy $E_p \to \infty$ and it is reasonable to assume that the growth of $\bar{\mathcal{J}}_q^E$ is slow enough that the contribution from $E_p = \infty$ is zero.[10] This is consistent with the fact that anomalies and their related transport processes are not UV-sensitive despite the fact that historically they were discovered in calculations where one should renormalize UV-sensitive quantities.

Now we turn to the IR contribution — there is no good reason for the boundary contribution from $E_p = 0$ to vanish and in fact we need it to balance the contribution from the anomaly. Equating the anomaly to this IR contribution, we get

$$\bar{\mathfrak{A}} = -\frac{1}{\mu} \sum_F g_q(E_p = 0) \, \bar{\mathcal{J}}_q^E(E_p = 0) \tag{5.6}$$

This is an interesting equation which is a version of the well-known relation between the spectral flow of chiral zero energy states and the anomaly in the system. To see how this might work, we will now write the anomaly term also as a sum. For a set of charged Weyl fermions in $d = (2n - 1) + 1$ dimensional flat spacetime, the covariant anomaly is given by

$$\bar{\mathfrak{A}}[F] = -\frac{1}{n!} \left(\frac{F}{2\pi} \right)^n \sum_{\text{species}} \chi_{d=2n} q^{n+1} \tag{5.7}$$

where the sum runs over every species with one term appearing in the sum for every particle-antiparticle couple.[11]

---

[10] In fact, if $\bar{\mathcal{J}}_q^E$ did grow exponentially with energy the various integrals we have been writing down would all be UV divergent and we would have to worry how to make sense out of them. We will assume that this is not the case and this assumption can be justified in explicit examples.

[11] It does not matter which among that couple is chosen, since $\chi_{d=2n} q^{n+1}$ is the same for both — this follows from the fact that if the particles charge and chirality is $(q, \chi_{d=2n})$ then the antiparticle's charge/chirality is $(-q, (-1)^{n-1}\chi_{d=2n})$.

Let us divide the field-strength F appearing in the above equations into electric and magnetic fields in the rest frame of the fluid. The rest frame electric field is $E_\mu = u^\nu F_{\mu\nu}$ which we can think of as a 1-form E. The rest frame magnetic field is a 2-form obtained by subtracting the electric part from F,

$$B \equiv F - u \wedge E$$

where $u = u_\mu dx^\mu$ is the velocity 1-form. Substituting $F = B - E \wedge u$ into the anomaly above we get

$$\mu\bar{\mathfrak{A}}[F] = \sum_{\text{species}} \chi_{d=2n}\, qE \wedge \frac{q\mu}{2\pi} \left(\frac{qB}{2\pi}\right)^{n-1} \wedge \frac{u}{(n-1)!} \tag{5.8}$$

Now use the fact that $g_q(E_p = 0) - g_{-q}(E_p = 0) = -q\mu$ to get

$$\bar{\mathfrak{A}}[F] = -\frac{1}{\mu}\sum_F g_q(E_p = 0)\chi_{d=2n}\frac{qE}{2\pi} \wedge \left(\frac{qB}{2\pi}\right)^{n-1} \wedge \frac{u}{(n-1)!} \tag{5.9}$$

where the sum now runs over all the fermions with particles and antiparticles counted separately. Comparing this with (5.6), we get for each particle

$$\bar{\mathcal{J}}_q^E(E_p = 0) = qE \wedge \bar{\mathcal{J}}_q(E_p = 0) = \frac{qE}{2\pi} \wedge \chi_{d=2n} \left(\frac{qB}{2\pi}\right)^{n-1} \wedge \frac{u}{(n-1)!}$$

or

$$\bar{\mathcal{J}}_q(E_p = 0) = \frac{\chi_{d=2n}}{2\pi} \left(\frac{qB}{2\pi}\right)^{n-1} \wedge \frac{u}{(n-1)!} \tag{5.10}$$

This flow of density of states at zero energy has a direct explanation in terms of Landau level physics in the local rest frame of the fluid. We review this connection in more detail in the appendix C and relate it to the chiral magnetic effect in ideal Weyl gas. Hence, to conclude the following equations

$$D\bar{\mathcal{J}}_q + \frac{\partial}{\partial E_p}\bar{\mathcal{J}}_q^E = 0 \quad \text{with}$$

$$\bar{\mathcal{J}}_q^E = (qE - E_p a) \wedge \bar{\mathcal{J}}_q \tag{5.11}$$

$$\bar{\mathcal{J}}_q(E_p = 0) = \frac{\chi_{d=2n}}{2\pi} \left(\frac{qB}{2\pi}\right)^{n-1} \wedge \frac{u}{(n-1)!}$$

implies for an ideal Weyl gas the macroscopic adiabaticity condition in eq. (5.2). Our next task is to solve these equations for an arbitrary fluid flow which we will do in the next section.

6 Solving the adiabaticity equation

We now seek a solution of eq. (5.11) for an arbitrary fluid flow of an ideal Weyl gas. We begin by decomposing the exterior derivative of the velocity into its vorticity part and the acceleration part

$$Du = 2\omega - u \wedge a \tag{6.1}$$

where ω is the vorticity 2-form with $\omega_{\mu\nu}u^\nu = 0$. We will now make an ansatz for the solution of the form

$$\bar{\mathcal{J}}_q = \frac{\chi_{d=2n}}{(2\pi)^n} \sum_{k=0}^{n-1} \alpha_k \frac{(2\omega E_p)^k}{k!} \wedge \frac{(qB)^{n-1-k}}{(n-1-k)!} \wedge u \tag{6.2}$$

This ansatz is motivated by the solution to the macroscopic adiabaticity equations presented in [4] which took the form of a polynomial in 2-forms B and ω. The powers of E_p are then fixed by dimensional analysis. The numerical coefficients are arranged such that the $E_p = 0$ boundary condition fixes $\alpha_0 = 1$. We now want to substitute this ansatz into the eq. (5.2) to fix other α_k s. Using the following identities [4]

$$Du = 2\omega + a \wedge u$$
$$D(qB) \wedge u = -qE \wedge 2\omega \wedge u \tag{6.3}$$
$$D(2\omega) \wedge u = a \wedge 2\omega \wedge u$$

we get

$$D\bar{\mathcal{J}}_q = -\frac{\chi_{d=2n}}{(2\pi)^n} \sum_{k=0}^{n-1} \left[k\alpha_{k-1}\frac{qE}{E_p} - (k+1)\alpha_k a \right] \wedge \frac{(2\omega E_p)^k}{k!} \frac{(qB)^{n-1-k}}{(n-1-k)!} \wedge u \tag{6.4}$$

where we have in addition used the fact that any $2n$ form made of purely spatial forms B and ω is zero. On the other hand

$$\frac{\partial}{\partial E_p} \bar{\mathcal{J}}_q^E = \frac{\chi_{d=2n}}{(2\pi)^n} \sum_{k=0}^{n-1} \left[k\alpha_k \frac{qE}{E_p} - (k+1)\alpha_k a \right] \wedge \frac{(2\omega E_p)^k}{k!} \frac{(qB)^{n-1-k}}{(n-1-k)!} \wedge u \tag{6.5}$$

and demanding that the sum of the last two equations should vanish sets $\alpha_k = \alpha_{k-1}$ for all $k \geq 1$. Along with the boundary condition at $E_p = 0$ which sets $\alpha_0 = 1$ this determines $\alpha_k = 1$ for all k. Substituting this into our ansatz, we finally get

$$\bar{\mathcal{J}}_q = \frac{\chi_{d=2n}}{2\pi} \left(\frac{qB + 2\omega E_p}{2\pi} \right)^{n-1} \wedge \frac{u}{(n-1)!} \tag{6.6}$$

This expression is the central result of the article - it is a formula for how the chiral states of a given energy flow when the fluid flows. Since we have not invoked any equations of motion for the fluid in our derivation, this is an 'off-shell' solution valid for arbitrary fluid flows — a microscopic analogue of the off-shell solution derived in [4]. In the rest of the article we will explore various consequences of the above formula.

7 Anomaly/transport in Ideal Weyl gas of arbitrary dimensions

We begin by substituting for $\bar{\mathcal{J}}_q$ in the expression for $\bar{\mathcal{G}}_{\text{anom}}$

$$\bar{\mathcal{G}}_{\text{anom}} = \sum_F \int_0^\infty dE_p \bar{\mathcal{J}}_q \, g_q$$
$$= \sum_F \chi_{d=2n} \int_0^\infty \frac{dE_p}{2\pi} \left(\frac{qB + 2\omega E_p}{2\pi} \right)^{n-1} \wedge \frac{u}{(n-1)!} \, g_q \tag{7.1}$$

To evaluate this integral, we will again pair the particles and anti-particles together and use the fact that if the charge/chirality of a particle is $(q, \chi_{d=2n})$ then the charge/chirality of the anti-particle is $(-q, (-1)^{n-1}\chi_{d=2n})$.

$$
\begin{aligned}
&\bar{\mathcal{G}}_{\text{anom}} \\
&= \sum_{\text{species}} \int_0^\infty \frac{dE_p}{2\pi} \left[g_q \left(\frac{qB + 2\omega E_p}{2\pi} \right)^{n-1} + (-1)^{n-1} g_{-q} \left(\frac{-qB + 2\omega E_p}{2\pi} \right)^{n-1} \right] \wedge \frac{\chi_{d=2n} u}{(n-1)!} \\
&= \sum_{\text{species}} \int_0^\infty \frac{dE_p}{2\pi} \left[g_q \left(\frac{qB + 2\omega E_p}{2\pi} \right)^{n-1} + g_{-q} \left(\frac{qB - 2\omega E_p}{2\pi} \right)^{n-1} \right] \wedge \frac{\chi_{d=2n} u}{(n-1)!} \qquad (7.2)
\end{aligned}
$$

To proceed further, it is convenient to employ a formal trick — the trick is to construct a generating function which will in one sweep contain in its Taylor expansion $\bar{\mathcal{G}}_{\text{anom}}$ of arbitrary even dimensions. To this end let us multiply the above expression by τ^{n-1} where τ is a formal parameter and perform a sum over all integers $n \geq 1$. We get

$$
\begin{aligned}
&\sum_{n=1}^\infty \tau^{n-1} \left(\bar{\mathcal{G}}_{\text{anom}} \right)_{d=2n} \\
&= \sum_{n=1}^\infty \sum_{\text{species}} \int_0^\infty \frac{dE_p}{2\pi} \left[g_q \left(\frac{qB + 2\omega E_p}{2\pi} \right)^{n-1} + g_{-q} \left(\frac{qB - 2\omega E_p}{2\pi} \right)^{n-1} \right] \wedge \frac{\tau^{n-1} \chi_{d=2n} u}{(n-1)!} \\
&= \sum_{\text{species}} e^{\frac{\tau}{2\pi} qB} \int_0^\infty \frac{dE_p}{2\pi} \left[g_q e^{\frac{\tau}{2\pi} 2\omega E_p} + g_{-q} e^{-\frac{\tau}{2\pi} 2\omega E_p} \right] \wedge \chi_d u \qquad (7.3)
\end{aligned}
$$

where we have used χ_d to represent the chirality in the appropriate dimension.

We evaluate this integral in the appendix A. The final result is

$$
\begin{aligned}
&\sum_{n=1}^\infty \tau^{n-1} \left(\bar{\mathcal{G}}_{\text{anom}} \right)_{d=2n} \\
&= - \sum_{\text{species}} e^{\frac{\tau}{2\pi} qB} \frac{2\pi}{(2\omega\tau)^2} \left[\frac{\frac{\tau}{2} 2\omega\beta^{-1}}{\sin \frac{\tau}{2} 2\omega\beta^{-1}} e^{\frac{\tau}{2\pi} 2\omega q\mu} - \left(1 + \frac{\tau}{2\pi} 2\omega q\mu \right) \right] \wedge \chi_d u
\end{aligned}
\qquad (7.4)
$$

or

$$
\begin{aligned}
&\sum_{n=1}^\infty \tau^{n+1} \left(\bar{\mathcal{G}}_{\text{anom}} \right)_{d=2n} \\
&= -\frac{1}{(2\omega)^2} \sum_{\text{species}} \left[2\pi \frac{\frac{\tau}{2} 2\omega\beta^{-1}}{\sin \frac{\tau}{2} 2\omega\beta^{-1}} e^{\frac{q\tau}{2\pi} (B + 2\omega\mu)} - 2\pi e^{\frac{\tau}{2\pi} qB} \left(1 + \frac{\tau}{2\pi} 2\omega q\mu \right) \right] \wedge \chi_d u
\end{aligned}
\qquad (7.5)
$$

We want to now compare this with the form eq. (1.3) derived in [4] which we reproduce for the convenience of the reader

$$
\begin{aligned}
\bar{\mathcal{G}}_{\text{anom}} = \frac{1}{(2\omega)^2} \Big\{ &\mathfrak{F}_{\text{anom}}^\omega[T(2\omega), B + \mu(2\omega)] - \left[\mathfrak{F}_{\text{anom}}^\omega[T(2\omega), B + \mu(2\omega)] \right]_{\omega=0} \\
&- \omega \left[\frac{\delta}{\delta\omega} \mathfrak{F}_{\text{anom}}^\omega[T(2\omega), B + \mu(2\omega)] \right]_{\omega=0} \Big\} \wedge u
\end{aligned}
\qquad (7.6)
$$

By comparing term by term, we get a simple expression

$$(\mathfrak{F}^\omega_{\text{anom}})_{d=2n} = -2\pi \sum_{\text{species}} \chi_{d=2n} \left[\frac{\frac{\tau}{2} T}{\sin \frac{\tau}{2} T} e^{\frac{\tau}{2\pi} q\mu} \right]_{\tau^{n+1}} \tag{7.7}$$

where the subscript τ^{n+1} denotes that one needs to Taylor-expand in τ and retain the coefficient of τ^{n+1}.

Our aim was to give a prescription to get to $\mathfrak{F}^\omega_{\text{anom}}$ from the corresponding anomaly polynomial. The anomaly polynomial of an Ideal Weyl gas is given by (see appendix B for explicit expressions in various dimensions)

$$(\mathcal{P}_{\text{anom}})_{d=2n} = -2\pi \sum_{\text{species}} \chi_{d=2n} \left[\hat{A}\left(\tau\mathfrak{R}\right) \, e^{\frac{\tau}{2\pi} qF} \right]_{\tau^{n+1}}$$

Now using (see appendix B for a derivation)

$$\left[\hat{A}\left(\tau\mathfrak{R}\right) \right]_{p_{k>1}(\mathfrak{R})=0} = \frac{\frac{\tau}{2}\sqrt{-p_1(\mathfrak{R})}}{\sin\left(\frac{\tau}{2}\sqrt{-p_1(\mathfrak{R})}\right)} \tag{7.8}$$

we can write

$$\mathfrak{F}^\omega_{\text{anom}} = \mathcal{P}_{\text{anom}}\left(F \mapsto \mu, \; p_1(\mathfrak{R}) \mapsto -T^2, \; p_{k>1}(\mathfrak{R}) \mapsto 0 \right) \tag{7.9}$$

in any dimension for arbitrary collection of free fermions. This is a remarkably elegant result which takes the anomaly polynomial and via simple substitutions converts it into the polynomial which governs the anomaly-induced transport.

8   Discussion

One of our main motivations in undertaking this study was to understand at a more microscopic level how the anomaly-induced transport comes about. While the transport proportional to the magnetic field can be understood relatively easily,[12] the microscopic origins of the vortical effect were relatively obscure. Having understood now the relevant microscopic dynamics as that of spectral flow, one might ask whether one can proceed away from equilibrium. In other words, how does this picture of chiral spectral current help us understand the non-equilibrium phenomena driven by anomaly.

While we do not have a complete answer to this question, let us give an idea of what the form the answer might take. Ideally to answer non-equilibrium questions, one would like to have a Boltzmann-type equation describing an ideal Weyl gas to which interactions can be added via collision terms. The Boltzmann equation is intimately tied to the flow of states in the semi-classical phase-space: in fact, it is just a conservation-type equation on the phase space. Let ζ^A be the co-ordinates on the phase-space and $\mathcal{J}^A_{\text{tot}}$ be the net flow of states (the total spectral current) in the phase-space. Then the Boltzmann equation takes the schematic form

$$\frac{\partial}{\partial \zeta^A} \left[f \mathcal{J}^A_{\text{tot}} \right] = C[f]$$

---

[12]By a Landau level argument which we review in appendix C.

where the r.h.s. is the collision term which takes into account the scattering from one point in the phase-space to the other point. Assuming the total number of states is conserved, we can assume $\partial_A \mathcal{J}_{\text{tot}}^A = 0$ so that the Boltzmann equation reduces to

$$\mathcal{J}_{\text{tot}}^A \partial_A f = C[f]$$

Hence, we see that a spectral current in the phase space leads us immediately to a Boltzmann type equation. To proceed further we need to figure out what the expression for $\mathcal{J}_{\text{tot}}^A$ is? Conventional kinetic theory would suggest that

$$\dot{x}^\mu \sim p^\mu \quad \text{and} \quad \dot{p}_\mu \sim q p^\nu F_{\mu\nu}$$
$$\implies \mathcal{J}_{naive}^A \partial_A \sim p^\mu \frac{\partial}{\partial x^\mu} + q p^\nu F_{\mu\nu} \frac{\partial}{\partial p_\mu} \tag{8.1}$$

which would be identical to the Boltzmann operator one would write down for a Dirac fermion. But, as we had argued in this article, in an ideal Weyl gas there is an *additional* contribution to the spectral current over and above the contribution in a Dirac gas. This should lead to the modification of the naive Boltzmann operator that we had written above. Unfortunately, the spectral current that we have written down in this article has no momentum information — it is only a function of $\{x^\mu, E_p\}$ and hence we cannot yet write down the exact modification of the Boltzmann operator. It would be nice to work out the momentum-resolved chiral spectral current in order to write down this modification.

We will now give an alternate argument on why we expect such a modification to the Boltzmann equation — as is known in the phenomenology of Weyl semi-metals [26, 27] (these are $3+1$d phases with Weyl fermionic excitations), the Weyl fermion in $3+1$d should be thought of as a source of Berry flux in the momentum space. The presence of such a Berry-flux is known to lead to exactly the kind of modifications of the Boltzmann equations [24, 25] that we propose above. It would be nice to clarify the relation between such Berry phase related ideas and the ideas presented in this article.

Let us recap: what we have done is to use adiabaticity as the basic idea which drives the anomaly-induced transport in Ideal Weyl gases. This has resulted in a simple rule given by eq. (7.9) which determines completely the polynomial introduced in [4]. The novel result is the way gravitational anomalies seem to enter into the temperature dependence as was noted by [11]. While we have reproduced their $4d$ results, we have done it by an entirely different method — they had employed the $4d$ Kubo formulae derived in [5] whereas our derivation relies on adiabaticity.

Kubo formulae for anomaly-induced transport in higher dimensions involve higher point functions and since the Kubo formalism is not well-developed beyond two-point functions, it is *a priori* more difficult to take that route for calculations in higher dimensions. In this article, rather than generalize the Kubo formulae in [5] we have taken an alternate route. But having got the answer, let us provide a guess as to how the results of [5] would generalize. Consider the small frequency/small momentum limit of the following n-point function in $d = 2n$ dimensions at finite T, μ:

$$\epsilon^{0i_1 i_2 \dots i_n j_1 j_2 \dots j_{n-1}} \langle\, T_{0i_1}(k_1^\alpha) T_{0i_2}(k_2^\alpha) \dots T_{0i_l}(k_l^\alpha) J_{i_{l+1}}(k_{l+1}^\alpha) J_{i_{l+2}}(k_{l+2}^\alpha) \dots J_{i_n}(k_n^\alpha) \,\rangle$$
$$\sim \xi_l(T, \mu)\, \delta^{2n}\,(k_1 + k_2 + \dots + k_n)\, k_1^{[j_1} k_2^{j_2} \dots k_{n-1}^{j_{n-1}]} \tag{8.2}$$

then generalizing the results of [5], it is tempting to conjecture that $\xi_l(T, \mu)$ is the transport coefficient associated with $(2\omega)^{l-1} \wedge B^{n-l} \wedge u$ term in $\bar{q}_{\mathrm{anom}}$ or equivalently $(2\omega)^l \wedge B^{n-l-1} \wedge u$ term in $\bar{J}_{\mathrm{anom}}$ since these two are related by the generalised Onsager type relation [4]

$$\frac{\delta \bar{q}_{\mathrm{anom}}}{\delta B} = \frac{\delta \bar{J}_{\mathrm{anom}}}{\delta (2\omega)} \tag{8.3}$$

This in particular would mean that for a free theory we expect the transport coefficient $\xi_l(T, \mu)$ to be related to a Fermi-Dirac integral of the type[13]

$$\xi_l(T, \mu) \sim \int_0^\infty \frac{dE_p}{2\pi} E_p^l \left[f_q - (-1)^l f_{-q} \right]$$

It would be nice to show that such a Kubo-type formula as conjectured above does hold,[14] since it then opens up the possibility that we can calculate such transport coefficients in any field theory. In particular, such a Kubo formula would be of enormous use in checking whether the substitution rule in eq. (7.9) holds beyond free theory. It would be interesting to see what happens to this rule as the effective anomaly polynomial is changed via Green-Schwarz mechanism.

There is already some evidence that this rule holds in strongly coupled holographic phases [23]. It would be interesting to generalize this by constructing and studying rotating black hole solutions in AdS_{2n+1} with pure/mixed gravitational Chern-Simons terms. This would presumably link the proposed modification of Wald entropy [12, 13] in the presence of such Chern-Simons terms to the anomaly-induced entropy transport in the dual CFT.

We do not yet have an intuitive field theory understanding why the higher Pontryagin classes do not contribute to the anomaly induced transport that we discuss in this article. One way to undertsand this statement better would be to see whether this is true in fluids with gravity duals via the calculation we just outlined. One way this might work out is if higher Pontryagin classes fall off too fast in the bulk to contribute to boundary transport while the first Pontryagin class falls off slowly and asymptotes to $-T^2(2\omega)^2$ near the AdS boundary.[15]

Another way in which the T^2 coefficients in $4d$ were calculated was by Vilenkin using rotating ensembles [15–20]. This method as Vilenkin used is plagued with technical subtleties which have to do with trying to define rigid rotation in a relativistic field theory in the Minkowski spacetime. But these subtleties can be avoided by studying rotating ensembles on a *sphere* which serves as an infrared regulator. We have done some preliminary calculations in Weyl fermions using this method and they seem to agree with the results presented here. The details of this calculation will be presented elsewhere.

Further it would be interesting to generalise all these arguments to spacetimes with non-zero curvature. A first step towards such a generalisation would be to first classify the various curvature dependent transport processes which are sensitive to anomalies. Since a free theory of Weyl fermions is conformal, we can restrict our attention to just the

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[13]We would like to thank Carlos Hoyos for suggesting such a relation.

[14]This would presumably involve generalising the Kubo formalism to higher point functions [31–36].

[15]We would like to thank Karl Landsteiner and Mukund Rangamani for discussions regarding this issue.

Weyl-covariatised curvature tensors [37] and the holographic computations that we had outlined above could be easily used to identify the transport processes which are sensitive to anomalies. It would be satisfying to understand the chiral spectral current in the strongly interacting systems via holography.[16]

Acknowledgments

A   Notes on Fermi-Dirac functions

Our objective in this section is to evaluate various moments associated with Fermi-Dirac distributions. We will start by defining some basic functions which would be useful later on. First we want to define the nth poly-logarithm $Li_n(x)$ defined via

$$
\begin{aligned}
Li_n(x) &\equiv \sum_{k=1}^{\infty} \frac{x^k}{k^n} = \sum_{k=1}^{\infty} \int_0^{\infty} dy (xe^{-y})^k \frac{y^{n-1}}{(n-1)!} \\
&= \int_0^{\infty} dy \, \frac{1}{x^{-1}e^y - 1} \frac{y^{n-1}}{(n-1)!} = \int_0^{\infty} dy \, \frac{d}{dy} \left[\ln \left(1 - xe^{-y} \right) \right] \frac{y^{n-1}}{(n-1)!} \\
&= \int_0^{\infty} dy \, \left[-\ln \left(1 - xe^{-y} \right) \right] \frac{y^{n-2}}{(n-2)!}
\end{aligned}
\tag{A.1}
$$

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[16]We would like to thank Jyotirmoy Bhattacharya for discussions regarding this issue.

the last line being valid for $n \geq 2$. The importance of polylogarithm lies in its relation to various moments of Fermi-Dirac integrals - it follows from above that

$$\int_0^\infty dE_p \frac{E_p^k}{k!} g_q = \frac{1}{\beta^{k+2}} Li_{k+2}\left(-e^{\beta q\mu}\right)$$
$$\int_0^\infty dE_p \frac{E_p^k}{k!} f_q = -\frac{1}{\beta^{k+1}} Li_{k+1}\left(-e^{\beta q\mu}\right)$$

(A.2)

We introduce next the nth Bernoulli Polynomials $B_n(x)$ - these are polynomials defined via the following generating function

$$\frac{t\, e^{xt}}{e^t - 1} \equiv \sum_{n=0}^\infty \frac{t^n}{n!}\, B_n(x)$$

(A.3)

We now write the identity relating poly-logarithms with Bernoulli polynomials (which itself is a special case of the relation between poly-logarithms and Hurwitz zeta function)

$$Li_n\left(-e^{\beta q\mu}\right) + (-1)^n Li_n\left(-e^{-\beta q\mu}\right) = -\frac{(2\pi i)^n}{n!} B_n\left(\frac{1}{2} + \frac{\beta q\mu}{2\pi i}\right)$$

(A.4)

which implies the following identity for the Fermi-Dirac moments

$$\int_0^\infty \frac{dE_p}{2\pi} \frac{1}{k!} \left(\frac{E_p}{2\pi}\right)^k \left[g_q + (-1)^k g_{-q}\right] = -\frac{2\pi}{(k+2)!} \left(\frac{i}{\beta}\right)^{k+2} B_{k+2}\left(\frac{1}{2} + \frac{\beta q\mu}{2\pi i}\right)$$

(A.5)

or more explicitly

$$\int_0^\infty \frac{dE_p}{2\pi} [g_q + g_{-q}] = -2\pi \left[\frac{1}{2!}\left(\frac{q\mu}{2\pi}\right)^2 + \frac{T^2}{4!}\right]$$
$$\int_0^\infty \frac{dE_p}{2\pi} \left(\frac{E_p}{2\pi}\right) [g_q - g_{-q}] = -2\pi \left[\frac{1}{3!}\left(\frac{q\mu}{2\pi}\right)^3 + \left(\frac{q\mu}{2\pi}\right)\frac{T^2}{4!}\right]$$
$$\int_0^\infty \frac{dE_p}{2\pi} \left(\frac{E_p}{2\pi}\right)^2 [g_q + g_{-q}] = -2\pi \left[\frac{1}{4!}\left(\frac{q\mu}{2\pi}\right)^4 + \frac{1}{2!}\left(\frac{q\mu}{2\pi}\right)^2\frac{T^2}{4!} + \frac{7}{8}\frac{T^4}{6!}\right]$$
$$\int_0^\infty \frac{dE_p}{2\pi} \left(\frac{E_p}{2\pi}\right)^3 [g_q - g_{-q}] = -2\pi \left[\frac{1}{5!}\left(\frac{q\mu}{2\pi}\right)^5 + \frac{1}{3!}\left(\frac{q\mu}{2\pi}\right)^3\frac{T^2}{4!} + \left(\frac{q\mu}{2\pi}\right)\frac{7}{8}\frac{T^4}{6!}\right]$$
$$\int_0^\infty \frac{dE_p}{2\pi} \left(\frac{E_p}{2\pi}\right)^4 [g_q + g_{-q}] = -2\pi \left[\frac{1}{6!}\left(\frac{q\mu}{2\pi}\right)^6 + \frac{1}{4!}\left(\frac{q\mu}{2\pi}\right)^4\frac{T^2}{4!} + \frac{1}{2!}\left(\frac{q\mu}{2\pi}\right)^2\frac{7}{8}\frac{T^4}{6!} + \frac{31}{24}\frac{T^6}{8!}\right]$$

(A.6)

Similarly, we can write

$$\int_0^\infty \frac{dE_p}{2\pi} \frac{1}{k!} \left(\frac{E_p}{2\pi}\right)^k \left[f_q - (-1)^k f_{-q}\right] = \frac{1}{(k+1)!} \left(\frac{i}{\beta}\right)^{k+1} B_{k+1}\left(\frac{1}{2} + \frac{\beta q\mu}{2\pi i}\right)$$

(A.7)

or more explicitly

$$\int_0^\infty \frac{dE_p}{2\pi} \left[f_q - f_{-q}\right] = \left(\frac{q\mu}{2\pi}\right)$$

$$\int_0^\infty \frac{dE_p}{2\pi} \left(\frac{E_p}{2\pi}\right) \left[f_q + f_{-q}\right] = \frac{1}{2!}\left(\frac{q\mu}{2\pi}\right)^2 + \frac{T^2}{4!}$$

$$\int_0^\infty \frac{dE_p}{2\pi} \left(\frac{E_p}{2\pi}\right)^2 \left[f_q - f_{-q}\right] = \frac{1}{3!}\left(\frac{q\mu}{2\pi}\right)^3 + \left(\frac{q\mu}{2\pi}\right)\frac{T^2}{4!}$$

$$\int_0^\infty \frac{dE_p}{2\pi} \left(\frac{E_p}{2\pi}\right)^3 \left[f_q + f_{-q}\right] = \frac{1}{4!}\left(\frac{q\mu}{2\pi}\right)^4 + \frac{1}{2!}\left(\frac{q\mu}{2\pi}\right)^2\frac{T^2}{4!} + \frac{7}{8}\frac{T^4}{6!}$$

$$\int_0^\infty \frac{dE_p}{2\pi} \left(\frac{E_p}{2\pi}\right)^4 \left[f_q - f_{-q}\right] = \frac{1}{5!}\left(\frac{q\mu}{2\pi}\right)^5 + \frac{1}{3!}\left(\frac{q\mu}{2\pi}\right)^3\frac{T^2}{4!} + \left(\frac{q\mu}{2\pi}\right)\frac{7}{8}\frac{T^4}{6!}$$

$$\int_0^\infty \frac{dE_p}{2\pi} \left(\frac{E_p}{2\pi}\right)^5 \left[f_q + f_{-q}\right] = \frac{1}{6!}\left(\frac{q\mu}{2\pi}\right)^6 + \frac{1}{4!}\left(\frac{q\mu}{2\pi}\right)^4\frac{T^2}{4!} + \frac{1}{2!}\left(\frac{q\mu}{2\pi}\right)^2\frac{7}{8}\frac{T^4}{6!} + \frac{31}{24}\frac{T^6}{8!}$$

(A.8)

Alternately, we can directly write down a generating function for these integrals by using the generating function of Bernoulli polynomials. Multiply eq. (A.5) by τ^k, sum over $k = 0$ to ∞ and then use eq. (A.3) to get

$$\int_0^\infty \frac{dE_p}{2\pi} \left[g_q e^{\frac{\tau}{2\pi}E_p} + g_{-q} e^{-\frac{\tau}{2\pi}E_p}\right] = -\frac{2\pi}{\tau^2}\left[\frac{\frac{\tau}{2}T}{\sin\left(\frac{\tau}{2}T\right)} e^{\tau\frac{q\mu}{2\pi}} - \left(1 + \tau\frac{q\mu}{2\pi}\right)\right] \qquad \text{(A.9)}$$

which we used in the main text to evaluate the integral in eq. (7.4). In a similar vein we get a generating function for the f_q integral

$$\int_0^\infty \frac{dE_p}{2\pi} \left[f_q e^{\frac{\tau}{2\pi}E_p} - f_{-q} e^{-\frac{\tau}{2\pi}E_p}\right] = \frac{1}{\tau}\left[\frac{\frac{\tau}{2}T}{\sin\left(\frac{\tau}{2}T\right)} e^{\tau\frac{q\mu}{2\pi}} - 1\right] \qquad \text{(A.10)}$$

These formulae can be used to explicitly calculate $\bar{\mathcal{G}}_{\text{anom}}$.

$$\left(\bar{\mathcal{G}}_{\text{anom}}\right)_{d=2} = -2\pi \sum_{\text{species}} \chi_{d=2} \left[\frac{1}{2!}\left(\frac{q\mu}{2\pi}\right)^2 + \frac{T^2}{4!}\right] u \qquad \text{(A.11)}$$

$$\left(\bar{\mathcal{G}}_{\text{anom}}\right)_{d=4} = -2\pi \sum_{\text{species}} \chi_{d=4} \left[\frac{1}{3!}\left(\frac{q\mu}{2\pi}\right)^3 + \left(\frac{q\mu}{2\pi}\right)\frac{T^2}{4!}\right] (2\omega) \wedge u$$

$$-2\pi \sum_{\text{species}} \chi_{d=4} \left[\frac{1}{2!}\left(\frac{q\mu}{2\pi}\right)^2 + \frac{T^2}{4!}\right] (qB) \wedge u \qquad \text{(A.12)}$$

$$\left(\bar{\mathcal{G}}_{\text{anom}}\right)_{d=6} = -2\pi \sum_{\text{species}} \chi_{d=6} \left[\frac{1}{4!}\left(\frac{q\mu}{2\pi}\right)^4 + \frac{1}{2!}\left(\frac{q\mu}{2\pi}\right)^2\frac{T^2}{4!} + \frac{7}{8}\frac{T^4}{6!}\right] (2\omega)^2 \wedge u$$

$$-2\pi \sum_{\text{species}} \chi_{d=6} \left[\frac{1}{3!}\left(\frac{q\mu}{2\pi}\right)^3 + \left(\frac{q\mu}{2\pi}\right)\frac{T^2}{4!}\right] (2\omega) \wedge (qB) \wedge u$$

$$-2\pi \sum_{\text{species}} \chi_{d=6} \left[\frac{1}{2!}\left(\frac{q\mu}{2\pi}\right)^2 + \frac{T^2}{4!}\right] \frac{(qB)^2}{2!} \wedge u \qquad \text{(A.13)}$$

$$
\begin{aligned}
(\bar{\mathcal{G}}_{\text{anom}})_{d=8} = &- 2\pi \sum_{\text{species}} \chi_{d=8} \left[\frac{1}{5!} \left(\frac{q\mu}{2\pi}\right)^5 + \frac{1}{3!} \left(\frac{q\mu}{2\pi}\right)^3 \frac{T^2}{4!} + \left(\frac{q\mu}{2\pi}\right) \frac{7}{8} \frac{T^4}{6!} \right] (2\omega)^3 \wedge u \\
&- 2\pi \sum_{\text{species}} \chi_{d=8} \left[\frac{1}{4!} \left(\frac{q\mu}{2\pi}\right)^4 + \frac{1}{2!} \left(\frac{q\mu}{2\pi}\right)^2 \frac{T^2}{4!} + \frac{7}{8} \frac{T^4}{6!} \right] (2\omega)^2 \wedge (qB) \wedge u \\
&- 2\pi \sum_{\text{species}} \chi_{d=8} \left[\frac{1}{3!} \left(\frac{q\mu}{2\pi}\right)^3 + \left(\frac{q\mu}{2\pi}\right) \frac{T^2}{4!} \right] (2\omega) \wedge \frac{(qB)^2}{2!} \wedge u \\
&- 2\pi \sum_{\text{species}} \chi_{d=8} \left[\frac{1}{2!} \left(\frac{q\mu}{2\pi}\right)^2 + \frac{T^2}{4!} \right] \frac{(qB)^3}{3!} \wedge u
\end{aligned}
\tag{A.14}
$$

$$
\begin{aligned}
(\bar{\mathcal{G}}_{\text{anom}})_{d=10} = &- 2\pi \sum_{\text{species}} \chi_{d=10} \left[\frac{1}{6!} \left(\frac{q\mu}{2\pi}\right)^6 + \frac{1}{4!} \left(\frac{q\mu}{2\pi}\right)^4 \frac{T^2}{4!} + \frac{1}{2!} \left(\frac{q\mu}{2\pi}\right)^2 \frac{7}{8} \frac{T^4}{6!} + \frac{31}{24} \frac{T^6}{8!} \right] (2\omega)^4 \wedge u \\
&- 2\pi \sum_{\text{species}} \chi_{d=10} \left[\frac{1}{5!} \left(\frac{q\mu}{2\pi}\right)^5 + \frac{1}{3!} \left(\frac{q\mu}{2\pi}\right)^3 \frac{T^2}{4!} + \left(\frac{q\mu}{2\pi}\right) \frac{7}{8} \frac{T^4}{6!} \right] (2\omega)^3 \wedge (qB) \wedge u \\
&- 2\pi \sum_{\text{species}} \chi_{d=10} \left[\frac{1}{4!} \left(\frac{q\mu}{2\pi}\right)^4 + \frac{1}{2!} \left(\frac{q\mu}{2\pi}\right)^2 \frac{T^2}{4!} + \frac{7}{8} \frac{T^4}{6!} \right] (2\omega)^2 \wedge \frac{(qB)^2}{2!} \wedge u \\
&- 2\pi \sum_{\text{species}} \chi_{d=10} \left[\frac{1}{3!} \left(\frac{q\mu}{2\pi}\right)^3 + \left(\frac{q\mu}{2\pi}\right) \frac{T^2}{4!} \right] (2\omega) \wedge \frac{(qB)^3}{3!} \wedge u \\
&- 2\pi \sum_{\text{species}} \chi_{d=10} \left[\frac{1}{2!} \left(\frac{q\mu}{2\pi}\right)^2 + \frac{T^2}{4!} \right] \frac{(qB)^4}{4!} \wedge u
\end{aligned}
\tag{A.15}
$$

The energy/charge/entropy currents can be obtained from these expressions via

$$
\begin{aligned}
\bar{J}_{\text{anom}} &= - \frac{\partial \bar{\mathcal{G}}_{\text{anom}}}{\partial \mu} \\
\bar{J}_{S,\text{anom}} &= - \frac{\partial \bar{\mathcal{G}}_{\text{anom}}}{\partial T} \\
\bar{q}_{\text{anom}} &= \bar{\mathcal{G}}_{\text{anom}} + T \bar{J}_{S,\text{anom}} + \mu \bar{J}_{\text{anom}}
\end{aligned}
\tag{A.16}
$$

B Anomaly polynomials

One of the most succinct ways to capture the anomalies in a system is via the anomaly polynomial $\mathcal{P}_{\text{anom}}(F, \mathfrak{R})$ of gauge field strengths F and the spacetime curvature $\mathfrak{R}$. We will not review how these polynomials are calculated or how covariant anomalies are obtained from them since these topics are covered well in various textbooks [38–40] and lecture notes [41, 42]. We will mainly state the results relevant for Weyl fermions in arbitrary dimensions to save the reader the effort of converting the standard results into our notation.

We will begin by reviewing various forms relevant for dealing with gravitational anomalies. Let $\mathfrak{R}_{ab}$ be the curvature 2-forms of the spacetime with

$$
\mathfrak{R}_{ab} \equiv \frac{1}{2!} R_{abcd} \, dx^c \wedge dx^d
\tag{B.1}
$$

In $d = 2n$ dimensions we can think of $\mathfrak{R}_{ab}$ as a real $2n \times 2n$ antisymmetric matrix of 2-forms. At a given point in the manifold, we can diagonalize it with the diagonal entries being 2-forms

$$\mathfrak{R}_{ab} = \begin{pmatrix} +ir_1 & 0 & \ldots \ldots & \ldots & \ldots \\ 0 & -ir_1 & \ldots \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \ldots & +ir_n & 0 \\ \ldots & \ldots & \ldots \ldots & 0 & -ir_n \end{pmatrix}_{ab} \tag{B.2}$$

where $r_{j=1,\ldots,n}$ are real 2-forms. Polynomials in these 2-forms can be used to construct various other useful forms. The rest of this section basically consists of various such polynomials, relations between them and their generating functions etc. We start with the most basic form

$$\mathfrak{R}_k \equiv \frac{1}{2} \mathfrak{R}_{a_1}{}^{a_2} \wedge \mathfrak{R}_{a_2}{}^{a_3} \ldots \wedge \mathfrak{R}_{a_k}{}^{a_1} = \frac{1 + (-1)^k}{2} \sum_j (ir_j)^k \tag{B.3}$$

so $\mathfrak{R}_k$ is a is a $2k$-form which is a kth-degree polynomial in the curvature 2-forms. As is evident from the expression above, it is non-zero only when k is even.

The next form which is we will introduce is called the kth-Pontryagin class denoted by $p_k(\mathfrak{R})$ which is $4k$-form using a specific $2k$-th degree polynomial in the curvature 2-forms. It is defined via the relation

$$\det \left[1 + \frac{\tau}{2\pi} \mathfrak{R} \right] = \prod_j \left[1 - \left(\frac{\tau}{2\pi} ir_j \right)^2 \right] \equiv \sum_k \tau^{2k} p_k(\mathfrak{R}) \tag{B.4}$$

This gives

$$p_1(\mathfrak{R}) = -\frac{1}{(2\pi)^2} \mathfrak{R}_2$$

$$p_2(\mathfrak{R}) = -\frac{1}{(2\pi)^4} \left[\frac{1}{2} \mathfrak{R}_4 - \frac{1}{2} \mathfrak{R}_2^2 \right] \tag{B.5}$$

$$p_3(\mathfrak{R}) = -\frac{1}{(2\pi)^6} \left[\frac{1}{3} \mathfrak{R}_6 - \frac{1}{2} \mathfrak{R}_2 \mathfrak{R}_4 + \frac{1}{6} \mathfrak{R}_2^3 \right]$$

which can be inverted to get

$$\frac{1}{(2\pi)^2} \mathfrak{R}_2 = -p_1(\mathfrak{R})$$

$$\frac{1}{(2\pi)^4} \mathfrak{R}_4 = -2p_2(\mathfrak{R}) + p_1^2(\mathfrak{R}) \tag{B.6}$$

$$\frac{1}{(2\pi)^6} \mathfrak{R}_6 = -3p_3(\mathfrak{R}) + 3p_1(\mathfrak{R})p_2(\mathfrak{R}) - p_1^3(\mathfrak{R})$$

We will express all the other polynomials in the basis of either $\mathfrak{R}_k$s or $p_k(\mathfrak{R})$.

An object that plays an important role in the anomalies generated by Weyl fermions is the Dirac genus (or A-roof genus) $\hat{A}_k(\mathfrak{R})$ which is defined via

$$\hat{A}(\tau\mathfrak{R}) \equiv \det^{1/2} \left[\frac{\frac{1}{2}\frac{\tau}{2\pi}\mathfrak{R}}{\sin\left(\frac{1}{2}\frac{\tau}{2\pi}\mathfrak{R}\right)} \right] = \prod_j \left[\frac{\frac{1}{2}\frac{\tau}{2\pi}r_j}{\sinh\left(\frac{1}{2}\frac{\tau}{2\pi}r_j\right)} \right] = \sum_k \tau^{2k}\hat{A}_k(\mathfrak{R}) \qquad \text{(B.7)}$$

which gives

$$\begin{aligned}
\hat{A}_1(\mathfrak{R}) &= \frac{1}{(2\pi)^2}\left[\frac{\mathfrak{R}_2}{4!}\right] = \frac{1}{4!}\left[-p_1(\mathfrak{R})\right] \\
\hat{A}_2(\mathfrak{R}) &= \frac{1}{(2\pi)^4}\left[\frac{1}{4}\left(\frac{\mathfrak{R}_4}{6!}\right) + \frac{1}{2}\left(\frac{\mathfrak{R}_2}{4!}\right)^2\right] = \frac{1}{6!(2\pi)^4}\frac{1}{8}\left(2\mathfrak{R}_4 + 5\mathfrak{R}_2^2\right) \\
&= \frac{1}{6!}\left[-\frac{1}{2}p_2(\mathfrak{R}) + \frac{7}{8}p_1^2(\mathfrak{R})\right] \\
\hat{A}_3(\mathfrak{R}) &= \frac{1}{(2\pi)^6}\left[\frac{2}{9}\left(\frac{\mathfrak{R}_6}{8!}\right) + \frac{1}{4}\left(\frac{\mathfrak{R}_2}{4!}\right)\left(\frac{\mathfrak{R}_4}{6!}\right) + \frac{1}{6}\left(\frac{\mathfrak{R}_2}{4!}\right)^3\right] \\
&= \frac{1}{8!(2\pi)^6}\frac{1}{24}\left(\frac{16}{3}\mathfrak{R}_6 + 14\,\mathfrak{R}_4\mathfrak{R}_2 + \frac{35}{3}\mathfrak{R}_2^3\right) \\
&= \frac{1}{8!}\left[-\frac{2}{3}p_3(\mathfrak{R}) + \frac{11}{6}p_1(\mathfrak{R})p_2(\mathfrak{R}) - \frac{31}{24}p_1^3(\mathfrak{R})\right]
\end{aligned} \qquad \text{(B.8)}$$

The kth-Dirac genus $\hat{A}_k(\mathfrak{R})$ is hence a $4k$-form using a specific $2k$-th degree polynomial in the curvature 2-forms.

We now want to calculate the Dirac genus in the special case where the only non-zero Pontryagin class is $p_1(\mathfrak{R})$ with $p_{k>1}(\mathfrak{R}) = 0$. It is clear that in this case $\hat{A}_k(\mathfrak{R}) \sim p_1^k(\mathfrak{R})$ but we want to calculate the numerical coefficient exactly. To do this, we use the following trick: we set the 2-forms $r_{j>1} = 0$ with a non-zero r_1. Then

$$\det\left[1 + \frac{\tau}{2\pi}\mathfrak{R}\right] = 1 - \left(\frac{\tau}{2\pi}ir_1\right)^2$$

which means

$$p_1(\mathfrak{R}) = \left(\frac{r_1}{2\pi}\right)^2, \quad p_{k>1}(\mathfrak{R}) = 0$$

The roof-genus is easily evaluated in this case as

$$\frac{\frac{1}{2}\frac{\tau}{2\pi}r_j}{\sinh\left(\frac{1}{2}\frac{\tau}{2\pi}r_j\right)} = \frac{\frac{\tau}{2}\sqrt{p_1(\mathfrak{R})}}{\sinh\left(\frac{\tau}{2}\sqrt{p_1(\mathfrak{R})}\right)} = \frac{\frac{\tau}{2}\sqrt{-p_1(\mathfrak{R})}}{\sin\left(\frac{\tau}{2}\sqrt{-p_1(\mathfrak{R})}\right)}$$

So this gives us the required answer

$$\left[\hat{A}(\tau\mathfrak{R})\right]_{p_{k>1}(\mathfrak{R})=0} = \frac{\frac{\tau}{2}\sqrt{-p_1(\mathfrak{R})}}{\sin\left(\frac{\tau}{2}\sqrt{-p_1(\mathfrak{R})}\right)} \qquad \text{(B.9)}$$

The anomaly polynomial for a bunch of Weyl fermions in a $d = (2n-1)+1$ dimensional spacetime is given by

$$(\mathcal{P}_{\text{anom}})_{d=2n} = -2\pi \sum_{\text{species}} \chi_{d=2n} \left[\hat{A}(\tau\mathfrak{R})\,e^{\frac{\tau}{2\pi}qF}\right]_{\tau^{n+1}}$$

where we have assumed that there is a single U(1) symmetry under which the charge of the Weyl fermion is denoted by the letter q. The sum is over the species which means that each particle/anti-particle pair contributes one term to the sum (the answer does not depend on whether we take the chirality/charge of a particle or the anti-particle). We now proceed to write down the explicit expressions for $\mathcal{P}_{\text{anom}}$ by using the formula above.

$$
(\mathcal{P}_{\text{anom}})_{d=1+1} = -2\pi \sum_{\text{species}} \chi_{d=2} \left[\frac{1}{2!} \left(\frac{qF}{2\pi} \right)^2 + \hat{A}_1(\mathfrak{R}) \right]
$$

$$
(\mathcal{P}_{\text{anom}})_{d=3+1} = -2\pi \sum_{\text{species}} \chi_{d=4} \left[\frac{1}{3!} \left(\frac{qF}{2\pi} \right)^3 + \left(\frac{qF}{2\pi} \right) \hat{A}_1(\mathfrak{R}) \right]
$$

$$
(\mathcal{P}_{\text{anom}})_{d=5+1} = -2\pi \sum_{\text{species}} \chi_{d=6} \left[\frac{1}{4!} \left(\frac{qF}{2\pi} \right)^4 + \frac{1}{2!} \left(\frac{qF}{2\pi} \right)^2 \hat{A}_1(\mathfrak{R}) + \hat{A}_2(\mathfrak{R}) \right]
$$

$$
(\mathcal{P}_{\text{anom}})_{d=7+1} = -2\pi \sum_{\text{species}} \chi_{d=8} \left[\frac{1}{5!} \left(\frac{qF}{2\pi} \right)^5 + \frac{1}{3!} \left(\frac{qF}{2\pi} \right)^3 \hat{A}_1(\mathfrak{R}) + \left(\frac{qF}{2\pi} \right) \hat{A}_2(\mathfrak{R}) \right]
$$

$$
(\mathcal{P}_{\text{anom}})_{d=9+1} = -2\pi \sum_{\text{species}} \chi_{d=10} \left[\frac{1}{6!} \left(\frac{qF}{2\pi} \right)^6 + \frac{1}{4!} \left(\frac{qF}{2\pi} \right)^4 \hat{A}_1(\mathfrak{R}) \right.
$$
$$
\left. + \frac{1}{2!} \left(\frac{qF}{2\pi} \right)^2 \hat{A}_2(\mathfrak{R}) + \hat{A}_3(\mathfrak{R}) \right] \tag{B.10}
$$

where explicit expressions for $\hat{A}_i(\mathfrak{R})$ are given by eq. (B.8).

C Chiral magnetic effect in arbitrary dimensions

In this appendix, our main aim is to understand in more physical terms how the boundary condition for the chiral spectral current arises. In particular, we would like to understand the rate at which the zero modes are injected into the fluid in terms of Landau level physics.

Let us consider a Weyl fermion in $(2n-1)+1$ dimensions. As before we have a finite temperature T, a chemical potential μ and a constant velocity u^μ. We will again work in the rest frame of u^μ. The main aim of this section is to understand the transport that arises when you turn on a magnetic field in this rest frame. To do this, let us divide the $2n-1$ spatial directions into a direction x_1 and $n-1$ planes where the kth plane is the $(x_{i_{2k}}, x_{i_{2k+1}})$ plane where $k = 1, \ldots, n-1$. We by consider the system with a uniform magnetic field strength turned on in all these $n-1$ spatial planes $b_k \equiv B_{i_{2k}i_{2k+1}} \neq 0$ for $k = 1, \ldots, n-1$ leaving out one spatial direction x_1.

We can solve the Weyl equation in this magnetic background and the solutions are the well-known Landau levels with their Landau degeneracies. The energy levels are determined by the charge q, the momentum along the x_1 direction p_1 and the spin along k^{th} plane $S_{i_{2k}i_{2k+1}} \equiv \frac{1}{2}\sigma_k$ (the allowed values are $\sigma_k = \pm 1$) and the Landau level number in the kth

plane being a non-negative integer n_k.

$$E^2 \equiv p_1^2 + 2 \sum_{k=1}^{n-1} \left[|qb_k| \left(n_k + \frac{1}{2} \right) - \frac{1}{2} qb_k \sigma_k \right] \tag{C.1}$$

where we see the free-particle dispersion along x_1 direction, the harmonic oscillator like spectrum of the covariant Laplacian along with the Zeeman splitting due to the magnetic moment. The Landau degeneracy in momentum between p_1 and $p_1 + dp_1$ (for a fixed Landau level number n_k and spins σ_k) is

$$\mathcal{D}_{\text{Landau}} = \frac{dp_1}{2\pi} \prod_{k=1}^{n-1} \frac{|qb_k|}{2\pi} \tag{C.2}$$

We notice that these are effectively a collection of $1+1$ dimensional free Dirac/Weyl Fermions with different masses. So one can repeat the analysis in section 3 and we conclude that there should be a current in this rest frame (along x_1 direction) which just depends on the chiral states.

To get a chiral state one has to set the effective $2d$ mass appearing above to zero: it is easy to see that the 2d-Weyl states are obtained only when $n_k = 0, \sigma_k = \text{sign}\,(qb_k)$. These states have $E^2 = p_1^2 \equiv E_p^2$ and a degeneracy

$$\frac{dp_1}{2\pi} \prod_{k=1}^{n-1} \frac{|qb_k|}{2\pi} = \frac{dE_p}{2\pi} \left(\prod_{k=1}^{n-1} \frac{qb_k}{2\pi} \right) \left(\prod_{k=1}^{n-1} \sigma_k \right)$$

If we denote the $1+1$-dimensional chirality as $\chi_{d=2}$ their total contribution to $\mathcal{G}_{\text{anom}}^\mu$ is

$$(e_1)^\mu \left(\prod_{k=1}^{n-1} \frac{qb_k}{2\pi} \right) \chi_{d=2} \left(\prod_{k=1}^{n-1} \sigma_k \right) \times \int_0^\infty \frac{dE_p}{2\pi} g_q \tag{C.3}$$

where $(e_1)^\mu$ is the unit vector along x_1 direction which ensures that the current is along the x_1 direction. We recognize the appearance of the $d = 2n$ chirality

$$\chi_{d=2n} = \chi_{d=2} \prod_{k=1}^{n-1} \sigma_k = \chi_{d=2} \prod_{k=1}^{n-1} \text{sign}\,(qb_k)$$

and this equation fixes the 2d chirality $\chi_{d=2}$ of the zero Landau-level in terms of the chirality $\chi_{d=2n}$ of the original Weyl fermion. This means we get one 2d-Weyl fermion of definite chirality for each Weyl fermion in higher dimensions.

As before, it is useful to take a Hodge dual and write this in terms of forms. Let $F_{\mu\nu}$ be the field strength which we can think of as a 2-form F. The rest frame electric field is $E_\mu = u^\nu F_{\mu\nu}$ which we can think of as a 1-form E. The rest frame magnetic field is a 2-form obtained by subtracting the electric part from F,

$$B \equiv F - u \wedge E$$

where $u = u_\mu dx^\mu$ is the velocity 1-form. In terms of these quantities, we finally get the total contribution from all fermions as

$$(\bar{\mathcal{G}}_{\text{anom}})_{u=\text{constant}} = \sum_F \int_0^\infty \frac{dE_p}{2\pi} g_q \, \chi_{d=2n} \left(\frac{qB}{2\pi}\right)^{n-1} \wedge \frac{u}{(n-1)!} \qquad (\text{C.4})$$

where the sum is performed over all the fermionic particles counting the particle and the anti-particle separately. We have denoted in the subscript our assumption that the velocity u^μ is constant. This gives the rate of injection of chiral zero modes by the magnetic field as

$$\bar{\mathcal{J}}_q(E_p = 0) = 1 \frac{\chi_{d=2n}}{2\pi} \left(\frac{qB}{2\pi}\right)^{n-1} \wedge \frac{u}{(n-1)!} \qquad (\text{C.5})$$

Certain comments are in order — when we turn on a mild magnetic field over a fluid, the gap between the Landau levels (and the Zeeman split) are small and hence one expects that any transport will get contribution from many Landau levels not just the lowest Landau level. This expectation is in fact correct in the case of conventional quantities like pressure or energy density — but remarkably enough the anomaly-induced contribution above does not get any contribution from the higher Landau levels even when the magnetic field is small. This is related to the statement that only chiral fermions contribute to the sum above. As a consistency check, one can easily check that Dirac fermions do not contribute to this sum (this follows since in a Dirac species for a given value of q, $\chi_{d=2n}$ takes both the values ± 1 thus canceling out in the above sum).

D Notation

We work in the $(- + + \ldots)$ signature. The dimensions of the spacetime in which the fluid lives is denoted by $d = 2n$. The Greek indices $\mu, \nu = 0, 1, \ldots, d-1$ are used as space-time indices.

We denote Hodge-duals by an overbar — for example, $\bar{J}$ is the 2n-1 form Hodge-dual to the 1-form J_μ. We mostly just use the Hodge-duality between 1-forms and 2n-1 forms and our conventions are completely defined by the following statement- given any $2n-1$ form $\bar{V}$ hodge-dual to V_μ and a 1-form A_μ, we have

$$D\bar{V} = (D_\mu V^\mu) \, \text{Vol}_{2n}$$
$$A \wedge \bar{V} = -\bar{V} \wedge A = A_\mu V^\mu \, \text{Vol}_{2n} \qquad (\text{D.1})$$

Given a 0-form α its Hodge-dual 2n-form is simply $\bar{\alpha} \equiv \alpha \, \text{Vol}_{2n}$.

We have included a table with other useful parameters used in the text. In the table 1, the relevant equations are denoted by their respective equation numbers appearing inside parentheses.

<table>
<tr><td colspan="4" align="center">Table of Notation</td></tr>
<tr><td align="right">Symbol</td><td>Definition</td><td align="right">Symbol</td><td>Definition</td></tr>
<tr><td align="right">ε</td><td>Energy density</td><td align="right">p</td><td>Pressure</td></tr>
<tr><td align="right">n</td><td>Charge density</td><td align="right">s</td><td>Entropy density</td></tr>
<tr><td align="right">μ</td><td>Chemical potential</td><td align="right">T</td><td>Temperature</td></tr>
<tr><td align="right">β</td><td>$1/T$</td><td align="right">$\chi_{d=2n}$</td><td>2n dimensional chirality</td></tr>
<tr><td align="right">q</td><td>Fermion charge</td><td align="right">E_p</td><td>Fermion energy</td></tr>
<tr><td align="right">g_q</td><td>$-\frac{1}{\beta}\ln\left[1+e^{-\beta(E_p-q\mu)}\right]$</td><td align="right">$f_q$</td><td>$\left[e^{\beta(E_p-q\mu)}+1\right]^{-1}$</td></tr>
<tr><td align="right">$\mathfrak{R}_{\mathfrak{ab}}$</td><td>Curvature 2-forms (B.1)</td><td align="right">$\mathfrak{R}_k$</td><td>See (B.3)</td></tr>
<tr><td align="right">$\mathcal{P}_{\mathrm{anom}}$</td><td>Anomaly polynomial</td><td align="right">$p_k(\mathfrak{R})$</td><td>Pontryagin class (B.4)</td></tr>
<tr><td align="right">$\mathfrak{F}^{\omega}_{\mathrm{anom}}$</td><td>See (1.3)</td><td align="right">$\hat{A}_k(\mathfrak{R})$</td><td>A-roof genus (B.8)</td></tr>
<tr><td align="right">u^μ, u</td><td>Fluid velocity,
1-form</td><td align="right">a_μ, a</td><td>Acceleration field
$(u.D)u_\mu$, 1-form</td></tr>
<tr><td align="right">$g_{\mu\nu}$</td><td>Spacetime metric</td><td align="right">$P_{\mu\nu}$</td><td>$g_{\mu\nu}+u_\mu u_\nu$</td></tr>
<tr><td align="right">$\sigma_{\mu\nu}$</td><td>Shear strain rate</td><td align="right">$\omega_{\mu\nu}, \omega$</td><td>Fluid vorticity, 2-form</td></tr>
<tr><td align="right">$T^{\mu\nu}$</td><td>Energy-momentum
tensor of the fluid</td><td align="right">J^μ</td><td>Charge currents
with anomalies</td></tr>
<tr><td align="right">J_S^μ</td><td>Entropy current</td><td align="right">D</td><td>Exterior derivative</td></tr>
<tr><td align="right">$\mathcal{G}^\mu_{\mathrm{anom}}$</td><td>Anomaly-induced
Gibbs current</td><td align="right">$\bar{\mathcal{G}}_{\mathrm{anom}}$</td><td>Hodge-dual of $\mathcal{G}^\mu_{\mathrm{anom}}$
$2n-1$ form</td></tr>
<tr><td align="right">q^μ_{anom}</td><td>Anomaly-induced
heat current</td><td align="right">$\bar{q}_{\mathrm{anom}}$</td><td>Hodge-dual of q^μ_{anom}
$2n-1$ form</td></tr>
<tr><td align="right">J^μ_{anom}</td><td>Anomaly-induced
Charge current</td><td align="right">$\bar{J}_{\mathrm{anom}}$</td><td>Hodge-dual of $J^{i\mu}_{\mathrm{anom}}$
$2n-1$ form</td></tr>
<tr><td align="right">$J^\mu_{S,\mathrm{anom}}$</td><td>Anomaly-induced
Entropy current</td><td align="right">$\bar{J}_{S,\mathrm{anom}}$</td><td>Hodge-dual of $J^\mu_{S,\mathrm{anom}}$
$2n-1$ form</td></tr>
<tr><td align="right">$F_{\mu\nu}, F$</td><td>non-dynamical gauge
field strength, 2-form</td><td align="right">E^μ, E</td><td>Rest frame electric
field $F_{\mu\nu}u^\nu$, 1-form</td></tr>
<tr><td align="right">$B_{\mu\nu}, B$</td><td>Rest frame magnetic
fields $F-u\wedge E$</td><td align="right">$\mathfrak{A}$</td><td>Anomaly
$D\bar{J}\equiv\bar{\mathfrak{A}}$</td></tr>
<tr><td align="right">$\mathcal{J}_q^\mu$</td><td>Chiral Spectral current</td><td align="right">$\bar{\mathcal{J}}_q$</td><td>Hodge-dual of $\mathcal{J}_q^\mu$</td></tr>
<tr><td align="right">$\mathcal{J}_q^E$</td><td>See eq. (4.5)</td><td align="right">$\bar{\mathcal{J}}_q^E$</td><td>Hodge-dual of $\mathcal{J}_q^E$</td></tr>
</table>

Table 1. Table of notation.

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