

Hall viscosity from gauge/gravity duality

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ABSTRACT: In (2+1)-dimensional systems with broken parity, there exists yet another transport coefficient, appearing at the same order as the shear viscosity in the hydrodynamic derivative expansion. In condensed matter physics, it is referred to as “Hall viscosity”. We consider a simple holographic realization of a (2+1)-dimensional isotropic fluid with broken spatial parity. Using techniques of fluid/gravity correspondence, we uncover that the holographic fluid possesses a nonzero Hall viscosity, whose value only depends on the near-horizon region of the background. We also write down a Kubo’s formula for the Hall viscosity. We confirm our results by directly computing the Hall viscosity using the formula.

KEYWORDS: AdS-CFT Correspondence, Holography and condensed matter physics (AdS/CMT)

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1 Introduction

There have been intensive efforts to use methods of gauge/gravity correspondence [1–3] in studying strongly interacting systems at finite temperature and/or chemical potential. One of the motivations for such efforts is to understand the state of strongly interacting quark-gluon plasma created at RHIC, which is known to have a shear viscosity to entropy density ratio not too far from the AdS/CFT value [4]. The hydrodynamic limit of AdS/CFT has attracted much attention in recent years. The emerging picture out of these investigations is that the dynamics of perturbations of a black-brane background is governed, in the long-wavelength limit, by the same hydrodynamic equations which describe relativistic fluids [5].

Phenomenologically, the hydrodynamic equations are written down based on general principles like symmetries and the second law of thermodynamics as well as the kinetic approach based on the Boltzmann equation. While the hydrodynamic equations have been known for a long time, the ability to derive them from holography provides fresh perspectives, complementary to those of the phenomenological approach. In particular, holography was instrumental in the discovery of new hydrodynamic effects in systems with triangle anomalies [5, 6]. It has been directly observed that in these systems, there exist additional terms in the currents of conserved charges, proportional to the vorticity of the fluid flow. It was further discovered that such contributions are required by the triangle anomalies and the second law of thermodynamics, and hence are not restricted to theories with a gravitational dual [7]. There have been some attempts to rederive these terms in the kinetic approach [8].

In this paper, we investigate parity-odd effects in (2+1)-dimensional relativistic hydrodynamics. In contrast to (3+1)-dimensions, there are no triangle anomalies. However

parity may be broken explicitly or spontaneously. One can ask whether there are new hydrodynamic effects which are disallowed in parity-invariant fluids.

It is easy to observe that indeed such an effect does exist, and is the relativistic generalization of what is called the “Hall viscosity” in the condensed matter literature [9, 10]. In condensed matter, there are variety of models where the Hall viscosity phenomenon is investigated [11]. One may ask whether Hall viscosity appears more generally in a relativistic context. Consider the stress tensor of a relativistic fluid

$$\tau^{\mu\nu} = (E + p)u^\mu u^\nu + pg^{\mu\nu} - \eta P^{\mu\alpha} P^{\nu\beta} V_{\alpha\beta}, \quad V_{\alpha\beta} \equiv \nabla_\alpha u_\beta + \nabla_\beta u_\alpha - g_{\alpha\beta} \nabla \cdot u, \quad (1.1)$$

where u^μ is the fluid velocity and $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection to the directions perpendicular to u^μ .¹ On general grounds, there exists a Hall viscosity contribution to the stress tensor

$$\tau_H^{\mu\nu} = -\frac{1}{2}\eta_A(\epsilon^{\mu\alpha\beta} u_\alpha V_\beta^\nu + \epsilon^{\nu\alpha\beta} u_\alpha V_\beta^\mu), \quad (1.2)$$

which by construction, is only allowed in (2+1)-dimensions. It is worth mentioning that the Hall viscosity term does not contribute to entropy production and hence is dissipationless. In the comoving frame, where the fluid velocity at a given point is $u^\mu = (1, \vec{0})$, the Hall viscosity contribution to the stress tensor τ^{ij} has the same form (up to a sign) as the one discussed in [9]

$$\tau_H^{ij} = \frac{1}{2}\eta_A^{ijkl} V_{kl}, \quad \eta_A^{ijkl} = -\frac{1}{2}\eta_A(\delta^{ik}\epsilon^{jl} + \delta^{jk}\epsilon^{il} + \delta^{il}\epsilon^{jk} + \delta^{jl}\epsilon^{ik}). \quad (1.3)$$

In this paper we write down a holographic theory which realizes the phenomenon of Hall viscosity. Our bulk theory is Anti de Sitter gravity coupled to a gravitational Chern-Simons term. It is similar to the model considered in [12]. We find that the corresponding boundary theory exhibits “Hall viscosity.” Moreover, we discover that the Hall viscosity is completely determined by the near horizon region of the black-brane.

The paper is organized as follows. In section two, we present our holographic setup in detail. In section three, the Hall viscosity contribution to the stress tensor is computed using fluid/gravity correspondence. Section four is where a Kubo’s formula is proposed and is further used to calculate the Hall viscosity coefficient. We conclude the presentation by the outlook section. Some details of the holographic renormalization of our model appears in the appendix.

2 The setup

It is interesting to construct gravity backgrounds dual to isotropic hydrodynamic flows with a non-vanishing Hall viscosity term. This is only possible in (2+1)-dimensions and in the presence of a broken time-reversal or parity. Here we propose a gravity dual which realizes this goal. We do not insist on deriving the gravity dual from string theory in a top-down approach. Rather we take a more phenomenological attitude. We introduce a parity-breaking interaction by turning on a gravitational θ -term in the bulk action. Parity

¹We assume the fluid is conformal so the bulk viscosity is zero.

breaking alternative gravity theories have been studied in the past [12] with phenomenology in mind. We consider a generalization of this class of models which includes a negative cosmological constant.

Let us begin by describing the setup. Throughout the main text, we use the uppercase Latin letters for the four spacetime coordinates. Lower case letters i, j are reserved for the spatial boundary field theory directions. Greek letters refer to the boundary coordinates both temporal and spatial. The convention we follow for the ϵ -tensor is $\epsilon^{vxy} = 1$, where v is null and x and y are space-like. .

Our bulk theory lives in four spacetime dimensions. The Lagrangian density is

$$\mathcal{L} = R - 2\Lambda - \frac{1}{2}(\partial\theta)^2 - V(\theta) - \frac{\lambda}{4}\theta^*RR, \tag{2.1}$$

where

$$\begin{aligned} *RR &= *R^M{}_N{}^{PQ} R^N{}_{MPQ}, \\ *R^M{}_N{}^{PQ} &= \frac{1}{2}\epsilon^{PQAB}R^M{}_{NAB}, \end{aligned} \tag{2.2}$$

and ϵ^{ABCD} is the four dimensional Levi-Civita tensor and λ is a coupling constant. We set $\Lambda = -3$ from now on. Note that in order for the gravitational θ -term to have a nontrivial effect on the field equations, the field θ must be spacetime-dependent.

Varying the action (2.1) with respect to θ and the metric leads to the following field equations

$$\begin{aligned} G_{MN} + \Lambda g_{MN} - \lambda C_{MN} &= T_{MN}(\theta), \\ \nabla^2\theta &= \frac{dV}{d\theta} + \frac{\lambda}{4}\theta^*RR, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} C^{MN} &= m_A\epsilon^{ABP(M}\nabla_P R^N)_{B} + m_{AB} *R^{B(MN)A}, \\ T_{MN} &= \frac{1}{2}\partial_M\theta\partial_N\theta - \frac{1}{4}g_{MN}(\partial\theta)^2 - \frac{1}{2}g_{MN}V(\theta), \end{aligned} \tag{2.4}$$

and $m_M = \nabla_M\theta, m_{MN} = \nabla_M\nabla_N\theta = \nabla_{(M}\nabla_{N)}\theta$. In the above the symmetric-traceless C -tensor is the analog of the Cotton tensor in three-dimensions. Similar equations of motion (2.3) were also derived in for example [12, 13]. Details of variation of the action and holographic renormalization are gathered and briefly discussed in the appendix A.

To write down a black-brane background solution of the equations of motion (2.3), we take the following ansatz

$$\begin{aligned} ds^2 &= g_{MN}^{(b)}dx^M dx^N = 2H(r)dvdr - r^2f(r)dv^2 + r^2dx \cdot dx, \\ \theta &= \theta^{(b)}(r). \end{aligned} \tag{2.5}$$

We note that [13] for any ansatz of the above form, the Pontryagin form $*RR$ is identically zero. In addition to this, for the ansatz (2.5), the C -tensor vanishes identically. These two

observations imply that any solution of the form (2.5) to the following system

$$\begin{aligned} G_{MN} + \Lambda g_{MN} &= T_{MN}(\theta), \\ \nabla^2 \theta &= \frac{dV}{d\theta}, \end{aligned} \tag{2.6}$$

will give rise to a solution of our system (2.3).² We also write a general formula for the Hall viscosity in terms of a general background solution (2.5), so one is not required to be more specific about the background solution.

3 Fluid dynamics/gravity correspondence

In this section, we perform the fluid/gravity procedure as appears in [5] (see also [14]). Before proceeding to the computation, let us outline briefly the algorithm.³ The idea is to systematically find the gravity background describing the boundary hydrodynamics in a derivative expansion. The background geometry

$$\begin{aligned} ds^2 &= -2H(r, b)u_\mu dx^\mu dr - r^2 f(r, b)u_\mu u_\nu dx^\mu dx^\nu + r^2(\eta_{\mu\nu} + u_\mu u_\nu)dx^\mu dx^\nu, \\ \theta &= \theta(r, b), \end{aligned} \tag{3.1}$$

describes the boundary hydrodynamics in (2+1)-dimensions at thermal equilibrium, where b is the black-brane's Hawking temperature and $u^\mu = (1 - \vec{\beta}^2)^{-1/2}(1, \vec{\beta})$. If one promotes u^μ and b to slowly varying functions of the boundary coordinates, the resulting inhomogeneous background (expanded up to first derivative) will cease to be a solution

$$\begin{aligned} ds_{(1)}^2 &= 2H(r)dvd r - r^2 f(r)dv^2 + r^2 dx \cdot dx + \epsilon[-r^2 \delta b \partial_b f dv^2 + 2\delta b \partial_b H dvdr \\ &\quad - 2r^2(1 - f(r))x^\mu \partial_\mu \beta_i^{(0)} dv dx^i - 2H(r)x^\mu \partial_\mu \beta_i^{(0)} dr dx^i], \\ \theta^{(1)} &= \theta^{(b)}(r) + \epsilon[u^\mu \frac{\partial \theta}{\partial u^\mu} + \delta b \partial_b \theta], \end{aligned} \tag{3.2}$$

where derivatives are evaluated on the background. In the above ϵ counts the number of derivatives along the boundary. The procedure is to correct the resulting geometry by adding an extra piece of order ϵ . We parametrize the correction as

$$\begin{aligned} ds_{corr}^2 &= \epsilon \left[\frac{k(r)}{r^2} dv^2 + 2h(r)dvd r - r^2 h(r)dx \cdot dx + \frac{2}{r} a^i(r)dv dx^i + r^2 \alpha_{ij}(r)dx^i dx^j \right], \\ \theta_{corr} &= \epsilon \Theta(r), \end{aligned} \tag{3.3}$$

where α_{ij} is taken to be symmetric and traceless. The trace-reversed form of the Einstein equation is more appropriate for the fluid/gravity correspondence

$$\begin{aligned} E_{MN} &= R_{MN} + 3g_{MN} - \lambda C_{MN} - d_{MN} = 0, \\ \nabla^2 \theta &= \frac{dV}{d\theta} + \frac{1}{4} {}^*RR, \end{aligned} \tag{3.4}$$

²The field θ is taken to be a relevant deformation by appropriately choosing the potential.

³We only need first order hydrodynamics to compute the Hall viscosity.

where

$$d_{MN} = \frac{1}{2}(\partial_M \theta \partial_N \theta + g_{MN} V(\theta)). \quad (3.5)$$

Before proceeding, the following simplifying observation will prove helpful. It turns out that the general structure of the perturbation theory is as follows

$$\mathfrak{F}_{MN}[\epsilon h, g_{MN}^{(b)}] = \epsilon \lambda C_{MN}^{(1)}(\theta^{(b)}, g_{MN}^{(b)}) + \epsilon d_{MN}^{(1)}, \quad (3.6)$$

where \mathfrak{F}_{MN} is a linear differential operator (containing only radial derivatives) acting on h , where h collectively refers to first order gravity fluctuations. The superscript “(1)” denotes first order (in ϵ) quantities. Also note that the C -tensor vanishes on the background as previously mentioned. Evidently, the Hall viscosity term can only originate from $C_{MN}^{(1)}$. Here we are only interested in computing the coefficient of Hall viscosity. Corrections (proportional to ϵ) to $\theta^{(b)}$ will generate higher order terms in ϵ on the right-hand side of (3.6). Similarly, corrections (proportional to ϵ) to $g_{MN}^{(b)}$ on the left hand side produce terms which are irrelevant to the Hall viscosity computation. Therefore, as long as first order hydrodynamics is concerned, we can only use the background solution for θ and g_{MN} .

As in [5, 14], various components of the equations of motion correspond to the constitutive relations and/or the hydrodynamic equations of motion. The Hall viscosity coefficient can be computed from T_{xy} (and/or $T_{xx} - T_{yy}$) component(s) of the stress tensor. For this, studying the tensor sector will suffice. Along the way, we observe that

$$C_{xy}^{(1)} = \frac{1}{4H} \frac{d}{dr} \left(\frac{r^4 f' \theta'}{H^2} \right) (\partial_x \beta_x - \partial_y \beta_y), \quad C_{xx}^{(1)} - C_{yy}^{(1)} = \frac{1}{2H} \frac{d}{dr} \left(\frac{r^4 f' \theta'}{H^2} \right) (-\partial_x \beta_y - \partial_y \beta_x). \quad (3.7)$$

From $E_{xy}^{(1)} = 0$

$$\begin{aligned} & \frac{1}{H} \frac{d}{dr} \left[-\frac{1}{2} \frac{r^4 f(r)}{H(r)} \frac{d}{dr} \alpha_{xy}(r) \right] - 2 \frac{r}{H} \sigma_{xy} \\ & + \left[\frac{r^3 H' f}{H^3} - r^3 \frac{f'}{H^2} - 3r^2 \frac{f}{H^2} + 3r^2 - \frac{r^2}{2} V(\theta) \right] \alpha_{xy}(r) = \frac{\lambda}{4H} \frac{d}{dr} \left(\frac{r^4 f' \theta'}{H^2} \right) (\partial_x \beta_x - \partial_y \beta_y). \end{aligned} \quad (3.8)$$

Background equations of motion imply

$$E_{xx}^{(b)} = 0 = \frac{r^3 H' f}{H^3} - r^3 \frac{f'}{H^2} - 3r^2 \frac{f}{H^2} + 3r^2 - \frac{r^2}{2} V(\theta). \quad (3.9)$$

Putting eqs. (3.8) and (3.9) together, for a general background solution (2.5), one obtains

$$\frac{1}{H} \frac{d}{dr} \left[-\frac{1}{2} \frac{r^4 f(r)}{H(r)} \frac{d}{dr} \alpha_{xy}(r) \right] = 2 \frac{r}{H} \sigma_{xy} + \frac{\lambda}{4H} \frac{d}{dr} \left(\frac{r^4 f' \theta'}{H^2} \right) (\partial_x \beta_x - \partial_y \beta_y), \quad (3.10)$$

where

$$\sigma_{ij} = \frac{1}{2}(\partial_i \beta_j + \partial_j \beta_i) - \frac{1}{2} \delta_{ij} \partial_k \beta_k. \quad (3.11)$$

Also from $E_{xx}^{(1)} - E_{yy}^{(1)} = 0$

$$\frac{1}{H} \frac{d}{dr} \left[-\frac{1}{2} \frac{r^4 f(r)}{H(r)} \frac{d}{dr} \alpha_{xx}(r) \right] = 2 \frac{r}{H} \sigma_{xx} + \frac{\lambda}{4H} \frac{d}{dr} \left(\frac{r^4 f' \theta'}{H^2} \right) (-\partial_x \beta_y - \partial_y \beta_x). \quad (3.12)$$

Having solved for α_{xx} from the above, using the traceless condition we have

$$\alpha_{yy}(r) = -\alpha_{xx}(r). \tag{3.13}$$

3.1 Tensor perturbations

Let us concentrate on equations (3.10) and (3.12). One can integrate (3.10) and solve for α_{xy}

$$\alpha_{xy}(r) = \int_r^\infty \frac{2H(s)ds}{s^4 f(s)} \int_{r_H}^s dw \mathfrak{S}_{xy}(w), \tag{3.14}$$

where

$$\mathfrak{S}_{xy}(r) = 2r\sigma_{xy} + \frac{\lambda}{4} \frac{d}{dr} \left(\frac{r^4 f' \theta'}{H^2} \right) (\partial_x \beta_x - \partial_y \beta_y), \tag{3.15}$$

where the source free solution is discarded since it is not normalizable. The following formula is useful when we compute the stress tensor

$$r^n \alpha_{xy}(r) \rightarrow -\frac{r^{n+1}}{n} \frac{d\alpha_{xy}(r)}{dr}, \tag{3.16}$$

$$r \rightarrow \infty.$$

In order to write a general formula we further assume

$$f(r) = 1 - \mathcal{O}(r_H^3/r^3), \quad H(r) = 1 - \mathcal{O}(r_H^3/r^3). \tag{3.17}$$

These assumptions are not necessarily the minimal requirements enabling one to write a general formula for the Hall viscosity. The xy -component of the holographic stress tensor is given by

$$T_{xy} = -\frac{1}{8\pi G_4} \frac{r^3 \sqrt{f}}{2H} \frac{d\alpha_{xy}}{dr} + \dots \tag{3.18}$$

where dots represent contributions which vanish near the boundary, therefore they are discarded. With these assumptions and the above formula, one can write the contribution of the Hall term to the stress tensor

$$T_{xy}^{Hall} = -\frac{\lambda}{8\pi G_4} \frac{r^4 f'(r) \theta'(r)}{4H^2(r)} \Big|_{r=r_H} (\partial_x \beta_x - \partial_y \beta_y). \tag{3.19}$$

Similarly, integrating (3.12), one obtains

$$\alpha_{xx}(r) = \int_r^\infty \frac{2H(s)ds}{s^4 f(s)} \int_{r_H}^s dw \mathfrak{S}_{xx}(w), \tag{3.20}$$

where

$$\mathfrak{S}_{xx}(r) = 2r\sigma_{xx} + \frac{\lambda}{4} \frac{d}{dr} \left(\frac{r^4 f' \theta'}{H^2} \right) (-\partial_x \beta_y - \partial_y \beta_x). \tag{3.21}$$

The gravity stress tensor will give the Hall contribution

$$T_{xx}^{Hall} - T_{yy}^{Hall} = \frac{\lambda}{8\pi G_4} \frac{r^4 f'(r) \theta'(r)}{2H^2(r)} \Big|_{r=r_H} (\partial_x \beta_y + \partial_y \beta_x). \tag{3.22}$$

Comparing the definition of the Hall viscosity (1.2) and our results (3.22) and (3.19), we obtain

$$\eta_A = -\frac{\lambda}{8\pi G_4} \frac{r^4 f'(r) \theta'(r)}{4H^2(r)} \Big|_{r=r_H}. \quad (3.23)$$

This is a general membrane-paradigm-type formula [18] for the Hall viscosity coefficient, in the sense that the Hall viscosity is entirely determined by the near horizon region of the black-brane geometry.

4 A Kubo's formula for η_A

Hall viscosity can also be computed at zero spatial momentum. Generally speaking, a transport coefficient could be viewed as parametrizing the response of a fluid to hydrodynamic perturbations. One way to induce these disturbances is by perturbing the non-dynamical boundary metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h^2)$. We work in the local rest frame of the fluid $u^\mu = (1, \vec{0})$ and at zero spatial momentum. The minimal set of background metric perturbations which needs to be switched on to compute the Hall viscosity is $h = \{h_{xy}(t), h_{xx}(t), h_{yy}(t)\}$. One gets

$$T^{xy} = -Ph_{xy} - \eta \frac{\partial h_{xy}}{\partial t} + \frac{\eta_A}{2} \frac{\partial}{\partial t} (h_{xx} - h_{yy}), \quad (4.1)$$

where P is the pressure. A similar formula for the shear viscosity has been written down before; see for example [16]. Here we have the extra term proportional to the Hall viscosity. One could write this in terms of the following retarded Green's function

$$\eta_A = -\frac{i}{2} \partial_\omega [G^{xy,xx}(\omega, k) - G^{xy,yy}(\omega, k)]|_{\omega=k=0}. \quad (4.2)$$

For detailed definitions used in writing this Kubo formula see [17].

4.1 Linearized gravity perturbation

In this section we study linearized gravitational perturbations of the background (2.5). We write down equations of motion (2.3) for the following linearized perturbations

$$g_{MN} = g_{MN}^{(b)} + g_{MN}^{(1)}, \quad g_{xy}^{(1)} = r^2 h_{xy}(r) e^{-i\omega v}, \quad g_{xx}^{(1)} = r^2 h_{xx}(r) e^{-i\omega v}, \quad g_{yy}^{(1)} = r^2 h_{yy}(r) e^{-i\omega v}, \quad (4.3)$$

where $h_{xx}(r) = -h_{yy}(r)$. Note that in this coordinate system, the horizon is a non-singular surface. Boundary condition at the horizon is incoming, since we are interested in the response. At infinity we impose $h_{yy}(r) \rightarrow H_{yy}^0$, $h_{xy}(r) \rightarrow H_{xy}^0$. The equations of motion are solved in an expansion in small frequencies. It turns out that one can solve the equations for a general background. We only care about the first order correction in ω . One obtains

$$\begin{aligned} h_{xx}(r) &= H_{xx}^0 + \frac{\omega}{2} \int_\infty^r \frac{-i\lambda H_{xy}^0 s^4 f'(s) \theta'(s) + 2iH_{xx}^0 s^2 H^2(s) + 2c_1 H^2(s)}{s^4 f(s) H(s)} ds, \\ h_{xy}(r) &= H_{xy}^0 + \frac{\omega}{2} \int_\infty^r \frac{i\lambda H_{xx}^0 s^4 f'(s) \theta'(s) + 2iH_{xy}^0 s^2 H^2(s) + 2c_2 H^2(s)}{s^4 f(s) H(s)} ds, \end{aligned} \quad (4.4)$$

where the integration constants c_1 and c_2 , are determined through demanding regularity for h_{xx} and h_{xy} at the horizon. This leads to

$$c_1 = -iH_{xx}^0 r_H^2 + i\lambda H_{xy}^0 \frac{s^4 f'(s)\theta'(s)}{2H^2(s)} \Big|_{s=r_H}, \quad c_2 = -iH_{xy}^0 r_H^2 - i\lambda H_{xx}^0 \frac{s^4 f'(s)\theta'(s)}{2H^2(s)} \Big|_{s=r_H}. \quad (4.5)$$

As shown in the appendix, the Chern-Simons term does not contribute to the stress tensor. Let us focus on the xy -component of the stress tensor. The relevant (to the Hall viscosity computation) part of $h_{xy}(r)$ is

$$h_{xy}(r) = H_{xy}^0 + \frac{i\lambda\omega}{3r^3} W H_{xx}^0 + \dots, \quad (4.6)$$

where

$$W = \frac{r^4 f'(r)\theta'(r)}{2H^2(r)} \Big|_{r=r_H}. \quad (4.7)$$

The gravity stress tensor gives

$$8\pi G_4 T_{xy} = \frac{i\lambda\omega}{2} W H_{xx}^0, \quad (4.8)$$

Comparing this with the Kubo's formula (4.2) and equation (4.1), we can read off the Hall viscosity coefficient as

$$\eta_A = -\frac{\lambda}{8\pi G_4} \frac{r^4 f'(r)\theta'(r)}{4H^2(r)} \Big|_{r=r_H}. \quad (4.9)$$

This is exactly equal to what we computed using fluid/gravity duality in the previous section.

5 Outlook

In this paper, we have constructed a holographic model exhibiting Hall viscosity. Although a non-zero Hall viscosity coefficient was not unexpected given the choice of interactions in our model, it is gratifying to find that the rules of gauge/gravity duality indeed lead to a Hall viscosity. It is interesting to note that the value of the Hall viscosity, in our model, depends only on the behavior of the scalar field θ at the horizon, which indicates that there exists a membrane paradigm principle that fixes the value of this kinetic coefficient at the black-brane horizon. It would be interesting to find out if that is true and how the Hall viscosity is communicated to the boundary (see, e.g., [19]). It is also interesting to explore the connection between the holographic approach developed here with the purely Lagrangian approach to Hall viscosity of [20].

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A Holographic renormalization

In this appendix lower case Latin indices a, b, c, \dots run over all four spacetime coordinates. The Latin indices i, j, k, \dots refer to the spatial coordinates. We set $16\pi G_4 = 1$. To compute the Hall viscosity contribution to the hydrodynamic flow of the boundary theory, one has to write down the stress-tensor associated with the theory (2.1). It is also crucial to make sure that the action principle is well-defined. This is to say, one should only be required to keep fields (and not their normal derivatives) fixed at the boundary. The general procedure is to add the analog of a Gibbons-Hawking term to the bulk action (2.1). For the Chern-Simons modified gravity (2.1), this term has been computed in [13]. Here we show that for the particular case of interest in this paper, i.e., when θ is a relevant deformation (vanishes asymptotically), on any solution to the equations of motion coming from the action (2.1), the stress-tensor is just that of asymptotically AdS₄ spaces. The only counter-terms needed are those of asymptotically AdS₄ spacetimes. Variation of the Chern-Simons term is straightforward

$$\begin{aligned} \delta S_{CS} &= -\frac{\lambda}{4} \delta \int d^4x \sqrt{-g} \theta^* R R, \\ &= -\lambda \int d^4x \sqrt{-g} \nabla_c (\theta^* R^b{}_{a{}cd} \delta \Gamma^a{}_{bd}) + \lambda \int d^4x \sqrt{-g} \nabla_b [\delta g_{ed} \nabla_c (\theta^* R^{bcd})] - \lambda S_1, \\ S_1 &= \int d^4x \sqrt{-g} \delta g_{ed} \nabla_b \nabla_c (\theta^* R^{bcd}). \end{aligned} \quad (\text{A.1})$$

On the second line of the above equation, there are two boundary terms which we come back to later. For the moment let us focus on S_1 . Using the second Bianchi identity $R^{be}{}_{[fg;c]} = 0$, its contracted form $R^{be}{}_{fg;b} = R^e{}_{g;f} - R^e{}_{f;g}$ and the fact that δg_{ed} is symmetric, S_1 can be rewritten as

$$S_1 = - \int d^4x \sqrt{-g} \delta g_{ed} [\nabla_{(b} \nabla_{c)} \theta^* R^{c(ed)b} + \nabla_c \theta \epsilon^{cgf(d} \nabla_f R^e{}_{g)}] = - \int d^4x \sqrt{-g} \delta g_{ed} C^{ed}, \quad (\text{A.2})$$

where the C -tensor (2.4) definition was utilized. Now let us collect all the boundary terms on the second line of (A.1)

$$\delta S_b = -\lambda \int d^4x \sqrt{-g} \nabla_c (\theta^* R^b{}_{a{}cd} \delta \Gamma^a{}_{bd}) + \lambda \int d^4x \sqrt{-g} \nabla_b [\delta g_{ed} \nabla_c (\theta^* R^{bcd})]. \quad (\text{A.3})$$

We show that the θ dependent boundary terms above will vanish when θ is a relevant perturbation. We were able to demonstrate this in the Gaussian normal coordinates

$$ds^2 = dr^2 + g_{ij} dx^i dx^j, \quad (\text{A.4})$$

using standard identities (see [15] for example). Let us focus on the first boundary term in (A.3)

$$S_2 = -\lambda \int d^4x \sqrt{-g} \nabla_c (\theta^* R^b{}_{a{}cd} \delta \Gamma^a{}_{bd}) = \lambda \int d^3x \theta (2 \epsilon^{rjkl} \nabla_k K^i{}_l \delta K_{ij} + \sqrt{-h} {}^* R_j{}^{kri} \delta \Gamma^j{}_{ki}), \quad (\text{A.5})$$

where Codazzi equation in the coordinates (A.4)

$${}^*R_a{}^{brd}\delta\Gamma_{bd}^a = {}^*R_j{}^{kri}\delta\Gamma_{ki}^j + 2 {}^*R_r{}^{jri}\delta\Gamma_{ji}^r, \quad (\text{A.6})$$

as well as the following two identities (true in the normal coordinates), $R_{kl}^{ir} = {}^{(3)}\nabla_k K_l^i - {}^{(3)}\nabla_l K_k^i$ and $K_{ij} = \frac{1}{2}\frac{\partial g_{ij}}{\partial r}$ were used. Note that the Fefferman-Graham expansion of the metric and extrinsic curvature are given by $g_{ij} = e^{2r}g_{ij}^{(0)} + g_{ij}^{(2)} + \dots$, $K_{ij} = e^{2r}g_{ij}^{(0)} + 0 + \dots$ and $K^i_j = \delta^i_j - e^{-2r}g_{(2)}^{il}g_{lj}^{(0)} + \dots$. Utilizing these, one concludes that S_2 vanishes at least as fast as

$$S_2 \sim \int_{\partial} d^3x \theta \rightarrow 0. \quad (\text{A.7})$$

The argument showing that the second θ -dependent boundary term also vanishes proceeds similarly. The last term in (A.3) can be further simplified (using the second Bianchi identity)

$$\begin{aligned} S_3 &= \frac{\lambda}{2} \int d^3x \varepsilon^{rdfg} \partial_r \theta R_{fg}^{re} \delta g_{ed}, \\ &= -\lambda \int d^3x \varepsilon^{rdfg} \partial_r \theta \nabla_f K_g^e \delta g_{ed} \rightarrow 0. \end{aligned} \quad (\text{A.8})$$

In sum, the only terms relevant to the variation of the action (including the conventional Gibbons-Hawking term) are

$$\delta S = - \int d^4x \sqrt{-g} \delta g_{ed} (G^{ed} + \Lambda g^{ed} - \lambda C^{ed}) - \int d^3x \sqrt{h} (K^{ed} - h^{ed} K) \delta g_{ed}. \quad (\text{A.9})$$

The only counter-term will be a boundary cosmological constant renormalization since our boundary is flat. The stress tensor is then

$$\delta S = \frac{1}{2} \int d^3x \sqrt{g_{ij}^{(0)}} T^{ij} \delta g_{ij}^{(0)} = - \int d^3x \sqrt{h} (K^{ij} - g^{ij} K - 2g^{ij}) \delta g_{ij}. \quad (\text{A.10})$$

The background can be written in an ADM form

$$\begin{aligned} ds^2 &= N^2 dr^2 + g_{\alpha\beta} (dx^\alpha + N^\alpha dr) (dx^\beta + N^\beta dr), \\ &= (N^2 + g_{\alpha\beta} N^\alpha N^\beta) dr^2 + 2N_\alpha dx^\alpha dr + g_{\alpha\beta} dx^\alpha dx^\beta, \end{aligned} \quad (\text{A.11})$$

where $N_\alpha = g_{\alpha\beta} N^\beta$. In our gauge $g_{rr} = 0$ so $N^2 + N_\alpha N^\alpha = 0$. The extrinsic curvature is calculated using the standard formula

$$K_{\alpha\beta} = \frac{1}{2N} \left(N_{\alpha;\beta} + N_{\beta;\alpha} - \frac{\partial g_{\alpha\beta}}{\partial r} \right). \quad (\text{A.12})$$

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