

Line operators on $S^1 \times \mathbb{R}^3$ and quantization of the Hitchin moduli space

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ABSTRACT: We perform an exact localization calculation for the expectation values of Wilson-'t Hooft line operators in $\mathcal{N} = 2$ gauge theories on $S^1 \times \mathbb{R}^3$. The expectation values are naturally expressed in terms of the complexified Fenchel-Nielsen coordinates, and form a quantum mechanically deformed algebra of functions on the associated Hitchin moduli space by Moyal multiplication. We propose that these expectation values are the Weyl transform of the Verlinde operators, which act on Liouville/Toda conformal blocks as difference operators. We demonstrate our proposal explicitly in $SU(N)$ examples.

KEYWORDS: Supersymmetric gauge theory, Wilson, 't Hooft and Polyakov loops, Solitons Monopoles and Instantons

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1 Introduction and summary

Wilson loops and their magnetic cousins, 't Hooft loops, are universal observables in gauge theories whose properties characterize the phases of each theory. They represent heavy probe particles with electric and magnetic charges moving along a closed trajectory in spacetime. When acting on the Hilbert space, these operators do not commute if the two loops are linked within the constant time slice [1]. Indeed 't Hooft successfully used their non-trivial commutation relations to classify the possible phases of non-Abelian gauge theories [1–3].

Noncommutativity is also a hallmark of quantization. The position and the momentum of a particle do not commute with each other; they cannot be simultaneously diagonalized or precisely measured. For physicists quantization is usually the process of obtaining a Hilbert space and noncommuting operators acting on it from a classical system. In certain situations (especially for mathematicians), however, one is primarily interested in the “operators” without a Hilbert space. In such a scheme, called deformation quantization, the product of two functions on a phase space (Poisson manifold) is continuously (in \hbar) deformed into a noncommutative associative product whose order $\mathcal{O}(\hbar)$ correction is given by the Poisson bracket. It is a non-trivial result that any Poisson structure admits a canonical deformation quantization [4].

In this paper we study Wilson-'t Hooft line operators in $\mathcal{N} = 2$ supersymmetric gauge theories on $S^1 \times \mathbb{R}^3$ in the Coulomb phase. We consider half-BPS line operators [5] extended along S^1 and perform an exact localization calculation of their expectation values (vevs) following [6]. The vev of the product of operators turns out to be given by the Moyal product of the vevs of the individual operators. For an $\mathcal{N} = 2$ theory characterized by a punctured Riemann surface [7], the line operators precisely realize the deformation quantization of the Hitchin moduli space, with respect to the Poisson structure specified by the complexified Fenchel-Nielsen coordinates [8, 9].

Let us summarize our main results in more detail. The vev of a line operator is a holomorphic function of a and b , which take values in the complexified Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ and its dual $\mathfrak{t}_{\mathbb{C}}^*$. The variable a is a combination of the electric Wilson line A_{τ} and a real vector multiplet scalar, while b combines the magnetic Wilson line and the other real scalar in the vector multiplet, all evaluated at the infinity of \mathbb{R}^3 . These variables a and b parametrize

the Coulomb branch of the gauge theory compactified on S^1 . Since the path integral for the vev defines a supersymmetric index (2.11), the electric and magnetic Wilson lines can be regarded as chemical potentials for electric and magnetic charges. The vev also depends holomorphically on the non-dynamical variables m_f , which are complex combinations of masses and chemical potentials for flavor symmetries. Importantly, a real parameter λ also enters in the vev. It is defined as the chemical potential for the simultaneous spatial and R-symmetry rotations.

We find that the vev of the Wilson operator in representation R is simply given by

$$\langle W_R \rangle = \text{Tr}_R e^{2\pi i a}, \tag{1.1}$$

where the trace is taken in R . The localization calculation is only non-trivial for 't Hooft and dyonic line operators. In particular, they have a non-trivial one-loop determinant as well as the non-perturbative contributions from Polyakov-'t Hooft monopoles screening the charges of the singular monopole [10, 11]. The 't Hooft operator specified by a coweight B has a vev of the form

$$\langle T_B \rangle = \sum_v e^{2\pi i v \cdot b} Z_{1\text{-loop}}(a, m_f, \lambda; v) Z_{\text{mono}}(a, m_f, \lambda; B, v). \tag{1.2}$$

For simplicity we often suppress the dependence on some of a , m_f and λ . The function $Z_{1\text{-loop}}(v = B)$ given in (4.37)–(4.39) is the one-loop determinant around the leading saddle point. The sum is over the magnetic charges v reduced from B due to monopole screening. There exists a number of non-perturbative saddle points that correspond to coweight v , and $Z_{1\text{-loop}}(v)Z_{\text{mono}}(B, v)$ is the sum of the fluctuation determinants around the saddle points in the sector v . The function $Z_{\text{mono}}(B, v)$, given in (5.27), (5.33), (5.34) for $G = \text{U}(N)$ and matter in the adjoint or fundamental representation, is a monopole analog of the Nekrasov instanton partition function [12]. For a more general dyonic Wilson-'t Hooft operator, we insert into (1.2) a Wilson operator in the subgroup of the gauge group unbroken by B .

Let us suppose that the spatial rotation associated with λ takes place in the 12-plane. It is useful to think of the 3-axis as the Euclidean time direction, and consider line operators L_i at various points on the axis. By the original argument of 't Hooft, we show that these operators form a noncommutative algebra, generalizing the standard 't Hooft commutation relation $W \cdot T = e^{2\pi i/N} T \cdot W$ for minimal Wilson (W) and 't Hooft (T) loops in $\text{SU}(N)$ gauge theories.¹ Moreover, operator multiplication is implemented by noncommutative associative products, i.e.,

$$\langle L_1 \cdot L_2 \cdot \dots \cdot L_n \rangle = \langle L_1 \rangle * \langle L_2 \rangle * \dots * \langle L_n \rangle, \tag{1.3}$$

where

$$(f * g)(a, b) \equiv e^{i \frac{\lambda}{4\pi} (\partial_b \cdot \partial_{a'} - \partial_a \cdot \partial_{b'})} f(a, b) g(a', b') \Big|_{a'=a, b'=b}. \tag{1.4}$$

¹Non-commutativity can also be understood by thinking of S^1 as time. The electric and magnetic fields produced by the Wilson and 't Hooft operators generate a non-zero Poyinting vector carrying angular momentum, and contribute non-trivially to the supersymmetric index with $\lambda \neq 0$ [13].

This is the Moyal product associated with the Poisson structure determined by the holomorphic symplectic form

$$da \wedge db \tag{1.5}$$

and $\hbar = \frac{\lambda}{2\pi}$, where a and b are contracted in the canonical way.

It is illuminating to be even more explicit, focusing on the $SU(2)$ $\mathcal{N} = 2^*$ theory. In this case the expectation values of the minimal Wilson (W), 't Hooft (T), and dyonic (D) operators are given by²

$$\begin{aligned} \langle W \rangle &= e^{2\pi ia} + e^{-2\pi ia}, & \langle T \rangle &= \left(e^{2\pi ib} + e^{-2\pi ib} \right) \left(\frac{\sin(2\pi a + \pi m) \sin(2\pi a - \pi m)}{\sin(2\pi a + \frac{\pi}{2}\lambda) \sin(2\pi a - \frac{\pi}{2}\lambda)} \right)^{1/2}, \\ \langle D \rangle &= \left(e^{2\pi i(b+a)} + e^{-2\pi i(b+a)} \right) \left(\frac{\sin(2\pi a + \pi m) \sin(2\pi a - \pi m)}{\sin(2\pi a + \frac{\pi}{2}\lambda) \sin(2\pi a - \frac{\pi}{2}\lambda)} \right)^{1/2}. \end{aligned} \tag{1.6}$$

For $\lambda = 0$, these expressions precisely appeared in [8] as the definition of Darboux coordinates (a, b) on the Hitchin moduli space for a one-punctured torus. In [9], they were identified as the complexification of the Fenchel-Nielsen coordinates, which are Darboux coordinates for the real symplectic structure on Teichmüller space. Their findings are consistent with our identification (1.5) of the symplectic structure. For $\lambda \neq 0$, our results provide quantum deformations.

Thus $\mathcal{N} = 2$ gauge theories on $S^1 \times \mathbb{R}^3$ produce a noncommutative algebra of operators quantizing the Hitchin moduli space. Is there a Hilbert space on which the line operators naturally act? We claim that the space of conformal blocks in Liouville or Toda conformal field theories is such a Hilbert space. This is demonstrated by showing that the Verlinde operators³ [15, 16], labeled by closed curves on the Riemann surface and corresponding to line operators in gauge theories [17], are exactly the Weyl transform (also known as the Weyl ordering) of the vevs of the line operators on $S^1 \times \mathbb{R}^3$, where a and b are treated as coordinates and momenta, respectively. The twist/quantization variable λ is related to the variable b that parametrizes the central charge $c = 1 + 6(b + b^{-1})^2$ as $\lambda = b^2$. This result is a concrete realization of the proposal that the algebra of line operators provide quantization of the Hitchin moduli space [9, 13, 18, 19]. The connection to Liouville/Toda theories provides a very strong check of our localization computations. Moreover, we conjecture that the connection should hold even when $\mathcal{N} = 2$ gauge theories have no Lagrangian description. Thus it is now possible to compute the line operator vevs on $S^1 \times \mathbb{R}^3$ for such theories as the inverse Weyl transform of the Verlinde operators.

Then the AGT relation [20] between Liouville/Toda theories and four-dimensional gauge theories would suggest that our analysis should intimately parallel the localization computation of 't Hooft loops on S^4 [11] corresponding to $\lambda = b^2 = 1$. Indeed $Z_{1\text{-loop}}(\lambda; B)$

²For $SU(2)$ we simplify notation by substituting $a \rightarrow \text{diag}(a, -a), b \rightarrow \text{diag}(b, -b)$. When there is only one mass parameter, $m \equiv m_{f=1}$.

³Verlinde operators are the difference operators that act on conformal blocks, and arise from the monodromy of extended conformal blocks with degenerate field insertions. Verlinde operators in Liouville theory coincide with the geodesic length operators in quantum Teichmüller theory [14].

and $Z_{\text{mono}}(\lambda; B, v)$ with $\lambda = 1$ appeared in [11] as the contributions from the equator S^1 of S^4 , where a 't Hooft loop was inserted.

Exactly the same physical system on $S^1 \times \mathbb{R}^3$ was considered in [13], where supersymmetric line operators were analyzed from the point of view of wall-crossing in the IR effective theories. Their twist parameter y is given by $y = -e^{\pi i \lambda}$. Based on the consistency of the wall-crossing formula in $\mathcal{N} = 2$ gauge theories and several other assumptions, they conjectured expressions for the line operator vevs in terms of the (commutative and noncommutative) Fock-Goncharov coordinates on the Hitchin moduli space. It would be desirable to perform more detailed comparisons. This should help create a bridge between the AGT correspondence [20] and the study of wall-crossing, perhaps along the line of [21].

This paper is organized as follows. Section 2 defines the gauge theory setup and the quantities we wish to compute. We begin our localization calculations in section 3, where we analyze the symmetries of the system and lay out our strategy. We also calculate the classical on-shell action in the supersymmetric background defining a 't Hooft operator. Section 4 is devoted to the one-loop analysis. In section 5 we compute the non-perturbative contributions due to monopole screening. Putting all together the classical, one-loop, and screening contributions, section 6 summarizes the results of our localization calculations and gives explicit expressions in several examples. We then turn to the quantization aspects of our results. In section 7 we study the noncommutative structure in the algebra formed by line operators and show that it implements the deformation quantization of the Hitchin moduli space. Next we discuss the relation to gauge theories on S^4 and Liouville/Toda theories in section 8. We conclude the paper in section 9 with a discussion on related works and future directions. Appendix A explains our convention for spinors and gamma matrices. In appendix C we review Kronheimer's correspondence between singular monopoles and U(1)-invariant instantons on a Taub-NUT space. This relation is used in sections 4 and 5. Appendix D contains technical computations in Liouville and Toda theories. In appendix E we compute the classical $\text{SL}(2, \mathbb{C})$ holonomies on the four-punctured sphere and compare them with gauge and Liouville calculations.

2 $\mathcal{N} = 2$ gauge theories on $S^1 \times \mathbb{R}^3$ and line operators

In this paper we study four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry on $S^1 \times \mathbb{R}^3$ in the Coulomb branch. For notational convenience, we will use the notation appropriate for $\mathcal{N} = 2^*$ theory, which can be thought of as a dimensional reduction of the ten-dimensional super Yang-Mills, though we will state general results applicable to other field contents [6, 11]. The ten-dimensional gauge field A_M ($M = 1, \dots, 9, 0$) gives rise to the four-dimensional gauge field A_μ ($\mu = 1, \dots, 4$), hypermultiplet scalars $A_i \equiv \Phi_i$ ($i = 5, \dots, 8$), and vector multiplet scalars $A_A \equiv \Phi_A$ ($A = 0, 9$). The ten-dimensional chiral spinor Ψ also decomposes into the gaugino $\psi \equiv \frac{1-\Gamma_{5678}}{2} \Psi$ and hypermultiplet fermion $\chi \equiv \frac{1+\Gamma_{5678}}{2} \Psi$. Our spinor and gamma matrix conventions are summarized in appendix A. Real fields are hermitian matrices, and the gauge covariant derivative is $D_\mu = \partial_\mu + iA_\mu$. In terms of the coordinates $x^\mu = (x^i, \tau)$ ($\mu = 1, \dots, 4$, $i = 1, 2, 3$), the metric is simply $ds^2 = d\tau^2 + dx^i dx^i$. We denote the radius of the Euclidean time circle by R .

The theory is defined by the physical action

$$S = S_{\text{vec}} + S_{\text{hyp}}, \tag{2.1}$$

where the two terms describing the vector and hypermultiplets are given by

$$S_{\text{vec}} = \frac{1}{g^2} \int_{S^1 \times \mathbb{R}^3} d^4x \text{Tr} \left(\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi_A D^\mu \Phi_A - [\Phi_0, \Phi_9]^2 - \psi \Gamma^\mu D_\mu \psi - i \psi \Gamma^A [\Phi_A, \psi] \right) + \frac{i\vartheta}{8\pi^2} \int_{S^1 \times \mathbb{R}^3} \text{Tr} (F \wedge F), \tag{2.2}$$

and

$$S_{\text{hyp}} = \frac{1}{g^2} \int_{S^1 \times \mathbb{R}^3} d^4x \text{Tr} \left(D_\mu \Phi_i D^\mu \Phi_i - \frac{1}{2} [\Phi_i, \Phi_j]^2 - ([\Phi_A, \Phi_i] - i M_{Aij} \Phi_j)^2 - \chi \Gamma^\mu D_\mu \chi - i \chi \Gamma^A \left([\Phi_A, \chi] - \frac{i}{4} M_{Aij} \Gamma^{ij} \chi \right) - i \chi \Gamma^i [\Phi_i, \chi] \right). \tag{2.3}$$

Here Tr denotes an invariant metric on the Lie algebra of the gauge group G , ϑ is the theta angle, and $i, j = 5, 6, 7, 8$ denote the hypermultiplet scalar directions. The two real anti-symmetric matrices $M_{ij} \equiv M_{0ij}$ and M_{9ij} are proportional to a single pure-imaginary anti-symmetric matrix F_{ij} ,⁴ which is normalized as $F_{ij} F_{ji} = 4$ and is taken to be anti-self-dual in the 5678 directions so that only the hypermultiplet fermions get massive. The flavor generator F is represented as F_{ij} on the scalars and as $\frac{1}{4} F_{ij} \Gamma^{ij}$ on spinors. The real mass parameters $M \equiv M_0$ and M_9 are defined by $M_{Aij} = i M_A F_{ij}$ ($A = 0, 9$). The massless limit is $\mathcal{N} = 4$ super Yang-Mills.

Our aim is to compute the expectation value of half-BPS line operators along S^1 , placed at a point on the 3-axis of \mathbb{R}^3 . The most basic line operator is the Wilson operator defined as

$$W_R = \text{Tr}_R P \exp \int_{S^1} (-iA + \Phi_0) d\tau. \tag{2.4}$$

This is labeled by the representation R of the gauge group, or equivalently its highest weight. The supersymmetric 't Hooft operator with charge B is defined by integrating over the fluctuations of the fields around the configuration

$$A \equiv A_\mu dx^\mu = \left(i g^2 \vartheta \frac{B}{16\pi^2} \frac{1}{r} + A_\tau^{(\infty)} \right) d\tau + \frac{B}{2} \cos \theta d\varphi \tag{2.5}$$

in the background.

$$\Phi_0 = -g^2 \vartheta \frac{B}{16\pi^2} \frac{1}{r} + \Phi_0^{(\infty)}, \quad \Phi_9 = \frac{B}{2r} + \Phi_9^{(\infty)}$$

We recall that $\tau \equiv x^4$, and we have introduced polar coordinates ($r \equiv |\vec{x}|, \theta, \varphi$) for \mathbb{R}^3 . Our choice of scalars in (2.4) and (2.5) ensures that the Wilson and 't Hooft operators preserve the same sets of supercharges. The action of the U(1) R-symmetry rotates $\Phi_0 + i\Phi_9$ and changes the set of preserved supercharges. Note that we define the electric Wilson line $A_\tau^{(\infty)}$ in the local trivialization such that the $d\varphi$ term is given by $(B/2) \cos \theta d\varphi$ rather than the

⁴The flavor symmetry generator F_{ij} ($i, j = 5, \dots, 7$) should not be confused with the field strength $F_{MN} = -i[D_M, D_N]$ ($M, N = 1, \dots, 9, 0$).

more familiar $-(B/2)(\pm 1 - \cos \theta)d\varphi$. Our choice guarantees that when $\lambda \neq 0$, the holonomy at the spatial infinity with $\theta = \pi/2$ is $\exp(-2\pi i R A_\tau^{(\infty)})$. This will play a role in section 7.

More general line operators are dyonic and carry both electric and magnetic charges. Such operators are defined by a path integral for a 't Hooft operator with charge B , with the insertion of a Wilson operator for the stabilizer of B in G . The dyonic charges are elements of the sum of coweight and weight lattices of G

$$\Lambda_{cw} \oplus \Lambda_w, \tag{2.6}$$

and the charges related by a simultaneous action of the Weyl group the two lattices are equivalent [5]. Due to Dirac quantization, the magnetic charge must be a coweight which has integer inner products with all the weights in the matter representation.⁵

Having defined the line operators whose vevs we wish to compute, let us explain the parameters of the theory those vevs will depend on. We are studying the theory in the Coulomb branch, so the real scalars in the vector multiplet have the expectation values

$$\langle \Phi_A \rangle \equiv \Phi_A^{(\infty)} \in \mathfrak{t} \quad A = 0, 9, \tag{2.7}$$

which are the asymptotic values at $|\vec{x}| = \infty$. Since we compactify the theory on S^1 , we also have the electric and magnetic Wilson lines. The electric Wilson line is the asymptotic value of the τ -component of the gauge field

$$A_\tau^{(\infty)} \in \mathfrak{t}. \tag{2.8}$$

Due to potential terms in the action (2.2), $\Phi_A^{(\infty)}$ and $A_\tau^{(\infty)}$ can be simultaneously diagonalized, i.e., they can take values in the Cartan subalgebra \mathfrak{t} .

We also need to consider the magnetic Wilson line. In the IR theory this is the vev of the scalar dual to the gauge field in three dimensions. In the UV theory we define it as follows. At a generic point of the Coulomb branch, the scalar vevs $\Phi_A^{(\infty)}$ classically breaks the gauge group G to the maximal torus T . The path integral includes infinitely many sectors classified by the magnetic charges at infinity. The general boundary condition is such that asymptotically as $|\vec{x}| \rightarrow \infty$, we allow $\Phi_A(\vec{x})$ to take any values that are gauge equivalent to $\Phi_A^{(\infty)}$, i.e., there is a map $g : S^2 \rightarrow G$ such that

$$\Phi_A(\vec{x}) \rightarrow g(\vec{n}) \cdot \Phi_A^{(\infty)} \cdot g^{-1}(\vec{n}) \quad \text{as} \quad |\vec{x}| \rightarrow \infty \tag{2.9}$$

with $\vec{n} \equiv \vec{x}/|\vec{x}| \in S^2$. Then the scalars $\Phi_A(\vec{x})|_{|\vec{x}|=\infty}$ themselves define a map from S^2 to the orbit $\{g(\Phi_0^{(\infty)}, \Phi_9^{(\infty)})g^{-1} | g \in G\}$, which is diffeomorphic to G/T because the stabilizer of a generic element of $\mathfrak{t} \times \mathfrak{t}$ is T . We can demand that $g = 1$ at the north pole of S^2 , so that Φ_A at $|\vec{x}| = \infty$ define a homotopy class in $\pi_2(G/T)$ with a base point at the north pole. If G is simply connected, the maximal torus can be identified with the quotient of the Cartan subalgebra by the coroot lattice⁶ $T \simeq \mathfrak{t}/\Lambda_{cr}$, so $\pi_2(G/T) \simeq \pi_1(T) = \Lambda_{cr}$. In fact G/T

⁵In the theories whose gauge group is a product of $SU(2)$'s, the electric and magnetic charges with these constraints and equivalence relations match the homotopy classes of non-self-intersecting curves on the corresponding Riemann surface [17].

⁶See [22] for a review of lattices in the Cartan subalgebra \mathfrak{t} and its dual \mathfrak{t}^* .

depends only on the Lie algebra of G , so $\pi_2(G/T) = \Lambda_{cr}$ for any G . The infinitely many topological sectors are therefore classified by Λ_{cr} . Physically this makes sense because Λ_{cr} is the lattice of magnetic charges carried by Polyakov-'t Hooft monopoles. This lattice is more coarse than the coweight lattice Λ_{cw} in which the magnetic charge B of the 't Hooft operator takes values, $\Lambda_{cr} \subset \Lambda_{cw}$. With generic matter representations, the lattice of 't Hooft charges B allowed by Dirac quantization would be smaller than Λ_{cw} .

Let us now insert a 't Hooft operator with magnetic charge $B \in \Lambda_{cw}$ at the origin. The insertion of the 't Hooft operator changes the topology of the vector bundles in which the fields take values, and in particular the structure of the boundary conditions at spatial infinity. One can classify the allowed configurations by the asymptotic magnetic charges taking values in the shifted lattice $\Lambda_{cr} + B \subset \Lambda_{cw}$. We define the magnetic Wilson line $\Theta \in \mathfrak{t}^*$ as the chemical potential for the magnetic charges. The expectation value of the 't Hooft operator is given by the sum

$$\langle T_B \rangle = \sum_{v \in \Lambda_{cr} + B} e^{iv \cdot \Theta} \int_v \mathcal{D}A \mathcal{D}\Psi e^{-S}, \tag{2.10}$$

where the path integral in each summand is performed with the boundary condition specified by v . In the three-dimensional Abelian gauge theory that arises via dimensional reduction, Θ is identified with the expectation values of scalars dual to the photons [23], and the UV and IR definitions of Θ are consistent.

Along the circle S^1 we can impose various twisted boundary conditions on the fields. It is convenient to exhibit them by representing the line operator vev as a supersymmetric index, taking S^1 as a time direction. The line operator L modifies the Hilbert space of the theory, rather than acts on the original Hilbert space as a linear transformation. We define our observable, the expectation value of the line operator L , to be a trace in the modified Hilbert space \mathcal{H}_L

$$\langle L \rangle = \text{Tr}_{\mathcal{H}_L} (-1)^F e^{-2\pi R H} e^{2\pi i \lambda (J_3 + I_3)} e^{2\pi i \mu_f F_f}, \tag{2.11}$$

where J_3 and I_3 are the generators of the Lorentz $SU(2)$ and the R-symmetry $SU(2)$. Here J_3 generates a rotation along the 3-axis: $iJ_3 = x^1 \partial_2 - x^2 \partial_1$ when acting on a scalar. As we will see below, the combination $J_3 + I_3$ commutes with the supercharge we use for localization. We have also included the twist by the flavor symmetries with generators F_f and chemical potentials μ_f , $f = 1, \dots, N_F$. The definition (2.11) of the line operator vev coincides with the one used in [13]. The system may be realized in terms of a path integral over the fields with appropriate twisted boundary conditions along S^1 . In this paper we adopt the equivalent formulation where everywhere in the action (2.1) on \mathbb{R}^4 the time derivative is shifted as

$$\partial_\tau \rightarrow \partial_\tau - \frac{i}{R} \lambda (J_3 + I_3) - \frac{i}{R} \sum_{f=1}^{N_F} \mu_f F_f \tag{2.12}$$

and the fields are periodic in τ . The electric and magnetic Wilson lines can also be regarded as the chemical potentials for the corresponding charges.

As we will see all the parameters except λ will enter the line operator vevs in specific complex combinations. These are the moduli

$$a \equiv R(A_\tau^{(\infty)} + i\Phi_0^{(\infty)}) \in \mathfrak{t}_\mathbb{C}, \quad b \equiv \frac{\Theta}{2\pi} - \frac{4\pi i R}{g^2} \Phi_9^{(\infty)} + \frac{\vartheta}{2\pi} a \in \mathfrak{t}_\mathbb{C}^*. \quad (2.13)$$

and the complexified mass parameters

$$m_f \equiv -\mu_f + iRM_f \in \mathbb{C} \quad f = 1, \dots, N_F. \quad (2.14)$$

We use the Lie algebra metric Tr in the action to regard $\Phi_9^{(\infty)}$ and a as elements of $\mathfrak{t}_\mathbb{C}^*$.

General $\mathcal{N} = 2$ theories have several mass parameters M_{Af} with $A = 0, 9$ and $f = 1, \dots, N_F$. These can be thought of as the vevs of the scalars in the vector multiplets that weakly gauge the flavor symmetries. Only $M_f \equiv M_{A=0,f}$, which are the analog of Φ_0 , will enter the line operator vevs.

3 Localization for gauge theories on $S^1 \times \mathbb{R}^3$

We apply the localization technique introduced for calculations in gauge theory on S^4 [6]. In this formalism, one adds a new term $tQ \cdot V$ to the action, so that the path integral takes the form

$$\int \mathcal{D}A \mathcal{D}\Psi e^{-S - tQ \cdot V}. \quad (3.1)$$

Here A and Ψ include all the bosons and fermions, respectively. We will also need to add ghost fields after gauge-fixing. For observables that are invariant under the supercharge Q of choice, the path integral is independent of the parameter t . The localization action is chosen to be $V = (\Psi, \overline{Q \cdot \Psi}) = (\psi, \overline{Q \cdot \psi}) + (\chi, \overline{Q \cdot \chi})$, where ψ and χ denote the fermions in the vector multiplet and the hypermultiplet. Since the bosonic part of $Q \cdot V$ is a positive definite term $\|Q \cdot \Psi\|^2$, the path integral is dominated by the solutions of $Q \cdot \Psi = 0$ in the limit $t \rightarrow +\infty$ and can be calculated exactly by summing the fluctuation determinants at all the saddle points.

3.1 Symmetries

For localization we need to close off-shell the relevant subalgebra of the whole superalgebra. For this we introduce seven auxiliary fields K_j as in [6]. The supersymmetry transformations in $\mathcal{N} = 2^*$ theory are given by

$$Q \cdot A_M = \epsilon \Gamma_M \Psi, \quad (3.2)$$

$$Q \cdot \Psi = \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon + iK^i \nu_i, \quad (3.3)$$

$$Q \cdot K_j = i\nu_j \Gamma^M D_M \Psi. \quad (3.4)$$

The gamma matrices and the constant spinors ν_i ($i = 1, \dots, 7$) are defined in appendix A. The gauge fields in F_{MN} and D_M include mass matrices $M_{Aij} = iM_A F_{ij}$ through the Scherk-Schwarz mechanism [6]. The spinor ϵ must be chosen so that the line operators are

invariant under the supersymmetry transformation Q . We will use the same spinor as used in [11]

$$\epsilon = \frac{1}{\sqrt{2}}(1, 0^7, 1, 0^7), \quad (3.5)$$

where the power indicates the number of repeated entries. It satisfies⁷

$$\Gamma_{5678}\epsilon = -\epsilon, \quad \Gamma_{04}\epsilon = -i\epsilon, \quad \Gamma_{1239}\epsilon = \epsilon, \quad (2\Gamma_{12} + \Gamma_{56} + \Gamma_{78})\epsilon = 0. \quad (3.6)$$

The last condition implies that the supercharge commutes with the combination $J_3 + I_3$ of spatial and R-symmetry rotations. This explains why this particular combination entered the definition (2.11) of the vev.

We will need later the square of the supersymmetry transformation given by the spinor ϵ in (3.5), Using the vector

$$v^M \equiv \epsilon \Gamma^M \epsilon = (i, 0^3, 1, 0^5) \quad M = 0, 1, \dots, 9, \quad (3.7)$$

we find that Q^2 generates time translation, minus the complexified gauge transformation G_Λ with gauge parameter $\Lambda = A_\tau + i\Phi_0$, and the flavor symmetry transformation iMF :

$$\begin{aligned} Q^2 \cdot A_M &= -F_{\tau M} - [i\Phi_0, D_M] - i\delta_M^i M_{ij} \Phi_j, \\ Q^2 \cdot \Psi &= -\partial_\tau \Psi - i[A_\tau + i\Phi_0, \Psi] - \frac{i}{4} M_{ij} \Gamma^{ij} \Psi, \\ Q^2 \cdot K_i &= -\partial_\tau K^i - i[A_\tau + i\Phi_0, K_i]. \end{aligned} \quad (3.8)$$

See appendix C and (2.27) of [6].

3.2 Localization equations

Let us study the localization equations $Q \cdot \Psi = 0$, whose solutions the path integral localizes to. We decompose Ψ as

$$\Psi = \sum_{M=1}^9 \Psi_M \tilde{\Gamma}^M \bar{\epsilon} + i \sum_{j=1}^7 \Upsilon_j \nu^j. \quad (3.9)$$

Noting that

$$\Psi_M = \epsilon \Gamma_M \Psi, \quad i\Upsilon_j = \bar{\nu}_j \Psi. \quad (3.10)$$

we obtain

$$0 = Q \cdot \Psi_M = \frac{1}{2} F_{PQ} \epsilon \Gamma_M \Gamma^{PQ} \epsilon \quad M = 1, \dots, 9, \quad (3.11)$$

$$0 = iQ \cdot \Upsilon_j = \frac{1}{2} F_{MN} \bar{\nu}_j \Gamma^{MN} \epsilon + iK_j \quad j = 1, \dots, 7. \quad (3.12)$$

⁷The third condition implies that Q corresponds to the fermionic symmetry for the Donaldson-Witten twist [24] in the 1239-directions. Thus $\langle L \rangle$ is a limit of the five-dimensional Nekrasov partition function [12] for a theory on $S^1 \times \mathbb{R}^4$ with a line operator insertion, where one of the equivariant parameter for the rotation in the 39 plane is set to zero and a direction in \mathbb{R}^4 is compactified on an infinitely small circle.

The equations (3.11) reduce to⁸

$$0 = Q \cdot \Psi_M = -v^N F_{NM}. \quad (3.13)$$

According to (3.8), these are equivalent to Q^2 -invariance, i.e., invariance under a combination of τ -translation, gauge transformations, and flavor transformations. Due to the replacement of the τ -derivative in (2.12), for generic λ the bosonic fields must also be invariant under the combination $J_3 + I_3$ of spatial and R-symmetry rotations. Among the various components of (3.12), the most important equations are⁹

$$0 = iQ \cdot \Upsilon_j = D_j \Phi_9 - \frac{1}{2} \sum_{k,l=1}^3 \epsilon_{jkl} F_{kl} + iK_j \quad j, k, l = 1, 2, 3. \quad (3.14)$$

The imaginary part sets K_j to zero. The real part is precisely the Bogomolny equations

$$*_3 F = D\Phi_9 \quad (3.15)$$

that describe monopoles on \mathbb{R}^3 ! Thus we conclude that the path integral localizes to the fixed points on the monopole moduli space with respect to spatial rotations and gauge transformations.

Four other components of (3.12) read

$$0 = iQ \cdot \Upsilon_j = \sum_{k=1}^3 \sum_{l=5}^8 (\bar{\nu}_j \Gamma^{kl} \epsilon) D_k \Phi_l + \sum_{l=5}^8 (\bar{\nu}_j \Gamma^{9l} \epsilon) i[\Phi_9, \Phi_l] + iK_j \quad j = 4, 5, 6, 7. \quad (3.16)$$

Again the imaginary part requires K_j to vanish. The real part of (3.16) is in fact the “realification” of the Dirac-Higgs equation

$$\sum_{i=1}^3 \sigma^i D_i q + [\Phi_9, q] = 0, \quad (3.17)$$

where the two-component “spinor” q is a linear combination of Φ_i with $i = 5, 6, 7, 8$. See appendix B for a related discussion. As in topological twist, the hypermultiplet scalars behave as a spinor under the combination $J_3 + I_3$. Though generically (3.17) itself admits non-zero solutions, the Q^2 -invariance, in particular the invariance under flavor transformations, requires q to vanish.

Thus localization on $S^1 \times \mathbb{R}^3$ leaves no bosonic zero-mode to be integrated over, and the final answer for the vev will be expressed as a finite sum. This is in contrast with the results for S^4 [6, 11] where the path integral reduced to a finite dimensional matrix integral.

⁸To show this we used the identities $\Gamma_M \tilde{\Gamma}_{[P} \Gamma_{Q]} = \Gamma_{[M} \tilde{\Gamma}_P \Gamma_{Q]} + 2\delta_{M[P} \Gamma_{Q]}$ and $\epsilon \Gamma_{[M} \tilde{\Gamma}_P \Gamma_{Q]} \epsilon = 0$.

⁹We used the following facts: $\bar{\nu}_j \Gamma^{kl} \epsilon = -\epsilon_{jkl}$ for $j, k, l \in \{1, 2, 3\}$, $\bar{\nu}_j \Gamma^{kl} \epsilon = 0$ for $j, k \in \{1, 2, 3\}$ and $l \in \{5, 6, 7, 8\}$, $\bar{\nu}_j \Gamma^{k9} \epsilon = \delta_{jk}$ for $j, k \in \{1, 2, 3\}$, and $\bar{\nu}_j \Gamma^{9l} \epsilon = 0$ for $j \in \{1, 2, 3\}$ and $l \in \{5, 6, 7, 8\}$. We also went ahead and set the hypermultiplets to zero. This is justified below by Q^2 -invariance.

3.3 On-shell action

Let us work out the classical contribution $e^{-S_{\text{cl}}}$, given by the on-shell action evaluated in the background (2.5). The on-shell action for the hypermultiplet simply vanishes, therefore we focus on the action (2.2) for the vector multiplet. For the background (2.5), we also have

$$F = ig^2\vartheta \frac{B}{16\pi^2} \frac{d\tau \wedge dr}{r^2} - \frac{B}{2} \sin\theta d\theta \wedge d\varphi, \quad (3.18)$$

$$*F = -\frac{B}{2r^2} d\tau \wedge dr + ig^2\vartheta \frac{B}{16\pi^2} \sin\theta d\theta \wedge d\varphi. \quad (3.19)$$

Our orientation is such that the volume form is $d\tau \wedge dx^1 \wedge dx^2 \wedge dx^3$. The action (2.2) is divergent in the presence of such a singular dyonic background. We can render the action finite by cutting off the spacetime at $\Sigma_3 \equiv \{r = \delta\}$ and by adding the boundary term [11, 25]

$$S_{\text{bdry}} = \frac{2}{g^2} \int_{\Sigma_3} \text{Tr}(\Phi_9 F - i\Phi_0 * F) \wedge d\tau. \quad (3.20)$$

We find that

$$\begin{aligned} S_{\text{vec}} &= \frac{1}{g^2\delta} \left(4\pi^2 R + \frac{g^2\vartheta^2 R}{16\pi^2} \right) \text{Tr} B^2 - i\vartheta R \text{Tr} \left(A_\tau^{(\infty)} B \right), \\ S_{\text{bdry}} &= -\frac{1}{g^2\delta} \left(4\pi^2 R + \frac{g^2\vartheta^2 R}{16\pi^2} \right) \text{Tr} B^2 - \frac{8\pi^2 R}{g^2} \text{Tr} \left(\Phi_9^{(\infty)} B \right) + \vartheta R \text{Tr} \left(\Phi_0^{(\infty)} B \right). \end{aligned} \quad (3.21)$$

Thus the classical on-shell action is given by

$$S_{\text{cl}}(B) \equiv S_{\text{vec}} + S_{\text{bdry}} = -\frac{8\pi^2 R}{g^2} \text{Tr} \left[\Phi_9^{(\infty)} B \right] - i\vartheta R \text{Tr} \left[\left(A_\tau^{(\infty)} + i\Phi_0^{(\infty)} \right) B \right]. \quad (3.22)$$

The on-shell action nicely combines with the weight $e^{iB \cdot \Theta}$ for the magnetic charge in (2.10) so that

$$\langle T_B \rangle \sim e^{iB \cdot \Theta} e^{-S_{\text{cl}}(B)} = e^{2\pi i B \cdot b}, \quad (3.23)$$

where b was defined in (2.13). This is the leading classical approximation to the 't Hooft operator vev. We will compute one-loop and non-perturbative corrections in the following sections.

4 One-loop determinants

Having computed the classical contribution to the 't Hooft operator vev, in this section we will compute the one-loop correction following [6] and in parallel with [11]. As we saw in the previous section, the path integral reduces to a sum over saddle points. For each saddle point we need to compute the fluctuation determinants. The methods here will also be used in section 5 for the computation of such non-perturbative corrections.

4.1 Gauge fixing

The gauge fixing action in the R_ξ -gauge is

$$S_{\text{gf}} = \int d^4x \text{Tr} \left(-i\tilde{c} \sum_{M=1,2,3,9} D_{(0)}^M D_M c + \tilde{b} \left(i \sum_{M=1,2,3,9} D_{(0)}^M \tilde{A}_M + \frac{\xi}{2} \tilde{b} \right) \right). \quad (4.1)$$

We have defined $\tilde{A}_M \equiv A_M - A_{(0)M}$ where $A_{(0)M}$ is the background configuration given in (2.5). The ghost fields c, \tilde{c} are fermionic, and \tilde{b} is bosonic. By defining the BRST transformations¹⁰

$$\begin{aligned} Q_B \cdot A_M &= -[c, D_M], & Q_B \cdot \Psi &= -i[c, \Psi], & Q_B \cdot K_i &= -i[c, K_i], \\ Q_B \cdot c &= -\frac{i}{2}[c, c], & Q_B \cdot \tilde{c} &= \tilde{b}, & Q_B \cdot \tilde{b} &= 0, \end{aligned} \quad (4.2)$$

we can write

$$S_{\text{gf}} = Q_B \cdot V_{\text{gh}}, \quad V_{\text{gh}} \equiv \int d^4x \text{Tr} \left(\tilde{c} \left(i \sum_{M=1,2,3,9} D_{(0)}^M \tilde{A}_M + \frac{\xi}{2} \tilde{b} \right) \right). \quad (4.3)$$

The BRST transformation squares to zero, $\{Q_B, Q_B\} = 0$. Unlike the case of S^4 [6] where the spacetime is compact, we do not need to introduce ghosts-for-ghosts to deal with constant gauge transformations.

We define the action of the supercharge Q on the ghosts by

$$\begin{aligned} Q \cdot c &= -v^M \tilde{A}_M \equiv -\tilde{\Phi} = -i\tilde{\Phi}_0 - \tilde{A}_\tau, & Q \cdot \tilde{c} &= 0, \\ Q \cdot \tilde{b} &= -v^M D_M \tilde{c} = -\partial_\tau \tilde{c} - i[A_\tau + i\Phi_0, \tilde{c}]. \end{aligned} \quad (4.4)$$

In the background Q annihilates all the fermions, therefore the background is supersymmetric. We have $\{Q, Q\}(\text{ghost}) = 0$.

4.2 One-loop determinants and the index theorem

After gauge fixing, the total fermionic symmetry we use for localization is

$$\hat{Q} \equiv Q + Q_B. \quad (4.5)$$

While Q^2 in (3.8) involves a gauge transformation G_Λ with a dynamical gauge parameter $\Lambda = A_\tau + i\Phi_0$, the gauge transformation that appears in $\hat{Q}^2 = Q^2 + \{Q, Q_B\}$ turns out to have a fixed parameter $\Lambda = A_{(0)\tau} + i\Phi_{(0)0} = A_\tau^{(\infty)} + i\Phi_0^{(\infty)}$.¹¹

$$\hat{Q}^2 = -\partial_\tau - i(A_\tau^{(\infty)} + i\Phi_0^{(\infty)}) + MF. \quad (4.6)$$

¹⁰To compare with Pestun's formalism in [6], set $\tilde{a}_0, b_0, c_0, \tilde{c}_0$ to zero. Then separate his BRST transformation δ into our Q_B and the part δ_0 proportional to a_0 : $\delta = Q_B + \delta_0$. Then our Q can be written as $s + \delta_0$ with $a_0 = -\Phi_{(0)0}$, where s denotes the supersymmetry transformation in [6].

¹¹For the gauge field $\hat{Q}^2 \cdot A_M = -\partial_\tau \tilde{A}_M - i[A_\tau^{(\infty)} + i\Phi_0^{(\infty)}, \tilde{A}_M]$.

Saddle points of the path integral remain the same after we replace $Q \cdot V$ by $\widehat{Q} \cdot \widehat{V}$. Recall that $M \equiv M_0$ is one of the mass parameters defined below (2.3) and that F is the flavor symmetry generator. The path integral to consider is

$$\int \mathcal{D}A \mathcal{D}\Psi \mathcal{D}K \mathcal{D}\tilde{b} \mathcal{D}c \mathcal{D}\tilde{c} e^{-S-t\widehat{Q} \cdot \widehat{V}}, \quad (4.7)$$

where

$$\widehat{V} = \left\langle \Psi, \overline{\widehat{Q} \cdot \Psi} \right\rangle + V_{\text{gh}}. \quad (4.8)$$

In order to evaluate the path integral in the limit $t \rightarrow \infty$, we need to compute the superdeterminant of the kinetic operator in $\widehat{Q}_{(0)} \cdot \widehat{V}^{(2)}$, where $\widehat{Q}_{(0)}$ is the linearization of \widehat{Q} , and $\widehat{V}^{(2)}$ is the quadratic part of \widehat{V} . Following [6] let us define

$$X_0 = (\tilde{A}_M)_{M=1}^9, \quad X_1 = (\Upsilon_i, c, \tilde{c}) \quad (4.9)$$

and their partners

$$\begin{aligned} X'_0 &\equiv \widehat{Q}_{(0)} \cdot X_0 = (\Psi_M - [c, D_{(0)M}])_{M=1}^9, \\ X'_1 &\equiv \widehat{Q}_{(0)} \cdot X_1 = \left(K_i - i(\bar{\nu}_i \Gamma^{MN} \epsilon) D_{(0)M} \tilde{A}_N, -\tilde{\Phi}, b \right). \end{aligned} \quad (4.10)$$

Now $\widehat{V}^{(2)}$ takes the form

$$V^{(2)} = \left\langle \left(X'_0 \ X_1 \right), \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X_0 \\ X'_1 \end{pmatrix} \right\rangle, \quad (4.11)$$

where D_{00} and others are certain differential operators. Then $\widehat{Q}_{(0)} \cdot V^{(2)}$ is given by

$$\begin{aligned} \widehat{Q}_{(0)} \cdot V^{(2)} &= \left\langle \left(X_0, X'_1 \right) \begin{pmatrix} -\mathcal{R}_{00} & \\ & 1 \end{pmatrix}, \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X_0 \\ X'_1 \end{pmatrix} \right\rangle \\ &+ \left\langle \left(X'_0, X_1 \right), \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} -1 & \\ & -\mathcal{R}_{11} \end{pmatrix} \begin{pmatrix} X'_0 \\ X_1 \end{pmatrix} \right\rangle, \end{aligned} \quad (4.12)$$

where $\widehat{Q}_{(0)}^2 \cdot X_0 = \mathcal{R}_{00} \cdot X_0$ and $\widehat{Q}_{(0)}^2 \cdot X_1 = \mathcal{R}_{11} \cdot X_1$. Thus the one-loop determinant is given by

$$\begin{aligned} Z_{1\text{-loop}} &= \frac{\det^{1/2} \left[\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} -1 & \\ & -\mathcal{R}_{11} \end{pmatrix} \right]}{\det^{1/2} \left[\begin{pmatrix} -\mathcal{R}_{00} & \\ & 1 \end{pmatrix} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \right]} = \frac{\det^{1/2} \mathcal{R}_{11}}{\det^{1/2} \mathcal{R}_{00}} \\ &= \frac{\det_{\text{Coker} D_{10}}^{1/2} \mathcal{R}}{\det_{\text{Ker} D_{10}}^{1/2} \mathcal{R}}. \end{aligned} \quad (4.13)$$

In the final line we have introduced notation $\mathcal{R} = \widehat{Q}_{(0)}^2$ and used the fact that \mathcal{R} commutes with D_{10} as guaranteed by \mathcal{R} -invariance of \widehat{V} . Thus we only need the differential operator

D_{10} , which can be obtained by explicitly computing $\widehat{V}^{(2)}$. It is easy to see what to expect from the results in section 3.2. There we saw that the localization equations are given by the Bogomolny and Dirac-Higgs equations. In appendix B, we will show that D_{10} involves the linearization of these equations as well as the dual of the gauge transformation.

The symmetry generator $\mathcal{R} = \widehat{Q}_{(0)}^2$ is given in (4.6). In a general $\mathcal{N} = 2$ theory, we replace that last term MF by $\sum_f M_f F_f$, where F_f are the flavor symmetry generators in (2.11). We also perform the shift (2.12) of the τ derivative. It is also useful to rescale \mathcal{R} as $\mathcal{R} \rightarrow -R\mathcal{R}$. This does not affect the value of the one-loop determinant (4.13) due to cancellations between the numerator and the denominator. Then \mathcal{R} takes a simple expression

$$\mathcal{R} = \varepsilon R \partial_\tau - i\lambda(J_3 + I_3) + ia + i \sum_{f=1}^{N_F} m_f F_f. \tag{4.14}$$

We have introduced a formal parameter ε that should be set to one at the end of calculation. A Fourier mode $e^{in\tau/R}$ along S^1 contributes $in\varepsilon$ to \mathcal{R} .

The form (4.13) of the one-loop determinant implies that it can be obtained from the equivariant index of the operator D_{10}

$$\text{ind } D_{10} \equiv \text{Tr}_{\text{Ker } D_{10}} e^{2\pi\mathcal{R}} - \text{Tr}_{\text{Coker } D_{10}} e^{2\pi\mathcal{R}}. \tag{4.15}$$

Indeed if it is given in terms of weights w_j and multiplicities c_j as

$$\text{ind } D_{10} = \sum c_j e^{w_j}, \tag{4.16}$$

the one-loop determinant is given by $Z_{1\text{-loop}} = \left(\prod_j w_j^{c_j}\right)^{-1/2}$. In the following we will separately define the indices for differential operators acting on vector and hypermultiplets. We will also adopt a normalization for ind that corresponds to $\text{ind}(D_{10}) \rightarrow -\frac{1}{2}\text{ind}(D_{10})$, so that the translation from the index to the one-loop determinant is simply given by the rule $\sum_j c_j e^{w_j} \rightarrow \prod_j w_j^{c_j}$. Then

$$Z_{1\text{-loop}} = \prod_j w_j^{c_j}. \tag{4.17}$$

Thus we need to compute the weights under the gauge transformation with parameter $a \equiv R(A_\tau^{(\infty)} + i\Phi_0^{(\infty)})$, a time translation by ε , and a spatial rotation along the 3-axis with angle $2\pi\lambda$, and flavor transformations with parameters m_f .

4.3 Calculation of the equivalent index

Before we delve into the details of the calculations, let us summarize our methodology that extends the techniques developed in [11], listing at the same time the relevant complexes and their interrelations. We showed above that the vector multiplet contribution to the one-loop determinant can be computed from the index of the complex that linearizes the Bogomolny equations in \mathbb{R}^3

$$D_{\text{Bogo}} : 0 \rightarrow \Omega^0(\text{ad } E) \xrightarrow{(D, [i\Phi_9, \bullet])} \Omega^1(\text{ad } E) \oplus \Omega^0(\text{ad } E) \rightarrow \Omega^1(\text{ad } E) \rightarrow 0, \tag{4.18}$$

where $\text{ad } E$ is the adjoint gauge bundle. The second arrow is the gauge transformation whose conjugate¹² appeared in (B.4), and the third is the map $(\delta A, \delta \Phi_9) \mapsto *D\delta A - D\delta \Phi_9 + i[\Phi_9, \delta A]$ in (B.3). As reviewed in appendix C, the Bogomolny equations with a single singularity on \mathbb{R}^3 are equivalent to the anti-self-duality equations on the (single-centered) Taub-NUT space with invariance under the action of the group that we call $U(1)_K$. Linearizing the correspondence, we will obtain the index of the Bogomolny complex¹³ (4.18) from the index of the self-dual complex

$$D_{\text{SD}} : 0 \rightarrow \Omega^0(\text{ad } E) \xrightarrow{D} \Omega^1(\text{ad } E) \xrightarrow{(1+*)D} \Omega^{2+}(\text{ad } E) \rightarrow 0 \quad (4.19)$$

on the four-dimensional space by taking an invariant part under the $U(1)_K$ action [11, 26]. Similarly the hypermultiplet contribution will be derived from index of the complex

$$D_{\text{DH},R} : 0 \rightarrow \Gamma(S \otimes R(E)) \xrightarrow{\sigma^j D_j + \Phi_9} \Gamma(S \otimes R(E)) \rightarrow 0, \quad (4.20)$$

where S is the spinor bundle over \mathbb{R}^3 , and Φ_9 acts on $q \in \Gamma(S \otimes R(E))$ in the matter representation R . Its index will be obtained from the $U(1)_K$ invariant part of the index of the twisted Dirac complex [11]

$$D_{\text{Dirac},R} : 0 \rightarrow \Gamma(S^+ \otimes R(E)) \xrightarrow{\bar{\sigma}^\mu D_\mu} \Gamma(S^- \otimes R(E)) \rightarrow 0 \quad (4.21)$$

in four dimensions.

Both the self-dual and Dirac complexes are related to the Dolbeault complex

$$\bar{D}_R : 0 \rightarrow \Omega^{0,0}(R(E)) \rightarrow \Omega^{0,1}(R(E)) \rightarrow \Omega^{0,2}(R(E)) \rightarrow 0. \quad (4.22)$$

To see this note that upon complexification we have $\Omega_{\mathbb{C}}^0 = \Omega^{0,0}$, $\Omega_{\mathbb{C}}^1 = \Omega^{1,0} \oplus \Omega^{0,1}$ and $\Omega_{\mathbb{C}}^{2+} = \Omega^{2,0} \oplus \Omega^{0,0}\omega \oplus \Omega^{0,2}$, where ω is the Kähler form. See, e.g., [27]. Since by Hodge duality $\Omega^{2,2} = \Omega^{0,0}$ and $\Omega^{2,1} = \Omega^{1,0}$, the complexification of the self-dual complex (4.19) is isomorphic to the Dolbeault complex (4.22) with $R = \text{ad}$ twisted by $\Omega^{0,0} \oplus \Omega^{2,0}$. For spinors recall that $\Omega^{p,q} = \Gamma(\Lambda^{p,q})$ and that $K = \Lambda^{2,0}$ is the canonical line bundle. We have

$$S^+ = K^{1/2} \otimes (\Lambda^{0,0} \oplus \Lambda^{0,2}), \quad S^- = K^{1/2} \otimes \Lambda^{0,1}. \quad (4.23)$$

Thus the Dirac complex (4.21) is isomorphic to the Dolbeault complex (4.22) twisted by $(\Omega^{2,0})^{1/2}$.

Let us now review the index of the Dolbeault complex. We will compute the index of the Dolbeault complex on Taub-NUT space by applying the Atiyah-Bott fixed point formula. Taub-NUT space is holomorphically isomorphic to flat \mathbb{C}^2 with local coordinates (z_1, z_2) , for which the $U(1) \times U(1)$ -equivariant index of the (untwisted) Dolbeault complex is given by

$$\text{ind}(\bar{\partial}) = \frac{t_1 t_2}{(1-t_1)(1-t_2)}. \quad (4.24)$$

¹²The equivariant index remains the same when we “fold” (4.18) into $0 \rightarrow \Omega^0 \oplus \Omega^1 \rightarrow \Omega^1 \oplus \Omega^0 \rightarrow 0$, where twisting by $\text{ad } E$ is implicit, and the second arrow is the linearized Bogomolny equations plus the dual of a gauge transformation (B.4). The same remark applies to the self-dual complex (4.19). It is the folded form of the complexes that naturally arises from gauge-fixing.

¹³We will refer to (4.18) and (4.20) as the Bogomolny and Dirac-Higgs (DH) complexes.

Let us denote by $U(1)_{J+R}$ the group generated by $J_3 + I_3$, the simultaneous spatial and R-symmetry rotations. The action of (t_1, t_2) on \mathbb{C}^2 is standard, $(z_1, z_2) \mapsto (t_1 z_1, t_2 z_2)$, and is related to $U(1)_K \times U(1)_{J+R}$ as

$$t_1 = e^{-2\pi i \nu + \pi i \lambda}, \quad t_2 = e^{2\pi i \nu + \pi i \lambda}, \quad (4.25)$$

as can be seen from (C.11). Here $e^{2\pi i \nu}$ parametrizes $U(1)_K$, while $2\pi \lambda$ is the angle of rotation along the 3-axis of \mathbb{R}^3 , which is the base of the circle fibration in Taub-NUT space (C.1). The $SU(2)$ R-symmetry action on the fields is also parametrized by λ .

For our purposes the best way to understand the formula (4.24) is to consider the group action on the basis of sections. For example an element of $\Omega^{0,0}$ can be expanded as

$$\sum_{k,l,m,n} c_{klmn} z_1^k \bar{z}_1^l z_2^m \bar{z}_2^n, \quad (4.26)$$

where $k, l, m, n \in \mathbb{Z}_{\geq 0}$ and the coefficients transform as $c_{klmn} \mapsto t_1^{-k+l} t_2^{-m+n} c_{klmn}$. Elements of $\Omega^{0,1}$ and $\Omega^{0,2}$ admit similar expansions. Summing up the weights with appropriate signs determined by the degrees in the complex, we obtain

$$\begin{aligned} \text{ind}_\delta(\bar{\partial}) &= \sum_{k,l,m,n \geq 0} (1 - t_1 - t_2 + t_1 t_2) t_1^{-k+l} t_2^{-m+n} \\ &= \frac{(1 - t_1)(1 - t_2)}{(1 - e^{-\delta} t_1^{-1})(1 - e^{-\delta} t_1)(1 - e^{-\delta} t_2^{-1})(1 - e^{-\delta} t_2)}. \end{aligned} \quad (4.27)$$

Factors $e^{-\delta}$ with small $\delta > 0$ are inserted to keep track of how we expand the numerator. We obtain (4.24) from the regularized index (4.27) by taking the limit $\delta \rightarrow 0$. Including the gauge group action, we obtain the index for the Dolbeault operator twisted by $R(E)$

$$\text{ind}_\delta(\bar{D}_R) = \frac{(1 - t_1)(1 - t_2)}{(1 - e^{-\delta} t_1^{-1})(1 - e^{-\delta} t_1)(1 - e^{-\delta} t_2^{-1})(1 - e^{-\delta} t_2)} \sum_{w \in R} e^{2\pi i w \cdot a}. \quad (4.28)$$

The relationships of the self-dual and Dirac complexes to the Dolbeault complex described above imply that

$$\text{ind}_\delta(D_{\text{SD},\mathbb{C}}) = (1 + t_1^{-1} t_2^{-1}) \text{ind}_\delta(\bar{D}_{\text{adj}}), \quad (4.29)$$

$$\text{ind}_\delta(D_{\text{Dirac},R}) = t_1^{-1/2} t_2^{-1/2} \text{ind}_\delta(\bar{D}_R). \quad (4.30)$$

Furthermore, the indices of the Bogomolny and Dirac-Higgs complexes are obtained by taking the $U(1)_K$ -invariant parts. This can be implemented by substituting (4.25) and $a \rightarrow a + B\nu$ and then integrating over ν :

$$\text{ind}(D_{\text{Bogo},\mathbb{C}}) = \lim_{\delta \rightarrow 0} \int_0^1 d\nu \text{ind}_\delta(D_{\text{SD},\mathbb{C}})|_{a \rightarrow a + B\nu}, \quad (4.31)$$

$$\text{ind}(D_{\text{DH},R}) = \lim_{\delta \rightarrow 0} \int_0^1 d\nu \text{ind}_\delta(D_{\text{Dirac},R})|_{a \rightarrow a + B\nu}. \quad (4.32)$$

The factors $e^{-\delta}$ in the integrands specify which poles to pick in the contour integrals. We also need to take into account the Fourier modes on S^1 that give rise to an infinite sum

$\sum_n e^{in\varepsilon}$. The formal parameter ε for time translation should be set to one at the end of the calculation.

Finally, the one-loop determinant $Z_{1\text{-loop}}^{\text{vm}}$ for the vector multiplet is obtained by the rule $\sum_j c_j e^{w_j} \rightarrow \prod_j w_j^{c_j}$ from

$$\text{ind}(D^{\text{vm}}) = \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{2\pi i n \varepsilon} \text{ind}(D_{\text{Bogomolny}, \mathbb{C}}). \quad (4.33)$$

The factor of 1/2 in (4.33) accounts for the complexification of the Bogomolny complex.

For the hypermultiplet, the one-loop determinant $Z_{1\text{-loop}}^{\text{hm}}$ arises if the same rule is applied to [11]

$$\text{ind}(D_R^{\text{hm}}) = -\frac{1}{2} \sum_{n \in \mathbb{Z}} e^{2\pi i n \varepsilon} \sum_{f=1}^{N_F} \left(e^{-2\pi i m_f} \text{ind}(D_{\text{DH}, R}) + e^{2\pi i m_f} \text{ind}(D_{\text{DH}, R})|_{a \rightarrow -a} \right). \quad (4.34)$$

Let us explain the meaning of this expression (4.34). The precise flavor symmetry of a massless theory is best described in terms of half-hypermultiplets. If an irreducible representation R is real, half-hypermultiplets can only appear in an even number $2N_F$, and the flavor symmetry G_F is $\text{Sp}(2N_F)$. The symplectic group $\text{Sp}(2N_F)$ has rank N_F in our convention. For a complex irreducible representation R , half-hypermultiplets always appear in conjugate pairs $R \oplus \bar{R}$. With N_F such pairs, the flavor symmetry is $\text{U}(N_F)$. When an irreducible representation R is pseudo-real, the theory is anomalous unless an even number $2N_F$ of half-hypermultiplets are present [28]. The flavor symmetry group in this case is $\text{SO}(2N_F)$. Parameters m_f in (4.34) are the equivariant parameters for the flavor group G_F of the massless theory, and are related to the physical masses M_f and the flavor chemical potentials μ_f as

$$m_f = -\mu_f + iRM_f. \quad (4.35)$$

The particular combination of terms in (4.34) was derived in [11] based on Higgsing which produces various types of matter representations.

The indices $\text{ind}(D_{\text{Bogomolny}, \mathbb{C}})$ and $\text{ind}(D_{\text{DH}, R})$ were computed in [11]:

$$\begin{aligned} \text{ind}(D_{\text{Bogomolny}, \mathbb{C}}) &= -\frac{e^{\pi i \lambda} + e^{-\pi i \lambda}}{2} \sum_{\alpha} e^{2\pi i \alpha \cdot a} \left(e^{(|\alpha \cdot B| - 1)\pi i \lambda} + e^{(|\alpha \cdot B| - 3)\pi i \lambda} + \dots + e^{-(|\alpha \cdot B| - 1)\pi i \lambda} \right), \\ \text{ind}(D_{\text{DH}, R}) &= -\frac{1}{2} \sum_{w \in R} e^{2\pi i w \cdot a} \left(e^{(|w \cdot B| - 1)\pi i \lambda} + e^{(|w \cdot B| - 3)\pi i \lambda} + \dots + e^{-(|w \cdot B| - 1)\pi i \lambda} \right). \end{aligned} \quad (4.36)$$

By applying the rule to (4.33) and (4.34), we find the one-loop determinant

$$\begin{aligned} & \prod_{n \in \mathbb{Z}} \prod_{\alpha} \prod_{k=0}^{|\alpha \cdot B| - 1} \left[n\varepsilon + \frac{1}{2}\lambda + \alpha \cdot a + \left(\frac{|\alpha \cdot B| - 1}{2} - k \right) \lambda \right]^{-1/2} \\ & \sim \prod_{\alpha > 0} \prod_{k=0}^{|\alpha \cdot B| - 1} \prod_{\pm} \sin^{-1/2} \left[\pi \left(\alpha \cdot a \pm \left(\frac{|\alpha \cdot B|}{2} - k \right) \lambda \right) \right] \\ & =: Z_{1\text{-loop}}^{\text{vm}}(a, \lambda; B), \end{aligned} \quad (4.37)$$

for the vector multiplet and

$$\begin{aligned}
 & \prod_{n \in \mathbb{Z}} \prod_{f=1}^{N_F} \prod_{w \in R} \prod_{k=0}^{|w \cdot B| - 1} \left[n \varepsilon + w \cdot a - m_f + \left(\frac{|w \cdot B| - 1}{2} - k \right) \lambda \right]^{1/2} \\
 & \sim \prod_{f=1}^{N_F} \prod_{w \in R} \prod_{k=0}^{|w \cdot B| - 1} \sin^{1/2} \left[\pi \left(w \cdot a - m_f + \left(\frac{|w \cdot B| - 1}{2} - k \right) \lambda \right) \right] \\
 & =: Z_{1\text{-loop}}^{\text{hm}}(a, m_f, \lambda; B)
 \end{aligned} \tag{4.38}$$

for the hypermultiplet. In the final expressions we set ε to one. When there is more than one matter irreducible representation we need to take a product over them. Combining the vector multiplet and hypermultiplet contributions, the one-loop factor is given by

$$Z_{1\text{-loop}}(a, m_f, \lambda; B) := Z_{1\text{-loop}}^{\text{vm}}(a, \lambda; B) Z_{1\text{-loop}}^{\text{hm}}(a, m_f, \lambda; B). \tag{4.39}$$

5 Contributions from monopole screening

In this section we calculate the contributions from non-perturbative saddle points of the localization action $Q \cdot V$. Since the bosonic part of $Q \cdot V$ is given by $\|Q \cdot \Psi\|^2$, these saddle points are the solutions of the equation $Q \cdot \Psi = 0$. As we saw in section 3.2, the solutions of $Q \cdot \Psi = 0$ are the fixed points of the Bogomolny equations with a prescribed singularity.

5.1 Definition of Z_{mono}

The moduli space of the solutions of the Bogomolny equations with a singularity prescribed by B has infinitely many components. For example, even for $B = 0$ there exist the components whose elements are smooth monopoles with charges labeled by all $v \in \Lambda_{cr}$. In our localization calculation only the components that contain fixed points of the $U(1)_{J+R} \times T$ -action are relevant, where T is the maximal torus of the gauge group. Invariance under $U(1)_{J+R} \times T$ -action is a strong constraint, because the T -invariance for generic $a \in \mathfrak{t}$ requires the adjoint fields to be Abelian, i.e., that they belong to \mathfrak{t} . The only Abelian solutions to the Bogomolny equations are the singular Dirac monopole solutions, and the singularity must be located at the point where the 't Hooft operator is inserted. This argument almost shows that the background configuration (2.5) is the only saddle point of the path integral. Abelian solutions of the Dirac form (2.5), where B is replaced by some other coefficient $v \in \Lambda_{cr} + B$, can however arise as a limit in the family of solutions whose singularity has coefficient B [10]. Such solutions represent smooth monopoles that approach the singular monopole and screen its charge. See [29] for an explicit example. For our calculation we only need to consider the components of the moduli space that contain such solutions. Under Kronheimer's correspondence, mentioned in section 4.3 and reviewed in appendix C, the Abelian solution specified by v uplift to a small instanton located at the point on Taub-NUT space where the S^1 fiber degenerates. Since our calculation needs only the local behavior of the fields near this point, we can replace Taub-NUT space by \mathbb{C}^2 . A more satisfying justification for this replacement is the fact that such a small instanton solution belongs

to a component of the instanton moduli space that is isomorphic as a complex variety to a component of the instanton moduli space for \mathbb{C}^2 [30]. See also [31]. We denote by $\mathcal{M}(B, v)$ the moduli space for the Bogomolny equations that descend from the component of the instanton moduli space. A generic point of $\mathcal{M}(B, v)$ is a solution that approaches the background (2.5) near the origin, and the same expression with B replaced by v asymptotically at infinity. It can be shown that we need $\|v\| \leq \|B\|$ for $\mathcal{M}(B, v)$ to be non-empty [11].

Since all the fixed points in (B, v) take the form of the 't Hooft background (2.5) except that B is replaced by v , each contributes a factor $e^{-S_{\text{cl}}(v)}$ computed in section 3.3. This classical contribution depends only on v and is universal among the fixed points in $\mathcal{M}(B, v)$. We also need to include the fluctuation determinant $\prod_j w_j^{c_j}$ from each fixed point, which can be computed from the indices of the Bogomolny and Dirac-Higgs complexes via the rule $\sum_j c_j e^{w_j} \rightarrow \prod_j w_j^{c_j}$, as in the one-loop analysis in section 4. By factoring out $Z_{1\text{-loop}}(v)$ that was computed in section 4, we denote the sum of such determinants by

$$Z_{1\text{-loop}}(v)Z_{\text{mono}}(B, v) \equiv \sum_{\substack{\text{fixed points} \\ \text{in } \mathcal{M}(B, v)}} \prod_j w_j^{c_j}. \tag{5.1}$$

This equation defines $Z_{\text{mono}}(B, v)$ as a function of B, v, a, b, m_f , and λ . As mentioned in footnote 7, $\langle L \rangle$ may be thought of as a dimensional reduction of the five-dimensional instanton partition function with an operator insertion. Thus $Z_{\text{mono}}(B, v)$ can be interpreted in terms of appropriate characteristic classes on $\mathcal{M}(B, v)$.

5.2 Monopole moduli space for $G = \text{U}(N)$

In order to compute $Z_{\text{mono}}(B, v)$ explicitly, we need a method to describe the component $\mathcal{M}(B, v)$ of the monopole moduli space and their fixed points. Let us now review the ADHM construction of $\mathcal{M}(B, v)$ in the case $G = \text{U}(N)$ [10].

We consider the flat space \mathbb{C}^2 parametrized by coordinates $z = (z_1, z_2)$. Let us set $W := \mathbb{C}^N$ and $V := \mathbb{C}^k$. The instanton bundle over \mathbb{C}^2 with instanton number k is described by a family of complexes

$$V \xrightarrow{\alpha(z)} \mathbb{C}^2 \otimes V \oplus W \xrightarrow{\beta(z)} V, \tag{5.2}$$

where the maps depend on z as

$$\alpha(z) = \begin{pmatrix} z_2 - B_2 \\ -z_1 + B_1 \\ -J \end{pmatrix}, \quad \beta(z) = \begin{pmatrix} z_1 - B_1 & z_2 - B_2 & -I \end{pmatrix}. \tag{5.3}$$

When the complex ADHM equation

$$[B_1, B_2] + IJ = 0 \tag{5.4}$$

which is equivalent to $\beta(z)\alpha(z) = 0$ is satisfied, the cohomology groups

$$H_z^0 = \text{Ker}[\alpha(z)], \quad H_z^1 = \text{Ker}[\beta(z)]/\text{Im}[\alpha(z)], \quad H_z^2 = V/\text{Im}[\beta(z)] \tag{5.5}$$

can be defined. If $H_z^0 = H_z^2 = 0$, $E_z = H_z^1$ describes the fiber of a smooth irreducible instanton bundle over \mathbb{C}^2 . We are also interested in singular configurations that arise as a limit of smooth ones, therefore we set $E_z = H_z^1 - H_z^0 - H_z^2$ in general. The Euler characteristic $\dim H_z^0 - \dim H_z^1 + \dim H_z^2 = -\dim E_z = -N$ is independent of z .

A monopole solution in $\mathcal{M}(B, v)$ descends from a $U(1)_K$ -invariant instanton. The group acts geometrically on (z_1, z_2) as $(z_1, z_2) \mapsto (e^{-2\pi i\nu} z_1, e^{2\pi i\nu} z_2)$ as in (4.25). Since (B_1, B_2) represent the positions of the instantons, they transform as $(B_1, B_2) \mapsto (e^{-2\pi i\nu} B_1, e^{2\pi i\nu} B_2)$. The group $U(1)_K$ also acts on the gauge bundle. The fiber E_0 at $z = 0$ if mapped to itself, and its character for $U(1)_K$ is given by $e^{2\pi i B\nu}$ where $e^{2\pi i\nu} \in U(1)_K$ and the charge B of the 't Hooft operator is regarded as a $N \times N$ diagonal matrix. The group $U(1)_K$ also acts on W and V . Since W represents the fiber E_∞ at $z = \infty$, its character is $\text{Tr} e^{2\pi i\nu\nu}$. The character of V can be written as $e^{2\pi i K\nu}$ with a $k \times k$ diagonal matrix K . The identification of E_z with $H_z^1 - H_z^0 - H_z^2$ implies that K is determined by¹⁴

$$\text{Tr} e^{2\pi i B\nu} = \text{Tr} e^{2\pi i\nu\nu} + (e^{2\pi i\nu} + e^{-2\pi i\nu} - 2)\text{Tr} e^{2\pi i K\nu} \quad e^{2\pi i\nu} \in U(1)_K \quad (5.6)$$

up to conjugation.

To describe $\mathcal{M}(B, v)$, we impose $U(1)_K$ invariance on the ADHM data. Namely the ADHM data must satisfy the conditions

$$-B_1 + [K, B_1] = 0, \quad B_2 + [K, B_2] = 0, \quad KI - Iv = 0, \quad vJ - JK = 0. \quad (5.7)$$

For the instanton moduli space, one would take a quotient by $GL(k, \mathbb{C})$. The matrix K breaks the $GL(k, \mathbb{C})$ into its commutant $\prod_r GL(k_r, \mathbb{C})$. Two combinations of such data are considered equivalent if they are related by an action of $\prod_r GL(k_r, \mathbb{C})$:

$$(B_1, B_2, I, J) \sim (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}) \quad g \in \prod_r GL(k_r, \mathbb{C}). \quad (5.8)$$

Thus the complex variety $\mathcal{M}(B, v)$ is given by the holomorphic quotient

$$\mathcal{M}(B, v) = \left\{ (B_1, B_2, I, J) \left| \begin{array}{l} -B_1 + [K, B_1] = 0 \\ B_2 + [K, B_2] = 0 \\ KI - IM = 0 \\ MJ - JK = 0 \end{array} \right. \right\} / \prod_r GL(k_r, \mathbb{C}). \quad (5.9)$$

The notion of fixed points requires a regularization of singularities in $\mathcal{M}(B, v)$. In this paper we do not attempt to describe the regularization in detail though we believe that this is important for the precise definition of the 't Hooft loop with a given magnetic charge B . See section 9 for a further discussion on this point. We will use a partial regularization that descends from the moduli space of non-commutative instantons that smooth the small instanton singularities. This led to a prescription, based on contour integrals, for how to take into account the fixed point contributions in [11]. Here we give an alternative prescription for the calculation of the fixed points and their contributions.

¹⁴A warning on notation. The “ K ” in $U(1)_K$ stands for Kronheimer. The matrix K is the weight of $U(1)_K$ acting on the k -dimensional vector space on which B_1 and B_2 act as endomorphisms.

5.3 Fixed points and their contributions

Next we turn to the description of fixed points. We need to know which fixed point \vec{Y} on the instanton moduli space descends to the specific component $\mathcal{M}(B, v)$ of the monopole moduli space. The fixed points are given by the ADHM data (B_1, B_2, I, J) that satisfy

$$\begin{aligned} \varepsilon_1 B_1 + [\phi, B_1] &= 0, & \varepsilon_2 B_2 + [\phi, B_2] &= 0, \\ \phi I - I a &= 0, & (\varepsilon_1 + \varepsilon_2) J + a J - J \phi &= 0. \end{aligned} \tag{5.10}$$

for any $(\varepsilon_1, \varepsilon_2, a) \in \text{Lie}[\text{U}(1) \times \text{U}(1) \times T]$ for some $\phi = \text{diag}(\phi_1, \dots, \phi_k)$ parametrizing the Cartan subalgebra of $\prod_r \text{U}(k_r) \subset \text{U}(k)$. Solutions to these equations are known [12, 32] and are expressed in terms of Young diagrams \vec{Y} . See [33] for explicit expressions for (B_1, B_2, I, J) at the fixed point \vec{Y} . Here we only need the expressions¹⁵ for ϕ_s [12]

$$\phi_s = (i_s - 1)\varepsilon_1 + (j_s - 1)\varepsilon_2 + a_{\alpha(s)} \text{ where } \alpha(s) \in \{1, \dots, N\} \text{ is such that } s \in Y_{\alpha(s)}. \tag{5.11}$$

Since the fixed point \vec{Y} in the instanton moduli space satisfy the general $\text{U}(1)^2 \times T$ -invariance condition (5.10) together with (5.11), it also satisfies the $\text{U}(1)_K$ invariance condition (5.7) if $\text{U}(1)_K$ is embedded in $\text{U}(1)^2 \times \text{U}(N)$ in such a way that their actions are compatible. Since the embedding is given by the substitution

$$\varepsilon_1 \rightarrow -\nu, \quad \varepsilon_2 \rightarrow \nu, \quad a \rightarrow a + v\nu, \tag{5.12}$$

the $\text{U}(1)_K$ -invariant fixed points correspond to

$$\vec{Y} \quad \text{such that} \quad K_s = v_{\alpha(s)} + j_{\alpha(s)} - i_{\alpha(s)} \tag{5.13}$$

up to a permutation of $s \in \{1, \dots, k\}$.

To obtain the weights w_j each fixed point contributes, we can combine the method in section 4 with the known result for the Dolbeault index at the fixed point. We recall from that section that the Dolbeault index on \mathbb{C}^2 , defined by a formal application of the Atiyah-Bott formula, is given by

$$\text{ind}(\bar{D}_{\text{adj}}) = \sum_{\alpha, \beta=1}^N e_{\alpha} e_{\beta}^{-1} \frac{1}{(1 - t_1^{-1})(1 - t_2^{-1})}, \tag{5.14}$$

where $e_{\alpha} = e^{2\pi i a_{\alpha}}$. Let us define

$$\chi(Y) = \sum_{(i,j) \in Y} t_1^{i-1} t_2^{j-1} \tag{5.15}$$

and the conjugate

$$\chi(Y)^* = \sum_{(i,j) \in Y} t_1^{1-i} t_2^{1-j}. \tag{5.16}$$

¹⁵In this subsection we use Greek alphabets α, β, \dots to denote the $\text{U}(N)$ indices, and use (i, j) to denote the location of a box in a Young diagram.

Then the local index for the Dolbeault operator at the fixed point \vec{Y} is given by

$$\begin{aligned} \text{ind}(\overline{D}_{\text{adj}})_{\vec{Y}} &= \sum_{\alpha, \beta=1}^N e_{\alpha} e_{\beta}^{-1} \left(\frac{1}{(1-t_1)(1-t_2)} - \chi(Y_{\alpha}) \right) \\ &\quad \times \left(\frac{1}{(1-t_1^{-1})(1-t_2^{-1})} - \chi(Y_{\beta})^* \right) (1-t_1)(1-t_2). \end{aligned} \quad (5.17)$$

As shown in section 4.3, the one-loop determinant is obtained from the non-polynomial part $\text{ind}(\overline{D}_{\text{adj}})^{1\text{-loop}} \equiv \sum_{\alpha, \beta} e_{\alpha} e_{\beta}^{-1} (1-t_1^{-1})^{-1} (1-t_2^{-1})^{-1}$ of (5.17). The rest of (5.17) is a Laurent polynomial, which we denote by $\text{ind}(\overline{D}_{\text{adj}})_{\vec{Y}}^{\text{inst}}$. It is nothing but (minus) the character of the tangent space to the moduli space, and can be rewritten as [32]

$$\text{ind}(\overline{D}_{\text{adj}})_{\vec{Y}}^{\text{inst}} = - \sum_{\alpha, \beta=1}^N e_{\alpha} e_{\beta}^{-1} \left(\sum_{s \in Y_{\alpha}} t_1^{-L_{Y_{\beta}}(s)} t_2^{A_{Y_{\alpha}}(s)+1} + \sum_{t \in Y_{\beta}} t_1^{L_{Y_{\alpha}}(t)+1} t_2^{-A_{Y_{\beta}}(t)} \right). \quad (5.18)$$

We have introduced the arm- and leg-lengths

$$A_Y(s) = \lambda_i - j, \quad L_Y(s) = \lambda_j^T - i, \quad (5.19)$$

where λ_i and λ_i^T are the numbers of boxes in the i -th row and column of Y , respectively.

Let us denote the $U(1)_K$ -invariant part of $\sum_{n \in \mathbb{Z}} e^{2\pi i n \varepsilon} ((1+t_1^{-1}t_2^{-1})/2) \text{ind}(\overline{D}_{\text{adj}})^{1\text{-loop}}$ by $\text{ind}(D^{\text{vm}})^{1\text{-loop}}$, where $U(1)_K$ acts on the gauge bundle with a generator v . It gives rise to $Z_{1\text{-loop}}^{\text{vm}}(v)$ via the rule $\sum_j c_j e^{w_j} \rightarrow \prod_j w_j^{c_j}$ as in section 4. Similarly we define

$$\text{ind}(D^{\text{vm}})_{\vec{Y}}^{\text{mono}} \equiv U(1)_K\text{-invariant part of } \sum_{n \in \mathbb{Z}} e^{2\pi i n \varepsilon} \frac{1+t_1^{-1}t_2^{-1}}{2} \text{ind}(\overline{D}_{\text{adj}})_{\vec{Y}}^{\text{inst}}. \quad (5.20)$$

The same rule applied to this gives a contribution to $Z_{\text{mono}}^{\text{vm}}(B, v)$.

The $U(1)_K$ -invariant terms arise from the triples

$$(\alpha, \beta, s \in Y_{\alpha}) \quad \text{such that} \quad v_{\alpha} - v_{\beta} + L_{Y_{\beta}}(s) + A_{Y_{\alpha}}(s) + 1 = 0 \quad (5.21)$$

which contribute

$$-\frac{1}{2} \sum_{n \in \mathbb{Z}} e^{2\pi i n \varepsilon} (1 + e^{-2\pi i \lambda}) e^{2\pi i (a_{\alpha} - a_{\beta})} e^{\pi i (A_{Y_{\alpha}}(s) - L_{Y_{\beta}}(s) + 1)\lambda}, \quad (5.22)$$

and also from the triples

$$(\alpha, \beta, t \in Y_{\beta}) \quad \text{such that} \quad v_{\alpha} - v_{\beta} - L_{Y_{\alpha}}(t) - A_{Y_{\beta}}(t) - 1 = 0 \quad (5.23)$$

which contribute

$$-\frac{1}{2} \sum_{n \in \mathbb{Z}} e^{2\pi i n \varepsilon} (1 + e^{-2\pi i \lambda}) e^{2\pi i (a_{\alpha} - a_{\beta})} e^{\pi i (L_{Y_{\alpha}}(s) - A_{Y_{\beta}}(s) + 1)\lambda} \quad (5.24)$$

to $\text{ind}(D^{\text{vm}})_{\vec{Y}}^{\text{mono}}$. By applying the rule $\sum_j c_j e^{w_j} \rightarrow \prod_j w_j^{c_j}$, we find the vector multiplet contribution to $Z_{\text{mono}}^{\text{vm}}$

$$z_{\vec{Y}}^{\text{vec}} = \prod_{(\alpha, \beta, s)} \prod_{\pm} \left(\sin \left[\pi \left(a_{\alpha} - a_{\beta} + \frac{1}{2} (A_{Y_{\alpha}}(s) - L_{Y_{\beta}}(s) \pm 1)\lambda \right) \right] \right)^{-1}. \quad (5.25)$$

We emphasize that the products are over the triples (α, β, s) satisfying (5.21). The contributions from (α, β, t) in (5.23) are identical to those from (α, β, s) in (5.21). Thus the power in (5.25) is -1 , not $-1/2$. The same remark applies to (5.27) below.

For a single hypermultiplet in the adjoint representation, we need to consider

$$\text{ind}(D_{\text{adj}}^{\text{hm}})^{\text{mono}}_{\vec{Y}} \equiv \text{U}(1)_K\text{-invariant part of} \\ - \frac{e^{2\pi im} + e^{-2\pi im}}{2} \sum_{n \in \mathbb{Z}} e^{2\pi in \varepsilon} t_1^{-1/2} t_2^{-1/2} \text{ind}(\overline{D}_{\text{adj}})_{\text{inst}}. \quad (5.26)$$

From this we get the contribution of an adjoint hypermultiplet

$$z_{\vec{Y}}^{\text{adj}} = \prod_{(\alpha, \beta, s)} \prod_{\pm} \sin \left[\pi \left(a_\alpha - a_\beta + \frac{1}{2} (A_{Y_\alpha}(s) - L_{Y_\beta}(s)) \lambda \pm m \right) \right]. \quad (5.27)$$

The product is over the same triples as above.

For a hypermultiplet in the fundamental representation, we need the Dolbeault index for the corresponding bundle¹⁶

$$\text{ind}(\overline{D}_{\text{fund}})_{\vec{Y}} = \sum_{\alpha=1}^N e_\alpha t_1 t_2 \left(\frac{1}{(1-t_1)(1-t_2)} - \chi(Y_\alpha) \right). \quad (5.28)$$

Thus the Dirac index is

$$\text{ind}(\overline{D}_{\text{Dirac, fund}})_{\vec{Y}} = \sum_{\alpha=1}^N e_\alpha t_1^{1/2} t_2^{1/2} \left(\frac{1}{(1-t_1)(1-t_2)} - \chi(Y_\alpha) \right). \quad (5.29)$$

For a pair

$$(\alpha, s \in Y_\alpha) \quad \text{such that} \quad v_\alpha - i_s + j_s = 0, \quad (5.30)$$

it contributes

$$- e^{2\pi i a_\alpha} e^{\pi i (i_s + j_s - 1) \lambda} \quad (5.31)$$

to $\text{ind}(D_{\text{DH, fund}})^{\text{mono}}_{\vec{Y}}$ and thus

$$\frac{1}{2} \sum_{n \in \mathbb{Z}} e^{2\pi in \varepsilon} (e^{2\pi i a_\alpha} e^{\pi i (i_s + j_s - 1) \lambda} e^{-2\pi im} + e^{-2\pi i a_\alpha} e^{-\pi i (i_s + j_s - 1) \lambda} e^{2\pi im}) \quad (5.32)$$

to $\text{ind}(D_{\text{fund}}^{\text{hm}})^{\text{mono}}_{\vec{Y}}$. Then

$$z_{\vec{Y}}^{\text{fund}}(a, m, \lambda; B, v) = \prod_{(\alpha, s)} \sin \left[\pi \left(a_\alpha - m + \frac{1}{2} (i_s + j_s - 1) \lambda \right) \right]. \quad (5.33)$$

Again we stress that the product is over the pairs (α, s) satisfying (5.30).

The total monopole screening contribution is then given by

$$Z_{\text{mono}}(a, m_f, \lambda; B, v) = \sum_{\vec{Y}} z_{\vec{Y}}^{\text{vec}}(a, \lambda; B, v) \prod_R \prod_f z_{\vec{Y}}^R(a, m_f; B, v), \quad (5.34)$$

where the sum is over N -tuples of Young diagrams \vec{Y} satisfying (5.13), and the product is over the matter representations R . Explicit expressions for $Z_{\text{mono}}(B, v)$ will appear as part of the operator vevs in section 6.

¹⁶When used in instanton counting, this leads to the contribution of a fundamental hypermultiplet.

6 Gauge theory results

For a Wilson operator in an arbitrary representation R , the on-shell action vanishes. The only saddle point in the path integral is the trivial one, and the one-loop determinant is 1 due to Bose-Fermi cancellations. Thus the expectation value is given by evaluating the holonomy (2.4) in the background:

$$\langle W_R \rangle = \text{Tr}_R \exp \left[2\pi i R \left(A_\tau^{(\infty)} + i\Phi_0^{(\infty)} \right) \right] = \text{Tr}_R e^{2\pi i a}, \quad (6.1)$$

where a was defined in (2.13).

For the 't Hooft operator, we combine the classical, one-loop, and monopole screening contributions from the previous sections:

$$\langle T_B \rangle = \sum_v e^{2\pi i v \cdot b} Z_{1\text{-loop}}(v) Z_{\text{mono}}(B, v). \quad (6.2)$$

6.1 $SU(2)$ $\mathcal{N} = 2^*$

For $SU(2)$, it is convenient to substitute

$$a \rightarrow \begin{pmatrix} a & \\ & -a \end{pmatrix}, \quad b \rightarrow \begin{pmatrix} b & \\ & -b \end{pmatrix} \quad (6.3)$$

with the understanding that in the following the symbols a and b are complex numbers rather than matrices. For this gauge group we can label the line operators by a pair of integers (p, q) , where p and q are magnetic and electric charges respectively [17, 34, 35], and they are related to the coweight and the highest weight of the representation as

$$\begin{aligned} B = (p/2, -p/2) &\equiv \text{diag}(p/2, -p/2) \in \Lambda_{cw}, \\ (q/2, -q/2) &\equiv \text{diag}(q/2, -q/2) \in \Lambda_w \leftrightarrow \text{spin } q/2 \text{ representation}. \end{aligned} \quad (6.4)$$

The most basic Wilson operator $W_{1/2} = L_{0,1}$ corresponding to spin 1/2 has an expectation value

$$\langle W_{1/2} \rangle = \langle L_{0,1} \rangle = e^{2\pi i a} + e^{-2\pi i a}. \quad (6.5)$$

For the minimal 't Hooft operator $T_{1/2} = L_{1,0}$ that is S-dual to $W_{1/2}$, we find

$$\langle T_{1/2} \rangle = \langle L_{1,0} \rangle = (e^{2\pi i b} + e^{-2\pi i b}) \left(\frac{\sin(2\pi a + \pi m) \sin(2\pi a - \pi m)}{\sin(2\pi a + \frac{\pi}{2}\lambda) \sin(2\pi a - \frac{\pi}{2}\lambda)} \right)^{1/2}. \quad (6.6)$$

For the minimal dyonic loops $L_{1,\pm 1}$,

$$\langle L_{1,\pm 1} \rangle = (e^{2\pi i(b \pm a)} + e^{-2\pi i(b \pm a)}) \left(\frac{\sin(2\pi a + \pi m) \sin(2\pi a - \pi m)}{\sin(2\pi a + \frac{\pi}{2}\lambda) \sin(2\pi a - \frac{\pi}{2}\lambda)} \right)^{1/2}. \quad (6.7)$$

The simplest example with monopole screening contribution is given by

$$\begin{aligned} \langle L_{2,0} \rangle = & (e^{4\pi i b} + e^{-4\pi i b}) \frac{\prod_{s_1, s_2 = \pm 1} \sin^{1/2}(2\pi a + s_1 \pi m + s_2 \frac{\pi}{2} \lambda)}{\sin^{1/2}(2\pi a + \pi \lambda) \sin^{1/2}(2\pi a - \pi \lambda) \sin(2\pi a)} \\ & + \sum_{s = \pm} \frac{\prod_{\pm} \sin \pi(2a \pm m + s \lambda / 2)}{\sin(2\pi a) \sin \pi(2a + s \lambda)}, \end{aligned} \quad (6.8)$$

where we used (4.39) and (5.34). We observe that this is the Moyal product of the minimal 't Hooft operator vev with it self,

$$\langle L_{2,0} \rangle = \langle L_{1,0} \rangle * \langle L_{1,0} \rangle. \quad (6.9)$$

In the SU(2) case $*$ is defined by

$$(f * g)(a, b) \equiv e^{i \frac{\lambda}{8\pi} (\partial_b \partial_{a'} - \partial_a \partial_{b'})} f(a, b) g(a', b')|_{a'=a, b'=b} \quad (6.10)$$

with a different coefficient due to the factor of 2 in the inner product

$$a \cdot b \rightarrow \text{Tr}[\text{diag}(a, -a) \cdot \text{diag}(b, -b)] = 2ab. \quad (6.11)$$

In section 7, we will explain how the Moyal product appears from the structure of the path integral.

The precise choice of signs and relative numerical normalizations among terms is difficult to fix purely in gauge theory without additional assumptions. In the examples considered in this paper we choose to be pragmatic and make the choice by assuming physically reasonable structures such as Moyal multiplication, correspondence with the Verlinde operators, as well as agreement with classical $\text{SL}(2, \mathbb{C})$ holonomies in the $\lambda \rightarrow 0$ limit.

6.2 $U(N)$ $\mathcal{N} = 2^*$

For the gauge group $U(N)$, the minimal 't Hooft operators, with charges¹⁷ $B = (\pm 1, 0^{N-1})$ (the power indicates the number of repeated entries) corresponding to the fundamental and anti-fundamental representations of the Langlands dual group, have the expectation values

$$\langle T_{B=(\pm 1, 0^{N-1})} \rangle = \sum_{l=1}^N e^{\pm 2\pi i b_l} \left(\prod_{\pm} \prod_{j \neq l} \frac{\sin \pi (a_l - a_j \pm m)}{\sin \pi (a_l - a_j \pm \lambda/2)} \right)^{1/2}. \quad (6.12)$$

For the magnetic charge $B = (1, -1, 0^{N-2})$, corresponding to the adjoint representation,

$$\begin{aligned} & \langle T_{B=(1, -1, 0^{N-2})} \rangle \\ &= \sum_{k \neq l} e^{2\pi i (b_k - b_l)} \left[\frac{\left[\prod_{\pm, \pm} \sin \pi (a_{kl} \pm m \pm \lambda/2) \right] \left[\prod_{\pm} \prod_{j \neq k, l} \sin \pi (a_{kj} \pm m) \sin \pi (a_{lj} \pm m) \right]}{\sin^2 \pi a_{kl} \prod_{\pm} \sin \pi (a_{kl} \pm \lambda) \left[\prod_{\pm} \prod_{j \neq k, l} \sin \pi (a_{kj} \pm \lambda/2) \sin \pi (a_{lj} \pm \lambda/2) \right]} \right]^{1/2} \\ &+ \sum_{l=1}^N \prod_{j \neq l} \frac{\prod_{\pm} \sin \pi (a_{lj} \pm m + \lambda/2)}{\sin \pi a_{lj} \sin \pi (a_{lj} + \lambda)}. \end{aligned} \quad (6.13)$$

¹⁷The Cartan subalgebra of $U(N)$ is spanned by real diagonal matrices. For $SU(N)$ they must be traceless. We often drop “diag” in $a = \text{diag}(a_1, \dots, a_N)$ to simplify notation. The inner product is defined by the trace $a \cdot a' = \text{Tr} aa'$, and this is used to identify the Cartan algebra with its dual.

From (6.12) and (6.13) we find that

$$\langle T_{B=(1,-1,0^{N-2})} \rangle = \langle T_{B=(-1,0^{N-1})} \rangle * \langle T_{B=(1,0^{N-1})} \rangle. \quad (6.14)$$

For $B = (2, 0^{N-1})$,

$$\begin{aligned} \langle T_{B=(2,0^{N-1})} \rangle &= \sum_{k=1}^N e^{4\pi i b_k} \left(\prod_{j \neq k} \frac{\prod_{\pm, \pm} \sin \pi(a_{kj} \pm m \pm \lambda/2)}{[\sin^2 \pi a_{kj} \prod_{\pm} \sin \pi(a_{kj} \pm \lambda)]} \right)^{1/2} \\ &+ \sum_{k \neq l} \frac{\prod_{\pm} \sin \pi(a_{kl} \pm m + \lambda/2)}{\sin \pi(a_{kl} + \lambda) \sin \pi a_{kl}}. \end{aligned} \quad (6.15)$$

For this we find

$$\langle T_{B=(2,0^{N-1})} \rangle = \langle T_{B=(1,0^{N-1})} \rangle * \langle T_{B=(1,0^{N-1})} \rangle. \quad (6.16)$$

Results for the gauge group $SU(N)$ can be obtained by taking a and b traceless.

6.3 $U(2)$ $N_{\mathbf{F}} = 4$

For the minimal 't Hooft operator in this theory, we have

$$\begin{aligned} \langle T \rangle &= e^{\pi i b_{12}} \left(\frac{\prod_{f=1}^4 \sin \pi(a_1 - m_f) \sin \pi(a_2 - m_f)}{\sin^2 \pi a_{12} \prod_{\pm} \sin \pi(a_{12} \pm \lambda)} \right)^{1/2} \\ &+ e^{-\pi i b_{12}} \left(\frac{\prod_{f=1}^4 \sin \pi(a_1 + m_f) \sin \pi(a_2 + m_f)}{\sin^2 \pi a_{12} \prod_{\pm} \sin \pi(a_{12} \pm \lambda)} \right)^{1/2} \\ &+ \frac{\prod_{f=1}^4 \sin \pi(a_1 - m_f + \frac{\lambda}{2})}{\sin \pi a_{12} \sin \pi(-a_{12} - \lambda)} + \frac{\prod_{f=1}^4 \sin \pi(a_2 - m_f + \frac{\lambda}{2})}{\sin \pi a_{21} \sin \pi(-a_{21} - \lambda)}. \end{aligned} \quad (6.17)$$

We have defined $a_{jk} = a_j - a_k$.

6.4 $U(N)$ $N_{\mathbf{F}} = 2N$

For the minimal 't Hooft operator given by the magnetic charge $B = \text{diag}(1, -1, 0^{N-2})$ corresponding to the adjoint representation, we obtain

$$\begin{aligned} &\langle T_B \rangle \\ &= \sum_{\substack{1 \leq k, l \leq N \\ k \neq l}} e^{\pi i (b_k - b_l)} \frac{\left[\prod_{f=1}^N \sin \pi(a_k - m_f) \sin \pi(a_l - m_f) \right]^{1/2}}{\sin \pi a_{kl} \prod_{\pm} \left[\sin \pi(a_{kl} \pm \lambda) \prod_{j \neq k, l} \sin \pi(a_{kj} \pm \lambda/2) \sin \pi(a_{jl} \pm \lambda/2) \right]^{1/2}} \\ &+ \sum_{l=1}^N \frac{\prod_{f=1}^{2N} \sin \pi(a_l - m_f + \frac{\lambda}{2})}{\prod_{j \neq l} \sin \pi a_{lj} \sin \pi(-a_{lj} - \lambda)}. \end{aligned} \quad (6.18)$$

We have introduced the notation $a_{jk} \equiv a_j - a_k$.

We emphasize that (6.18) and (6.17) are the vev of the 't Hooft operator in the $U(N)$ and $U(2)$ theories, not in the $SU(N)$ and $SU(2)$ theories. We will compare (6.18) and (6.17) with the Verlinde operators in Toda and Liouville theories in section 8 that we will propose to be related to the line operators in the $SU(N)$ and $SU(2)$ theories. While we do not have a computational method intrinsic to $SU(N)$, we will see that (6.18) and (6.17), when a is restricted to be traceless, do reproduce a -dependent terms in the CFT results.

7 Noncommutative algebra and quantization

By using the structure of the path integral we have found, in this section we show that the vevs of the line operators on $S^1 \times \mathbb{R}^3$, inserted on the 3-axis ($x^1 = x^2 = 0$), form a non-commutative algebra, when the axis is considered as time and the operators are time-ordered. We will begin with the $U(1)$ case and then discuss the general gauge group.

7.1 Maxwell theory

Let us explain how non-commutativity arises in the algebra of Wilson-'t Hooft operators in Maxwell theory on $S^1 \times \mathbb{R}^3$ upon twisting by a spatial rotation along the S^1 .

We begin with an intuitive explanation based on classical fields [13]. By taking S^1 as time, the expectation value of the product of Wilson (W) and 't Hooft (T) operators can be thought of as the trace

$$\langle W \cdot T \rangle = \text{Tr}_{\mathcal{H}(W \cdot T)} (-1)^F e^{-2\pi R H} e^{2\pi i \lambda J_3} \tag{7.1}$$

taken in the Hilbert space $\mathcal{H}(W \cdot T)$ defined by the line operators. The space $\mathcal{H}(W \cdot T)$ differs from the simple product $\mathcal{H}(W) \otimes \mathcal{H}(T)$ because when both W and T are present, their electric and magnetic fields produce the Poynting vector $\vec{E} \times \vec{B}$ that carries a non-zero angular momentum. The orientation of the Poynting vector, and therefore the phase $e^{2\pi i \lambda J_3}$, depends on the relative positions of the operators on the 3-axis.

Next we present an approach suitable for localization. For simplicity let us turn off the theta angle. The line operator $L_{p,q}$ with magnetic and electric charges (p, q) at the origin $\vec{x} = 0$ is defined by the path integral over the fluctuations around the singular background

$$A = A_\tau^{(\infty)} d\tau + p \frac{\cos \theta}{2} d\varphi \tag{7.2}$$

with the insertion of the holonomy

$$e^{-iq \oint_{S^1} A}. \tag{7.3}$$

We note here that the expression for the monopole field in (7.2) has Dirac strings in two directions ($\theta = 0, \pi$). The expectation value $\langle L_{p,q} \rangle$ is a function of (a, b) , which are normalized electric and magnetic background Wilson lines

$$a \equiv R A_\tau^{(\infty)}, \quad b \equiv \frac{\Theta}{2\pi}. \tag{7.4}$$

We claim that the path integral yields the expectation value

$$\langle L_{p,q} \rangle = e^{-2\pi i (qa + pb)}. \tag{7.5}$$

The magnetic part is essentially the definition of the magnetic Wilson line Θ , which is defined as the chemical potential for the magnetic charge at infinity. The electric part arises because the holonomy (7.3) is evaluated against the background Wilson line.

Let us introduce a twist along the S^1 . If we think of the circle as the time direction, we can write

$$\langle L_{p,q} \rangle = \text{Tr}_{\mathcal{H}(L_{p,q})} (-1)^F e^{-2\pi R H} e^{2\pi i \lambda J_3}, \quad (7.6)$$

where J_3 is the Cartan generator of the spatial rotation group $SU(2)$. The twist by J_3 means that we rotate the system by angle $2\pi\lambda$ as we go along S^1 , i.e., we introduce the identification

$$(\tau + 2\pi R, \varphi) \sim (\tau, \varphi + 2\pi\lambda). \quad (7.7)$$

In terms of the new coordinates $(\tau', \varphi') = (\tau, \varphi + \frac{\lambda}{R}\tau)$, the identification is simply

$$(\tau' + 2\pi R, \varphi') \sim (\tau', \varphi'). \quad (7.8)$$

The components of the gauge field are related as

$$A_{\tau'} = A_\tau - \frac{\lambda}{R} A_\varphi, \quad A_{\varphi'} = A_\varphi. \quad (7.9)$$

Note that A_φ represents a holonomy around the Dirac strings. In our choice of local trivialization $A_\varphi(\theta = \pi/2) = 0$, so we have a simple relation

$$\oint_{S^1} A = 2\pi a \quad \text{at} \quad \theta = \pi/2. \quad (7.10)$$

Thus the monopole field does not contribute to the holonomy as claimed above, in fact even after twisting. The holonomies at $\theta \neq \pi/2$ are, however, shifted from a . Indeed we find

$$\oint_{S^1} A = \int_0^{2\pi R} d\tau' A_{\tau'} = 2\pi \left(a \mp \frac{p}{2} \lambda \right) \quad \text{at} \quad \theta = \begin{cases} 0, \\ \pi. \end{cases} \quad (7.11)$$

One can picture the shift as arising from the holonomy winding around the Dirac strings. Then for the product of Wilson and 't Hooft operators $W \equiv L_{0,1}, T \equiv L_{1,0}$

$$W(\vec{x} = (0, 0, z)) \cdot T(\vec{x} = 0), \quad (7.12)$$

its expectation value is given by

$$\langle W(\vec{x} = (0, 0, z)) \cdot T(\vec{x} = 0) \rangle = e^{-2\pi i (a \mp \frac{1}{2} \lambda)} e^{-2\pi i b} \quad \text{for} \quad \begin{cases} z > 0, \\ z < 0. \end{cases} \quad (7.13)$$

The Wilson line operator (7.3) for $z > 0$ is evaluated at $\theta = 0$, and for $z < 0$ at $\theta = \pi$. The difference λ between the shifts in a at $z > 0$ and $z < 0$ is independent of the choice of local trivialization. We can also see that the expectation value of the product of operators is given by the Moyal product of the expectation values:

$$\langle W(z) \cdot T(0) \rangle = \begin{cases} \langle W \rangle * \langle T \rangle & \text{for } z > 0, \\ \langle T \rangle * \langle W \rangle & \text{for } z < 0, \end{cases} \quad (7.14)$$

where the Moyal product $*$ is defined by

$$(f * g)(a, b) \equiv \lim_{a' \rightarrow a, b' \rightarrow b} e^{i \frac{\lambda}{4\pi} (\partial_b \partial_{a'} - \partial_a \partial_{b'})} f(a, b) g(a', b'). \quad (7.15)$$

This is the special case of the more general result for an arbitrary gauge group that we now turn to.

7.2 Non-Abelian gauge theories

Here we consider a general $\mathcal{N} = 2$ gauge theory with arbitrary matter content. Let us suppose that we have multiple line operators $L_i \equiv L_{B_i, R_i}(\vec{x} = (0, 0, z_i))$ located at various points $\vec{x} = (0, 0, z_i)$ on the 3-axis, ordered so that

$$z_1 > z_2 > \dots > z_n. \quad (7.16)$$

In the localization calculation, it suffices to consider the Abelian configurations with magnetic charges v_i associated with B_i as only these contribute to the path integral. As is clear from the Maxwell case, the holonomy at z_i around S^1 is shifted by the magnetic fields $\propto v_j$ created by L_j for $j \neq i$:

$$a \rightarrow a + \frac{\lambda}{2} \left(\sum_{j < i} v_j - \sum_{j > i} v_j \right). \quad (7.17)$$

Let us assume that the individual operator vevs are given by

$$\langle L \rangle = \sum_{v, w} Z_{L, \text{total}}(a, b; v, w) \equiv \sum_{v, w} e^{2\pi i (w \cdot a + v \cdot b)} Z_L(a, m_f, \lambda; v, w) \quad (7.18)$$

for some functions $Z_L(a, m_f, \lambda; v, w)$. Then localization calculation yields

$$\langle L_1 \cdot L_2 \cdot \dots \cdot L_n \rangle = \prod_{i=1}^n \sum_{w_i} \sum_{v_i} Z_{L_i, \text{total}} \left(a + \frac{\lambda}{2} \left(\sum_{j < i} v_j - \sum_{j > i} v_j \right), b; v_i, w_i \right), \quad (7.19)$$

One can easily see that (7.19) is the Moyal product of the expectation values of individual operators

$$\langle L_1 \cdot L_2 \cdot \dots \cdot L_n \rangle = \langle L_1 \rangle * \langle L_2 \rangle * \dots * \langle L_n \rangle, \quad (7.20)$$

where $*$ is defined by

$$(f * g)(a, b) \equiv e^{i \frac{\lambda}{4\pi} (\partial_b \cdot \partial_{a'} - \partial_a \cdot \partial_{b'})} f(a, b) g(a', b') \Big|_{a'=a, b'=b} \quad (7.21)$$

with the natural product \cdot between the derivatives inside the exponential.

As a concrete example, let us consider $SU(2)$ $\mathcal{N} = 2^*$ theory. We computed the vev of the charge-two 't Hooft operator in (6.8). As explained in [11], this operator corresponds to the product of two minimal 't Hooft operators. This is because the resolution of the singular moduli space corresponds to separating the charge-two 't Hooft operator into two minimal ones [10]. Indeed one can check that the expression (6.8) is precisely the Moyal product of (6.6) with itself.

7.3 Deformation quantization of the Hitchin moduli space

We are now going to explain that the noncommutative algebra structure given by the Moyal multiplication above realizes a deformation quantization of the Hitchin moduli space associated with the gauge theory.

In [7], a correspondence between certain $\mathcal{N} = 2$ gauge theories and punctured Riemann surfaces C was discovered. The correspondence is a main ingredient of the relation [20] between gauge theories and two-dimensional conformal field theories. The correspondence is also manifested in the relation between the gauge theories and the Hitchin systems on the Riemann surfaces. This made it possible to study the integrable structure [36–38] as well as the low-energy dynamics of these theories using the Hitchin system on the Riemann surfaces [39], generalizing [40].

Let $A = A_z dz + A_{\bar{z}} d\bar{z}$ be a connection of a G -bundle over C , and $\varphi = \varphi_z dz + \bar{\varphi}_{\bar{z}} d\bar{z}$ an adjoint-valued 1-form. They are assumed to possess prescribed singularities at the punctures. The Hitchin moduli space is the space of solutions to

$$\begin{aligned} F_{z\bar{z}} &= [\varphi_z, \bar{\varphi}_{\bar{z}}], \\ D_{\bar{z}}\varphi_z &= 0, \quad D_z\bar{\varphi}_{\bar{z}} = 0, \end{aligned} \tag{7.22}$$

up to G -gauge transformations. The Hitchin moduli space is hyperKähler, and therefore has a $\mathbb{C}\mathbb{P}^1$ of complex structures \mathcal{J} , each being a linear combination of three complex structures $\mathcal{J} = I, J$, and K . Each complex structure \mathcal{J} is associated with a real symplectic form $\omega_{\mathcal{J}} := g\mathcal{J}$, as well as a holomorphic symplectic form $\Omega_{\mathcal{J}}$. For $\mathcal{J} = I, J, K$, these are given by $\Omega_I = \omega_J + i\omega_K, \Omega_J = \omega_K + i\omega_I, \Omega_K = \omega_I + i\omega_J$.

In the original assignment of I, J, K by Hitchin [41], we are particularly interested in the complex structure J . The combination $\mathcal{A} \equiv A + i\varphi$ is then holomorphic, and (7.22) implies that¹⁸ \mathcal{A} is a flat $G_{\mathbb{C}}$ connection. In terms of \mathcal{A} , Ω_J is given by

$$\Omega_J \propto \int_C \text{Tr} \delta\mathcal{A} \wedge \delta\mathcal{A}. \tag{7.23}$$

The $U(1)$ R-symmetry rotates the phases of $\varphi_z, \bar{\varphi}_{\bar{z}}$, and $\Phi_0 + i\Phi_9$, and Ω_J transforms accordingly [39].

We focus on the one-punctured torus, which corresponds to $SU(2)$ $\mathcal{N} = 2^*$ theory. Let us define generators of the first homology so that the holonomy matrices (A, B, M) along them satisfy the relation

$$AB = MBA. \tag{7.24}$$

Here M is the holonomy around a small circle surrounding the puncture, and A and B are the holonomy matrices for the usual A- and B-cycles. Dehn’s theorem [42, 43] allows us to label the non-self-intersecting closed curves by two integers (p, q) with equivalence

¹⁸More precisely, the first of (7.22) combined with the difference of the second and the third is equivalent to the flatness of \mathcal{A} . The J -holomorphic structure of the Hitchin moduli space can be described by dropping the sum of the second and the third equations, and by taking the quotient with respect to $G_{\mathbb{C}}$ gauge transformations.

$(p, q) \sim (-p, -q)$. They can be naturally identified with the charges of line operators in (6.4) [17]. In particular, we have the correspondence

$$\langle L_{0,1} \rangle \leftrightarrow \text{Tr} A, \tag{7.25}$$

$$\langle L_{1,0} \rangle \leftrightarrow \text{Tr} B, \tag{7.26}$$

$$\langle L_{1,\pm 1} \rangle \leftrightarrow \text{Tr} A^{\pm 1} B. \tag{7.27}$$

Let us consider the case $\lambda = 0$. From (6.5)–(6.7) we find that

$$\langle L_{0,1} \rangle_{\lambda=0} = e^{2\pi i a} + e^{-2\pi i a}, \tag{7.28}$$

$$\langle L_{1,0} \rangle_{\lambda=0} = (e^{2\pi i b} + e^{-2\pi i b}) \left(\frac{\sin(2\pi a + \pi m) \sin(2\pi a - \pi m)}{\sin^2(2\pi a)} \right)^{1/2}, \tag{7.29}$$

$$\langle L_{1,\pm 1} \rangle_{\lambda=0} = (e^{2\pi i(b \pm a)} + e^{-2\pi i(b \pm a)}) \left(\frac{\sin(2\pi a + \pi m) \sin(2\pi a - \pi m)}{\sin^2(2\pi a)} \right)^{1/2}. \tag{7.30}$$

Replacing the arrows in (7.27) by equalities, these expressions were exactly given as the definition of the Darboux coordinates¹⁹ (a, b) on the Hitchin moduli space with respect to the symplectic structure Ω_J ! Later in [9], (a, b) were identified with the complexification of the Fenchel-Nielsen coordinates of Teichmüller space. Here we see that both the coordinates (a, b) and the symplectic structure Ω_J arise naturally from the gauge theory on $S^1 \times \mathbb{R}^3$.

For $SU(2)$ $N_F = 4$ theory, our gauge theory calculation of the 't Hooft and dyonic operator vevs is not complete due to the difficulty with monopole screening contributions. The relation with Liouville theory and the formula (8.27) below suggests, however, that (a, b) are the complexified Fenchel-Nielsen coordinates on the Hitchin moduli space associated with the four-punctured sphere [62].

8 Gauge theory on S^4 and Liouville/Toda theories

In this section we propose a precise relation between the line operator vevs on $S^1 \times \mathbb{R}^3$ and the corresponding difference operators that act on the conformal blocks of Liouville and Toda field theories. We first motivate the correspondence by gauge theory considerations. Then we will give an algorithm for computing the line operator vevs on $S^1 \times \mathbb{R}^3$ using two-dimensional CFT.

Let us consider the Liouville theory on a genus g Riemann surface with n punctures $C_{g,n}$. The correlation function of primary fields V_{α_e} ($e = 1, \dots, n$) with momenta α_e , inserted at the punctures, takes the form

$$\left\langle \prod V_{\alpha_e} \right\rangle_{C_{g,n}} = \int \left[\prod d\alpha_i \right] \mathcal{C}(\alpha_i; \alpha_e) |\mathcal{F}(\alpha_i; \alpha_e)|^2, \tag{8.1}$$

where the integral is over internal momenta α_i ($i = 1, \dots, 3g - 3 + n$) and the function $\mathcal{C}(\alpha_i; \alpha_e)$ is a product of DOZZ three-point functions [44, 45]. The conformal block

¹⁹In [8] the Darboux coordinates were denoted by (α, β) , and are related to our (a, b) by a trivial rescaling. We also have $\text{Tr} M = 2 \cos \pi m$.

$\mathcal{F}(\alpha_i; \alpha_e)$ depends on α_i, α_e , and the gluing parameters q_i holomorphically. The central charge c of Liouville theory is parametrized as

$$c = 1 + 6Q^2, \quad Q = \mathbf{b} + \mathbf{b}^{-1}. \tag{8.2}$$

The AGT correspondence [20] states that for $\mathbf{b} = 1$ the correlation function (8.1) coincides with the partition function of the corresponding $\mathcal{N} = 2$ gauge theory on S^4 as defined by Pestun in [6]. The gluing parameters q_i are related to the complexified couplings $\tau_i = \frac{\theta_i}{2\pi} + \frac{4\pi i}{g_i^2}$ as $q_i = e^{2\pi i \tau_i}$. Pestun's partition function contains as the north and south pole contributions the Nekrasov instanton partition functions defined in the Omega background [12]. The parameter \mathbf{b} is related to the equivariant parameters $\varepsilon_1, \varepsilon_2$ of the Omega background as $\mathbf{b}^2 = \varepsilon_1/\varepsilon_2$. The path integral formulation of the deformation $S^4_{\mathbf{b}}$ to $\mathbf{b} \neq 1$ is unknown at the time of writing. Even for $\mathbf{b} \neq 1$, it is expected that (8.1) will be reproduced by the partition function of the $\mathcal{N} = 2$ gauge theory on S^4 , deformed in a certain way by a parameter \mathbf{b} .

For an $\mathcal{N} = 2$ gauge theory with $SU(2)$ gauge groups associated with a punctured Riemann surface $C_{g,n}$, there is a correspondence between the charges (B, R) of Wilson-'t Hooft operators and a collection γ of non-self-intersecting closed curves on $C_{g,n}$ [17]. In [15, 16] it was shown that there exists a difference operator Λ_γ , the Verlinde operator, whose action on $\mathcal{F}(\alpha_i; \alpha_e)$ we denote by²⁰

$$\mathcal{F}(\alpha_i; \alpha_e) \rightarrow [\Lambda_\gamma \cdot \mathcal{F}](\alpha_i; \alpha_e). \tag{8.3}$$

The same papers demonstrated that, for $\mathbf{b} = 1$, the expectation value of the Verlinde operator defined as

$$\int \left[\prod d\alpha_i \right] \mathcal{C}(\alpha_i; \alpha_e) \overline{\mathcal{F}(\alpha_i; \alpha_e)} [\Lambda_\gamma \cdot \mathcal{F}](\alpha_i; \alpha_e) \tag{8.4}$$

reproduces the expectation value of the Wilson loop on S^4 computed by Pestun [6].

The agreement of (8.4) with the 't Hooft loop expectation value, again for $\mathbf{b} = 1$, was more recently verified in [11]. This was done by performing a localization calculation for 't Hooft loops placed along a large circle, called the equator, of S^4 . The neighborhood of the equator is approximately $S^1 \times \mathbb{R}^3$, therefore much of the analysis overlaps the present paper. Because of the curvature, however, in the orthonormal frame of the metric such that the conformal Killing spinor is periodic, the hypermultiplet becomes antiperiodic. We expect that this property persists for $\mathbf{b} \neq 1$. The effect of antiperiodicity is to multiply the right hand side of (4.34) by an extra factor $e^{i\varepsilon/2}$. Thus we conjecture, and the AGT correspondence suggests, that the 't Hooft loop vev on S^4 for general \mathbf{b} is given by

$$\begin{aligned} \langle T_B \rangle_{S^4} &= \int_{it} da \sum_v Z_{\text{pole}}(a + \mathbf{b}^2 v/2, \bar{q}) Z_{\text{equator}}(a; B, v) Z_{\text{north}}(a - \mathbf{b}^2 v/2, q) \\ &= \int_{it} da Z_{\text{pole}}(a, \bar{q}) \sum_v e^{-(\mathbf{b}^2/2)v \cdot \partial_a} Z_{\text{equator}}(a; B, v) e^{-(\mathbf{b}^2/2)v \cdot \partial_a} Z_{\text{north}}(a, q), \end{aligned} \tag{8.5}$$

²⁰For the Verlinde operator in Liouville theory, corresponding to a connected closed curve on the Riemann surface, our normalization of the operator agrees with [16]. Our operator is $2 \cos \mathbf{b}Q$ times those in [15, 46].

where

$$Z_{\text{pole}}(a, q) = e^{-S_{\text{cl}}(a)} Z_{1\text{-loop}}(a) Z_{\text{inst}}(a, q) \tag{8.6}$$

is the Nekrasov partition function including the classical, one-loop, and instanton contributions on \mathbb{C}^2 ,²¹ and the equator contribution

$$Z_{\text{equator}}(a; B, v) = Z_{1\text{-loop}}(a, \mathbf{m}_f + 1/2, \mathbf{b}^2; v) Z_{\text{mono}}(a, \mathbf{m}_f + 1/2, \mathbf{b}^2; B, v). \tag{8.7}$$

is given in terms of the one-loop determinant in (4.39) and the monopole screening contribution in (5.34). The shift in mass is due to the antiperiodicity of hypermultiplets mentioned above. For $\mathbf{b} = 1$ (8.5) was established in [11], where the definitions of $Z_{1\text{-loop}}$ and Z_{mono} were slightly different due to the the shift in the mass. The second equality in (8.5) involves a shift of integration contours and integration by parts. For some examples in Liouville theory, it was checked that the shift of contours does not encounter poles [11].

Based on the conjectured relation (8.5) between the line operator vevs on S^4 and $S^1 \times \mathbb{R}^3$ for general $\lambda = \mathbf{b}^2$, we propose that the vev in the theory on $S^1 \times \mathbb{R}^3$ can be obtained from the Verlinde operator in Liouville theory by the following algorithm. This algorithm was used in [11] in the case $\mathbf{b} = 1$ to read off Z_{equator} from the Verlinde operator. First we change the normalization of the conformal block and define²²

$$\mathcal{B}(\alpha_i; \alpha_e) \equiv \mathcal{C}(\alpha_i; \alpha_e)^{1/2} \mathcal{F}(\alpha_i; \alpha_e) \tag{8.8}$$

using the square root of the function $\mathcal{C}(\alpha_i; \alpha_e)$ that appears in the correlation function (8.1). With the one-loop factor in (8.6) whose precise definition was given in [11], we expect that $\mathcal{B}(\alpha) = Z_{\text{pole}}(a)$ with the identification $\alpha_i = Q/2 + a_i/\mathbf{b}$. In this normalization, the Liouville correlation function is simply given by

$$\left\langle \prod V_{\alpha_e} \right\rangle_{C_{g,n}} = \int \left[\prod d\alpha_i \right] |\mathcal{B}(\alpha_i; \alpha_e)|^2, \tag{8.9}$$

where we used the fact that in the physical range of Liouville momenta, the function $\mathcal{C}(\alpha_i; \alpha_e)$ is real. The Verlinde operator acts on $\mathcal{B}(\alpha)$ as the difference operator defined by

$$[\mathcal{L}_\gamma \cdot \mathcal{B}](\alpha_i; \alpha_e) \equiv \mathcal{C}(\alpha_i; \alpha_e)^{1/2} [\Lambda_\gamma \cdot \mathcal{F}](\alpha_i; \alpha_e). \tag{8.10}$$

Its vev is then given by

$$\int \left[\prod d\alpha_i \right] \overline{\mathcal{B}(\alpha_i; \alpha_e)} [\mathcal{L}_\gamma \cdot \mathcal{B}](\alpha_i; \alpha_e). \tag{8.11}$$

The operator algebra of \mathcal{L}_γ is isomorphic to that of Λ_γ . In the case γ is purely magnetic, we conjecture that \mathcal{L}_γ is related to the 't Hooft loop T_B above as

$$\mathcal{L}_\gamma = \sum_v e^{-(\mathbf{b}^2/2)v \cdot \partial_a} Z_{\text{equator}}(a; B, v) e^{-(\mathbf{b}^2/2)v \cdot \partial_a} \tag{8.12}$$

²¹Since we are primarily interested in the equator contributions we suppress the dependence on $\varepsilon_1, \varepsilon_2, \mathbf{m}_f$, and λ in (8.6). The mass parameters \mathbf{m}_f as well as the Coulomb moduli a are pure imaginary.

²²The Verlinde operator in [16] was computed in the standard normalization for conformal blocks [47]. Yet another normalization introduced in [48] was used to calculate the Verlinde operators in [15], and in this basis the operators are free of square-roots [9].

up to an overall constant. For more general dyonic charges the Verlinde operator takes the form

$$\mathcal{L} = \sum_{v,w} e^{-(b^2/2)v \cdot \partial_a} e^{2\pi i w \cdot a} Z_L \left(a, m_f + \frac{1}{2}, b^2; v, w \right) e^{-(b^2/2)v \cdot \partial_a}, \quad (8.13)$$

with some functions $Z_L(a, m_f, \lambda; v, w)$. For a product of $SU(2)$'s, our Lie algebra convention is such that $v \cdot \partial_a = \sum_i v_i \frac{\partial}{\partial a_i}$ and $w \cdot a = \sum_i \text{Tr} [\text{diag}(w_i, -w_i) \text{diag}(a_i, -a_i)] = 2 \sum_i w_i a_i$, with v_i and w_i being half-integers. The ‘‘highest’’ v (corresponding to the highest weight of the Langlands-dual representation) and w have $v_i = p_i/2$ and $w_i = q_i/2$, where (p_i, q_i) are the Dehn-Thurston parameters [17]. We conjecture that the line operator vevs are given in terms of these functions as

$$\langle L \rangle_{S^4} = \int_{it} da Z_{\text{pole}}(a, \bar{q}) \sum_{v,w} e^{-\frac{b^2}{2}v \cdot \partial_a} e^{2\pi i w \cdot a} Z_L \left(a, m_f + \frac{1}{2}, b^2; v, w \right) e^{-\frac{b^2}{2}v \cdot \partial_a} Z_{\text{pole}}(a, q) \quad (8.14)$$

on S^4 and

$$\langle L \rangle_{S^1 \times \mathbb{R}^3} = \sum_{v,w} e^{2\pi i(w \cdot a + v \cdot b)} Z_L(a, m_f, \lambda; v, w) \quad (8.15)$$

on $S^1 \times \mathbb{R}^3$.

We have focused so far on the correspondence [20] between the gauge theories whose gauge group is a product of $SU(2)$'s and Liouville theory on the corresponding Riemann surface, but we also propose that the relation (8.5) should hold for more general gauge groups and Toda theories [20, 49]. Some examples of Verlinde operators in Toda theories were computed in [46, 50]. We conjecture that the Verlinde operators in Toda theories are precisely related to the line operator vevs on S^4 and $S^1 \times \mathbb{R}^3$ via the equations (8.13), (8.14), and (8.15).

We observe that the Verlinde operator (8.13) is related to the vev (8.15) on $S^1 \times \mathbb{R}^3$ precisely by the Weyl transform (ordering)²³

$$\langle L \rangle_{S^1 \times \mathbb{R}^3} \xrightarrow{\text{Weyl}} \mathcal{L}. \quad (8.16)$$

The parameter $-b$ plays the role of the canonical momentum:

$$\begin{aligned} b &\leftrightarrow i \frac{\lambda}{2\pi} \frac{\partial}{\partial a} && \text{in general,} \\ b_i &\leftrightarrow i \frac{\lambda}{4\pi} \frac{\partial}{\partial a_i} && \text{for } SU(2) \text{ and Liouville.} \end{aligned} \quad (8.17)$$

Thus our proposal (8.5) implies that the Verlinde operators are the Weyl transform of the line operator vevs on $S^1 \times \mathbb{R}^3$, when the gauge theory has a Lagrangian description. It is very natural to conjecture that this relation should hold even when the gauge theory does not admit a Lagrangian description [7, 52].

²³For a 2-dimensional phase space parametrized by (q, p) , the operator \mathcal{O} and its inverse Weyl transform f are related by $f(q, p) = \int d\sigma e^{-\frac{i}{\hbar} p \sigma} \langle q | e^{\frac{i}{2\hbar} \sigma \hat{p}} \mathcal{O}(\hat{q}, \hat{p}) e^{\frac{i}{2\hbar} \sigma \hat{p}} | q \rangle$, $\mathcal{O} = \frac{1}{(2\pi)^2 \hbar} \int d\sigma d\tau d q d p e^{-i\tau(\hat{q}-q) - \frac{i}{\hbar} \sigma(\hat{p}-p)} f(q, p)$, where $[\hat{q}, \hat{p}] = i\hbar$, $\hat{q}|q\rangle = q|q\rangle$, $\langle q|q'\rangle = \delta(q - q')$ [51].

The mass shift in (8.7) and (8.12) is consistent with a somewhat confusing aspect of the correspondence [7] between $\mathcal{N} = 2$ gauge theories and Riemann surfaces. Namely the massless limit of a gauge theory corresponds to removing a puncture in the Hitchin system [13, 39], while it corresponds to tuning external momenta to special values in Liouville/Toda theories [20, 49, 53] keeping the puncture. In gauge theories the shift arises due to the difference in the geometries where the theories live.

Below we demonstrate our proposal with several examples.

8.1 $SU(2)$ $\mathcal{N} = 2^*$

This theory corresponds to the Liouville theory on the one-punctured torus [20]. Let $C(\alpha_1, \alpha_2, \alpha_3)$ be the DOZZ three-point function of Liouville theory [44, 45]. We denote the internal and external Liouville momenta by α and α_e respectively. The Verlinde loop operator that corresponds to the minimal 't Hooft operators acts on the conformal block as

$$[\mathcal{L}_{1,0} \cdot \mathcal{F}](\alpha, \alpha_e) = \sum_{\pm} H_{\pm}(\alpha) \mathcal{F}(\alpha \pm \mathbf{b}/2, \alpha_e) . \tag{8.18}$$

This implies the following expression, conjectural for $\mathbf{b} \neq 1$, of the minimal 't Hooft operator vev in the $\mathcal{N} = 2^*$ theory on S^4 :

$$\langle L_{1,0} \rangle_{S^4} = \int_{Q/2+i\mathbb{R}} d\alpha C(\alpha, \alpha_e, Q - \alpha) \sum_{\pm} \overline{\mathcal{F}(\alpha, \alpha_e)} H_{\pm}(\alpha) \mathcal{F}(\alpha \pm \mathbf{b}/2, \alpha_e) . \tag{8.19}$$

The map between the Liouville and gauge theory parameters is given by

$$\alpha = \frac{Q}{2} + \frac{a}{\mathbf{b}}, \quad \alpha_e = \frac{Q}{2} + \frac{\mathbf{m}}{\mathbf{b}} . \tag{8.20}$$

The coefficients H_{\pm} are known to be [16]

$$H_{\pm}(\alpha) = \frac{\Gamma(\pm 2a)\Gamma(\pm 2a + \mathbf{b}Q)}{\Gamma(\pm 2a + \mathbf{m} + \mathbf{b}Q/2)\Gamma(\pm 2a - \mathbf{m} + \mathbf{b}Q/2)} . \tag{8.21}$$

By performing the manipulations explained above, in appendix D.1 we obtain

$$\mathcal{L}_{1,0} = \sum_{\pm} e^{\pm \frac{1}{4}\mathbf{b}^2\partial_a} \left(\prod_{\pm} \frac{\cos(2\pi a \pm \pi \mathbf{m})}{\sin(2\pi a \pm \frac{\pi}{2}\mathbf{b}^2)} \right)^{1/2} e^{\pm \frac{1}{4}\mathbf{b}^2\partial_a} . \tag{8.22}$$

This is indeed related to the 't Hooft operator vev (6.6) by the Weyl transform above.

8.2 $SU(N)$ $\mathcal{N} = 2^*$

The Verlinde operator corresponding to the 't Hooft operator with charge $B = (1, 0^{N-1})$, acting on the Toda conformal block for the one-punctured torus, was computed in [46] in the standard normalization. In appendix D.2 we convert it to a difference operator acting on the block \mathcal{B} in the normalization that absorbs the square root of the three-point function. We find

$$\mathcal{L}_{B=(1,0^{N-1})} = \sum_{l=1}^N e^{-\frac{\mathbf{b}^2}{2}h_l \cdot \partial_a} \left(\prod_{\pm} \prod_{j \neq l} \frac{\cos \pi(a_{lj} \pm \mathbf{m})}{\sin \pi(a_{lj} \pm \lambda/2)} \right) e^{-\frac{\mathbf{b}^2}{2}h_l \cdot \partial_a} , \tag{8.23}$$

where $(h_l)_j = \delta_{jl} - 1/N$. Note that h_l are the coweights that correspond to the weights in the fundamental representation of the Langlands dual group. The Verlinde operator (8.23) is the Weyl transform of the vev (6.12) on $S^1 \times \mathbb{R}^3$ as expected.

8.3 $SU(2)$ $N_F = 4$

To compare with gauge theory calculations, we relate a and m_f to α and α_e by

$$\begin{aligned} \alpha &= \frac{Q}{2} + \frac{a}{b}, & \alpha_1 &= \frac{Q}{2} + \frac{m_1 - m_2}{2b}, & \alpha_2 &= \frac{Q}{2} + \frac{m_1 + m_2}{2b}, \\ \alpha_3 &= \frac{Q}{2} + \frac{m_3 + m_4}{2b}, & \alpha_4 &= \frac{Q}{2} + \frac{m_3 - m_4}{2b}. \end{aligned} \tag{8.24}$$

For the minimal Wilson operator, the corresponding Verlinde operator is

$$\mathcal{L}_{0,1} = e^{2\pi i a} + e^{-2\pi i a}. \tag{8.25}$$

In appendix (D.3), we show that

$$\begin{aligned} \mathcal{L}_{2,0} &= \sum_{\pm} e^{\pm \frac{1}{2} b^2 \partial_a} \left(\frac{\prod_{f=1}^4 \prod_{s=\pm} \cos(\pi a + s\pi m_f)}{\sin(2\pi a + \pi b^2) \sin^2(2\pi a) \sin(2\pi a - \pi b^2)} \right)^{1/2} e^{\pm \frac{1}{2} b^2 \partial_a} \\ &\quad - \frac{1}{2} \cos \pi(b^2 - \sum_f m_f) + \sum_{s=\pm} \frac{\prod_{f=1}^4 \cos \pi(-b^2/2 + sa + m_f)}{\sin(2\pi a) \sin \pi(sb^2 - 2a)}. \end{aligned} \tag{8.26}$$

In view of (8.13) and (8.15), this is related to the 't Hooft operator vev (6.17) in the $U(2)$ theory with $a_1 = -a_2 = a$ by the Weyl transform, up to an a -independent term $-\frac{1}{2} \cos \pi(b^2 - \sum m_f)$. The whole expression is invariant under $a \rightarrow -a$ as well as under the action of the $SO(8)$ Weyl group.²⁴ We see that Liouville theory in fact fixes the a -independent additive constant. This constant is such that the whole second line of (8.26) vanishes in the limit $a \rightarrow \pm i\infty$. By the argument given above, then, Liouville theory predicts that the $SU(2)$ theory with $N_F = 4$ has

$$Z_{\text{mono}}(a, m_f; 2, 0) = -\frac{1}{2} \cos \pi \left(\lambda - \sum_f m_f \right) - \sum_{s=\pm} \frac{\prod_{f=1}^4 \sin \pi(sa - m_f + \lambda/2)}{\sin(2\pi a) \sin \pi(s\lambda + 2a)}. \tag{8.27}$$

Thus the minimal 't Hooft operator vev on $S^1 \times \mathbb{R}^3$ should be²⁵

$$\begin{aligned} \langle L_{2,0} \rangle &= (e^{4\pi i b} + e^{-4\pi i b}) \left(\frac{\prod_{\pm} \prod_{f=1}^4 \sin \pi(a \pm m_f)}{\sin^2 2\pi a \prod_{\pm} \sin \pi(2a \pm \lambda)} \right)^{1/2} \\ &\quad - \frac{1}{2} \cos \pi \left(\lambda - \sum_f m_f \right) - \sum_{s=\pm} \frac{\prod_{f=1}^4 \sin \pi(sa - m_f + \lambda/2)}{\sin(2\pi a) \sin \pi(s\lambda + 2a)}. \end{aligned} \tag{8.28}$$

In appendix E, we show that the $\lambda = 0$ limit of this expression coincides with the classical holonomy on the four-punctured sphere written in terms of the complexified Fenchel-Nielsen coordinates.

²⁴Four generators of the Weyl group of the $SO(8)$ flavor group act on the masses as $m_1 \leftrightarrow m_2$, $m_2 \leftrightarrow m_3$, $m_3 \leftrightarrow m_4$, and $m_3 \leftrightarrow -m_4$ respectively. Agreement up to an additive constant is almost as much as one can hope for. Without $SO(8)$ Weyl invariance, however, (6.17) with $a_1 = a = -a_2$ cannot be the answer for $SU(2)$ gauge theory.

²⁵Essentially the same expression has been obtained purely from quantization of the Hitchin system [54].

8.4 $SU(N)$ $N_F = 2N$

The Verlinde operator in the $SU(N)$ superconformal QCD corresponding to the 't Hooft operator with $B = (1, -1, 0^{N-2})$ was computed in [46]. This was done in the standard normalization, and in appendix D.3, we convert the operator to the difference operator acting on the block \mathcal{B} . It is given up to a multiplicative constant by

$$\begin{aligned}
 & \mathcal{L}_{B=(1,-1,0^{N-2})} \\
 &= \sum_{\substack{1 \leq j, k \leq N \\ j \neq k}} e^{-\frac{b^2}{4} e_{jk} \cdot \partial_a} \frac{\left[\prod_{f=1}^N \cos \pi(a_j - m_f) \cos \pi(a_k - m_f) \right]^{\frac{1}{2}}}{\sin \pi a_{jk} \prod_{\pm} \left[\sin \pi(a_{jk} \pm b^2) \prod_{i \neq j, k} \sin \pi(a_{ji} \pm \frac{b^2}{2}) \sin \pi(a_{ik} \pm \frac{b^2}{2}) \right]^{\frac{1}{2}}} e^{-\frac{b^2}{4} e_{jk} \cdot \partial_a} \\
 &+ \sum_{k=1}^N \frac{\prod_{f=1}^{2N} \cos \pi \left(a_k - m_f + \frac{b^2}{2} \right)}{\prod_{i \neq k} \sin \pi a_{ki} \sin \pi(-a_{ki} - b^2)} \\
 &+ (-1)^{N-1} \frac{e^{N\pi i(\sum_{f>N} m_f - \sum_{f \leq N} m_f)/N} \sin \pi b^2}{\sin(\pi(N-2)b^2)} - \frac{1}{2} \cos \pi \left(b^2 - \sum_f m_f \right). \tag{8.29}
 \end{aligned}$$

Here $e_{jk} \equiv h_j - h_k$ are the coroots. This implies that for the minimal 't Hooft operator in the $SU(N)$ theory with $N_F = 2N$, the vev on $S^1 \times \mathbb{R}^3$ is given by

$$\begin{aligned}
 & \langle T_{B=(1,-1,0^{N-2})} \rangle \\
 &= \sum_{\substack{1 \leq j, k \leq N \\ j \neq k}} e^{\pi i(b_j - b_k)} \frac{\left[\prod_{f=1}^N \sin \pi(a_j - m_f) \sin \pi(a_k - m_f) \right]^{1/2}}{\sin \pi a_{jk} \prod_{\pm} \left[\sin \pi(a_{jk} \pm \lambda) \prod_{i \neq j, k} \sin \pi(a_{ji} \pm \lambda/2) \sin \pi(a_{ik} \pm \lambda/2) \right]^{1/2}} \\
 &+ \sum_{k=1}^N \frac{\prod_{f=1}^{2N} \sin \pi(a_k - m_f + \lambda/2)}{\prod_{i \neq k} \sin \pi a_{ki} \sin \pi(-a_{ki} - \lambda)} \\
 &+ (-1)^{N-1} \frac{e^{\pi i(\sum_{f>N} m_f - \sum_{f \leq N} m_f)} \sin \pi \lambda}{\sin(\pi(N-2)\lambda)} + \frac{(-1)^{N-1}}{2} \cos \pi \left(\lambda - \sum_f m_f \right). \tag{8.30}
 \end{aligned}$$

This is identical to the $U(N)$ result (6.18) up to the terms independent of a and b . The expression (8.30) is a prediction of Toda theory for the $SU(N)$ gauge theory.

9 Discussion

Let us conclude with remarks on future directions and related works.

We focused on conformal $\mathcal{N} = 2$ gauge theories because localization calculations are the cleanest for them. Line operators in non-conformal asymptotically free theories also exhibit rich dynamics [13] and the spectrum of BPS states is often simpler. The easiest way to compute correlation functions in such theories would be to start with a conformal theory and decouple some matter fields by sending their mass to infinity. It would be interesting to study this limit in detail.

Our calculation of the line operator vevs, or the supersymmetric index (2.11), in terms of the complexified Fenchel-Nielsen coordinates made use of the equivariant index theorem. It is amusing to note that the calculation of the supersymmetric index in terms of the Fock-Goncharov coordinates can also be formulated in terms of an index theorem, but applied to the moduli space constructed from the Seiberg-Witten prepotential governing the IR dynamics [55, 56].

In our computational scheme for the monopole screening contributions $Z_{\text{mono}}(B, v)$ in $\mathcal{N} = 2^*$ theory, the 't Hooft operator is S-dual to the Wilson operator in a product of fundamental representations. As such the 't Hooft operator is reducible, i.e., it can be written as a linear combination of other line operators with positive coefficients. Related to this is the fact that the 't Hooft operator vev (6.17) in the $G = \text{U}(2)$ theory with $N_F = 4$ fundamental hypermultiplets becomes $\text{SO}(8)_F$ Weyl-invariant not just by substituting $(a_1, a_2) \rightarrow (a, -a)$, but only after adding an a -independent term in (8.26). It is important to develop a method intrinsic to irreducible line operators for gauge group $\text{SU}(N)$ rather than $\text{U}(N)$. This may involve decomposing the cohomology of the monopole moduli space into irreducible representations of the Langlands dual gauge group [57] and incorporate the computation of operator product expansions [10, 35, 58].

We found that the line operators in $\mathcal{N} = 2$ theories on $S^1 \times \mathbb{R}^3$ realize a deformation quantization of the Hitchin moduli space. We expect that this can be explained in the framework of [18], by dimensionally reducing the theory on the circle parametrized by τ as well as the one parametrized by the polar angle in the 12-plane. We would obtain a (4, 4) sigma model on a half plane whose target space is the Hitchin moduli space, and the boundary condition would correspond to the canonical coisotropic brane. Liouville/Toda conformal blocks arise as open string states by including another boundary mapped to the brane of opers. It would be interesting to study these systems in more detail and understand the appearance of the Weyl transform.

Some of the results in [39] obtained by the wall-crossing formula can be reproduced from our results that are obtained directly by localization calculations. It would be interesting to further explore the relation between the UV and IR theories as well as the integrability aspects of the line operators.

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A Spinors and gamma matrices

Chiral spinors Ψ and ϵ transform in a representation of Spin(10), whose generators are constructed from 32×32 matrices Γ^M obeying

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN} \quad M = 0, 1, \dots, 9. \quad (\text{A.1})$$

We use the Euclidean signature $\eta^{MN} = \delta^{MN}$. We can take Γ^M in the form

$$\Gamma^M = \begin{pmatrix} 0 & \tilde{\Gamma}^M \\ \Gamma^M & 0 \end{pmatrix}, \quad (\text{A.2})$$

where $\Gamma^M \equiv (\Gamma^0, \Gamma^1, \dots, \Gamma^9)$ and $\tilde{\Gamma}^M \equiv (-\Gamma^0, \Gamma^1, \dots, \Gamma^9)$ are 16×16 matrices that satisfy

$$\tilde{\Gamma}^M \Gamma^N + \tilde{\Gamma}^N \Gamma^M = 2\delta^{MN}, \quad \Gamma^M \tilde{\Gamma}^N + \Gamma^N \tilde{\Gamma}^M = 2\delta^{MN}. \quad (\text{A.3})$$

We also use notation $\Gamma^{MN} \equiv \tilde{\Gamma}^{[M} \Gamma^{N]}$, $\tilde{\Gamma}^{MN} \equiv \Gamma^{[M} \tilde{\Gamma}^{N]}$, and $\Gamma^{MNPQ} \equiv \tilde{\Gamma}^{[M} \Gamma^N \tilde{\Gamma}^P \Gamma^{Q]}$. Our spinors have positive chirality with respect to the chirality matrix

$$\Gamma \equiv -i\Gamma^1 \dots \Gamma^9 \Gamma^0 = \begin{pmatrix} -i\tilde{\Gamma}^1 \Gamma^2 \dots \tilde{\Gamma}^9 \Gamma^0 & 0 \\ 0 & -i\Gamma^1 \tilde{\Gamma}^2 \dots \Gamma^9 \tilde{\Gamma}^0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.4})$$

In ten dimensions with Euclidean signature the chiral spinor representation is complex. We take $\Gamma^1, \dots, \Gamma^9$ to be real and $\Gamma^0 = i$ pure imaginary. As in [11], for the explicit expressions we use matrices as defined in appendix A of [6] with a permutation of spacetime indices. Let $\underline{\Gamma}^M$ be the gamma matrices in [6]. Then our Γ^M are given by

$$\begin{aligned} \Gamma^M &= \underline{\Gamma}^{M+1} \quad \text{for } M = 1, 2, 3, 5, 6, 7, \\ \Gamma^4 &= \underline{\Gamma}^1, \quad \Gamma^8 = \underline{\Gamma}^5, \quad \Gamma^9 = \underline{\Gamma}^9, \quad \Gamma^0 = i\underline{\Gamma}^0. \end{aligned} \quad (\text{A.5})$$

The factor of i in the relation to Γ^0 arises because our present conventions use the Euclidean metric $\eta^{MN} = \delta^{MN}$, while [6] used the Lorentz metric with $\eta^{00} = -1$.

For off-shell supersymmetry, we need a set of spinors ν^i ($i = 1, \dots, 7$) that satisfy the relations [6, 59]

$$\begin{aligned} \epsilon \Gamma^M \nu_i &= 0, \\ \frac{1}{2}(\epsilon \Gamma_N \epsilon) \tilde{\Gamma}_{\alpha\beta}^N &= \nu_\alpha^i \nu_\beta^i + \epsilon_\alpha \epsilon_\beta, \\ \nu_i \Gamma^M \nu_j &= \delta_{ij} \epsilon \Gamma^M \epsilon. \end{aligned} \quad (\text{A.6})$$

Explicitly, we take

$$\begin{aligned} \nu_j &= \Gamma^{8,j+4} \epsilon \quad j = 1, 2, 3, \\ \nu_4 &= \Gamma^{89} \epsilon, \\ \nu_j &= \Gamma^{8,j-4} \epsilon \quad j = 5, 6, 7. \end{aligned} \quad (\text{A.7})$$

We also use the standard Pauli matrices σ^i , $i = 1, 2, 3$, defined as

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \quad (\text{A.8})$$

B Differential operators for the one-loop determinants

In this appendix we derive the differential operators whose indices enter the one-loop calculations.

We will need the relations inverse to (4.9) and (4.10):

$$\begin{aligned}
 \tilde{A}_M &= X_{0M} & M &= 1, \dots, 9, \\
 \tilde{\Phi}_0 &\equiv \tilde{A}_0 = iX'_{18} + iX_{04} \\
 \Psi_M &= X'_{0M} - D_{(0)M}X_{18} & M &= 1, \dots, 9, \\
 K_j &= X'_{1j} + i \sum_{M=1}^9 \sum_{N=1}^9 (\bar{\nu}_j \Gamma^{MN} \epsilon) D_{(0)M} X_{0N} & j &= 1, \dots, 7, \\
 \Upsilon_j &= X_{1j} & j &= 1, \dots, 7 \\
 c &= X_{18}, & \tilde{c} &= X_{19}, \quad \tilde{b} = X'_{19}.
 \end{aligned} \tag{B.1}$$

Then the quadratic part of \hat{V} is given by

$$\begin{aligned}
 \hat{V}^{(2)} &= \int d^4x \operatorname{Tr} \left(\sum_{M=1,2,3,9} (X'_{0M} - D_{(0)M}X_{18})(-D_{(0)\tau}X_{0M} + D_{(0)M}X_{04}) \right. \\
 &\quad \left. + \sum_{j=1}^7 X_{1j} \left(X'_{1j} + 2i(\nu_j \tilde{\Gamma}^{NP} \bar{\epsilon}) D_{(0)N} X_{0P} \right) + X_{19} \left(i \sum_{M=1,2,3,9} D_{(0)}^M X_{0M} + \frac{\xi}{2} X'_{19} \right) \right).
 \end{aligned} \tag{B.2}$$

From this we read off D_{10} :

$$\begin{aligned}
 &(D_{10} \cdot X_0)_{j=1}^7 \\
 &= 2i \sum_{k=1}^3 \sum_{l=1}^3 (\nu_j \tilde{\Gamma}^{kl} \bar{\epsilon}) D_{(0)k} X_{0l} + 2i \sum_{k=1}^3 \sum_{l=5}^8 (\nu_j \tilde{\Gamma}^{kl} \bar{\epsilon}) D_{(0)k} X_{0l} + 2i \sum_{k=1}^3 (\nu_j \tilde{\Gamma}^{k9} \bar{\epsilon}) D_{(0)k} X_{09} \\
 &\quad + 2i \sum_{l=1}^3 (\nu_j \tilde{\Gamma}^{9l} \bar{\epsilon}) D_{(0)9} X_{0l} + 2i \sum_{l=5}^8 (\nu_j \tilde{\Gamma}^{9l} \bar{\epsilon}) D_{(0)9} X_{0l}.
 \end{aligned}$$

The differential operator D_{10} splits into the vector and hypermultiplet parts. Let us begin with the vector multiplet. For $j = 1, 2, 3$, we have

$$\begin{aligned}
 &(D_{10} \cdot X_0)_{j=1}^3 \\
 &= -2i\epsilon_{jkl} D_{(0)k} X_{0l} + 2iD_{(0)j} X_{09} - 2iD_{(0)9} X_{0j} \\
 &= -2i(D_{\text{Bogo}} \cdot X_0)_j
 \end{aligned} \tag{B.3}$$

where we used that $\nu_j \tilde{\Gamma}^{kl} \bar{\epsilon} = -\epsilon_{jkl}$ for $j, k, l \in \{1, 2, 3\}$, $\nu_j \tilde{\Gamma}^{kl} \bar{\epsilon} = 0$ for $j, k \in \{1, 2, 3\}$ and $l \in \{5, 6, 7, 8\}$, $\nu_j \tilde{\Gamma}^{k9} \bar{\epsilon} = \delta_{jk}$ for $j, k \in \{1, 2, 3\}$, $\nu_j \tilde{\Gamma}^{9l} \bar{\epsilon} = 0$ for $j \in \{1, 2, 3\}$ and $l \in \{5, 6, 7, 8\}$. The differential operator D_{Bogo} is the linearization of the Bogomolny equations. For $j = 9$, we get

$$(D_{10} \cdot X_0)_9 = i \sum_{M=1,2,3,9} D_{(0)}^M X_{0M}. \tag{B.4}$$

This is the conjugate of the linearized gauge transformation and has its origin in the gauge-fixing condition. We also have

$$(D_{10} \cdot X_0)_8 = \sum_{M=0}^9 D_{(0)}^M (D_{(0)M} X_{04} - D_{(0)\tau} X_{0M}) . \tag{B.5}$$

As in [6], the computation of the symbol shows that (B.5) can be dropped by neglecting X_{04} and X_{18} , and that D_{10} acting on the vector multiplet fails to be elliptic, though we have checked that D_{10} is transversally elliptic, i.e., it is elliptic in the directions other than τ . Since we work in a non-compact space, our application of the localization formula for the index is done formally, as in the calculation of the instanton partition function.

For the hypermultiplet, we need to consider the components $j = 4, 5, 6, 7$ of (B.3):

$$\begin{aligned} (D_{10} \cdot X_0)_{j=4}^7 &= 2i \sum_{k=1}^3 \sum_{l=5}^8 (\nu_j \tilde{\Gamma}^{kl} \tilde{\epsilon}) D_{(0)k} X_{0l} + 2i \sum_{l=5}^8 (\nu_j \tilde{\Gamma}^{9l} \tilde{\epsilon}) D_{(0)9} X_{0l} . \end{aligned}$$

This differential operator is the “realification” of the Dirac-Higgs operator

$$D_{\text{DH}} \equiv \sigma^i D_{(0)i} + [\Phi_{(0)9}, \cdot] \tag{B.6}$$

acting on the “spinor” $2^{-1/2}(X_{05} - iX_{06} + iX_{07} + X_{08}, iX_{05} - X_{06} - X_{07} - iX_{08})^T$ and mapping to another $2^{-1/2}(iX_{11} + iX_{12} + X_{13} - X_{14}, X_{11} - X_{12} - iX_{13} - iX_{14})^T$.

C Monopoles on \mathbb{R}^3 and instantons on Taub-NUT

Let us review Kronheimer’s correspondence [60] between several singular monopoles on \mathbb{R}^3 and $U(1)_K$ -invariant instantons on a multi-centered Taub-NUT space, which has the metric

$$ds^2 = V d\vec{x}^2 + V^{-1} (d\psi + \omega)^2, \quad V = l + \sum_j \frac{1}{2|\vec{x} - \vec{x}_j|}, \quad d\omega = - *_3 dV, \tag{C.1}$$

where $l > 0$ is a constant. From the three-dimensional fields (A, Φ) with singularities

$$A \sim \frac{B_j}{2} \cos \theta d\varphi, \quad \Phi \sim \frac{B_j}{2r} \quad \text{near } \vec{x} = \vec{x}_j, \tag{C.2}$$

where (r, θ, φ) are the spherical coordinates on a 3-ball centered at $\vec{x} = \vec{x}_j$, we construct a four-dimensional gauge connection

$$\mathcal{A} \equiv g \left(A + \Phi \frac{d\psi + \omega}{V} \right) g^{-1} - ig dg^{-1} \tag{C.3}$$

and its curvature $\mathcal{F} = d\mathcal{A} + i\mathcal{A} \wedge \mathcal{A}$. The singularities in A and Φ cancel in (C.3) to define a smooth four-dimensional gauge field \mathcal{A} . Here g is a suitable singular gauge transformation that locally behaves as $g \sim e^{iB_j\psi}$ near $\vec{x} = \vec{x}_j$ so that \mathcal{A} is smooth there.²⁶

²⁶For a single singular monopole we can take $g = e^{iB\psi}$. In the present paper this is all we need even when there is more than one singularity because the index calculation is local.

The four-dimensional field \mathcal{A} is invariant under the $U(1)_K$ action $\psi \rightarrow \psi + \nu$, which rotates the circle fiber as well as acts on the gauge bundle as a gauge transformation. The claim is that the Bogomolny equations

$$*_3 F = D\Phi \tag{C.4}$$

are equivalent to the anti-self-dual equations

$$*_4 \mathcal{F} + \mathcal{F} = 0. \tag{C.5}$$

To show this, let us use the fact that \mathcal{A} is obtained by a singular gauge transformation from

$$\tilde{\mathcal{A}} = A + \Phi \frac{d\psi + \omega}{V}, \tag{C.6}$$

therefore $\mathcal{F} = g\tilde{\mathcal{F}}g^{-1}$. Then, for the orientation (volume form) $\propto (d\psi + \omega)dx^1 dx^2 dx^3$,

$$\mathcal{F} = g \left(F + D\Phi \wedge \frac{d\psi + \omega}{V} - \Phi \frac{*_3 dV}{V} + \Phi (d\psi + \omega) \wedge \frac{dV}{V^2} \right) g^{-1}, \tag{C.7}$$

and

$$*_4 \mathcal{F} = g \left(- *_3 F \wedge \frac{d\psi + \omega}{V} - *_3 D\Phi - \Phi \frac{(d\psi + \omega) \wedge dV}{V^2} + \Phi \frac{*_3 dV}{V} \right) g^{-1}, \tag{C.8}$$

so $\mathcal{F} + *_4 \mathcal{F} = 0$ if and only if $F = *_3 D\Phi$.

The holonomy of the four-dimensional field at infinity $|\vec{x}| = \infty$ is related to the scalar expectation value as

$$P e^{-i \oint \mathcal{A}} \rightarrow e^{-2\pi i \Phi(\infty)/l} \text{ as } |\vec{x}| \rightarrow \infty \tag{C.9}$$

up to conjugation.

In the single-center case, the metric (C.1) approaches twice the metric of \mathbb{C}^2 in the limit $l \rightarrow 0$, $ds^2 \rightarrow 2ds_{\mathbb{C}^2}^2$, where

$$\begin{aligned} ds_{\mathbb{C}^2}^2 &= (2r)^{-1} [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)] + 2r(d\psi + \omega)^2 \\ &= |dz_1|^2 + |dz_2|^2 \end{aligned} \tag{C.10}$$

and

$$z_1 = r^{1/2} \cos \frac{\theta}{2} e^{-i\psi + i\varphi/2}, \quad z_2 = r^{1/2} \sin \frac{\theta}{2} e^{i\psi + i\varphi/2}, \quad \omega = -\frac{1}{2} \cos \theta d\varphi. \tag{C.11}$$

For general l , Taub-NUT space is isomorphic as a complex manifold to $\mathbb{C}^2 = \{(z_1, z_2)\}$ with the same parametrization. See for example [61].

D $Z_{1\text{-loop}}$ from Liouville/Toda theories

In this appendix we rewrite the Verlinde operators for 't Hooft loops [15, 16, 46] into the form (8.10) acting on the normalized conformal block (8.8). We will first do this in the simplest case $G = SU(2)$ $\mathcal{N} = 2^*$ theory for illustration, and then perform more complicated calculations for $\mathcal{N} = 2^*$ Yang-Mills and super conformal QCD with gauge group $G = SU(N)$.

D.1 $SU(2)$ $\mathcal{N} = 2^*$

We shift the argument in (8.19) as $\alpha \rightarrow \alpha \mp \mathbf{b}/4$. Then the expectation value of the Verlinde operator for the minimal 't Hooft loop $T = L_{1,0}$ becomes

$$\langle T \rangle = \sum_{\pm} \int d\alpha C(\alpha \mp \mathbf{b}/4, \alpha_e, Q - \alpha \pm \mathbf{b}/4) \overline{\mathcal{F}(\alpha \pm \mathbf{b}/4)} H_{\pm}(\alpha \mp \mathbf{b}/4) \mathcal{F}(\alpha \pm \mathbf{b}/4). \quad (\text{D.1})$$

The monodromy factors are

$$H_{\pm} \left(\alpha \mp \frac{\mathbf{b}}{4} \right) = \frac{\Gamma(\pm 2a - \mathbf{b}^2/2) \Gamma(\mathbf{b}Q \pm 2a - \mathbf{b}^2/2)}{\prod_{s=\pm} \Gamma(\frac{1}{2} \pm 2a + sm)}. \quad (\text{D.2})$$

The a -dependent part of the three-point function $C(\alpha, \alpha_e, Q - \alpha)$ reads

$$C(\alpha, \alpha_e, Q - \alpha) \propto \frac{\prod_{s_1, s_2 = \pm} \Gamma_{\mathbf{b}}(Q/2 + s_1 2a/\mathbf{b} + s_2 m/\mathbf{b})}{\prod_{s=\pm} \Gamma_{\mathbf{b}}(Q + 2sa/\mathbf{b}) \Gamma_{\mathbf{b}}(2sa/\mathbf{b})}, \quad (\text{D.3})$$

where $\Gamma_{\mathbf{b}}(z)$ is the double gamma function. For its full definition, see for example [15]. For the present purpose we only need the relations

$$\Gamma_{\mathbf{b}}(z) = \Gamma_{1/\mathbf{b}}(z), \quad \Gamma_{\mathbf{b}}(z + \mathbf{b}) = \frac{\sqrt{2\pi} \mathbf{b}^{bz-1/2}}{\Gamma(\mathbf{b}x)} \Gamma_{\mathbf{b}}(z). \quad (\text{D.4})$$

According to (8.10) and (8.12) the equator contribution is

$$\begin{aligned} Z_{\text{equator}} &= \left(\frac{C(\alpha \mp \mathbf{b}/4, \alpha_e, Q - \alpha \pm \mathbf{b}/4)}{C(\alpha \pm \mathbf{b}/4, \alpha_e, Q - \alpha \mp \mathbf{b}/4)} \right)^{1/2} H_{\pm}(\alpha \mp \mathbf{b}/4) \\ &= \left(\frac{\prod_{\pm} \cos(2\pi a \pm \pi m)}{\prod_{\pm} \sin(2\pi a \pm \pi \mathbf{b}^2/2)} \right)^{1/2}. \end{aligned} \quad (\text{D.5})$$

Thus the Verlinde operator (8.12) acting on $\mathcal{B}(\alpha; \alpha_e)$ is given by (8.22).

D.2 $SU(N)$ $\mathcal{N} = 2^*$

Let us generalize the calculation for $SU(2)$ above to $SU(N)$. The weights in the fundamental representation are given by

$$h_i = (0, \dots, 0, 1, 0, \dots, 0) - \frac{1}{N}(1, \dots, 1), \quad N = 1, \dots, N. \quad (\text{D.6})$$

The roots are

$$e_{ij} = h_i - h_j, \quad 1 \leq i, j \leq N, \quad (\text{D.7})$$

and the simple roots are $e_i := e_{i, i+1}$, $i = 1, \dots, N-1$. The fundamental weights ω_i ($i = 1, \dots, N-1$) are defined as the dual basis, $\omega_i \cdot e_j = \delta_{ij}$, since we identify roots and coroots by the metric. Let $\rho = \sum_i \omega_i$ be the Weyl vector, and keep the same notation $Q = \mathbf{b} + \mathbf{b}^{-1}$ as in the Liouville case. The three-point function with two generic momenta $\alpha_1 = i\hat{a}_1 + Q\rho$, $\alpha_2 = i\hat{a}_2 + Q\rho$ and one semi-degenerate momentum $\alpha_3 = \kappa\omega_{N-1}$ is given by

$$C^{(1)}(\alpha_1, \alpha_2, \kappa) := C(\alpha_1, \alpha_2, \alpha_3 = \kappa\omega_{N-1}) \propto \frac{\prod_{i < j} \Upsilon(-i\hat{a}_1 \cdot e_{ij}) \Upsilon(-i\hat{a}_2 \cdot e_{ij})}{\prod_{i, j=1}^N \Upsilon(\kappa/N + i\hat{a}_1 \cdot h_i + i\hat{a}_2 \cdot h_j)}. \quad (\text{D.8})$$

When $\alpha_1 = i\hat{a}_1 + Q\rho$, $\alpha_2 = i\hat{a}_2 + Q\rho$, $\alpha_3 = \kappa\omega_1$, the three-point function is

$$C^{(2)}(\alpha_1, \alpha_2, \kappa) := C(\alpha_1, \alpha_2, \alpha_3 = \kappa\omega_1) \propto \frac{\prod_{i < j} \Upsilon(-i\hat{a}_1 \cdot e_{ij}) \Upsilon(-i\hat{a}_2 \cdot e_{ij})}{\prod_{i,j=1}^N \Upsilon(\kappa/N - i\hat{a}_1 \cdot h_i - i\hat{a}_2 \cdot h_j)}. \quad (\text{D.9})$$

The two-dimensional theory corresponding to $\mathcal{N} = 2^*$ is the $SU(N)$ Toda theory on the torus with one semi-degenerate puncture. With the parametrization

$$\alpha = Q + i\hat{a}, \quad \alpha_e = \left(\frac{Q}{2} + i\hat{m}\right) N\omega_{N-1}, \quad (\text{D.10})$$

the vev of the Verlinde operator corresponding to the minimal 't Hooft operator $T = T_{B=(1,0^{N-1})}$ is

$$\langle T \rangle = \int d\alpha C(2Q\rho - \alpha, \alpha_e, \alpha) \overline{\mathcal{F}(\alpha)} \sum_{l \neq k} H_k(\alpha) \mathcal{F}(\alpha - \mathbf{b}h_k), \quad (\text{D.11})$$

where

$$H_k(\alpha) = \prod_{j \neq k} \frac{\Gamma(ib\hat{a}_{jk}) \Gamma(\mathbf{b}Q + ib\hat{a}_{jk})}{\Gamma(\mathbf{b}Q/2 + ib\hat{a}_{jk} - ib\hat{m}) \Gamma(\mathbf{b}Q/2 + ib\hat{a}_{jk} + ib\hat{m})}. \quad (\text{D.12})$$

In order to relate this to the vev of 't Hooft operator on $S^1 \times \mathbb{R}^3$ we set

$$\alpha = Q\rho - \frac{a}{\mathbf{b}}, \quad \alpha_e = \left(\frac{Q}{2} - \frac{\mathbf{m}}{\mathbf{b}}\right) N\omega_{N-1}. \quad (\text{D.13})$$

Let us define

$$\tilde{\Upsilon}(x) := \frac{\Upsilon(x + \mathbf{b})}{\Upsilon(x)} = \frac{\Gamma(\mathbf{b}x)}{\Gamma(1 - \mathbf{b}x)} \mathbf{b}^{1-2\mathbf{b}x}. \quad (\text{D.14})$$

Then

$$\begin{aligned} Z_k(\alpha) &:= \left(\frac{C(\alpha + \mathbf{b}h_k/2, \alpha_e, 2Q\rho - \alpha - \mathbf{b}h_k/2)}{C(\alpha - \mathbf{b}h_k/2, \alpha_e, 2Q\rho - \alpha + \mathbf{b}h_k/2)} \right)^{1/2} H_k(\alpha + \mathbf{b}h_k/2) \\ &= \left(\prod_{j < l, \pm} \frac{\Upsilon(\pm(a_{jl}/\mathbf{b} - \mathbf{b}(\delta_{jk} - \delta_{kl})/2))}{\Upsilon(\pm(a_{jl}/\mathbf{b} + \mathbf{b}(\delta_{jk} - \delta_{kl})/2))} \prod_{j \neq l} \frac{\Upsilon(Q/2 - \mathbf{m}/\mathbf{b} - a_{jl}/\mathbf{b} - \mathbf{b}(\delta_{jk} - \delta_{kl})/2)}{\Upsilon(Q/2 - \mathbf{m}/\mathbf{b} - a_{jl}/\mathbf{b} + \mathbf{b}(\delta_{jk} - \delta_{kl})/2)} \right)^{1/2} \\ &\quad \times \prod_{j \neq k} \frac{\Gamma(-a_{jk} - \mathbf{b}^2/2) \Gamma(\mathbf{b}Q - a_{jk} - \mathbf{b}^2/2)}{\prod_{\pm} \Gamma(\mathbf{b}Q/2 - a_{jk} - \mathbf{b}^2/2 \pm \mathbf{m})} \quad (\text{D.15}) \\ &= \left(\prod_{j < k} \frac{\tilde{\Upsilon}(a_{jk}/\mathbf{b} - \mathbf{b}/2)}{\tilde{\Upsilon}(-a_{jk}/\mathbf{b} - \mathbf{b}/2)} \prod_{k < l} \frac{\tilde{\Upsilon}(-a_{kl}/\mathbf{b} - \mathbf{b}/2)}{\tilde{\Upsilon}(a_{kl}/\mathbf{b} - \mathbf{b}/2)} \prod_{j \neq k} \frac{\tilde{\Upsilon}(Q/2 - \mathbf{m}/\mathbf{b} - a_{jk}/\mathbf{b} - \mathbf{b}/2)}{\tilde{\Upsilon}(Q/2 - \mathbf{m}/\mathbf{b} - a_{kl}/\mathbf{b} - \mathbf{b}/2)} \right)^{1/2} \\ &\quad \times \prod_{j \neq k} \frac{\Gamma(-a_{jk} - \mathbf{b}^2/2) \Gamma(1 - a_{jk} + \mathbf{b}^2/2)}{\prod_{\pm} \Gamma(1/2 - a_{jk} \pm \mathbf{m})} \\ &= \left(\prod_{j \neq k} \frac{\tilde{\Upsilon}(a_{jk}/\mathbf{b} - \mathbf{b}/2)}{\tilde{\Upsilon}(-a_{jk}/\mathbf{b} - \mathbf{b}/2)} \frac{\tilde{\Upsilon}(Q/2 - \mathbf{m}/\mathbf{b} - a_{jk}/\mathbf{b} - \mathbf{b}/2)}{\tilde{\Upsilon}(Q/2 - \mathbf{m}/\mathbf{b} + a_{jk}/\mathbf{b} - \mathbf{b}/2)} \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
 & \times \prod_{j \neq k} \frac{\Gamma(-a_{jk} - \mathbf{b}^2/2)\Gamma(1 - a_{jk} + \mathbf{b}^2/2)}{\prod_{\pm} \Gamma(1/2 - a_{jk} \pm \mathbf{m})} \\
 = & \left(\prod_{j \neq k} \frac{\Gamma(a_{jk} - \mathbf{b}^2/2)\Gamma(1 + a_{jk} + \mathbf{b}^2/2) \prod_{\pm} \Gamma(1/2 \pm \mathbf{m} - a_{jk})}{\Gamma(1 - a_{jk} + \mathbf{b}^2/2)\Gamma(-a_{jk} - \mathbf{b}^2/2) \prod_{\pm} \Gamma(1/2 \pm \mathbf{m} + a_{jk})} \right)^{1/2} \\
 & \times \prod_{j \neq k} \frac{\Gamma(-a_{jk} - \mathbf{b}^2/2)\Gamma(1 - a_{jk} + \mathbf{b}^2/2)}{\prod_{\pm} \Gamma(1/2 - a_{jk} \pm \mathbf{m})} \\
 = & \left(\prod_{j \neq k} \frac{\Gamma(a_{jk} - \mathbf{b}^2/2)\Gamma(1 + a_{jk} + \mathbf{b}^2/2)\Gamma(-a_{jk} - \mathbf{b}^2/2)\Gamma(1 - a_{jk} + \mathbf{b}^2/2)}{\prod_{s_1, s_2 = \pm} \Gamma(1/2 + s_1 \mathbf{m} + s_2 a_{jk})} \right)^{1/2} \\
 = & \left(\prod_{j \neq k} \prod_{\pm} \frac{\cos \pi(a_{jk} \pm \mathbf{m})}{\sin \pi(a_{jk} \pm \mathbf{b}^2/2)} \right)^{1/2}. \tag{D.16}
 \end{aligned}$$

It follows that the Verlinde operator is given by (8.23).

D.3 $SU(N)$ $N_{\mathbf{F}} = 2N$

We use the notation in section D.2 for Toda theory. For the $SU(N)$ theory with $N_{\mathbf{F}} = 2N$ fundamentals corresponding to the sphere with two full and two semi-degenerate punctures, we set the parameters as

$$\begin{aligned}
 \alpha &= i\hat{a} + Q\rho \\
 \hat{\mathbf{m}}_2 &= \left(\frac{Q}{2} + i\hat{\mathbf{m}}_2 \right) N\omega_{N-1}, \quad \hat{\mathbf{m}}_3^* = \left(\frac{Q}{2} + i\hat{\mathbf{m}}_3 \right) N\omega_1, \\
 \mathbf{m}_1 &= Q\rho + i\tilde{\mathbf{m}}_1, \quad \mathbf{m}_4^* = Q\rho + i\tilde{\mathbf{m}}_4^*, \\
 \tilde{\mathbf{m}}_f &= \begin{cases} \hat{\mathbf{m}}_2 + i\tilde{\mathbf{m}}_1 \cdot h_i & \text{for } f = i = 1, \dots, N, \\ \hat{\mathbf{m}}_3 + i\tilde{\mathbf{m}}_4 \cdot h_i & \text{for } f = N + i = N + 1, \dots, 2N, \end{cases} \tag{D.17}
 \end{aligned}$$

where $h_i^* := -h_{N+1-i}$. We slightly abuse notation; $\tilde{\mathbf{m}}_1$ and $\tilde{\mathbf{m}}_4$ differ from $\tilde{\mathbf{m}}_{f=1}$ and $\tilde{\mathbf{m}}_{f=4}$. Similar remarks apply for \mathbf{m}_f below. The nonzero coweight term in the vev of 't Hooft operator on S^4 is given as

$$\begin{aligned}
 \langle T \rangle &= \int d\alpha C(\mathbf{m}_4^*, \hat{\mathbf{m}}_3^*, \alpha) C(2Q - \alpha, \hat{\mathbf{m}}_2, \mathbf{m}_1) \overline{\mathcal{F}(\alpha)} \sum_{l \neq k} H_{l,k}(\alpha) \mathcal{F}(\alpha - \mathbf{b}h_{lk}) \\
 &+ \text{zero-coweight terms}, \tag{D.18}
 \end{aligned}$$

where

$$H_{l,k}(\alpha) = \pi^2 \frac{\prod_{j \neq l} \Gamma(ib\hat{a}_{jl})\Gamma(\mathbf{b}Q + ib\hat{a}_{jl}) \prod_{j \neq k} \Gamma(\mathbf{b}^2\delta_{jl} + ib\hat{a}_{kj})\Gamma(\mathbf{b}Q + \mathbf{b}^2\delta_{lj} + ib\hat{a}_{kj})}{\prod_f \Gamma(\mathbf{b}Q/2 - ib\hat{a}_l + ib\tilde{\mathbf{m}}_f)\Gamma(\mathbf{b}Q/2 + ib\hat{a}_k - ib\tilde{\mathbf{m}}_f)} \tag{D.19}$$

and $h_{lk} = h_l - h_k$, $a_i \equiv a \cdot h_i$ and $a_{ij} \equiv a \cdot (h_i - h_j)$. In order to relate this to the vev of the 't Hooft operator on $S^1 \times \mathbb{R}^3$, we introduce a slightly different parametrization

$$\begin{aligned}
 \alpha &= Q\rho - \frac{a}{b}, \\
 \alpha_2 &= \left(\frac{Q}{2} - \frac{m_2}{b}\right) N\omega_{N-1}, \quad \alpha_3 = \left(\frac{Q}{2} - \frac{m_3}{b}\right) N\omega_1, \\
 \alpha_1 &= Q\rho - \frac{m_1}{b}, \quad \alpha_4 = Q\rho - \frac{m_4}{b}, \\
 \mathbf{m}_f &= \begin{cases} m_2 + m_1 \cdot h_i \equiv m_2 + m_{1,i} & \text{for } f = i, \\ m_3 - m_4 \cdot h_i \equiv m_3 - m_{4,i} & \text{for } f = N + i. \end{cases}
 \end{aligned} \tag{D.20}$$

We define $(h_{lk})_{ij} := (h_l - h_k) \cdot (h_i - h_j) = \delta_{lj} - \delta_{li} - \delta_{ki} + \delta_{kj}$ and $\tilde{\Upsilon}^{(2)} := \frac{\Upsilon(x+2b)}{\Upsilon(x)} = \frac{\Gamma(bx)\Gamma(bx+b^2)}{\Gamma(1-bx)\Gamma(1-bx-b^2)}$, which is analogous to $\tilde{\Upsilon}(x) := \frac{\Upsilon(x+b)}{\Upsilon(x)} = \frac{\Gamma(bx)}{\Gamma(1-bx)}$. Let us calculate

$$\begin{aligned}
 Z_{l,k}(\alpha) &:= \left(\frac{C(\alpha_4, \alpha_3, \alpha + bh_{lk}/2)C(2Q\rho - \alpha - bh_{lk}/2, \alpha_2, \alpha_1)}{C(\alpha_4, \alpha_3, \alpha - bh_{lk}/2)C(2Q\rho - \alpha + bh_{lk}/2, \alpha_2, \alpha_1)} \right)^{1/2} H_{l,k}(\alpha + bh_{lk}/2) \\
 &= \left(\frac{C^{(2)}(Q\rho - m_4/b, Q\rho - a/b + bh_{lk}/2, \kappa = Q\rho/2 - m_3/b)}{C^{(2)}(Q\rho - m_4/b, Q\rho - a/b - bh_{lk}/2, \kappa = Q\rho/2 - m_3/b)} \right. \\
 &\quad \left. \times \frac{C^{(1)}(Q\rho - m_1/b, Q\rho + a/b - bh_{lk}/2, \kappa = Q/2 - m_2/b)}{C^{(1)}(Q\rho - m_1/b, Q\rho + a/b + bh_{lk}/2, \kappa = Q/2 - m_2/b)} \right)^{1/2} H_{l,k}(\alpha + bh_{lk}/2) \\
 &= \pi^2 \left(\prod_{i < j} \frac{\Upsilon(a_{ij}/b - b(h_{lk})_{ij}/2)}{\Upsilon(a_{ij}/b + b(h_{lk})_{ij}/2)} \prod_{i,j} \frac{\Upsilon(Q/2 - m_3/b + m_{4,i}/b + a_j/b + b(\delta_{lj} - \delta_{kj})/2)}{\Upsilon(Q/2 - m_3/b + m_{4,i}/b + a_j/b - b(\delta_{lj} - \delta_{kj})/2)} \right. \\
 &\quad \left. \times \prod_{i < j} \frac{\Upsilon(-a_{ij}/b + b(h_{lk})_{ij}/2)}{\Upsilon(-a_{ij}/b - b(h_{lk})_{ij}/2)} \prod_{i,j} \frac{\Upsilon(Q/2 - m_2/b - m_{1,i}/b + a_j/b + b(\delta_{lj} - \delta_{kj})/2)}{\Upsilon(Q/2 - m_2/b - m_{1,i}/b + a_j/b - b(\delta_{lj} - \delta_{kj})/2)} \right)^{1/2} \\
 &\quad \times \frac{\prod_{j \neq l} \Gamma(-a_{jl} - b^2(1 + \delta_{kj})/2) \Gamma(bQ - a_{jl} - b^2(1 + \delta_{kj})/2)}{\prod_f \Gamma(bQ/2 + a_l - b^2/2 - m_f) \Gamma(bQ/2 - a_k - b^2/2 + m_f)} \\
 &\quad \times \prod_{j \neq k} \Gamma(-a_{kj} + b^2(\delta_{lj} - 1)/2) \Gamma(bQ - a_{kj} + b^2(\delta_{lj} - 1)/2) \\
 &= \pi^2 \left(\prod_{i \neq j} \frac{\Upsilon(a_{ij}/b - b(h_{lk})_{ij}/2)}{\Upsilon(a_{ij}/b + b(h_{lk})_{ij}/2)} \prod_{i,j} \frac{\Upsilon(Q/2 - m_3/b + m_{4,i}/b + a_j/b + b(\delta_{lj} - \delta_{kj})/2)}{\Upsilon(Q/2 - m_3/b + m_{4,i}/b + a_j/b - b(\delta_{lj} - \delta_{kj})/2)} \right. \\
 &\quad \left. \times \prod_{i,j} \frac{\Upsilon(Q/2 - m_2/b - m_{1,i}/b + a_j/b + b(\delta_{lj} - \delta_{kj})/2)}{\Upsilon(Q/2 - m_2/b - m_{1,i}/b + a_j/b - b(\delta_{lj} - \delta_{kj})/2)} \right)^{1/2} \\
 &\quad \times \frac{\Gamma(-a_{kl} - b^2) \Gamma(bQ - a_{kl} - b^2) \Gamma(-a_{kl}) \Gamma(bQ - a_{kl})}{\prod_f \Gamma(1/2 + a_l - m_f) \Gamma(1/2 - a_k + m_f)} \\
 &\quad \times \prod_{j \neq l,k} \Gamma(-a_{jl} - b^2/2) \Gamma(bQ - a_{jl} - b^2/2) \Gamma(-a_{kj} - b^2/2) \Gamma(bQ - a_{kj} - b^2/2) \\
 &= \pi^2 \left(\frac{\Upsilon(a_{lk}/b - b) \Upsilon(a_{kl}/b + b)}{\Upsilon(a_{lk}/b + b) \Upsilon(a_{kl}/b - b)} \right)
 \end{aligned}$$

$$\begin{aligned}
 & \times \prod_{i \neq l, k} \frac{\Upsilon(a_{il}/b + b/2)\Upsilon(a_{ik}/b - b/2)}{\Upsilon(a_{il}/b - b/2)\Upsilon(a_{ik}/b + b/2)} \prod_{j \neq l, k} \frac{\Upsilon(a_{lj}/b - b/2)\Upsilon(a_{kj}/b + b/2)}{\Upsilon(a_{lj}/b + b/2)\Upsilon(a_{kj}/b - b/2)} \\
 & \times \prod_i \frac{\Upsilon(Q/2 - m_3/b + m_{4,i}/b + a_l/b + b/2)}{\Upsilon(Q/2 - m_3/b + m_{4,i}/b + a_l/b - b/2)} \frac{\Upsilon(Q/2 - m_3/b + m_{4,i}/b + a_k/b - b/2)}{\Upsilon(Q/2 - m_3/b + m_{4,i}/b + a_k/b + b/2)} \\
 & \times \left(\frac{\Upsilon(Q/2 - m_2/b - m_{1,i}/b + a_l/b + b/2)}{\Upsilon(Q/2 - m_2/b - m_{1,i}/b + a_l/b - b/2)} \frac{\Upsilon(Q/2 - m_2/b - m_{1,i}/b + a_k/b - b/2)}{\Upsilon(Q/2 - m_2/b - m_{1,i}/b + a_k/b + b/2)} \right)^{1/2} \\
 & \times \frac{\Gamma(-a_{kl} - b^2)\Gamma(1 - a_{kl})\Gamma(-a_{kl})\Gamma(bQ - a_{kl})}{\prod_f \Gamma(1/2 + a_l - m_f)\Gamma(1/2 - a_k + m_f)} \prod_{j \neq l, k} \prod_{\pm} \Gamma(-a_{jl} \pm b^2/2)\Gamma(-a_{kj} \pm b^2/2) \\
 = & \pi^2 \left(\frac{\tilde{\Upsilon}^{(2)}(a_{kl}/b - b)}{\tilde{\Upsilon}^{(2)}(a_{lk}/b - b)} \prod_{i \neq l, k} \frac{\tilde{\Upsilon}(a_{il}/b - b/2)}{\tilde{\Upsilon}(a_{ik}/b - b/2)} \prod_{j \neq l, k} \frac{\tilde{\Upsilon}(a_{kj}/b - b/2)}{\tilde{\Upsilon}(a_{lj}/b - b/2)} \right. \\
 & \times \left. \prod_i \frac{\tilde{\Upsilon}(Q/2 - m_3/b + m_{4,i}/b + a_l/b - b/2)}{\tilde{\Upsilon}(Q/2 - m_3/b + m_{4,i}/b + a_k/b - b/2)} \frac{\tilde{\Upsilon}(Q/2 - m_2/b - m_{1,i}/b + a_l/b - b/2)}{\tilde{\Upsilon}(Q/2 - m_2/b - m_{1,i}/b + a_k/b - b/2)} \right)^{1/2} \\
 & \times \frac{\Gamma(-a_{kl} - b^2)\Gamma(1 - a_{kl})\Gamma(-a_{kl})\Gamma(bQ - a_{kl})}{\prod_f \Gamma(1/2 + a_l - m_f)\Gamma(1/2 - a_k + m_f)} \prod_{j \neq l, k} \prod_{\pm} \Gamma(-a_{jl} \pm b^2/2)\Gamma(-a_{kj} \pm b^2/2) \\
 = & \pi^2 \left(\frac{\Gamma(a_{kl} - b^2)\Gamma(a_{kl})\Gamma(1 - a_{lk} + b^2)\Gamma(1 - a_{lk})}{\Gamma(1 - a_{kl} + b^2)\Gamma(1 - a_{kl})\Gamma(a_{lk} - b^2)\Gamma(a_{lk})} \prod_f \frac{\Gamma(1/2 - m_f + a_l)\Gamma(1/2 + m_f - a_k)}{\Gamma(1/2 + m_f - a_l)\Gamma(1/2 - m_f + a_k)} \right. \\
 & \times \left. \prod_{i \neq l, k} \frac{\Gamma(a_{il} - b^2/2)\Gamma(1 - a_{ik} + b^2/2)\Gamma(a_{ki} - b^2/2)\Gamma(1 - a_{li} + b^2/2)}{\Gamma(1 - a_{il} + b^2/2)\Gamma(a_{ik} - b^2/2)\Gamma(1 - a_{ki} + b^2/2)\Gamma(a_{li} - b^2/2)} \right)^{1/2} \\
 & \times \frac{\Gamma(-a_{kl} - b^2)\Gamma(1 - a_{kl})\Gamma(-a_{kl})\Gamma(bQ - a_{kl})}{\prod_f \Gamma(1/2 + a_l - m_f)\Gamma(1/2 - a_k + m_f)} \\
 & \times \prod_{j \neq l, k} \Gamma(-a_{jl} - b^2/2)\Gamma(1 - a_{jl} + b^2/2)\Gamma(-a_{kj} - b^2/2)\Gamma(1 - a_{kj} + b^2/2) \\
 = & \frac{\prod_f [\cos \pi(a_l - m_f) \cos \pi(a_k - m_f)]^{1/2}}{\prod_{\pm} \left[\sin \pi(\pm a_{lk}) \sin \pi(\pm a_{lk} - b^2) \prod_{j \neq l, k} \sin \pi(\pm a_{jl} - b^2/2) \sin \pi(\pm a_{jk} - b^2/2) \right]^{1/2}}. \quad (\text{D.21})
 \end{aligned}$$

This gives the one-loop factors for the terms with non-zero coweights in (8.28) and (8.30). The terms with zero coweight given in [46] appear in (8.28) and (8.30) without modification because their expressions are independent of the normalization of the conformal block.

E SU(2) holonomies on the four-punctured sphere

The Hitchin moduli space on the four-punctured sphere as a complex manifold is described by four $SL(2, \mathbb{C})$ holonomy matrices M_e ($e = 1, \dots, 4$) satisfying $M_1 M_2 M_3 M_4 = 1$ up to conjugation with fixed conjugacy classes for M_e . We set

$$W = \text{Tr} M_1 M_2, \quad T = \text{Tr} M_1 M_4, \quad D = \text{Tr} M_1 M_3. \quad (\text{E.1})$$

They satisfy the identity

$$\begin{aligned}
 0 = & D^2 + (WT - \text{Tr}M_1\text{Tr}M_3 - \text{Tr}M_2\text{Tr}M_4)D \\
 & + (W - \text{Tr}M_1\text{Tr}M_2)(W - \text{Tr}M_3\text{Tr}M_4) + (T - \text{Tr}M_2\text{Tr}M_3)(T - \text{Tr}M_1\text{Tr}M_4) \\
 & + \sum_{e=1}^4 (\text{Tr}M_e)^2 - \prod_{e=1}^4 \text{Tr}M_e - 4.
 \end{aligned}
 \tag{E.2}$$

We expect that the quantities W , T , and D correspond to Wilson, 't Hooft, and dyonic operators [17] in the $SU(2)$ theory with $N_F = 4$ fundamental hypermultiplets. Anticipating a match with the results of localization, we make an ansatz

$$W = x + 1/x, \quad T = -(y^2 + 1/y^2)Z(x) + C_1, \quad D = \left(xy^2 + \frac{1}{xy^2}\right)Z(x) + C_2, \tag{E.3}$$

where $Z(x)$ is a function of $x \equiv e^{2\pi ia}$, and C_1 and C_2 are independent of $y \equiv e^{2\pi ib}$. The ansatz is motivated by the localization computation, where we expect a common one-loop factor $Z(x)$ for T and D . Let us substitute these into (E.2) and organize the equation in powers of y . The minus sign in the first term in T was put by hand to ensure that there are no terms proportional to y^4 or $1/y^4$. We can choose C_1 and C_2 such that terms proportional to y^2 and $1/y^2$ also vanish. Then y drops out of the equation (E.2), which can then be solved for Z . The result is

$$\begin{aligned}
 Z &= 4 \frac{\prod_{\pm} \prod_{i=1}^4 \sin^{1/2} \pi(a \pm m_f)}{\sin^2 2\pi a}, \\
 C_1 &= 2 \frac{\prod_{f=1}^4 \cos \pi m_f}{\cos^2 \pi a} + 2 \frac{\prod_{f=1}^4 \sin \pi m_f}{\sin^2 \pi a}, \\
 C_2 &= 2 \frac{\prod_{f=1}^4 \cos \pi m_f}{\cos^2 \pi a} - 2 \frac{\prod_{f=1}^4 \sin \pi m_f}{\sin^2 \pi a},
 \end{aligned}
 \tag{E.4}$$

where $\text{Tr}M_e = e^{2\pi i\gamma_e} + e^{-2\pi i\gamma_e}$ and

$$2\gamma_1 = m_1 - m_2, \quad 2\gamma_2 = m_1 + m_2, \quad 2\gamma_3 = m_3 + m_4, \quad 2\gamma_4 = m_3 - m_4. \tag{E.5}$$

Then $-T/4$ is precisely the $\lambda = 0$ limit of (8.28).²⁷ These expressions for W , T , and D were given in [8] as the definition of Darboux coordinates a and b .

References

- [1] G. 't Hooft, *On the phase transition towards permanent quark confinement*, *Nucl. Phys. B* **138** (1978) 1 [INSPIRE].
- [2] G. 't Hooft, *A property of electric and magnetic flux in non-Abelian gauge theories*, *Nucl. Phys. B* **153** (1979) 141 [INSPIRE].
- [3] G. 't Hooft, *Topology of the gauge condition and new confinement phases in non-Abelian gauge theories*, *Nucl. Phys. B* **190** (1981) 455 [INSPIRE].

²⁷It would be nice to understand the origin of several minus signs that seem unavoidable.

- [4] M. Kontsevich, *Deformation quantization of Poisson manifolds. 1*, *Lett. Math. Phys.* **66** (2003) 157 [[q-alg/9709040](#)] [[INSPIRE](#)].
- [5] A. Kapustin, *Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality*, *Phys. Rev. D* **74** (2006) 025005 [[hep-th/0501015](#)] [[INSPIRE](#)].
- [6] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, [arXiv:0712.2824](#) [[INSPIRE](#)].
- [7] D. Gaiotto, *$N = 2$ dualities*, [arXiv:0904.2715](#) [[INSPIRE](#)].
- [8] N. Nekrasov, A. Rosly and S. Shatashvili, *Darboux coordinates, Yang-Yang functional and gauge theory*, *Nucl. Phys. Proc. Suppl.* **216** (2011) 69 [[arXiv:1103.3919](#)] [[INSPIRE](#)].
- [9] T. Dimofte and S. Gukov, *Chern-Simons theory and S-duality*, [arXiv:1106.4550](#) [[INSPIRE](#)].
- [10] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, [hep-th/0604151](#) [[INSPIRE](#)].
- [11] J. Gomis, T. Okuda and V. Pestun, *Exact results for 't Hooft loops in gauge theories on S^4* , [arXiv:1105.2568](#) [[INSPIRE](#)].
- [12] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831 [[hep-th/0206161](#)] [[INSPIRE](#)].
- [13] D. Gaiotto, G.W. Moore and A. Neitzke, *Framed BPS states*, [arXiv:1006.0146](#) [[INSPIRE](#)].
- [14] J. Teschner, *An analog of a modular functor from quantized Teichmüller theory*, in *Handbook of Teichmüller theory. Vol. I*, vol. 11 of *IRMA Lect. Math. Theor. Phys.*, Eur. Math. Soc., Zürich Switzerland (2007), pg. 685 [[math/0510174](#)] [[INSPIRE](#)].
- [15] N. Drukker, J. Gomis, T. Okuda and J. Teschner, *Gauge theory loop operators and Liouville theory*, *JHEP* **02** (2010) 057 [[arXiv:0909.1105](#)] [[INSPIRE](#)].
- [16] L.F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, *Loop and surface operators in $N = 2$ gauge theory and Liouville modular geometry*, *JHEP* **01** (2010) 113 [[arXiv:0909.0945](#)] [[INSPIRE](#)].
- [17] N. Drukker, D.R. Morrison and T. Okuda, *Loop operators and S-duality from curves on Riemann surfaces*, *JHEP* **09** (2009) 031 [[arXiv:0907.2593](#)] [[INSPIRE](#)].
- [18] N. Nekrasov and E. Witten, *The omega deformation, branes, integrability and Liouville theory*, *JHEP* **09** (2010) 092 [[arXiv:1002.0888](#)] [[INSPIRE](#)].
- [19] J. Teschner, *Quantization of the Hitchin moduli spaces, Liouville theory and the geometric Langlands correspondence I*, [arXiv:1005.2846](#) [[INSPIRE](#)].
- [20] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, *Lett. Math. Phys.* **91** (2010) 167 [[arXiv:0906.3219](#)] [[INSPIRE](#)].
- [21] T. Dimofte, D. Gaiotto and S. Gukov, *Gauge theories labelled by three-manifolds*, [arXiv:1108.4389](#) [[INSPIRE](#)].
- [22] S. Gukov and E. Witten, *Gauge theory, ramification, and the geometric Langlands program*, [hep-th/0612073](#) [[INSPIRE](#)].
- [23] I. Affleck, J.A. Harvey and E. Witten, *Instantons and (super)symmetry breaking in $(2 + 1)$ -dimensions*, *Nucl. Phys. B* **206** (1982) 413 [[INSPIRE](#)].

- [24] E. Witten, *Topological Quantum Field Theory*, *Commun. Math. Phys.* **117** (1988) 353 [[INSPIRE](#)].
- [25] S. Giombi and V. Pestun, *The 1/2 BPS 't Hooft loops in $N = 4$ SYM as instantons in 2d Yang-Mills*, [arXiv:0909.4272](#) [[INSPIRE](#)].
- [26] M. Pauly, *Monopole moduli spaces for compact 3-manifolds*, *Math. Ann.* **311** (1998) 125.
- [27] S.K. Donaldson and P.B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York U.S.A. (1990).
- [28] E. Witten, *An $SU(2)$ anomaly*, *Phys. Lett. B* **117** (1982) 324 [[INSPIRE](#)].
- [29] S.A. Cherkis and B. Durcan, *The 't Hooft-Polyakov monopole in the presence of an 't Hooft operator*, *Phys. Lett. B* **671** (2009) 123 [[arXiv:0711.2318](#)] [[INSPIRE](#)].
- [30] S.A. Cherkis, *Moduli spaces of instantons on the Taub-NUT space*, *Commun. Math. Phys.* **290** (2009) 719 [[arXiv:0805.1245](#)] [[INSPIRE](#)].
- [31] E. Witten, *Branes, instantons, and Taub-NUT spaces*, *JHEP* **06** (2009) 067 [[arXiv:0902.0948](#)] [[INSPIRE](#)].
- [32] H. Nakajima and K. Yoshioka, *Instanton counting on blowup. 1*, [math/0306198](#) [[INSPIRE](#)].
- [33] U. Bruzzo, F. Fucito, J.F. Morales and A. Tanzini, *Multiinstanton calculus and equivariant cohomology*, *JHEP* **05** (2003) 054 [[hep-th/0211108](#)] [[INSPIRE](#)].
- [34] A. Kapustin, *Holomorphic reduction of $N = 2$ gauge theories, Wilson-'t Hooft operators and S -duality*, [hep-th/0612119](#) [[INSPIRE](#)].
- [35] A. Kapustin and N. Saulina, *The algebra of Wilson-'t Hooft operators*, *Nucl. Phys. B* **814** (2009) 327 [[arXiv:0710.2097](#)] [[INSPIRE](#)].
- [36] N.A. Nekrasov and S.L. Shatashvili, *Supersymmetric vacua and Bethe ansatz*, *Nucl. Phys. Proc. Suppl.* **192-193** (2009) 91 [[arXiv:0901.4744](#)] [[INSPIRE](#)].
- [37] N. Nekrasov and S. Shatashvili, *Bethe ansatz and supersymmetric vacua*, in *American Institute of Physics Conference Series*, V. Lebedev and M. Feigel'Man eds., *AIP Conf. Ser.* **1134** (2009) 154 [[INSPIRE](#)].
- [38] N.A. Nekrasov and S.L. Shatashvili, *Quantum integrability and supersymmetric vacua*, *Prog. Theor. Phys. Suppl.* **177** (2009) 105 [[arXiv:0901.4748](#)] [[INSPIRE](#)].
- [39] D. Gaiotto, G.W. Moore and A. Neitzke, *Wall-crossing, Hitchin systems and the WKB approximation*, [arXiv:0907.3987](#) [[INSPIRE](#)].
- [40] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, *Nucl. Phys. B* **460** (1996) 299 [[hep-th/9510101](#)] [[INSPIRE](#)].
- [41] N.J. Hitchin, *The selfduality equations on a Riemann surface*, *Proc. Lond. Math. Soc.* **55** (1987) 59 [[INSPIRE](#)].
- [42] M. Dehn, *On curve systems on two-sided surfaces, with application to the mapping problem*, in *Papers on group theory and topology*, translated from the german by J. Stillwell, Springer-Verlag, New York U.S.A. (1987), pg. 234.
- [43] W.P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, *Bull. Amer. Math. Soc.* **19** (1988) 417.
- [44] H. Dorn and H. Otto, *Two and three point functions in Liouville theory*, *Nucl. Phys. B* **429** (1994) 375 [[hep-th/9403141](#)] [[INSPIRE](#)].

- [45] A.B. Zamolodchikov and A.B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, *Nucl. Phys. B* **477** (1996) 577 [[hep-th/9506136](#)] [[INSPIRE](#)].
- [46] J. Gomis and B. Le Floch, *'T Hooft operators in gauge theory from Toda CFT*, *JHEP* **11** (2011) 114 [[arXiv:1008.4139](#)] [[INSPIRE](#)].
- [47] A. Belavin, A.M. Polyakov and A. Zamolodchikov, *Infinite conformal symmetry in two-dimensional Quantum Field Theory*, *Nucl. Phys. B* **241** (1984) 333 [[INSPIRE](#)].
- [48] B. Ponsot and J. Teschner, *Liouville bootstrap via harmonic analysis on a noncompact quantum group*, [hep-th/9911110](#) [[INSPIRE](#)].
- [49] N. Wyllard, *A_{N-1} conformal Toda field theory correlation functions from conformal $N = 2$ $SU(N)$ quiver gauge theories*, *JHEP* **11** (2009) 002 [[arXiv:0907.2189](#)] [[INSPIRE](#)].
- [50] F. Passerini, *Gauge theory Wilson loops and conformal Toda field theory*, *JHEP* **03** (2010) 125 [[arXiv:1003.1151](#)] [[INSPIRE](#)].
- [51] J.A. Harvey, *Komaba lectures on noncommutative solitons and D-branes*, [hep-th/0102076](#) [[INSPIRE](#)].
- [52] P.C. Argyres and N. Seiberg, *S-duality in $N = 2$ supersymmetric gauge theories*, *JHEP* **12** (2007) 088 [[arXiv:0711.0054](#)] [[INSPIRE](#)].
- [53] T. Okuda and V. Pestun, *On the instantons and the hypermultiplet mass of $N = 2^*$ super Yang-Mills on S^4* , *JHEP* **03** (2012) 017 [[arXiv:1004.1222](#)] [[INSPIRE](#)].
- [54] J. Teschner, *Quantization of moduli of flat connections, Liouville theory, and integrable models webpage*, <http://online.itp.ucsb.edu/online/integral11/teschner/>.
- [55] S. Lee and P. Yi, *Framed BPS states, moduli dynamics and wall-crossing*, *JHEP* **04** (2011) 098 [[arXiv:1102.1729](#)] [[INSPIRE](#)].
- [56] H. Kim, J. Park, Z. Wang and P. Yi, *Ab initio wall-crossing*, *JHEP* **09** (2011) 079 [[arXiv:1107.0723](#)] [[INSPIRE](#)].
- [57] E. Witten, *Geometric Langlands and the equations of Nahm and Bogomolny*, [arXiv:0905.4795](#) [[INSPIRE](#)].
- [58] N. Saulina, *A note on Wilson-'t Hooft operators*, *Nucl. Phys. B* **857** (2012) 153 [[arXiv:1110.3354](#)] [[INSPIRE](#)].
- [59] N. Berkovits, *A ten-dimensional super Yang-Mills action with off-shell supersymmetry*, *Phys. Lett. B* **318** (1993) 104 [[hep-th/9308128](#)] [[INSPIRE](#)].
- [60] P. Kronheimer, *Monopoles and Taub-NUT metrics*, MSc. thesis, available on the author's home page, Oxford University, Oxford U.K. (1986).
- [61] G. Gibbons and P. Rychenkova, *Hyper-Kähler quotient construction of BPS monopole moduli spaces*, *Commun. Math. Phys.* **186** (1997) 585 [[hep-th/9608085](#)] [[INSPIRE](#)].
- [62] S.A. Cherkis and A. Kapustin, *Periodic monopoles with singularities and $N = 2$ super QCD*, *Commun. Math. Phys.* **234** (2003) 1 [[hep-th/0011081](#)] [[INSPIRE](#)].