

# Heavy flavour kinetic equilibration in the confined phase

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**ABSTRACT:** By making use of a non-perturbative definition of a momentum diffusion coefficient as well as Heavy Meson Chiral Perturbation Theory, we investigate the Brownian motion and kinetic equilibration of heavy quark flavours deep in the confined phase. It appears that the momentum diffusion coefficient can be expressed in terms of known low-energy constants; it increases rapidly at temperatures above 50 MeV, behaving as  $\sim T^7/F_\pi^4$  for  $\frac{m_\pi}{\pi} \ll T \ll F_\pi$ , where  $m_\pi$  and  $F_\pi$  are the pion mass and decay constant, respectively. The early increase may suggest a broad peak in  $\kappa/T^3$  around the QCD crossover. For a more detailed understanding the computation could be generalized in a number of ways.

**KEYWORDS:** Thermal Field Theory, Heavy Quark Physics, Chiral Lagrangians

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## 1 Introduction

An ideal scenario for what could happen in a heavy ion collision is that light quarks and gluons form a rapidly expanding thermalized medium, and that there are some “probes” available, whose properties are affected by the medium in a significant yet tractable way. Among the most attractive probe candidates are heavy quarks (charm and bottom quarks as well as their antiparticles), which can be copiously produced in an initial hard process. After a while the heavy quarks decay, but modifications on their behaviour caused by the thermal medium could conceivably be deduced from the experimentally observed transverse momentum distributions and azimuthal anisotropies of the leptonic decay products [1, 2].

With such motivations in mind, a significant body of work has been carried out during the last 20 years or so, concerning the effects that a thermal medium can have on the propagation of heavy quarks [3, 4]. Roughly, the initial stages are characterized by radiative energy loss (bremsstrahlung), which may slow down the heavy quarks; the final stages are characterized by elastic scatterings, which cause collisional energy loss but also produce random kicks corresponding to Brownian motion. Assuming, idealistically, that the thermal system is spatially large enough for all of these processes to take place, they have been described by a number of related physical concepts and observables, such as energy loss ( $dE/dx$ ), stopping distance, jet quenching, (momentum) diffusion, drag, or kinetic equilibration.

In the present paper, we focus on a late stage of the above scenario, in which the heavy quarks are practically at rest with respect to the thermal medium but also undergo Brownian motion. (Possible extensions to more general kinematics are outlined in the conclusions.) More specifically, we are interested in an observable called the momentum diffusion coefficient,  $\kappa$ , which characterizes the random force acting on the heavy quarks and, through a fluctuation-dissipation relation, also determines their kinetic equilibration rate.

Recently, a number of theoretical works have addressed the same problem. For instance,  $\kappa$  has been computed to leading [5] and next-to-leading [6] order in the weak-coupling expansion. It has also been given a non-perturbative definition [7] in the framework of Heavy Quark Effective Theory [8]–[12]; this was partly inspired by computations through AdS/CFT techniques in the large- $N_c$  limit of strongly coupled  $\mathcal{N} = 4$  Super-Yang-Mills theory, which suggest a larger  $\kappa$  than in leading-order QCD [13–15]. (Similar conclusions have also been reached for AdS models resembling QCD; see e.g. ref. [16] and references therein.) Numerical simulations have been carried out within so-called classical lattice gauge theory, suggesting again a value larger than indicated by the weak-coupling expansion [17]. All of this makes a strong case for attacking the problem with lattice simulations, a challenge that may be less daunting than the determination of many other “transport coefficients”, such as viscosities, because of the simpler structure of the pertinent spectral function [18]; indeed the very first numerical attempts look rather promising [19]. The inherent uncertainties related to analytic continuation from Euclidean numerical data have also been looked into, and it appears that it might be feasible to carry through the program at least on the qualitative level [20]. Once a value of  $\kappa$  is available, it can be incorporated in hydrodynamical simulations to yield results relevant for experiment (cf. e.g. ref. [5]); there is a growing body of such works under way.

The purpose of the present paper is to make use of the non-perturbative definition of  $\kappa$  introduced in ref. [7], and to evaluate it deep in the confined phase. This is possible in QCD because, due to chiral symmetry breaking,<sup>1</sup> the infrared dynamics of the confined phase can be parametrized with a small number of “low-energy constants”. In fact, in a certain limit, we find that  $\kappa$  is fully determined in terms of the pion decay constant and the pion mass. There is, of course, a long history to applying Chiral Perturbation Theory (not to mention hadronic models) for the computation of various thermodynamic properties of the pion gas, see e.g. ref. [21]; past developments and the non-trivial theoretical challenges that are related particularly to transport coefficients have recently been discussed in the context of the bulk viscosity of strongly interacting matter in ref. [22].

The paper is organized as follows. The non-perturbative definition of the momentum diffusion coefficient is reviewed in section 2; the chiral effective theory relevant for handling heavy-light mesons in the confined phase is described in section 3; and the computation of the momentum diffusion coefficient is presented in section 4. Some conclusions comprise section 5.

## 2 Basic physics of momentum diffusion

Considering time scales short compared with the life-time of the heavy quarks (or heavy-light mesons) but long compared with those of microscopic processes involving gluons and light quarks (or light mesons), which have a momentum  $p \sim T \sim 100$  MeV, where  $T$  denotes the temperature, the dynamics of the heavy degrees of freedom, with a mass

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<sup>1</sup>The number of light flavours is assumed non-zero,  $N_f > 0$ ; it is unclear whether  $\kappa$  can be given a sensible meaning in the confined phase for  $N_f = 0$  [7] even if the related Euclidean correlator appears to exist [18].

$M \gg T$ , presumably resembles Brownian motion. If so, it can be described by the Langevin equation, with the role of the stochastic noise being played by a force induced by QCD-mediated collisions with gluons and light quarks (or light mesons), which are present with an abundant number density  $n \sim T^3$ . The corresponding equations of motion have the form

$$\dot{p}_k(t) = -\eta_D p_k(t) + \xi_k(t), \tag{2.1}$$

$$\langle\langle \xi_k(t) \xi_l(t') \rangle\rangle = \kappa \delta_{kl} \delta(t-t'), \quad \langle\langle \xi_k(t) \rangle\rangle = 0, \tag{2.2}$$

where  $p_k$  is the momentum of the heavy objects ( $k = 1, 2, 3$ );  $\xi_k$  is a Gaussian stochastic noise; and  $\langle\langle \dots \rangle\rangle$  denotes an average over the noise. According to eq. (2.2) the momentum diffusion coefficient,  $\kappa$ , characterizes the auto-correlator of the force,

$$\kappa = \frac{1}{3} \int_{-\infty}^{\infty} dt \sum_k \langle\langle \xi_k(t) \xi_k(0) \rangle\rangle, \tag{2.3}$$

whereas the coefficient  $\eta_D$  appearing in eq. (2.1) is referred to as the “kinetic equilibration rate” or the “drag coefficient”. As is well-known, in classical statistical physics the two can be fluctuation-dissipation related to each other:  $\eta_D \simeq \kappa/(2TM)$ , where we interpret  $M$  as a specific mass definition (more precisely the heavy quark “kinetic” mass; we assume a regularization scheme respecting Lorentz symmetry, and then  $M$  is equal to the “rest” or “pole” mass; it should be fixed non-perturbatively).

Now, it was argued in ref. [7] that the classical definition of  $\kappa$  as an autocorrelator of forces, eq. (2.3), can “naturally” be extended to QCD. Suppose that we know the heavy quark Hamiltonian,  $\hat{H}$ , as well as the Noether current associated with the U(1) flavour symmetry,  $\hat{\mathcal{J}}^\mu$ . More specifically,  $\hat{H}$  and  $\hat{\mathcal{J}}^0$  are needed up to  $\mathcal{O}(M^0)$  in an expansion in a large  $M$ , whereas the  $\hat{\mathcal{J}}^k$  are needed up to  $\mathcal{O}(1/M)$ . With these operators we can define a “susceptibility” related to the conserved charge,

$$\chi^{00} \equiv \beta \int_{\mathbf{x}} \langle\langle \hat{\mathcal{J}}^0(t, \mathbf{x}) \hat{\mathcal{J}}^0(t, \mathbf{0}) \rangle\rangle_T, \quad \beta \equiv \frac{1}{T}, \quad \int_{\mathbf{x}} \equiv \int d^3\mathbf{x}, \tag{2.4}$$

and the “acceleration” associated with the spatial components,

$$\frac{d\hat{\mathcal{J}}^k}{dt} = i[\hat{H}, \hat{\mathcal{J}}^k] + \frac{\partial \hat{\mathcal{J}}^k}{\partial t}, \tag{2.5}$$

where the partial derivative acts on possible background fields. Consequently,

$$\kappa \equiv \frac{\beta}{3} \sum_{k=1}^3 \lim_{\omega \rightarrow 0} \left[ \lim_{M \rightarrow \infty} \frac{M^2}{\chi^{00}} \int_{-\infty}^{\infty} dt e^{i\omega(t-t')} \int_{\mathbf{x}} \left\langle \frac{1}{2} \left\{ \frac{d\hat{\mathcal{J}}^k(t, \mathbf{x})}{dt}, \frac{d\hat{\mathcal{J}}^k(t', \mathbf{0})}{dt'} \right\} \right\rangle_T \right]. \tag{2.6}$$

Since  $M \int_{\mathbf{x}} d\hat{\mathcal{J}}^k/dt$  represents, according to Newton’s law, a force, this definition is indeed a generalization of eq. (2.3). The ordering of the various limits requires, however, a careful analysis [7]; the situation simplifies only if a dependence  $d\hat{\mathcal{J}}^k/dt \sim 1/M$  can be factored out and cancelled. We also note that, as dictated by standard relations between various time orderings at finite temperatures [23],

$$\kappa = \lim_{\omega \rightarrow 0} \frac{2T \rho_E(\omega)}{\omega}, \tag{2.7}$$

where the spectral function  $\rho_E$  is defined as in eq. (2.6) but with a commutator replacing the anticommutator. (The notation  $\langle\langle \dots \rangle\rangle_T$  refers to the usual thermal average.)

### 3 Heavy meson chiral perturbation theory

The goal now would be to evaluate eq. (2.6) at low temperatures. This can be achieved by making use of an effective field theory that is valid in the regime considered and allows us to define the operators entering the definition. Given that the framework may be unfamiliar within the finite-temperature community, we give a few ingredients in the following, although no attempt is made at a comprehensive review (see refs. [24–26] for introductions).

The “usual” chiral Lagrangian describing the pseudo Nambu-Goldstone bosons of chiral symmetry breaking has the form [27]

$$\mathcal{L}_{\chi\text{PT}} = \frac{F^2}{4} \text{Tr} (\partial^\mu U \partial_\mu U^\dagger) + \frac{\Sigma}{2} \text{Tr} (\mathcal{M}^\dagger U + U^\dagger \mathcal{M}) + \dots, \quad (3.1)$$

where  $F$  is the pion decay constant in the chiral limit (naively  $F \simeq 93 \text{ MeV}$ ),  $\Sigma$  is the chiral condensate,  $\mathcal{M}$  is the quark mass matrix, and  $U \in \text{SU}(N_f)$  is the Goldstone field. For simplicity we take the mass matrix to be of the form  $\mathcal{M} = m \mathbb{1}$  in the following, with  $m \in \mathbb{R}$ ; then the pion mass squared is  $m_\pi^2 = 2m\Sigma/F^2 + \mathcal{O}(1/F^4)$ . Parametrizing  $U = \exp(\frac{2i\xi}{F})$ , the tree-level propagator is

$$\langle \xi_{ab}(x) \xi_{cd}(y) \rangle^{(0)} = \frac{1}{2} \left( \delta_{ad} \delta_{bc} - \frac{1}{N_f} \delta_{ab} \delta_{cd} \right) \Delta(x-y; m_\pi^2), \quad (3.2)$$

where  $\Delta$  is a scalar propagator. It is important to keep in mind that for the  $\mathcal{M}$  chosen, eq. (3.1) contains no terms cubic in  $\xi$ , i.e. no three-pion interactions. (Interactions among odd numbers of pions are contained in the Wess-Zumino-Witten term, but it is suppressed by a large number of derivatives.)

The next step is to supplement the chiral Lagrangian with a piece describing heavy-light mesons. The resulting effective description is called Heavy Meson Chiral Perturbation Theory (HM $\chi$ PT) [28–30].

To start with we note that, unlike Heavy Quark Effective Theory (HQET) [8]–[12], which describes all states containing a single heavy quark or antiquark (provided that the associated gluons and light quarks are soft), the mesonic HM $\chi$ PT only contains a specific subset of such states, namely parity-odd pseudoscalar and vector mesons ( $D$  and  $D^*$  for charm,  $B$  and  $B^*$  for bottom), with total spin 0 or 1. The standard convention is to denote the scalar by  $P_a$  and the vector by  $P_a^{*\mu}$ , where  $a$  is a light flavour index ( $a = 1, \dots, N_f$ ); to streamline the notation we define  $Q_a^\mu \equiv P_a^{*\mu}$  in the following. Although  $Q_a^\mu$  is written as a four-vector, it only contains three independent components, cf. eq. (3.10) below.

The form of HM $\chi$ PT is dictated by symmetries, of which there are many. First of all there is a U(1) symmetry related to the conserved heavy quark number; this yields the Noether current  $\mathcal{J}^\mu$  alluded to above. Second, HQET at  $\mathcal{O}(M^0)$  displays a heavy quark spin symmetry, since Pauli matrices first appear at  $\mathcal{O}(1/M)$ ; this implies a relation between the fields  $P_a$  and  $Q_a^\mu$  (cf. eq. (4.2) below). Third, the light-quark index  $a$  enjoys a specific transformation property under the  $\text{SU}(N_f)_L \times \text{SU}(N_f)_R$  chiral symmetry. Fourth, even though the effective Lagrangian is non-relativistic, its origin in relativistic QCD implies that it also remembers something about proper Lorentz symmetry, provided that this

has not been broken through regularization. This is usually implemented by introducing an arbitrary parameter, a four-velocity  $v^\mu$ , and an associated “velocity reparametrization invariance” [31]. Finally, there are the usual discrete C, P, and T symmetries of QCD.

It turns out to be non-trivial to implement all the symmetries in a convenient way. In fact, in order to define a simple parity transformation it is not practical to use the Goldstone field  $U$  in the part of the Lagrangian involving the heavy mesons, but rather a “coset field”  $\sqrt{U}$ .<sup>2</sup> (General formal considerations can be found in refs. [33, 34].) If  $U$  transforms in  $SU(N_f)_L \times SU(N_f)_R$  as

$$U \rightarrow LUR^\dagger, \tag{3.3}$$

then the field  $\sqrt{U}$  can be assigned the transformation

$$\sqrt{U} \rightarrow L\sqrt{U}W^\dagger \quad \text{and} \quad \sqrt{U} \rightarrow W\sqrt{U}R^\dagger, \tag{3.4}$$

where the complicated (space-time dependent) field  $W$  is defined through eqs. (3.4). We can subsequently introduce the traceless and Hermitean fields

$$\mathcal{V}_\mu \equiv \frac{i}{2} \left[ \sqrt{U}^\dagger \partial_\mu \sqrt{U} + \sqrt{U} \partial_\mu \sqrt{U}^\dagger \right], \tag{3.5}$$

$$\mathcal{A}_\mu \equiv \frac{i}{2} \left[ \sqrt{U}^\dagger \partial_\mu \sqrt{U} - \sqrt{U} \partial_\mu \sqrt{U}^\dagger \right], \tag{3.6}$$

which transform as

$$\mathcal{V}_\mu \rightarrow W\mathcal{V}_\mu W^\dagger + iW\partial_\mu W^\dagger, \quad \mathcal{A}_\mu \rightarrow W\mathcal{A}_\mu W^\dagger. \tag{3.7}$$

Note that in the chiral expansion, inserting  $\sqrt{U} = \exp(\frac{i\xi}{F})$ , we get

$$\mathcal{V}_\mu = \frac{i}{2F^2} \left( \xi \partial_\mu \xi - \partial_\mu \xi \xi \right) + \mathcal{O}\left(\frac{\xi^4}{F^4}\right), \quad \mathcal{A}_\mu = -\frac{1}{F} \partial_\mu \xi + \mathcal{O}\left(\frac{\xi^3}{F^3}\right), \tag{3.8}$$

i.e.  $\mathcal{V}_\mu$  couples to an even number of pions and  $\mathcal{A}_\mu$  to an odd number.

The heavy meson fields  $P_a$  and  $Q_a^\mu$  are normally assembled into a  $4 \times 4$  matrix  $H_a$  and its conjugate  $\bar{H}_a \equiv \gamma^0 H_a^\dagger \gamma^0$ :

$$H_a \equiv \frac{1 + \not{v}}{2} \left( Q_a + i\gamma_5 P_a \right), \quad \bar{H}_a = \left( Q_a^\dagger + i\gamma_5 P_a^\dagger \right) \frac{1 + \not{v}}{2}, \tag{3.9}$$

where, following a frequent convention,  $(\dots)^\dagger$  denotes complex conjugation acting on  $P_a$ ,  $Q_a^\mu$ . Here  $v^\mu$  is a four-velocity with  $v \cdot v = 1$ . The vector meson field is constrained by

$$v \cdot Q_a = 0, \tag{3.10}$$

which implies for instance that  $\not{v} Q_a = -Q_a \not{v}$ . The chiral transformation can be assigned as

$$H \rightarrow H W^\dagger, \quad \bar{H} \rightarrow W \bar{H}. \tag{3.11}$$

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<sup>2</sup>We follow the notation of e.g. ref. [32], where HM $\chi$ PT was used in connection with a lattice investigation.

Defining

$$\mathcal{D}_{ba}^\mu \equiv \partial^\mu \delta_{ba} + i\mathcal{V}_{ba}^\mu, \quad (3.12)$$

the effective Lagrangian at  $\mathcal{O}(M^0)$  has the form

$$\begin{aligned} \mathcal{L}_{\text{HM}\chi\text{PT}}^{(0)} = & -i \text{Tr} (\bar{H}_a v \cdot \mathcal{D}_{ba} H_b) + M_P \text{Tr} (\bar{H}_a H_a) + g_\pi \text{Tr} (\bar{H}_a H_b \mathcal{A}_{ba} \gamma_5) \\ & + \sigma_1 \text{Tr} (\bar{H}_a H_b) (\sqrt{U} \mathcal{M}^\dagger \sqrt{U} + \sqrt{U}^\dagger \mathcal{M} \sqrt{U}^\dagger)_{ba} \\ & + \sigma'_1 \text{Tr} (\bar{H}_a H_a) (\sqrt{U} \mathcal{M}^\dagger \sqrt{U} + \sqrt{U}^\dagger \mathcal{M} \sqrt{U}^\dagger)_{bb} + \dots, \end{aligned} \quad (3.13)$$

where the trace is over Dirac matrices, and “...” denotes terms of higher order in the chiral expansion. The coefficient  $g_\pi$  is a new dimensionless low-energy constant, estimated from e.g.  $D^* \rightarrow D\pi$  decays [26] or lattice determinations (e.g. refs. [35, 36]) to be around  $g_\pi \simeq 0.5$ .

The mass term on the first row of eq. (3.13), proportional to  $M_P$ , is often not shown, because it can be shifted away by a time-dependent phase transformation. However, for the finite-temperature application that will be discussed in the next section, it is useful to keep  $M_P$  explicit. The terms proportional to the low-energy constants  $\sigma_1, \sigma'_1$  play a role, at leading order, for the mass spectrum:  $\sigma'_1$  shifts  $M_P$  by a flavour-independent amount, whereas  $\sigma_1$  would break the flavour symmetry if  $\mathcal{M}$  were non-degenerate. In our case both of these effects can be accounted for through  $M_P$ , so we omit  $\sigma_1, \sigma'_1$  in the following. The spin symmetry between  $P_a$  and  $Q_a^\mu$  is only broken at  $\mathcal{O}(1/M)$ ; although we presently turn to such effects, the “trivial” mass splitting that they induce will be considered to already be accounted for by the present remarks, and we denote the vector meson mass by  $M_Q$ .

Now, we turn to terms of  $\mathcal{O}(1/M)$ . Some of these can be deduced from eq. (3.13) through reparametrization invariance [31]. This is a transformation which leaves invariant the “original” momentum operator  $P_\mu \equiv M v_\mu + i\mathcal{D}_\mu$  as well as scalar products such as  $v \cdot v = 1$ . Writing  $v_\mu \rightarrow v_\mu + \epsilon_\mu/M$  with  $v \cdot \epsilon = 0$ , which implies  $i\mathcal{D}_\mu \rightarrow i\mathcal{D}_\mu - \epsilon_\mu$ , we observe in particular that the combination

$$\mathcal{O}_{ba} = iv \cdot \mathcal{D}_{ba} - \frac{a_1}{2M} (\mathcal{D} \cdot \mathcal{D})_{ba} \quad (3.14)$$

is invariant only for  $a_1 = 1$ . This is clearly a reflection of Lorentz symmetry, and means that the coefficient of a particular  $\mathcal{O}(1/M)$ -operator is fixed in terms of that of  $iv \cdot \mathcal{D}_{ba}$  at  $\mathcal{O}(M^0)$ . (Whether it is  $M$  or  $M_P$  that appears in the denominator of eq. (3.14) plays no role, because their difference amounts to an effect of  $\mathcal{O}(1/M^2)$ .) So,

$$\mathcal{L}_{\text{HM}\chi\text{PT}}^{(1)} = \frac{1}{2M} \text{Tr} [\bar{H}_a (\mathcal{D} \cdot \mathcal{D})_{ba} H_b] + \dots \quad (3.15)$$

The term proportional to  $g_\pi$  in eq. (3.13) also implies the existence of terms of  $\mathcal{O}(g_\pi/M)$  with specific coefficients, but the argument is not quite as transparent as that with eq. (3.14); we postpone a discussion to after eq. (4.2). Analyses of all terms of  $\mathcal{O}(1/M)$ , also those not fixed by reparametrization invariance, have been carried out in refs. [37, 38].

## 4 Main computation

We now apply the theory defined by eqs. (3.13), (3.15) to estimate the momentum diffusion coefficient as defined by eq. (2.6). The computation carried out is rather straightforward and simple-minded, but the result is interesting so we present the steps in some detail.

### 4.1 Force operator

We start by rewriting the sum of eqs. (3.13), (3.15) after carrying out the Dirac traces, setting  $v \rightarrow (1, \mathbf{0})$ , and adding terms of  $\mathcal{O}(g_\pi/M)$  through an argument to be discussed presently (the indices  $k, l, m$  are spatial; indices are raised and lowered with the metric  $(+---)$ ; repeated indices are summed over; and “...” indicates terms of higher order in the chiral expansion):

$$\begin{aligned}
 \mathcal{L}_{\text{HM}\chi\text{PT}} = & 2 \left[ P_a^\dagger i\mathcal{D}_{ba}^0 P_b - M_P P_a^\dagger P_a \right] + 2 \left[ Q_{ak}^\dagger i\mathcal{D}_{ba}^0 Q_{bk} - M_Q Q_{ak}^\dagger Q_{ak} \right] \\
 & + 2ig_\pi \left[ P_a^\dagger Q_{bl} - Q_{al}^\dagger P_b + \epsilon_{klm} Q_{ak}^\dagger Q_{bm} \right] \mathcal{A}_{ba}^l \\
 & + \frac{1}{M} \left[ P_a^\dagger (\mathcal{D}^l \mathcal{D}^l)_{ba} P_b + Q_{ak}^\dagger (\mathcal{D}^l \mathcal{D}^l)_{ba} Q_{bk} \right] \\
 & + \frac{g_\pi}{M} \left[ 2P_a^\dagger \mathcal{D}_{cb}^k Q_{ck} + 2Q_{ck}^\dagger \overleftarrow{\mathcal{D}}_{ac}^k P_b + \epsilon_{klm} \left( Q_{ak}^\dagger \mathcal{D}_{cb}^l Q_{cm} - Q_{ck}^\dagger \overleftarrow{\mathcal{D}}_{ac}^l Q_{bm} \right) \right] \mathcal{A}_{ba}^0 \\
 & + \mathcal{O}\left(\frac{1}{M}\right) + \dots .
 \end{aligned} \tag{4.1}$$

In the part of  $\mathcal{O}(1/M)$  only terms with spatial derivatives acting on  $P$  and  $Q$  have been shown, because these are the only ones contributing to  $\mathcal{J}^k$ . Also,  $Q_c^\dagger \overleftarrow{\mathcal{D}}_{ac}^k \equiv (\mathcal{D}_{ca}^k Q_c)^\dagger$ .

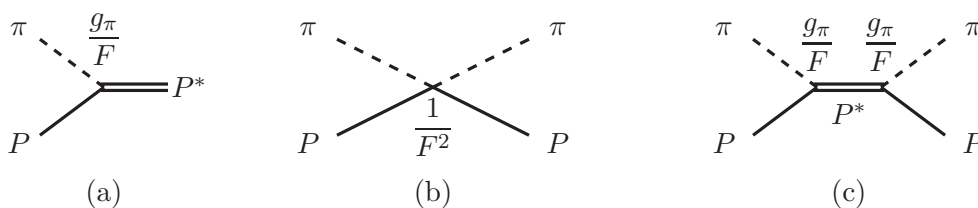
The heavy quark spin symmetry amounts to the transformations

$$\delta P_a = -\alpha_l Q_{al} , \quad \delta Q_{ak} = \alpha_k P_a + \epsilon_{klm} \alpha_l Q_{am} , \quad \alpha_k \in \mathbb{R} , \quad k, l, m \in \{1, 2, 3\} . \tag{4.2}$$

Terms of  $\mathcal{O}(M^0)$  in eq. (4.1) are easily seen to be invariant under this transformation; the mass terms are invariant only if  $M_Q = M_P$ , i.e. the difference  $M_Q - M_P$  must be of  $\mathcal{O}(1/M)$ , as mentioned above. Even though this is not the case in general, the terms of  $\mathcal{O}(1/M)$  shown explicitly in eq. (4.1) are also invariant under eq. (4.2) (the term on the fourth row is invariant only up to total derivatives and terms containing covariant derivatives acting on  $\mathcal{A}_{ba}^0$ , which have been omitted because they do not contribute to  $\mathcal{J}^k$ ). The invariance exists because these terms are related to invariant terms of  $\mathcal{O}(M^0)$ .

The fourth row of eq. (4.1), with a coefficient  $g_\pi/M$ , is new with respect to eq. (3.15). It is connected to the second row of eq. (4.1) through reparametrization invariance; to be concrete, one could imagine that the vector field of an original relativistic theory, let us denote it with  $\tilde{Q}$ , satisfy a transversality constraint of the type  $P_\mu \tilde{Q}^\mu = 0$  rather than eq. (3.10). So, recalling that  $P_\mu \equiv Mv_\mu + i\mathcal{D}_\mu$ , every appearance of  $Q^\mu$  can be replaced with  $\tilde{Q}^\mu \equiv Q^\mu - v^\mu i\mathcal{D} \cdot Q/M$ , where  $Q^\mu$  does satisfy eq. (3.10) [31]. This yields terms of  $\mathcal{O}(g_\pi/M)$  with fixed coefficients. The  $\mathcal{O}(1/M)$  structures shown in eq. (4.1) that have been determined through reparametrization invariance are the only ones from the full list [38] that contain spatial derivatives acting on  $P_a, Q_{al}$ , and therefore contribute to  $\mathcal{J}^k$ . (An





**Figure 1.** Some of the scatterings experienced by  $P$ -mesons almost at rest with respect to a pion gas ( $P$  and  $P^*$  could represent  $B$  and  $B^*$ -mesons, respectively). The process (a) is kinematically allowed for  $D$ -mesons, but not for  $B$ -mesons. There is no  $t$ -channel scattering on thermal pions, because the Lagrangian in eq. (3.1) contains no three-pion vertex and that in eq. (4.1) contains no  $PP\pi$ -vertex. Electromagnetic interactions, such as scatterings on blackbody photons, have been omitted.

explicit crosscheck on the logic presented could be obtained by starting from a genuinely relativistic formulation, such as the ones in refs. [37, 39], and taking the non-relativistic limit only in the end, because then all consequences of Lorentz symmetry are automatically respected. In the following the terms of  $\mathcal{O}(g_\pi/M)$  are included for illustration but are omitted from the final result.)

Now, all terms proportional to  $g_\pi$  couple to  $\mathcal{A}_{ba}^\mu$  and thus to an odd number of pions, whereas terms without  $g_\pi$  couple to an even number of pions (cf. eq. (3.8)). Also,  $g_\pi$  necessarily mixes the fields  $P$  and  $Q^\dagger$ , or  $Q$  and  $Q^\dagger$ . This makes the terms proportional to  $g_\pi$  in general the most important ones from the point of view of zero-temperature phenomenology, such as the decays of the  $D^*$  and  $B^*$  mesons [26]; a relevant amplitude is illustrated in figure 1(a). However, if all particles are on-shell, this process is kinematically forbidden for  $B$ -mesons, because the mass difference between  $B$  and  $B^*$ , scaling as  $1/M$ , is below the pion mass. Therefore the dominant processes for  $B$ -mesons are of the types shown in figures 1(b) and 1(c), and the scattering amplitude is  $\mathcal{O}(1/F^2)$ .

Next, we obtain from eq. (4.1) the U(1) Noether current:

$$\mathcal{J}^0 = 2P_a^\dagger P_a + (P_a \leftrightarrow Q_{ak}) + \mathcal{O}\left(\frac{1}{M}\right), \tag{4.3}$$

$$\begin{aligned} \mathcal{J}^k &= \frac{1}{M} \left[ i(P_a^\dagger \partial^k P_a - \partial^k P_a^\dagger P_a) - 2P_a^\dagger \mathcal{V}_{ba}^k P_b \right] + (P_a \leftrightarrow Q_{al}) \\ &\quad + \frac{2ig_\pi}{M} \left[ P_a^\dagger Q_{bk} - Q_{ak}^\dagger P_b + \epsilon_{lkm} Q_{al}^\dagger Q_{bm} \right] \mathcal{A}_{ba}^0 + \mathcal{O}\left(\frac{1}{M^2}\right). \end{aligned} \tag{4.4}$$

In order to determine the acceleration from eq. (2.5), we would need the Hamiltonian; however, equivalent information should be contained in classical equations of motion, which read

$$i\partial_0 P_a = M_P P_a + \mathcal{V}_{ba}^0 P_b - ig_\pi \mathcal{A}_{ba}^l Q_{bl} + \mathcal{O}\left(\frac{1}{M}\right), \tag{4.5}$$

$$i\partial_0 Q_{ak} = M_Q Q_{ak} + \mathcal{V}_{ba}^0 Q_{bk} + ig_\pi \left( \mathcal{A}_{ba}^k P_b - \epsilon_{klm} \mathcal{A}_{ba}^l Q_{bm} \right) + \mathcal{O}\left(\frac{1}{M}\right). \tag{4.6}$$

Note that when acting on  $\mathcal{J}^0$ ,  $\mathcal{J}^k$ , the terms of  $\mathcal{O}(M_P, M_Q)$  from here cancel out (apart

from a possible remainder proportional to  $M_Q - M_P \sim 1/M$ ), so that effectively the time derivatives count as terms of  $\mathcal{O}(M^0)$ .

Combining eqs. (4.4)–(4.6), and noting that in eq. (2.6) a spatial average can be taken over the currents and that therefore partial integrations are allowed with respect to  $\partial^k$ , we finally obtain  $\partial^0 \int_{\mathbf{x}} \mathcal{J}^k$ . Terms do proliferate quite a bit: defining

$$iF_{ab}^{\mu\nu} \equiv [\mathcal{D}^{\mu T}, \mathcal{D}^{\nu T}]_{ab} = i \left( \partial^\mu \mathcal{V}_{ba}^\nu - \partial^\nu \mathcal{V}_{ba}^\mu + i \mathcal{V}_{ca}^\mu \mathcal{V}_{bc}^\nu - i \mathcal{V}_{ca}^\nu \mathcal{V}_{bc}^\mu \right), \quad (4.7)$$

$$iG_{ab}^{\mu\nu} \equiv [\mathcal{D}^{\mu T}, i\mathcal{A}^{\nu T}]_{ab} = i \left( \partial^\mu \mathcal{A}_{ba}^\nu + i \mathcal{V}_{ca}^\mu \mathcal{A}_{bc}^\nu - i \mathcal{A}_{ca}^\nu \mathcal{V}_{bc}^\mu \right), \quad (4.8)$$

$$iH_{ab}^{\mu\nu} \equiv [i\mathcal{A}^{\mu T}, i\mathcal{A}^{\nu T}]_{ab} = i \left( i \mathcal{A}_{ca}^\mu \mathcal{A}_{bc}^\nu - i \mathcal{A}_{ca}^\nu \mathcal{A}_{bc}^\mu \right), \quad (4.9)$$

$$i\tilde{H}_{ab}^{\mu\nu} \equiv \{i\mathcal{A}^{\mu T}, i\mathcal{A}^{\nu T}\}_{ab} = i \left( i \mathcal{A}_{ca}^\mu \mathcal{A}_{bc}^\nu + i \mathcal{A}_{ca}^\nu \mathcal{A}_{bc}^\mu \right), \quad (4.10)$$

where  $(\dots)^T$  denotes a transpose with respect to flavour indices, we obtain the structure

$$\begin{aligned} \partial^0 \int_{\mathbf{x}} \mathcal{J}^k &= \frac{2}{M} \int_{\mathbf{x}} \left\{ P^\dagger (F^{k0} - g_\pi^2 H^{k0}) P + Q_l^\dagger (F^{k0} - g_\pi^2 H^{k0}) Q_l \right. \\ &\quad + i g_\pi \left[ P^\dagger G^{lk} Q_l - Q_l^\dagger G^{lk} P + \epsilon_{mln} Q_m^\dagger G^{lk} Q_n \right] \\ &\quad + i g_\pi \left[ P^\dagger (G^{00} - i \Delta M \mathcal{A}^{0T}) Q_k - Q_k^\dagger (G^{00} + i \Delta M \mathcal{A}^{0T}) P + \epsilon_{mkn} Q_m^\dagger G^{00} Q_n \right] \\ &\quad \left. + g_\pi^2 \left[ Q_k^\dagger \tilde{H}^{l0} Q_l - Q_l^\dagger \tilde{H}^{l0} Q_k + \epsilon_{klm} \left( Q_m^\dagger \tilde{H}^{l0} P - P^\dagger \tilde{H}^{l0} Q_m \right) \right] \right\} + \mathcal{O}\left(\frac{1}{M^2}\right), \end{aligned} \quad (4.11)$$

where  $\Delta M \equiv M_Q - M_P$  and flavour indices have been suppressed. For  $\Delta M = 0$  each row is separately invariant under the heavy quark spin symmetry, eq. (4.2). We focus on two of the operators in the following, namely  $\sim P^\dagger F P$  (cf. eq. (4.24)) and  $\sim i g_\pi P^\dagger G Q$  (cf. eq. (4.19)).

## 4.2 Euclidean formulation

Before proceeding with the main line of the computation, we need to confront the specific time ordering appearing in eq. (2.6). We do this by going through the imaginary-time formalism, because this produces an intermediate result which could in principle be compared with lattice simulations. Denoting by  $\tau = it$  the Euclidean time coordinate, the propagator of  $P_a$  is then determined by the Euclidean action (corresponding to a weight  $\exp(-S_E)$ )

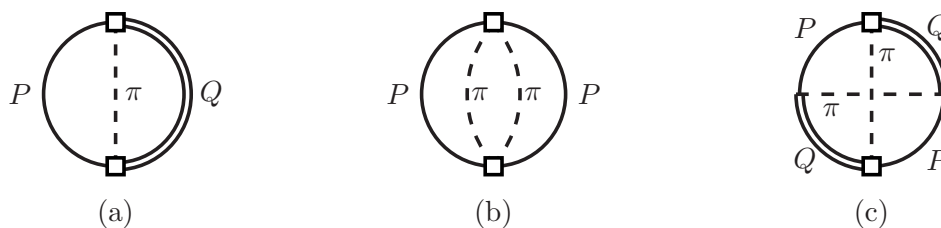
$$S_E^{(0)} = \int_0^\beta d\tau \int_{\mathbf{x}} 2 P_a^\dagger (\partial_\tau + M_P) P_a + (P_a \leftrightarrow Q_{ak}). \quad (4.12)$$

Due to charge conservation the susceptibility of eq. (2.4) can be re-expressed as

$$\chi^{00} = \int_0^\beta d\tau \int_{\mathbf{x}} \langle [2P_a^\dagger P_a](\tau, \mathbf{x}) [2P_b^\dagger P_b](0, \mathbf{0}) \rangle_T + (P_a \leftrightarrow Q_{ak}) + \mathcal{O}\left(\frac{1}{M}\right), \quad (4.13)$$

and, taking into account the Wick rotation of the time coordinate, the Euclidean correlator related to eq. (2.6) can be defined as

$$G_E(\tau) \equiv \frac{\beta}{3} \lim_{M \rightarrow \infty} \frac{M^2}{\chi^{00}} \int_{\mathbf{x}} \langle \partial^0 \hat{\mathcal{J}}^k(\tau, \mathbf{x}) \partial^0 \hat{\mathcal{J}}^k(0, \mathbf{0}) \rangle_T \Big|_{it \rightarrow \tau}. \quad (4.14)$$



**Figure 2.** Euclidean correlators corresponding to the scattering processes shown in figure 2. The solid (single or double-lined) circle represents the Euclidean time interval; open squares correspond to force operators; and  $P$  and  $Q$  stand for the pseudoscalar and vector fields, respectively (in particle language the  $Q$ -field is denoted by  $P^*$ ). The pion lines do *not* meet in process (c).

The subscript in  $G_E$  could refer to “electric”. In the subsequent steps we again omit hats, because eq. (4.14) can be evaluated through normal Euclidean path integrals.

The correlator of eq. (4.14) is periodic across the Euclidean time interval,  $G_E(\tau + k\beta) = G_E(\tau)$ ,  $k \in \mathbb{Z}$ , and can be Fourier analyzed in the Matsubara formalism. In particular, after a Fourier transform,

$$\tilde{G}_E(\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} G_E(\tau), \quad (4.15)$$

where  $\omega_n = 2\pi nT$ ,  $n \in \mathbb{Z}$ , the spectral function is obtained from an imaginary part [23]:

$$\rho_E(\omega) = \text{Im} \tilde{G}_E(\omega_n \rightarrow -i[\omega + i0^+]). \quad (4.16)$$

The momentum diffusion coefficient then follows from eq. (2.7).

The Euclidean action in eq. (4.12) implies that, for  $0 < |\tau - \sigma| < \beta$ , the free propagator is

$$\begin{aligned} & \langle P_a(\tau, \mathbf{x}) P_b^\dagger(\sigma, \mathbf{y}) \rangle_T^{(0)} \\ &= \frac{1}{2} \delta_{ab} \delta^{(3)}(\mathbf{x} - \mathbf{y}) T \sum_{\omega_n} \frac{e^{i\omega_n(\tau - \sigma)}}{i\omega_n + M_P} \\ &= \frac{1}{2} \delta_{ab} \delta^{(3)}(\mathbf{x} - \mathbf{y}) n_B(M_P) \left[ \theta(\sigma - \tau) e^{(\sigma - \tau)M_P} + \theta(\tau - \sigma) e^{(\beta - \tau + \sigma)M_P} \right], \end{aligned} \quad (4.17)$$

where  $n_B(M_P) \equiv 1/(e^{\beta M_P} - 1)$  is the Bose distribution. The propagator  $\langle Q_{ak} Q_{bl}^\dagger \rangle_T^{(0)}$  has the same structure, with an additional  $\delta_{kl}$ . The susceptibility of eq. (4.13) then evaluates to

$$\chi^{00} = \beta N_f \delta^{(3)}(\mathbf{0}) \left( e^{-\beta M_P} + 3e^{-\beta M_Q} \right) + \mathcal{O}\left(\frac{1}{F^2}, \frac{1}{M}\right), \quad (4.18)$$

where we also took the limit  $M_P, M_Q \gg T$ , replacing  $n_B(M)$  through  $\exp(-\beta M)$ .

### 4.3 Leading order

We now move on to consider the correlator  $G_E$ , defined by eq. (4.14), with the force inserted from eq. (4.11) and the susceptibility from eq. (4.18). Some of the relevant Feynman graphs

are shown in figure 2; we start by considering the process (a). It was already argued in connection with figure 1 that this should give no contribution to  $\kappa$  but, as a preparation for the next-to-leading order computation, we recall briefly how the vanishing can be seen.

Making use of eqs. (3.8), (4.8) and the 2nd and 3rd rows of eq. (4.11), the process in figure 2(a) corresponds to

$$\langle \partial^0 \mathcal{J}_k(\tau, \mathbf{x}) \partial^0 \mathcal{J}_k(0, \mathbf{0}) \rangle_T \sim \frac{g_\pi^2}{M^2 F^2} \left\langle [Q_a^\dagger \partial \partial \xi_{ba} P_b](\tau, \mathbf{x}) [P_c^\dagger \partial \partial \xi_{dc} Q_d](0, \mathbf{0}) \right\rangle_T, \quad (4.19)$$

where space-time indices have been omitted. Contracting the heavy-meson fields as in eq. (4.17), we note that there is one appearance of “forward” and one of “backward” propagation; in the imaginary-time formalism, these amount to circling the Euclidean time interval. If we also insert the Goldstone boson propagator from eq. (3.2), the result becomes

$$\int_{\mathbf{x}} \langle \partial^0 \mathcal{J}_k(\tau, \mathbf{x}) \partial^0 \mathcal{J}_k(0, \mathbf{0}) \rangle_T \sim \frac{g_\pi^2 (N_f^2 - 1)}{M^2 F^2} \delta^{(3)}(\mathbf{0}) n_B(M_P) n_B(M_Q) e^{(\beta-\tau)M_P} e^{\tau M_Q} \partial \partial \partial \Delta(\tau, \mathbf{0}), \quad (4.20)$$

where

$$\Delta(\tau, \mathbf{0}) = T \sum_{\omega_n} \int_{\mathbf{p}} \frac{e^{i\omega_n \tau + i\mathbf{p} \cdot \mathbf{0}}}{\omega_n^2 + E_p^2} = \int_{\mathbf{p}} \frac{n_B(E_p)}{2E_p} [e^{\tau E_p} + e^{(\beta-\tau)E_p}] e^{i\mathbf{p} \cdot \mathbf{0}}, \quad (4.21)$$

with  $E_p \equiv \sqrt{m_\pi^2 + p^2}$  and  $\int_{\mathbf{p}} \equiv \int d^3\mathbf{p}/(2\pi)^3$ . A Fourier transform (cf. eq. (4.15)) can now be taken:

$$\begin{aligned} & \int_0^\beta d\tau e^{i\omega_n \tau} e^{(\beta-\tau)M_P} e^{\tau M_Q} [e^{\tau E_p} \pm e^{(\beta-\tau)E_p}] \\ &= \frac{e^{\beta(M_Q+E_p)} - e^{\beta M_P}}{i\omega_n + M_Q - M_P + E_p} \pm \frac{e^{\beta M_Q} - e^{\beta(M_P+E_p)}}{i\omega_n + M_Q - M_P - E_p}. \end{aligned} \quad (4.22)$$

The subsequent discontinuity, eq. (4.16), turns these into energy conservation constraints:

$$\begin{aligned} \rho_E(\omega) &\propto \int_{\mathbf{p}} \text{Im} \left[ \frac{e^{\beta(M_Q+E_p)} - e^{\beta M_P}}{\omega + M_Q - M_P + E_p + i0^+} \pm \frac{e^{\beta M_Q} - e^{\beta(M_P+E_p)}}{\omega + M_Q - M_P - E_p + i0^+} \right] \\ &= -\pi \int_{\mathbf{p}} \left[ \delta(\omega + M_Q - M_P + E_p) e^{\beta E_p} \pm \delta(\omega + M_Q - M_P - E_p) \right] e^{\beta M_Q} (1 - e^{\beta\omega}), \end{aligned} \quad (4.23)$$

where the  $\delta$ -functions were made use of for rewriting the exponentials. (Using Hermitean conjugate operators in eq. (4.19) yields the same but with  $M_Q \leftrightarrow M_P$ .)

It can now be seen that the limit of eq. (2.7) can be non-trivial only if there are pion momenta with  $E_p = \pm(M_Q - M_P)$  (in fact this is a necessary but not a sufficient condition; in principle it could happen that the numerator of eq. (4.20), which has been left implicit here, vanishes at the same point, however this does not appear to be the case). This corresponds indeed to figure 1(a), and cannot be realized if  $|M_Q - M_P| < m_\pi$ , as is the case with  $B$ -mesons. Let us end by remarking that the same is expected to be the case also for many corrections of  $\mathcal{O}(g_\pi^2/F^4)$ , obtained from the topology of figure 2(a) by dressing either the pion propagator or the force operator by a closed “bubble”, which does not affect momentum flow.

#### 4.4 Next-to-leading order

Moving on to next-to-leading order, we need to consider graphs like (b) and (c) in figure 2. Due to the large number of terms proportional to  $g_\pi$  in eq. (4.11), as well as the possibly associated uncertainty as discussed below eq. (4.2), we simplify the task here by setting  $g_\pi \rightarrow 0$ , omitting thereby contributions of  $\mathcal{O}(g_\pi^2)$  and  $\mathcal{O}(g_\pi^4)$  from  $\kappa$ , which would otherwise be of the same order in the chiral expansion. Recalling that phenomenologically  $g_\pi \simeq 0.5$ , this cannot change the overall magnitude of the result. In this limit only the process (b) (as well as the same topology with  $P_a \rightarrow Q_{ak}$ ) is left over. Apart from “trivial” mass effects, spin symmetry implies that the terms containing  $P_a$  and  $Q_{ak}$  have identical structures at  $\mathcal{O}(g_\pi^0)$ . Therefore in the following only the part involving  $P_a$  is displayed explicitly.

As a first step we rewrite the relevant term of eq. (4.11):

$$\partial^0 \int_{\mathbf{x}} \mathcal{J}^k = \frac{2i}{M} \int_{\mathbf{x}} P_a^\dagger (\mathcal{D}_{ca}^0 \mathcal{D}_{bc}^k - \mathcal{D}_{ca}^k \mathcal{D}_{bc}^0) P_b + \mathcal{O}\left(\frac{g_\pi}{M}\right). \quad (4.24)$$

This can be contrasted with eq. (2.17) of ref. [7]: the colour-electric field strength of HQET has been replaced by a kind of “chiral-electric” field strength in HM $\chi$ PT.

Going then over to the Euclidean correlator of eq. (4.14), we obtain

$$G_E(\tau) = \frac{\beta}{3} \lim_{M \rightarrow \infty} \frac{4}{\chi_{00}} \int_{\mathbf{x}} \left\langle \left\{ P_a^\dagger [\mathcal{D}_\tau^T, \mathcal{D}_k^T]_{ab} P_b \right\}(\tau, \mathbf{x}) \left\{ P_c^\dagger [\mathcal{D}_\tau^T, \mathcal{D}_k^T]_{cd} P_d \right\}(0, \mathbf{0}) \right\rangle_T + \mathcal{O}(g_\pi^2). \quad (4.25)$$

Inserting the heavy meson propagator from eq. (4.17) and the susceptibility from eq. (4.18), this can be re-expressed as

$$G_E(\tau) = \frac{1}{3N_f} \left\langle [\mathcal{D}_\tau^T, \mathcal{D}_k^T]_{ab}(\tau, \mathbf{0}) [\mathcal{D}_\tau^T, \mathcal{D}_k^T]_{ba}(0, \mathbf{0}) \right\rangle_T + \mathcal{O}\left(\frac{g_\pi^2}{F^4}, \frac{1}{F^6}\right). \quad (4.26)$$

With the contribution of  $Q_{ak}$  added as has already been done here, the Boltzmann weights (cf. eq. (4.18)) have duly cancelled between the numerator and the denominator.

Having obtained eq. (4.26), the problem has reduced to one within normal Chiral Perturbation Theory, eq. (3.1). Noting from eqs. (3.8), (4.7) that

$$[\mathcal{D}_\tau^T, \mathcal{D}_k^T]_{ab} = i(\partial_\tau \mathcal{V}_k - \partial_k \mathcal{V}_\tau)_{ba} + \mathcal{O}\left(\frac{1}{F^4}\right) \quad (4.27)$$

$$= \frac{1}{F^2} (\partial_k \xi \partial_\tau \xi - \partial_\tau \xi \partial_k \xi)_{ba} + \mathcal{O}\left(\frac{1}{F^4}\right), \quad (4.28)$$

and inserting the propagator from eq. (3.2), we obtain after some algebra that

$$G_E(\tau) = \frac{N_f^2 - 1}{6F^4} \left[ \partial_\tau \partial_k \Delta(\tau, \mathbf{0}) \partial_\tau \partial_k \Delta(\tau, \mathbf{0}) - \partial_\tau^2 \Delta(\tau, \mathbf{0}) \nabla^2 \Delta(\tau, \mathbf{0}) \right] + \mathcal{O}\left(\frac{g_\pi^2}{F^4}, \frac{1}{F^6}\right). \quad (4.29)$$

Employing  $\Delta(\tau, \mathbf{0})$  from eq. (4.21), the first term of eq. (4.29) is seen not to contribute, because the  $\mathbf{p}$ -integrand is odd in  $\mathbf{p} \rightarrow -\mathbf{p}$ . The second term of eq. (4.29) does contribute; the Fourier transform (cf. eq. (4.15)) can be carried out and, apart from a contact term

$\propto \delta(\tau)$ , yields a sum of four terms, with  $\omega_n$ -dependence in structures of the type  $\sim 1/(i\omega_n \pm E_p \pm E_q)$ , in analogy with eq. (4.22). The spectral function (cf. eq. (4.16)) is obtained by replacing these with  $-\pi\delta(\omega \pm E_p \pm E_q)$ , in analogy with eq. (4.23). Thereby we end up with

$$\begin{aligned} \rho_E(\omega) &= \frac{\pi(N_f^2 - 1)}{6F^4} \int_{\mathbf{p}, \mathbf{q}} \frac{p^2 E_q}{4E_p} \\ &\times \left\{ \left[ 1 + n_B(E_p) + n_B(E_q) \right] \left[ \delta(\omega - E_p - E_q) - \delta(\omega + E_p + E_q) \right] \right. \\ &\quad \left. + \left[ n_B(E_p) - n_B(E_q) \right] \left[ \delta(\omega + E_p - E_q) - \delta(\omega - E_p + E_q) \right] \right\} \\ &+ \mathcal{O}\left(\frac{g_\pi^2}{F^4}, \frac{1}{F^6}\right). \end{aligned} \tag{4.30}$$

The final step is to take the limit defined in eq. (2.7). Due to vanishing phase space, only the second structure of eq. (4.30) gives a contribution linear in  $\omega$  at  $\omega \rightarrow 0$ ; this corresponds to the process in figure 1(b) in the limit of vanishing energy transfer. The terms linear in  $\omega$  can be extracted by using the  $\delta$ -functions to rewrite the arguments of the Bose distributions, and by then Taylor-expanding the latter. There are two terms, amounting to a symmetrization  $\mathbf{p} \leftrightarrow \mathbf{q}$ ; afterwards we can set  $E_q \rightarrow E_p$ . Given that  $n'_B(E) = -\beta n_B(E)[1 + n_B(E)]$ , this yields

$$\begin{aligned} \kappa &= \frac{\pi(N_f^2 - 1)}{3F^4} \int_{\mathbf{p}, \mathbf{q}} \frac{p^2 + q^2}{4} n_B(E_p) \left[ 1 + n_B(E_p) \right] \delta(E_p - E_q) + \mathcal{O}\left(\frac{g_\pi^2}{F^4}, \frac{1}{F^6}\right) \\ &= \frac{(N_f^2 - 1)T}{24\pi^3 F^4} \int_0^\infty dp^2 p^2 (3p^2 + 2m_\pi^2) n_B(E_p) + \mathcal{O}\left(\frac{g_\pi^2}{F^4}, \frac{1}{F^6}\right), \end{aligned} \tag{4.31}$$

where a partial integration was carried out. In the limit  $\pi T \gg m_\pi$  the integral can be performed explicitly and the result reads

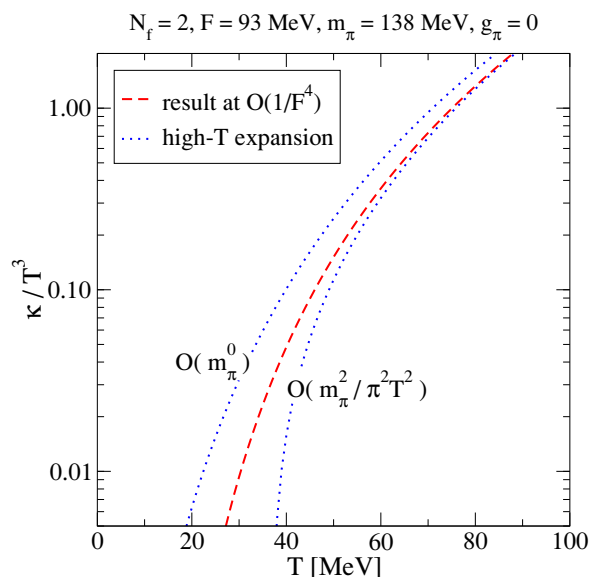
$$\kappa \stackrel{\pi T \gg m_\pi}{\approx} \frac{2(N_f^2 - 1)\pi^3 T^7}{63F^4} \left[ 1 - \frac{7m_\pi^2}{10\pi^2 T^2} + \mathcal{O}\left(\frac{m_\pi^4}{\pi^4 T^4}\right) \right] + \mathcal{O}\left(\frac{g_\pi^2 T^7}{F^4}, \frac{T^9}{F^6}\right). \tag{4.32}$$

A numerical evaluation is shown in figure 3. This constitutes our final result.

## 5 Conclusions and outlook

The purpose of this paper has been to estimate the overall magnitude of the momentum diffusion coefficient,  $\kappa$ , of a heavy-light pseudoscalar meson almost at rest with respect to a heat bath, which has a temperature in the range of some tens of MeV. The result, eq. (4.31) and figure 3, is given in terms of low-energy constants of two-flavour Chiral Perturbation Theory.

Although the result obtained is not immediately applicable to heavy ion collision experiments, in which the initial temperature may be in the range of hundreds of MeV and much of heavy quark energy loss could take place in the deconfined phase, the hope is that the analysis is nevertheless of theoretical interest. After all, physical QCD is believed to



**Figure 3.** An illustration of  $\kappa/T^3$  as a function of the temperature. The dashed line shows the result from eq. (4.31); the dotted lines show two orders of the high-temperature expansion from eq. (4.32). At high temperatures,  $T \gtrsim 80$  MeV, the chiral expansion breaks down because higher order corrections of relative magnitude  $\mathcal{O}(T^2/F^2)$  become large; at low temperatures,  $T \lesssim 30$  MeV, the high-temperature expansion breaks down because the expansion parameter  $\mathcal{O}(m_\pi^2/\pi^2 T^2)$  exceeds unity (the unexpanded result of eq. (4.31) remains valid but is exponentially small). In the deconfined phase a value  $\kappa/T^3 \gtrsim 2$  might be phenomenologically acceptable (see e.g. ref. [15]; an oft-cited “diffusion coefficient” is  $D \simeq 2T^2/\kappa \lesssim 1/T$ ). The weak-coupling expansion suggests in general values  $\kappa/T^3 \lesssim 2$  [6], and at very high temperatures  $\kappa/T^3$  decreases like  $\sim 1/\ln^2(T/T_0)$ .

possess a crossover rather than a genuine phase transition between low and high temperatures, so that analyses in the former regime may yield qualitative information also for the latter. More concretely, our analysis combined with existing weak-coupling computations [6] suggest a picture in which the momentum diffusion coefficient exhibits a broad peak around the QCD crossover. So, heavy quark jets produced in an initial hard process may continue to approach kinetic equilibrium all the way until their final decoupling within the hadronic phase.

Despite the qualitative nature of our study, it seems that a number of potentially interesting extensions can be envisaged. Most obviously, the result for  $\kappa$  of  $B$ -mesons could be completed with terms of  $\mathcal{O}(g_\pi^2)$  and  $\mathcal{O}(g_\pi^4)$ . Although no qualitative changes are expected, the corrections may be numerically important, perhaps in the range of  $\sim 50\%$ . In addition the analysis appears formally interesting: as discussed in the text, it should probably be carried out both within the non-relativistic theory as well as within a relativistic extension thereof, taking the non-relativistic limit only afterwards in the latter case, in order to have a crosscheck.

Another interesting topic is the “extension” of the study to  $D$ -mesons. This immediately leads to the dramatic effect that resonant contributions, such as the process shown in figure 1(a), are allowed, whereby the  $D$ -mesons can change their identity (a further com-

plication is that for  $D^*$  decays electromagnetic processes are important). The handling of the related rich physics may suggest an excursion away from systematic computations, into the realm of hadronic models; an overview of recent works in this direction can be found e.g. in ref. [40]. Nevertheless, it can be noted that resonant phenomena could also play a role in theoretical considerations of the  $B$ -sector, if we took the chiral limit  $m_\pi \rightarrow 0$  while keeping the mass difference  $M_{B^*} - M_B$  fixed and non-zero.

Yet another line is that whereas only the “intercept”,  $\kappa = \lim_{\omega \rightarrow 0} 2T\rho_E(\omega)/\omega$ , was addressed here, the full spectral function  $\rho_E(\omega)$  (eq. (4.16)), and even the Euclidean correlator  $G_E(\tau)$  (eq. (4.14)), are also interesting objects. For instance, they may help in the interpretation of the corresponding lattice measurements *à la* ref. [19] (if these were unquenched); it is in this spirit that both functions have been computed also in the deconfined phase [18], and in fact  $G_E(\tau)$  even in the confined phase in the presence of a large lattice spacing, through the use of the lattice strong-coupling expansion [18]. The terms of  $\mathcal{O}(g_\pi^2/F^2)$  (cf. figure 2(a)) *do* contribute significantly to  $G_E(\tau)$ , even if they do not contribute to  $\kappa$  as has been discussed in the text; it might turn out to be useful to gain understanding on how hidden the information about  $\kappa$  is in the directly measurable  $G_E(\tau)$ .

A further topic is not to consider the spectral function  $\rho_E(\omega)$  yielding  $\kappa$ , but rather the *full* spectral function  $\rho(\omega, \mathbf{k}) \equiv \int_{t, \mathbf{x}} e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}} \langle \frac{1}{2} [\hat{\mathcal{J}}^\mu(t, \mathbf{x}), \hat{\mathcal{J}}_\mu(0, \mathbf{0})] \rangle_T$ , of which  $\rho_E(\omega)$  is a specific limit [7]. Restricting first to vanishing momentum,  $\mathbf{k} = \mathbf{0}$ , and to small frequencies, the analysis can still be carried out within the non-relativistic framework. This spectral function is expected to show a narrow and prominent “transport peak” at frequencies  $|\omega| \lesssim \eta_D \sim \kappa/(2TM)$ . A direct analysis of such infrared features tends to be difficult, necessitating complicated resummations (see e.g. ref. [22]), but at least the theory in question is not a gauge theory, whereby these may be more tractable than in the deconfined phase.

The most ambitious goal would be to consider  $\rho(\omega, \mathbf{k})$  also for  $\mathbf{k} \neq \mathbf{0}$ , perhaps even for  $|\mathbf{k}| \gtrsim M$ , corresponding to heavy mesons moving at a relativistic speed with respect to a pionic plasma. Although this could take us away from the range of validity of systematic  $\text{HM}\chi\text{PT}$ , simple relativistic extensions can be written down (see e.g. refs. [37, 39]). Within such a framework one might also try to understand phenomena such as radiative energy loss, perhaps making contact with classic pion gas computations that have recently been revived through the AdS/CFT setup [41].

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**Note added.** After the submission of this paper three works appeared [42–44] in which the kinetic equilibration of  $D$ -mesons is studied in a spirit similar to that suggested in section 5.



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